A Hypergraph Analog of Dirac's Theorem for Long Cycles in 2-Connected graphs, II: Large Uniformities

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Abstract

Dirac proved that each n-vertex 2-connected graph with minimum degree k contains a cycle of length at least $\min\{2k, n\}$. We obtain analogous results for Berge cycles in hypergraphs. Recently, the authors proved an exact lower bound on the minimum degree ensuring a Berge cycle of length at least $\min\{2k, n\}$ in n-vertex r-uniform 2-connected hypergraphs when $k \geqslant r + 2$. In this paper we address the case $k \leqslant r + 1$ in which the bounds have a different behavior. We prove that each n-vertex r-uniform 2-connected hypergraph H with minimum degree k contains a Berge cycle of length at least $\min\{2k, n, |E(H)|\}$. If $|E(H)| \geqslant n$, this bound coincides with the bound of the Dirac's Theorem for 2-connected graphs.

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1 Introduction and Results

1.1 Definitions and known results for graphs

A hypergraph H is a family of subsets of a ground set. We refer to these subsets as the *edges* of H and to the elements of the ground set as the *vertices* of H. We use $E(H)$ and $V(H)$ to denote the set of edges and the set of vertices of H respectively. We say H is *r-uniform* (r-graph, for short) if every edge of H contains exactly r vertices. A *graph* is a 2-graph.

The *degree* $d_H(v)$ of a vertex v in a hypergraph H is the number of edges containing v. The *minimum degree*, $\delta(H)$, is the minimum over degrees of all vertices of H.

A *hamiltonian cycle* in a graph is a cycle which visits every vertex. Degree conditions providing that a graph has a hamiltonian cycle were first studied by Dirac in 1952.

Theorem 1 (Dirac [4])**.** *Let $n \geqslant 3$. If G is an n-vertex graph with minimum degree $\delta(G) \geqslant n/2$, then G has a hamiltonian cycle.*

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In the same paper, Dirac observed that if $\delta(G) \geqslant 2$, then G has a cycle of length at least $\delta(G)+1$. This is best possible: examples are many copies of $K_{\delta(G)+1}$ sharing a single vertex. Each of these graphs has a cut vertex. Dirac [4] showed that each 2-connected graph has a much longer cycle.

Theorem 2 (Dirac [4])**.** *Let $n \geqslant k \geqslant 2$. If G is an n-vertex, 2-connected graph with minimum degree $\delta(G) \geqslant k$, then G has a cycle of length at least* $\min\{2k, n\}$*.*

Both theorems are sharp. For example, when $2 \leqslant k \leqslant n/2$, the complete bipartite graph $K_{k-1,n-(k-1)}$ has minimum degree $k-1$ and the length of a longest cycle $2k-2$. Note that $K_{k-1,n-(k-1)}$ is $(k-1)$-connected, so for large k demanding 3-connectedness instead of 2-connectedness in Theorem 2 will not strengthen the bound.

Stronger statements for bipartite graphs have been proved by Voss and Zuluaga [24], and then refined by Jackson [16]:

Theorem 3 (Jackson [16])**.** *Let G be a 2-connected bipartite graph with bipartition (A, B), where $|A| \geqslant |B|$. If each vertex of A has degree at least a and each vertex of B has degree at least b, then G has a cycle of length at least* $2\min\{|B|, a+b-1, 2a-2\}$*. Moreover, if $a = b$ and $|A| = |B|$, then G has a cycle of length at least* $2\min\{|B|, 2a-1\}$*.*

A sharpness example for Theorem 3 is a graph $G_3 = G_3(a, b, a', b')$ for $a' \geqslant b' \geqslant a+b-1$ obtained from disjoint complete bipartite graphs $K_{a'-b,a}$ and $K_{b,b'-a}$ by joining each vertex in the a part of $K_{a'-b,a}$ to each vertex in the b part of $K_{b,b'-a}$.

1.2 Definitions and known results for uniform hypergraphs

We consider *Berge cycles* in hypergraphs.

Definition 4. A **Berge cycle of** length c in a hypergraph is an alternating list of c distinct vertices and c distinct edges $C = v_1, e_1, v_2, \ldots, e_{c-1}, v_c, e_c, v_1$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leqslant i \leqslant c$ (we always take indices of cycles of length c modulo c). We call vertices $v_1, \ldots, v_c$ **the defining vertices** of C and the pairs $v_1v_2, v_2v_3, \ldots, v_cv_1$ **the defining edges** of C. Given some edge $e_i \in E(C)$, we also call the defining edge v_iv_{i+1} the **the projection of** e_i. We write $V(C) = \{v_1, \ldots, v_c\}$, $E(C) = \{e_1, \ldots, e_c\}$.

Notation for Berge paths is similar. In addition, a **partial Berge path** is an alternating sequence of distinct edges and vertices beginning with an edge and ending with a vertex $e_0, v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ such that $v_1 \in e_0$ and for all $1 \leqslant i \leqslant k$, $\{v_i, v_{i+1}\} \subseteq e_i$.

The *circumference*, $c(H)$, of a (hyper)graph H is the length of a longest (Berge) cycle in H.

A series of variations of Theorem 1 for Berge cycles in a number of classes of r-graphs were obtained by Bermond, Germa, Heydemann and Sotteau [1], Clemens, Ehrenmüller and Person [2], Coulson and Perarnau [3], Ma, Hou, and Gao [22], the present authors [18], and Salia [23].

In particular, exact bounds for all values of $3 \leqslant r < n$ are as follows.

Theorem 5 (Theorem 1.7 in [18])**.** *Let $t = t(n) = \lfloor \frac{n-1}{2} \rfloor$, and suppose $3 \leqslant r < n$. Let H be an r-graph. If (a) $r \leqslant t$ and $\delta(H) \geqslant \binom{t}{r-1} + 1$ or (b) $r \geqslant n/2$ and $\delta(H) \geqslant r$, then H contains a hamiltonian Berge cycle.*

For an analog of Theorem 2, we define the connectivity of a hypergraph with the help of its *incidence bipartite graph*:

Definition 6. Let H be a hypergraph. The **incidence graph $\boldsymbol{I_H}$ of $\boldsymbol{H}$** is the bipartite graph with $V(I_H) = X \cup Y$ such that $X = V(H), Y = E(H)$ and for $x \in X, y \in Y$, $xy \in E(I_H)$ if and only if the vertex x is contained in the edge y in H.

It is easy to see that if H is an r-graph with minimum degree $\delta(H)$, then each $x \in X$ and each $y \in Y$ satisfy $d_{I_H}(x) \geqslant \delta(H), d_{I_H}(y) = r$. Furthermore, there is a natural bijection between the set of Berge cycles of length c in H and the set of cycles of length $2c$ in I_H: such a Berge cycle $v_1, e_1, \dots, v_c, e_c, v_1$ can also be viewed as a cycle in I_H with the same sequence of vertices.

Using the notion of the incidence graph, we define connectivity in hypergraphs.

Definition 7. A hypergraph H is ***k*-connected** if its incidence graph I_H is a k-connected graph.

Theorem 3 of Jackson applied to I_H of a 2-connected r-graph H yields the following approximation of an analog of Theorem 2 for $k \leqslant r - 1$:

Corollary 8. *Let n, k, r be positive integers with $2 \leqslant k \leqslant r - 1$. If H is an n-vertex 2-connected r-graph H with $\delta(H) \geqslant k + 1$, then $c(H) \geqslant \min\{2k, n, |E(H)|\}$.*

The following construction shows that the bound of Corollary 8 is not far from exact.

Construction 1.1. For $m \geqslant 2$, we let $V(H_k) = A_1 \cup \ldots \cup A_m \cup \{x, y\}$ where $A_i = \{a_{i,1}, \dots, a_{i,r-1}\}$ for $1 \leqslant i \leqslant m$, and let $E(H_k) = E_1 \cup \ldots \cup E_m$ where for each $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k - 1$, $E_i = \{e_{i,1}, \dots, e_{i,k-1}\}$ and $e_{i,j} = (A_i - a_{i,j}) \cup \{x, y\}$. By construction, H_k is 2-connected and $\delta(H_k) = k - 2$. Each Berge cycle in H_k can contain edges from at most two E_is, and $|E_i| = k - 1$ for all $1 \leqslant i \leqslant m$. So, $c(H_k) = 2k - 2$.

Very recently, the authors [21] proved an exact analog of Theorem 2 for r-graphs when $k \geqslant r + 2$:

Theorem 9. *Let n, k, r be positive integers with $3 \leqslant r \leqslant k - 2 \leqslant n - 2$. If H is an n-vertex 2-connected r-graph with*

$$\delta(H) \geqslant \binom{k-1}{r-1} + 1, \tag{1}$$

then $c(H) \geqslant \min\{2k, n\}$.

Observe that the minimum degree required to guarantee a Berge cycle of length at least $2k$ in a 2-connected r-graph is roughly of the order $2^{r-1}/r$ times smaller than the sharp bound guaranteed in Theorem 5(b). The following constructions show the sharpness of Theorem 9.

Construction 1.2. Let $q \geqslant 2$ be an integer and $4 \leqslant r+1 \leqslant k \leqslant n/2$. For $n = q(k-2)+2$, let $H_1 = H_1(k)$ be the r-graph with $V(H_1) = \{x,y\} \cup V_1 \cup V_2 \cup \ldots \cup V_q$ where for all $1 \leqslant i \leqslant q$, $|V_i| = k-2$ and $V_i \cup \{x,y\}$ induces a clique. Thus $c(H_1) \leqslant 2(k-2)+2 = 2k-2$.

Construction 1.3. Let $4 \leqslant r+1 \leqslant k \leqslant n/2$. Let $H_2 = H_2(k)$ be the r-graph with $V(H_2) = X \cup Y$ where $|X| = k-1$, $|Y| = n-(k-1)$, and $E(H_2)$ is the set of all hyperedges containing at most one vertex in Y. No Berge cycle can contain consecutive vertices in Y, so $c(H_2) \leqslant 2k-2$.

One can check that both H_1 and H_2 have minimum degree $\binom{k-1}{r-1}$ and H_2 is $(k-1)$-connected and is well defined for all $n \geqslant k$.

It was also proved in [21] that for $3 \leqslant r < n$ and every n-vertex 2-connected r-graph H, $c(H) \geqslant \min\{4, n, |E(H)|\}$.

1.3 The main result and structure of the paper

The ideas of [21] were insufficient to prove exact results for $3 \leqslant k \leqslant r+1$, because in this case the conditions providing a cycle of length at least $2k$ are weaker. For $3 \leqslant k \leqslant r+1$, the restrictions on the minimum degree are linear in k while for $k \geqslant r+2$ they are at least quadratic. Our main result is

Theorem 10. *Let n, k, r be positive integers with $3 \leqslant k \leqslant r+1 \leqslant n$. If H is an n-vertex 2-connected r-graph with*

$$\delta(H) \geqslant k, \tag{2}$$

then $c(H) \geqslant \min\{2k, n, |E(H)|\}$.

Note that under the conditions of the above theorem it might occur that $|E(H)| < \min\{2k, n\}$, which could not happen in Theorem 9. The bound of the theorem coincides with the bound of Theorem 2 for 2-connected graphs when $|E(H)| \geqslant \min\{2k, n\}$. It is sharp when $k = 3$ and when $k = r+1$. For $4 \leqslant k \leqslant r$ the theorem improves the bound of Corollary 8, and Construction 1.1 shows that the bound either is exact or differs from the exact by 1.

The proof of our main result, Theorem 10 is by contradiction. We consider a counterexample H to the theorem and study its properties to show that such an example cannot exist. In Section 2 we introduce notation, define special substructures of H, so called *lollipops* and *disjoint cycle-path pairs*, and define when a structure in H is *better* than another structure. In these terms, we explain the structure of the paper in more detail and state our main lemmas. In Section 3 we derive some properties of "good" lollipops and disjoint cycle-path pairs. In the remaining four sections we prove the four main lemmas stated in Section 2.

2 Notation and setup

For a hypergraph H, and a vertex $v \in V(H)$,

$$N_H(v) = \{u \in V(H) : \text{there exists } e \in E(H) \text{ such that } \{u,v\} \subset e\}$$

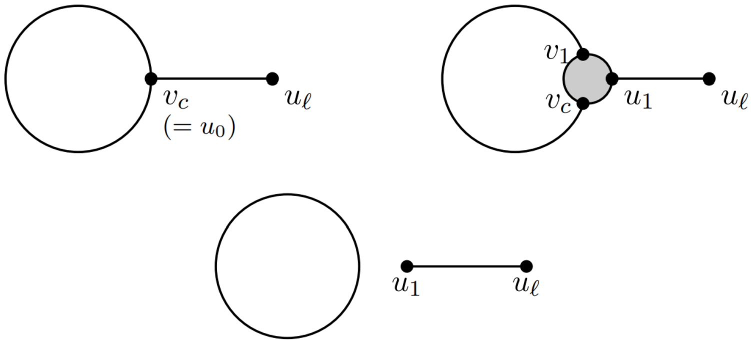

Figure 1: An o-lollipop, a p-lollipop, and a dcp-pair.

is the H*-neighborhood* of v.

When G is a subhypergraph of a hypergraph H and $u, v \in V(H)$, we say that u and v are G*-neighbors* if there exists an edge $e \in E(G)$ containing both u and v.

If P is a (Berge) path and a and b are two elements of P, we use $P[a, b]$ to denote the unique segment of P from a to b.

Let $r \geqslant 3$. Let H be a 2-connected, n-vertex, r-uniform hypergraph satisfying (2). Suppose

$$H \textit{ does not contain a Berge cycle of length at least } \min\{2k, n, |E(G)|\}. \tag{3}$$

Recall that this minimum is not n by Theorem 5.

A *lollipop* (C, P) is a pair where C is a Berge cycle and P is a Berge path or a partial Berge path that satisfies one of the following:

— P is a Berge path starting with a vertex in C, $|V(C) \cap V(P)| = 1$, and $|E(C) \cap E(P)| = 0$. We call such a pair (C, P) an **ordinary lollipop** (or o-lollipop for short). See Fig. 1 (left).

— P is a partial Berge path starting with an edge in C, $|V(C) \cap V(P)| = 0$, and $|E(C) \cap E(P)| = 1$. We call such a pair (C, P) a **partial lollipop** (or p-lollipop for short). See Fig. 1 (middle).

A lollipop (C, P) *is better* than a lollipop (C', P') if

(R1) $|V(C)| > |V(C')|$, or

(R2) Rule (R1) does not distinguish (C, P) from (C', P'), and $|V(P) - V(C)| > |V(P') - V(C')|$; or

(R3) Rules (R1) and (R2) do not distinguish (C, P) from (C', P'), and the total number of vertices of $V(P) - V(C)$ contained in the edges of $E(C) - E(P)$ counted with multiplicities is larger than the total number of vertices of $V(P') - V(C')$ contained in the edges of $E(C') - E(P')$; or

(R4) Rules (R1)–(R3) do not distinguish (C,P) from (C',P'), and the number of edges in $E(P)-E(C)$ fully contained in $V(P)-V(C)$ is larger than the number of edges in $E(P')-E(C')$ fully contained in $V(P')-V(C')$.

The criteria (R1)–(R4) define a partial ordering on the (finite) set of lollipops. For $1 \leqslant j \leqslant 4$, we will say that a lollipop is *j-good* if it is best among all lollipops according to the rules (R1)–(Rj). For example, a lollipop is 1-good if the cycle in it is a longest cycle in H. Clearly if $i < j$ and a lollipop is j-good, then it is also i-good. We call a 4-good lollipop a *best lollipop*.

Let a lollipop (C,P) be j-good for some $1 \leqslant j \leqslant 4$. Say $C = v_1, e_1, \ldots, v_c, e_c, v_1$. If (C,P) is an o-lollipop then let $P = u_0, f_0, u_1, \ldots, f_{\ell-1}, u_\ell$, where $u_0 = v_c$. If (C,P) is a p-lollipop then let $P = f_0, u_1, f_1, \ldots, f_{\ell-1}, u_\ell$ where $f_0 = e_c$. With this notation, we have $|E(P)| = \ell$, $|V(P)| = \ell + 1$ if P is a Berge path, and $|V(P)| = \ell$ if P is a partial Berge path. Assume $c < \min\{2k, |E(G)|, n\}$.

A big part of the proof is to show that the case $\ell \geqslant k$ is impossible. After we prove this, we assume that $\ell \leqslant k-1$ and consider a somewhat weaker structure than lollipop, we call it a **disjoint cycle-path pair** or *dcp-pair* for short. It is a pair (C,P) of a cycle C and a path P without common edges and common defining vertices. Similarly to lollipops, we say that a dcp-pair (C,P) *is better* than another dcp-pair (C',P') following the Rules (R1)–(R4) above. We say a dcp-pair (C,P) is j-good for some $1 \leqslant j \leqslant 4$ if it is best among all dcp-pairs according to rules (R1)–(Rj). A small simplification in the definition for dcp-pairs is that $V(P) - V(C) = V(P)$ and $E(C) - E(P) = E(C)$.

With the help of dcp-pairs, we will show that there is a 2-good lollipop (C',P') with $|E(P)| \geqslant k$, thus obtaining the final contradiction.

So, the four big pieces of the proof are the following.

Lemma 11. *Let (C,P) be a best lollipop in H. If $|V(C)| = c$, $|E(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell < k$.*

Lemma 12. *Let (C,P) be a best lollipop in H with $|V(C)| = c$, $|E(P)| < k$ and $c < \min\{2k, n, |E(H)|\}$. Then every 2-good dcp-pair (C',P') in H has $|V(P')| < k$.*

Lemma 13. *Let (C,P) be a 3-good dcp-pair in H. If $|V(C)| = c$, $|V(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell \geqslant 2$.*

Lemma 14. *Let (C,P) be a 4-good dcp-pair in H. If $|V(C)| = c$, $|V(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell \geqslant k$.*

To prepare the proofs of Lemmas 11–14, in the next section we describe some useful properties of good lollipops and dcp-pairs. After that, in the next four sections we prove Lemmas 11–14.

3 Simple properties of best lollipops

In this subsection we consider j-good lollipops and dcp-pairs (C,P) and prove some basic claims to be used throughout the rest of the paper. We start from a simple but useful observation on graph cycles.

Claim 15. *Suppose $c < 2k$ and $C = v_1, e_1, v_2, \dots, v_c, e_c, v_1$ is a graph cycle. Let $1 \leqslant i < j \leqslant c$.*

(a) The longer of the two subpaths of C connecting $\{v_i, v_{i+1}\}$ with $\{v_j, v_{j+1}\}$ and using neither e_i nor e_j has at least $\lceil c/2 \rceil$ vertices. In particular, this path omits at most $k-1$ vertices in C. We call it a long e_i, e_j-segment of C.

(b) The longer of the two subpaths of C connecting $\{v_i, v_{i+1}\}$ with v_j and not using e_i has at least $\lceil (c+1)/2 \rceil$ vertices. This path omits at most $k-1$ vertices in C. We call it a long e_i, v_j-segment of C.

(c) The longer of the two subpaths of C connecting v_i with v_j has at least $\lceil (c+2)/2 \rceil$ vertices. In particular, this path omits at most $k-2$ vertices in C. We call it a long v_i, v_j-segment of C.

We will need the following useful notions. Given an r-graph H and a lollipop or a dcp-pair (C, P) in H, let $H' = H'(C, P)$ denote the subhypergraph of H with $V(H') = V(H)$ and $E(H') = E(H) - E(C) - E(P)$.

Claim 16. *Let (C, P) be a 1-good lollipop or a 1-good dcp-pair in H. For each $1 \leqslant i \leqslant c$ and $1 \leqslant m \leqslant \ell$, if some edge $g \notin E(C)$ contains $\{u_m, v_i\}$, then*
(a) neither e_{i-1} nor e_i intersects $V(P) - u_0$, and
(b) if (C, P) is 3-good, then no edge in H' intersects both $V(P) - u_0$ and $\{v_{i-1}, v_{i+1}\}$ (indices count modulo c). In particular, the set $N_{H'}(V(P) - u_0) \cap V(C)$ does not contain two consecutive vertices of C.

Proof. Let $g \notin E(C)$ contain $\{u_m, v_i\}$. If $g \in E(P)$, say $g = f_q$, then we may assume $u_m = u_{q+1}$.

Suppose e_{i-1} contains u_j for some $1 \leqslant j \leqslant \ell$. If either $j \geqslant m$ or $g \neq f_m$, then we may replace the segment v_{i-1}, e_{i-1}, v_i in C with the path $v_{i-1}, e_{i-1}, u_j, P[u_j, u_m], u_m, g, v_i$. Otherwise we replace the segment with the path $v_{i-1}, e_{i-1}, u_j, P[u_j, u_{m-1}], u_{m-1}, g, v_i$. We obtain a longer cycle, contradicting the choice of C. The case with $u_j \in e_i$ is symmetric. This proves (a).

Suppose now some $e \in E(H')$ contains $\{u_j, v_{i-1}\}$ for some $1 \leqslant j \leqslant \ell$ (the case when $e \supset \{u_j, v_{i+1}\}$ is symmetric). If $e \neq g$, then as in the proof of (a) we may replace the segment v_{i-1}, e_{i-1}, v_i in C with $v_{i-1}, e, u_j, P[u_j, u_m], u_m, g, v_i$ or $v_{i-1}, e, u_j, P[u_j, u_{m-1}], u_{m-1}, g$ to get a longer cycle.

If $e = g$, then by (a), $e_{i-1} \cap (V(P) - u_0) = \emptyset$. Note that by the case $g = e \in E(H')$. Let C' be obtained from C by replacing the edge e_{i-1} with g and let $P' = P$. If $i \neq 1$ or (C, P) is a dcp-pair, then (C', P') is better than (C, P) by Rule (R3). If $i = 1$ and (C, P) is an o-lollipop or a p-lollipop, then the cycle obtained from C by replacing subpath v_c, e_c, v_1 with the path $v_c, f_1, u_1, \dots, u_j, g, v_1$ is longer than C. □

Claim 17. *Let (C, P) be a 1-good lollipop or a 1-good dcp-pair in H. For $1 \leqslant q \leqslant \ell - 1$ and $1 \leqslant i, j \leqslant c$, the following hold:*

(a) If $u_q \in e_i$ and $u_\ell \in e_j$ then $j = i$ or $|j - i| \geqslant \ell - q + 1$.

(b) If $\{v_i, u_q\} \subset e$ for some edge $e \in (E(H') \cup \{f_0\}) - E(C)$ and if $u_\ell \in e_j$, then either $j > i$ and $j - i \geqslant \ell - q + 1$, or $i > j$ and $i - j \geqslant \ell - q + 2$.

(c) If there exist distinct edges $e, f \in (E(H') \cup \{f_0\}) - E(C)$ such that $\{v_i, u_q\} \subset e$ and $\{v_j, u_\ell\} \subset f$, then $j = i$ or $|j - i| \geqslant \ell - q + 2$.

Proof. We will prove (a). If $j \neq i$, then we can replace the segment of C from e_i to e_j containing $|j - i|$ vertices with the path $e_j, u_\ell, P[u_\ell, u_q], u_q, e_i$ which contains $\ell - q + 1$ vertices. The new cycle cannot be longer than C, thus (a) holds. The proofs for (b) and (c) are similar so we omit them. □

Claim 18. *Suppose (C, P) is a 2-good lollipop or 2-good dcp-pair. Then all H'-neighbors of u_ℓ are in $V(C) \cup V(P)$. Moreover,*

(a) if (C, P) is an o-lollipop, then u_ℓ has no H'-neighbors in $\{v_1, v_2, \ldots, v_\ell\} \cup \{v_{c-1}, v_{c-2}, \ldots, v_{c-\ell}\}$, and u_ℓ is not in any edge in the set $\{e_1, e_2, \ldots, e_{\ell-1}\} \cup \{e_c, e_{c-1}, \ldots, e_{c-\ell}\}$,

(b) if (C, P) is a p-lollipop, then u_ℓ has no H'-neighbors in $\{v_1, v_2, \ldots, v_\ell\} \cup \{v_c, v_{c-1}, \ldots, v_{c-\ell+1}\}$, and u_ℓ is not in any edge in the set $\{e_1, \ldots, e_{\ell-1}\} \cup \{e_{c-1}, \ldots, e_{c-(\ell-1)}\}$.

Proof. Let $g \in E(H')$ contain u_ℓ. Suppose first there is a vertex $y \in V(H) - (V(C) \cup V(P))$ such that $y \in g$. Let P' be the path obtained from P by adding edge g and vertex y to the end of P. Then (C, P') is a lollipop with $|V(P')| > |V(P)|$, a contradiction.

Part (a) follows from Claim 17(b,c) for $q = 1$ since $f_0 \notin E(C)$ and contains $\{u_1, v_c\}$. Part (b) follows from Claim 17(a,b) for $q = 1$ since $f_0 = e_c$ contains $\{u_1, v_c, v_1\}$. □

Claim 19. *Let (C, P) be a j-good lollipop or j-good dcp-pair for some $1 \leqslant j \leqslant 4$.*

(a) If $u_\ell \in f_m$ for some $1 \leqslant m \leqslant \ell - 2$ and P_{m+1} is obtained from P by replacing the subpath $u_m, f_m, u_{m+1}, \ldots, u_\ell$ with the subpath $u_m, f_m, u_\ell, f_{\ell-1}, u_{\ell-1}, \ldots, u_{m+1}$, then (C, P_{m+1}) also is a j-good lollipop or dcp-pair.

(b) If some edge $g \in E(H')$ contains $V(P) - V(C)$ or is contained in $V(P) - V(C)$ and contains $\{u_\ell, u_m\}$ for some $1 \leqslant m \leqslant \ell - 2$, and if P'_{m+1} is obtained from P by replacing the subpath $u_m, f_m, u_{m+1}, \ldots, u_\ell$ with the subpath $u_m, g, u_\ell, f_{\ell-1}, u_{\ell-1}, \ldots, u_{m+1}$, then (C, P'_{m+1}) also is a j-good lollipop or dcp-pair.

Proof. Let us check the definition of a j-good lollipop or dcp-pair. Part (a) holds because the vertex set and edge set of P_{m+1} are the same as those of P.

In Part (b), $V(P'_{m+1}) - V(C) = V(P) - V(C)$, and $E(P'_{m+1})$ is obtained from $E(P)$ by deleting f_m and adding g. But since g contains $V(P) - V(C)$ or is contained in $V(P) - V(C)$, (C, P) cannot be better than (C, P'_{m+1}). □

4 Proof of Lemma 11: paths in lollipops must be short

Let (C, P) be a best lollipop with $C = v_1, e_1, v_2, \ldots, v_c, e_c, v_1$. Say $P = u_0, f_0, u_1, f_1, \ldots, u_\ell$ with $u_0 = v_c$ if (C, P) is an o-lollipop, and $P = f_0, u_1, f_1, \ldots, u_\ell$ with $f_0 = e_c$ if it is a p-lollipop.

In this section we prove that if P is long, then we can find a longer cycle than C. For this we will use a modification of a lemma from Dirac's original proof of Theorem 1 in [4].

Let Q and Q' be two (graph) paths in a graph G. We say Q and Q' are *aligned* if for every $x, y \in V(Q) \cap V(Q')$, x appears before y in Q if and only if x appears before y in Q'.

Lemma 20 (Lemma 5 in [20])**.** *Let Q be an x, y-path in a 2-connected graph G, and let $z \in V(G) - \{y\}$. Then there exists an x, z-path P_1 and an x, y-path P_2 such that (a) $V(P_1) \cap V(P_2) = \{x\}$, and (b) each of P_1 and P_2 is aligned with Q.*

We call a vertex $x \in V(H)$ *eligible* if there exists a best lollipop (C', P') where x is the end vertex of P' that is not contained in C'. In particular, by Claim 19 in our best lollipop (C, P), for any $f_m \in E(P)$ containing u_ℓ, u_{m+1} is eligible by considering the best lollipop (C, P_{m+1}), where P_{m+1} is defined as in Claim 19.

Set $u_0 = v_c$ if (C, P) is a p-lollipop. Recall if (C, P) is a p-lollipop, then $E(C) \cap E(P) = e_c = f_0$. Define

$$S_1 = N_{H'}(u_\ell) \cap V(P) \text{ and } S_2 = \{u_m : 0 \leqslant m \leqslant \ell - 1, u_\ell \in f_m, u_m \notin S_1\}.$$

Observe that $(S_1 \cup S_2) \cap V(C) \subseteq \{u_0\}$.

We will prove the following for (C, P) and eligible vertex u_ℓ, but all proofs will work for *any* best lollipop and corresponding eligible vertex.

Lemma 21. *If $\delta(H) \geqslant k$ and $|E(P)| = \ell \geqslant k$, then (A) $|S_1 \cup S_2| \leqslant k - 1$ and (B) $|S_1| \leqslant k - 2$.*

Proof. Throughout this proof we will use i_1 to denote the smallest index such that $u_{i_1} \in S_1 \cup S_2$ and j_1 to denote the smallest index such that $u_{j_1} \in S_1$.

Since $\ell \geqslant k$, u_ℓ has no H'-neighbors in $V(C) - V(P)$ by Claim 18. That is, $S_1 = N_{H'}(u_\ell)$.

If (C, P) is an o-lollipop, then define $X = V(C) - \{v_c\}$. Otherwise set $X = V(C)$.

Claim 22. *(A) If $|S_1 \cup S_2| \geqslant k$ and some edge $g \in E(C) \cup E(H')$ intersects both X and $\{u_{i_1+1}, \dots, u_\ell\}$, then (C, P) is a p-lollipop and $g = e_c (= f_0)$.*

(B) If $|S_1| \geqslant k - 1$, then no edge in $E(H')$ intersects both X and $\{u_{j_1+1}, \dots, u_\ell\}$.

Proof. Suppose $g \in E(C) \cup E(H')$ is an edge intersecting X and $\{u_{i_1+1}, \dots, u_\ell\}$, say $\{v_a, u_b\} \in g$ where $b \geqslant i_1 + 1$, and without loss of generality $g = e_{a-1}$ if $g \in E(C)$. By symmetry, if $g \in E(H')$ then we may assume $a \leqslant \lfloor c/2 \rfloor$ when (C, P) is an *o*-lollipop, and $a \leqslant \lceil c/2 \rceil$ when (C, P) is a p-lollipop, and if $g = e_{a-1}$, then we may assume $a - 1 \leqslant \lfloor c/2 \rfloor$ or $a - 1 = c$ (in this case, (C, P) is an o-lollipop, otherwise we have proven the claim).

By Claim 18, $u_\ell \notin g$, so $b \leqslant \ell - 1$. Let j be the largest index smaller than b such that $u_j \in S_1 \cup S_2$. Since $i_1 < b$, such j exists. Let h be an edge in $E(H') \cup f_j$ that contains $\{u_j, u_\ell\}$. Set

$$C' = v_c, e_{c-1}, v_{c-1}, \dots, v_a, g, u_b, f_b, \dots, u_\ell, h, u_j, f_{j-1}, \dots, u_0 (= v_c).$$

The cycle C' contains all vertices in $S_1 \cup S_2 \cup \{u_\ell\}$, and among these vertices, only u_0 may belong to C. Moreover, if $a - 1 \neq c$, then C' contains at least $c - (a - 1) \geqslant c - (\lfloor c/2 \rfloor) \geqslant c - (k - 1)$ vertices in C. Therefore

$$|C'| \geqslant c - (a - 1) + |S_1 \cup S_2 \cup \{u_\ell\}| - |(S_1 \cup S_2) \cap V(C)| \geqslant c - (k - 1) + k + 1 - 1 > c = |C|,$$

contradicting the choice of C. If $a-1=c$, then C' contains all the vertices of C along with all vertices in $S_1 \cup S_2 \cup \{u_\ell\}$, so we also have $|C'| > |C|$. This proves (A).

The proof for (B) is similar but we replace $S_1 \cup S_2$ with S_1 and i_1 with j_1. Define the cycle C' as before, and observe that if (C, P) is an o-lollipop, then C' contains at least $c-(a-1) \geqslant c-(\lfloor c/2 \rfloor -1) \geqslant c-(k-2)$ vertices in C, and if (C, P) is a p-lollipop, then C' contains at least $c-(k-1)$ vertices in C, but $S_1 \cap V(C) = \emptyset$. In either case, $|C'| \geqslant c-(k-1)+(k-1)+1 > |C|$. □

Recall that i_1 is the smallest index with $u_{i_1} \in S_1 \cup S_2$. If $u_{i_1} \in S_1$ set $\beta = i_1$; otherwise set $\beta = i_1 + 1$.

Claim 23. *Suppose* $|S_1 \cup S_2| \geqslant k$. *Then*

(a) there exists an edge f_j *with* $j \geqslant \beta$ *that intersects* $V(C)$*,*

(b) $\{u_j, \dots, u_{\ell-1}\} \subseteq S_1 \cup S_2$*,*

(c) (C, P) *is a p-lollipop, and*

(d) $u_\ell \in f_0$ *and* $u_\ell \in f_j$ *for every* f_j *satisfying (a).*

Proof. Consider the 2-connected incidence graph I_H of H and the (graph) path

$$P' = v_1, e_1, v_2, \dots, v_c, f_0, u_1, \dots, f_\ell, u_\ell$$

in I_H. We apply Lemma 20 to P' with $z = u_\beta$ to obtain two internally disjoint (graph) paths P_1 and P_2 such that P_1 is a v_1, z-path, P_2 is a v_1, u_ℓ-path, and each P_i is aligned with P'.

We modify P_i as follows: if $P_i = a_1, a_2, \dots, a_{j_i}$, let q_i be the last index such that $a_{q_i} \in X' := \{v_1, e_1, \dots, v_c, e_c\}$ and let p_i be the first index such that $a_{p_i} \in Y' := \{u_\beta, f_\beta, u_{\beta+1}, \dots, f_\ell, u_\ell\}$.

If $a_{p_i} = u_s$ for some s, then set $P'_i = P_i[a_{q_i}, a_{p_i}]$. If $a_{p_i} = f_s$ for some s, then set $P'_i = P_i[a_{q_i}, a_{p_i}], u_{s+1}$.

Observe that P'_1 and P'_2 are either Berge paths or partial Berge paths in H. Moreover, P'_1 ends with vertex $z = u_\beta$ and contains no other elements of Y' since it is aligned with P'.

If both P'_1 and P'_2 begin with v_1, then some P'_i avoids f_0 and first intersects the set $\{u_1, f_1, \dots, u_\ell\}$ at some element a_m. Then replacing the segment v_1, e_c, v_c in C with the longer segment $v_1, P'_i[v_1, a_m], a_m, P[a_m, f_0], f_0, v_c$ yields a cycle in H that is longer than C, a contradiction. Therefore we may assume that P'_1 and P'_2 are vertex-disjoint and edge-disjoint in H. Let u_g be the last vertex of P'_2. We have $g > \beta$.

If $f_{g-1} \notin E(P'_2)$, or if $u_{g-1} \in S_1$, let g' be the largest index less than g such that $u_{g'} \in S_1 \cup S_2$. Otherwise if $f_{g-1} \in E(P'_2)$ and $u_{g-1} \notin S_1$, let g' be the largest index less $g-1$ such that $u_{g'} \in S_1 \cup S_2$. If $u_{g'} \in S_1$, let $h \in E(H')$ contain $\{u_{g'}, u_\ell\}$; otherwise, let $h = f_{g'}$. We claim

$$h \notin E(P'_1 \cup P'_2). \tag{4}$$

Suppose not. If $h \in E(H')$, then h contains a vertex outside of $\{u_\beta, \dots, u_\ell\}$. But this violates either Claim 22 (A) or the definition of $S_1 = N_{H'}(u_\ell) \subseteq \{u_\beta, \dots, u_\ell\}$. If $h = f_{g'}$, then $f_{g'} \in E(P_2')$. By construction of P_2' and the choice of g' less than g, we must have $f_{g'} = a_{p_2}$ and hence $u_g = u_{g'+1}$. However, in this case ($f_{g'} = f_{g-1} \in E(P_2')$), we chose g' such that $g' < g-1$, a contradiction. This proves (4).

By Claim 15, there exists a long a_{q_1}, a_{q_2}-segment Q of C such that $|V(Q)| \geqslant c-(k-1)$ with equality only if at least one of a_{q_1} or a_{q_2} is an edge of C.

By (4) we may define the cycle

$$C' = a_{q_1}, Q, a_{q_2}, P_2', u_g, f_g, \ldots, u_\ell, h, u_{g'}, f_{g'-1}, \ldots, u_\beta, P_1', a_{q_1}.$$

Observe that C' contains the set $U = \{u_\beta, \ldots, u_{g'}\} \cup \{u_g, \ldots, u_\ell\}$. If $f_{g-1} \notin E(P_2')$ or $g' = g-1$, then $S_1 \cup S_2 \cup \{u_\ell\} - \{u_{i_1}\} \subseteq U - V(C)$. Otherwise, $S_1 \cup S_2 \cup \{u_\ell\} - \{u_{g-1}, u_{i_1}\} \subseteq U - V(C)$. Therefore $|U - V(C)| \geqslant |S_1 \cup S_2| + 1 - 2 \geqslant k-1$ with equality only if $f_{g-1} \in E(P_2')$ and $u_{g-1} \in S_2$. We have

$$|C'| \geqslant |V(Q)| + |V(P_2' \cup P_1') - V(C) - U| + |U - V(C)|. \tag{5}$$

If P_1' or P_2' contains a vertex outside of $V(C) \cup U$, then $|C'| \geqslant c-(k-1)+1+(k-1) > |C|$, contradicting the choice of C. Thus by construction, we may assume that

$$\text{each of } P_1' \text{ and } P_2' \text{ contains at most one edge.} \tag{6}$$

Similarly, if $|V(Q)| \geqslant c-(k-1)+1$ or if $|U - V(C)| \geqslant k$ then $|C'| \geqslant c-(k-1)+0+(k-1)+1 > |C|$, a contradiction. So $f_{g-1} \in E(P_2')$ and $u_{g-1} \in S_2$, proving (a). By (6) and Claim 22, $P_2' = v_i, f_{g-1}, u_g$ for some $v_i \in V(C)$. Also, we must have $U - V(C) = S_1 \cup S_2 \cup \{u_\ell\} - \{u_{g-1}, u_{i_1}\}$ which has size exactly $k-1$. In particular, $\{u_g, u_{g+1}, \ldots, u_{\ell-1}\} \subseteq S_1 \cup S_2$, so (b) holds.

In order to have $|V(Q)| = c-(k-1)$ by Claim 15, P_1' must begin with an edge of C. Then by (6) and Claim 22 (A), we must have (C, P) is a p-lollipop and $P_1' = e_c, u_\beta$. Therefore (c) holds. We have shown that $u_\ell \in f_{g-1}$. If f_s is another edge with $s \geqslant \beta$ that intersects $V(C)$, say at vertex $v_{s'}$, then we may substitute $P_2' = v_{s'}, f_s, u_{s+1}$ (which is disjoint from P_1') and symmetrically obtain that $u_\ell \in f_s$ as well. If $u_\ell \notin f_0$ (so $i_1 \geqslant 1$ by Claim 18), then the cycle

$$C'' = a_{q_1}, Q, a_{q_2}, f_{g-1}, u_g, \ldots, u_\ell, h, u_{g'}, \ldots, u_1, f_0 (= e_c = a_{q_1})$$

contains all of $S_1 \cup S_2 \cup \{u_\ell\} - \{u_{g-1}\}$, and this set is disjoint from $V(C)$. Therefore $|C''| \geqslant c-(k-1)+k > |C|$. This proves (d). □

Claim 24. $f_{\ell-1} \cap V(C) \subseteq \{v_{\lceil c/2 \rceil}\}$.

Proof. If $f_{\ell-1}$ contains a vertex $v_j \in V(C) - \{v_{\lceil c/2 \rceil}\}$, without loss of generality we may assume $j \geqslant \lceil c/2 \rceil + 1$. Then $C' = v_1, e_1, \ldots, v_j, f_{\ell-1}, u_{\ell-1}, f_{\ell-2}, \ldots, u_1, f_0, v_1$ has length at least $\lceil c/2 \rceil + 1 + |V(P) - V(C)| - 1 \geqslant c-(k-2)+\ell-1 > c$, a contradiction. □

Claim 25. *If $|S_1 \cup S_2| \geqslant k$ and $u_i \in S_1 \cup S_2$, then u_{i+1} is eligible.*

Proof. By Claim 23, (C, P) is a p-lollipop. If $u_i \in S_1$ then let $h \in E(H')$ contain $\{u_i, u_\ell\}$. By Claim 18, $h \subseteq V(P) - V(C)$. The result follows from Claim 19. □

Claim 26. $|S_1 \cup S_2| \leqslant k - 1$.

Proof. Suppose $|S_1 \cup S_2| \geqslant k$. By Claim 23, (C, P) is a p-lollipop and $u_\ell \in f_0$. Let f_j be an edge with $j \geqslant \beta \geqslant 1$ intersecting $V(C)$. By Claims 23 and 25, $u_{j+1}, u_{j+2}, \ldots, u_\ell$ are also eligible vertices. Thus applying Claim 23 to these vertices and their corresponding best lollipops (C, P'_{i+1}) (where P'_{i+1} is defined as in Claim 19) imply that $\{u_{j+1}, \ldots, u_\ell, u_\ell\} \subseteq f_0$.

Symmetrically consider $P' = f_0, u_\ell, f_{\ell-1}, u_{\ell-1}, \ldots, u_1$ and observe that (C, P') is a best lollipop and P' has first edge f_0. Applying Claims 23 and 25 to (C, P') and the eligible vertex u_1, we obtain that $u_1 \in f_0$ and vertices $u_2, u_3, \ldots, u_j$ are eligible and therefore contained in f_0. Thus $V(P) \subseteq f_0$ and so $r = |f_0| \geqslant |V(P) \cup \{v_1, v_c\}|$.

No edge in C may contain u_ℓ by Claim 18. If a vertex $w \in f_{\ell-1}$ is outside of $V(C) \cup V(P)$, then we can replace u_ℓ with w in P and get another best lollipop. So again by Claim 23, $w \in f_0$. By Claim 24, $f_{\ell-1}$ contains at most one vertex in $V(C)$. Then f_0 contains all of $f_{\ell-1} - V(C)$ as well as two vertices v_1 and v_c in C. This contradicts that H is r-uniform. □

We are now ready to prove Lemma 21. It remains to prove (B), so suppose towards contradiction that $|S_1| \geqslant k-1$. By Claim 26, $|S_1| = |S_1 \cup S_2| = k-1$ and $S_1 = S_1 \cup S_2$. As in the Proof of Claim 26, let $P' = v_1, e_1, v_2, \ldots, v_c, f_0, u_1, \ldots, f_\ell, u_\ell$, and apply Lemma 20 to the incidence graph I_H with P' and $z = u_{j_1}$. We obtain two aligned with P' paths and modify them to get Berge paths or partial Berge paths P'_1 and P'_2 such that P'_1 starts in C and ends in u_{j_1}, and P'_2 starts in C and ends in a vertex u_g with $g > j_1$. Also, if P'_2 contains some edge f_i with $i \geqslant j_1$, then we may assume $g = i + 1$.

Let a_{q_1} and a_{q_2} be the first elements in P'_1 and P'_2 respectively. Let Q be a long a_{q_1}, a_{q_2}-segment in C. As before, $|V(Q)| \geqslant c - (k-1)$ with equality only if at least one of a_{q_1} and a_{q_2} is an edge.

Let g' be the largest index less than g such that $u_{g'} \in S_1$ and let $h \in E(H')$ contain $\{u_{g'}, u_\ell\}$. Define

$$C' = a_{q_1}, Q, a_{q_2}, P'_2, u_g, f_g, \ldots, u_\ell, h, u_{g'}, \ldots, u_{j_1}, P'_1, a_{q_1}.$$

The cycle C' contains the set $U = \{u_{j_1}, \ldots, u_{g'}\} \cup \{u_g, \ldots, u_\ell\}$ which contains $S_1 \cup \{u_\ell\}$ and intersects $V(C)$ in at most one vertex, u_0. Therefore

$$|U - V(C)| \geqslant k - 1 + 1 - 1 = k - 1 \text{ with equality only if } S_1 \cup \{u_\ell\} = U \text{ and } u_0 \in S_1. \tag{7}$$

If P'_1 or P'_2 contain any internal vertices, if $|U - V(C)| \geqslant k$, or if $|V(Q)| \geqslant c-(k-1)+1$, then $|C'| \geqslant c - (k-1) + (k-1) + 1 > |C|$, a contradiction. Thus P'_1 and P'_2 contain at most one edge, $|U - V(C)| = k - 1$, and $|V(Q)| = c - (k-1)$. Then the equality in (7) holds. In particular, $u_0 \in S_1$ implies (C, P) is an o-lollipop and $P'_1 = u_0$. By Claim 15 and the fact $|V(Q)| = c - (k-1)$, $P'_2 = e_j, u_g$ for some $e_j \in E(C)$. Moreover we must have $c = 2k - 1$ and $e_j = e_{k-1}$, otherwise $|V(Q)| \geqslant c - (k-1) + 1$. By Claim 18 and the fact $\ell \geqslant k$, $g \leqslant \ell - 1$ and therefore $u_{\ell-1} \in S_1$.

Claim 27. *For all $u_s \in S_1$, then $f_s \subseteq V(P)$.*

Proof. Let $h' \in E(H')$ contain $\{u_s, u_\ell\}$ and set $P'_{s+1} = u_0, f_0, \ldots, u_s, h', u_\ell, f_{\ell-1}, \ldots, u_{s+1}$. The lollipop (C, P'_{s+1}) is a 3-good lollipop that omits edge f_s which is incident to the end vertex u_{s+1} of P_{s+1}. By Claim 18 applied to (C, P'_{s+1}), $f_s \subseteq V(P'_{s+1}) = V(P)$. □

Claim 28. *For every $s \geqslant g$, $f_s \subseteq S_1 \cup \{u_\ell\}$.*

Proof. By Claim 27 and (7), $f_s \subseteq V(P)$. Suppose f_s contains a vertex $u_{s'} \in V(P) - U = \{u_{g'+1}, \ldots, u_{g-1}\}$.

Let $h' \in E(H')$ be an edge in $E(H')$ containing $\{u_s, u_\ell\}$. Then

$$C'' = v_c, e_c, v_1, \ldots, v_{k-1}, e_{k-1}, u_g, \ldots, u_s, h', u_\ell, f_{\ell-1}, \ldots, u_{s+1}, f_s, u_{s'}, \ldots, u_0 (= v_c)$$

contains $U \cup \{u_\ell\} \cup \{u_{g'+1}, \ldots, u_{s'}\}$ and therefore is longer than C' (and C). □

Recall that $S_1 = S_1 \cup S_2$, and so if $u_\ell \in f_s$, then $u_s \in S_1$. If every f_s containing u_ℓ is a subset of $S_1 \cup \{u_\ell\}$, then $d_H(u_\ell) \leqslant \binom{|S_1|}{r-1} = \binom{k-1}{r-1} < k = \delta(H)$, a contradiction. So by the previous Claim, u_ℓ must be contained in some f_s with $s \leqslant g'$. If there exists an edge $h' \in E(H')$ containing $\{u_s, u_{g+1}\}$, then the cycle

$$C'' = v_c, e_c, v_1, \ldots, v_{k-1}, e_{k-1}, u_g, f_{g-1}, \ldots, u_{s+1}, f_s, u_\ell, f_{\ell-1}, \ldots, u_{g+1}, h', u_s, \ldots, u_0$$

contains all of $V(P)$ and is longer than C, a contradiction. Otherwise every edge in $E(H')$ that intersects u_ℓ contains $r-1 \geqslant k-2$ vertices in the $(k-1)$-set S_1 but does not contain both u_s and u_{g+1}. This implies $r = k-1$ and since $|S_1 \cup \{u_\ell\}| = k > r$, u_ℓ is contained in at least two edges in H'. But then there can be only two such edges, one that avoids u_s and one that avoids u_{g+1}. Let h'' be the one avoiding u_s. By Claim 28, $f_{\ell-1}$ also contains $k-2$ vertices in S_1 and therefore must contain both u_s and u_{g+1}. Then we replace h' in C'' with $f_{\ell-1}$ and $f_{\ell-1}$ with h'' to obtain another longer cycle. □

As a corollary we have the following.

Corollary 29. *Suppose (C', P') is a best lollipop with $C' = v'_1, e'_1, v'_2, \ldots, v'_c, e'_c, v'_1$ and $P' = u'_1, f'_1, \ldots, f'_\ell, u'_\ell$. If $\ell = |V(P')| \geqslant k$, then*

(i) $d_{P'}(u'_\ell) \leqslant k-1$,

(ii) $d_{H-C'-P'}(u'_\ell) \leqslant 1$ with equality only if $r = k-1$, and

(iii) $d_{C'-P'}(u'_\ell) = 0$.

Moreover if $d_{H-C'-P'}(u'_\ell) = 1$ and $d_{P'-C'}(u'_\ell) = k-1$, then $u'_\ell \in f'_i$ for every $u'_i \in N_{H-C'-P'}(u'_\ell)$.

Proof. Part (i) follows from Lemma 21 (A). By Lemma 21 (B), $|S_1| \leqslant k-2 \leqslant r-1$. Thus if $N_{H-C'-P'}(u_\ell)$ is nonempty, then $|N_{H-C'-P'}(u_\ell)| = r-1$. That is, $d_{H-C'-P'}(u_\ell) \leqslant 1$ and we obtain Part (ii). Part (iii) follows from Claim 18.

The "moreover" part comes from the fact that that $\{u'_i : u'_\ell \in f'_i\} \subseteq S_1 \cup S_2$. But if $d_{P'}(u'_\ell) = k-1 = |S_1 \cup S_2|$ then $S_1 \subseteq \{u'_i : u'_\ell \in f'_i\}$. □

Finally we are ready to show that the paths in lollipops cannot be long. Recall the statement of Lemma 11.

Lemma 11. *Let (C, P) be a best lollipop in H. If $|V(C)| = c$, $|E(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell < k$.*

Proof of Lemma 11. Suppose that $\ell \geqslant k$. If $r \geqslant k$, then by Corollary 29,

$$d_H(u_\ell) \leqslant d_{H'}(u_\ell) + d_{C-P}(u_\ell) + d_P(u_\ell) \leqslant 0 + 0 + k - 1 < \delta(H).$$

So assume $r = k-1$. Let u_{i+1} be an eligible vertex. By Corollary 29 applied to (C, P_{i+1}) (note that $E(P) = E(P_{i+1})$), $d_{C-P}(u_{i+1}) = 0$, and therefore $d_P(u_{i+1}) = k-1$ and $d_{H'}(u_{i+1}) = 1$. Let $e \in E(H')$ contain u_ℓ. Say $e = \{u_{j_1}, \ldots, u_{j_r}\}$ where $j_1 < \ldots < j_r = \ell$. Also by Corollary 29, u_ℓ belongs to each of the edges $f_{j_1}, \ldots, f_{j_r}$.

If possible, we choose a best lollipop (C, P) such that $j_1 > 0$.

Case 1: $j_1 > 0$. Consider the incidence graph I_H. Set $X = C(= V(C) \cup E(C))$ and $Y = P[u_{j_1}, u_\ell]$. As in the proof of Lemma 21 we apply the modification of Dirac's Lemma, Lemma 20, to the (graph) path

$$P' = v_1, e_1, \ldots, v_c, f_0, \ldots, f_\ell, u_\ell$$

in I_H with $z = u_{j_1}$. After modification as detailed in the proof of Lemma 21, we obtain two disjoint, aligned (Berge or partial Berge) paths P_1' and P_2' in H such that P_1' starts in X, ends in u_{j_1} and is internally disjoint from $X \cup Y$, and P_2' starts in X, ends in a vertex $u_i \in \{u_{j_1+1}, \ldots, u_\ell\}$ and is internally disjoint from $X \cup Y$ except possibly in its last edge, only if that edge is f_{i-1}.

Suppose a_1 and b_1 are the first elements of P_1' and P_2' respectively, and let Q be a long a_1, b_1-segment of C provided by Claim 15. If a_1 and b_1 are both vertices, then $|V(Q)| \geqslant c-(k-2)$, otherwise if at least one of a_1 or b_1 is in $E(C)$, then $|V(Q)| \geqslant c-(k-1)$.

Let α be the largest index smaller than i such that $u_\alpha \in e$. Consider the cycle

$$C^* = a_1, Q, b_1, P_2', u_i, P[u_i, u_\ell], u_\ell, e, u_\alpha, P[u_\alpha, u_{j_1}], u_{j_1}, P_1', a_1.$$

This cycle contains the set $U = \{u_{j_1}, \ldots, u_\alpha\} \cup \{u_i, \ldots, u_\ell\}$, and $e \subseteq U$. Since $j_1 > 1$, $|U \cap V(C)| = 0$. If $|U| > |e| = k-1$, then $|C^*| \geqslant c - (k-1) + |U| > c$, contradicting the choice of C as a longest cycle. Therefore we may assume $e = U$.

We will show that

$$e \text{ contains } r \text{ consecutive vertices in } P. \tag{8}$$

If not then $i > \alpha + 1 \geqslant j_1 + 1$, and so $u_{i-1} \notin e$. Recall that $u_\ell \in f_{j_1}$, and $\{u_i, u_{i+1}\} \subset U = e$.

If $f_{i-1} \notin E(P_2')$, set

$$C' = a_1, Q, b_1, P_2', u_i, f_{i-1}, \ldots, u_{j_1+1}, f_{j_1}, u_\ell, f_{\ell-1}, \ldots, u_{i+1}, e, u_{j_1}, P_1', a_1.$$

Otherwise set

$$C' = a_1, Q, b_1, P_2'[b_1, f_{i-1}], f_{i-1}, u_{i-1}, f_{i-2}, \ldots, u_{j_1+1}, f_{j_1}, u_\ell, f_{\ell-1}, \ldots, u_i, e, u_{j_1}, P_1', a_1.$$

The cycle C' contains $e \cup \{u_{i-1}\}$. Therefore $|C'| \geqslant c-(k-1)+(k-1+1) > c$, a contradiction. This proves (8). In particular, $u_{\ell-1} \in e$.

By the choice of (C, P) as a best lollipop, if we swap edge e with $f_{\ell-1}$ in P we obtain another lollipop with $f_{\ell-1}$ taking the place of $e \in E(H')$. By the case, $e \subseteq V(P) - V(C)$ and so by Rule (R4) in the choice of (C, P), $f_\ell \subseteq V(P) - V(C)$ as well. In particular, the first vertex of $f_{\ell-1}$ in P is not u_0. Thus by symmetry, $f_{\ell-1}$ also contains r consecutive vertices in P. But this implies $f_{\ell-1} = e$ since both edges end in u_ℓ, a contradiction.

Case 2: In every best lollipop, $j_1 = 0$. By Claim 18, all best lollipops are o-lollipops.

Recall that for every eligible u_{i+1}, the lollipop (C, P_{i+1}) is a best lollipop and the path P_{i+1} begins with u_0, f_0. By the case, $u_0 \in N_{H'}(u_{i+1})$, and so by the "moreover" part of Corollary 29 applied to (C, P_{i+1}), $u_{i+1} \in f_0$ for every eligible u_{i+1}. Thus

$$r = |f_0| \geqslant |\{u_0\} \cup \{u_{i+1} : u_\ell \in f_i\}| \geqslant 1 + d_P(u_\ell) = k,$$

which contradicts that $r = k-1$. $\square$

5 Proof of Lemma 12: short paths in lollipops imply short paths in dcp-pairs

We have shown that the paths in best lollipops must be short. Recall that a disjoint cycle-path pair, or a dcp-pair, is a cycle and a path that share no defining vertices or edges. We now show that all 2-good dcp-pairs must also have short paths. This will be useful for us because it implies that since r is large, edges intersecting the path must "stick out" of the path. We restate the contrapositive of Lemma 12 in the following slightly stronger form.

Lemma 12′. *Suppose $2 \leqslant s \leqslant k \leqslant r+1$. Let (C, Q) be a 2-good dcp-pair in a 2-connected n-vertex r-graph H with $c(H) < \min\{n, 2k, |E(H)|\}$ and $\delta(H) \geqslant k$. If $|V(Q)| \geqslant s$, then H contains a lollipop (C, P) with $|E(P)| \geqslant s$.*

Proof. Suppose (C, Q) is a 2-good dcp-pair, say $Q = u_1, f_1, u_2, \ldots, f_{q-1}, u_q$ with $q \geqslant s$. Let H' be the subhypergraph with edge set $E(H) - E(C) - E(Q)$. If the lemma does not hold, then

$$\textit{no edge in } E(H) - E(Q) \textit{ containing } u_1 \textit{ or } u_q \textit{ intersects } V(C), \tag{9}$$

otherwise we already obtain a desired o-lollipop or p-lollipop. By the maximality of $|V(Q)|$,

$$\textit{each } g \in E(H') \textit{ containing } u_1 \textit{ or } u_q \textit{ does not meet } V(H) - V(C) - V(Q), \textit{ thus by (9), } g \subseteq V(Q)\ . \tag{10}$$

Let $R = w_1, g_1, w_2, \ldots, g_{t-1}, w_t$ be a shortest path in H from $V(C)$ to $V(Q)$. If $g_{t-1} = f_j \in E(Q)$ for some j then we let $w_t = u_j$. Rename the vertices of C so that $w_1 = v_c$ and $w_t = u_j$ for some $1 \leqslant j \leqslant q-1$. Combining with R any path with s vertices starting with

u_j that is otherwise disjoint from C and R yields a lollipop as desired. If $j \geqslant s$, we take the path $Q[u_j, u_1]$. So, $j \leqslant s-1$.

Case 1: $d_{H'}(u_1) = 0$. Let j' denote the maximum i such that $u_1 \in f_i$. By the case, $j' \geqslant d_H(u_1) \geqslant k \geqslant s > j$, so path $u_j, f_{j-1}, \ldots, f_1, u_1, f_{j'}, u_{j'}, f_{j'-1}, \ldots, u_{j+1}$ has at least $k \geqslant s$ vertices and does not use f_j.

Case 2: $d_{H'}(u_1) \geqslant 2$ or the unique edge $h \in H'$ containing u_1 is not $\{u_1, \ldots, u_r\}$. Let j' denote the maximum i such that some edge $h \in H'$ contains u_1 and u_i. Then $j' \geqslant r+1 > j$, so the path $u_j, f_{j-1}, \ldots, f_1, u_1, h, u_{j'}, f_{j'-1}, \ldots, u_{j+1}$ has at least $r+1 \geqslant k$ vertices and does not use f_j.

Case 3: The unique edge $h \in H'$ containing u_1 is $\{u_1, \ldots, u_r\}$. Now, $f_1 \neq \{u_1, \ldots, u_r\}$, and by (10), $f_1 \subset V(Q)$. So, switching h with f_1, we get Case 2. ☐

6 Proof of Lemma 13: Disjoint cycle-path pairs have non-trivial paths

In this section, we show that the path P in a 3-good dcp-pair must have at least 2 vertices. Let (C, P) be such a dcp-pair. If $|C| < n$, there is a vertex outside of C, thus $\ell \geqslant 1$.

We will show that $\ell \geqslant 2$ using the notion of *expanding* sets that can be used to modify C into a longer cycle. Indeed, suppose $\ell = 1$ and $P = u_1$.

Let C' be any longest cycle in H (i.e., $|C'| = |C| = c$), and let $u \in V(H) - V(C')$. Say that a set $W \subseteq V(C')$ is (u, C')-*expanding* if for every distinct $v_i, v_j \in W$, there is a v_i, v_j-path $R(v_i, v_j)$ whose internal vertices are disjoint from $V(C') \cup \{u\}$ and all edges are in $E(H) - E(C')$. One example of a (u, C')-expanding set is $V(C') \cap g$ where g is any edge in $E(H) - E(C')$.

Another example is a set of the form $N_{H-C'}(w) \cap V(C')$ for a vertex $w \in V(H) - V(C') - \{u\}$. (11)

Claim 30. *Let (C', u) be a 2-good dcp-pair (so C' is a longest cycle in H) with $C' = v_1, e_1, \ldots, v_c, e_c, v_1$. For any (u, C')-expanding set W, u is contained in at most one edge of $\{e_j : v_j \in W\}$ and in at most one edge of $\{e_{j-1} : v_j \in W\}$.*

Moreover, suppose $v_j \in W$ and $u \in e_j$. For every $v_i \in W - \{v_j\}$, if $e \in E(H) - E(C') - E(R(v_i, v_j))$ contains u, then $v_{i+1} \notin e$. Similarly, if $u \in e_{j-1}$, then $v_{i-1} \notin e$.

Proof. Suppose $v_i, v_j \in W$ and $u \in e_i, e_j$. By symmetry we may assume $i < j$. The cycle

$$C'' = v_1, \ldots, v_i, R(v_i, v_j), v_j, e_{j-1}, \ldots, v_{i+1}, e_i, u, e_j, v_{j+1}, e_{j+1}, \ldots, v_c, e_c, v_1$$

has length $|C'| + 1 > c$, a contradiction. The case for $u \in e_{i-1}, e_{j-1}$ is symmetric.

For the "moreover" part, let us suppose $e \in E(H) - E(C') - E(R(v_i, v_j))$ contains u and v_{i+1}. Then replacing edge e_i in C'' with e also yields a cycle longer than C'. ☐

We now apply this claim to the 3-good dcp-pair (C, P) with $C = v_1, e_1, \ldots, v_c, e_c, v_1, P = u_1$. Define $A = N_{H'}(u_1)$, $a = |A|$, $B = \{e_j \in E(C), u_1 \in e_j\}$, and $b = |B|$. We obtain the following.

Claim 31. *Suppose W is a (u_1, C)-expanding set.*

(i) If the edges of B form exactly q intervals in C, then $|W| \leqslant c - (b + q) + 2$.

(ii) If the vertices in W form exactly q' intervals in C, then $b \leqslant c - (|W| + q') + 2$.

Proof. Set $V = \bigcup_{e_i \in B}\{v_i, v_{i+1}\}$. We have $|V| = b + q$. By Claim 30, W contains at most one vertex v_i for which $e_i \in B$ and at most one vertex v_i for which $e_{i-1} \in B$. Thus $|W \cap V| \leqslant 2$ and $|W| \leqslant c - |V| + 2$. This proves (i). The proof of (ii) is similar. □

Now we are ready to prove Lemma 13. Recall the statement.

Lemma 13. *Let (C, P) be a 3-good dcp-pair in H. If $|V(C)| = c$, $|V(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell \geqslant 2$.*

Proof. Suppose $P = u_1$. By Rule (R3), u_1 is a vertex in $V(H) - V(C)$ with the largest degree in $E(C)$. If there exists $e \in E(H) - E(C)$ containing at least 2 vertices $u, u' \notin V(C)$, then the path $P' = u, e, u'$, the pair (C, P') is a better dcp-pair than (C, P). It follows that

$$\text{for each } e \in E(H) - E(C),\ |e \cap V(C)| \geqslant r - 1. \tag{12}$$

In particular, this yields $N_{H'}(u_1) \subseteq V(C)$. By Claim 16, A does not intersect the set $\bigcup_{e_i \in B}\{v_i, v_{i+1}\}$ and

$$\textit{no two vertices of } A \textit{ are consecutive on } C\textit{; thus } 2k - 1 \geqslant c \geqslant 2a + b. \tag{13}$$

It follows that

$$a \leqslant \lfloor c/2 \rfloor \text{ and } k \leqslant d_H(u_1) \leqslant \binom{a}{r-1} + b \leqslant \binom{a}{r-1} + c - 2a. \tag{14}$$

Case 1: $d_{H'}(u_1) \geqslant 2$. Then $a \geqslant (r - 1) + 1$. If $r \geqslant k$, then $\lfloor c/2 \rfloor \geqslant a \geqslant r \geqslant k$, contradicting to $c < 2k$. Thus we may assume $k = r + 1$. Now (13) yields $a = r$, $c = 2k - 1$ and $b \leqslant 1$. On the other hand, (14) yields $b \geqslant 1$. So, $b = 1$ and all r-tuples of vertices containing u_1 and contained in $A \cup \{u_1\}$ are edges of H'. By symmetry, we may assume that $B = \{e_{2k-1}\}$ and $A = \{v_2, v_4, \ldots, v_{2k-2}\}$. Since $k = r + 1$, for every $1 \leqslant j \leqslant k - 1$ we can choose an edge $g_{2j} \in E(H')$ containing u_1 and v_{2j} so that all g_{2j} are distinct. We also let $g_1 = g_{2k-1} = e_{2k-1}$.

Claim 32. *Let $1 < i < 2k - 1$ be odd. Then (i) the only edge in $E(C) - \{e_i, e_{i-1}\}$ that may contain v_i is e_{2k-1}, and (ii) $N_{H'}(v_i) \cap V(C) \subseteq A$.*

Proof. Suppose $v_i \in e_j$ for some j. By symmetry, we may assume $j > i$. If j is even then set

$$C_j = v_i, e_i, \ldots, v_j, g_j, u_1, g_{i-1}, v_{i-1}, e_{i-1}, \ldots, v_{j+1}, e_j, v_i,$$

and if j is odd, then set

$$C'_j = v_i, e_{i-1}, v_{i-1}, \ldots, v_{j+1}, g_{j+1}, u_1, g_{i+1}, v_{i+1}, e_{i+1}, \ldots, v_j, e_j, v_i.$$

Then each of C_j and C'_j contains $c + 1$ vertices, contradicting the choice of C. This proves (i). For (ii), suppose some $h \in E(H')$ contains v_i and v_j where j is odd. Then replace edge e_j in the cycle C'_j above with h to obtain a cycle longer than C. □

Since $|e_{2k-1} - \{u_1\}| = r - 1 = k - 2$, there exists $v_i \notin e_{2k-1}$ where $1 < i < 2k - 1$ is odd. By Claim 32, $k \leqslant \delta(H) \leqslant d_H(v_i) \leqslant 2 + d_{H'}(v_i)$, and therefore $d_{H'}(v_i) \geqslant k - 2 = (r + 1) - 2 \geqslant 2$.

Set $C' = v_1, e_1, \ldots, v_{i-1}, g_{i-1}, u_1, g_{i+1}, v_{i+1}, e_{i+1}, \ldots, v_1$. If there exists $e \in E(H')$ containing v_i and a vertex $w \notin V(C)$, then (C', v_i, e, w) is a better dcp-pair than (C, P). Thus $N_{H'}(v_i) \subseteq V(C)$. It follows that $(r - 1) + 1 \leqslant |N_{H'}(v_i)| \leqslant |A| = k - 1$, and so $N_{H'}(v_i) = A$.

Now we consider (C', v_i) which is a 2-good dcp-pair that does not use the edges e_{i-1}, e_i. Similarly, if one of these edges, say e_i, contains a vertex $w \notin V(C') \cup \{v_i\}$, then (C', v_i, e_i, w) is a better p-lollipop than (C, P). On the other hand, if $e_i \cup e_{i-1} \subseteq A$, then $d_H(v_i) \leqslant \binom{|A|}{r-1} < k \leqslant \delta(H)$, a contradiction. By the previous claim, the only vertices in $V(C) - A - \{v_i\}$ that may belong to e_i or e_{i-1} are v_1 and v_{2k-1}. Without loss of generality, let v_1 belong to $e_j \in \{e_{i-1}, e_i\}$. Since $v_2 \in A$, there exists $g \in E(H')$ containing $\{v_2, v_i\}$ (and note that $g \neq g_{i-1}, g_{i+1}$ because $v_i \notin N_{H'}(u_1)$). Then replacing in C' the segment v_1, e_1, v_2 with v_1, e_j, v_i, g, v_2 yields a longer cycle than C.

Case 2: $d_{H'}(u_1) = 1$, say $u_1 \in f \in E(H')$. Then $d_H(u_1) = 1 + b$, $A = f \cap V(C)$ and $a = r - 1$. By (14),

$$k \leqslant d_H(u_1) \leqslant 1 + c - 2(r - 1) \leqslant 2k - 2r + 2.$$

When $k > 2$, this is possible only if $k = r + 1$, $r = 3$, $c = 2r + 1 = 7$ and $d_H(u_1) = k = 4$. In this case we have $c = 2k - 1 = 7$. Let $e \in E(H')$ contain u_1, say $e = \{u_1, v_i, v_j\}$. Since by Claim 16, e cannot contain consecutive vertices in C, by symmetry we may assume that $i = 1$ and $j \in \{5, 6\}$. By Claim 16, u_1 cannot be contained in the 4 distinct edges e_1, e_7, e_j, e_{j-1} incident to v_1 and v_j. Thus as $d_H(u_1) \geqslant k = 4$ and $d_{H'}(u_1) = 1$, u_1 is contained in the remaining $7 - 4 = 3$ edges of C. That is, u_1 belongs to the edges e_2, e_3, and e_k where $k = 6$ if $j = 5$ and $k = 4$ if $j = 6$. Let C' be the cycle obtained by swapping v_3 with u_1 and observe that (C', v_3) is a 2-good dcp-pair.

The set $e = \{v_1, v_j, u_1\}$ forms a (v_3, C')-expanding set. Because $u_1 \in e$ now takes the place of v_3 in C' and $v_3 \in e_2, e_3$, by Claim 30 v_3 cannot belong to the edges e_1, e_7, e_j, e_{j-1} and cannot be an H'-neighbor to v_2, v_7, v_{j-1}, or v_{j+1} (here we use the fact that $v_3 \notin e$). Moreover by Claim 16, $N_{H'}(v_3)$ also does not contain vertices u_1 and v_4 which are incident to e_2 and e_3 respectively.

This yields the following two facts: the only edge of C' other than e_2 and e_3 that may contain v_3 is e_k, and the only H'-edge containing v_3 may be $\{v_3, v_1, v_j\}$. Recall that $u_1 \in e_k$ and therefore $e_k = \{v_k, v_{k+1}, u_1\}$ which does not contain v_3. It follows that $d_H(v_3) \leqslant d_C(v_3) + d_{H'}(v_3) \leqslant 2 + 1 < k = \delta(H)$, a contradiction.

Case 3: $d_{H'}(u_1) = 0$ for some $u_1 \in V(H) - V(C)$. So, $d_H(u_1) = b$. If u_1 is the only vertex outside of C, then $r \geqslant \lfloor \frac{n-1}{2} \rfloor$, so by Theorem 5, H has a Hamiltonian cycle, a contradiction. Hence,

$$n = c + x \qquad \textit{for some } x \geqslant 2. \tag{15}$$

Let $w \in V(H) - (V(C) \cup \{u_1\})$. Let us show that

$$d_{H'}(w) \leqslant 1. \tag{16}$$

Indeed, suppose $g_1, g_2 \in E(H')$ and $w \in g_1 \cap g_2$. Let $W = V(C) \cap (g_1 \cup g_2)$. As observed in (11), this W is u_1-expanding. Since $g_2 \neq g_1$, by (12), $|W| \geqslant r$. Also, by Claim 17, vertices in g_2 could not be next to vertices in g_1 on C. Thus if $|W| = r$, then $|g_1 \cap g_2| = r - 1$; hence no two vertices of W are consecutive on C. In this case, by Claim 31(ii), $b \leqslant c - |W| - q + 2$ where $q = |W| = r$. Since $k \leqslant r + 1$, we get $k \leqslant d_H(u_1) \leqslant (2k-1) - 2r + 2 \leqslant 2k - r - 2$ contradicting $k \leqslant r + 1$.

Thus $|W| \geqslant r + 1$. But still since vertices in g_2 could not be next to vertices in g_1 on C, $q \geqslant 2$. So, $k \leqslant d_H(u_1) \leqslant (2k-1) - (r+1+2) + 2 = 2k - r - 2$ contradicting $k \leqslant r+1$. This proves (16).

Let $n = c + x$ and $|E(H)| = c + y$. Then considering the sum of degrees of vertices in H, we have

$$k(c + x) = k \cdot n \leqslant \sum\nolimits_{v \in V(H)} d_H(v) = r(c + y). \tag{17}$$

By (15), if $k \geqslant r$, then

$$y \geqslant x \geqslant 2. \tag{18}$$

Moreover, if $k = r + 1$ then (17) yields

$$y \geqslant \frac{(r+1)(c+x)}{r} - c = \frac{c+x}{r} + x = \frac{n}{r} + x > 2 + x.$$

So, if $Z = \{g_1, \ldots, g_z\}$ is the set of edges in $E(H')$ contained in $V(C)$, then (16) together with $d_{H'}(u_1) = 0$ yields

$$y \leqslant z + x - 1, \textit{ and hence } z \geqslant 1 \textit{ when } k \geqslant r \textit{ and } z \geqslant 4 \textit{ when } k = r + 1. \tag{19}$$

Suppose the edges of B form exactly q intervals in C. Since $b = d_H(u_1) \geqslant k$, by Claim 31(i)

$$r \leqslant |W| \leqslant c - (b + q) + 2 \leqslant 2k - 1 - (k + q) + 2 = k - q + 1.$$

As $k \leqslant r + 1$, $q \leqslant 2$ with equality only if $k = r + 1$ and $d_H(u_1) = k$. Let $J_B = \{j \in [c] : e_j \in B\}$. By symmetry we may assume $1 \in J_B$.

Case 3.1: $J_B = \{1, \ldots, i_1\} \cup \{i_2, \ldots, i_3\}$ where $i_1 + 2 \leqslant i_2 \leqslant i_3 \leqslant c - 1$. In this case $q = 2$, so $k = r + 1$ and $|J_B| = d_H(u_1) = k$. If some $g_j \in Z$ contains v_i for some i in the $(k-2)$-element subset $\{2, \ldots, i_1\} \cup \{i_2 + 1, \ldots, i_3\}$, then in order not to create a longer cycle, the indices of the remaining $r - 1$ vertices of g_j cannot be in $J'_B = \{1, 2, \ldots, i_1 + 1\} \cup \{i_2, \ldots, i_3 + 1\}$ by Claim 30. Since $|J'_B| = k + 2$, we need $c \geqslant (r - 1) + (k + 2) = 2k$, a contradiction. So, each $g_j \in Z$ is contained in the set $F = \{v_{i_1+1}, \ldots, v_{i_2}\} \cup \{v_{i_3+1}, \ldots, v_c, v_1\}$ which has size $c - (k-2) = k + 3 = r + 2$. Also, each g_j contains at most one of v_1, v_{i_2} and at most one of v_{i_1+1}, v_{i_3+1}. There are exactly four such possibilities: $F - \{v_1, v_{i_1+1}\}, F - \{v_1, v_{i_3+1}\}, F - \{v_{i_2}, v_{i_1+1}\}, F - \{v_{i_2}, v_{i_3+1}\}$. So, $z = 4$ and each g_j is one of these possibilities, say $g_1 = F - \{v_{i_2}, v_{i_1+1}\}$. But then g_1 contains $\{v_c, v_1\}$, and we can replace e_c with g_1, and again get a 3-good dcp-pair. Since $g_1 \neq e_c$, this is a contradiction.

Case 3.2: J_B is a single interval for each 3-good lollipop. Let $J_B = \{1, \dots, b\}$ for $b \geqslant k$.

First suppose $k \leqslant r$. Then $z \geqslant 1$, say $g_1 \in Z$. If g_1 contains a vertex v_j with $2 \leqslant j \leqslant b$, then $g_1 - v_j \subseteq \{v_{b+2}, \dots, v_c\}$ by Claim 30. It follows that $r - 1 = |g_1 - v_j| \leqslant c - (b+1) \leqslant (2k-1) - (k+1) = k - 2$, contradicting $k \leqslant r$. Hence $g_1 \subseteq \{v_{b+1}, v_{b+2}, \dots, v_c, v_1\}$, and so $r = |g_1| \leqslant c - (b-1) \leqslant (2k-1) - (k-1) = k$. This is possible only if $c = 2k-1$, $b = k$, $k = r$ and $g_1 = \{v_{b+1}, v_{b+2}, \dots, v_c, v_1\}$. Let C' be obtained from C by replacing e_{c-1} with g_1. Then (C', P) is also a 3-good dcp-pair. We apply (19) to (C', P) to obtain that there exists some $g_1' \in E(H) - E(C')$ contained in $V(C)$ (possibly $g_1' = e_{c-1}$). Since $V(C) = V(C')$, and $g_1' \neq g_1$, we use Claim 30 and the fact that $V(C) \cap g_1$ and $V(C') \cap g_1'$ are both u_1-expanding to obtain that there are at least r edges of C which cannot contain u_1. Then $k = b \leqslant c - r = k - 1$, a contradiction.

Finally consider the case $k = r + 1$. By (19), $z \geqslant 4$. First suppose $b \geqslant k + 1$. If some $g_i \in Z$ contains a vertex v_j with $2 \leqslant j \leqslant b$, then $g_i - v_j \subseteq \{v_{b+2}, \dots, v_c\}$ by Claim 30. That is, $r = |g_i| \leqslant 1 + (c - (b+2) + 1) \leqslant k - 2$, a contradiction. Thus each $g_i \in Z$ is a subset of $\{v_{b+1}, \dots, v_c, v_1\}$ which has size at most $k - 1 = r$. We get a contradiction to $z \geqslant 4$.

So we may assume $J_B = \{1, \dots, k\}$. Let $F = \{v_{k+2}, \dots, v_c\}$, $B' = E(C) - E(B)$ and $B'' = B' - e_{k+1} - e_c = \{e_{k+2}, \dots, e_{c-1}\}$.

Among all c-cycles C with the same cyclic sequence $v_1, \dots, v_c$ of vertices and containing edges $e_1, \dots, e_k$ in the same positions, choose one in which

(a) most edges in B' are subsets of $F' = F \cup \{v_{k+1}, v_1\} = \{v_{k+1}, \dots, v_c, v_1\}$ and

(b) given that (a) holds, most edges in B'' contain F.

We claim that

$$\textit{no edges in } E(H') - Z \textit{ intersect } \{v_2, \dots, v_k\}. \tag{20}$$

Indeed, suppose some $g_0 \in E(H') - Z$ contains v_i for some $2 \leqslant i \leqslant k$. By (12), there is a unique $w \in g_0 - V(C)$. By (16), the set B_w of the edges of C containing w has at least $k - 1$ edges. By Claim 30, there is no $i' \in \{1, \dots, k+1\} - \{i\}$ such that $v_{i'} \in g_0$. Hence $g_0 - v_i - w \subseteq F$. Since $|F| = r - 1 = |g_0 - v_i - w| + 1$, we may assume by symmetry that $v_c \in g_0$. Recall that if $v_j \in g_0$ then $e_{j-1}, e_j \notin B_w$. This yields that $B_w \subseteq \{e_1, \dots, e_{k+1}\}$. So, there is $1 \leqslant i' \leqslant k$ such that $e_{i'} \in B_w$. As above, $i' \notin \{i-1, i\}$. By symmetry we may assume $i' > i$. Then the cycle

$$v_1, e_1, v_2, \dots, v_{i-1}, e_{i-1}, u_1, e_{i'-1}, v_{i'-1}, e_{i'-2}, v_{i'-2}, \dots, v_i, g_0, w, e_{i'}, v_{i'+1}, e_{i'+1}, \dots, v_c, e_c, v_1$$

omits $v_{i'}$ but goes through w and u_1, thus is longer than C. This contradiction proves (20).

Next, we claim that

$$\textit{no edges in } Z \textit{ intersect } \{v_2, \dots, v_k\}. \tag{21}$$

Indeed, suppose some $g_1 \in Z$ contains v_i for some $2 \leqslant i \leqslant k$. Then the indices of the remaining $r - 1$ vertices of g_1 cannot be in $J_B' = \{1, 2, \dots, k+1\}$. So, $g_1 - v_i \subseteq F$, which yields $c = 2k - 1$, $k = r + 1$, and $g_1 = F + v_i$. Now, by choice of C, since we can switch

g_1 with any edge in B'', each edge in B'' either is contained in F' or contains F. Since only one edge containing F may also contain v_i,

$$v_i \textit{ does not belong to any edge in } B'' \cup (Z - g_1). \tag{22}$$

Let cycle C' be formed by swapping u_1 with v_i and g_1 with e_{c-1}, that is,

$$C' = u_1, e_i, v_{i+1}, e_{i+1}, v_{i+2}, \ldots, e_{c-2}, v_{c-1}, g_1, v_c, e_c, v_1, \ldots, e_{i-1}, u_1.$$

By (20) and (22), v_i is contained only in edges of C', so C' is an optimal choice of cycle under the same conditions as C. If the edges of C' containing v_i are not all consecutive along C', then we get Case 3.1 and arrive at a contradiction. So they are consecutive. As v_i is contained in g_1 (which plays the role of e_{c-1} in C'), either $v_i \in e_c$ or $v_i \in e_{c-2}$. If $v_i \in e_c$ then the cycle

$$C'' = v_1, e_1, v_2, \ldots, v_{i-1}, e_{i-1}, u_1, e_i, v_{i+1}, e_{i+1}, v_{i+2}, \ldots, v_{c-1}, e_{c-1}, v_c, g_1, v_i, e_c, v_1$$

is longer than C. The case for $v_i \in e_{c-2}$ is similar. This proves (21).

By (21), all edges in Z are contained in F'. Since $|Z| \geqslant 4$, each edge in B'' can be replaced with an edge in Z. Thus by Rule (a), each edge in B'' also is contained in F'. Then the $(r+1)$-element set F' contains at least $|B''| + |Z| \geqslant r - 2 + 4 = r + 2$ subsets of size r, a contradiction. □

7 Disjoint cycle-path pairs and cycle-cycle pairs

We conclude the proof of Theorem 10 in this section by proving Lemma 14. Recall the statement.

Lemma 14. *Let (C, P) be a 4-good dcp-pair in H. If $|V(C)| = c$, $|V(P)| = \ell$ and $c < \min\{2k, n, |E(H)|\}$, then $\ell \geqslant k$.*

When handling dcp-pairs, we will also consider structures which are a bit richer, called *disjoint cycle-cycle pairs* or *dcc-pairs* for short. A **disjoint cycle-cycle pair** (C, P) is obtained from a dcp-pair (C, P') with $P' = u_1, f_1, u_2, \ldots, u_\ell$ by adding edge $f_\ell \notin E(C) \cup E(P')$ containing $\{u_1, u_\ell\}$. In other words, in a dcc-pair (C, P), P is a cycle $P = u_1, f_1, u_2, \ldots, u_\ell, f_\ell, u_1$ whose defining elements are disjoint from the defining elements of C.

We partially order dcp- and dcc-pairs together: We say a pair (C, P) *is better* than a pair (C', P') if

(S1) $|V(C)| > |V(C')|$, or

(S2) Rule (S1) does not distinguish (C, P) from (C', P'), and $|V(P)| > |V(P')|$; or

(S3) Rules (S1) and (S2) do not distinguish (C, P) from (C', P'), and the total number of vertices of $V(P)$ contained in the edges of $E(C)$ counted with multiplicities is larger than the total number of vertices of $V(P')$ contained in the edges of $E(C')$; or

(S4) Rules (S1)–(S3) do not distinguish (C, P) from (C', P'), P is a cycle and P' is a path; or

(S5) Rules (S1)–(S4) do not distinguish (C, P) from (C', P'), and the number of edges in $E(P)$ fully contained in $V(P)$ is larger than the number of edges in $E(P')$ fully contained in $V(P')$.

Since a dcc-pair is a dcp-pair (C, P) with an additional edge added to P and rules (S1)–(S3) are the same as (R1)–(R3), all claims in Section 3 that hold for i-good lollipops and dcp-pairs when $i \leqslant 3$ also hold for i-good dcc-pairs.

Let (C, P) be a best dcp- or dcc-pair. Let $C = v_1, e_1, v_2, \ldots, v_c$, and let $P = u_1, f_1, u_2, \ldots u_\ell$ when P is a path and $P = u_1, f_1, u_2, \ldots u_\ell, f_\ell, u_1$ when P is a cycle. To prove Lemma 14, we consider the case $2 \leqslant \ell \leqslant k-1$, $3 \leqslant k \leqslant r+1$, and $\delta(H) \geqslant k$.

Since $\ell \leqslant k-1 \leqslant r$, at most one edge can be contained entirely in $V(P)$, and this is possible only if $k = r+1$. Moreover, by (S5) in this case such an edge is in $E(P)$. Thus any H'-edge containing u_1 or u_ℓ must be contained in $V(C) \cup V(P)$ and must intersect $V(C)$. One benefit of dcc-pairs is that when P is a cycle, then any two consecutive vertices can play the role of u_1 and u_ℓ. In particular,

$$\text{if } P \text{ is a cycle, then each } h \in E(H') \cup E(P) \text{ intersecting } V(P) \text{ is contained in } V(P) \cup V(C). \tag{23}$$

For all $j \in [\ell]$, we denote by B_j the set of edges of C containing u_j, and let $b_j = |B_j|$.

In the next subsection we prove a series of claims about vertices and edges in graph paths and cycles. We will apply these claims to the projections of the Berge cycle C (that is, the graph cycle $v_1, e_1, \ldots, v_c, e_c, v_1$ where $e_i = v_i v_{i+1}$). We complete the proof of Lemma 14 in the final two subsections, breaking into cases of dcp-pairs and dcc-pairs.

7.1 Useful facts on graph cycles

Claim 33. *Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a graph cycle. Let A and B be nonempty subsets in $V(C)$ such that*

$$\text{for each } v_i \in A \text{ and } v_j \in B, \text{ either } i = j \text{ or } |i-j| \geqslant q \geqslant 2. \tag{24}$$

(i) If $A = B$, then $s \geqslant q|A|$.
(ii) If $B \neq A$, then $s \geqslant |A| + |B| + 2q - 3$ with equality only if $A \subset B$ or $B \subset A$.

Proof. Part (i) is obvious. We prove (ii) by induction on $|A \cap B|$ and take indices modulo s.

If $A \cap B = \emptyset$, then C contains $|A| + |B|$ vertices in $A \cup B$ and at least $q-1$ vertices outside of $A \cup B$ between A and a closest to A vertex in B in either direction along the cycle.

Suppose now that (ii) holds for all A' and B' with $|A' \cap B'| < t$ and that $|A \cap B| = t > 0$, say $v_i \in A \cap B$. By symmetry, we may assume $|A| \leqslant |B|$. If $A = \{v_i\}$, then C has $|B|-1$ vertices in $B - A$ and at least $q-1$ vertices between v_i and a closest to v_i vertex in $B - A$, in either direction along the cycle. Thus, $s \geqslant 1+|B-A|+2(q-1) = |A|+|B|-1+2q-2 = |A|+|B|+2q-3$, as claimed.

Finally, suppose $|A| \geqslant 2$. By definition, $(A \cup B) \cap \{v_{i-q+1}, v_{i-q+2}, \dots, v_{i+q-1}\} = \{v_i\}$. Define $e = v_{i-q}v_{i+1}$ and let $A' = A - v_i$, $B' = B - v_i$, and $C' = v_1, e_1, \dots, v_{i-q}, e, v_{i+1}, e_{i+1}, \dots, v_s, e_s, v_1$. Then A' and B' satisfy (24). So, by induction, $|V(C')| \geqslant |A'| + |B'| + 2q - 3$ with equality only if $A' \subset B'$. Hence

$$s \geqslant q + |V(C')| \geqslant q + (|A|-1) + (|B|-1) + 2q - 3 = |A| + |B| + 3q - 5$$

with equality only if $A \subset B$. Since $q \geqslant 2$, this proves (ii). □

The line graph of cycle C_s of length s is again C_s in which the vertices play the roles of the edges of the original graph. Thus Claim 33 implies the following.

Claim 34. *Let $C = v_1, e_1, \dots, v_s, e_s, v_1$ be a graph cycle. Let A and B be nonempty subsets in $E(C)$ such that*

$$\text{for each } e_i \in A \text{ and } e_j \in B, \text{ either } i = j \text{ or } |i-j| \geqslant q \geqslant 2. \tag{25}$$

(i) If $A = B$, then $s \geqslant q|A|$.
(ii) If $B \neq A$, then $s \geqslant |A| + |B| + 2q - 3$ with equality only if $A \subset B$ or $B \subset A$.

Claim 35. *Let $C = v_1, e_1, \dots, v_s, e_s, v_1$ be a graph cycle. Suppose $F \subset V(C)$. Let A and B be nonempty subsets of $V(C)$ that are disjoint from F and such that*

$$\text{for each } v_i \in A \text{ and } v_j \in B, \text{ either } i = j \text{ or } |i-j| \geqslant q \geqslant 2. \tag{26}$$

$$\text{for each } v_i \in F \text{ and } v_j \in F \cup A \cup B, \text{ either } i = j \text{ or } |i-j| \geqslant q \geqslant 2. \tag{27}$$

(i) If $A = B$, then $s \geqslant q|A| + q|F|$.
(ii) If $B \neq A$, then $s \geqslant |A|+|B|+2q-3+q|F|$ with equality only if $A \subset B$ or $B \subset A$.

Proof. Let $C' = v'_1, e'_1, v'_2, \dots, v'_{s'}, e'_{s'}, v'_1$ be a cycle obtained from C by iteratively contracting $e_i, e_{i+1}, \dots e_{i+q-1}$ for each $v_i \in F$. In particular, $s' = s - q|F|$. Due to (27), each $v_i \in A \cup B$ was unaffected by the edge contractions and hence still exists as some $v'_{i'}$ in C'. Moreover, by (27), (26) still holds for A and B in C'.

So, Claim 33 applied to A, B and C' yields that if $A = B$, then $s' \geqslant q|A|$, and if $B \neq A$, then $s' \geqslant |A| + |B| + 2q - 3$ with equality only if $A \subset B$ or $B \subset A$. Since $s' = s - q|F|$, this proves our claim. □

Claim 36. *Let $q \geqslant 1$. Let $C = v_1, e_1, \dots, v_s, e_s, v_1$ be a graph cycle. Let I be an independent nonempty subset of $V(C)$ and $\emptyset \neq B \subset E(C)$ be such that*

$$\text{for each } v_i \in I \text{ and } e_j \in B, \text{ the distance on } C \text{ from } v_i \text{ to } \{v_j, v_{j+1}\} \text{ is at least } q. \tag{28}$$

Then $s \geqslant 2|I| + |B| + 2(q-1)$.

Proof. The vertices of I partition the edges of C into $|I|$ intervals of length at least 2. If such an interval Q contains an $e_j \in B$, then by (28), apart from edges of B it also contains at least $2q$ edges from the ends of the interval to the closest edges in B. This proves the claim. □

Claim 37. *Let $q \geqslant 1$, $q' \geqslant q-1$. Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a graph cycle. Let A be a nonempty subset of $V(C)$ and $\emptyset \neq B \subset E(C)$ be such that*

$$\text{for each } v_i \in A \text{ and distinct } e_j, e_k \in B, \text{ the distance on } C \text{ from } v_i \text{ to } \{v_j, v_{j+1}\} \text{ is at least } q', \text{ and } |j-k| \geqslant q. \tag{29}$$

Then $s \geqslant |A| + q|B| + 2q' - q$.

Proof. Removing the edges of B from C yields B path segments. Any segment not containing a vertex in A has length at least $q-1$. Any segment containing a vertex from A contains at least $2(q'-1) \geqslant q-1$ edges not incident to a vertex in A, and there are at least $|A|+1$ edges incident to A. Thus $s \geqslant |B|(q-1)+|B|+2(q'-1)-(q-1)+|A|+1$. □

Claim 38. *Let $q \geqslant 3$. Let $C = v_1, e_1, \ldots, v_s, e_s, v_1$ be a graph cycle. Let I be an independent nonempty subset of $V(C)$ and $\emptyset \neq A \subset V(C)$ be such that*

$$\text{for each } v_i \in I \text{ and } v_h \in A, \text{ the distance on } C \text{ from } v_i \text{ to } v_h \text{ is either } 0 \text{ or at least } q. \tag{30}$$

(i) If $A = I$ then $s \geqslant q|I|$.
(ii) If $A \subsetneq I$ then $s \geqslant 2|I| + (q-2)(|A|+1)$.
(iii) If $A \setminus I \neq \emptyset$, then $s \geqslant 2|I| + |A| + 2q - 3$.

Proof. Call a subpath of C connecting two verties in I and not containing other vertices in I an *I-interval.* If $A = I$, then each I-interval has length at least q, which yields (i). To prove (ii), observe that each I-interval has at least 2 edges, and the I-intervals with at least one end in A have length at least q. Since $A \neq I$, the number of such intervals is at least $|A|+1$. This proves (ii).

We prove (iii) by induction on $|A \cap I|$. If $|A \cap I| = \emptyset$, then each I-interval Q containing $m > 0$ vertices of A by (30) contains at least $2q$ edges from the ends of the interval to the closest vertices in A and a path of at least $m-1$ edges connecting these extremal vertices. Together, Q has at least $2q + m - 1$ edges which is by $m + 2q - 3$ more than 2. This proves the base of induction.

Suppose now that $v_i \in A \cap I$. Consider C' obtained from C by deleting vertices $v_i, v_{i+1}, \ldots, v_{i+q-1}$ and adding edge $v_{i-1}v_{i+q}$. Let $I' = I - v_i$ and $A' = A - v_i$. By induction assumption, $|V(C')| \geqslant 2|I'| + |A'| + 2q - 3 = 2|I| + |A| + 2q - 6$. So, $s \geqslant 2|I| + |A| + 2q - 6 + q \geqslant 2|I| + |A| + 2q - 3$, as claimed. □

7.2 Proof of Lemma 14 for dcp-pairs

Proof of Lemma 14 when (C, P) is a dcp-pair. Suppose that, among all dcp- and dcc-pairs, a best (C, P) is a dcp-pair.

Case 1: u_1 is in an H'-edge g. If $u_\ell \in g$, then we can add g to P as f_ℓ, a contradiction to (S4). So, $|g\cap V(C)| \geqslant r-\ell+1$. If $b_\ell > 0$, then by Claims 17 and 36 with $I = g\cap V(C)$, $B = B_\ell$ and $q = \ell$, we have

$$c \geqslant 2(r-\ell+1)+2(\ell-1)+b_\ell = 2r+b_\ell \geqslant 2k-2+b_\ell.$$

So $b_\ell \leqslant 1$, and $d_{H'}(u_\ell) \geqslant d_H(u_\ell)-b_\ell-d_P(u_\ell) \geqslant k-1-(\ell-1) \geqslant 1$. Let $g' \in E(H')$ contain u_ℓ. Since P does not extend to a cycle, $u_1 \notin g'$, so $g' \neq g$ and $|g'\cap V(C)| \geqslant r-\ell+1$.

For $j \in \{1,\ell\}$, let $A_j = N_{H'}(u_j)\cap V(C)$. By Claim 17, if $v_i, v_j \in A_1\cap A_\ell$ then $i = j$ or $|i-j| \geqslant \ell+1$. We apply Claim 35 with $F = A_1\cap A_\ell$, $A = A_1 - F$, $B = A_\ell - F$ and $q = \ell+1$. If $A \not\subseteq B$ and $B \not\subseteq A$, then Claim 35(ii) gives

$$c \geqslant 2(r-\ell+1-|F|)+2(\ell+1)-2+(\ell+1)|F| \geqslant 2r-2\ell+2-2|F|+2\ell+3|F| \geqslant 2r+2 \geqslant 2k,$$

a contradiction. If say $A \subsetneq B$ but $|A\cup F|+|B\cup F| \geqslant 2r-2\ell+3$, then

$$c \geqslant 2r-2\ell+3-2|F|+2(\ell+1)-3+(\ell+1)|F| \geqslant 2r-2\ell+2+2\ell \geqslant 2k,$$

again. So, assume $A_1 = A_\ell$.

If $k \leqslant r$, then by Claim 35(i), $c \geqslant (\ell+1)(r-\ell+1) \geqslant 2r \geqslant 2k$ for $2 \leqslant \ell \leqslant k-1 \leqslant r-1$, a contradiction. Thus, let $k = r+1$. Also, if $|A_1| \geqslant r-\ell+2$, then we similarly have $c \geqslant (\ell+1)(r-\ell+2) \geqslant 2(r+1) \geqslant 2k$. Thus $|A_1| = r-\ell+1$, which means $g\cap V(P) = V(P)-u_\ell$, $g'\cap V(P) = V(P)-u_1$, and u_1, u_ℓ are not contained in other edges of H'. It follows that there are $e_i \in E(C)$ containing u_1 and $e_j \in E(C)$ containing u_ℓ.

If $i \neq j$, then Claim 36 with $I = A_1$, $B = \{e_i, e_j\}$ and $q = \ell$ implies $c \geqslant 2(r-\ell+1)+2+2(\ell-1) = 2r+2 \geqslant 2k$. Thus $e_i = e_j$ and $d_C(u_1) = d_C(u_\ell) = 1$ and $r = |e_i| \geqslant |\{u_1, u_\ell, v_i, v_{i+1}\}| \geqslant 4$. At least one of $f_1, f_{\ell-1}$ is not $V(P)$, say $f_1 \neq V(P)$. Then switching g with f_1 in P we get another dcp-pair that is 2-good. Hence every vertex in $f_1\cap V(C)$ is distance at least $\ell+1$ from any vertex in $A_\ell = A_1$. Either $f_1\cap V(P) \neq \{u_1,\ldots,u_{\ell-1}\}$ or $A_1 \neq f_1\cap V(C)$. In both cases, we get a case we have considered.

Case 2: u_1, u_ℓ are not in any H'-edges. Then $d_C(u_1) \geqslant k-(\ell-1) \geqslant 2$ and $d_C(u_\ell) \geqslant k-(\ell-1) \geqslant 2$.

We apply Claim 34 with $A = B_1$, $B = B_\ell$ and $q = \ell$. If $A \not\subseteq B$ and $B \not\subseteq A$, then the claim gives $c \geqslant 2(k-\ell+1)+2\ell-2 = 2k$, a contradiction. If say $A \subseteq B$ but $|A|+|B| \geqslant 2k-2\ell+3$, then $c \geqslant 2k-2\ell+3+2\ell-3 = 2k$, again. So assume $B_1 = B_\ell$ and $|B_1| = k-\ell+1$.

Then $d_P(u_\ell) \geqslant \ell-1 = |E(P)|$. So $u_\ell \in f_1$. For any $2 \leqslant i \leqslant \ell-1$ Consider $P_i = u_1, f_1, \ldots, u_{i-1}, f_i, u_\ell, f_{\ell-1}, \ldots, u_i$, and observe that (C, P_i) is another best dcp-pair. If $d_{H'}(u_i) \geqslant 1$, then we get Case 1. Otherwise as above, we get $B_i = B_1$ and $d_P(u_i) = |E(P)|$. In other words,

$$V(P) \subseteq \bigcap\nolimits_{j=1}^{\ell} f_j \text{ and } V(P) \subseteq e_i \text{ for each } e_i \in B_1 = \cdots = B_\ell. \tag{31}$$

By Claim 17(a), for distinct $e_i, e_j \in B_1$, $|j-i| \geqslant \ell$.

Since $V(P) \subseteq f_j$ for each $j \in [\ell]$, we can construct any path covering $V(P)$ using edges $f_1, \ldots, f_{\ell-1}$ in arbitrary order. Let $I = \left(\bigcup_{j=1}^{\ell} f_j\right) \cap V(C)$ and $J = \left(\bigcup_{j=1}^{\ell} f_j\right) - (V(P) \cup V(C))$.

Suppose $w \in J$, by symmetry say $w \in f_1$. Assume first that there is $g \in E(H')$ such that $w \in g$. By the case, $g \cap V(P) = \emptyset$.

Let P' be obtained by replacing u_1 with w in P. Then (C, P') is a 2-good dcp-pair and so by Claims 17 and 18, $g \subseteq V(C) \cup \{w\}$ and the distance from any $v_i \in g$ to some edge in B_1 is at least ℓ. Apply Claim 37 with $A = g \cap V(C)$, $B = B_1$, and $q = q' = \ell$ to obtain $c \geqslant (r-1) + \ell(k - (\ell - 1)) + \ell \geqslant (r-1) + 2k > c$, a contradiction. Thus $d_{H'}(w) = 0$.

If there is $e_j \in E(C) - B_1$ such that $w \in e_{i_0}$, then in view of the path P' above, then for any $e_i \in B_\ell$, $|j-i| \geqslant \ell$ by Claim 17. Therefore $c \geqslant \ell(|B_1|+1) = \ell(k-(\ell-1)+1) \geqslant 2k$, a contradiction again. Since $d_H(w) \geqslant k$, $w \in \bigcap_{j=1}^{\ell-1} f_j$ and $w \in e_i$ for each $e_i \in B_1$. Then (31) holds not only for $V(P)$ but for $V(P) \cup J$, which implies $\ell \leqslant |V(P) \cup J| \leqslant r - 2$ and therefore $|I| \geqslant 2$. If $v_i \in I$, say $v_i \in f_1$, $i \leqslant i' \leqslant i + \ell - 2$ and $e_{i'} \in B_\ell$, then the cycle obtained from C by replacing subpath $v_i, e_i, v_{i+1}, \ldots, e_{i'}, v_{i'+1}$ with the path $v_i, f_1, u_2, f_2, \ldots, u_\ell, e_{i'}, v_{i'+1}$ is longer than C, a contradiction.

Applying Claim 37 with $A = I, B = B_1, q' = \ell - 1, q = \ell$ we obtain $c \geqslant 2 + \ell(k - (\ell - 1)) + 2(\ell - 1) - \ell \geqslant \ell(k - \ell + 2) \geqslant 2k$, a contradiction. □

7.3 Proof of Lemma 14 for dcc-pairs

Proof of Lemma 14 when (C, P) is a dcc-pair. **Case 1:** There is $g \in E(H')$ with $g \cap V(P) \neq \emptyset$ and $|g \cap V(C)| \geqslant k - \ell + 1$. By symmetry, we may assume $u_1 \in g$ and $|f_\ell \cap V(P)| \leqslant |f_1 \cap V(P)|$. Since $\ell \leqslant r$, at most one f_j is contained in $V(P)$. So

$$|f_\ell \cap V(C)| \geqslant r - \ell \textit{ and if } r = \ell, \textit{ then } |f_\ell \cap V(C)| \geqslant 1. \tag{32}$$

By Claim 17, if $v_i \in g$ and $v_j \in f_\ell$, then $i = j$ or $|i - j| \geqslant \ell + 1$. By Claim 38 with $I = g \cap V(C)$, $A = f_\ell \cap V(C)$ and $q = \ell + 1$, we have 3 possibilities. If $A = I$, then $c \geqslant (\ell + 1)(k - \ell + 1) \geqslant 2k$. If $A \subsetneq I$ then $c \geqslant 2(k - \ell + 1) + (\ell - 1)(|f_\ell \cap V(C)| + 1)$. By (32), when $2 \leqslant \ell \leqslant r - 1$ this gives $c \geqslant 2(k - \ell + 1) + (\ell - 1)(r - \ell + 1) \geqslant 2k$, and when $\ell = r$ (and hence $k = r + 1$) this gives $c \geqslant 2(k - r + 1) + (k - 2)2 = 2k$, again. If $A \setminus I \neq \emptyset$, then $c \geqslant 2(k - \ell + 1) + |A| + 2\ell - 1 > 2k$. In all cases we get a contradiction.

Case 2: There is $g \in E(H')$ such that $g \supseteq V(P)$. Then by Rule (S5), each $f_j \in E(P)$ contains $V(P)$. Since $|E(P)| \geqslant 2$, $\ell \leqslant r - 1$. Let $E' = E(P) \cup \{g\}$. By Claim 23, every edge in E' is a subset of $V(C) \cup V(P)$. Set $I = (\bigcup_{f \in E'} f) \cap V(C)$. Since the sets $f_j \cap V(C)$ are distinct for distinct j, $|I| \geqslant r - \ell + 1$. Moreover, if $r = \ell + 1$ then $|I| \geqslant |E'| = \ell + 1 \geqslant 3 > r - \ell + 1$. By the case, for any $u_i, u_{i'} \in V(P)$ and any f, f', there is a $u_i, u_{i'}$-path of length $\ell - 1$ whose set of edges is $E' - \{f, f'\}$. Hence if $v_i \in f \in E'$ and $v_{i'} \in f' \in E'$ where $f' \neq f$, then $|i' - i| \geqslant \ell + 1$. Since $|E'| \geqslant 3$, there are some $t \geqslant 3$ segments in C of length at least $\ell + 1$ that are internally disjoint from I. The vertices of I are nonconsecutive on C thus the edges $\{e_i, e_{i+1} : v_i \in I\}$ are distinct (but may intersect the segments of length at least $\ell + 1$). We get

$$c \geqslant 2|I| + t(\ell - 1) \geqslant 2(r - \ell + 1) + t(\ell - 1) = 2r + \ell - 1.$$

This is at least $2k$, unless $t = 3$, $k = r + 1$, $\ell = 2$ and $|I| = r - \ell + 1$. If $|I| = r - \ell + 1$, then for each $f \in E', f \cap V(C)$ is an $r - \ell$-subset of I, and it follows that every distinct $v_i, v_j \in I$ satisfies $|i - j| \geqslant \ell + 1$. In order to have $t = 3$, we must then have $|I| = 3$. Thus, the unresolved situation is $\ell = 2$, $|I| = 3 = r - 2 + 1 = r - 1$ and $k = r + 1 = 5$. In this case, $c = 9$ and up to symmetry, $I = \{v_1, v_4, v_7\}$, so we may assume $f_1 = \{u_1, u_2, v_1, v_4\}, f_2 = \{u_1, u_2, v_1, v_7\}, g = \{u_1, u_2, v_4, v_7\}$. But $d_H(u_2) \geqslant 5$. If some e_i contains u_2, then by symmetry we may assume $i = 1$, contradicting Claim 16. Thus $d_C(u_2) = 0$, so $d_{H'}(u_2) \geqslant 3$. One such edge is g, another is possibly $\{u_2, v_1, v_4, v_7\}$, but then the third must contain a vertex in $V(C) - \{v_1, v_4, v_7\}$, again contradicting Claim 16 (maybe switching g with f_1).

Case 3: There are edges in H' intersecting $V(P)$, but for each $g \in E(H')$, $|g \cap V(P)| \leqslant \ell - 1$ and $|g \cap V(C)| \leqslant k - \ell$. Then $k = r + 1$ and for each $g \in E(H')$, $|g \cap V(P)| = \ell - 1$. Let $A = \bigcup_{i=1}^{\ell} f_i \cap V(C)$. Since $|E(P)| \geqslant 2$, $|A| \geqslant r - \ell + 1$.

Suppose some distinct i, j, $v_i \in g$ and $v_j \in A$. Without loss of generality, $v_j \in f_\ell$ (because P is a cycle). Since $|g \cap V(P)| = \ell - 1$, we may also assume $u_1 \in g$. By Claim 17, $|i - j| \geqslant \ell + 1$. So, we can apply Claim 38 with $I = g \cap V(C)$ and $q = \ell + 1$. Since $|A| \geqslant |I|$, if $A \neq I$, then

$$c \geqslant 2|I| + |A| + 2q - 3 \geqslant 2(r - \ell + 1) + (r - \ell + 1) + 2\ell - 1 = 3r + 2 - \ell \geqslant 2r + 2, \text{a contradiction.}$$

Thus $A = I$ and the vertices of A partition $V(C)$ into $r - \ell + 1$ intervals of length at least $\ell + 1$. If some e_i contains a vertex of P, say $u_\ell \in e_i$, then $g \cap \{u_{\ell-1}, u_1\} \neq \emptyset$, say $u_1 \in g$. So, in this case, if $v_j \in I$, then by Claim 17, v_j is distance at least ℓ from $\{v_i, v_{i+1}\}$ in C. An interval containing m edges of C that intersects $V(P)$ has at least $2\ell + m$ edges, and

$$c \geqslant (\ell + 1)(r - \ell + 1) - (\ell + 1) + 2\ell + m \geqslant (\ell + 1)(r - \ell + 1) + \ell$$

with equality only when exactly one edge of C intersects $V(P)$. We get a contradiction, unless $\ell = r$ and exactly one edge of C intersects $V(P)$. In this case, $|I| = 1$ and we can rename the vertices of C so that $I = \{v_c\}$ and e_r is the edge intersecting $V(P)$. Then each edge $f \in E(H') \cup E(P)$ intersecting $V(P)$ is contained in $V(P) \cup \{v_c\}$. So, the sum of degrees of vertices in P is at most $\binom{\ell}{r-1}(r - 1) \leqslant r(r - 1)$ (for edges containing v_c) plus r (for a possible edge $V(P)$) plus $r - 2$ (for e_r). This totals $r^2 + r - 2$ which is less than $k\ell \leqslant \delta(H)|V(P)|$, a contradiction.

The remaining possibility is that no edges of C intersect $V(P)$. Since $A = I$ for any choice of $g \in E(H')$ and $|I| = r - \ell + 1$, $g \cap V(C)$ is the same for all $g \in E(H')$ intersecting $V(P)$, and $f \cap V(C) \subset I$ for all $f \in E(P)$. So each edge $f \in E(H') \cup E(P)$ intersecting $V(P)$ is contained in $V(P) \cup I$ and $|V(P) \cup I| = r + 1$. Therefore, $\sum_{u \in V(P)} d_H(u)$ is at most $\ell(r - \ell + 1)$ (for the at most $r - \ell + 1$ edges containing $V(P)$) plus $\ell(\ell - 1)$ (for the at most ℓ edges containing I), which sums to $\ell r < \ell k$, a contradiction.

Case 4: No edges in H' intersect $V(P)$ and $f_j \supseteq V(P)$ for each $j \in [\ell]$. Let $B(P)$ be the set of the edges in C intersecting $V(P)$. Since $|E(P)| \geqslant 2$, $\ell \leqslant r - 1$. Let $I = \bigcup_{i=1}^{\ell} f_i \cap V(C)$. By (23), each f_i is contained in $V(C) \cup V(P)$, therefore the sets

$f_i \cap V(C)$ are distinct for distinct i. It follows that

$$|I| \geqslant r - \ell + 1 \geqslant k - \ell. \tag{33}$$

By the case, for any $u_i, u_{i'} \in V(P)$ and any $f_j \in E(P)$, there is a $u_i, u_{i'}$-path of length $\ell - 1$ whose set of edges is $E(P) - f_j$. Hence if v_j belongs to an edge of $E(P)$ and $e_{j'} \in B(P)$ where $j \neq j'$, then by Claim 17 v_j has distance at least ℓ from $\{v_{j'}, v_{j'+1}\}$. Also the vertices of I are nonconsecutive on C. So, by Claim 36 with $B = B(P)$ and $q = \ell$, we have $c \geqslant 2|I| + 2(\ell - 1) + |B(P)|$. If $|I| \geqslant k - \ell + 1$ or $|I| = k - \ell$ and $|B(P)| \geqslant 2$, then we get a contradiction. Otherwise, by (33), $k = r + 1$, $|I| = k - \ell$ and $|B(P)| \leqslant 1$. Since each $u_j \in V(P)$ is in at least $k - \ell \geqslant 1$ edges of C, $|B(P)| = 1$, say $e_i \supseteq V(P)$, and $r - 2 = |e_i| - 2 \geqslant |V(P)| = \ell$. So, $\sum_{u \in V(P)} d_H(u) \leqslant \ell|E(P)| + \ell(1) = \ell(\ell + 1) \leqslant \ell(r - 1) < \ell k$, a contradiction.

Case 5: No edges in H' intersect $V(P)$ and there is $j \in [\ell]$ such that either $|f_j \cap V(C)| \geqslant k - \ell + 1$ or $|f_j \cap V(C)| = k - \ell$ and $|B_j \cup B_{j+1}| \geqslant 2$. By Claim 36 with $I = f_j \cap V(C)$ $B = B_j \cup B_{j+1}$ and $q = \ell$, we have $c \geqslant 2|I| + 2(\ell - 1) + |B|$. Since by the case $2|I| + |B| \geqslant 2k - 2\ell + 2$, we get a contradiction.

Case 6: All other possibilities. This means (a) no edges in H' intersect $V(P)$, (b) there is $j_0 \in [j]$ with $|f_{j_0} \cap V(P)| \leqslant \ell - 1$, and (c) for each $j \in [\ell]$, $|f_j \cap V(C)| \leqslant k - \ell$, and if $|f_j \cap V(C)| = k - \ell$ then $|B_j \cup B_{j+1}| \leqslant 1$. Since $|f_{j_0} \cap V(C)| \geqslant r - (\ell - 1) \geqslant k - \ell$, in order for (c) to hold, we need $k = r + 1$ and $|B_{j_0} \cup B_{j_0+1}| \leqslant 1$. For all $u_j \in V(P)$, $d_C(u_j) \geqslant k - \ell \geqslant 1$. In view of u_{j_0}, we need $k = \ell + 1$. Thus $r = \ell = k - 1$. Then as an edge in C cannot contain all ℓ vertices in P, at least two edges of C intersect $V(P)$, so there are two distinct j_1, j_2 such that $|B_{j_i} \cup B_{j_i+1}| \geqslant 2$ for $i = 1, 2$. But since $\ell = r$, at most one of f_{j_1} and f_{j_2} contains $V(P)$, so the other satisfies Case 5, a contradiction. □

8 Concluding remarks

1. It would be interesting to understand whether for $4 \leqslant k \leqslant r$, the bound on $\delta(H)$ in Theorem 10 may be lowered to $k - 1$ to match Construction 1.1 or there is a better construction.

2. Observe that there are sharpness examples for Theorems 1, 2, 5 and 9 with connectivity 3 and greater (see, e.g. Construction 1.3 for Theorem 9). But we do not know 3-connected extremal examples for Theorem 9. Moreover, it well may be that hypergraphs with higher connectivity require a smaller minimum degree to force the existence of a long Berge cycle. A natural question to study would be the following.

Question. Let $n \geqslant r + 1 \geqslant k \geqslant 3$. What is the minimum number $f(k, r)$ such that every n-vertex, r-uniform, 3-connected hypergraph with minimum degree at least $f(k, r)$ necessarily has $c(H) \geqslant \min\{n, |E(H)|, 2k\}$?

3. Another interesting question is: How the bound will change if instead of r-graphs we consider the hypergraphs in which the size of each edge is at least r?

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