Discrete Mathematics 347 (2024) 114153

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On a property of 2-connected graphs and Dirac's Theorem

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ARTICLE INFO

Article history:
Received 15 December 2023
Accepted 29 June 2024
Available online 14 July 2024

Keywords:
Berge cycles
Extremal hypergraph theory
Minimum degree

ABSTRACT

We refine a property of 2-connected graphs described in the classical paper of Dirac from 1952 and use the refined property to somewhat shorten Dirac's proof of the fact that each 2-connected n-vertex graph with minimum degree at least k has a cycle of length at least $\min\{n, 2k\}$.

1. Introduction

One of the basic facts on 2-connected graphs is their characterization by Whitney [5] from 1932:

Theorem 1 *(Whitney [5]). A graph G with at least 3 vertices is 2-connected if and only if for any distinct $u, v \in V(G)$ there are internally disjoint u, v-paths.*

Given paths P and P' with the common origin in a graph, we say P' is *aligned with* P if for all $u, v \in V(P) \cap V(P')$ if u appears before v in P, then u also appears before v in P'.

In his thesis [1] and classical paper [2], Dirac refined (the main part of) Theorem 1 as follows.

Lemma 2 *(Dirac, Lemma 2 in [2]). If x and y are two distinct vertices of a graph without cut vertices, and if W is any given path connecting x and y, then the graph contains two paths connecting x and y and having the following properties: (i) they are internally disjoint; (ii) each of them is aligned with W.*

He also says that this lemma can be further refined as follows.

Corollary 3 *(Dirac, Corollary on p.73 in [2]). If x is adjacent to a vertex z of W, then the graph contains two paths connecting x and y such that they have the properties (i) and (ii), and one of them goes through z.*

Dirac used this corollary to prove the following famous theorem:

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1 Research is supported in part by NSF Grant DMS-2153507.
2 Research is supported in part by NSF RTG grant DMS-1937241.

https://doi.org/10.1016/j.disc.2024.114153

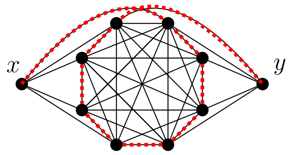

Fig. 1. A 5-connected construction.

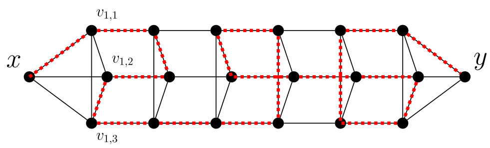

Fig. 2. Construction H_3.

Theorem 4 *(Dirac, Theorem 4 in [2]). Let $n > k \geq 2$. Each 2-connected n-vertex graph with minimum degree at least k has a cycle of length at least* $\min\{n, 2k\}$.

Pósa [4] used Lemma 2 to derive an extension of Theorem 4. In this note, we refine Corollary 3 (with an almost the same proof) as follows.

Lemma 5. *Let P be an x, y-path in a 2-connected graph G, and let $z \in V(G)$. Then there exists an x, z-path P_1 and an x, y-path P_2 such that*

(a) P_1 and P_2 are internally disjoint, and (b) each of P_1 and P_2 is aligned with P.

We then use this lemma to give a somewhat shorter and logically simpler proof of Theorem 4.

Remark 1. The authors [3] used Lemma 5 to prove an analog of Theorem 4 for Berge cycles in r-uniform 2-connected hypergraphs.

Remark 2. Douglas West pointed out how to easily derive Lemma 5 from Lemma 2. We still present the full proof in order to show a shorter proof from scratch for Theorem 4.

Remark 3. The straightforward generalization of Lemma 5 or Lemma 2 to k-connected graphs is not true. In fact, for *each* positive integer k, there exists a k-connected graph G and an x, y-path P such that G has no 3 x, y-paths aligned with P.

Set $A = \{a_1, \dots, a_{k-1}\}$ and $B = \{b_1, \dots, b_{k-1}\}$, and let G_k be the graph with vertex set $A \cup B \cup \{x, y\}$ such that $G_k[A \cup B]$ induces a clique, $N(x) = A \cup \{b_1\}$ and $N(y) = B \cup \{a_1\}$. Define the path $P = x, b_1, b_2, \dots, b_{k-1}, a_{k-1}, a_{k-2}, \dots, a_1, y$. Observe that G_k is k-connected and any x, y-path aligned in P in G must use edge xb_1 or edge a_1y. Hence G_k has at most 2 internally disjoint paths aligned with P. Fig. 1 displays the construction for $k = 5$.

Remark 4. Let x and y be distinct vertices in a graph G. If G is 2-connected, then Lemma 2 provides two internally disjoint x, y-paths aligned with any fixed x, y-path P in G. If G is k-connected for some $k \geq 3$, then Menger's Theorem guarantees k internally disjoint x, y-paths. A natural question is: Given $k \geq 3$, a k-connected graph G and an x, y-path P in G, how many of the k paths in Menger's Theorem always can be chosen to be aligned with P?

Somewhat surprisingly, the answer is "zero". We construct an example H_3 for $k = 3$ is as follows (see Fig. 2). For $1 \leq i \leq 6$, let $W_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$. Let $V(H_3) = \{x, y\} \cup \bigcup_{i=1}^{6} W_i$. For $1 \leq j \leq 3$, let $P_j = x, v_{1,j}, v_{2,j}, \dots, v_{6,j}, y$. The edge set of H_3 contains the edges of these three paths plus $H_3[W_i] = K_3$ for all $1 \leq i \leq 6$. By construction, H_3 is 3-connected, and the only triple of internally disjoint x, y-paths in H_3 is $\{P_1, P_2, P_3\}$. But none of these paths is aligned with the x, y-path

$$P_0 = x, v_{1,1}, v_{2,1}, v_{2,2}, v_{1,2}, v_{1,3}, v_{2,3}, v_{3,3}, v_{4,3}, v_{4,1}, v_{3,1}, v_{3,2}, v_{4,2}, v_{5,2}, v_{6,2}, v_{6,3}, v_{5,3}, v_{5,1}, v_{6,1}, y :$$

the edge $v_{4,1}v_{3,1}$ in P_0 is opposite to an edge in P_1, the edge $v_{2,2}v_{1,2}$ is opposite to an edge in P_2 and the edge $v_{6,3}v_{5,3}$ is opposite to an edge in P_3.

The examples for $k \geq 4$ are very similar.

Remark 5. In the statement of Lemma 5, we cannot replace an x, y-path P_2 with an x, z'-path for an arbitrary z': If $n \geq 5$, G is an n-cycle $v_1, v_2, \dots, v_n, v_1$ and $P = v_1, v_2, \dots, v_n$, then G has no two v_1, v_3-paths both aligned with P.

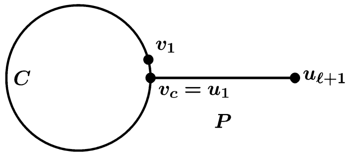

Fig. 3. A lollipop (C, P).

Remark 6. After the proof of Theorem 4, Dirac [2] thanks a referee for simplifying his original proof. Bjarne Toft[3] suggests that this referee possibly was Harold A. Stone.

2. Proofs

We view paths as having one of the two possible orientations. For a path P and $u, v, w \in V(P)$, let $P[u, v]$ denote the subpath of P from u to v, and let $P^+(w)$ denote the part of P starting from w.

2.1. Proof of Lemma 5

We induct on $|V(P)|$. The base case $P = x, y$ follows from the fact that the connected graph $G - xy$ has an x, y-path P_1, so we can take $P_2 = P$.

Induction step: Let the lemma hold for all paths with fewer than s vertices and let $P = v_1, v_2, \ldots, v_s$ with $x = v_1$ and $y = v_s$. By the induction hypothesis, the lemma holds for $P' = v_2, v_3, \ldots, v_s$ with $x' = v_2$ and $y' = v_s$. Let P'_1 and P'_2 be the corresponding v_2, z-path and v_2, v_s-path, respectively.

Case 1. $v_1 \in P'_i$ for some $i \in \{1, 2\}$. Then let $P_i = (P'_i)^+(v_1)$, and let P_{3-i} be obtained P'_{3-i} by adding edge v_1v_2 at the start.

Case 2. $v_1 \notin P'_i$ for $i = 1, 2$. Since G is 2-connected, $G - v_2$ has a path from v_1 to $P \cup P'_1 \cup P'_2$. Let Q be a shortest such path and u be the end of Q distinct from v_1.

If $u \in P'_i$, then replace P'_i with $Q, (P'_i)^+(u)$, and then extend P'_{3-i} by adding v_1v_2. Suppose now that $u \in P - P'_1 - P'_2$, say $u = v_j$. Choose the minimum $j' \geq j$ such that $v_{j'} \in P'_1 \cup P'_2$. It exists, since $v_s \in P_2$, say $v_{j'} \in P'_i$. In this case, let $P_i = Q, P[v_j, v_{j'}], (P'_i)^+(v_{j'})$, and $P_{3-i} = v_1, v_2, P'_{3-i}$. This proves the lemma.

2.2. Proof of Theorem 4

Suppose graph G is a counter-example to the theorem and its minimum degree, $\delta(G)$, is k. Since G is 2-connected, $k \geq 2$.

A *lollipop* in G is a pair (C, P) where C is a cycle and P is a path such that $V(C) \cap V(P)$ is one vertex that is an end of P (see Fig. 3). A lollipop (C, P) is *better than* a lollipop (C', P') if $|V(C)| > |V(C')|$ or $|V(C)| = |V(C')|$ and $|V(P)| > |V(P')|$.

Let (C, P) be a best lollipop in G. For definiteness, let $C = v_1, v_2, \ldots, v_c, v_1$ and $P = u_1, u_2, \ldots, u_{\ell+1}$, where $u_1 = v_c$. Since G is a counterexample, $c < n$. So, since G is 2-connected, $\ell \geq 1$.

By the maximality of ℓ, $N(u_{\ell+1}) \subseteq V(P) \cup V(C)$.

Case 1: There is $v_i \in N(u_{\ell+1})$. If $1 \leq i \leq \ell$, then the cycle $v_i, v_{i+1}, \ldots, v_c, u_2, u_3, \ldots, u_{\ell+1}, v_i$ is longer than C, a contradiction. Thus $i \geq \ell + 1$. Symmetrically, $i \leq c - \ell - 1$. On the other hand, if $N(u_{\ell+1}) \cap V(C)$ contains two consecutive vertices v_i and v_{i+1}, then replacing edge v_iv_{i+1} in C with the path $v_i, u_{\ell+1}, v_{i+1}$, we again get a cycle longer than C. Since $|N(u_{\ell+1}) \cap V(P)| \leq \ell$, we get

$$k \leq d(u_{\ell+1}) = |N(u_{\ell+1}) \cap V(P)| + |N(u_{\ell+1}) \cap V(C) - v_c| \leq \ell + \left\lceil \frac{c-1-2\ell}{2} \right\rceil = \left\lceil \frac{c-1}{2} \right\rceil < k,$$

a contradiction.

Case 2: $N(u_{\ell+1}) \subseteq V(P)$. Let $N(u_{\ell+1}) = \{u_{j_1}, \ldots, u_{j_s}\}$ with $j_1 < j_2 < \ldots, < u_s$. Let $P' = v_1, v_2, \ldots, v_c, u_2, \ldots, u_{\ell+1}$, $x = v_1$, $y = u_{\ell+1}$ and $z = u_{j_1}$. By Lemma 5 for these P', x, y and z, there exists an x, z-path P_1 and an x, y-path P_2 that are internally disjoint and aligned with P'.

For $h \in \{1, 2\}$, let a_h be the last vertex of P_h in C and b_h be the first vertex of P_h in $Y = \{u_{j_1}, u_{j_1+1}, \ldots, u_{\ell+1}\}$. Since P_1 is aligned with P', $b_1 = u_{j_1}$.

If $a_2 = a_1$, then since P_1 and P_2 are internally disjoint, $a_2 = a_1 = v_1$, and one of P_1 and P_2, say P_h, does not contain $v_c (= u_1)$ and first intersects P at some vertex u_i with $i \geq 2$. Then deleting from C edge v_1v_c and adding instead paths $P_h[v_1, u_i]$ and $P[u_1, u_i]$, we get a longer cycle, a contradiction. Thus, $P_1[a_1, b_1]$ and $P_2[a_2, b_2]$ are disjoint.

[3] Private communication.

Let Q be the longer of the two subpaths of C connecting a_1 with a_2. Then $|V(Q)| \geq 1 + \lceil \frac{c}{2} \rceil$. Let $b_2 = u_j$. Since P_1 and P_2 are internally disjoint, $j > j_1$. Let j' be the largest index in $\{j_1, \dots, j_s\}$ that is less than j. Since $j > j_1$, j' is well defined and $j' \geq j_1$.

Consider the closed walk

$$C' = a_2, Q, a_1, P_1[a_1, u_{j_1}], u_{j_1}, u_{j_1+1}, \dots, u_{j'}, u_{\ell+1}, u_\ell, \dots, u_j(= b_2), P_2[b_2, a_2], a_2.$$

Since $P_1[a_1, u_{j_1}]$ and $P_2[b_2, a_2]$ are disjoint and both are internally disjoint from $V(C) \cup Y$, C' is a cycle. It has at least $|V(Q)| \geq 1 + \lceil \frac{c}{2} \rceil = 1 + c - \lfloor \frac{c}{2} \rfloor$ vertices in C and at least $1 + d(u_{\ell+1}) \geq 1 + k$ vertices in P. Since $|V(C) \cap V(P)| = 1$, we have

$$|V(C')| \geq (1 + c - \left\lfloor \frac{c}{2} \right\rfloor) + (1 + k) - 1 > c.$$

This contradiction with the maximality of C proves the theorem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

We thank Bjarne Toft and Douglas West for their valuable comments.

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