

Mean-field Potts and random-cluster dynamics from high-entropy initializations

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Abstract

A common obstruction to efficient sampling from high-dimensional distributions is the multimodality of the target distribution because Markov chains may get trapped far from stationarity. Still, one hopes that this is only a barrier to the mixing of Markov chains from *worst-case* initializations and can be overcome by choosing high-entropy initializations, e.g., a product or weakly correlated distribution. Ideally, from such initializations, the dynamics would escape from the saddle points separating modes quickly and spread its mass between the dominant modes.

In this paper, we study convergence from high-entropy initializations for the random-cluster and Potts models on the complete graph—two extensively studied high-dimensional landscapes that pose many complexities like discontinuous phase transitions and asymmetric metastable modes. We study the Chayes–Machta and Swendsen–Wang dynamics for the mean-field random-cluster model and the Glauber dynamics for the Potts model. We sharply characterize the set of product measure initializations from which these Markov chains mix rapidly, even though their mixing times from worst-case initializations are exponentially slow. Our proofs require careful approximations of projections of high-dimensional Markov chains (which are not themselves Markovian) by tractable 1-dimensional random processes, followed by analysis of the latter’s escape from saddle points separating stable modes.

1 Introduction

A ubiquitous and generically hard computational task is to minimize a high-dimensional function f over a discrete space $\{1, \dots, q\}^n$; closely connected is the problem of sampling from the probability distribution with mass proportional to $e^{-\beta f}$ for β large. The function f is often viewed as an energy landscape in statistical physics or as a loss function in machine learning, and the β -large setting is referred to as the low-temperature regime. The difficulty is induced by the possibility of f having several minima with large basins of attraction (or, equivalently, by the multi-modality of the induced distribution); this poses a barrier to traditional optimization/sampling algorithms like gradient descent and Markov chain sampling, at least when initialized from a worst-case state, e.g., in a sub-optimal mode.

Still, a black box approach to these tasks would select the initial state randomly from a product measure or more general high-entropy distribution and would hope that this can circumvent (some of) the bottlenecks in the space. Ideally, a high-entropy initialization spreads its mass across the space in such a way that dynamics are primarily driven by diffusion away from the saddles separating dominant modes, picking the basin to fall into with the correct probability. However, rigorous studies of convergence rates to stationarity from high-entropy initializations are severely lacking and generically difficult.

We focus on the theoretical study of high-entropy initializations in the context of Markov chains for spin system models, like the Ising, Potts, and random-cluster models. In the statistical physics literature,

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questions of dynamics from high-entropy initializations have a long (empirical) history. A paradigmatic version of this is in the Ising model on the integer lattice graph \mathbb{Z}^d at low temperatures, where it is widely expected (though entirely open to prove) that the Glauber dynamics (the natural reversible local Markov chain) initialized from a uniform-at-random assignment of ± 1 mixes in polynomial time: see, e.g., the review paper [9] for the rich physics literature about this process. Even in other geometries, for instance on random graphs, bounding the speed of convergence of the low-temperature Ising Glauber dynamics initialized uniformly at random from $\{\pm 1\}^n$ seems to be mathematically very challenging. (See, e.g., [20, 25, 45, 46] for recent progress on zero-temperature $\beta = \infty$ versions of this problem.) In the special case of the Ising Glauber dynamics on the complete graph, known as the mean-field model, the process reduces perfectly to a 1-dimensional birth-death Markov chain. Here [18, 38] showed the escape from the saddle corresponding to balanced configurations at low temperatures is fast, implying $O(n \log n)$ mixing when initialized from a product of fair coin tosses, despite slow mixing from worst-case.

When one generalizes from the Ising model to the q -state Potts model, the above questions become significantly more complicated due to the presence of a higher-dimensional space of spin counts, the possibility of slow mixing at intermediate temperatures, and the lack of symmetry between the modes. The closely related random-cluster model faces similar difficulties, as well as the non-locality of its interactions and update rules. Together, these yield a rich class of models for investigating the above-described expected benefits from high-entropy (e.g., product) initializations to overcome the slow mixing of standard Markov chains.

Let us note that purely from an approximate sampling perspective, it has been known that sampling from the ferromagnetic Ising model is tractable since [35, 44], but general sampling in the ferromagnetic Potts model is #BIS hard [23, 28]. This has led to much recent work towards finding general criteria (on q , the underlying graph, and the temperature) under which sampling is algorithmically tractable, e.g., [1, 3, 8, 11, 12, 30–32, 34]. Of particular relevance to our work is a series of recent results using special initializations of Markov chains to overcome bottlenecks, specifically using a priori knowledge of the global minimizers of the energy landscape to initialize near them (e.g., initializing an Ising model in the all-plus or all-minus states with probability 1/2 each), and showing that mixing locally in those modes is rapid: see [10, 21, 26, 27] for recent works in this direction for the Ising, random-cluster, and exponential random graph models.

By contrast, the interest in initializations from product measures (and similar high-entropy distributions), is towards understanding success/failure of black-box approaches that do not require prior knowledge of the target distribution's modes. In particular, such initializations do not rely on, nor hope for, the convexity of the landscape near the initialization, instead relying on the non-convexity to allow access to multiple modes. We also mention that there are close connections between such high-entropy initializations and simulated annealing and tempering schemes, which we expand on in Section 1.3.

In this paper, we study the convergence of Markov chains in spin systems in the presence of metastability and phase coexistence from high-entropy initializations (predominantly, product measures). By phase coexistence, we mean the presence of multiple modes of roughly equal weights, and by metastability, we mean the presence of multiple modes of vastly different weights; in both situations, Markov chains get trapped for exponential time scales far from stationarity. We study two canonical Markov chains, the Chayes–Machta (CM) dynamics for the mean-field random-cluster model and the Glauber dynamics for the mean-field Potts model. These chains are known to be slow mixing from worst-case initializations for ranges of intermediate, near-critical, and low temperatures. We study their equilibration times in their slow-mixing regimes from product measure initializations and sharply characterize the families of product initializations that lead to optimally fast mixing.

Already on the complete graph, the understanding of mixing from such initializations poses significant challenges, and traditionally, the mean-field setting is a natural starting point for the study of random processes in other families of graphs of significant interest. Our analysis requires a careful understanding of low-dimensional projections of the Markov chains near unstable saddle points of the landscape separating

its dominant modes; namely, in regions where the fluctuations of the projected chain compete on the same scale with its drift as well as regions of strict non-convexity. Unlike the case of the mean-field Ising Glauber dynamics where the projection onto the count of plus spins is itself a 1-dimensional birth-and-death Markov chain, in our setting, these projections are not themselves Markov chains. Instead, we approximate the projections of truly high-dimensional Markov chains on a “good” set of configurations by more tractable 1-dimensional chains whose diffusion away from saddle points separating modes we then study. Altogether, this amounts to a significantly more refined analytic control of the Markov chains than previous works on these dynamics. See Section 1.4 for more on the proof ideas.

1.1 Main results

The (ferromagnetic) Potts model on the n -vertex complete graph $G = ([n], \binom{[n]}{2})$ at inverse temperature $\beta > 0$ is the probability distribution π_β over spin assignments $\sigma \in \{1, \dots, q\}^n$ to vertices of G such that

$$\pi_\beta(\sigma) = \frac{1}{Z_{\beta,q}} \cdot \exp \left(-\frac{\beta}{n} \sum_{i \neq j} \mathbf{1}\{\sigma_i \neq \sigma_j\} \right), \quad (1)$$

where Z_β is the normalizing constant known as the partition function and σ_i denotes the spin of vertex i .

Closely related is the random-cluster model on G with parameters q , now allowed to be in $(0, \infty)$, and $p \in [0, 1]$. This is a model that assigns to edge-subsets A of G probability

$$\mu_\beta(A) = \frac{1}{Z_{p,q}} \cdot \left(\frac{p}{1-p} \right)^{|A|} q^{k([n], A)}, \quad (2)$$

where $k([n], A)$ is the number of connected components in the subgraph induced by A , and $Z_{p,q}$ is the corresponding partition function. Note that when $q = 1$, the mean-field random-cluster model corresponds exactly to the Erdős–Rényi random graph model, but when $q \neq 1$ the cluster weighting can change the phenomenology significantly. When $q \geq 2$ is integer and $p = 1 - e^{-\beta/n}$, the random-cluster model is closely linked to the Potts model; in particular, if one assigns spins to the components of $A \sim \mu_\beta$ independently and uniformly at random among $\{1, \dots, q\}$, the result is a sample from the Potts distribution at inverse temperature β . To unify the discussion, via the reparametrization $p = 1 - e^{-\beta/n} \approx \beta/n$, we henceforth only work with a temperature parameter β , even when discussing the random-cluster model.

The two canonical Markov chains we consider are the Glauber dynamics for the Potts model and the Chayes–Machta (CM) dynamics for the random-cluster model. The Glauber dynamics is the Markov chain which at each time-step picks a vertex i uniformly at random among $[n]$ and resamples its spin conditionally on the remainder of the configuration; namely, resamples it to take spin $k \in \{1, \dots, q\}$ with probability proportional to $\exp(\frac{\beta}{n} \sum_j \mathbf{1}\{\sigma_j = k\})$. The Potts Glauber dynamics is typically exponentially slow to equilibrate at low temperatures due to bottlenecks between spin configurations where each of the q colors dominates. The CM dynamics is an appealing alternative that uses the connection between the Potts and random-cluster models to overcome the low-temperature bottlenecks of the Glauber dynamics. More precisely, the CM dynamics is the following Markov chain defined for general $q \geq 1$ as follows: from an edge-subset X_t generate X_{t+1} via

1. *Activation step:* Independently for each connected component C of X_t , with probability $\frac{1}{q}$ label all vertices of C *active* and otherwise label all vertices of C *inactive*.
2. *Percolation step:* Independently for each edge e both of whose endpoints are active, include e in X_{t+1} with probability β/n . For all other edges e , let $X_{t+1}(e) = X_t(e)$.

The CM dynamics is a generalization to non-integer values of q of the famous Swendsen–Wang (SW) dynamics for the Potts model. Indeed for integer q , if the activation step of the CM dynamics is performed by coloring the components of X_t independently among $[q]$, and activating one of the color classes, then this produces a Markov chain on Potts configurations which is basically equivalent to the SW dynamics. (Technically, the SW chain treats each color class as an activated set and does percolation steps within each of them simultaneously before recoloring.) As far as speeds of convergence are concerned, the CM and SW are thus closely related [4, 19]; our results all also apply to the SW dynamics without significant modifications; see Remark 1.

The standard way to quantify the speed of convergence of a Markov chain is the mixing time, i.e., the time it takes to reach total-variation distance ε to stationarity, either from a prescribed initialization, or from a worst-case initialization. For the worst-case mixing, as soon as $\varepsilon < 1/2$, the TV-distance decays exponentially fast, so one usually takes as a convention $\varepsilon = 1/4$. In the context of this paper, we are considering mixing times from a specified initial distribution, and often in settings where the worst-case initialization mixing time is exponentially large. This does not satisfy the same boosting property (i.e., exponential decay of TV-distance once it is $< 1/2$) as the worst-case mixing time does, so when we refer to bounds on mixing time from an initial distribution, we mean that in that time, we can attain any TV-distance $\varepsilon > 0$ independent of n . Note that when we discuss mixing from an initial distribution, the law of the process is over both the initialization and the run of the Markov chain.

To describe what is known about the mixing time behavior of the CM and Glauber dynamics, we first recall the static phase diagram of the mean-field Potts and random-cluster models. These have been extensively analyzed and are controlled by three threshold values $\beta_u \leq \beta_c \leq \beta_s$ of β . The middle one, β_c , is the critical point marking the distinction between order (a dominant spin class in the Potts setting or, equivalently, a “giant” connected component in the random-cluster) and disorder (balanced spin classes or small connected components) in a typical sample from the distribution. When $q > 2$, β_u and β_s are two additional critical points marking the onset of metastability for the ordered phases and the end of metastability of the disordered phase, respectively (see Figure 2) and [6, 15, 40].

The above static description of these mean-field models is closely related to the worst-case initialization mixing time of its dynamics. For instance, the Potts Glauber dynamics transitions from fast $O(n \log n)$ convergence when $\beta < \beta_u$ to exponentially slow as soon as $\beta > \beta_u$; first, when $\beta \in (\beta_u, \beta_c)$, due to the metastability of q basins corresponding to ordered configurations (one for each spin) preventing equilibration to the disordered phase, then when $\beta \geq \beta_c$ due to the coexistence of the q ordered phases: see [16]. The CM dynamics is similarly fast when $\beta < \beta_u$, and still exponentially slow in the window (β_u, β_s) . This is because, even though the q ordered phases are equivalent as far as the CM dynamics is concerned, there is still metastability between the ordered or disordered phases. The benefit of CM dynamics is seen once $\beta > \beta_s$, where it becomes fast again: see [2, 4, 5, 22, 24, 39] which together give the worst-case mixing behavior of the mean-field CM and SW dynamics.

We start by stating our main result for the CM dynamics demonstrating the sharp families of product initializations that circumvent the exponential slowdown throughout the metastability regime $\beta \in (\beta_u, \beta_s)$. Let $\bigotimes \text{Ber}(\lambda_0/n)$ be the product distribution where each edge is included with probability λ_0/n . (Note that $\bigotimes \text{Ber}(\lambda_0/n)$ corresponds to the classical Erdős–Rényi $G(n, \lambda_0/n)$ random graph model.)

Theorem 1.1. *For every $q > 2$ and $\beta \in (\beta_u, \beta_s)$, there exist $\lambda_*(\beta, q)$ and $c_*(q)$ such that the CM dynamics initialized from $\bigotimes \text{Ber}(\lambda_0/n)$ mixes in $O(\log n)$ steps whenever*

1. $\beta \in (\beta_u, \beta_c)$ and $\lambda_0 < \lambda_*(\beta, q) - \omega(n^{-1/2})$,
2. $\beta = \beta_c$ and $\lambda_0 = \lambda_*(\beta, q) + c_*(q)n^{-1/2} + o(n^{-1/2})$,
3. $\beta \in (\beta_c, \beta_s)$ and $\lambda_0 > \lambda_*(\beta, q) + \omega(n^{-1/2})$.

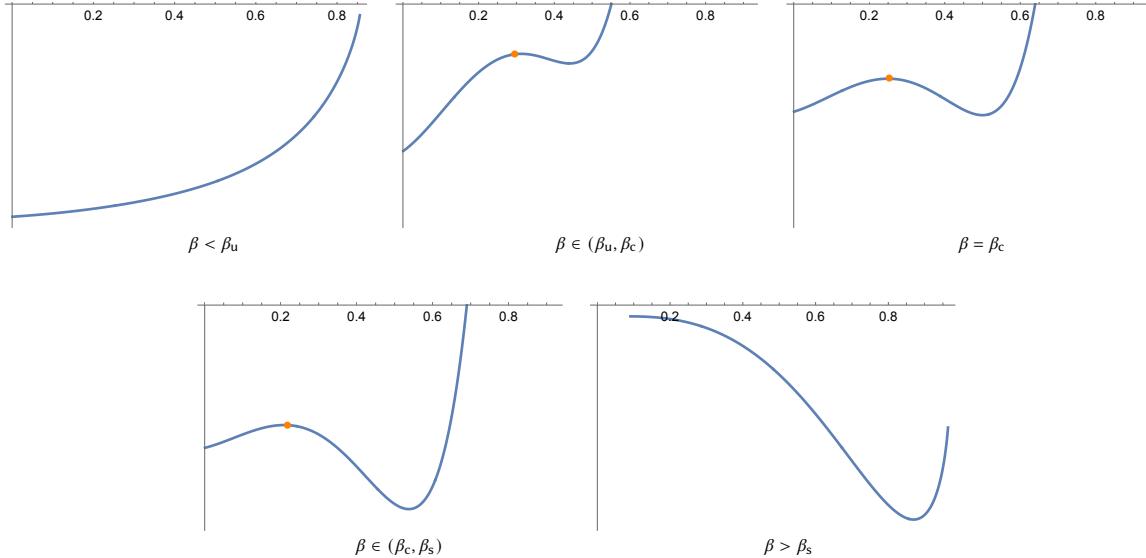


Figure 1: The negative log probability (i.e., free energy) of the largest connected component size in the n -vertex mean-field random-cluster model with $q = 3$ (for large n). The x -axes are the fraction of vertices in the largest component, and the y -axes represent the logarithm of the total weight of the configurations with such largest component sizes divided by n . We emphasize that the landscape is a function of all the component sizes and can exhibit more complexity in the other directions, but our results show this 1-dimensional projection approximately governs the dynamics from product initializations.

Recall that the CM dynamics is fast from every initialization when $\beta \notin (\beta_u, \beta_s)$ per [2, 4], so this covers all temperatures where the worst-case mixing time is slow. The constant λ_* is explicit and is such that the expected size of the giant in an Erdős–Rényi $(n, \lambda_*/n)$ is $\theta_* n$ where θ_* is the saddle point in the landscape of the random-cluster model projected onto its giant component size, separating the ordered and disordered phases: see Figure 1. The constant c_* is defined in terms of left vs. right exit probabilities for a 1-dimensional Gaussian process. This is described detail in a detailed proof overview in Section 1.4.

As hinted by fact that λ_* places the giant exactly at the saddle point separating the ordered and disordered phases, the thresholds of Theorem 1.1 are sharp. Namely, if λ_0 is not in the delineated regimes, then there is an $\varepsilon > 0$ such that the ε -mixing time is exponentially slow: see Section 1.2. This demonstrates that even from the perspective of product measure initializations, it is still important that some parameter of the product measure be tuned according to the model from which one is sampling.

Remark 1. When $q \geq 3$ is integer, the results of Theorem 1.1 apply without change to the SW dynamics, in which instead of one activated set per step, there are q activated sets corresponding to each of the q color classes and independent percolation steps within each. Because of this distinction, all components are updated in every step, and this strictly simplifies the proofs in the SW case, not having to distinguish the analyses between situations where the giant component is activated or not. Interestingly, the c_* parameter in the initialization at $\beta = \beta_c$ might differ between the SW and CM dynamics. This is because the 1-dimensional Gaussian process approximations have different variances.

We now turn to our results for the Potts Glauber dynamics that show the sharp families of high-entropy initializations that overcome the exponential bottlenecks in the metastable and phase-coexistence regimes. Let $\hat{v}^\otimes(m_0)$ denote the distribution over $\{1, \dots, q\}^{[n]}$, where first, a dominant spin s is selected uniformly at random from $\{1, \dots, q\}$, and then, independently, each vertex is assigned spin s with probability $m_0 \geq 1/q$ and each of the remaining $q - 1$ spins with probability $\frac{1-m_0}{q-1}$.

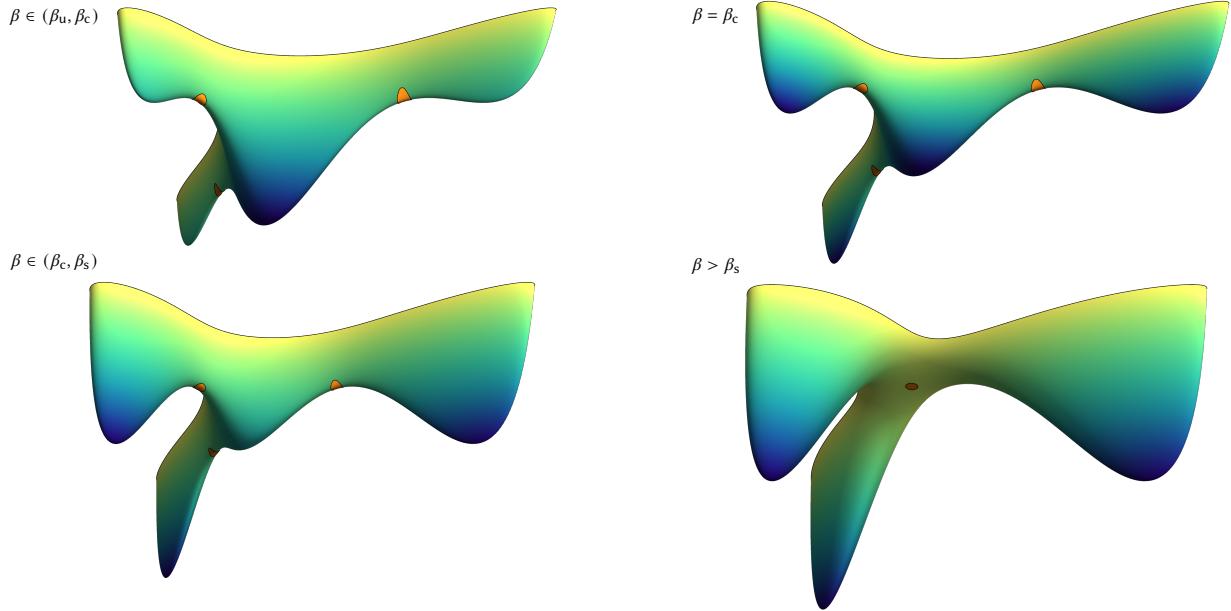


Figure 2: The negative log probability (i.e., free energy) of the n -vertex mean-field Potts model with $q = 3$, projected down onto the fractional spin counts; i.e., the xy -axes are the fraction of vertices assigned spin 1 and 2 respectively (which also determines the fraction of vertices with spin 3), and the z -axis is the negative logarithm of the configuration weights divided by n . The orange-marked spots are the saddle points near which Glauber dynamics may be initialized in Theorem 1.2.

Theorem 1.2. *For every $\beta > \beta_u$, there exist $m_*(\beta, q)$ and $\hat{c}_*(q)$ such that the Potts Glauber dynamics initialized from $\hat{v}^\otimes(m_0)$ mixes in $O(n \log n)$ steps whenever*

1. $\beta \in (\beta_u, \beta_c)$ and $m_0 < m_*(\beta, q) - \omega(n^{-1/2})$,
2. $\beta = \beta_c$ and $m_0 = m_*(\beta, q) + \hat{c}_*(q)n^{-1/2} + o(n^{-1/2})$,
3. $\beta \in (\beta_c, \beta_s]$ and $m_0 > m_*(\beta, q) + \omega(n^{-1/2})$,
4. $\beta > \beta_s$ and $m_0 \geq 1/q$ arbitrary.

When $\beta \leq \beta_u$ the mixing time is fast from arbitrary initialization per [16], so the above covers all temperature regimes where the worst-case mixing time is slow. The constant m_* is explicit and corresponds to initializing exactly at the saddle point(s) separating the q ordered phases from the disordered phase (see Figure 2). We note that in the low-temperature $\beta > \beta_s$ regime, our result includes the special case $m_0 = 1/q$, which is the fully uniform-at-random assignment of spins to vertices, and requires understanding the delicate competition between multiple simultaneously unstable directions around the saddle. Finally, the ranges of m_0 in the theorem statement are sharp for fast mixing from initializations of the form \hat{v}^\otimes ; the following subsection expands on what we mean by this sharpness.

1.2 Sharpness of the initialization parameter regimes

The parameter values λ_0 and m_0 in Theorems 1.1–1.2 are such that in Figures 1–2, the initializations can be microscopically near the saddle, so long as they are on the correct side of it. We also obtain slow mixing even from the families of high-entropy initializations considered, when the initialization parameters are not in the windows in Theorems 1.1–1.2. Namely, for every $\beta \in (\beta_u, \beta_s)$,

- Theorem 3.24 establishes that the CM dynamics are exponentially slow to attain $o(1)$ TV-distance to stationarity when initialized from $\bigotimes \text{Ber}(\lambda_0/n)$ for λ_0 outside the range specified in Theorem 1.1;
- Theorem 4.22 establishes that Potts dynamics are exponentially slow to attain $o(1)$ TV-distance to stationarity when initialized from $\hat{\nu}^\otimes(m_0)$ for m_0 outside the range specified in Theorem 1.2.

These results are a consequence of our proof strategy, which shows that whichever side of the saddles at λ_* or m_* the initialization starts on, it quasi-equilibrates to the corresponding phase, and then takes exponential time to leave that phase. E.g., if $\beta \in (\beta_c, \beta_s)$ and $\lambda_0 < \lambda_* + O(n^{-1/2})$ then the CM dynamics has an $\Omega(1)$ chance of quasi-equilibrating to the disordered phase. This implies slow mixing when combined with the fact that for $\beta \in (\beta_c, \beta_s)$, initialized from the disordered phase, mixing is exponentially slow.

1.3 Initializations at other temperatures and relations to simulated annealing

Another central purpose of understanding mixing times of high-entropy initializations for Markov chains is that it is closely related to understanding simulated annealing schemes. Simulated annealing, introduced in [36] (and variants like simulated tempering [41]) lowers the temperature over the run of the Markov chain, so that when run at low-temperatures, one is effectively initializing from the stationary measure at a nearby, but higher temperature (and therefore higher-entropy) distribution. This is one of the most commonly implemented modifications to vanilla MCMC to make it better able to sample from multimodal low-temperature landscapes. Simulated annealing and simulated tempering have been analyzed in the context of mean-field Potts dynamics by e.g., [1, 47] and it is known that the discontinuous phase transition at β_c presents a serious obstruction to fast mixing from such schemes, for instance because the high-temperature initialization is metastable when $\beta \in (\beta_c, \beta_s)$.

Still, our results lend insight into the approach by precisely classifying the set of temperatures β_0 for which initialization from μ_{β_0} or π_{β_0} would be fast for the CM and Glauber dynamics. Because the properties of the initialization used in our proofs are only quantities like the size of the giant component, the sum of squares of non-giant component sizes for random-cluster and the proportions vector in the Potts Glauber case, applying them to initializations from μ_{β_0} or π_{β_0} instead of the product measures yields the following: there exists an explicit $b_*(\beta, q) \geq \beta_c$ such that

- When $\beta \in (\beta_u, \beta_c)$, the CM dynamics initialized from μ_{β_0} for $\beta_0 < b_*(\beta, q)$ will mix in $O(\log n)$ steps, but will be slow from any $\beta > b_*(\beta, q)$.
- When $\beta = \beta_c$, the CM dynamics initialized from μ_{β_0} will be slow to equilibrate for every $\beta_0 \neq \beta_c$.
- When $\beta \in (\beta_c, \beta_s)$, the CM dynamics initialized from μ_{β_0} mixes in $O(\log n)$ steps for every $\beta_0 > b_*(\beta, q)$ but not from $\beta_0 < b_*(\beta, q)$.

Similar results hold for simulated annealing for the Potts Glauber dynamics, where per item 4 of Theorem 1.2 one also gets $O(n \log n)$ mixing at low-temperature $\beta > \beta_s$ when initialized from π_{β_0} for any $\beta_0 > 0$.

1.4 Proof outlines

We focus our proof overview on the CM dynamics at the critical point β_c , where the equilibrium measure is roughly a $(1 - \xi, \xi)$ -mixture of an ordered and a disordered phase. This regime is the most technically involved part of the proof and contains most of the ideas used in other parameter regimes as well.

In what follows, let $f(\theta) = f_{\beta, q}(\theta)$ be the expected drift for the size of the giant component in one step of CM dynamics. When $\beta \in (\beta_u, \beta_s)$, let θ_* be its unstable fixed point separating the two stable fixed points of 0 (the disordered phase) and θ_r (the ordered phase); see Figure 3 and Section 2.3 for the precise definitions. For intuition, when comparing to Figure 1, the drift is roughly the derivative of the

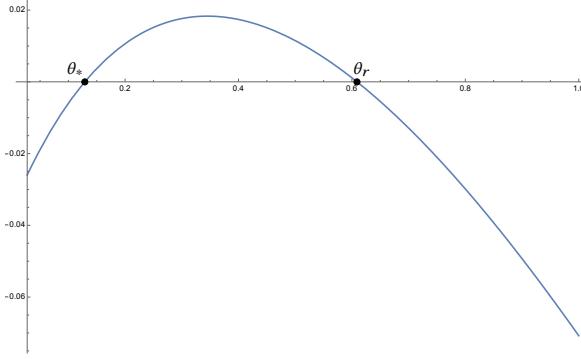


Figure 3: The drift function $f(\theta)$ for the fractional size of the giant component under the mean-field CM dynamics in $\beta \in (\beta_u, \beta_s)$ with its repulsive and attractive fixed points θ_* and θ_r , respectively, marked.

log-probability landscape: a giant component of size θ_*n corresponds to the saddle point separating the modes whose minima are at no giant component, and at a giant component of size θ_rn .

The overall strategy of the proof can be described in the following two step manner.

1. *Escape from a local neighborhood of the unstable fixed point:* We show that locally near, i.e., in an $O(\sqrt{n})$ window around the unstable fixed point θ_*n , the evolution of the size of its giant component, though not literally a Markov process, is well-approximated by a 1-dimensional Gaussian Markov chain (so long as the dynamics maintains a certain typical structure).

The 1-dimensional Gaussian Markov chain is monotone, and for each ξ has a unique initialization c_ξ such that in $O(1)$ time, it exits the unstable fixed point to the right (towards the ordered phase) with probability $1 - \xi$, and to the left (towards the disordered phase) with probability ξ . This behavior transfers to the CM dynamics by the approximation, and by picking the initialization's edge-probability λ_0/n so that the initial giant component size has mean $\theta_*n + c_\xi\sqrt{n} + o(\sqrt{n})$.

2. *Convergence to the stable fixed point:* Once the giant component size is $\omega(\sqrt{n})$ away from θ_*n (say to the right), its drift towards the stable fixed point θ_r , dominates the fluctuations. By a dyadic argument, we show that in a further $O(\log n)$ steps, the CM dynamics gets $\Omega(n)$ away from θ_*n . From there, the landscape is effectively strictly convex and there is a macroscopic drift towards the stable fixed point. At this point, up to small modifications to handle the unlikely event that the dynamics leaves the region of convexity, the fast quasi-equilibration to the ordered phase follows from arguments very similar to those employed in [4].

We outline the above two items in more detail in what follows.

1.4.1 Escaping the unstable fixed point separating phases

Typical fast mixing arguments in the analysis of SW and CM chains rely on a uniform drift (dominating the fluctuations) to take them to a local neighborhood of a stable fixed point. In our setting, near the unstable fixed point at θ_*n for the giant component, the drift and fluctuations are on the same order, and together give constant probabilities to exiting the neighborhood of θ_*n to the right (and subsequently quasi-equilibrating to the ordered phase), or to the left (and subsequently quasi-equilibrating to the disordered phase). Understanding this competition to pinpoint the right initialization for fast mixing to occur requires pinpointing not just the order, but the variance, and even the exact distribution of the fluctuations.

Towards this, we show that the evolution of the giant component, which we denote by $|\mathcal{L}_1(X_t)|$, is well approximated near its unstable fixed point θ_*n , by an explicit 1-dimensional Gaussian Markov chain. To be precise, there exist explicit constants $a, (b_t^2)_{t \geq 0} > 0$ such that if we define

$$\bar{Z}_{t+1} - \bar{Z}_t = \begin{cases} a\bar{Z}_t + \mathcal{N}(0, b_t^2) & \text{with prob., } 1/q \\ 0 & \text{otherwise} \end{cases}, \quad (3)$$

then the giant component, centered by θ_*n and divided by \sqrt{n} , tracks this process closely. I.e., if we denote by $\bar{L}_t = n^{-1/2}(|\mathcal{L}_1(X_t)| - \theta_*n)$, by Theorem 3.19 there exists a coupling \mathbf{P} such that for every $T = O(1)$,

$$\mathbf{P}\left(\sup_{t \leq T} |\bar{L}_t - \bar{Z}_t| \geq O\left(\frac{\log n}{\sqrt{n}}\right)\right) = o(1), \quad (4)$$

where the initializations are $X_0 \sim G(n, \lambda_0/n)$ for $\lambda_0 = \lambda_* + c_*n^{-1/2} + o(n^{-1/2})$, and $\bar{Z}_0 \sim \mathcal{N}(0, F(c_*))$ for an explicit increasing function F .

To show this, we first argue that $(X_t)_{t \in [0, T]}$ stays in some good set of configurations \mathcal{G}_T , consisting of having a specific (time-dependent) sum-of-squares of its non-giant component sizes, and a sufficient number of singleton components. Lemma 3.14 shows that with high probability, $(X_t)_{t \in [0, T]} \in \mathcal{G}_T$ for $T = O(1)$ times when initialized from a supercritical Erdős–Rényi, but we emphasize that if these a priori regularity estimates do not hold for the configuration, the approximation by the 1-dimensional chain can fail completely. Once on the set \mathcal{G}_T , the approximation of the evolution of the giant by a Gaussian Markov chain boils down to concentration estimates for the set of vertices not activated in the past k steps for each k , and local limit theorems both for the activation and percolation steps of the CM dynamics.

With (4) in hand, we translate exit probabilities for \bar{Z}_t to the right and left of an interval $[-\gamma, \gamma]$ with γ large, to exit probabilities on \bar{L}_t with only $o(1)$ error. A subtle technical point is that, although the variances of $\bar{Z}_t - \bar{Z}_{t-1}$'s steps are time-dependent, they are notably neither n -dependent, nor \bar{Z}_{t-1} -dependent, so this is a tractable and monotone chain on \mathbb{R} .

1.4.2 Mixing within a phase away from the fixed-point

From the above we deduce that for a well-chosen initialization parameter λ_0 , after $O(1)$ many steps, $|\mathcal{L}_1(X_t)|$ has exited $[\theta_*n - \gamma\sqrt{n}, \theta_*n + \gamma\sqrt{n}]$ to the right with probability $1 - \xi + o(1)$ and to the left with probability $\xi - o(1)$. The next step is to show that if it exited to the right, with probability $1 - O(\gamma^{-2})$, in a further $O(\log n)$ steps, it quasi-equilibrates from there to the ordered phase (the random-cluster measure conditioned on having a giant), and if it exited to the left similarly to the disordered phase. Without loss of generality, let us discuss the first setting of exiting to the right.

This step can be broken into three parts:

- (i) initially, the drift away from the saddle for $|\mathcal{L}_1(X_t)|$ is proportional to its distance to θ_*n , and this can be used to get it from $\theta_*n + \gamma\sqrt{n}$ to $\theta_*n + \Omega(n)$ in $O(\log n)$ steps, except with probability $1 - O(\gamma^{-2})$ (which covers the possibility it goes back to the saddle in which case quasi-equilibration fails).
- (ii) from $\theta_*n + \Omega(n)$, the distance to $\theta_r n$ contracts exponentially fast, and in $O(\log n)$ steps, we have $|\mathcal{L}_1(X_t) - \theta_r n| = O(\sqrt{n})$;
- (iii) any two configurations having $|\mathcal{L}_1(X_t) - \theta_r n| = O(\sqrt{n})$ and a few other easy to maintain properties can be coupled with probability $1 - \varepsilon$ in $O_\varepsilon(1)$ further steps; since the ordered phase measure also satisfies these properties, we can couple X_t to a sample from the ordered phase with high probability.

The last two stages here are not so different from those arguments carried out in [4]. However, we emphasize a point of care in all these arguments is that unlike mixing guarantees from worst-case initializations, couplings cannot be restarted arbitrarily if they fail, and a single bad move (which can be correlated with failure of the coupling) could destroy mixing.

1.4.3 The Potts Glauber dynamics proof

At a very high level, the proofs of the Potts Glauber dynamics results in Theorem 1.2 for $\beta \in (\beta_u, \beta_s)$ follow a similar strategy to the CM dynamics proof. We first argue a priori that from the initializations considered in the theorem, all color counts except the one dominant color stay within $\tilde{O}(\sqrt{n})$ of one another for long times: this plays the analogue of the good set of configurations \mathcal{G}_T on which a 1-dimensional chain approximating the evolution of the dominant color class, is sufficient. Within this good set of configurations, we first bound the time to escape the local neighborhood of the unstable fixed point, and then show that whichever side the dynamics exits out of, in $O(n \log n)$ steps, it quasi-equilibrates to the corresponding phase (disordered, or dominated by one of the q colors). The technical tools in these proofs, in addition to those appearing in the CM proofs, are martingale concentration estimates adapted from [16]. This allows us to roughly treat every n steps of the Potts Glauber updates like a single step of the CM dynamics.

Some notable further challenges arise in the low-temperature regime of $\beta > \beta_s$ that are not present in the CM analysis, due to the fact that the saddle at $(1/q, \dots, 1/q)$ proportions vector is unstable in all q directions simultaneously, while when $\beta < \beta_s$, the saddles we encounter are stable in all but one direction. To handle these difficulties, we obtain a better quantitative understanding of the q -dimensional drift landscape (for general q), and use refined approximations of the behavior of the Glauber dynamics' proportions vector away from its fixed points by a q -dimensional gradient dynamical system. For the reader only interested in the Potts Glauber dynamics, this is found in Section 4, and its presentation is pretty much self-contained.

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2 Preliminaries

In this section, we introduce notations and gather known facts about Erdős–Rényi random graphs, the mean-field random-cluster measure, and the mean-field Potts measure that we will appeal to in our proofs.

Let us begin with some discussion of the notation used throughout the paper. We will always think of β, q fixed, and therefore may drop their dependencies from subscripts. All our results should be thought of as applying for n sufficiently large. When we use big- O , little- o , and Θ notation, the implicit constants may depend on β, q but not on n , and also not on the threshold γ which will appear frequently for the window around the saddle point we are zooming in around.

Throughout, for a random-cluster configuration (i.e., a subset of the edges of the complete graph) X , we use $\mathcal{L}_i(X)$ to denote the i -the largest component (breaking ties according to an arbitrary ordering on components). We write $|\mathcal{L}_i(X)|$ for the number of vertices in $\mathcal{L}_i(X)$. Furthermore, we define the following:

- $R_k(X) := \sum_{i \geq 1} |\mathcal{L}_i(X)|^k$ is the sum of k 'th powers of component sizes; and
- $R_k^-(X) := \sum_{i \geq 2} |\mathcal{L}_i(X)|^k$ is like $R_k(X)$ but excludes the largest component of X .

2.1 Structure of the mean-field Potts and random-cluster equilibrium measures

We begin by recapping the typical behavior of the mean-field Potts and random-cluster measures, describing their phase transitions as the inverse-temperature parameter β varies. As described in the introduction, when $q > 2$, there are three critical points $\beta_u < \beta_c < \beta_s$ of relevance, given by

$$\begin{aligned}\beta_u &:= \sup \left\{ \beta \geq 0 : \left(1 + (q-1)e^{\beta \cdot \frac{1-qx}{q-1}} \right)^{-1} - x \neq 0, \quad \forall x \in (1/q, 1) \right\}, \\ \beta_c &:= \frac{2(q-1)}{q-2} \log(q-1), \quad \text{and} \quad \beta_s := q.\end{aligned}$$

To characterize the giant component appearing at $\beta \geq \beta_c$, we define $\theta_r = \theta_r(\beta, q)$ as the largest $x > 0$ satisfying the equation

$$e^{\beta x} = \frac{1-x}{1+(q-1)x}. \quad (5)$$

The following lemma characterizes the mean-field random-cluster distribution at each β . To state the lemma, including the form of coexistence at $\beta = \beta_c$, we define the constant $\xi = \xi(q)$, as

$$\xi := \frac{1}{1+\xi'}, \quad \text{where} \quad \xi' := \frac{1}{q-1} \left(\frac{q-\beta_c/(q-1)}{q-\beta_c} \right)^{(2-q)/2} \exp \left(\frac{\beta_c^2(q-2)(q^2-4q+2)}{4q(q-1)^2} \right). \quad (6)$$

Lemma 2.1. *Let $q > 2$ and consider the mean-field random-cluster model at fixed β . Let $X \sim \mu_\beta$.*

- If $\beta < \beta_c$, with probability $1 - o(1)$, it has

$$|\mathcal{L}_1(X)| = O(\log n), \quad \text{and} \quad R_2(X) = O(n).$$

- If $\beta = \beta_c$, then for ξ defined as in (6), with probability $\xi - o(1)$, it has

$$|\mathcal{L}_1(X)| = O(\log n) \quad \text{and} \quad R_2(X) = O(n),$$

and with probability $(1 - \xi) - o(1)$, it has

$$|\mathcal{L}_1(X)| - \theta_r n = o(n), \quad |\mathcal{L}_2(X)| = O(\log n), \quad \text{and} \quad R_2^-(X) = O(n).$$

- If $\beta > \beta_c$, with probability $1 - o(1)$, it has

$$|\mathcal{L}_1(X)| - \theta_r n = o(n), \quad |\mathcal{L}_2(X)| = O(\log n), \quad \text{and} \quad R_2^-(X) = O(n).$$

The bounds on $|\mathcal{L}_1(X)|$ in Lemma 2.1 appeared in [40] and [6], whereas the bounds for $|\mathcal{L}_2(X)|$ and $R_2^-(X)$ follow from the analogous bound for sub-critical random graphs (see, e.g., [33]) and the machinery from [6] to transfer such results to the random-cluster model.

To describe the corresponding phase transition for the Potts measure, we let $S(\sigma)$ be its proportions vector, i.e.,

$$S(\sigma) = \left(\frac{1}{n} \sum_{v \in [n]} \mathbf{1}\{\sigma(v) = 1\}, \dots, \frac{1}{n} \sum_{v \in [n]} \mathbf{1}\{\sigma(v) = q\} \right).$$

We sometimes consider S as close to another vector even if it is only so up to permutation of the q coordinates in the vector; in this case we say “up to a permutation of the q spins”, noting that the measure, and the law of the Glauber dynamics are invariant under such permutations. In the analysis of the Potts Glauber dynamics, also, we typically consider S permuted so that its largest count is always its first coordinate. By Lemma 2.1 and the coupling between the random-cluster model and the ferromagnetic Potts model (coloring components independently), we obtain the following.

Corollary 2.2. *Let $q > 2$ and consider the mean-field Potts model at fixed β . Let $\sigma \sim \pi_{G,\beta}$.*

- *If $\beta < \beta_c$, with probability $1 - o(1)$, it has*

$$\|S(\sigma) - (\frac{1}{q}, \dots, \frac{1}{q})\|_1 = O(\sqrt{n}).$$

- *If $\beta = \beta_c$, then for ξ defined as in (6), with probability $\xi - o(1)$, it has*

$$\|S(\sigma) - (\frac{1}{q}, \dots, \frac{1}{q})\|_1 = O(\sqrt{n}).$$

With probability $(1 - \xi) - o(1)$, (up to a permutation of the q spins) it has

$$\|S(\sigma) - (\frac{\theta_r(q-1)+1}{q}, \frac{1-\theta_r}{q}, \dots, \frac{1-\theta_r}{q})\|_1 = O(\sqrt{n}).$$

- *If $\beta > \beta_c$, with probability $1 - o(1)$, (up to a permutation of the q spins) it has*

$$\|S(\sigma) - (\frac{\theta_r(q-1)+1}{q}, \frac{1-\theta_r}{q}, \dots, \frac{1-\theta_r}{q})\|_1 = O(\sqrt{n}).$$

2.2 Random graph preliminaries

Due to the percolation step in the definition of the CM dynamics, precise random graph estimates are essential to careful understanding of the CM dynamics. We begin with central limit and local limit theorems for the size of the giant in a super-critical Erdős–Renyi random graph.

2.2.1 Limit theorems for the size of the giant component

For $\lambda > 1$, the expected size of the giant component in a $G(n, \lambda/n)$ random graph is roughly $\alpha(\lambda)n$ where

$$\alpha(\lambda) := \max\{x > 0 : e^{-\lambda x} = 1 - x\}. \quad (7)$$

In particular, Theorem 5 of [13] showed the following: if $\lambda > 1$ uniformly in n , and $G \sim G(n, \lambda/n)$, then

$$|\mathbb{E}[|\mathcal{L}_1(G)|] - \alpha(\lambda)n| \leq \tilde{O}(1). \quad (8)$$

It is easy to check that the $e^{-\lambda x} - 1 + x$ has strictly positive x -derivative at $(\lambda, \alpha(\lambda))$ as long as $\lambda > 1$, and moreover it is analytic in x . By the analytic implicit function theorem, this implies $\alpha(\lambda)$ is analytic in λ for all $\lambda > 1$. Moreover, the size is concentrated about $\alpha(\lambda)n$ with variance approximately $\sigma^2(\lambda)n$, where

$$\sigma^2(\lambda) := \frac{\alpha(\lambda)(1 - \alpha(\lambda))}{(1 - \lambda(1 - \alpha(\lambda)))^2}. \quad (9)$$

In fact, it is known to satisfy the following central and local limit theorems.

Theorem 2.3 ([43]). *Let $G(n, \lambda/n)$ with $\lambda > 1$. Let $\alpha = \alpha(\lambda)$ and $\sigma = \sigma(\lambda)$. For any compact interval $J \subset \mathbb{R}$ and any $\delta > 0$, for all large n and any integer $k \in \mathbb{N}$ satisfying $\frac{k-\alpha n}{\sigma\sqrt{n}} \in J$, we have*

$$\frac{1-\delta}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{(k-\alpha n)^2}{2\sigma^2 n}\right) \leq \mathbb{P}(|\mathcal{L}_1(G)| = k) \leq \frac{1+\delta}{\sigma\sqrt{2\pi n}} \exp\left(-\frac{(k-\alpha n)^2}{2\sigma^2 n}\right). \quad (10)$$

Theorem 2.4 ([7]). *Let $G(n, \lambda/n)$ with a fixed $\lambda = O(1)$ with $(\lambda - 1)^3 n \rightarrow \infty$ as $n \rightarrow \infty$. Let $|\mathcal{L}_1(G)|$ denote the number of vertices in the largest component of $G \sim G(n, \lambda/n)$. We have as $n \rightarrow \infty$, that*

$$\frac{|\mathcal{L}_1(G)| - \alpha(\lambda)n}{\sigma(\lambda)\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, 1), \quad (11)$$

where \xrightarrow{D} denotes convergence in distribution, and $\mathcal{N}(0, 1)$ is a standard normal.

2.2.2 Discrete duality

We also want sharp understanding of other statistics of the component counts for $G(n, \lambda/n)$ random graphs in different regimes of λ . A well-known tool to translate bounds between sub-critical and super-critical λ is the discrete duality principle. We start with an observation using the definition of total-variation distance.

Observation 2.5. *If $\mu(A) \geq 1 - \varepsilon$, and $\nu = \mu(\cdot \mid A)$, then $\|\mu - \nu\|_{TV} \leq \frac{2\varepsilon}{1-\varepsilon}$.*

Lemma 2.6. *Suppose $\lambda > 1$ uniformly in n and consider $X \sim G(n, \lambda/n)$. For any $U : |U| - \alpha(\lambda)n| = o(n)$, conditional on $\{\mathcal{L}_1(X) = U\}$, the law of the graph induced by X on U^c has total-variation distance at most $e^{-\Omega(n)}$ to $G(n - |U|, \lambda/n)$.*

Proof. Reveal the edge-set on U . Conditioning on $\mathcal{L}_1(X) = U$ is equivalent to conditioning on all edges between U and U^c being absent, the vertices of U being connected within $E(U)$, and the event $\mathcal{E}_{|U|}$ that the subgraph induced on U^c has no component larger $|U|$. The first two parts are measurable with respect to edges in $E \setminus E(U^c)$, so the law induced on $E(U^c)$ is exactly that of $G(n - |U|, \lambda/n)$ conditioned on $\mathcal{E}_{|U|}$.

By a standard calculation, if $|U| = \alpha(\lambda)n + o(n)$, then $\frac{n-|U|}{n}\lambda < 1$ is uniformly bounded away from 1, so that the resulting graph is uniformly sub-critical and the probability of $\mathcal{E}_{|U|}$ is exponentially small in n . The result follows from Observation 2.5. \square

2.2.3 Refined random graph statistics

We now summarize estimates for R_2, R_3 and R_2^- and R_3^- for sub and super-critical random graphs.

Lemma 2.7 ([33]). *Let $G \sim G(n, \lambda/n)$. For $n \geq 1$ and $\lambda < 1$,*

$$\mathbb{E}[R_2(G)] = \frac{n}{1-\lambda} \cdot \left(1 + O(n^{-1}(1-\lambda)^{-3})\right), \quad \mathbb{E}[R_3(G)] = \frac{n}{(1-\lambda)^3} \cdot \left(1 + O(n^{-1}(1-\lambda)^{-3})\right), \quad (12)$$

and

$$\text{Var}(R_2(G)) = O\left(\frac{n}{(1-\lambda)^5}\right), \quad \text{Var}(R_3(G)) = O\left(\frac{n}{(1-\lambda)^9}\right). \quad (13)$$

The following is a list of typical structural properties of sub-critical random graphs that we deduce. In what follows, $I_1(X) := |\{i : |\mathcal{L}_i(X)| = 1\}|$ denotes the number of isolated vertices in X .

Lemma 2.8. *Let $G \sim G(n, \lambda/n)$ where $\lambda < 1$. Then G satisfies*

1. $|R_2(G) - \mathbb{E}[R_2(G)]| \leq \sqrt{n} \log^2 n$, with probability $1 - O((\log n)^{-4})$;
2. $I_1(G) = \Omega(n)$ with probability $1 - O(n^{-1})$;
3. $|\mathcal{L}_1(G)| = O(\log n)$ with probability $1 - O(n^{-1})$;
4. $R_3(G) = O(n), R_2(G) = O(n)$ with probability $1 - O(n^{-1})$.

Proof. Items (1) and (4) follow from Chebyshev's inequality and Lemma 2.7; items (2) and (3) can be found in standard literature of random graphs (see, e.g., Lemma 5.7 in [39] and Lemma 7 in [14]). \square

Given Lemma 2.6, we can also deduce the analogous properties for super-critical random graphs.

Corollary 2.9. *Let $G \sim G(n, \lambda/n)$, where $\lambda > 1$ uniformly in n . Then $\mathbb{E}[R_2^-(G)], \mathbb{E}[R_3^-(G)], \text{Var}(R_2^-(G))$ and $\text{Var}(R_3^-(G))$ are all $O(n)$.*

Finally, the following concerns stability of the size of the giant under perturbations of the vertex count.

Lemma 2.10 (Lemmas 2.7 & 2.8 in [2]). *Let G_{d_n} be distributed as a $G(n + m, d_n/n)$ random graph where $|m| = o(n)$ and $\lim_{n \rightarrow \infty} d_n = d$. Assume $1 < d_n$ and d_n is bounded away from 1 for all n . Then,*

1. $|\mathcal{L}_2(G_{d_n})| = O(\log n)$ with probability $1 - O(n^{-1})$,
2. $\text{Var}(|\mathcal{L}_1(G_{d_n})|) = \Theta(n)$ and
3. For $A = o(\log n)$ and sufficiently large n , there exists a constant c such that

$$\mathbb{P}(|\mathcal{L}_1(G_{d_n})| - \alpha(d)n| > |m| + A\sqrt{n}) \leq e^{-cA^2}.$$

We arrive at the following by combining Lemmas 2.6, 2.8, and 2.10.

Lemma 2.11. *Let $G \sim G(n, \lambda/n)$ where $\lambda > 1$. Then with high probability, G satisfies*

1. $|R_2^-(G) - \mathbb{E}[R_2^-(G)]| \leq \sqrt{n} \log^2 n$, with probability $1 - O((\log n)^{-4})$;
2. $I_1(G) = \Omega(n)$ with probability $1 - O(n^{-1})$;
3. $R_3^-(G) = O(n)$, $R_2^-(G) = O(n)$ with probability $1 - O(n^{-1})$.

2.3 Drift of the giant component in the CM dynamics

We end this preliminaries section by describing properties of the drift function for the size of the giant component in the CM dynamics when $\beta \in (\beta_u, \beta_s)$. This function will govern the expected change to the (fractional) size of the giant component in the CM dynamics on the event that the giant is activated. To start, for $\theta \in [0, 1]$, let

$$k_a(\theta) = \theta + \frac{1}{q}(1 - \theta), \quad \text{and let} \quad \phi(\theta) := \alpha(\beta k_a(\theta)) \cdot k_a(\theta),$$

so that $\phi(\theta)$ is the expected fractional size of the giant component from a configuration with fractional size θ . Define the drift function

$$f(\theta) := \phi(\theta) - \theta.$$

Note that $f(\theta)$ is well-defined if $\theta \in (\theta_s, 1]$, where $\theta_s = \inf\{\theta : \beta k_a(\theta) > 1\} = \frac{q-\beta}{\beta(q-1)}$; below θ_s , the percolation step will typically be sub-critical and in one step the fractional size of the giant would become $o(1)$, so we can formally set $\alpha(\theta) = 0$ for $\theta < 1$.

We compile a set of useful facts about f , and refer to Figure 3 for a visual aid.

Lemma 2.12 ([2]). *For every $\beta > 0$, the following properties hold for f :*

1. The function f is continuous, differentiable and strictly concave in $(\theta_s, 1]$.
2. The function ϕ is continuous, differentiable, and strictly increasing in $(\theta_s, 1]$.
3. Let $f(\theta_s^+) := \lim_{\theta \rightarrow \theta_s^+} f(\theta)$. Then $\text{sgn}(f(\theta_s^+)) = \text{sgn}(\beta - q)$.
4. If $\beta \in (\beta_u, \beta_s)$, then f has exactly 2 roots in $(\theta_s, 1]$; θ_* is the first root and θ_r is the second root.

Notice that θ_r defined in this way, matches the solution to (5) when $\beta \geq \beta_c$.

The next lemma is a minor extension of Lemma 3.9 in [2].

Lemma 2.13. *Let $\beta \in (\beta_u, \beta_s)$ and $q > 2$.*

1. For $\theta \in (\theta_*, \theta_r)$, we have $\theta \leq \phi(\theta) \leq \theta_r$.
2. For a fixed $\vartheta \in (\theta_*, \theta_r]$, there exists $\delta = \delta(\vartheta) \in (0, 1)$ such that $\delta|\theta - \theta_r| \leq |\phi(\theta) - \theta|$ for all $\theta \in [\vartheta, 1]$.

Proof. First, let $\theta_* < \theta < \theta_r$. By Lemma 2.12 (2) and (4) we have $\phi(\theta) < \phi(\theta_r) = \theta_r$. Moreover, by Lemma 2.12 (1) and (4), f is strictly concave and $f(\theta_r) = 0 = f(\theta_*)$. Hence $f(\theta) > 0$, or equivalently $\theta < \phi(\theta)$. This establish the first part, and in particular, shows that $f(\vartheta) > 0$ for a fixed $\vartheta \in (\theta_*, \theta_r)$. The second part follows from the fact $f(\vartheta) > 0$ and the concavity of f as in the proof in Lemma 3.9 in [2] without modification. \square

We will also use the following lemma, which extends Lemma 3.7 in [2].

Lemma 2.14. *Let $s \in (0, 1)$ be a fixed constant. If $\beta \in (\beta_u, \beta_s)$, then for all $\theta \in (\theta_s, \theta_* - s]$ there exists a $\delta > 0$ such that $f(\theta) \leq -\delta$.*

Proof. By Lemma 2.12, we have $f(\theta_s^+) < 0$ and $f(\theta) < 0$ for $\theta \in (\theta_s, \theta_* - s]$. Hence, f as a continuous function must attain a negative maximum on $(\theta_s, \theta_* - s]$ and the lemma follows. \square

3 Mixing from product initializations for the CM dynamics

Our aim in this section is to establish the mixing time results of Theorem 1.1. Recall that f is the drift function for the giant component in the CM dynamics, i.e., if a configuration X_0 has a giant component of size θn , on the event of activation of X_0 , the expected size of the giant in X_1 is roughly $(\theta + f(\theta))n$. For $\beta \in (\beta_u, \beta_s)$, this function has an unstable fixed point θ_* , i.e., $f(\theta_*) = 0$ and $f'(\theta_*) > 0$.

Section 3.1 will focus on quasi-equilibration to the measure constrained to the ordered (resp., disordered) phase when initialized with a giant that is $\omega(n^{-1/2})$ away from θ_* . This is the core of all the results of Theorem 1.1 except the $\beta = \beta_c$ case where we need to study the small-drift diffusion in the $O(n^{-1/2})$ neighborhood of θ_* to pick the ordered vs. disordered phases with the correct relative probabilities; this latter step is done in Section 3.2. Finally, the results are combined to prove Theorem 1.1 in Section 3.3.

3.1 Quasi-equilibrating away from the unstable fixed point

Our main goal for this subsection is to show that initialized from a random graph with a giant component of fractional size $\omega(n^{-1/2})$ away from θ_* , and a “reasonable” structure on the complement of the giant, the CM dynamics quasi-equilibrates to the random-cluster measure conditioned on the phase corresponding to the side of θ_* it initializes. To be more precise, for fixed $\beta \in (\beta_u, \beta_s)$, define

$$\mu^{\text{ord}} = \mu_\beta(\cdot \mid \Omega^{\text{ord}}), \quad \text{and} \quad \mu^{\text{dis}} = \mu_\beta(\cdot \mid \Omega^{\text{dis}}),$$

where

$$\Omega^{\text{ord}} = \{A : |\mathcal{L}_1(A)| \geq \theta_* n\}, \quad \text{and} \quad \Omega^{\text{dis}} = \{A : |\mathcal{L}_1(A)| < \theta_* n\}.$$

Note that μ^{ord} has $|\mathcal{L}_1(A)|$ concentrated around $\theta_r > \theta_*$, where θ_r is the (stable) fixed point of f to the right of θ_* . On the other hand μ^{dis} has $|\mathcal{L}_1(A)|$ that is concentrated around 0.

Throughout the paper, we use $\mathbb{P}_{X_0}(\cdot)$ for the law of the Markov chain initialized at X_0 and \mathbb{P}_v when it is initialized from the distribution v . The main result of this subsection is the following:

Lemma 3.1. *Suppose that $q > 2$, $\gamma > 0$ and $\beta \in (\beta_u, \beta_s)$. If X_0 is such that $|\mathcal{L}_1(X_0)| \geq \theta_* n + \gamma \sqrt{n}$, $|\mathcal{L}_2(X_0)| = O(\log n)$, and $R_2^-(X_0) = O(n)$. There exists $C > 0$ such that if $T = C \log n$, we have*

$$\|\mathbb{P}_{X_0}(X_T \in \cdot) - \mu^{\text{ord}}\|_{\text{TV}} = O(\gamma^{-2}).$$

An analogous statement holds w.r.t. μ^{dis} if $|\mathcal{L}_1(X_0)| \leq \theta_* n - \gamma \sqrt{n}$.

The main part of the proof of Lemma 3.1 is showing that any initialization whose giant is $\omega(n^{1/2})$ away from the unstable critical point θ_*n gets to $\Omega(n)$ away from it in $O(\log n)$ steps. Let us introduce a few more notational shorthands for this section: we let

- $L_t := |\mathcal{L}_1(X_t)|$ be the process tracking the giant component size of X_t ;
- Λ_t be the event that $\mathcal{L}_1(X_{t-1})$ is activated in the t 'th step.
- A_t be the number of activated vertices at the t 'th step, and let $A_t^- = A_t - L_t \mathbf{1}\{\Lambda_t\}$.

Quasi-equilibrating to the ordered phase

We start with the case where we are equilibrating towards the ordered phase.

Lemma 3.2. *Let $q > 2$, $\gamma > 0$ and $\beta \in (\beta_u, \beta_s)$. If X_0 is any configuration such that $L_0 \geq \theta_*n + \gamma\sqrt{n}$, $|\mathcal{L}_2(X_0)| = O(\log n)$ and $R_2^-(X_0) = O(n)$, then there exist constants $s \in (0, \theta_r - \theta_*)$ and $C > 0$ so that the first time that $L_t \geq (\theta_* + s)n$, $|\mathcal{L}_2(X_t)| = O(\log n)$, and $R_2^-(X_t) = O(n)$, all hold is at most $T = C \log n$ with probability $1 - O(\gamma^{-2})$.*

To prove Lemma 3.2, we use the following lemma to estimate the drift after 1 step.

Lemma 3.3 (Lemma 3.19 [2]). *Suppose $0 < \beta < \beta_s$ and that X_t has at most one large component whose size is at least $2n^{11/12}$. Let $\varepsilon > 0$ be a small constant. Then for sufficiently large n : if $L_t/n \geq \theta_s + \varepsilon$, then*

$$\mathbb{E}[L_{t+1} \mid X_t, \Lambda_{t+1}] \leq L_t + f(L_t/n)n + 3n^{1/4},$$

and

$$\frac{f(L_t/n)n}{q} - 2n^{1/4} \leq \mathbb{E}[L_{t+1} - L_t \mid X_t] \leq \frac{f(L_t/n)n}{q} + 2n^{1/4}.$$

If $L_t/n \in (\theta_s - \varepsilon, \theta_s + \varepsilon)$, then

$$\mathbb{E}[L_{t+1} - L_t \mid X_t] \leq \frac{f(\theta_s + \varepsilon)n}{q} + \frac{2\varepsilon n}{q} + 2n^{1/4}.$$

Proof. Let τ be the first time t when $L_t \geq (\theta_* + s)n$ where $s > 0$ is a small constant that will be decided. Let $(t_i)_{i \geq 0}$ be the subset of times in $[T] = \{1, \dots, T\}$ at which the largest component is activated. We can condition on this sequence, generating σ -algebra \mathcal{T} , and notice that this can be generated by a sequence of T independent $\text{Ber}(1/q)$ random variables which are also independent of the remaining randomness of the dynamics (say by reserving the first activation coin to always be used for the largest component).

Let $c_* = f'(\theta_*)/2 > 0$, and let M, M_0 and M_1 be sufficiently large constants. Our goal in this proof is to show the following by induction on t :

$$\begin{aligned} \mathbb{P}\left(\bigcap_{k=1}^{t \wedge \tau} \left\{ (L_k - \theta_*n) \geq \left(1 + \frac{c_*}{4}\right)^{\mathbf{1}\{\Lambda_k\}} (L_{k-1} - \theta_*n), R_2^-(X_k) \leq Mn, |\mathcal{L}_2(X_k)| \leq M_1n \right\} \mid \mathcal{T}\right) \\ \geq \prod_{i: t_i \leq t \wedge \tau} \left(1 - \frac{16M_0^2}{\gamma^2(1 + \frac{c_*}{4})^{2i-2}c_*^2}\right) \cdot \left(1 - O(T/n)\right). \end{aligned} \quad (14)$$

In words, (14) shows that with the stated probability, L_k drifts away from θ_*n while preserving the control on $R_2^-(X_k)$ and $|\mathcal{L}_2(X_k)|$. Let us first conclude the proof assuming (14). For any given $s > 0$, if $T = C \log n$

with sufficiently large $C > 0$, then standard binomial tail bounds yield that $|\{t_i : t_i \leq T\}| \geq T/2q$ with probability at least $1 - n^{-1}$, and

$$\left(1 + \frac{c_*}{4}\right)^{|\{t_i : t_i \leq T\}|} (L_0 - \theta_* n) \geq \left(1 + \frac{c_*}{4}\right)^{(C/2q) \log n} \gamma \sqrt{n} \geq s n.$$

Thus, on any such realization of \mathcal{T} where $|\{t_i : t_i \leq T\}| \geq T/2q$, the event in (14) implies $\{\tau \leq T\} \cap \{R_2^-(X_\tau) \leq Mn\} \cap \{|\mathcal{L}_2(X_\tau)| \leq M_1 n\}$. As such, we can conclude that

$$\begin{aligned} \mathbb{P}(\{\tau \leq T\} \cap \{R_2^-(X_\tau) \leq Mn\} \cap \{|\mathcal{L}_2(X_\tau)| \leq M_1 n\}) \\ \geq \left[\prod_{i=1}^T \left(1 - \frac{16M_0^2}{\gamma^2 c_*^2} \frac{1}{(1 + \frac{c_*}{4})^{2i-2}}\right) \right] \cdot \left[1 - O\left(\frac{T}{n}\right)\right] - \frac{1}{n} \geq 1 - \frac{A}{\gamma^2}, \end{aligned}$$

for some $A(c_*, M_0, \gamma, s) > 0$.

We now proceed to prove (14) inductively. The event in the probability holds at initialization deterministically; now suppose (14) holds at time t and let us show it holds at $t+1$. If $\tau = t$, then the event in the probability is unchanged at $t+1$, and the right-hand side is at least its value at t . Now suppose that $t < \tau$, and fix $(X_k)_{k \leq t}$ such that the events in (14) hold for it.

If $t+1 \notin \{t_i\}$ and Λ_{t+1}^c occurs, then $L_{t+1} \geq L_t$ deterministically. In this case, it suffices to consider only the probability of $\{R_2^-(X_{t+1}) \leq Mn, |\mathcal{L}_2(X_{t+1})| \leq M_1 n\}$. On Λ_{t+1}^c , the decrease in R_2^- as a result of the dissolution of active components is $R_2^-(X_t)/q$ in expectation, and at least $R_2^-(X_t)/q - o(n)$ with probability $1 - O(n^{-1})$ by Hoeffding's inequality and the fact that $R_2^-(X_t) \leq Mn$ and $|\mathcal{L}_2(X_t)| \leq M_1 n$. Moreover, with probability $1 - O(n^{-1})$, $A_{t+1} \in [\frac{n-L_t}{q} - \sqrt{n \log n}, \frac{n-L_t}{q} + \sqrt{n \log n}]$. Thus, $A_t \cdot \frac{\beta}{n} < 1$ is bounded away from 1 uniformly in n . Lemma 2.8 implies that the increase in R_2^- as a result of the creation of new components in the sub-critical percolation step is at most $C_1 n$ with probability $1 - O(n^{-1})$, for some constant $C_1 > 0$; besides, with probability $1 - O(n^{-1})$, the sizes of new components in the sub-critical percolation step is at most $M_1 \log n$. By a union bound,

$$\mathbb{P}(|\mathcal{L}_2(X_{t+1})| \leq M_1 n, R_2^-(X_{t+1}) \leq R_2^-(X_t) \left(1 - \frac{1}{q}\right) + C_1 n + o(n) \mid X_t, \Lambda_{t+1}^c) = 1 - O(n^{-1}).$$

When $R_2^-(X_t) > 8C_1 q n$, we have $R_2^-(X_{t+1}) \leq R_2^-(X_t)$ with probability $1 - O(n^{-1})$ and when $R_2^-(X_t) \leq 8C_1 q n$, we have $R_2^-(X_{t+1}) \leq (8q+1)C_1 n$ with probability $1 - O(n^{-1})$. Therefore, in both cases we have $R_2^-(X_{t+1}) \leq Mn$ by setting $M \geq (8q+1)C_1$.

Now suppose $t+1 = t_i$ for some $i \geq 1$ so that the events $\Lambda_{t+1}, R_2^-(X_t) \leq Mn, |\mathcal{L}_2(X_t)| \leq M_1 n$ and

$$L_t - \theta_* n \geq \left[1 + \frac{c_*}{4}\right]^{i-1} (L_0 - \theta_* n), \quad (15)$$

occur. Following a similar argument, this time the activated component being with high probability supercritical, we derive that

$$\mathbb{P}(\{R_2^-(X_{t+1}) \leq Mn, |\mathcal{L}_2(X_{t+1})| \leq M_1 n\} \mid X_t, \Lambda_{t+1}) = 1 - O(n^{-1}). \quad (16)$$

Next, we focus on the change of L_{t+1} on Λ_{t+1} . By Taylor expansion, for small enough $s > 0$ we obtain that for $\Theta \in [\theta_* + \frac{\gamma}{\sqrt{n}}, \theta_* + s]$,

$$f(\Theta) = f(\theta_*) + f'(\theta_*)(\Theta - \theta_*) + O((\Theta - \theta_*)^2) \geq c_* \cdot (\Theta - \theta_*). \quad (17)$$

By Lemma 3.3, we have

$$\mathbb{E}[L_{t+1} \mid X_t, \Lambda_{t+1}] \geq L_t + f(L_t/n)n - 2n^{1/4}. \quad (18)$$

Set $s_t = f\left(\frac{L_t}{n}\right) \cdot \frac{n}{4}$. For $L_t > \theta_* n + \gamma\sqrt{n}$, we have $f\left(\frac{L_t}{n}\right) \geq \frac{c_* \gamma}{\sqrt{n}}$, so for large enough n ,

$$f\left(\frac{L_t}{n}\right)n - 2n^{\frac{1}{4}} - s_t \geq f\left(\frac{L_t}{n}\right) \cdot \frac{n}{2}. \quad (19)$$

Then by (18), (19) and Chebyshev's inequality, we obtain that

$$\begin{aligned} \mathbb{P}\left(L_{t+1} \leq L_t + f(L_t/n) \cdot \frac{n}{2} \mid X_t, \Lambda_{t+1}\right) &\leq \mathbb{P}\left(\left|L_{t+1} - \mathbb{E}[L_{t+1} \mid X_t, \Lambda_{t+1}]\right| \geq s_t \mid X_t, \Lambda_{t+1}\right) \\ &\leq \frac{\text{Var}(L_{t+1} \mid X_t, \Lambda_{t+1})}{s_t^2} = \frac{16M_0^2 n}{f(L_t/n)^2 n^2}, \end{aligned}$$

where the last inequality follows from the Fact 3.26 that there exist constants $M_0 > 0$ and $s \in (0, 1)$ such that if $\theta_* n - sn \leq L_t \leq \theta_* n + sn$, then

$$\text{Var}(L_{t+1} \mid X_t, \Lambda_{t+1}) \leq M_0^2 n.$$

By (15) and (17), we get

$$\frac{16M_0^2 n}{f(L_t/n)^2 n^2} \leq \frac{16M_0^2 n}{c_*^2 \cdot (L_t - \theta_* n)^2} \leq \frac{16M_0^2}{c_*^2 \gamma^2 (1 + c_*/4)^{2(i-1)}}.$$

Hence, by combining the inequalities above,

$$\begin{aligned} \mathbb{P}\left(L_{t+1} - \theta_* n \geq \left(1 + \frac{c_*}{4}\right) \cdot (L_t - \theta_* n) \mid X_t, \Lambda_{t+1}\right) &\geq \mathbb{P}\left(L_{t+1} \geq L_t + f(L_t/n) \cdot \frac{n}{2} \mid X_t, \Lambda_{t+1}\right) \\ &\geq 1 - \frac{16M_0^2}{c_*^2 \gamma^2 (1 + c_*/4)^{2(i-1)}}. \end{aligned}$$

By a union bound, this inequality and (16) conclude the induction step of (14) for the case of Λ_{t+1} . \square

We now proceed to show that once the giant's size is macroscopically away from the unstable fixed point at θ_* , in a further $O(\log n)$ steps, it equilibrates (to the corresponding phase) quickly. This part of the proof follows closely those of [4] for $\beta > \beta_s$, with a little care due to the rare event that the dynamics crosses to the other side of the unstable fixed point. The argument goes in two stages, the first getting the giant to within $O(\sqrt{n})$ of the stable fixed point at θ_r , and the second quasi-equilibrating from there.

Lemma 3.4. *Let $q > 2$ and $\beta > \beta_u$. Let $s \in (0, \theta_r - \theta_*)$ be a fixed constant. Suppose X_0 satisfies that $L_0 \geq (\theta_* + s)n$, $|\mathcal{L}_2(X_0)| = O(\log n)$, and $R_2^-(X_0) = O(n)$. Then, for any $\varepsilon > 0$ there exists $T = O(\log n)$ such that X_T satisfies all of the following properties with probability $1 - \varepsilon$:*

1. $|L_T - \theta_r n| = O(\sqrt{n})$;
2. $I_1(X_T) = \Omega(n)$;
3. $|\mathcal{L}_2(X_T)| = O(\log n)$;
4. $R_2^-(X_T) = O(n)$.

Proof. This lemma was established in [2] (see Lemma 3.26 there) for the case when $\beta \geq q$; the same argument can be carried over to the more general setting where $\beta > \beta_u$ with only minor modifications, as we detailed next. Let $\Delta_t = |L_t - \theta_r n|$. For $\beta \geq q$ and a configuration X_t such that $L_t \geq (\theta_* + s)n$, $|\mathcal{L}_2(X_t)| = O(\log n)$ and $R_2^-(X_t) = O(n)$, it was shown in [2, Eq. (3.22)] that

$$\mathbb{E}[\Delta_{t+1} \mid X_t] \leq \left(1 - \frac{1}{q}\right)\Delta_t + \frac{|\theta_r - \phi(L_t/n)|}{q}n + O(\sqrt{n}). \quad (20)$$

It can be readily checked that the same inequality holds under the weaker condition that $q > 2$ and $\beta > \beta_u$. Specifically, Fact 3.28 from [2] holds in this setting since under the assumption that $L_t \geq (\theta_* + s)n$, the percolation step of the CM dynamics is subcritical (resp. supercritical) when the largest component of X_t is inactive (resp., active), and this is essentially all that is required to establish (20).

Lemma 2.13 then implies that there exists a constant $\delta \in (0, 1)$ such that

$$|\theta_r - \phi(L_t/n)| = \frac{1}{n}|\theta_r n - L_t| - \frac{1}{n}|L_t - \phi(L_t/n)n| \leq (1 - \delta)|\theta_r - (L_t/n)|;$$

(note that Lemma 2.13 extends Lemma 3.9 from [2] to the $\beta > \beta_u$ regime). Plugging this bound into (20), we obtain

$$\mathbb{E}[\Delta_{t+1} | X_t] \leq (1 - \frac{\delta}{q})\Delta_t + O(\sqrt{n}). \quad (21)$$

Now, let Ω_{good} be the set of all random-cluster configurations X such that $|\mathcal{L}_1(X)| \geq (\theta_* + s)n$, $|\mathcal{L}_2(X)| = O(\log n)$ and $R_2^-(X) = O(n)$. Lemma 3.25 from [2] shows that if $X_t \in \Omega_{\text{good}}$, then $X_{t+1} \in \Omega_{\text{good}}$ with probability $1 - O(n^{-1})$. (Again, this result from [2] is stated for $\beta \geq q$, but it extends to the $q > 2$ and $\beta > \beta_u$ setting by the same observations made above about the subcriticality/supercriticality of the percolation step; in addition, Lemma 3.25 only states a $1 - o(1)$ but its proofs yields a $1 - O(n^{-1})$ bound on the probability.) By averaging over all the configurations on Ω_{good} , we get from (21) that

$$\mathbb{E}[\Delta_{t+1}] \leq (1 - \frac{\delta}{q})\mathbb{E}[\Delta_t] + O(\sqrt{n}) + n \cdot \mathbb{P}(X_t \notin \Omega_{\text{good}}) \leq (1 - \frac{\delta}{q})\mathbb{E}[\Delta_t] + O(\sqrt{n}). \quad (22)$$

Iterating this bound, we obtain

$$\mathbb{E}[\Delta_{t+1}] \leq (1 - \frac{\delta}{q})^t \Delta_0 + O(\sqrt{n}).$$

Since $\Delta_0 = O(n)$, there exists some $T = O(\log n)$ so that $\mathbb{E}[\Delta_T] \leq C\sqrt{n}$, and by Markov's inequality we have $\Delta_T \leq 2C\sqrt{n}/\varepsilon$ with probability $1 - \varepsilon/2$ for any fixed $\varepsilon > 0$. Finally, note that $X_T \in \Omega_{\text{good}}$ with probability $1 - o(1)$ and in the percolation step of the last step, Lemma 2.8 and 2.11 imply that $I_1(X_T) = \Omega(n)$ with probability $1 - o(1)$. The result then follows from a union bound. \square

The next lemma shows that once the giant's size is within $O(\sqrt{n})$ of $\theta_r n$, mixing happens in at most $O(\log n)$ further steps.

Lemma 3.5. *Let $q > 2$ and $\beta > \beta_u$. Suppose X_0 is a configuration satisfying all the following conditions*

1. $|\mathcal{L}_1(X_0)| - \theta_r n = O(\sqrt{n})$;
2. $I_1(X_0) = \Omega(n)$;
3. $|\mathcal{L}_2(X_0)| = O(\log n)$;
4. $R_2^-(X_0) = O(n)$.

Suppose Y_0 also satisfies all these conditions. Then for any constant $\varepsilon > 0$, there exists $T = O(\log n)$ and a coupling of (X_t, Y_t) such that $Y_T = X_T$ with probability at least $1 - \varepsilon$.

Proof. This lemma essentially follows from Lemmas 3.16 and 3.27 in [2] but requires a slight generalization of the latter. Specifically, Lemma 3.27 from [2] provides a coupling from two configurations satisfying conditions 1 to 4 in the lemma statement to two configurations with the same component structure but assumes that $\beta \geq q$ and only provides an $\Omega(1)$ bound on the probability of success of the coupling. Lemma 3.16 from [2] provides a coupling from two configurations with the same component structure to the same configurations with high probability and holds for any $q > 1$ and $\beta > 0$.

Our first observation is that Lemma 3.27 and Corollary 3.33 from [2] hold when $q > 2$ and $\beta > \beta_u$. (This is a byproduct of the percolation step of the CM dynamics being subcritical (resp. supercritical) when the largest component of the configuration is inactive (resp., active).) To boost the probability of success of the coupling, we note that at a suitable $T_0 = O(\log n)$, either the coupling to the same component structure succeeds with probability at least $\alpha = \Omega(1)$, or, by Corollary 3.33 from [2], both X_{T_0} and Y_{T_0} satisfy conditions 2, 3 and 4 from the lemma statement with probability $1 - o(1)$ and also $|\mathcal{L}_1(X_{T_0})| - \theta_r n = O(\sqrt{n} \log^2 n)$, $|\mathcal{L}_1(Y_{T_0})| - \theta_r n = O(\sqrt{n} \log^2 n)$. Then, by Lemma 3.3, at time $T_1 = T_0 + O(\log n)$, we have that all four conditions from the lemma statement hold with probability $1 - 1/A$ for any desired constant $A > 0$. Iterating this reasoning, we obtain a coupling for which $X_{kT_1} = Y_{kT_1}$ with probability at least $1 - (1 - \alpha)^k - k/A - o(1)$ for any constant k . Letting $T = kT_1$ with k and A sufficiently large, we obtain a coupling under which X_T and Y_T have the same component structure with probability at least $1 - \varepsilon/2$ for any $\varepsilon > 0$. The result then follows from Lemma 3.16 in [2] and a union bound. \square

The above lemmas are the key ingredients to establish the first part of Lemma 3.1.

Quasi-equilibrating to the disordered phase

We require analogous lemmas to establish the second part, namely the quasi-equilibration to the disordered phase if initialized with a largest component of size at most $\theta_* n - \gamma \sqrt{n}$.

Lemma 3.6. *Let $q > 2$, $\gamma > 0$ and $\beta \in (\beta_u, \beta_s)$. If X_0 is a configuration such that $L_0 \leq \theta_* n - \gamma \sqrt{n}$, $|\mathcal{L}_2(X_0)| = O(\log n)$ and $R_2^-(X_0) = O(n)$, then there exist constants $s \in (0, \theta_* - \theta_s)$, $C > 0$ such that for $T = C \log n$ we have $L_T \leq (\theta_* - s)n$, $|\mathcal{L}_2(X_T)| = O(\log n)$, and $R_2^-(X_T) = O(n)$, with probability $1 - O(\gamma^{-2})$.*

The proof of Lemma 3.6 is essentially identical to that of Lemma 3.2 and is thus omitted. The next lemma shows how the giant's size goes from $(\theta_* - s)n$ to $O(\log n)$; a little care is needed here compared to the ordered side because on its way the giant may approach θ_s where it can take one step as a critical random graph. To deal with this, we recall the following two lemmas from [2].

Lemma 3.7 (Fact 3.18 in [2]). *Let $0 < \beta < \beta_s$ and X_0 has a unique component that is of size at least $2n^{11/12}$, then $|\mathcal{L}_2(X_t)| < 2n^{11/12}$ for all $0 \leq t \leq T$ with probability $1 - O(T \cdot n^{-1/12})$ for any $T = O(\log n)$.*

Lemma 3.8 (Fact 3.20 [2]). *Let $0 < \beta < \beta_s$. If X_0 is a configuration such that $L_0 \leq (\theta_s - \varepsilon)n$ and X_0 has at most one large component whose size is at least $2n^{11/12}$, then there exists $T = O(\log n)$ such that $L_T = O(\log n)$ with high probability.*

Note that although originally Fact 3.20 in [2] guarantees only $\Omega(1)$ probability for the statement to hold, it can be improved to, for example $1 - n^{-1/2}$, with a careful look.

With the above lemmas recalled, we show how the size of the giant goes from at most $(\theta_* - s)n$ to $o(n)$.

Lemma 3.9. *Let $q > 2$, $s > 0$ and $\beta \in (\beta_u, \beta_s)$. If X_0 is a configuration such that $L_0 \leq (\theta_* - s)n$, $|\mathcal{L}_2(X_0)| = O(\log n)$ and $R_2^-(X_0) = O(n)$, then there exists $T = O(\log n)$ such that $L_T = O(\log n)$ and $I_1(X_T) = \Omega(n)$ with probability $1 - o(1)$.*

Proof. Recall we set $\theta_s := \frac{q-\beta}{\beta(q-1)}$. Fix a small $\varepsilon > 0$, and define $\tau = \min\{\tau_\varepsilon, \tau_2\}$ where

$$\tau_\varepsilon := \min\{t > 0 : L_t/n \notin [\theta_s - \varepsilon, \theta_* - s/2]\} \quad \text{and} \quad \tau_2 = \min\{t > 0 : |\mathcal{L}_2(X_t)| > 2n^{11/12}\}.$$

We bound the drift of the giant in two cases of L_t : (i) $L_t/n \in [\theta_s + \varepsilon, \theta_* - s/2]$, and (ii) $L_t/n \in [\theta_s - \varepsilon, \theta_s + \varepsilon]$ to upper bound τ_ε . Lemmas 3.3 and 2.14 imply that in case (i) with $X_t : |\mathcal{L}_2(X_t)| < 2n^{11/12}$, there exists a constant $\delta > 0$ such that

$$\mathbb{E}[L_{t+1} - L_t \mid X_t] \leq \frac{f(L_t/n)n}{q} + O(n^{1/4}) \leq -\frac{\delta n}{q} + O(n^{1/4}). \quad (23)$$

Similarly, in case (ii), Lemmas 3.3 and 2.14 imply that

$$\mathbb{E}[L_{t+1} - L_t \mid X_t] \leq \frac{f(\theta_s + \varepsilon)n}{q} + \frac{2\varepsilon n}{q} + O(n^{1/4}) \leq -\frac{\delta n}{q} + \frac{2\varepsilon n}{q} + O(n^{1/4}). \quad (24)$$

By choosing ε small enough, we see from (23) and (24) that there exists $\eta > 0$ such that if $t < \tau$, then

$$\mathbb{E}[L_{t+1} - L_t \mid X_t] \leq -\eta n.$$

By a standard application of the optional stopping theorem (see, e.g., Lemma 2.20 in [2]), we have $\mathbb{E}[\tau] \leq 4/\eta$. By Markov's inequality, $\mathbb{P}(\tau > \frac{4\log n}{\eta}) \leq 1/\log n$, so for $T_1 = \frac{4\log n}{\eta}$, $\tau < T_1$ with high probability.

At the same time, by Lemma 3.7, $|\mathcal{L}_2(X_t)| < 2n^{11/12}$ holds for all $t \leq T$ with probability $1 - O(Tn^{-1/12})$, so with high probability it is τ_ε that is attained and $\tau_\varepsilon < T_1$. Moreover, while $L_t/n < \theta_* - s/2$ and $|\mathcal{L}_2(X_t)| < 2n^{11/12}$, the conditional variance of $L_{t+1} - L_t$ is at most $O(n^{23/24})$ so by Chebyshev's inequality and a union bound, the probability that $\tau_\varepsilon < T_1$ is attained by $L_t/n > \theta_* - s/2$ is at most $T_1 n^{-1/24} = o(1)$. Altogether, with high probability $L_{T_1} \leq (\theta_s - \varepsilon)n$ and $|\mathcal{L}_2(X_{T_1})| \leq 2n^{11/12}$.

At this point, Lemma 3.8 implies that after $T_2 = O(\log n)$ additional steps, the largest component in the configuration has size $O(\log n)$ with high probability. Finally, in the percolation step of the very last step, Lemma 2.8 and 2.11 imply that $I_1(X_{T_1+T_2}) = \Omega(n)$ with high probability. The result follows from a union bound. \square

To quasi-equilibrate to μ^{dis} from here, we appeal to the following lemma which lower bounds the probability of coupling two shattered configurations in $O(\log n)$ steps.

Lemma 3.10 ([2, Lemmas 3.15–3.16 and Fact 3.17]). *Let $q > 1$ and $0 < \beta < \beta_s$. Let X_0 be a random-cluster configuration such that $L_0 = O(\log n)$ and $I_1(X_0) = \Omega(n)$. Suppose Y_0 also satisfies these conditions. Then for any $\varepsilon > 0$ there exist $T = O(\log n)$ and a coupling of (X_t, Y_t) such that $X_T = Y_T$ with probability at least $1 - \varepsilon$.*

We are now in position to put all the above ingredients together to establish Lemma 3.1.

Proof of Lemma 3.1. Suppose $(X_t)_{t \geq 0}$ is initialized from X_0 with $L_0 \geq \theta_* n + \gamma\sqrt{n}$, $|\mathcal{L}_2(X_0)| = O(\log n)$, and $R_2^-(X_0) = O(n)$. Let $(Y_t)_{t \geq 0}$ be initialized from μ^{ord} and restricted to Ω^{ord} by rejecting any update that would take it out of Ω^{ord} ; by Lemma 2.1 and a union bound, with probability $1 - o(1)$, $(Y_t)_{t \geq 0}$ does not feel the restriction to Ω^{ord} (i.e., doesn't attempt to leave Ω^{ord}) for exponential in n many steps, so for all $e^{o(n)}$ time steps, we can treat Y_t as the (unrestricted) CM chain initialized from μ^{ord} . We claim that there is $T = O(\log n)$ such that with probability $1 - O(\gamma^{-2})$, we have coupled $X_T = Y_T$, which would imply the claim. In order to see this, notice that Lemma 3.2 can be stitched with Lemma 3.4 to get, with probability $1 - O(\gamma^{-2})$, a configuration X_{T_1} on which Lemma 3.5 can be applied. At the same time, Y_{T_1} satisfies the conditions of Lemma 3.5, being a sample from μ^{ord} (which satisfies those conditions by Lemma 2.1).

The proof for $L_0 \leq \theta_* n - \gamma\sqrt{n}$ is analogous (Lemmas 3.2–3.5 are replaced by Lemmas 3.6–3.10). \square

3.2 Getting away from the unstable fixed point at criticality

We now focus on the critical point $\beta = \beta_c$ where it is essential to understand the diffusion away from the fixed point $\theta_* n$; here, the drift and fluctuations of the giant component process compete on the same scale.

3.2.1 Staying in a nice set of configurations

We begin with certain a priori estimates guaranteeing that for a sufficiently long period of time ($O(1)$ times will suffice), the near-saddle dynamics has largest component size, and sum of squares of other

components, that are concentrated around explicit quantities. Moreover, the sum of cubes of components, and isolated vertices (things needed for sharp local limit theorems in updates) stay on the right order.

We begin by defining the following good set that captures all the a priori estimates except the concentration of the sum of squares of component sizes, which will come subsequently.

Definition 3.11. For a constant $K > 0$, let \mathcal{G}_K be all random-cluster configurations U satisfying:

1. The largest component has $|\mathcal{L}_1(U)| - \theta_* n| \leq K n^{1/2} \log n$,
2. The second largest component has $|\mathcal{L}_2(U)| \leq K \log n$,
3. The sum of squares of non-giant components has $R_2^-(U) \leq K n$,
4. The sum of cubes of non-giant components has $R_3^-(U) \leq K n$,
5. The singletons have $I_1(U) \geq \frac{n}{K}$.

We introduce some notations that will be useful in what follows. Recall for $\lambda > 1$, that $\alpha(\lambda)$ is defined to be the largest positive root of the equation (7) (i.e., $\alpha(\lambda)n$ is approximately the expected size of the giant in a $G(n, d/n)$), and recall $\sigma^2(\lambda)$ as defined in (9) which approximates the variance of the giant component in $G(n, \lambda/n)$. Also, for $\theta \in [0, 1]$, let $k_a(\theta)$ be the expected fraction of activated vertices if a giant of fractional size θ is activated, and $k_{ia}(\theta)$ be the same quantity if it is not activated, i.e.,

$$k_a(\theta) = \theta + \frac{1}{q}(1 - \theta), \quad \text{and} \quad k_{ia}(\theta) = \frac{1}{q}(1 - \theta). \quad (25)$$

Moreover, for $i \geq 1$ we use Λ_i to denote the event that the largest component is activated in step i of the CM dynamics; we use A_i to denote the total number of activated vertices in step i , and on Λ_i use A_i^- to denote the activated number minus the giant. Lastly, let $G_0 \sim G(n, p_0)$ and denote by G_i the random graph resampled in step i of $\{X_t\}$, that is, $G_i \sim G(A_i, \beta/n)$.

Let λ_* be the solution to $\alpha(\lambda_*) = \theta_*$.

Lemma 3.12. *Let $\beta = \beta_c$. For all $T \geq 0$ fixed independent of n , if $X_0 \sim G(n, \frac{p_0}{n})$ with $p_0 = \lambda_* + O(n^{-1/2})$, then for $K = K(T)$, $X_T \in \mathcal{G}_K$ with probability $1 - o(1)$.*

Proof of Lemma 3.12. Let $X_0 \sim G(n, \frac{p_0}{n})$. For some $K_0 > 0$, Lemmas 2.11 implies that, $X_0 \in \mathcal{G}_{K_0}$ with $1 - o(1)$ probability. Let $K_t = tK_0$. Now suppose $X_t \in \mathcal{G}_{K_t}$ for some $t \geq 0$. We will inductively show that $X_{t+1} \in \mathcal{G}_{K_{t+1}}$ with probability $1 - o(1)$. Then a union bound over $t \in [0, T]$ implies $X_T \in \mathcal{G}_{K_T}$.

We consider the two cases, Λ_{t+1} and Λ_{t+1}^c . First, $\mathbb{E}[A_{t+1} \mid \Lambda_{t+1}, X_t] = k_a(L_t/n)n$. Moreover, since property 3 holds at time t , by Hoeffding's inequality, we have

$$\mathbb{P}\left(|A_{t+1} - k_a(L_t/n)n| \geq \sqrt{n} \log n \mid \Lambda_{t+1}, X_t\right) \leq 2 \exp\left(-\frac{n \log^2 n}{R_2^-(X_t)}\right) = \exp(-C_1 \log^2 n), \quad (26)$$

where $C_1 > 0$ is some constant. In the percolation step, $G_{t+1} \sim G(A_{t+1}, \beta_c/n)$, and when the concentration in (26) holds, the random graph G_{t+1} is supercritical. By Lemma 2.10, $|\mathcal{L}_2(G_{t+1})| = O(\log n)$ with probability $1 - O(n^{-1})$ and $|\mathcal{L}_1(G_{t+1})| - \theta_* n| \leq n^{1/2} \log n$ with probability $1 - o(n^{-1})$. Since $|\mathcal{L}_2(X_t)| \leq K_t \log n$ by property 2 and the largest component of X_t has been activated, if $|\mathcal{L}_1(G_{t+1})| = \Omega(n)$, then $\mathcal{L}_1(X_{t+1}) = \mathcal{L}_1(G_{t+1})$. Moreover, with probability $1 - o(1)$, we have $G_{t+1} \in \mathcal{G}_{K_0}$ by Lemma 2.11. Consequently, we also establish property 2-5 by noting that

1. $|\mathcal{L}_2(X_{t+1})| \leq \max\{|\mathcal{L}_2(X_t)|, |\mathcal{L}_2(G_{t+1})|\} = \max\{K_0, K_t\} \log n$.

2. $R_2^-(X_{t+1}) \leq R_2^-(X_t) + R_2^-(G_{t+1}) \leq (K_0 + K_t)n$.
3. $R_3^-(X_{t+1}) \leq R_3^-(X_t) + R_3^-(G_{t+1}) = (K_0 + K_t)n$.
4. $I_1(X_{t+1}) \geq I_1(G_{t+1}) \geq \frac{n}{K_0} \geq \frac{n}{K_0 + K_t}$.

Next, suppose Λ_{t+1} does not happen. Then $\mathbb{E}[A_{t+1} \mid \Lambda_{t+1}^c, X_t] = k_{\text{ia}}(L_t/n)n$. Again, by property 3 and Hoeffding's inequality, we have the following concentration for A_{t+1} :

$$\mathbb{P}\left(|A_{t+1} - k_{\text{ia}}(L_t/n)n| \geq \sqrt{n} \log n \mid \Lambda_{t+1}^c, X_t\right) \leq 2 \exp\left(-\frac{n \log^2 n}{R_2^-(X_t)}\right) = \exp(-C_2 \log^2 n), \quad (27)$$

where $C_2 > 0$ is a constant. In the percolation step, $G_{t+1} \sim G(A_{t+1}, \beta_c/n)$ and if the estimate in (27) holds then G_{t+1} is subcritical. In this case, by Lemma 2.8, $|\mathcal{L}_1(G_{t+1})| \leq K_0 \log n$ with probability $1 - o(1)$, so $L_{t+1} = L_t$, satisfying the first property of $\mathcal{G}_{(t+1)K_0}$. In addition, with probability $1 - o(1)$, G_{t+1} satisfies that $R_2(G_{t+1}) \leq K_0 n, R_3(G_{t+1}) \leq K_0 n, I_1(G_{t+1}) \geq n/K_0$. Similar to the case Λ_{t+1} , these properties combined with the hypothesis $X_t \in \mathcal{G}_{K_t}$ imply that $X_{t+1} \in \mathcal{G}_{K_{t+1}}$. Therefore, by a union bound over all the above, we have $X_{t+1} \in \mathcal{G}_{K_{t+1}}$ with probability $1 - o(1)$. \square

At this point, we need one more property to hold throughout the process, which is concentration of $R_2^-(X_t)$ around an explicit deterministic quantity. To define that quantity, in addition to $G_0 \sim G(n, p_0/n)$, let $G_* \sim G(nk_{\text{ia}}(\theta_*), \beta_c/n)$. Then define a sequence of variances

$$\sigma_s^2 := \frac{1}{q}(1 - \frac{1}{q}) \left((1 - \frac{1}{q})^s \mathbb{E}[R_2^-(G_0)] + \sum_{i=1}^s (1 - \frac{1}{q})^{s-i} \mathbb{E}[R_2(G_*)] \right). \quad (28)$$

We show that $\frac{1}{q}(1 - \frac{1}{q})R_2^-(X_t)$ is concentrated around σ_t^2 for all $O(1)$ times.

Definition 3.13. For $K(t)$ as in Lemma 3.12, let \mathcal{G}_t be the event that for every $s \leq t$, $X_s \in \mathcal{G}_{K(s)}$ and furthermore

$$\left| \frac{1}{q}(1 - \frac{1}{q})R_2^-(X_s) - \sigma_s^2 \right| \leq \sqrt{n} \log^2 n \quad \text{for all } s \leq t.$$

Lemma 3.14. Let $\beta = \beta_c$. For all $T \geq 0$ fixed independent of n , if $X_0 \sim G(n, \frac{p_0}{n})$ with $p_0 = \lambda_* + O(n^{-1/2})$, then $(X_s)_{s \leq T} \in \mathcal{G}_T$, with probability $1 - o(1)$.

Proof. Fix $T \geq 0$ We will show that for each $s \leq T$, the event in Definition 3.13 holds with probability $1 - o(1)$; the lemma follows from a union bound over all $s \leq T$. First, Lemma 3.12 shows that $X_0, \dots, X_T \in \mathcal{G}_K$ with probability $1 - o(1)$. Working on this event, when $X_0, \dots, X_s \in \mathcal{G}_K$ we can express

$$R_2^-(X_s) = R_2^-(G_0^s) + \sum_{i=1}^s [1\{\Lambda_i\}R_2^-(G_i^s) + 1\{\Lambda_i^c\}R_2(G_i^s)], \quad (29)$$

where we are using G_i^t to denote the (random) subgraph induced on the vertex set of G_i (the resampled portion of X_i), consisting of vertices that are not re-activated up through time t (i.e., do not belong to $\bigcup_{j=i+1}^t G_j$). That is, the vertices of G_i that survive $t - i$ independent activation steps. Eq. (29) is a consequence of the fact that X_t is partitioned by $\bigcup_{i=0}^t G_i^t$ (G_t^t is the activated subgraph at time t).

Similarly, denote by G_*^t the induced subgraph of G_* on the vertices that have not been activated for t activation steps of the CM dynamics starting at G_* . We will show that for each $i \geq 1$,

$$\mathbb{P}\left(\left|1\{\Lambda_i\}R_2^-(G_i^s) + 1\{\Lambda_i^c\}R_2(G_i^s) - \mathbb{E}[R_2(G_*^{s-i})]\right| = O(\sqrt{n} \log n)\right) = 1 - o(1), \quad (30)$$

and

$$\mathbb{P}\left(\left|R_2^-(G_0^s) - \mathbb{E}[R_2^-(G_0^s)]\right| = O(\sqrt{n} \log n)\right) = 1 - o(1). \quad (31)$$

By (29) and a union bound of (30)–(31) over $i = 1, \dots, s$, together with Observation 3.25, we get the claimed bound for $R_2^-(X_s)$ with probability $1 - o(1)$.

Now we fix $i \in [1, s]$ and show (30). Up to an error of $o(1)$, we assume $X_{i-1} \in \mathcal{G}_K$. Suppose first the giant of X_{i-1} is activated in the step i , namely Λ_i . We define the activated window,

$$W_a := [nk_a(\theta_*) - 2\sqrt{n} \log n, nk_a(\theta_*) + 2\sqrt{n} \log n].$$

By (26), we have $|A_i - k_a(L_{i-1}/n)n| \leq \sqrt{n} \log n$ with probability $1 - o(1/n)$; by the first property of \mathcal{G}_K , we also have $|k_a(L_{i-1}/n)n - k_a(\theta_*)n| \leq \sqrt{n} \log n$. Let us work on the event that $A_i \in W_a$. We will rely on the following lemma which establishes concentration of the number of vertices not yet activated after $s - i$ steps; its proof follows from a simple calculation and Chebyshev's inequality, and is deferred.

Lemma 3.15. *Suppose $X \sim G(n, p)$ with np bounded away from 1 uniformly in n , and let X^r be the sub-graph of X that does not get activated in r activation steps. If $np > 1$ then*

$$\mathbb{P}(|R_2^-(X^r) - \mathbb{E}[R_2^-(X^r)]| > \sqrt{n} \log n) = O\left(\frac{1}{\log^2 n}\right),$$

and if $np < 1$ then the same concentration holds for $R_2(X^r)$.

We also use the following lemma that compares the expectation of G_i^s to that of G_*^{s-i} (the latter having deterministic parameter, while the former has a random parameter dictated by A_i , though close to the deterministic parameter when $A_i \in W_a$).

Lemma 3.16. *Suppose $G \sim G(nk_a(\theta_*) + m, \frac{\beta_c}{n})$, where $|m| = O(\sqrt{n} \log n)$, and $G_* \sim G(nk_{ia}(\theta_*), \frac{\beta_c}{n})$. Then for any integer $r \geq 0$,*

$$|\mathbb{E}[R_2^-(G^r)] - \mathbb{E}[R_2(G_*^r)]| = O(\sqrt{n} \log n).$$

Note that $A_i \in W_a$ implies that $G(A_i, \beta_c/n)$ is supercritical (uniformly in n). Combining Lemmas 3.15–3.16, the above inequality ensures that if $A_i \in W_a$, we have

$$\mathbb{P}(|R_2^-(G_i^s) - \mathbb{E}[R_2(G_*^{s-i})]| = O(\sqrt{n} \log n), A_i \in W_a) = 1 - O\left(\frac{1}{\log^2 n}\right).$$

Next suppose that that Λ_i does not happen. Set $W_{ia} := [nk_{ia}(\theta_*) - \sqrt{n} \log n, nk_{ia}(\theta_*) + \sqrt{n} \log n]$. By (27) we have $A_i \in W_{ia}$ with probability $1 - o(1/n)$. In this case, $G(A_i, \frac{\beta_c}{n})$ is subcritical (uniformly in n). Combining Observation 3.25 and (12) in Lemma 2.7, we get that

$$|\mathbb{E}[R_2(G_i^s)] - \mathbb{E}[R_2(G_*^{s-i})]| = O(\sqrt{n} \log n).$$

Then by Lemma 3.15 we have

$$\mathbb{P}(|R_2(G_i^s) - \mathbb{E}[R_2(G_*^{s-i})]| = O(\sqrt{n} \log n), A_i \in W_{ia}) = O\left(\frac{1}{\log^2 n}\right). \quad (32)$$

Hence, we get (30) by a union bound. The argument for (31) is only easier than the above, following immediately from Lemma 3.15 and the super-criticality of the initialization parameter. \square

3.2.2 Upper bound on escape time from the unstable fixed point

Let τ_γ denote the first time $X_t > \theta_* n + \gamma \sqrt{n}$ and let $\tau_{-\gamma}$ denote the first time $X_t < \theta_* n - \gamma \sqrt{n}$. We show an upper bound of the exit time from the window $I_\gamma := [\theta_* n - \gamma \sqrt{n}, \theta_* n + \gamma \sqrt{n}]$, where γ is a sufficiently large constant, to be chosen depending on the ε -total-variation distance to stationarity we aim for.

Lemma 3.17. *Let $\tilde{\tau} := \min\{\tau_\gamma, \tau_{-\gamma}\}$. If $X_0 \sim G(n, \frac{p_0}{n})$ with $p_0 = \lambda_* + O(n^{-1/2})$ then there exists a constant $C > 0$ such that for every $T \geq 1$ fixed independent of n ,*

$$\mathbb{P}(\tilde{\tau} \leq 2T e^{C\gamma^2}) \geq 1 - \frac{1}{T}.$$

Proof. If $L_0 \notin I_\gamma$, then $\tilde{\tau} = 0$ and the lemma holds trivially. We assume therefore that $L_0 \in I_\gamma$. Also we assume $X_0 \in \mathcal{G}_K$, for a large constant K , which happens with probability $1 - o(1)$ by Lemma 3.12. Let $\tau' := \min\{t : X_t \notin \mathcal{G}_K\}$ and let $\tau = \tilde{\tau} \wedge \tau'$. Observe that τ is stochastically dominated from above by a geometric random variable $\text{Geo}(y)$, where

$$y_\gamma := \min_{X_t \in \mathcal{G}_K : L_t \in I_\gamma} \mathbb{P}(L_{t+1} > \theta_* n + \gamma \sqrt{n} \mid X_t).$$

Our aim in what follows is to show existence of $\kappa > 0$ such that

$$y_\gamma \geq e^{-\kappa\gamma^2}. \quad (33)$$

Indeed, assuming (33), $\mathbb{E}[\tau \mid X_0 \in I_\gamma] \leq \mathbb{E}[\text{Geo}(y)] \leq e^{\kappa\gamma^2}$, and by Markov's inequality, we would have $\mathbb{P}(\tau > 2T e^{\kappa\gamma^2}) \leq \frac{1}{2T}$ for every T . Then we have

$$\mathbb{P}(\tilde{\tau} \geq 2T e^{\kappa\gamma^2}) \leq \mathbb{P}(\tau > 2T e^{\kappa\gamma^2}) + \mathbb{P}(\tau' \leq 2T e^{\kappa\gamma^2}) \leq \frac{1}{2T} + 2T e^{\kappa\gamma^2} \cdot o(1) \leq \frac{1}{T},$$

by Lemma 3.12 and a union bound.

To show (33), note that $\mathbb{P}(\Lambda_{t+1}) = 1/q$, and we suppose Λ_{t+1} happens. Then L_{t+1} is distributed like the giant component of $G(A_{t+1}, \frac{\beta_c}{n})$, where A_{t+1} is the number of activated vertices in step $t+1$. By property 3 of \mathcal{G}_K ,

$$\text{Var}(A_{t+1} \mid X_t, \Lambda_{t+1}) = \frac{1}{q} \left(1 - \frac{1}{q}\right) R_2^-(X_t) \leq Kn. \quad (34)$$

Suppose $L(X_t) = x_0 \in I_\gamma$. By Chebyshev's inequality we have

$$A_{t+1} \geq x_0 + \frac{n - x_0}{q} - 2\sqrt{Kn} \geq k_a(\theta_* - \gamma n^{-1/2})n - 2\sqrt{Kn} =: m,$$

with probability at least $1/4$. On this event, $G(A_{t+1}, \frac{\beta_c}{n}) \geq G(m, \frac{\beta_c}{n})$ so in order to show (33), it suffices to show that if $X \sim G(m, \beta_c/n)$,

$$\mathbb{P}(|\mathcal{L}_1(X)| \geq \theta_* n + \gamma \sqrt{n}) \geq 4qe^{-\kappa\gamma^2}. \quad (35)$$

Let $L = |\mathcal{L}_1(X)|$ for $X \sim G(m, \beta_c/n)$. Then, Let $\lambda = m \cdot \frac{\beta_c}{n}$. By subtracting $\alpha(\lambda)m$ on both sides and dividing by $\sigma(\lambda)\sqrt{m}$, we rewrite the left-hand side of (35) as

$$\mathbb{P}(L \geq \theta_* n + \gamma \sqrt{n}) = \mathbb{P}\left(\frac{L - \alpha(\lambda)m}{\sigma(\lambda)\sqrt{m}} \geq \frac{\theta_* n + \gamma \sqrt{n} - \alpha(\lambda)m}{\sigma(\lambda)\sqrt{m}}\right). \quad (36)$$

By Theorem 2.4, we have

$$\mathbb{P}\left(\frac{L - \alpha(\lambda)m}{\sigma(\lambda)\sqrt{m}} \geq \frac{\theta_*n + \gamma\sqrt{n} - \alpha(\lambda)m}{\sigma(\lambda)\sqrt{m_1}}\right) = \mathbb{P}\left(Z \geq \frac{\theta_*n + \gamma\sqrt{n} - \alpha(\lambda)m}{\sigma(\lambda)\sqrt{m}}\right) + o(1), \quad (37)$$

where Z is a standard Gaussian random variable. Also, observe using the facts that $m = \Theta(n)$ and $|\alpha(\lambda)m - \theta_*n - \gamma\sqrt{n}| = O(\gamma\sqrt{n})$ by definition of θ_* ,

$$\frac{\theta_*n + \gamma\sqrt{n} - \alpha(\lambda)m_1}{\sigma(\lambda)\sqrt{m_1}} \leq C_1\gamma, \quad (38)$$

for some constant $C_1 > 0$. By standard Gaussian tail estimates, we get for some $C_2 > 0$ and $\gamma \geq 1$ say,

$$\mathbb{P}(L \geq \theta_*n + \gamma\sqrt{n}) \geq e^{-C_2\gamma^2} + o(1).$$

This implies (35) for some $\kappa > 0$, for large n . \square

3.2.3 Approximating by a 1-dimensional Markov process

In order for mixing to be fast from the near-saddle initialization at $\beta = \beta_c$, we need the diffusion near the unstable fixed point to exit out the right and left with probabilities corresponding to the relative weights of the ordered and disordered phases. Towards capturing this, our goal in this subsection is to approximate the giant component process $(L_t)_{t \geq 0}$ near θ_*n by a monotone 1-dimensional Markov process.

In order to define this process, recall $\alpha(\lambda)$ and $\sigma^2(\lambda)$ from (7)–(9). For $\lambda > 1$, let

$$h_1(\lambda) := \alpha(\lambda) + \alpha'(\lambda) \cdot \frac{\lambda}{\beta_c} \quad \text{and} \quad h_2(\lambda) := \sigma^2(\lambda) \cdot \frac{\lambda}{\beta_c} = \frac{\alpha(\lambda)(1 - \alpha(\lambda))}{(1 - \lambda(1 - \alpha(\lambda)))^2} \cdot \frac{\lambda}{\beta_c}. \quad (39)$$

Since $\alpha(\lambda)$ is twice differentiable for $\lambda > 1$, h_1, h_2 exist, and are differentiable.

Definition 3.18. Define the 1-dimensional Markov process initialized from Z_0 and given Z_t ,

$$Z_{t+1} \sim Z_t + \varepsilon_{t+1} \left(f'(\theta_*)Z_t + \sqrt{n}\mathcal{N}(0, (h_1(\beta_c k_a(\theta_*)))^2 \sigma_t^2 + h_2(\beta_c k_a(\theta_*))) \right), \quad (40)$$

where $\varepsilon_{t+1} \sim \text{Ber}(1/q)$ independently of the normal.

Theorem 3.19. Suppose $X_0 \sim G(n, \frac{p_0}{n})$ with $p_0 = \lambda_* + O(n^{-1/2})$ and let $(Z_t)_t$ be the process of Definition 3.18 initialized from $Z_0 \sim \mathcal{N}((p_0 - \lambda_*)\alpha'(\lambda_*)n, \sigma^2(\lambda_*)n)$. There exists a coupling such that with probability $1 - T \cdot o(1)$, for all $t \leq T$,

$$|(L_t - \theta_*n) - Z_t| = O(\log n).$$

We consider the following intermediate process that stays within distance 1 of L_t w.h.p. and is easier to compare to Z_t : let $Y_0 \sim \mathcal{N}(\theta_*n + (p_0 - \lambda_*)\alpha'(\lambda_*)n, \sigma^2(\lambda_*)n)$, and given $Y_t \in \mathbb{R}$ with $t \geq 0$, with probability $1 - \frac{1}{q}$ set $Y_{t+1} = Y_t$ and with the remaining probability do the following:

1. generate a random real number $B_{t+1} \sim \mathcal{N}\left(\frac{n - Y_t}{q}, \sigma_t^2\right)$,
2. generate a random number

$$Y_{t+1} \sim \mathcal{N}\left(\alpha\left(\frac{\beta_c}{n} \cdot (Y_t + B_{t+1})\right) \cdot (Y_t + B_{t+1}), \sigma^2\left(\frac{\beta_c}{n} \cdot (Y_t + B_{t+1})\right) \cdot (Y_t + B_{t+1})\right).$$

Here, recall σ_t^2 was defined in (28). Besides the centering by $\theta_* n$, the key difference between Y_t and Z_t is that the variance of its increments are functions of Y_t themselves rather than simply functions of time.

Theorem 3.19 will follow from the following two lemmas. Recall that $\tilde{\tau} := \min\{\tau_Y, \tau_{-Y}\}$.

Lemma 3.20. *Let $X_0 \sim G(n, \frac{p_0}{n})$ with $p_0 = \lambda_* + O(n^{-1/2})$. For all $T \geq 0$ fixed independent of n , there exists a coupling \mathbf{P} of $\{(X_t, Y_t)\}_{t \geq 0}$ such that for all $t \leq T$, $\mathbf{P}(|L_t - Y_t| \leq 1) = 1 - o(1)$.*

Lemma 3.21. *There exists a coupling $\{(Y_t, Z_t)\}_{t \geq 0}$ such that for $t \leq \tilde{\tau}$, $|Y_t - (Z_t + \theta_* n)| = O(\log n)$ with probability $1 - t \cdot n^{-\Omega(1)}$.*

Proof of Lemma 3.20. We will couple (L_t, Y_t) inductively. For the base case, consider $t = 0$; by Lemma 2.3, (X_0, Y_0) can be coupled such that with probability $1 - o(1)$, $|L_0 - Y_0| \leq 1$.

Now suppose there exists a coupling of the first t steps of $\{(X_t, Y_t)\}$ such that $|L_t - Y_t| \leq 1$ with probability $1 - o(1)$. By Lemma 3.14, we have $(X_s)_{s \leq t} \in \mathcal{G}_t$ with probability $1 - o(1)$, so in what follows we work on that event. In case $(X_s)_{s \leq t} \notin \mathcal{G}_t$, we stop the coupling. For the induction step, we construct a coupling \mathbf{P}_{t+1} of (X_{t+1}, Y_{t+1}) .

On Λ_{t+1}^c , we let $Y_{t+1} = Y_t$. At the same time, on Λ_{t+1}^c and \mathcal{G}_t , with high probability the percolation step is sub-critical and therefore $L_{t+1} = L_t$ with probability $1 - o(1)$ (a similar argument was made in the proof of Lemma 3.12). Hence, we have $|L_{t+1} - Y_{t+1}| = |L_t - Y_t| \leq 1$ with probability $1 - o(1)$ in this case.

Next we consider the case when Λ_{t+1} occurs. Let A_{t+1}^- be the number of activated vertices in step $t+1$ that are not in $\mathcal{L}_1(X_t)$. Note that $X_t \in \mathcal{G}_t$ satisfies all the conditions of Lemma 3.27, which implies that there exists a coupling \mathbf{P} of (A_{t+1}^-, N_{t+1}) such that $\mathbf{P}(|A_{t+1}^- - N_{t+1}| > 1) = O(n^{-1/8})$, where $N_{t+1} \sim \mathcal{N}(\mathbb{E}[A_{t+1}^- | X_t], \text{Var}(A_{t+1}^- | X_t))$. In addition, we want to show that B_{t+1} and N_{t+1} are typically close, where B_{t+1} is the random variable used in generation of Y_{t+1} . Clearly, $\mathbb{E}[A_{t+1}^- | X_t] = \frac{n-L_t}{q}$, and note that $\text{Var}(A_{t+1}^- | X_t) = \frac{1}{q}(1 - \frac{1}{q})R_2^-(X_t)$. Hence, the inductive assumption ensures $|\mathbb{E}[A_{t+1}^- | X_t] - \mathbb{E}[B_{t+1} | Y_t]| \leq 1$, and since we're on the event \mathcal{G}_t , $|\text{Var}(A_{t+1}^- | X_t) - \sigma_t^2| \leq \sqrt{n} \log^2 n$. We appeal to the following standard bound on the TV-distance between 1-dimensional Gaussians, which is an easy calculation (see e.g., the univariate case of [17] which is focused on the more difficult multivariate case): if $N_X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $N_Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Then

$$\|N_X - N_Y\|_{\text{TV}} \leq \frac{3|\sigma_X^2 - \sigma_Y^2|}{2\sigma_Y^2} + \frac{|\mu_X - \mu_Y|}{2\sigma_Y}. \quad (41)$$

Applying (41)

$$\|B_{t+1} - N_{t+1}\|_{\text{TV}} \leq \frac{3|\text{Var}(A_{t+1}^- | X_t) - \sigma_t^2|}{2\text{Var}(A_{t+1}^- | X_t)} + \frac{|\mathbb{E}[A_{t+1}^- | X_t] - \mathbb{E}[B_{t+1} | Y_t]|}{2\sqrt{\text{Var}(A_{t+1}^- | X_t)}} = O\left(\frac{\sqrt{n} \log^2 n}{n}\right) = O\left(\frac{\log^2 n}{\sqrt{n}}\right).$$

So by the optimal coupling lemma, there exists a coupling \mathbf{P}_{t+1} of $(A_{t+1}^-, N_{t+1}, B_{t+1})$ such that

$$\mathbf{P}_{t+1}(\{B_{t+1} \neq N_{t+1}\} \cup \{|A_{t+1}^- - N_{t+1}| > 1\} | \mathcal{F}_t) = O(n^{-1/8}).$$

From now on, we assume $|B_{t+1} - A_{t+1}^-| \leq 1$ under the coupling \mathbf{P}_{t+1} . The inductive assumption implies

$$|(L_t + A_{t+1}^-) - (Y_t + B_{t+1})| \leq 2.$$

On the percolation step, Lemma 3.29 implies that there exists a coupling of $(|\mathcal{L}_1(G_{t+1})|, Y_{t+1})$ such that $|Y_{t+1} - |\mathcal{L}_1(G_{t+1})|| \leq 1$ with probability $1 - o(1)$. Since $|\mathcal{L}_2(X_t)| = O(\log n)$ and the giant component has been activated, with high probability the percolation step is super-critical and, $L_{t+1} = |\mathcal{L}_1(G_{t+1})|$.

Therefore, it follows a union bound over all the probabilistic estimates used so far that the coupling of (X_{t+1}, Y_{t+1}) satisfies all the desired properties with probability $1 - o(1)$. \square

Proof of Lemma 3.21. Under the identity coupling of the initial normal random variables, with probability one, $Y_0 = Z_0 + \theta_* n$. For $t \geq 0$, assume $|Y_t - (Z_t + \theta_* n)| = O(\log n)$ and $t < \tilde{\tau}$. Now we couple the step (Y_{t+1}, Z_{t+1}) . When $\varepsilon_{t+1} = 0$, we couple (Y_{t+1}, Z_{t+1}) such that both Y_{t+1} and Z_{t+1} stay idle. Now assume $\varepsilon_{t+1} = 1$. Let $\Theta_t := Y_t/n$. Taking the two sub-steps of $\{Y_t\}$ into one step, we have

$$Y_{t+1} = \alpha \left(\beta_c \left(\frac{1}{q} + \frac{q-1}{q} \Theta_t + \frac{W_{t+1}}{n} \right) \right) \cdot \left(\frac{n}{q} + \frac{q-1}{q} Y_t + W_{t+1} \right) \\ + \mathcal{N} \left(0, \sigma^2 \left(\beta_c \left(\frac{1}{q} + \frac{q-1}{q} \Theta_t + \frac{W_{t+1}}{n} \right) \right) \cdot \left(\frac{n}{q} + \frac{q-1}{q} Y_t + W_{t+1} \right) \right),$$

where $W_{t+1} \sim \mathcal{N}(0, \sigma_t^2)$. Let $\tilde{k}(\theta) = k_a(\theta) + W_{t+1}/n$. Then

$$Y_{t+1} = \alpha(\beta_c \cdot \tilde{k}(\Theta_t)) \cdot \tilde{k}(\Theta_t) n + \mathcal{N}(0, \sigma^2(\beta_c \cdot \tilde{k}(\Theta_t)) \cdot \tilde{k}(\Theta_t) n). \quad (42)$$

By Lemma 2.7 and Observation 3.25, $\sigma_t^2 = \Theta(n)$, so for the rest of the proof, we assume $W_{t+1} = O(\sqrt{n \log n})$, which happens with probability $1 - n^{-2}$ for every t . We first Taylor expand $\alpha(\beta_c \tilde{k}(\Theta_t))$ about $\beta_c k_a(\Theta_t)$. Using twice-differentiability of α , we get

$$\alpha(\beta_c \cdot \tilde{k}(\Theta_t)) = \alpha(\beta_c \cdot k_a(\Theta_t)) + \alpha'(\beta_c \cdot k_a(\Theta_t)) \cdot \frac{W_{t+1}}{n} + O\left(\frac{\log n}{n}\right). \quad (43)$$

Multiplying this with $\tilde{k}(\Theta_t) n = k_a(\Theta_t) n + W_{t+1}$, we get

$$\alpha(\beta_c \cdot \tilde{k}(\Theta_t)) \cdot \tilde{k}(\Theta_t) n = \phi(\Theta_t) n + h_1(d_t) W_{t+1} + O(\log n),$$

where

$$\phi(\theta) := \alpha(\beta_c k_a(\theta)) k_a(\theta) \quad \text{and} \quad d_t := \beta_c k_a(\Theta_t).$$

Let $d_* := \beta_c k_a(\theta_*)$. Since h'_1 is a bounded function and $|\Theta_t - \theta_*| \leq \frac{\gamma}{\sqrt{n}}$, $|h_1(d_t) - h_1(d_*)| = O(\frac{\gamma}{\sqrt{n}})$. Hence, by (41), we can couple $h_1(d_t) \cdot \mathcal{N}(0, \sigma_t^2)$ and $h_1(d_*) \cdot \mathcal{N}(0, \sigma_t^2)$ with probability at least

$$1 - \|\mathcal{N}(0, h_1(d_t)^2 \sigma_t^2) - \mathcal{N}(0, h_1(d_*)^2 \sigma_t^2)\|_{\text{TV}} \geq 1 - \frac{|h_1^2(d_t) - h_1^2(d_*)| \sigma_t^2}{2h_1(d_*) \sigma_t^2} = 1 - O\left(\frac{\gamma}{\sqrt{n}}\right).$$

Hence, with probability $1 - O(n^{-1/2})$ we can replace $h_1(d_t) W_{t+1}$ by $h_1(d_*) W_{t+1}$.

We proceed with Taylor expansion on $f(\cdot)$ about θ_* . By noting that $f(\theta_*) = 0$, $f'(\theta_*) = O(1)$, $f''(\theta_*) = O(1)$ and $(\Theta_t - \theta_*)^2 n \leq \gamma^2$, we have

$$\phi(\Theta_t) n = [\Theta_t + f(\Theta_t)] n = \Theta_t n + [f(\theta_*) + f'(\theta_*)(\Theta_t - \theta_*) + f''(\theta_*)(\Theta_t - \theta_*)^2] n \\ = \theta_* n + (1 + f'(\theta_*))(\Theta_t - \theta_*) n + O(\gamma^2).$$

Therefore, the following equation holds with probability $1 - O(n^{-1/2})$,

$$\alpha(\beta_c \cdot \tilde{k}(\Theta_t)) \cdot \tilde{k}(\Theta_t) n = \theta_* n + (1 + f'(\theta_*))(\Theta_t - \theta_*) n + h_1(d_*) W_{t+1} + O(\log n). \quad (44)$$

Finally we handle the normal random variable in (42). Let $\tilde{d}_* := \beta_c \tilde{k}(\theta_*)$ and $\tilde{d}_t := \beta_c \cdot \tilde{k}(\Theta_t)$. Since h'_2 is a bounded function and $|\Theta_t - \theta_*| \leq \frac{\gamma}{\sqrt{n}}$, $|h_2(\tilde{d}_t) - h_2(\tilde{d}_*)| = O(\frac{\gamma}{\sqrt{n}})$. Again, since h'_2 is a bounded function and $|\tilde{d}_* - d_*| = O(\sqrt{\frac{\log n}{n}})$, $|h_2(\tilde{d}_*) - h_2(d_*)| = O(\sqrt{\frac{\log n}{n}})$. By triangle inequality,

$$|h_2(\tilde{d}_t) - h_2(d_*)| \leq |h_2(\tilde{d}_*) - h_2(d_*)| + |h_2(\tilde{d}_t) - h_2(\tilde{d}_*)| = O(\sqrt{\frac{\log n}{n}}).$$

Hence, it follows from (41) that we can couple $\mathcal{N}(0, h_2(\tilde{d}_t) n)$ and $\mathcal{N}(0, h_2(d_*) n)$ to agree with probability $1 - O(\sqrt{\frac{\log n}{n}})$. The result follows from (42), (44) and this coupling. \square

3.2.4 Analysis of the limiting 1-dimensional process

Now that we have shown the giant component process near the fixed point θ_*n is well-approximated by the 1-dimensional process Z_t of (40), we show here that this simplified process's exit probabilities to the right and left are monotone, and oscillate on the \sqrt{n} scale. In what follows, let τ_γ^Z be the hitting time of $\gamma\sqrt{n}$ for Z_t from (40) and let $\tau_{-\gamma}^Z$ be the hitting time of $-\gamma\sqrt{n}$.

Lemma 3.22. *The Markov chain of (40) is monotone. In particular, for every $p \in (0, 1)$, there is a unique $c_* \in \mathbb{R}$ such that if $p_0 = \lambda_* + c_*n^{-1/2} + o(n^{-1/2})$ and $Z_0 \sim \mathcal{N}((p_0 - \lambda_*)\alpha'(\lambda_*)n, \sigma^2(\lambda_*)n)$ then*

$$\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z) = p + o_{\gamma,n}(1),$$

where $o_{\gamma,n}(1)$ means it goes to zero either as $n \rightarrow \infty$ or as $\gamma \rightarrow \infty$.

Proof. For the monotonicity, suppose $Z_t \leq Z'_t$ and consider Z_{t+1} and Z'_{t+1} generated per (40) using the same Bernoulli random variable ε_{t+1} and the same pair of normal random variables. If $\varepsilon_{t+1} = 0$ then $Z_{t+1} = Z_t \leq Z'_t = Z'_{t+1}$ and the monotonicity is preserved. If $\varepsilon_{t+1} = 1$, then under this coupling

$$Z'_{t+1} - Z_{t+1} = (Z'_t - Z_t)(1 + f'(\theta_*)).$$

This will be positive because $Z'_t - Z_t \geq 0$ and $f'(\theta_*) > 0$.

To see the other consequences, notice first that the process $(\bar{Z}_t)_{t \geq 0} = (Z_t n^{-1/2})_{t \geq 0}$ is n -independent (as all of $h_1, h_2, \sigma_t, f', \beta_u$ are n -independent). This implies that $\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z)$ are n -independent from n -independent initializations, which will be the case if $p_0 = \lambda_* + c_*n^{-1/2}$. As $Z_0 \rightarrow \gamma\sqrt{n}$, the probability $\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z)$ is easily checked to go to 1, and as $Z_0 \rightarrow -\gamma\sqrt{n}$, it goes to zero. Finally, to see that c_* is γ -independent with the $o_\gamma(1)$, similarly observe that

$$|\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z) - \mathbb{P}(\tau_{(1+f'(\theta_*)/2)\gamma}^Z < \tau_{-(1+f'(\theta_*)/2)\gamma}^Z)| = o_\gamma(1).$$

Together, these yield the lemma. \square

We can use the monotonicity above, together with the closeness of the Z_t process with X_t to translate the right initialization from $(Z_t)_t$ to $(X_t)_t$.

Lemma 3.23. *Suppose $X_0 \sim G(n, p_0/n)$ for $p_0 = \lambda_* + O(n^{-1/2})$ and let $(Z_t)_t$ be the process of Definition 3.18 initialized from $Z_0 \sim \mathcal{N}((p_0 - \lambda_*)\alpha'(\lambda_*)n, \sigma^2(\lambda_*)n)$. Then the hitting probabilities $\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z)$ and $\mathbb{P}(\tau_\gamma^X < \tau_{-\gamma}^X)$ are within $1 - o_{\gamma,n}(1)$ of one another.*

Proof. By Theorem 3.19, there exists a coupling such that the processes Z_t and $L_t - \theta_*n$ are within $O(\log n)$ for all $O(1)$ times, except with $o(1)$ probability. Now fix any $\varepsilon > 0$ admissible difference between the probabilities in the lemma. For any such ε , there exists a γ such that uniformly over t , if $L_t - \theta_*n$ is in $\gamma\sqrt{n} - O(\log n)$ the probability that in the next step it exceeds $\gamma\sqrt{n}$ is at least $1 - \varepsilon$; the uniformity over t uses the fact that the variance σ_t^2 is uniformly bounded, while the drift increases linearly with γ . This implies that if Z_t has hit $\gamma\sqrt{n}$ then with probability $1 - \varepsilon - o(1)$, the process $L_t - \theta_*n$ either has hit $\gamma\sqrt{n}$, or will hit it in the next step.

The same holds for hitting $-\gamma\sqrt{n}$, as well as for the converse implications, i.e., that if X has hit one side or the other then so has Z or will in the next step. Putting these together, we deduce that for every ε , there exists γ sufficiently large such that $|\mathbb{P}(\tau_\gamma^Z < \tau_{-\gamma}^Z) - \mathbb{P}(\tau_\gamma^X < \tau_{-\gamma}^X)| \leq \varepsilon + o(1)$. \square

3.3 Proof of Theorem 1.1

Off-criticality. We begin by concluding the bounds of mixing when initialized on the appropriate side of the unstable fixed point, taking care of all off-critical portions of Theorem 1.1.

For item 1, when $\beta \in (\beta_u, \beta_c)$, we let $\lambda_0 = \lambda_* - \omega(n^{-1/2})$, where λ_* is the solution to $\alpha(\lambda_*) = \theta_*$; by differentiability of α with strictly positive derivative in the super-critical regime, if $\lambda_0 = \lambda_* - \omega(n^{-1/2})$, then $\alpha(\lambda_0) = \theta_* - \omega(n^{-1/2})$. In particular, by Lemma 2.10, X_0 satisfies the conditions of Lemma 3.1 with $\gamma = \omega(1)$. Furthermore, by Lemma 2.1 and Observation 2.5, when $\beta < \beta_c$, we have $\|\mu^{\text{dis}} - \mu\|_{\text{TV}} = o(1)$. The result follows by the triangle inequality. For item 3, the proof is symmetrical, with the observation that when $\beta > \beta_c$, we have $\|\mu^{\text{ord}} - \mu\|_{\text{TV}} = o(1)$.

Criticality. We now turn to the mixing time at the critical point $\beta = \beta_c$. Fix an ε total-variation distance we are trying to achieve, and in turn take γ sufficiently large. By Lemma 2.1, the stationary distribution μ is a $\xi, 1 - \xi$ mixture of μ^{dis} and μ^{ord} for ξ defined in (6). By Lemma 3.22, there is a unique $c_*(\xi)$ such that the escape probabilities of Lemma 3.22 are within $o(1)$ of $\xi, 1 - \xi$. By Lemma 3.23, if $X_0 \sim G(n, \lambda_0)$ with $\lambda_0 = \lambda_* + c_* n^{-1/2} + o(n^{-1/2})$, then $|\mathbb{P}(\tau_Y^X < \tau_{-Y}^X) - \xi| \leq \varepsilon + o(1)$. Moreover, by Lemma 3.17, the minimum of these two exit times is $O(1)$. Finally, by Lemma 3.14, with probability $1 - o(1)$, at exit, the configuration satisfies the necessary conditions to apply Lemma 3.1 and quasi-equilibrate to the phase-restricted measure on the side the dynamics exits with probability $1 - O(\gamma^{-2})$. Combined, these imply there exists $T = O(\log n)$ such that if $X_0 \sim G(n, \lambda_0)$, then

$$\|\mathbb{P}(X_T \in \cdot) - ((1 - \xi)\mu^{\text{ord}} + \xi\mu^{\text{dis}})\|_{\text{TV}} \leq O(\gamma^{-2}) + \varepsilon + o(1),$$

which will be less than 2ε for n large and γ large.

3.4 Lower bound on the mixing time with different choices of λ_0

By combining the above quasi-equilibration results with slow mixing results of [24] for the CM dynamics, we show that if the initialization is the product measure with parameters not satisfying the conditions of Theorem 1.1, then mixing is slow.

Theorem 3.24. *For every $q > 2$ and $\beta \in (\beta_u, \beta_s)$, if $\lambda_*(\beta, q)$ and $c_*(q)$ are as in Theorem 1.1, then the CM dynamics initialized from $\bigotimes \text{Ber}(\lambda_0/n)$ with*

1. $\beta \in (\beta_u, \beta_c)$ and $\lambda_0 > \lambda_*(\beta, q) - O(n^{-1/2})$,
2. $\beta = \beta_c$ and $\lambda_0 \neq \lambda_*(\beta, q) + c_*(q)n^{-1/2} + o(n^{-1/2})$,
3. $\beta \in (\beta_c, \beta_s)$ and $\lambda_0 < \lambda_*(\beta, q) + O(n^{-1/2})$,

takes $\exp(\Omega(n))$ time to reach $o(1)$ TV-distance to stationarity.

Proof. We provide the details for item (1), the other cases following by similar reasoning. For any initialization parameter $\lambda_0 > \lambda_* - Kn^{-1/2}$ for some $K = O(1)$, by Lemma 3.22 and 3.23, there is a positive probability $c_K > 0$ that the process L_t hits $\theta_* + \gamma n^{-1/2}$ before $\theta_* - \gamma n^{-1/2}$ (for sufficiently large γ) in some $t \leq C_\gamma$ many steps. By Lemma 3.14, with probability $1 - o(1)$, at exit, the configuration satisfies the necessary conditions to apply Lemma 3.1 and quasi-equilibrate to the right to the ordered phase μ^{ord} . Putting these together, we find that for some $T_0 = O(\log n)$,

$$\|\mathbb{P}(X_{T_0} \in \cdot) - \mu^{\text{ord}}\|_{\text{TV}} \leq 1 - c_K + o_{\gamma, n}(1). \quad (45)$$

By using the optimal coupling on these, and then the identity coupling of CM chains after time T_0 ,

$$\|\mathbb{P}(X_{t+T_0} \in \cdot) - \mathbb{P}_{\mu^{\text{ord}}}(X_t \in \cdot)\|_{\text{TV}} \leq 1 - c_K + o_{\gamma, n}(1), \quad (46)$$

where the first chain here is initialized from the product measure, and the second from μ^{ord} .

Next, we claim that by the results of [24], a CM dynamics chain initialized from μ^{ord} retains total-variation distance $1 - o(1)$ for exponentially many steps to μ when $\beta \in (\beta_u, \beta_c)$. To see this, note from [24, Lemma 4.7 and the proof of Theorem 2] the existence of a bottleneck set A (that the giant component is at least $(\theta_* + \varepsilon)n$ and the number of vertices in components of size larger than M is at most ρn) such that it takes $\exp(\Omega(n))$ steps for a CM dynamics initialized from $\mu(\cdot \mid A)$ to leave A . The initialization from $\mu(\cdot \mid A)$ can be seen to be within $e^{-\Omega(n)}$ total-variation distance of an initialization from μ^{ord} by an application of Lemma 2.1 and Observation 2.5. From this, we deduce that for every t ,

$$\mathbb{P}_{\mu^{\text{ord}}}(X_t \in A) \geq 1 - te^{-\Omega(n)}. \quad (47)$$

By definition of total-variation distance, (46) together with (47), implies

$$\mathbb{P}(X_{t+T_0} \in A) \geq c_K - o_{Y,n}(1) - te^{-\Omega(n)}.$$

On the other hand, by Lemma 2.1, when $\beta \in (\beta_u, \beta_c)$, one has $\mu(A) = o(1)$, so the above bound implies that for γ large, and some $t = e^{\Omega(n)}$, the total-variation to μ is at least $c_K/2$, say. \square

3.5 Deferred proofs: concentration and local limit theorem for the activation step

We now include proofs of concentration of the activation steps and local limit theorems that were deferred.

3.5.1 Concentration properties of activation steps

We begin by describing some easy estimates on the activation step of the CM dynamics. For a graph X , let X^r be the sub-graph of X that does not get activated in r activation steps.

Observation 3.25. Suppose $X \sim G(n, p)$. For any integer $r \geq 0$,

$$\mathbb{E}[R_2(X^r)] = \left(1 - \frac{1}{q}\right)^r \mathbb{E}[R_2(X)], \quad \mathbb{E}[R_2^-(X^r)] = \left(1 - \frac{1}{q}\right)^r \mathbb{E}[R_2^-(X)].$$

Proof. Let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be independent Bernoulli random variables with parameter $(1 - \frac{1}{q})^r$. Firstly,

$$\mathbb{E}[R_2(X^r) \mid X] = \mathbb{E}\left[\sum_{j \geq 1} |\mathcal{L}_j(X)|^2 \cdot \mathcal{B}_j \mid X\right] = \sum_{j \geq 1} \mathbb{E}[|\mathcal{L}_j(X)|^2 \cdot \mathcal{B}_j \mid X] = \left(1 - \frac{1}{q}\right)^r R_2(X). \quad (48)$$

The first equality follows by taking expected values; the second equality is analogous. \square

Proof of Lemma 3.15. Let $\mathcal{B}_1, \dots, \mathcal{B}_n$ be i.i.d. Bernoulli random variables with parameter $(1 - \frac{1}{q})^r$. First we compute a conditional variance.

$$\text{Var}(R_2^-(X^r) \mid X) = \sum_{j \geq 2} \text{Var}(|\mathcal{L}_j(X)|^2 \cdot \mathcal{B}_j \mid X) \leq \sum_{j \geq 2} |\mathcal{L}_j(X)|^2 = R_2^-(X).$$

Then, by the law of total variance and Observation 3.25, we obtain

$$\text{Var}(R_2^-(X^r)) = \mathbb{E}[\text{Var}(R_2^-(X^r) \mid X)] + \text{Var}(\mathbb{E}[R_2^-(X^r) \mid X]) \leq \mathbb{E}[R_2^-(X)] + \text{Var}(R_2^-(X)). \quad (49)$$

If $n \cdot p > 1$ uniformly in n , then by Corollary 2.9 and (49), the right-hand side is $O(n)$ and the result follows by Chebyshev's inequality. The case $np < 1$ uniformly in n follows analogous reasoning. \square

We also prove Lemma 3.16 showing the approximability of $\mathbb{E}[R_2^-(X^r)]$ by $\mathbb{E}[R_2(G_*^r)]$.

Proof of Lemma 3.16. First, observe that $\tilde{\beta} := (nk_a(\theta_*) + m) \cdot \frac{\beta_c}{n} > 1$ uniformly in n . By Lemma 2.10,

$$\mathbb{P}(|\mathcal{L}_1(G)| - \alpha(\tilde{\beta})(nk_a(\theta_*) + m) | > \sqrt{n}(\log n)^{2/3}) \leq \frac{1}{2n^{10}}.$$

Moreover, for each U such that $||U| - \alpha(\tilde{\beta})(nk_a(\theta_*) + m)| \leq \sqrt{n}(\log n)^{2/3}$, Lemma 2.6 implies if $U = \mathcal{L}_1(G)$ then $G \setminus \mathcal{L}_1(G)$ can be coupled with $G_- \sim G(nk_a(\theta_*) + m - |\mathcal{L}_1(G)|, \frac{\beta_c}{n})$ with probability $1 - e^{-\Omega(n)}$. Hence, with probability at least $1 - n^{-10}$,

$$m' := ||\mathcal{L}_1(G)| - \alpha(\tilde{\beta})(nk_a(\theta_*) + m)| \leq \sqrt{n}(\log n)^{2/3},$$

and $G \setminus \mathcal{L}_1(G) = G_-$. It follows from Observation 3.25 that

$$|\mathbb{E}[R_2^-(G')] - \mathbb{E}[R_2(G_-)]| \leq |\mathbb{E}[R_2^-(G)] - \mathbb{E}[R_2(G_-)]| = O(n^{-8}). \quad (50)$$

Next we show when $m' \leq \sqrt{n}(\log n)^{2/3}$,

$$|\mathbb{E}[R_2(G'_*)] - \mathbb{E}[R_2(G'_-)]| = O(\sqrt{n} \log n), \quad (51)$$

and the lemma follows from (50) and (51). To see (51), we give an estimate of the number of vertices M in G_- . From the arguments above, we know

$$M = nk_a(\theta_*) + m - \alpha(\tilde{\beta})(nk_a(\theta_*) + m) + m'.$$

By algebraic manipulation and Taylor expansion of α , we obtain

$$\begin{aligned} M &= nk_{ia}(\theta_*) + (\theta_* n - \alpha(\tilde{\beta}) \cdot k_a(\theta_*) n) + O(\sqrt{n}(\log n)^{2/3}) \\ &= nk_{ia}(\theta_*) + [\theta_* n - \phi(\theta_*) n + m \beta_c \alpha'(k_a(\theta_*) \cdot \beta_c) \cdot k_a(\theta_*) + o(\sqrt{n})] + O(\sqrt{n}(\log n)^{2/3}), \end{aligned}$$

which gives (51) since $\phi(\theta_*) = \theta_*$. \square

Finally, we use the variance of the activation steps to get a bound on the variance of the giant component after one step of the CM dynamics.

Fact 3.26. For $\beta \in (\beta_u, \beta_s)$, there exist constants $M_0, M_1 > 0$ and $s > 0$ such that if a configuration X_t satisfies that $L_t \in (\theta_* n - sn, \theta_* n + sn)$ and $R_2^-(X_t) \leq M_1 n$, then

$$\text{Var}(L_{t+1} | X_t, \Lambda_{t+1}) \leq M_0^2 n$$

Proof. Let G_{t+1} be the random graph in the percolation step of step $t+1$. In other words, $G_{t+1} \sim G(A_{t+1}, \beta/n)$. By the law of total variance, we have

$$\text{Var}(L_{t+1} | X_t, \Lambda_{t+1}) = \mathbb{E}[\text{Var}(|\mathcal{L}_1(G_{t+1})| | A_{t+1}) | X_t, \Lambda_{t+1}) + \text{Var}(\mathbb{E}[|\mathcal{L}_1(G_{t+1})| | A_{t+1}] | X_t, \Lambda_{t+1})]. \quad (52)$$

If $A_{t+1} \cdot \frac{\beta}{n} > 1$ uniformly in n , then by Lemma 2.10, we have $\text{Var}(|\mathcal{L}_1(G_{t+1})| | A_{t+1}) \leq M_2 n$, where M_2 depends only on A_{t+1} . By our assumption and the computation in (34), $\text{Var}(A_{t+1} | X_t, \Lambda_{t+1}) \leq R_2^-(X_t) \leq M_1 n$. Thus, by Chebyshev's inequality, A_{t+1} concentrates around its mean $k_a(L_t/n)$ with sufficiently small $\delta = \delta(M_1, s) > 0$ deviation, with probability $1 - M_3 n^{-1}$, where M_3 depends only on δ and M_1 . Also, we know that for $\beta \in (\beta_u, \beta_s)$, $k_a(\theta_*)\beta > k_a(\theta_s)\beta \geq 1$. Thus, by continuity, for small enough s , $k_a(L_t/n)\beta \geq k_a(\theta_* - s)\beta > 1$, and A_{t+1} is such that the percolation step is strictly supercritical. Hence, we obtain

$$\mathbb{E}[\text{Var}(|\mathcal{L}_1(G_{t+1})| | A_{t+1}) | X_t, \Lambda_{t+1})] \leq (1 - M_3 n^{-1}) \cdot M_2 n + M_3 n, \quad (53)$$

where the M_3n contribution comes from the M_3n^{-1} probability event that $A_{t+1} \cdot \frac{\beta}{n} > 1$ does not hold uniformly in n .

Next we upper bound the second term in right-hand-side of (52). By (8), so long as $A_{t+1}\beta/n$ is bounded away from 1, we have

$$\mathbb{E}[|\mathcal{L}_1(G_{t+1})| \mid A_{t+1}] = \alpha(A_{t+1} \cdot \frac{\beta}{n}) \cdot A_{t+1} + \tilde{O}(1).$$

Using that $0 < \alpha(\cdot) < 1$, and taking the variance of the above, we get

$$\text{Var}(\mathbb{E}[|\mathcal{L}_1(G_{t+1})| \mid A_{t+1}] \mid X_t, \Lambda_{t+1}) \leq \text{Var}(A_{t+1} \mid X_t, \Lambda_{t+1}) + M_3n + \tilde{O}(1)$$

As already claimed, the conditional variance is at most $R_2(X_t) \leq M_1n$. Putting the above bounds together, we conclude. \square

3.5.2 Local limit theorem for the number of activated vertices

We start with the necessary local limit theorem for the activation step of the CM dynamics.

Lemma 3.27. *Let $q \geq 2$. Suppose X is a graph satisfying that*

1. $|\mathcal{L}_1(X)| = \Omega(n)$;
2. $R_2^-(X) = \Theta(n)$;
3. $I_1(X) = \Omega(n)$;
4. $R_3^-(X) = O(n)$.

Let A^- be the number of non-giant activated vertices of X , i.e.,

$$A^- = \sum_{i \geq 2} |\mathcal{L}_i(X)| \cdot B_i, \quad B_i \sim \text{Ber}(1/q) \quad \text{independently.}$$

Then there exists a coupling \mathbf{P} of (A^-, Z) such that $\mathbf{P}(|A^- - Z| > 1) = O(n^{-1/8})$, where $Z \sim \mathcal{N}(\mathbb{E}[A^-], \text{Var}(A^-))$.

We prove Lemma 3.27 by showing that it fits the criteria of the following classical local limit theorem.

Lemma 3.28 ([42]). *Let X_1, \dots, X_n be independent integer-valued random variables with mean μ_1, \dots, μ_n and let $S_n = \sum_{j=1}^n X_j$. Let μ and σ^2 be the mean and variance of S_n . Suppose the following conditions hold:*

1. $\sigma^2 \rightarrow \infty$ as $n \rightarrow \infty$.
2. $\sum_{j=1}^n \mathbb{E}[|X_j - \mu_j|^3] = O(\sigma^2)$.
3. For all j and $r \neq 0$, $\mathbb{P}(X_j = 0) \geq \mathbb{P}(X_j = r)$.
4. $\gcd\{M \in \mathbb{Z} : \frac{1}{\log n} \sum_{j=1}^n \mathbb{P}(X_j = 0) \mathbb{P}(X_j = M) \rightarrow \infty \text{ as } n \rightarrow \infty\} = 1$.

Then there exists a universal constant C_1 such that for $k \in \mathbb{Z}$ we have

$$\left| \mathbb{P}(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{C_1}{\sigma^2}.$$

Proof of Lemma 3.27. Suppose X has $m + 1$ components, where $m = \Theta(n)$ by assumption 3. For $j = 1, \dots, m$, let $V_j = |\mathcal{L}_{j+1}(X)|$ if $\mathcal{L}_{j+1}(X)$ is activated (when $B_{j+1} = 1$), and let $V_j = 0$ otherwise. Note that $A^- = \sum_{j=1}^m V_j$. Let μ and σ^2 be the mean and variance of A^- .

To apply Lemma 3.28, we verify its conditions hold in our setting. First, the fact that the variance of A^- goes to infinity follows from our second supposition that $R_2^-(X) = \Theta(n) \rightarrow \infty$. Item 2 of Lemma 3.28 follows from our fourth supposition that $R_3^-(X) = O(n)$. Moreover, by our definition of V_j , for each $j = 1, \dots, m$, we have for any $r \neq 0$,

$$\mathbb{P}(V_j = 0) = 1 - \frac{1}{q} \geq \frac{1}{2} \geq \frac{1}{q} \geq \mathbb{P}(V_j = r).$$

Finally, we analyze the fourth condition in Lemma 3.28. Note that in our case, if $M \in \mathbb{Z}$,

$$\frac{1}{\log m} \sum_{j=1}^m \mathbb{P}(V_j = 0) \mathbb{P}(V_j = M) = \frac{1}{\log m} \sum_{j=1}^m \left[\left(1 - \frac{1}{q}\right) \frac{1}{q} \cdot \mathbf{1}[L_{j+1}(X) = M] \right] = O\left(\frac{1}{\log m}\right) \sum_{j=1}^m \mathbf{1}[L_{j+1}(X) = M].$$

The third assumption of the current lemma states that for $M = 1$, we have $\sum_{j=1}^m \mathbf{1}[L_{j+1}(X) = M] = \Omega(n)$. Hence, as $m \rightarrow \infty$, for $M = 1$,

$$\frac{1}{\log m} \sum_{j=1}^m \mathbb{P}(V_j = 0) \mathbb{P}(V_j = M) = \Omega\left(\frac{m}{\log m}\right) \rightarrow \infty.$$

Since the gcd of any number with 1 is 1, we get the fourth condition. Therefore, Lemma 3.28 implies that for each $k \in \mathbb{Z}$,

$$\varepsilon_n(k) := \left| \mathbb{P}(A^- = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{C_1}{\sigma^2} = O\left(\frac{1}{n}\right).$$

In particular, for each integer $k \in [\mu - \sigma \cdot n^{1/4}, \mu + \sigma \cdot n^{1/4}]$, we have $\varepsilon_n(k) = O(1/n)$. Also, for each $k \in \mathbb{Z}$, by integrating the normal density, we obtain

$$\mathbb{P}\left(k - \frac{1}{2} \leq Z \leq k + \frac{1}{2}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} + O\left(\frac{1}{\sigma^2}\right). \quad (54)$$

Also, the probability that it doesn't lie in $[\mu - \sigma n^{1/4}, \mu + \sigma n^{1/4}]$ is at most $n^{-1/8}$, say, by Chebyshev's inequality. Using the fact that $\sigma^2 = \Theta(n)$, this implies that there exists a coupling \mathbf{P} of (A^-, Z) such that $\mathbf{P}(|A^- - Z| > 1) = O(n^{-1/8})$ as desired. \square

The next lemma provides us with the necessary local limit theorem in the percolation step.

Lemma 3.29. Suppose that $G \sim G(m, \frac{\beta_c}{n})$ where $m = \Omega(n)$ satisfying $m \cdot \frac{\beta_c}{n} > 1$ is bounded away from 1. Let $m' \in \mathbb{R}$ such that $|m - m'| = O(1)$. Suppose $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, where $\mu_Y := \alpha(\frac{\beta_c m'}{n}) \cdot m'$ and $\sigma_Y^2 := \sigma^2(\frac{\beta_c m'}{n}) \cdot m'$. Then there exists a coupling of $(Y, L(G))$ such that $\mathbb{P}(|Y - |L_1(G)|| \leq 1) = 1 - o(1)$.

Proof. Let $\delta > 0$ be an arbitrary number. By definition, $G \sim G(m, \frac{\beta_c}{n})$. Let $\mu_X = \alpha(\beta_c \cdot \frac{m}{n}) \cdot m$, and $\sigma_X^2 = \sigma^2(\beta_c \cdot \frac{m}{n}) \cdot m$. Suppose $W \sim \mathcal{N}(\mu_X, \sigma_X^2)$. Since $m \cdot \beta_c > n$ uniformly in n , Lemma 2.3 shows that for each $k \in [\mu - L\sigma_X, \mu + L\sigma_X]$ and any $L > 0$,

$$\varepsilon_G(k) := \left| \mathbb{P}(|\mathcal{L}_1(G)| = k) - \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(k-\mu_X)^2}{2\sigma_X^2}} \right| \leq \frac{\delta}{4\sqrt{2\pi}\sigma_X L}.$$

Then by (54) and Chebyshev's inequality, for L and m large, there is a coupling of $(|\mathcal{L}_1(G)|, \mathcal{W})$ such that

$$\mathbb{P}(|\mathcal{W} - |\mathcal{L}_1(G)|| > 1) \leq \frac{2\sigma_X L \delta}{4\sqrt{2\pi}\sigma_X L} + O\left(\frac{2L\sigma_X}{\sigma_X^2}\right) + \frac{\delta}{4} \leq \frac{\delta}{2}.$$

On the other hand, via our assumption on $|m - m'|$ and 41, we obtain that

$$\|Y - \mathcal{W}\|_{\text{TV}} \leq \frac{3|\sigma_X^2 - \sigma_Y^2|}{2\sigma_Y^2} + \frac{|\mu_X - \mu_Y|}{2\sigma_Y} = O(n^{-1/2}).$$

Therefore, for every $\delta > 0$, there exists a coupling of $(\mathcal{L}_1(G), \mathcal{W}, Y)$ such that $|Y - |\mathcal{L}_1(G)|| \leq 1$ with probability at least $1 - \delta$. \square

4 The Potts Glauber dynamics

Throughout this section, let \mathcal{S} be the simplex $\mathcal{S} := \{s \in [0, 1]^q : s_1 + \dots + s_q = 1\}$. For $s \in \mathcal{S}$, we denote by s_i the i -th coordinate of s . Let $\sigma_t \in \Omega$ be the Potts configuration at the t -th step of the Glauber dynamics. Let $S(\sigma_t) = (S_{t,1}, S_{t,2}, \dots, S_{t,q}) \in \mathcal{S}$ be the proportion vector of σ_t such that there are $nS_{t,i}$ spins of color $i \in [q]$ in σ_t , i.e.,

$$S_{t,k} = \frac{1}{n} \sum_{v \in [n]} \mathbf{1}\{\sigma_t(v) = k\}.$$

We denote by $\{S_t\}_t := \{S(\sigma_t)\}_t$ this Markov chain on the state space $\mathcal{S} \cap \frac{1}{n}\mathbb{Z}^q$.

4.1 Preliminaries for the Potts Glauber dynamics

For any $\beta \geq 0$, define $g_\beta : \mathcal{S} \rightarrow \mathcal{S}$ as

$$g_\beta(s) = (g_{\beta,1}(s), \dots, g_{\beta,q}(s)) \quad \text{where} \quad g_{\beta,k}(s) := \frac{e^{\beta \cdot s_k}}{\sum_{j=1}^q e^{\beta \cdot s_j}}.$$

This vector approximates the expected proportion vector after 1 step of Glauber dynamics initialized at s . Namely, the drift satisfies

$$\mathbb{E}[S_{t+1,k} - S_{t,k} \mid \mathcal{F}_t] = \frac{1}{n} g_{\beta,k}(S_t) - \frac{1}{n} S_{t,k} + O(n^{-2}); \quad (55)$$

see Eq. (3.1) of [16]. Without loss of generality, we will be taking the first coordinate as a distinguished one tracking the dominant color class (when there is one). Given this, it is natural to define a drift function for the first coordinate,

$$d_\beta(s) := g_{\beta,1}(s) - s_1. \quad (56)$$

We also define $D_\beta : [0, 1] \rightarrow \mathbb{R}$ as

$$D_\beta(x) := \max_{s: s_1=x} d_\beta(s) = d_\beta\left(x, \frac{1-x}{q-1}, \dots, \frac{1-x}{q-1}\right). \quad (57)$$

Equivalently, we can express (57) as

$$D_\beta(x) = \left(1 + (q-1) \exp\left(\beta \cdot \frac{1-qx}{q-1}\right)\right)^{-1} - x. \quad (58)$$

It is easy to see that D_β is a continuously differentiable function with derivative,

$$D'_\beta(x) := \frac{d}{dx} D_\beta(x) = \frac{q\beta \exp\left(\beta \cdot \frac{1-qx}{q-1}\right)}{\left(1 + (q-1) \cdot \exp\left(\beta \cdot \frac{1-qx}{q-1}\right)\right)^2} - 1. \quad (59)$$

The following lemma characterizes the roots of D_β when $\beta \in (\beta_u, \beta_s)$.

Lemma 4.1. *Suppose $q > 2$ and $\beta \in (\beta_u, \beta_s)$. Then $D_\beta(\frac{1}{q}) = 0$; there are exactly two solutions for $D_\beta(x) = 0$ in $(\frac{1}{q}, 1)$, denoted by m_* and m_r , where $m_* < m_r$. Moreover, $D'_\beta(m_*) > 0$ and $D'_\beta(m_r) < 0$.*

Proof. The definition of β_u is equivalent to $\beta_u = \sup\{\beta \geq 0 : D_\beta(x) < 0, \forall x \in (1/q, 1]\}$. From (58), we know that if $1/q < x \leq 1$, $D_\beta(x)$ for a fixed x is a strictly increasing in β . Hence, by continuity of D_β , for $\beta > \beta_u$, there exists $x^+ \in (1/q, 1]$ such that $D_\beta(x^+) > 0$. Next, by direct computation, for any $\beta \geq 0$ we have

$$D_\beta(\frac{1}{q}) = \frac{1}{1 + (q-1) \cdot \exp[\beta \cdot 0]} - \frac{1}{q} = 0, \quad D_\beta(1) = \frac{1}{1 + (q-1) \cdot \exp(-\beta)} - \frac{1}{q} < 0,$$

and if $\beta < q = \beta_s$ then

$$\frac{d}{dx} D_\beta(x) \Big|_{x=1/q} = \frac{q\beta}{[1 + (q-1)]^2} - 1 = \frac{\beta}{q} - 1 < 0.$$

This means for a sufficiently small $\varepsilon > 0$, $D_\beta(\frac{1}{q} + \varepsilon) < 0$ and $\frac{1}{q} + \varepsilon < x^+$. Since $D_\beta(\frac{1}{q} + \varepsilon) < 0$, $D_\beta(x^+) > 0$ and $D_\beta(1) < 0$, by continuity of D_β , there are at least two roots for $D_\beta(x) = 0$ in $(\frac{1}{q}, 1)$, among which there are two roots $m_* < m_r$ such that $D'_\beta(m_*) > 0$, $D'_\beta(m_r) < 0$.

Lastly we show that m_* and m_r are exactly the two roots for $D_\beta(x) = 0$ in $(0, \infty)$. To see this, note that roots of $D_\beta(x) = 0$ are roots of

$$\frac{1}{1 - \psi(x)} = x, \quad \text{where} \quad \psi(x) = (1-q) \cdot \exp\left(\beta \cdot \frac{1-qx}{q-1}\right).$$

Since $\psi(x)$ is strictly monotone in x , there are at most two such zeros. \square

The last preliminary estimates we require in order to analyze the Potts Glauber dynamics are the following two helpful sub/super-martingale concentration estimates we borrow from [16].

Lemma 4.2 (Lemma 2.1 in [16]). *For $x_0 \in \mathbb{R}$, let $(X_t)_{t \geq 0}$ be a discrete time process initialized from x_0 , with law \mathbf{P}_{x_0} , adapted to $(\mathcal{F}_t)_{t \geq 0}$, and satisfying*

1. $\exists \delta \geq 0 : \mathbf{E}_{x_0}[X_{t+1} - X_t \mid \mathcal{F}_t] \leq -\delta$ on $\{X_t \geq 0\}$ for all $t \geq 0$.
2. $\exists R > 0 : |X_{t+1} - X_t| \leq R$ for all $t \geq 0$.

Let $\tau_x^- = \inf\{t : X_t \leq x\}$ and $\tau_x^+ = \inf\{t : X_t \geq x\}$. The following holds.

1. If $\delta > 0$ then for any $t_1 \geq 0$:

$$\mathbf{P}_{x_0}(\tau_x^- > t_1) \leq \exp\left(-\frac{(\delta t_1 - x_0)^2}{8t_1 R^2}\right).$$

2. If $x_0 \leq 0$ then for any $x_2 > 0$ and $t_2 \geq 0$,

$$\mathbf{P}_{x_0}(\tau_{x_2}^+ \leq t_2) \leq 2 \exp\left(-\frac{(x_2 - R)^2}{8t_2 R^2}\right).$$

3. If $x_0 \leq 0, \delta > 0$ then for any $x_1 > 0$ and $t_1 \geq 0$,

$$\mathbb{P}_{x_0}(\tau_{x_1}^+ \leq t_3) \leq t_3^2 \exp\left(-\frac{(x_1 - R)\delta^2}{8R^3}\right).$$

Lemma 4.3 (Lemma 2.2 in [16]). *Let $\{X_t\}_{t \geq 0}$ be a process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and satisfying the following conditions for some $0 \leq 2\delta < a$:*

1. $X_{t+1} - X_t \in \{-1, 0, 1\}$.
2. $\mathbb{E}[X_{t+1} - X_t \mid \mathcal{F}_t] \geq -\delta$.
3. $\text{Var}(X_{t+1} \mid \mathcal{F}_t) \geq a$.
4. $X_0 \geq 0$.

Let $\tau_r^+ = \inf\{t : X_t \geq r\}$. Then

$$\mathbb{P}(\tau_r^+ \leq t) \geq C_1 \exp\left(-C_2\left(\frac{r}{\sqrt{t}} + \delta\sqrt{t}\right)^2\right) + O(t^{-1/2}),$$

where C_1, C_2 are positive constants which depend only on a .

4.2 Mixing away from the saddle when $\beta \in (\beta_u, \beta_s)$

We first establish that as soon as the proportion vector has reached an $\omega(n^{-1/2})$ distance from the unstable fixed point of $(m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})$ in the first coordinate, with probability going to 1 as $n \rightarrow \infty$, the Glauber dynamics rapidly quasi-equilibrates to the corresponding phase. We need to handle the regimes $\beta \in (\beta_u, \beta_s)$ and $\beta > \beta_s$ separately, with the latter introducing additional complications; this subsection is focused on the former. Define the $q+1$ stable Potts phases when $\beta \in (\beta_u, \beta_s)$ as

$$\begin{aligned} \Omega^{\text{dis}} &= \{\sigma : \|S(\sigma) - (\frac{1}{q}, \dots, \frac{1}{q})\|_1 \leq o(n)\}, \\ \Omega^{\text{ord},i} &= \{\sigma : \|S(\sigma) - (\frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1}, m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 \leq o(n)\}, \quad \text{and} \\ \Omega^{\text{ord}} &= \Omega^{\text{ord},1} \cup \dots \cup \Omega^{\text{ord},q}, \end{aligned}$$

where in $\Omega^{\text{ord},i}$, m_r is the i 'th coordinate. Let $\pi^{\text{dis}} = \pi(\cdot \mid \Omega^{\text{dis}})$, $\pi^{\text{ord},i} = \pi(\cdot \mid \Omega^{\text{ord},i})$ and $\pi^{\text{ord}} = \pi(\cdot \mid \Omega^{\text{ord}})$.

The initialization $\hat{v}^\otimes(m_0)$ in Theorem 1.2 has one distinguished coordinate (randomly chosen on $\{1, \dots, q\}$) which at least at time zero is the dominant color. By permutation symmetry, it is sufficient for us to assume that it is the first coordinate and we do so in what follows.

Theorem 4.4. *Let $q > 2$ and $\beta \in (\beta_u, \beta_s)$. Let $\gamma > 0$ be a large constant. Suppose σ_0 is a configuration such that $S_{0,1} = m$ for $m > 1/q$, $m \leq m_* - \gamma n^{-1/2}$ and $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$. Suppose $\sigma'_0 \sim \pi^{\text{dis}}$. Then there exists $T = O(n \log n)$ such that*

$$\|\mathbb{P}_{\sigma_0}(S(\sigma_T) \in \cdot) - \mathbb{P}_{\sigma'_0}(S(\sigma'_T) \in \cdot)\|_{\text{TV}} = e^{-\Omega(\gamma^2)}.$$

An analogous statement holds with σ_0 and σ'_0 such that $S_{0,1} = m$ for $m \geq m_* + \gamma n^{-1/2}$, $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$ and $\sigma'_0 \sim \pi^{\text{ord},1}$.

The following allows us to treat the $q-1$ non-dominant coordinates as constant $\frac{1-S_{t,1}}{q-1}$ and approximate the analysis by a 1-dimensional process for the first coordinate.

Lemma 4.5. *Let $q > 2$ and $\beta < \beta_s$. For any $\varepsilon > 0$, let $\tau_\varepsilon := \inf\{t : S_{t,1} \leq \frac{1}{q} + \frac{\varepsilon}{2}\}$. Suppose $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \cdot \log n)$ for some $m > \frac{1}{q} + \varepsilon$. Then for any $T = A_1 n \log n$, there exists $A_2 > 0$ such that $\max_{i,j \neq 1} |S_{T \wedge \tau_\varepsilon, i} - S_{T \wedge \tau_\varepsilon, j}| < \frac{2A_2 \log n}{\sqrt{n}}$ with probability $1 - O(n^{-2})$.*

Proof. Let $K > 0$ be a large constant such that $\max_{i,j \neq 1} |S_{0,i} - S_{0,j}| \leq Kn^{-1/2} \log n$. Let $T = A_1 n \log n$, and define the stopping times

$$\tau_{i,j}^+ = \min \left\{ t : S_{t,i} - S_{t,j} - \frac{K \log n}{\sqrt{n}} \geq \frac{A_2 \log n}{\sqrt{n}} \right\}, \quad \text{and} \quad \tau^+ = \min_{i,j \neq 1} \tau_{i,j}^+.$$

Our aim is to show existence of a large constant $A_2 > K$ such that the probability that $\{\tau^+ < T \wedge \tau_\varepsilon\}$ is at most $O(n^{-2})$.

Fix any pair of distinct (i, j) where $i, j \neq 1$. Let $W_t := S_{t \wedge \tau^+ \wedge \tau_\varepsilon, i} - S_{t \wedge \tau^+ \wedge \tau_\varepsilon, j} - Kn^{-1/2} \log n$ for all $t \geq 0$. Clearly $W_0 \leq 0$ and $|W_{t+1} - W_t| \leq \frac{2}{n}$ for all $t \geq 0$. We will show that on $\{W_t \geq 0\}$, if $t < \tau^+ \wedge \tau_\varepsilon$, then there exists a constant $\eta = \eta(\varepsilon, q, \beta) > 0$ such that

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -\frac{\eta K n^{-1/2} \log n}{2n}, \quad (60)$$

and $\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] = 0$ if $t \geq \tau^+ \wedge \tau_\varepsilon$. Given those, item (2) of Lemma 4.2 would imply that

$$\mathbb{P}(\tau_{i,j}^+ \leq T \wedge \tau_\varepsilon) \leq 2 \exp \left(-\frac{(\frac{A_2 \log n}{\sqrt{n}} - 2/n)^2}{8T \cdot (2/n)^2} \right) \leq 2 \exp \left(-\frac{(\frac{A_2 \log n}{2\sqrt{n}})^2}{\frac{2A_1 n \log n}{n^2}} \right) = 2 \exp \left(-\frac{A_2^2 \log n}{8A_1} \right).$$

For sufficiently large A_2 , this is at most n^{-2} . By a union bound over all pairs of (i, j) where $i, j \neq 1$, $\mathbb{P}(\tau^+ \leq T \wedge \tau_\varepsilon) \leq (q-1)^2/n^2$, concluding the proof. To show (60), recalling (55), note that for any $t < \tau^+ \wedge \tau_\varepsilon$,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] = \frac{1}{n} [(-S_{t,i} + g_{\beta,i}(S_t)) - (-S_{t,j} + g_{\beta,j}(S_t))] + O(n^{-2}). \quad (61)$$

In what follows, for a proportions vector S_t , define

$$\hat{S}_t := (S_{t,1}, \frac{1-S_{t,1}}{q-1}, \dots, \frac{1-S_{t,1}}{q-1}).$$

Since $|S_{t,i} - S_{t,j}| \leq \frac{2A_2 \log n}{\sqrt{n}}$ for all $i, j \neq 1$ so long as $t < \tau^+$, $\|S_t - \hat{S}_t\|_2^2 = O(\frac{\log^2 n}{n})$, and so by Taylor expansion of $g_{\beta,i}(S_t)$ and $g_{\beta,j}(S_t)$ about \hat{S}_t ,

$$\begin{aligned} g_{\beta,i}(S_t) - g_{\beta,j}(S_t) &= g_{\beta,i}(\hat{S}_t) - g_{\beta,j}(\hat{S}_t) + \langle S_t - \hat{S}_t, \nabla g_{\beta,i}(\hat{S}_t) - \nabla g_{\beta,j}(\hat{S}_t) \rangle + O\left(\|S_t - \hat{S}_t\|_2^2\right) \\ &= \langle S_t - \hat{S}_t, \nabla g_{\beta,i}(\hat{S}_t) - \nabla g_{\beta,j}(\hat{S}_t) \rangle + O\left(\frac{\log^2 n}{n}\right) \\ &= (S_{t,i} - S_{t,j}) \cdot \left(\frac{d}{dx_i} g_{\beta,i} \Big|_{x=\hat{S}_t} - \frac{d}{dx_i} g_{\beta,j} \Big|_{x=\hat{S}_t} \right) + O\left(\frac{\log^2 n}{n}\right). \end{aligned} \quad (62)$$

We now show there exists $\eta > 0$ for which $\frac{d}{dx_i} g_{\beta,i} \Big|_{x=\hat{S}_t} - \frac{d}{dx_i} g_{\beta,j} \Big|_{x=\hat{S}_t} < 1 - \eta$ when $\beta < \beta_s$ and $t < \tau_\varepsilon$.

Indeed, when $\beta < \beta_s = q$, $i \neq 1$

$$\begin{aligned} \left(\frac{d}{dx_i} g_{\beta,i} \Big|_{x=\hat{S}_t} - \frac{d}{dx_i} g_{\beta,j} \Big|_{x=\hat{S}_t} \right) &= \frac{\beta e^{\beta \cdot \hat{S}_{t,i}}}{\sum_{k=1}^q e^{\beta \cdot \hat{S}_{t,k}}} < \frac{qe^{\beta \cdot \hat{S}_{t,i}}}{\sum_{k=1}^q e^{\beta \cdot \hat{S}_{t,k}}} \\ &= 1 - \frac{e^{\beta \cdot \hat{S}_{t,1}} - e^{\beta \cdot \hat{S}_{t,i}}}{\sum_{k=1}^q e^{\beta \cdot \hat{S}_{t,k}}} \\ &\leq 1 - \frac{e^{\beta/q + \beta\varepsilon/2} - e^{\beta/q - \varepsilon\beta/(2q-2)}}{e^{\beta/q + \beta\varepsilon/2} + (q-1)e^{\beta/q - \varepsilon\beta/(2q-2)}} =: 1 - \eta. \end{aligned}$$

Hence,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] = -\frac{\eta}{n} \cdot (S_{t,i} - S_{t,j}) + O\left(\frac{\log^2 n}{n^2}\right).$$

If $W_t \geq 0$, then the first term is at most $-\frac{\eta K n^{-1/2} \log n}{n}$ and the second term is lower order, yielding (60). \square

We now turn to the convergences to the disordered and q ordered phases when the initialization is $\omega(n^{-1/2})$ away from the saddle point $(m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})$. Their proofs are analogous; we'll present the full proofs for the former, then mention any modifications that need to be made in the other direction.

Lemma 4.6. *Let $q > 2$ and $\beta \in (\beta_u, \beta_s)$. For a large constant $\gamma > 0$, suppose there exists $m \leq m_* - \gamma n^{-1/2}$ such that $S_{0,1} = m$ and $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \cdot \log n)$. Then there exist $T = O(n \log n)$ and $s = \Omega(1)$ such that with probability $1 - e^{-\Omega(\gamma^2)}$, the hitting time of $\{S_{t,1} \leq m_* - s\}$ is less than T .*

Proof. Lemma 4.1 implies that $c_* = D'_\beta(m_*) > 0$. We set $s > 0$ to be a sufficiently small constant to be chosen later.

We define several stopping times that will be useful: let $\tau_0 = 0$, for $i \geq 0$,

$$\tau_{i+1} := \inf \left\{ t \geq \tau_i : S_{t,1} < m_* - \left(1 + \frac{c_*}{16}\right)^{i+1} \frac{\gamma}{\sqrt{n}} \right\},$$

and

$$\tilde{\tau}_{i+1} := \inf \left\{ t > \tau_i : S_{t,1} > m_* - \left(1 + \frac{c_*}{16}\right)^i \frac{\gamma}{2\sqrt{n}} \right\}.$$

Let $k = C_1 \log n$ be the least positive integer such that $\gamma(1 + \frac{c_*}{16})^k > s\sqrt{n}$. Note that $\tau_{k+1} \geq \tau_k \geq \dots \geq \tau_1 \geq \tau_0$. If $\tau_k = \tau_0$, the lemma holds trivially. Thus we assume $\tau_k > \tau_0$ and will show that for all $i = 0, 1, \dots, k-1$,

$$\mathbb{P}(\tau_{i+1} < \tau_i + n, \tilde{\tau}_{i+1} > \min\{\tau_{k+1}, \tau_i + n\} \mid S_{\tau_i}) \geq 1 - 3r_i, \quad (63)$$

where $r_i := \exp(-C_2 \gamma^2 (1 + c_*/16)^{2i})$ for some constant $C_2 > 0$.

By averaging over S_{τ_i} and taking a union bound over $i = 0, 1, \dots, k$,

$$\mathbb{P}\left(\bigcap_{i=0}^k \{\tau_i \leq i \cdot n\}\right) \geq 1 - \sum_{i=0}^{k-1} 3r_i \geq 1 - e^{-\Omega(\gamma^2)}.$$

Since $S_{\tau_k,1} < m_* - s$, the lemma follows.

Now we proceed to prove (63) by showing that the two inequalities hold with probability $1 - 2r_i$ and $1 - r_i$ respectively. First we show that

$$\mathbb{P}(\tilde{\tau}_{i+1} \leq \tau_{k+1} \wedge (\tau_i + n) \mid S_{\tau_i}) \leq 2r_i. \quad (64)$$

Consider the process $\{Z_t\}_{t \geq 0}$ given by

$$Z_t := S_{(t+\tau_i) \wedge \tilde{\tau}_{i+1} \wedge \tau_{k+1}, 1} - S_{\tau_i, 1}.$$

It can be verified that $Z_0 = 0$ and $|Z_{t+1} - Z_t| \leq n^{-1}$ for all $t \geq 0$. We will also show that

$$\mathbb{E}_0[Z_{t+1} - Z_t \mid \mathcal{F}_t] \leq 0, \quad (65)$$

so $\{Z_t\}_{t \geq 0}$ satisfies all conditions of Lemma 4.2. We defer the proof of (65) momentarily and conclude the proof of (64) using Lemma 4.2. By the second part of Lemma 4.2,

$$\begin{aligned} \mathbb{P}(\tilde{\tau}_{i+1} \leq (\tau_i + n) \wedge \tau_{k+1}) &\leq \mathbb{P}(\tilde{\tau}_{i+1} \leq \tau_i + n) \leq 2 \exp\left(-\frac{[(1 + \frac{c_*}{16})^i \cdot \frac{\gamma}{2\sqrt{n}} - \frac{1}{n}]^2}{8n \cdot n^{-2}}\right) \\ &\leq 2 \exp\left(-\frac{\gamma^2}{33} \cdot (1 + \frac{c_*}{16})^{2i}\right) \\ &\leq \exp(-C_2 \gamma^2 (1 + c_*/16)^{2i}) \\ &= 2r_i, \end{aligned}$$

where the last inequality holds for a small $C_2 > 0$.

To see (65), we recall the drift function d_β . If $t + \tau_i \geq \tilde{\tau}_{i+1} \wedge \tau_{k+1}$, then $Z_{t+1} = Z_t$ and (65) clearly holds. It suffices to consider $t \geq 0$ such that $t + \tau_i < \tilde{\tau}_{i+1} \wedge \tau_{k+1}$. In this case, $Z_{t+1} - Z_t = S_{t+1+\tau_i, 1} - S_{t+\tau_i, 1}$, and so by (55)–(57) and Taylor expansion,

$$\mathbb{E}_0[Z_{t+1} - Z_t \mid S_{t+\tau_i}] \leq \frac{1}{n} D_\beta(S_{t+\tau_i, 1}) + O(n^{-2}) \quad (66)$$

$$= \frac{1}{n} \cdot [D_\beta(m_*) + (S_{t+\tau_i, 1} - m_*) \cdot c_* + O(|S_{t+\tau_i, 1} - m_*|^2)] + O(n^{-2}) \quad (67)$$

$$\leq \frac{S_{t+\tau_i, 1} - m_*}{2n} \cdot c^* < 0, \quad (68)$$

where the steps in the last line of (68) hold since $-s(1 + \frac{c_*}{16})^2 < S_{t+\tau_i, 1} - m_* < 0$, and s is sufficiently small in terms of c_* that the second order term is at most $c_*(S_{t+\tau_i, 1} - m_*)/2$, say.

Next we show the other inequality of (63) using an auxiliary process. Consider the process S'_t defined with $S_{\tau_i} = S'_{\tau_i}$ but such that at any step $t \geq \tau_i$, S'_t rejects the update at time $t+1$ if the resulting state would be such that $S'_{t+1, 1} > m_* - (1 + \frac{c_*}{16})^i \cdot \frac{\gamma}{2\sqrt{n}}$. We have $S'_t = S_t$ for all $t \in [\tau_i, \tilde{\tau}_{i+1})$. For S'_t we use τ'_i for its corresponding analog of τ_i . Let $\{W_t\}_{t \geq 0}$ be the process given by

$$W_t := S'_{t+\tau_i, 1} - \left[m_* - \left(1 + \frac{c_*}{16}\right)^{i+1} \cdot \frac{\gamma}{\sqrt{n}}\right].$$

In addition, $W_0 = w_0 > 0$ and $|W_{t+1} - W_t| \leq n^{-1}$ for all $t \geq 0$. Moreover, for all $t \geq 0$ on $\{W_t \geq 0\}$, by a bound analogous to (68), we have

$$\mathbb{E}_{w_0}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq \frac{S'_{t+\tau_i, 1} - m_*}{2n} \cdot c_* \leq -(1 + \frac{c_*}{16})^i \cdot \frac{\gamma}{2\sqrt{n}} \cdot \frac{c_*}{2n} =: -\delta. \quad (69)$$

Item (1) of Lemma 4.2 implies that

$$\mathbb{P}(\tau'_{i+1} > \tau'_i + n \mid S'_{\tau_i}) \leq \exp\left(-\frac{(\delta n - w_0)^2}{8n \cdot n^{-2}}\right) \leq \exp\left(-\frac{C_3[(1 + \frac{c_*}{16})^i \cdot \frac{\gamma}{\sqrt{n}}]^2}{8n^{-1}}\right) \leq r_i, \quad (70)$$

holds for suitable C_2 and C_3 . In addition, for $i < k$, we know $\mathbb{P}(\tau'_{i+1} > (\tau'_i + n) \wedge \tau'_{k+1} \mid S_{\tau_i}) = \mathbb{P}(\tau'_{i+1} > \tau'_i + n \mid S_{\tau_i})$.

Finally, (63) follows from a union bound of (70) and (64). \square

By the same reasoning as in the proof of Lemma 4.6, replacing the drift function by a uniform bound that holds between $m_* - s$ and $\frac{1}{q} + \rho_0$ (by continuity and the fact that $\frac{1}{q}, m_*$ are the only two zeros of D_β), we arrive at the following. (Since the proof is otherwise completely analogous, we omit it.)

Lemma 4.7. *Let $q > 2$ and $\beta \in (\beta_u, \beta_s)$. Suppose $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$ for some $m \leq m_* - s$ where $s = \Omega(1)$. Then with probability $1 - e^{-\Omega(n)}$, the followings hold:*

1. *For all $t = O(n \log n)$, $S_{t,1} \leq m_* - \frac{s}{2}$.*
2. *For any constant $\rho_0 > 0$, there exists $T = O(n \log n)$ such that $S_{T,1} \leq \frac{1}{q} + \rho_0$.*

Finally, it is known from the essential mixing results of [16] that contraction for the distance to equipropportionality holds when a configuration starts close enough to being equiproportional and that mixing to the disordered phase follows from the equipropportionality.

Lemma 4.8 (Lemma 4.1 in [16]). *Suppose $q > 2$ and $\beta < \beta_s$. There exists $\rho_0 = \rho_0(\beta, q) > 0$ such that for all $r > 0$: if $\|S_0\|_\infty \leq \frac{1}{q} + \rho_0$, there exists a constant $\alpha > 0$ such that*

$$\mathbb{P}\left(\left\|S_{\alpha n \log n} - \left(\frac{1}{q}, \dots, \frac{1}{q}\right)\right\|_1 > \frac{r}{\sqrt{n}}\right) = O(r^{-1}),$$

Lemma 4.9 (Corollary 4.4 and Lemma 4.5 in [16]). *Suppose $q > 2$, $r > 0$ and $\beta < \beta_s$. Let $\{\sigma_t\}_{t \geq 0}$ and $\{\sigma'_t\}_{t \geq 0}$ be two instances of Potts Glauber dynamics satisfying*

$$\|S(\sigma_0) - \left(\frac{1}{q}, \dots, \frac{1}{q}\right)\|_1 \leq \frac{r}{\sqrt{n}}, \quad \text{and} \quad \|S(\sigma'_0) - \left(\frac{1}{q}, \dots, \frac{1}{q}\right)\|_1 \leq \frac{r}{\sqrt{n}}.$$

Then there exist a coupling \mathbf{P} of $\{(\sigma_t, \sigma'_t)\}_{t \geq 0}$ and $T = O(n)$ such that $\mathbf{P}(S(\sigma_T) = S(\sigma'_T)) = 1 - o(1)$.

Combining the above results, we are ready to prove the first part of Theorem 4.4 regarding the disordered phase.

Proof of Theorem 4.4: the $m \leq m_* - \gamma n^{-1/2}$ case. Suppose σ_0 is a configuration such that $S_{0,1} = m$ for $m \leq m_* - \gamma n^{-1/2}$ and $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$. Let ρ_0 be as in Lemma 4.8. It follows from combining Lemma 4.5, Lemma 4.6, Lemma 4.7 that with probability $1 - e^{-\Omega(\gamma^2)}$ there exists $T_1 = O(n \log n)$ such that $\|S_{T_1}\|_\infty \leq \frac{1}{q} + \rho_0$. Then Lemma 4.8 implies that there exists $T_2 = T_1 + \alpha n \log n$ such that with probability $1 - e^{-\Omega(\gamma^2)}$, $\|S(\sigma_{T_2}) - \left(\frac{1}{q}, \dots, \frac{1}{q}\right)\|_1 = O(n^{-1/2})$. The same holds for σ'_{T_2} , and therefore we can now conclude the first part of the theorem from that point by Lemma 4.9. \square

In order to handle the case $m \leq m_* - \gamma n^{-1/2}$, we require the following four lemmas.

Lemma 4.10. *Let $q > 2$ and $\beta \in (\beta_u, \beta_s)$. For a large constant $\gamma > 0$, suppose there exists $m \geq m_* + \gamma n^{-1/2}$ such that $S_{0,1} = m$ and $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \cdot \log n)$. Then there exists $T = O(n \log n)$ and $s = \Omega(1)$ such that $S_{T,1} \geq m_* + s$ and $\|S_T - (S_{T,1}, \frac{1-S_{T,1}}{q-1}, \dots, \frac{1-S_{T,1}}{q-1})\|_1 = O(n^{-1/2} \cdot \log n)$ with probability $1 - e^{-\Omega(\gamma^2)}$.*

Lemma 4.11. *Let $q > 2$ and $\beta \in (\beta_u, \beta_s)$. Suppose $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$ for some $m \geq m_* + s$ where $s = \Omega(1)$. Then with probability $1 - n^{-\Omega(1)}$, the followings hold:*

1. *For all $t = O(n \log n)$, $S_{t,1} \geq m_* + \frac{s}{2}$.*
2. *For any constant $\rho_1 > 0$, there exists $T = O(n \log n)$ such that $S_{T,1} \geq m_* - \rho_1$.*

Lemma 4.12. Suppose $q > 2$ and $\beta > \beta_u$. There exists $\rho_1 = \rho_1(\beta, q) > 0$ such that if

$$\|S_0 - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_\infty \leq \rho_1,$$

then for every $r > 0$ there exists $T = O(n \log n)$ such that

$$\mathbb{P}\left(\|S_T - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 > \frac{r}{\sqrt{n}}\right) = O(r^{-1}).$$

Lemma 4.13. Suppose $q > 2$, $r > 0$ and $\beta > \beta_u$. Let $\{\sigma_t\}_{t \geq 0}$ and $\{\sigma'_t\}_{t \geq 0}$ be two instances of Potts Glauber dynamics satisfying

$$\|S(\sigma_0) - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 \leq \frac{r}{\sqrt{n}}, \quad \text{and} \quad \|S(\sigma'_0) - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 \leq \frac{r}{\sqrt{n}}.$$

Then there exist a coupling \mathbf{P} of $\{(\sigma_t, \sigma'_t)\}_{t \geq 0}$ and $T = O(n \log n)$ such that $\mathbf{P}(S(\sigma_T) = S(\sigma'_T)) = 1 - o(1)$.

The proofs of these four lemmas closely follow those of their analogs, Lemmas 4.6–4.9. For Lemma 4.13, we note that the proof for Lemma 4.9 in [16] only relies on the estimates of variance and drift around a stable fixed point, and it is not specific to the disordered phase. The main change to note is that for Lemma 4.10 instead of maximizing the drift for the first coordinate by D_β as done in (66)–(69), the drift is simply approximated by D_β using its Taylor expansion and the fact that the other coordinates are roughly proportional to each other by Lemma 4.5. Due to this change, we include below the details for the proof of Lemma 4.10.

Proof of Lemma 4.10. The proof is analogous to that of Lemma 4.6. We borrow the definition of c_* , s , τ_0 , k and r_i from Lemma 4.6. and explain the main differences here. First, we introduce the notations that are new or different from the previous proof. For $i \geq 0$,

$$\tau_{i+1} := \inf \left\{ t \geq \tau_i : S_{t,1} > m_* + \left(1 + \frac{c_*}{16}\right)^{i+1} \frac{\gamma}{\sqrt{n}} \right\},$$

$$\tilde{\tau}_{i+1} := \inf \left\{ t > \tau_i : S_{t,1} < m_* + \left(1 + \frac{c_*}{16}\right)^i \frac{\gamma}{2\sqrt{n}} \right\},$$

and

$$\tau_{i+1}^+ := \inf \left\{ t \geq \tau_i : \max_{j,l \neq 1} |S_{t,j} - S_{t,l}| > \frac{A \log n}{\sqrt{n}} \right\},$$

where $A > 0$ is a large constant depending on S_{τ_i} . Note that $\tau_{k+1} \geq \tau_k \geq \dots \geq \tau_0$, and $\tilde{\tau}_i < \tau_\varepsilon$, where τ_ε is as in Lemma 4.5 with sufficiently small ε .

Assume $\tau_k > \tau_0$ again. In lieu of (63), we shall prove that for all $i = 0, \dots, k-1$, if $\max_{j,l \neq 1} |S_{\tau_i,j} - S_{\tau_i,l}| \leq \frac{K \log n}{\sqrt{n}}$, then there exists a constant $A > K$ such that

$$\mathbb{P}(\tau_{i+1} \leq \tau_i + n, \tilde{\tau}_{i+1} > \min\{\tau_{k+1}, \tau_i + n\}, \tau_{i+1}^+ > \min\{\tau_{k+1}, \tau_i + n\} \mid S_{\tau_i}) \geq 1 - 3r_i - O(n^{-2}). \quad (71)$$

Then the lemma follows from (71). We show (71) in three steps:

$$\mathbb{P}(\tau_{i+1}^+ \leq (\tau_i + n) \wedge \tilde{\tau}_{i+1} \mid S_{\tau_i}) = O(n^{-2}), \quad (72)$$

$$\mathbb{P}(\tilde{\tau}_{i+1} \leq \tau_{k+1} \wedge (\tau_i + n) \wedge \tau_{i+1}^+ \mid S_{\tau_i}) \leq 2r_i, \quad (73)$$

and

$$\mathbb{P}(\tau'_{i+1} > \tau'_i + n \mid S'_{\tau_i}) \leq r_i, \quad (74)$$

where the stopping time τ'_i is the analog of τ_i for each i with regard to the auxiliary process S'_t that agrees with S_t except that any step $t \geq \tau_i$, S'_t rejects the update at $t+1$ if $S'_{t+1,1} < m_* + ((1 + \frac{c_*}{16})^i \cdot \frac{\gamma}{2\sqrt{n}})$ or $\max_{j,l \neq 1} |S_{t+1,j} - S_{t+1,l}| > \frac{A \log n}{\sqrt{n}}$. Observe that (71) follows from a union bound over (72), (73) and (74). Moreover, (72) is a consequence of Lemma 4.5; (73) and (74) are themselves analogs of (64) and (70) respectively.

To illustrate the main difference in the current proof, it suffices for us to prove (73). Define the process $\{Z_t\}_{t \geq 0}$ given by

$$Z_t = S_{\tau_i,1} - S_{(t+\tau_i) \wedge \tilde{\tau}_{i+1} \wedge \tau_{k+1} \wedge \tau_{i+1}^+, 1}.$$

Clearly we have $Z_0 = 0$, $|Z_{t+1} - Z_t| \leq n^{-1}$ for all $t \geq 0$. Once we show in addition that

$$\mathbb{E}_0[Z_{t+1} - Z_t \mid \mathcal{F}_t] \leq 0, \quad (75)$$

we can conclude (73) by Lemma 4.2(2). Since $Z_{t+1} = Z_t$ when $t + \tau_i \geq \tilde{\tau}_{i+1} \wedge \tau_{k+1} \wedge \tau_{i+1}^+$, it remains to show (75) for this process when $0 \leq t$ such that $t + \tau_i < \tilde{\tau}_{i+1} \wedge \tau_{k+1} \wedge \tau_{i+1}^+$. In this case, $Z_{t+1} - Z_t = -S_{t+1+\tau_i,1} + S_{t+\tau_i,1}$. Recall that for any S_t we set $\hat{S}_t := (S_{t,1}, \frac{1-S_{t,1}}{q-1}, \dots, \frac{1-S_{t,1}}{q-1})$. By (55)–(56) and Taylor expansion we have

$$\begin{aligned} \mathbb{E}_0[Z_{t+1} - Z_t \mid S_{t+\tau_i}] &= -\frac{1}{n} d_\beta(S_{t+\tau_i}) + O(n^{-2}) \\ &= -\frac{1}{n} [d_\beta(\hat{S}_{t+\tau_i}) + \langle S_{t+\tau_i} - \hat{S}_{t+\tau_i}, \nabla d_\beta(\hat{S}_{t+\tau_i}) \rangle] + O(\|S_{t+\tau_i} - \hat{S}_{t+\tau_i}\|_2^2) + O(n^{-2}) \end{aligned} \quad (76)$$

Note that $\langle S_{t+\tau_i} - \hat{S}_{t+\tau_i}, \nabla d_\beta(\hat{S}_{t+\tau_i}) \rangle = 0$ and $\|S_{t+\tau_i} - \hat{S}_{t+\tau_i}\|_1 = O(\frac{\log n}{\sqrt{n}})$. Hence, by (57) and (66)–(68)

$$\mathbb{E}_0[Z_{t+1} - Z_t \mid S_{t+\tau_i}] = -\frac{1}{n} D_\beta(S_{t+\tau_i,1}) + O\left(\frac{\log^2 n}{n^2}\right) \leq -\frac{S_{t+\tau_i,1} - m_*}{2n} \cdot c_* < 0, \quad (77)$$

concluding the proof. \square

Proof of Theorem 4.4: the $m \geq m_* + \gamma n^{-1/2}$ case. On the other side of m_* , if $S_{0,1} = m$ for $m \geq m_* + \gamma n^{-1/2}$ and $\|S_0 - (m, \frac{1-m}{q-1}, \dots, \frac{1-m}{q-1})\|_1 = O(n^{-1/2} \log n)$, then the coalescence of the proportion vectors can be proved analogously to the $m \leq m_* + \gamma n^{-1/2}$ case, with Lemmas 4.10–4.13 in place of Lemmas 4.6–Lemma 4.9. \square

4.3 Getting away from the saddle point when $\beta = \beta_c$

In order to handle the critical point itself, we need to also show that in $\Omega(n)$ number of steps the proportions chain gets $\gamma n^{-1/2}$ away from $(m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})$ with high probability, and furthermore it does so to the right and to the left with the correct relative probabilities.

To upper bound the exit time of the $O(n^{-1/2})$ window around the saddle point $(m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})$, our proof goes by considering a batch of $\gamma^2 n$ updates, after which there is a constant chance that the process gained exited the $\gamma n^{-1/2}$ window using the variance alone (even taking a worst-case bound on the drift functions). Iterating this ensures that in $O(e^{\gamma^4} \cdot \gamma^3 n)$ time the process will likely have escaped.

We use the following notations in this section. For a constant γ , let $\tau_\gamma^- = \inf\{t > 0 : S_{t,1} < m_* - \gamma/\sqrt{n}\}$ and $\tau_\gamma^+ = \inf\{t > 0 : S_{t,1} > m_* + \gamma/\sqrt{n}\}$.

Lemma 4.14. *If $\|S_0 - (m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})\|_1 = O(n^{-1/2})$, then for all large γ , after $T = \gamma^3 e^{O(\gamma^4)} n$ many steps, $S_{T,1} \notin [m_* - \gamma/\sqrt{n}, m_* + \gamma/\sqrt{n}]$ with probability $1 - O(\gamma^{-1})$.*

Proof. Suppose $S_{0,1} = x_0 \in [m_* - \gamma/\sqrt{n}, m_* + \gamma/\sqrt{n}]$. Denote by $\{W_t\}_{t \geq 0}$ the simple random walk on \mathbb{Z} ; define $\{Z_t\}_{t \geq 0}$ to be a process given by

$$Z_t := (m_* n + \gamma \sqrt{n}) - S_{t,1} n \cdot \mathbf{1}\{t < \tau_\gamma^+\} + W_{t-\tau_\gamma^+} \cdot \mathbf{1}\{t \geq \tau_\gamma^+\}.$$

We now verify that $\{Z_t\}_{t \geq 0}$ satisfies the conditions in Lemma 4.3. Indeed, by definition, $Z_0 = (m_* n + \gamma \sqrt{n}) - x_0 n \geq 0$, $Z_{t+1} - Z_t \in \{-1, 0, 1\}$. Moreover, when $t < \tau_\gamma^+$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] = n \mathbb{E}[S_{t,1} - S_{t+1,1} \mid \mathcal{F}_t] = -d_\beta(S_t) + O(n^{-1}) \geq -D_\beta(S_{t,1}) + O(n^{-1}) \geq -C_1 \gamma n^{-1/2},$$

and $\text{Var}(Z_{t+1} \mid \mathcal{F}_t) \geq C_2$; when $t \geq \tau_\gamma^+$, $\mathbb{E}[Z_{t+1} - Z_t \mid \mathcal{F}_t] = \mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] = 0$ and $\text{Var}(Z_{t+1} \mid \mathcal{F}_t) = \text{Var}[W_{t+1} \mid \mathcal{F}_t] = 1$. Define $\tau := \inf\{t : Z_t > 2\gamma\sqrt{n}\}$ and note that $\min\{\tau^+, \tau^-\} \leq \tau$. Lemma 4.3 implies that

$$\begin{aligned} \mathbb{P}(\tau \leq \gamma^2 n) &\geq C_3 \cdot \exp\left(-C_4 \left(\frac{2\gamma\sqrt{n}}{\sqrt{\gamma^2 n}} + C_1 \gamma n^{-1/2} \cdot \sqrt{\gamma^2 n}\right)^2\right) + O(1/\sqrt{\gamma^2 n}) \\ &\geq C_3 \exp\left(-C_4 \left(2 + C_1 \gamma^2\right)^2\right), \end{aligned}$$

where the constants $C_3 > 0$ and $C_4 > 0$ depend only on C_2 .

On the event that $\tau > \gamma^2 n$, then we can restart the process from the value of $S_{\gamma^2 n,1} \in [m_* - \gamma n^{-1/2}, m_* + \gamma n^{-1/2}]$ whence there is a fresh attempt of probability at least $e^{-\Omega(\gamma^4)}$ of exiting the window in the next $\gamma^2 n$ steps. Repeating this argument $K = \gamma^3 e^{\Omega(\gamma^4)}$ many times, each consisting of $\gamma^2 n$ steps, boosts the probability of having exited up to $1 - O(\gamma^{-1})$. \square

The next lemma ensures there exists an initialization perturbation of m_* by order $n^{1/2}$ such that from there, we get the correct relative probabilities for exiting the saddle to the right vs. the left to ensure convergence.

Lemma 4.15. *There exists a unique constant \hat{c}_* such that initialized from $\hat{v}^\otimes(m_0)$ with $m_0 = m_* + \hat{c}_* n^{-1/2} + o(n^{-1/2})$, we have*

$$\mathbb{P}(\tau_\gamma^- < \tau_\gamma^+) = \xi - o_{\gamma,n}(1).$$

Proof. Consider the time and space rescaled process $\bar{S}_t = \sqrt{n}(S_{tn} - (m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1}))$. By standard results regarding limits of discrete stochastic dynamics as stochastic differential equations (see, e.g., [37]), \bar{S}_t converges to the solution of a stochastic differential equation (SDE) Z_t on \mathcal{S} with drift and volatility functions with bounded Lipschitz constants. On a compact space, like \mathcal{S} , such convergence results only require $O(1)$ -Lipschitz bounds on the drift function (which hold in our setting for $g_\beta - s$) and moment estimates on the step-wise increments (for which ours are uniformly bounded by $\pm \frac{1}{\sqrt{n}}$).

In our context, the limiting SDE we end up with for the first coordinate of the rescaled process, $\bar{S}_{t,1}$, is a 1-dimensional SDE Z_t^1 . This is because the effect of the other coordinates $(\bar{S}_{t,2}, \dots, \bar{S}_{t,q})$ cancels to first order when they are in a neighborhood of equipropportionality, as seen in the cancellation of the first order dependence in between (76)–(77). The limiting volatility is constant because the effect of corrections to the vector $(m_*, \frac{1-m_*}{q-1}, \dots, \frac{1-m_*}{q-1})$ on the variance are vanishing. In total, we get that $\bar{S}_{t,1}$ converges to an SDE Z_t^1 solving

$$dZ_t^1 = D_{\beta_c}(Z_t^1) dt + A dB_t^1,$$

where B_t^1 is a standard Brownian motion, $A = A(q)$ is a constant, and this is initialized from $Z_0^1 \sim \mathcal{N}(\hat{c}_*, d^2)$ for some variance $d^2(\beta, q)$.

If for each initial \hat{c}_* we define $f(\hat{c}_*)$ to be the probability that $\bar{S}_{t,1}$ hits $-\gamma$ before γ and we define $f^Z(\hat{c}_*)$ to be the same probability for Z_t^1 then by the convergence described above,

$$|f(\hat{c}_*) - f^Z(\hat{c}_*)| = o_n(1)$$

for all \hat{c}_* . Furthermore, f^Z is easily checked to be n -independent, continuous, monotone, and going to 0 as $\hat{c}_* \rightarrow \gamma$ and 1 as $\hat{c}_* \rightarrow -\gamma$. Therefore, there is a unique \hat{c}_* where f^Z is the desired ξ . Finally, to see that the choice of \hat{c}_* is γ -independent up to $o_\gamma(1)$, note that if $\tau_\gamma^- < \tau_\gamma^+$, then with probability $1 - o_\gamma(1)$, also $\tau_{2\gamma}^- < \tau_{2\gamma}^+$ (by the argument in Lemma 4.6). \square

4.4 Escaping the unstable fixed point at low temperatures $\beta > \beta_s$

In the regime of $\beta > \beta_s$, the saddle becomes at $1/q$ and the landscape's geometry changes somewhat so that this saddle is separating all q ordered phases (the disordered phase no longer being metastable). This introduces some additional complications, particularly because the other coordinates besides the first one do not drift towards equiproportionality as they do in Lemma 4.5. This necessitates more understanding of the full q -dimensional landscape. We establish the following quasi-equilibration result.

Theorem 4.16. *Let $q > 2$ and $\beta > \beta_s$. Suppose σ_0 is a configuration such that $S_{0,1} > S_{0,i} + \gamma n^{-1/2}$ for every $i = 2, \dots, q$ and for a large constant $\gamma > 0$. Then there exists $T = O(n \log n)$ such that*

$$\|\mathbb{P}_{\sigma_0}(S(\sigma_T) \in \cdot) - \mathbb{P}_{\pi^{\text{ord},1}}(S(\sigma'_T) \in \cdot)\|_{\text{TV}} = o_\gamma(1).$$

The proof of Theorem 4.16 breaks up into several parts. We begin with some preliminary lemmas about the drift function's behavior, and classification of its fixed points and their attractive/repulsive directions. Using that, in Lemma 4.19, we show that as soon as one coordinate has a macroscopically larger fraction than the other coordinates, (even in the absense of equiproportionality of the other coordinates), the Glauber dynamics quickly quasiequilibrates to the corresponding ordered phase. Lemma 4.20 shows that if the dominant coordinate has $\omega(n^{-1/2})$ larger proportion than all the others, this gets boosted to a macroscopic bias. Finally, we use anti-concentration of the proportions vector to argue that if the initialization is the fully uniform-at-random initialization, i.e., $m_0 = \frac{1}{q}$, after $O(n)$ steps, with high probability, one coordinate is $\omega(n^{-1/2})$ larger than all the others.

4.4.1 Fixed point analysis of the q -dimensional drift function

We first provide a lemma that characterizes the 1-dimensional drift function D_β in this regime.

Lemma 4.17. *Suppose $q > 2$ and $\beta > \beta_s$. Then $D_\beta(\frac{1}{q}) = 0$ and there is a unique root for $D_\beta(x) = 0$ in $(\frac{1}{q}, 1)$, denoted m_r . Moreover, $D'_\beta(1/q) > 0$ and $D'_\beta(m_r) < 0$.*

Proof. We recall the following facts regarding D_β from the proof of Lemma 4.1,

$$D_\beta(1/q) = 0, \quad D_\beta(1) < 0 \quad \text{and} \quad D'_\beta(1/q) = -1 + \frac{\beta}{q}.$$

Since $\beta > \beta_s = q$, we have $D'_\beta(1/q) > 0$. Thus, for small enough $\varepsilon > 0$, $D_\beta(1/q + \varepsilon) > 0$, and it follows from continuity that there exists a zero of D_β between $1/q + \varepsilon$ and 1. Let m_r be the smallest zero that is greater than $1/q$. Using (58) and (59), one can verify that if $D_\beta(x) = 0$ then $D'_\beta(x) \neq 0$ for $x \in (1/q, 1]$ so $D'_\beta(m_r) < 0$. By continuity of D'_β , there exists a point $x^+ \in (1/q, m_r)$ such that $D'_\beta(x^+) = 0$ and $D_\beta(x^+) > 0$.

Now, by (59), $D'_\beta(x) = 0$ has at most two zeros. Aside from x^+ , let x^{++} denote the other zero (if it exists). If x^{++} does not exist or $x^{++} \leq m_r$, then $D'_\beta(x) < 0$ for all $x > m_r$ so m_r is the unique zero in $(1/q, 1)$. Now

assume $x^{++} > m_r$. By continuity of D'_β , $D'_\beta(x) < 0$ for all $x \in (m_r, x^{++})$, so by integration $D_\beta(x) < 0$ for $x \in (m_r, x^{++}]$. Moreover, note that D_β is a monotone function in $(x^{++}, 1]$ since D'_β does not longer change sign in this interval. Hence, D_β has no zero on $[x^{++}, 1]$ and m_r is the unique root in $(1/q, 1]$. \square

Unlike the $\beta \in (\beta_u, \beta_s)$ regime, though, we need a more refined understanding of the full q -dimensional landscape, establishing that the system driven by d_β has its only stable fixed points at $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$. This will be used to show that as long as the proportions vector is $\Omega(1)$ away from a fixed point of the system, it rapidly reaches a small neighborhood of a fixed point of the form $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$. The following classifies the fixed points of the dynamical system $ds_t = d_\beta(s_t)dt$.

Lemma 4.18. *The set of solutions of $d_\beta(s) = 0$, or equivalently $g_\beta(s) = s$ are classified (up to permutations) as the following. If $g_\beta(s) = s$, then for some $k \in \{1, \dots, q\}$, the vector s must be*

$$\underbrace{(a, \dots, a)}_k, \underbrace{\frac{1-ka}{q-k}, \dots, \frac{1-ka}{q-k}}_{q-k}$$

where $a(\beta, k) \geq 1/q$.

Moreover, when $\beta > \beta_s$, the only stable solution is the one where $k = 1$, and $a = m_r$, and the other fixed points are specifically unstable in the direction of increasing the first coordinate and decreasing another.

Proof. We first reason that all solutions are of the form of perturbations of $(a, \dots, a, \frac{1-ka}{q-k}, \dots, \frac{1-ka}{q-k})$. In order to see this, we suppose by way of contradiction there exists a solution \bar{s} having three distinct values a, b, c appearing in its proportions vector (wlog as $\bar{s}_1 = a, \bar{s}_2 = b$ and $\bar{s}_3 = c$; let $\bar{Z} = \sum_j e^{\beta \bar{s}_j}$). Then, $e^{\beta a} = a\bar{Z}$ and similarly for b and c , and $a \neq b \neq c$. But this is impossible because for any $C, \beta > 0$, the equation $e^{\beta x} = Cx$ has at most two solutions on $[0, 1]$. When $k = 1$, the only possible such solution is the one which has $D_\beta = 0$ (because the other coordinates are equal, which is where $D_\beta = d_\beta$), and we can apply Lemma 4.17 to read off that in that case $a = m_r$.

Now fix any vector \bar{s} of the form $(a, \dots, a, \frac{1-ka}{q-k}, \dots, \frac{1-ka}{q-k})$ for $k \geq 2$ with $g_\beta(\bar{s}) = \bar{s}$. We wish to show that

$$\langle \nabla d_{\beta,1}(\bar{s}), (1, -1, 0, \dots, 0) \rangle > 0$$

as that would say that such a fixed point is unstable with a drift towards the first coordinate increasing if we perturb in the $(1, -1, 0, \dots, 0)$ direction. To see this, differentiating $g_{\beta,1}$ and plugging in $g_{\beta,1}(\bar{s}) = \bar{s}$ and $\bar{s}_1 = \bar{s}_2 = \bar{a}$,

$$\frac{d}{dx_1} d_{\beta,1}(\bar{s}) - \frac{d}{dx_2} d_{\beta,1}(\bar{s}) = \beta(\bar{s}_1 - \bar{s}_1^2) - 1 + \beta\bar{s}_1\bar{s}_2 = \beta\bar{a} - 1.$$

Since $\bar{a} > 1/q$ and $\beta > \beta_s = q$, this is strictly positive as claimed. \square

4.4.2 Getting to the stable fixed point once one coordinate dominates

Using the above fixed point analysis, we can show that initialized macroscopically away from the fixed points of the drift function, the dynamics rapidly equilibrates to the right phase.

Lemma 4.19. *Suppose S_0 is such that $S_{0,1} \geq S_{0,i} + \delta$ for all $i \geq 2$ for some $\delta > 0$. Then, for every $\rho > 0$, there exists a time $T = O(n)$ such that $\|S_T - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 \leq \rho$ with probability $1 - o(1)$.*

Proof. We argue that away from the fixed points of the drift function $d_\beta(s) = g_\beta(s) - s$, the evolution of the proportions vector is well-approximated by the deterministic process defined by

$$S_t = S_0 + \frac{1}{n} \sum_{s \leq t} d_\beta(S_s). \quad (78)$$

By convergence of Euler discretization for ODEs (see e.g., [29]), since d_β has bounded Lipschitz constant, as $n \rightarrow \infty$, \mathbf{S}_{tn} converges in $C^\infty[0, T]$ to the solution to the ODE system $d\mathbf{S}_t = d_\beta(\mathbf{S}_t)dt$ initialized from \mathbf{S}_0 . This latter process is easily checked to be a gradient dynamical system, i.e., the gradient flow for

$$F_\beta(\mathbf{S}) = -\frac{1}{\beta} \log \sum_{i \in \{1, \dots, q\}} e^{\beta S_i} + \frac{1}{2} \|\mathbf{S}\|_2^2;$$

therefore it has no closed orbits, and by the bounded Lipschitz constant of d_β , converges exponentially fast to the fixed point in whose basin it is initialized. Moreover, by Lemma 4.18, if initialized with $\mathbf{S}_{0,1} \geq \mathbf{S}_{0,i} + \delta$ for all $i = 2, \dots, q$ for some $\delta > 0$, then it is in the basin of attraction of the stable fixed point $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$. In particular, for every $\delta > 0$, and $\rho > 0$, there is a $T = O(1)$ such that under the assumptions on \mathbf{S}_0 of the lemma, the system of (78) attains

$$\|\mathbf{S}_{Tn} - (m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})\|_1 \leq \rho. \quad (79)$$

It remains to show that for linear times $t = O(n)$, we have $\|\mathbf{S}_t - \mathbf{S}_0\|_1 = o(1)$ with high probability. For this, we can write

$$\|\mathbf{S}_{tn} - \mathbf{S}_{tn}\|_1 \leq \frac{1}{n} \sum_{l \leq tn} \|d_\beta(S_l) - d_\beta(\mathbf{S}_l)\|_1 + \left\| \sum_{l < tn} ((S_{l+1} - S_l) - d_\beta(S_l)) \right\|_1. \quad (80)$$

Since d_β has bounded Lipschitz coefficient (by some $C_{\beta,q}$), the first sum is at most $\frac{C_{\beta,q}}{n} \sum_{l \leq tn} \|\mathbf{S}_l - S_l\|_1$. By (55), the second sum is a sum of martingale increments (up to an error of $n \cdot O(n^{-2}) = O(n^{-1})$), each of which take values in $\{-\frac{1}{n}, 0, \frac{1}{n}\}$, so by standard martingale concentration estimates (Doob's maximal inequality and Azuma–Hoeffding bound), with probability $1 - o(1)$, the maximum over all $t \leq Tn$ of the second term above is $O(n^{-1/2} \log n)$. Applying the discrete Gronwall inequality, we get

$$\sup_{t \leq T} \|\mathbf{S}_{tn} - \mathbf{S}_t\|_1 \leq O(n^{-1/2} \log n) \cdot e^{C_{\beta,q} T}. \quad (81)$$

Combined with (79), and reparametrizing Tn to $T = O(n)$, we deduce the lemma. \square

4.4.3 Getting one coordinate to dominate

Once one of the color classes has a bias of at least γ/\sqrt{n} , we can call it the dominant color class and without loss of generality, label it the first coordinate. From there, the Potts Glauber dynamics gradually shifts $\Omega(1)$ away from a fixed point.

Lemma 4.20. *Let $q > 2$ and $\beta > \beta_s$. Suppose σ_0 is a configuration such that $S_{0,1} > S_{0,i} + \gamma n^{-1/2}$ for every $k = 2, \dots, q$ and a large $\gamma > 0$. Then there exist $T = O(n \log n)$ and $\delta = \Omega(1)$ such that with probability $1 - e^{-\Omega(\gamma^2)}$,*

$$S_{T,1} \geq S_{T,k} + \delta, \quad \text{for all } k = 2, \dots, q.$$

Proof. We first make several useful definitions. Set δ and ε be sufficiently small positive constants such that $q(1+\varepsilon)^2 \leq \beta$ and $4\delta\beta < \ln(1+\varepsilon)$, and set $L = O(\log n)$ be the least integer such that $\gamma(1+\frac{\varepsilon}{16})^L > 2\delta\sqrt{n}$. For every $k \geq 2$, we define $\tau_{0,k} = 0$,

$$\tau_{i+1,k} := \inf \left\{ t \geq \tau_{i,k} : S_{t,1} > S_{t,k} + \left(1 + \frac{\varepsilon}{16}\right)^{i+1} \frac{\gamma}{\sqrt{n}} \right\},$$

and

$$\tilde{\tau}_{i+1,k} := \inf \left\{ t > \tau_{i,k} : S_{t,1} < S_{t,k} + \left(1 + \frac{\varepsilon}{16}\right)^i \frac{\gamma}{2\sqrt{n}} \right\},$$

for every $i \geq 0$. Moreover, define $\tau_k^- := \inf \left\{ t \geq 0 : \frac{\beta e^{\beta S_{t,k}}}{\sum_{j=1}^q e^{\beta S_{t,j}}} \leq 1 + \varepsilon \right\}$, and $\tau_1^+ := \inf \{t \geq 0 : S_{t,1} \leq \max_{k \neq 1} S_{t,k}\}$. Note that by the choice of δ and ε , $\tau_k^- \geq \tau_{L+1,k} \wedge \tau_1^+$ for each k , and that $\tau_1^+ > \bigwedge_{i,k} \tilde{\tau}_{i,k}$. Define S'_t to be the auxiliary process such that $S_t = S'_t$ for $t < \bigwedge_{i,k} \tilde{\tau}_{i,k}$. Define $\{\tau'_{i,k}\}$ as the stopping times for S'_t in place of $\{\tau_{i,k}\}$. For $i \geq 0$, set $r_i = \exp \left[-\Omega(\gamma^2(1 + \frac{\varepsilon}{16})^{2i}) \right]$. We will show that

$$\mathbb{P} \left(\bigcap_{k=2}^q \left\{ \bigcap_{i=0}^{L-1} \{ \tau'_{i+1,k} \leq \tau'_{i,k} + n, \tilde{\tau}_{i+1,k} > \min\{\tau_{i,k} + n, \tau_1^+, \tau_{L+1,k}\} \} \cap \{ \tilde{\tau}_{L+1,k} > \tau_L + Ln \} \right\} \right) \geq 1 - \sum_{i=0}^{L-1} 3qr_i - o(1). \quad (82)$$

The events in (82) together imply that at time $T = Ln$, $S_{T,1} \geq S_{T,k} + \delta$ for all $k = 2, \dots, q$.

First we show that for each k and each $i \in [0, L-1]$, $\tau'_{i+1,k} \leq \tau'_{i,k} + n$ with probability $1 - r_i$. Define the process $\{W_t\}_{t \geq 0}$ given by

$$W_t = \left(1 + \frac{\varepsilon}{16}\right)^{i+1} \frac{\gamma}{\sqrt{n}} - S'_{t,1} + S'_{t,k}.$$

Clearly, $|W_{t+1} - W_t| \leq 2/n$ for $t \geq 0$. We will show that on $\{W_t \geq 0\}$,

$$\mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] \leq -\frac{\varepsilon}{2n} \cdot \left(1 + \frac{\varepsilon}{16}\right)^i \cdot \frac{\gamma}{2\sqrt{n}}, \quad (83)$$

and obtain $\mathbb{P}(\tau'_{i+1,k} > \tau'_{i,k} + n \mid \tau'_{i,k}) \leq r_i$ following Lemma 4.2(1). To show (83), recalling (61), we have

$$\begin{aligned} \mathbb{E}[W_{t+1} - W_t \mid \mathcal{F}_t] &= -\frac{1}{n} [g_{\beta,1}(S'_t) - g_{\beta,k}(S'_t) - (S'_{t,1} - S'_{t,k})] + O(n^{-2}) \\ &= -\frac{1}{n} \left[\frac{e^{\beta S'_{t,1}} - e^{\beta S'_{t,k}}}{\sum e^{\beta S'_{t,j}}} - (S'_{t,1} - S'_{t,k}) \right] + O(n^{-2}) \\ &\leq -\frac{1}{n} \left(\frac{\beta e^{\beta S'_{t,k}}}{\sum e^{\beta S'_{t,j}}} - 1 \right) (S'_{t,1} - S'_{t,k}) + O(n^{-2}). \end{aligned}$$

Observe that $\frac{\beta e^{\beta S'_{t,k}}}{\sum_{j=1}^q e^{\beta S'_{t,j}}} > 1 + \varepsilon$ for $t \leq \tau'_{L+1,k}$, since otherwise

$$e^{\beta S'_{t,k}} \leq \frac{1+\varepsilon}{\beta} \sum_{j=1}^q e^{\beta S'_{t,j}} \leq \frac{1}{(1+\varepsilon)q} \sum_{j=1}^q e^{\beta S'_{t,j}} \leq \frac{1}{1+\varepsilon} e^{\beta S'_{t,1}},$$

and $S'_{t,k} \leq S'_{t,1} - \beta^{-1} \ln(1 + \varepsilon) < S'_{t,1} - 4\delta$. Hence, we establish (83) by further noting that $S'_{t,1} - S'_{t,k} \geq \left(1 + \frac{\varepsilon}{16}\right)^i \frac{\gamma}{2\sqrt{n}}$.

Moreover, using positive drift of the process $S_{t+\tau_{i,k},1} - S_{t+\tau_{i,k},k}$ and analyzing an associated process as done in the proof for (64), we get that $\mathbb{P}(\tilde{\tau}_{i+1,k} \leq \min\{\tau_{i,k} + n, \tau_1^+, \tau_{L+1,k}\} \mid \tau_{i,k}) \leq 2r_i$. Finally, when the positive drift is constantly large, the probability that $\tilde{\tau}_{L+1,k} \leq \tau_L + Ln$ is diminishing. Therefore, we obtain (82) by a union bound and conclude the proof. \square

Proof of Theorem 4.16. By Lemma 4.20, after $T = O(n \log n)$ steps, with probability $1 - o_Y(1)$, the proportions chain has $S_{T,1} \geq S_{T,k} + \delta$ for all $k = 2, \dots, q$. From there, Lemma 4.19 ensures that in a further $O(n)$ steps, the proportions vector is within an arbitrarily small ρ distance from the fixed point corresponding to that phase, $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$. Finally, Lemma 4.12 and Lemma 4.13 imply coupling of the proportions chain from there to that of a dynamics initialized from $\pi^{\text{ord},1}$. \square

Let us finally describe how one obtains the case of $m_0 = 1/q$, where we are using a fully uniform at random initialization for the low-temperature Potts model.

Lemma 4.21. *Suppose $\beta > \beta_s$ and $\|S_0 - (\frac{1}{q}, \dots, \frac{1}{q})\|_1 = O(n^{-1/2})$. Then for all γ large, there exists a $C_\gamma > 0$ such that the hitting time to having $S_{0,i} > S_{0,j} + \gamma n^{-1/2}$ for some i and all $j \neq i$ is at most $C_\gamma n$ except with probability $1 - o_\gamma(1)$.*

Proof. Applying Lemma 4.20, it is sufficient to show that in some $O(n)$ steps possibly depending on γ , the process attains $S_{0,1} > S_{0,i} + \gamma n^{-1/2}$ (up to permutation of the coordinates). In order to show this, we show that uniformly over any initialization, one obtains some coordinate which is $\gamma n^{-1/2}$ larger than the rest in some n steps, with some $c_\gamma > 0$ probability. This then gets boosted to $1 - o_\gamma(1)$ probability after $C_\gamma n$ steps. There are two cases of initial proportions vector S_0 to consider:

1. Starting from S_0 and evolving S_t according only to the drift d_β as in (78) for n steps (taking the martingale increments to be zero), one of the coordinates becomes $2\gamma n^{-1/2}$ larger than all the others;
2. The complement, in which case there are some $k \geq 2$ coordinates which are within $2\gamma n^{-1/2}$ of the maximal coordinate under the drift of (78) after n steps.

In the first case, by Azuma–Hoeffding, after n steps, the second term in (80) is at most $\gamma\sqrt{n}/C$ with probability $1 - e^{-\Omega(\gamma^2)}$, whence following the reasoning between (80)–(81), we have $\|S_{t,1} - S_{t,1}\| \leq \gamma n^{-1/2}$ for all $t \leq n$ and we will have attained a configuration having $S_{T,1} > S_{T,i} + \gamma n^{-1/2}$ after $T = n$ steps.

Now consider the second case and let S_t be defined as in (78). Firstly, by comparison of

$$S_{t,1} - S_{t,1} = \sum_{l < t} (S_{t,1} - S_t - d_\beta(S_t)),$$

to a random walk with a variance strictly bounded away from zero, taking n steps that are $O(1/n)$ sized, there is a uniformly positive probability $p_{\gamma,1}$ that this is in $[2q\gamma n^{-1/2}, 3q\gamma n^{-1/2}]$. Also, conditionally on a typical realization of such a sequence, $S_{t,1}$ only changed $n/q + o(n)$ many times. On the remaining $(1 - \frac{1}{q})n - o(n)$ steps, the increments of $S_{t,2} - S_{t,2}$ still have a uniformly lower bounded variance. This leads to another uniformly positive probability $p_{\gamma,2}$ that it has $S_{t,2} - S_{t,2} \in [\gamma n^{-1/2}, -\gamma n^{-1/2}]$ (the conditioning on the event for $S_{t,1}$ only has a negligible effect on the drift for $S_{t,2}$ because of Lipschitz continuity of d_β). Repeating this for the next $q - 1$ coordinates, with the very last one having no variance remaining but deterministically having decreased because of what happened for the other coordinates, we find that there is at least probability $p_{\gamma,1} \cdots p_{\gamma,q-1}$ such that

$$S_{t,1} - S_{t,i} \geq -2\gamma n^{-1/2} - (q-1)\gamma n^{-1/2} + 2q\gamma n^{-1/2} = (q-1)\gamma n^{-1/2} \geq \gamma n^{-1/2},$$

for all $i \geq 2$. Importantly, by continuity and compactness of the space, the variances are uniformly bounded from above and below, and so the above lower bound $p_{\gamma,1} \cdots p_{\gamma,q-1}$ is independent of the initial proportions vector. This argument can thus be repeated some C_γ times to ensure a probability $1 - o_\gamma(1)$ that one coordinate has $\gamma n^{-1/2}$ larger proportion than any of the other coordinates. The permutation symmetry of the initialization ensures the q coordinates are equally likely to become this dominant one. \square

4.5 Proof of Theorem 1.2

With all the above ingredients at hand, we are in position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. For item 1 of the theorem, fix $\beta \in (\beta_u, \beta_c)$, and suppose $m_0 = m_* - \omega(n^{-1/2})$, where m_* is specified in Lemma 4.1. If σ_0 is generated according to $\hat{v}^\otimes(m_0)$, with probability $1 - o(1)$, by

concentration of multinomial random variables, S_0 has one coordinate that is within $O(n^{-1/2})$ of m_0 and the other coordinates are all within $O(n^{-1/2})$ of $\frac{1-m_0}{q-1}$. Without loss of generality, permute the coordinates so that it is the first coordinate that is close to m_0 . Then, by Theorem 4.4, $\|\mathbb{P}_{\sigma_0}(S(\sigma_T) \in \cdot) - \mathbb{P}_{\sigma'_0}(S(\sigma'_T) \in \cdot)\|_{\text{TV}} = o(1)$, where $\sigma'_0 \sim \pi^{\text{dis}}$. Furthermore, by Lemma 2.2, when $\beta < \beta_c$, we have $\|\pi^{\text{dis}} - \pi\|_{\text{TV}} = o(1)$, so it follows from the triangle inequality that

$$\begin{aligned} \|\mathbb{P}_{\sigma_0 \sim \hat{\nu}^{\otimes}(m_0)}(S(\sigma_T) \in \cdot) - \pi(S(\sigma) \in \cdot)\|_{\text{TV}} &\leq \|\mathbb{P}_{\sigma_0 \sim \hat{\nu}^{\otimes}(m_0)}(S(\sigma_T) \in \cdot) - \mathbb{P}_{\sigma'_0 \sim \pi^{\text{dis}}}(S(\sigma'_T) \in \cdot)\|_{\text{TV}} + \|\pi^{\text{dis}} - \pi\|_{\text{TV}} \\ &= o(1). \end{aligned}$$

Since $\hat{\nu}^{\otimes}(m_0)$ and π are invariant under permutation of vertices, we obtain that $\|\mathbb{P}_{\hat{\nu}^{\otimes}(m_0)}(\sigma_T \in \cdot) - \pi\|_{\text{TV}} = o(1)$. For item 3, the proof also follows from Theorem 4.4 and Lemma 2.2 by a symmetrical argument, upon noticing that $\|\pi - \pi^{\text{ord}}\|_{\text{TV}} = o(1)$, and π^{ord} is a $(1/q, \dots, 1/q)$ mixture of $(\pi^{\text{ord},i})_{i \in [q]}$, and each of the q coordinates are equally likely under the initialization $\hat{\nu}^{\otimes}(m_0)$ to dominate.

We proceed to item 2 where $\beta = \beta_c$. For any target $\varepsilon > 0$, we take $\gamma = \gamma(\varepsilon)$ to be sufficiently large. By Lemma 2.2, the stationary distribution π is within $o(1)$ total-variation distance of a $(\xi, 1 - \xi)$ mixture of π^{dis} and π^{ord} for ξ defined in (6). By Lemma 4.15, there is a constant $\hat{c}_*(\xi)$ such that if $m_0 = m_* + \hat{c}_* n^{-1/2} + o(n^{-1/2})$, then $\|\mathbb{P}_{\hat{\nu}^{\otimes}(m_0)}(\tau_Y^- < \tau_Y^+) - \xi\|_{\text{TV}} \leq \varepsilon$. Moreover, by Lemma 4.14, the minimum of τ_Y^- and τ_Y^+ is $O(n)$ with probability $1 - O(\gamma^{-1})$; by Lemma 4.5, if $T_1 = \tau_Y^- \wedge \tau_Y^+$ and $m_1 = S_{T_1,1}$ then $\|S_{T_1} - (m_1, \frac{1-m_1}{q-1}, \dots, \frac{1-m_1}{q-1})\|_1 = O(n^{-1/2} \log n)$ with probability $1 - O(n^{-2})$. Finally, applying Theorem 4.4 and Lemma 2.2 to the configuration at time T_1 as in the off-critical case, together with the spin symmetry for convergence to ordered phases, implies that there exists $T = T_1 + O(n \log n)$ such that

$$\|\mathbb{P}_{\hat{\nu}^{\otimes}(m_0)}(\sigma_T \in \cdot) - ((1 - \xi)\pi^{\text{ord}} + \xi\pi^{\text{dis}})\|_{\text{TV}} \leq \exp(-\Omega(\gamma^2)) + \varepsilon + O(\gamma^{-1}) + O(n^{-2}),$$

which will be less than 2ε for γ large.

It remains to discuss Item 4. If $m_0 \geq \frac{1}{q} + \omega(n^{-1/2})$, then together with spin symmetry and the vertex-permutation invariant of the initialization, Theorem 4.16 implies that $\|\mathbb{P}_{\hat{\nu}^{\otimes}(m_0)}(\sigma_T \in \cdot) - \pi^{\text{ord}}\|_{\text{TV}} = o(1)$ for $T = O(n \log n)$. If $\frac{1}{q} \leq m_0 \leq \frac{1}{q} + \omega(n^{-1/2})$ then we first apply Lemma 4.21 to get a coordinate (by symmetry a uniform at random one) to obtain $\gamma n^{-1/2}$ separation from the rest, then apply Theorem 4.16 to get the same. On the other hand, by Lemma 2.2 we have $\|\pi^{\text{ord}} - \pi\|_{\text{TV}} = o(1)$, and thus item 4 follows from the triangle inequality. \square

4.6 Lower bound on the mixing time with different choices of m_0

By combining the above quasi-equilibration results with metastability of the ordered and disordered phases, we show that if the initialization is the product measure with parameters not satisfying the conditions of Theorem 1.2, then mixing is exponentially slow.

Theorem 4.22. *For every $q > 2$ and $\beta \in (\beta_u, \beta_s)$, if $m_*(\beta, q)$ and $\hat{c}_*(q)$ are as in Theorem 1.2, then the CM dynamics initialized from $\hat{\nu}(m_0)$ with*

1. $\beta \in (\beta_u, \beta_c)$ and $m_0 > m_*(\beta, q) - O(n^{-1/2})$,
2. $\beta = \beta_c$ and $m_0 \neq m_*(\beta, q) + \hat{c}_*(q)n^{-1/2} + o(n^{-1/2})$,
3. $\beta \in (\beta_c, \beta_s)$ and $m_0 < m_*(\beta, q) + O(n^{-1/2})$,

takes $\exp(\Omega(n))$ time to reach $o(1)$ total-variation distance to stationarity.

Proof. We provide the details for the proof of item 1, the other cases following by similar reasoning. For any initialization parameter $m_0 > m_* - Kn^{-1/2}$ for some $K = O(1)$, by Lemma 4.15 and 4.14, there is a positive probability $c_K > 0$ that the process $S_{t,1}$ hits $m_* + \gamma n^{-1/2}$ before $m_* - \gamma n^{-1/2}$ (for sufficiently large γ) in some $t \leq C_\gamma n$ many steps. By Lemma 4.5, with probability $1 - o(1)$, at exit, the configuration satisfies the necessary conditions to apply Theorem 4.4 and quasi-equilibrate to the ordered phase π^{ord} . Putting these together, we find that for some $T_0 = O(n \log n)$,

$$\|\mathbb{P}(X_{T_0} \in \cdot) - \pi^{\text{ord}}\|_{\text{tv}} \leq 1 - c_K + o_{\gamma,n}(1). \quad (84)$$

Next, we claim that a Potts dynamics chain initialized from π^{ord} retains total-variation distance $1 - o(1)$ for exponentially many steps to π when $\beta \in (\beta_u, \beta_c)$. To see this, we use that by Corollary 2.2 the initialization from π^{ord} has proportions vector within $O(n^{-1/2})$ distance of $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$ (up to permutations). This is a stable fixed point of the drift function d_β when $\beta > \beta_u$ by Lemma 4.1. Following the Taylor expansion and martingale argument used in Lemma 4.5, in particular the application of Lemma 4.2, it takes exponential in n time to leave an ε -neighborhood of the stable fixed point $(m_r, \frac{1-m_r}{q-1}, \dots, \frac{1-m_r}{q-1})$. (Note that the details of this last stage of reasoning are provided in the proof of [16, Theorem 3].) Combining the above, we find that for some $C > 0$, for all $Cn \log n \leq t \leq e^{n/C}$,

$$\mathbb{P}(|\max_i S_{t,i} - m_r| \leq \varepsilon) \geq c_K - o_{\gamma,n}(1).$$

By Lemma 2.1, when $\beta \in (\beta_u, \beta_c)$, since $m_r \neq 1/q$, one has $\mu(|\max_i S_{t,i} - m_r| < \varepsilon) = o(1)$ for small ε , so the above bound implies that for γ large, the total-variation to stationarity is at least $c_K/2$. \square

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