DE GRUYTER Adv. Nonlinear Stud. 2024; 24(1): 103–140

Research Article
Special Issue: In honor of Joel Spruck

Georgios Daskalopoulos and Chikako Mese*

Infinite energy harmonic maps from quasi-compact Kähler surfaces

https://doi.org/10.1515/ans-2023-0122
Received March 26, 2023; accepted October 8, 2023; published online March 12, 2024

Abstract: We construct infinite energy harmonic maps from a quasi-compact Kähler surface with a Poincaré-type metric into an NPC space. This is the first step in the construction of pluriharmonic maps from quasiprojective varieties into symmetric spaces of non-compact type, Euclidean and hyperbolic buildings and Teichmüller space.

Keywords: harmonic maps; quasi-compact Kähler surfaces; infinite energy; non-positively curved

1 Introduction

In this paper, we prove the existence of harmonic maps of possibly infinite energy from quasi-compact Kähler surfaces with a Poincare-type metric to NPC spaces. Infinite energy harmonic maps between manifolds previously appeared in the work of Lohkamp and Wolf. Lohkamp [1] proved the existence of a harmonic map in a given homotopy class of maps between two non-compact manifolds, provided that a certain simplicity condition near infinity of the domain is satisfied. Wolf [2] studied harmonic maps of infinite energy when the domain is a nodal Riemann surface and applied it to describe degenerations of surfaces in the Riemann moduli space (see also [3]). A few years later, Jost and Zuo (cf. [4], [5]) sketched a proof of the existence of infinite energy maps from non-compact Kähler manifolds. The purpose of this paper is to provide a complete proof of the existence of harmonic maps from quasi-compact Kähler surfaces to a certain class of NPC targets. This is the first step in the construction of pluriharmonic maps from quasiprojective varieties into symmetric spaces of non-compact type, Euclidean and hyperbolic buildings and Teichmüller space which will be dealt in our upcoming paper, (cf. [6]).

Theorem 1. *Let M, $\tilde{X}$ and a $\rho\colon \pi_1(M) \to Isom(\tilde{X})$ be as follows:*
- *$M = \bar{M}\backslash\Sigma$ where Σ is a normal crossing divisor is a quasi-compact Kähler manifold of dimension 2 with universal cover $\tilde{M}$*
- *$\tilde{X}$ is an NPC space*
- *ρ is proper (cf. Definition* 2.7*).*
- *$\rho(\pi_1(M))$ satisfies Property $(\star)$ defined in Section* 2.4*.*

Then there exists a Poincaré-type Kähler metric g (cf. Section 3.4*) and ρ-equivariant harmonic map $\tilde{u}\colon \tilde{M} \to \tilde{X}$.*

GD supported in part by NSF DMS-2105226 and a Simons collaboration grant, CM supported in part by NSF DMS-2304697.

***Corresponding author: Chikako Mese**, Department of Mathematics, Johns Hopkins University, Baltimore, MD, USA, E-mail: cmese@math.jhu.edu
Georgios Daskalopoulos, Department of Mathematics, Brown University, Providence, RI, USA, E-mail: daskal@math.brown.edu

Many interesting examples satisfy Property ($\star$). These include homomorphisms into semisimple algebraic groups defined over $\mathbb{R}$, $\mathbb{C}$ or p-adic fields. See Remark 2.9.

Remark 1.1. We will also prove logarithmic energy estimates near infinity (cf. Theorem 6.6 and Theorem 6.7). This means that, for any transverse holomorphic disk to the divisor, the energy density behaves $\sim \frac{1}{|z|^2}$.

The main idea of the proof of Theorem 1 is to construct a *prototype map* which almost minimizes energy near infinity. This map is used to construct a Dirichlet solution defined on a compact subset of the domain. Because of the energy control, the sequence of harmonic maps corresponding to a compact exhaustion converges to an infinite energy harmonic map defined on the whole surface. This idea goes back to Lohkamp [1]. In our situation, the normal bundle of the divisor Σ may be non-trivial and the divisor may consist of more than one irreducible component. In other words, a quasi-compact Kähler surface $M = \bar{M}\backslash\Sigma$ does not necessarily satisfy the simplicity condition of Lohkamp.

In [4], Jost and Zuo sketched a construction of harmonic maps from quasi-projective manifolds. The point of this paper is to provide the details of this argument for a quasi-compact Kähler surface. We felt that a careful presentation of this argument is necessary because all the constructions in our future papers (e.g. [6]) depend on this result.

In a remarkable paper, Mochizuki [7] proved the existence of pluriharmonic metrics on flat vector bundles over quasi-projective manifolds of any dimension. These metrics correspond to pluriharmonic maps into the symmetric space $\mathrm{GL}(r,\mathbb{C})/U(r)$ by the Donaldson-Corlette theorem (cf. [8], [9]). In the forthcoming papers, we will generalize Mochizuki's result when the target is a symmetric space of non-compact type, a Euclidean or a hyperbolic building and Teichmüller space. Indeed, we will first prove that the harmonic map of Theorem 1 is actually pluriharmonic in these special cases. This is derived by adopting Mochizuki's version of the Siu-Sampson Bochner formula (cf. [10], [11]). We then prove the existence of a pluriharmonic map from a quasi-projective manifold of any dimension by an induction argument.

One of our main applications of the existence of pluriharmonic maps is the construction of logarithmic symmetric differential forms over quasi-project manifolds. Using this, we prove a logarithmic version of a conjecture by Esnault in the linear case [6].

We provide an outline of the paper below.

In Section 3, we discuss neighborhoods of the divisor and a Poincaré-type metric g (cf. Definition 3.4) due to Cornalba and Griffiths [12]. This is a complete metric which puts the divisor at infinity.

In Section 4, we construct a *prototype section* $v\colon M \to \tilde{M}\times_\rho\tilde{X}$ with controlled growth near infinity. The crucial tool is the Dirichlet solution on the punctured disk (cf. [10], [11] or Theorem 2.16). This enables us to construct a fiber-wise harmonic map on the normal bundle of the divisor. This map defined near the divisor is then extended to all of M.

In Section 5, we give precise estimates for energy growth of the prototype section near the divisor at infinity. These are important because they imply the estimates for the harmonic section.

In Section 6, we use the prototype section $v\colon M \to \tilde{M}\times_\rho\tilde{X}$ in order to construct a harmonic section $u\colon M \to \tilde{M}\times_\rho\tilde{X}$. We end with some energy estimates of the harmonic section.

In Section 7, we sketch a proof in the case of higher dimensional quasi-compact Kähler manifolds. Note that in our upcoming papers, *we will only use the two-dimensional case* and this is why we gave the details only for Kähler surface domains.

2 Preliminaries

2.1 NPC spaces

We refer to [13] for more details.

Definition 2.1. A curve $c: [a, b] \to \tilde{X}$ into a metric space is called a geodesic if length$(c([\alpha, \beta])) = d(c(\alpha), c(\beta))$ for any subinterval $[\alpha, \beta] \subset [a, b]$. (Note that a identically constant map from an interval is a geodesic.) A metric space $\tilde{X}$ is a geodesic space if there exists a geodesic connecting every pair of points in $\tilde{X}$.

Definition 2.2. An NPC space $\tilde{X}$ is a complete geodesic space that satisfies the following condition: For any three points $P, R, Q \in \tilde{X}$ and an arclength parameterized geodesic $c: [0, l] \to \tilde{X}$ with $c(0) = Q$ and $c(l) = R$,

$$d^2(P, Q_t) \le (1-t)d^2(P, Q) + td^2(P, R) - t(1-t)d^2(Q, R)$$

where $Q_t = c(tl)$.

Notation 2.3. It follows immediately from Definition 2.2 that, given $P, Q \in \tilde{X}$ and $t \in [0, 1]$, there exists a unique point with distance from P equal to $td(P, Q)$ and the distance from Q equal to $(1-t)d(P, Q)$. We denote this point *by*

$$(1-t)P + tQ.$$

Definition 2.4. Let $\tilde{X}$ be an NPC space. We say that two geodesics rays $c, c': [0, \infty) \to \tilde{X}$ are equivalent if there exists a constant K such that $d(c(t), c'(t)) \le K$ for all $t \in [0, \infty)$. Denote the equivalence class of a geodesic ray c by $[c]$. The set $\partial\tilde{X}$ of boundary points of $\tilde{X}$ is the set of equivalence classes of non-constant geodesic rays. Note that an isometric action on $\tilde{X}$ induces an action on $\partial\tilde{X}$.

2.2 Maps into NPC spaces

In this paper, we consider harmonic maps into NPC spaces. Important examples are when the target space $\tilde{X}$ is a smooth Riemannian manifold of non-positive sectional curvature. In this case, the energy of a smooth map $f: \Omega \to \tilde{X}$ is

$$E^f = \int_\Omega |df|^2 d\mathrm{vol}_g$$

where (Ω, g) is a Riemannian domain and $d\mathrm{vol}_g$ is the volume form of Ω.

In the case when the target is an arbitrary NPC space, we use the following definition of energy due to Korevaar-Schoen. We refer to [14] for more details.

Let (Ω, g) be a bounded Lipschitz Riemannian domain. Let Ω_ϵ be the set of points in Ω at a distance least ϵ from $\partial\Omega$. Let $B_\epsilon(x)$ be a geodesic ball centered at x and $S_\epsilon(x) = \partial B_\epsilon(x)$. We say $f: \Omega \to X$ is an L^2-map (or that $f \in L^2(\Omega, X)$) if

$$\int_\Omega d^2(f, P) d\mathrm{vol}_g < \infty.$$

For $f \in L^2(\Omega, X)$, define

$$e_\epsilon: \Omega \to \mathbf{R}, \quad e_\epsilon(x) = \begin{cases} \displaystyle\int_{y \in S_\epsilon(x)} \frac{d^2(f(x), f(y))}{\epsilon^2} \frac{d\sigma_{x,\epsilon}}{\epsilon} & x \in \Omega_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_{x,\epsilon}$ is the induced measure on $S_\epsilon(x)$. We define a family of functionals

$$E_\epsilon^f: C_c(X) \to \mathbf{R}, \quad E_\epsilon^f(\varphi) = \int_\Omega \varphi e_\epsilon d\mathrm{vol}_g.$$

We say f has finite energy (or that $f \in W^{1,2}(\Omega, X)$) if

$$E^f := \sup_{\varphi \in C_c(\Omega), 0 \le \varphi \le 1} \limsup_{\epsilon \to 0} E_\epsilon^f(\varphi) < \infty.$$

It is shown in [14] that if f has finite energy, the measures $e_\epsilon(x)\mathrm{dvol}_g$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e(x)$, which we call the energy density, so that $e_\epsilon(x)\mathrm{dvol}_g \rightharpoonup e(x)\mathrm{dvol}_g$. In analogy to the case of smooth targets, we write $|\nabla f|^2(x)$ in place of $e(x)$. In particular, the (Korevaar-Schoen) energy of f in Ω is

$$E^f[\Omega] = \int_\Omega |\nabla f|^2 \mathrm{dvol}_g.$$

Definition 2.5. We say a continuous map $u\colon \Omega \to \tilde{X}$ from a Lipschitz domain Ω is harmonic if it is locally energy minimizing; more precisely, at each $p \in \Omega$, there exists a neighborhood Ω of p so that all continuous comparison maps which agree with u outside of this neighborhood have no less energy.

For $V \in \Gamma\Omega$ where $\Gamma\Omega$ is the set of Lipschitz vector fields on Ω, $|f_*(V)|^2$ is similarly defined. The real valued L^1 function $|f_*(V)|^2$ generalizes the norm squared on the directional derivative of f. The generalization of the pull-back metric is the continuous, symmetric, bilinear, non-negative and tensorial operator

$$\pi_f(V, W) = \Gamma\Omega \times \Gamma\Omega \to L^1(\Omega, \mathbf{R})$$

where

$$\pi_f(V, W) = \frac{1}{2}|f_*(V+W)|^2 - \frac{1}{2}|f_*(V-W)|^2.$$

We refer to [14] for more details.

Let $(x^1, \dots, x^n)$ be local coordinates of (Ω, g) and $g = (g_{ij})$, $g^{-1} = (g^{ij})$ be the local metric expressions. Then energy density function of f can be written (cf. [14, (2.3vi)])

$$|\nabla f|^2 = g^{ij}\pi_f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

Next assume (Ω, g) is a 2-dimensional Hermitian domain and let $(z^1 = x^1 + ix^2, z^2 = x^3 + ix^4)$ be local complex coordinates. We extend π_f linearly cover $\mathbb{C}$ and denote

$$\frac{\partial f}{\partial z^i} \cdot \frac{\partial f}{\partial \bar{z}^j} = \pi_f\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$$

and

$$\left|\frac{\partial f}{\partial z^i}\right|^2 = \pi_f\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\right).$$

Thus,

$$\frac{1}{4}|\nabla f|^2 = g^{i\bar{j}}\frac{\partial f}{\partial z^i} \cdot \frac{\partial f}{\partial \bar{z}^j}.$$

2.3 Isometries of an NPC space

Throughout this paper, we denote the group of isometries of an NPC space $\tilde{X}$ by $\mathrm{Isom}(\tilde{X})$. Isometries of an NPC space are classified as follows.

Definition 2.6. For $I \in \mathrm{Isom}(\tilde{X})$, let

$$\Delta_I := \inf_{P \in \tilde{X}} d(I(P), P)$$

denote its translation length and define

$$\mathrm{Min}(I) := \{P \in \tilde{X} : d(I(P), P) = \Delta_I\}.$$

The isometry I is *elliptic* if $\Delta_I = 0$ and $\mathrm{Min}(I) \neq \emptyset$. It is *hyperbolic* if $\Delta_I > 0$ and $\mathrm{Min}(I) \neq \emptyset$. If I is elliptic or hyperbolic, then we say I is *semisimple*. Otherwise, I is said to be *parabolic*.

Definition 2.7. Let Γ be a finitely generated group, Λ be a finite set of generators of Γ, $\tilde{X}$ be an NPC space and $\rho\colon \Gamma \to \mathrm{Isom}(\tilde{X})$ be a homomorphism. Define $\delta\colon \tilde{X} \to [0,\infty)$ to be the function

$$\delta(P) = \max\{d(\rho(\lambda)P, P) : \lambda \in \Lambda\}.$$

We say ρ is proper if the sublevel sets of the function δ are bounded in $\tilde{X}$; i.e. given $c > 0$, there exists $P_0 \in X$ and $R_0 > 0$ such that

$$\{P \in \tilde{X} : \delta(P) \leq c\} \subset B_{R_0}(P_0).$$

Remark 2.8. If $\tilde{X}$ is locally compact and ρ does not fix a point at infinity, then ρ is proper by [15, Theorem 2.2.1].

2.4 Property ($\star$)

Given a homomorphism $\rho\colon \pi_1(M) \to \mathrm{Isom}(\tilde{X})$, we say $\rho(\pi_1(M))$ satisfies Property ($\star$) if following holds:
- Every $I \in \rho(\pi_1(M))$ has *exponential decay to its translation length.* In other words, either
  - (i) I is semisimple, or
  - (ii) I is parabolic, fixes $\xi \in \partial\tilde{X}$ and there exists a geodesic ray $c\colon [0,\infty) \to \tilde{X}$ and $a, b > 0$ such that

$$d^2(I(c(t)), c(t)) \leq \Delta_I^2 + be^{-at}.$$

- For any commuting pair of isometries $I_1, I_2 \in \rho(\pi_1(M))$, either
  - (i) I_1, I_2 do not fix a common point of $\partial\tilde{X}$, or
  - (ii) I_1, I_2 fix a common point $\xi \in \partial\tilde{X}$ and there exist an arclength parameterized geodesic ray $c\colon [0,\infty) \to \tilde{X}$ in the equivalence class ξ and $a, b > 0$ such that

$$d^2(I_i(c(t)), c(t)) \leq \Delta_{I_i}^2 + be^{-at}, \quad i = 1, 2.$$

Remark 2.9. Let G be a semisimple algebraic group defined over $\mathbb{R}$ or $\mathbb{C}$ acting on a symmetric space G/K of non-compact type or let G be a semisimple algebraic group defined over some non-archimedean local field K acting on a Bruhat-Tits building without a Euclidean factor. If $\rho\colon \pi_1(M) \to G$ is a homomorphism, then $\rho(\pi_1(M))$ satisfies Property ($\star$).

Lemma 2.10. *Let C be a closed convex set in $\tilde{X}$ and $\pi\colon \tilde{X} \to C$ a closest point projection map; i.e. $\pi(x)$ is the unique point of C such that $d(x, \pi(x)) = \min_{y\in C} d(x, y)$. If $I \in \mathrm{Isom}(\tilde{X})$ is such that $I(C) = C$, then $I \circ \pi(x) = \pi \circ I(x)$.*

Proof. Since $I \circ \pi(x), I^{-1} \circ \pi \circ I(x) \in C$, the definition of π implies

$$\begin{aligned} d(I(x), \pi \circ I(x)) &\leq d(I(x), I \circ \pi(x)) = d(x, \pi(x)) \\ &\leq d(x, I^{-1} \circ \pi \circ I(x)) = d(I(x), \pi \circ I(x)). \end{aligned}$$

Thus, $d(I(x), \pi \circ I(x)) = d(I(x), I \circ \pi(x))$ which implies $\pi \circ I(x) = I \circ \pi(x)$. ☐

Lemma 2.11. *Let γ_1 and γ_2 be generators of the abelian group $2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$ acting on $\mathbb{R} \times \mathbb{R}$ by translations $\gamma_1 \cdot (x, y) = (x + 2\pi, y)$ and $\gamma_2 \cdot (x, y) = (x, y + 2\pi)$ respectively. For a commuting pair of isometries I_1 and I_2, let $\langle I_1, I_2\rangle \subset \mathrm{Isom}(\tilde{X})$ be the subgroup generated by I_1, I_2 and $\rho\colon 2\pi\mathbb{Z} \times 2\pi\mathbb{Z} \simeq \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) \to \langle I_1, I_2\rangle$ be the homomorphism defined by $\gamma_1 \mapsto I_1, \gamma_2 \mapsto I_2$. If I_1, I_2 satisfy either (i) or (ii) of the second bullet point of Property ($\star$), then there exist constants $a, b > 0$ and a ρ-equivariant map*

$$\tilde{h}\colon [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \tilde{X}$$

such that

$$\left|\frac{\partial \tilde{h}}{\partial t}\right|^2 \leq 1, \quad \left|\frac{\partial \tilde{h}}{\partial x}\right|^2 \leq \frac{\Delta_{I_1}^2}{4\pi^2} + be^{-at}, \quad \left|\frac{\partial \tilde{h}}{\partial y}\right| \leq \frac{\Delta_{I_2}^2}{4\pi^2} + be^{-at}.$$

Proof. Assume that (i) of the second bullet point holds. Then by [15, Theorem 2.2.1 and Corollary 1.5.3), there exists a totally geodesic ρ-equivariant map $\tilde{f}: \mathbb{R} \times \mathbb{R} \to \tilde{X}$. In particular, $c(t) := \tilde{f}(t, y)$ maps to a point or is a constant speed reparameterization of a geodesic. Since $I_1 \circ c(t) = I_1 \circ \tilde{f}(t, y) = \tilde{f}(t + 2\pi, y) = c(t + 2\pi)$, the isometry I_1 fixes $c(t)$. If $c(t)$ maps to a point, then I_1 is an elliptic isometry fixing that point. If $c(t)$ is a geodesic line, then I_1 is a hyperbolic isometry fixing $C := c(\mathbb{R})$. Since I_1 commutes with the closest point projection map $\pi: \tilde{X} \to C$ by Lemma 2.10, $d(I_1(x), x) \geq d(\pi \circ I_1(x), \pi(x)) \geq d(I_1 \circ \pi(x), \pi(x))$ for all $x \in \tilde{X}$. Thus, $\Delta_{I_1} = \Delta_{I_1|_C}$ which implies $d(\tilde{f}(t, y), \tilde{f}(t + 2\pi, y)) = d(c(t), c(t + 2\pi)) = \Delta_{I_1}$. Similarly, $d(\tilde{f}(x, t), \tilde{f}(x, t + 2\pi)) = \Delta_{I_2}$. Thus,

$$\left|\frac{\partial \tilde{f}}{\partial x}\right|^2 = \frac{\Delta_{I_1}^2}{4\pi^2}, \quad \left|\frac{\partial \tilde{f}}{\partial y}\right| = \frac{\Delta_{I_2}^2}{4\pi^2}.$$

Thus, the map $\tilde{h}(t, x, y) = \tilde{f}(x, y)$ satisfies the desired inequalities.

Next, assume that (ii) of the second bullet point holds. Define

$$\tilde{h}: [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \tilde{X}$$

as follows: Fix $t \in [0, \infty)$. For $\theta \in [0, 2\pi)$, let $\theta \mapsto \tilde{h}(t, \theta, 0)$ be a geodesic from $c(t)$ to $I_1 \circ c(t)$ and $\theta \mapsto \tilde{h}(t, \theta, 2\pi)$ be a geodesic from $I_2(c(t))$ to $I_1 \circ I_2(c(t)) = I_2 \circ I_1(c(t))$. Next, let $\theta \mapsto \tilde{h}(t, x, \theta)$ be a geodesic from $\tilde{h}(t, x, 0)$ to $\tilde{h}(t, x, 2\pi)$. Finally, ρ-equivariantly extend to define this map on $t \times \mathbb{R} \times \mathbb{R}$. The NPC condition implies the assertion. □

2.5 Equivariant maps and sections of the associated flat $\tilde{X}$-bundle

Following Donaldson [8], we will replace equivariant maps with sections of an associated fiber bundle. Assume we have the following:

- a complete Riemannian manifold (M, g) with universal covering $\Pi: \tilde{M} \to M$
- an NPC space $\tilde{X}$
- an action of $\pi_1(M)$ on $\tilde{M}$ by deck transformations
- a homomorphism $\rho: \pi_1(M) \to \mathrm{Isom}(\tilde{X})$

Definition 2.12. A map $\tilde{f}: \tilde{M} \to \tilde{X}$ is said to be ρ-equivariant if

$$\tilde{f}(\gamma p) = \rho(\gamma)\tilde{f}(p), \quad \forall \gamma \in \pi_1(M), \; p \in \tilde{M}.$$

Remark 2.13. Assume $\rho: \pi_1(M) \to \mathrm{Isom}(\tilde{X})$ is proper (cf. Definition 2.7). If there exists a finite energy ρ-equivariant map $f: \tilde{M} \to \tilde{X}$, then there exists a Lipschitz harmonic map $u: \tilde{M} \to \tilde{X}$ (cf. [15, Theorem 2.1.3, Remark 2.1.5]). In this paper, we are trying to establish the existence of a harmonic map without assuming that there exists a finite energy map to start with.

The quotient under the action of $\pi_1(M)$ of the product $\tilde{M} \times \tilde{X}$ is the *twisted product*

$$\tilde{M} \times_\rho \tilde{X}.$$

In other words, $\tilde{M} \times_\rho \tilde{X}$ is the set of orbits $[(p, x)]$ of a point $(p, x) \in \tilde{M} \times \tilde{X}$ under the action of $\gamma \in \pi_1(M)$ via the deck transformation on the first component and the isometry $\rho(\gamma)$ on the second component. The fiber bundle

$$\tilde{M} \times_\rho \tilde{X} \to M$$

is called the *flat $\tilde{X}$-bundle* over M defined by ρ.

There is a one-to-one correspondence between sections of this fibration and ρ-equivariant maps

$$\tilde{f}: \tilde{M} \to \tilde{X} \quad \longleftrightarrow \quad f: M \to \tilde{M} \times_\rho \tilde{X}$$

satisfying the relationship

$$[(\tilde{p}, \tilde{f}(\tilde{p}))] \leftrightarrow f(p) \text{ where } \Pi(\tilde{p}) = p.$$

Since the energy density function $|\nabla \tilde{f}|^2$ of $\tilde{f}$ is a ρ-invariant function, we can define

$$|\nabla f|^2(p) := |\nabla \tilde{f}|^2(\tilde{p}).$$

We can similarly define the pullback inner product and directional energy density functions of f by using the corresponding notions for $\tilde{f}$ given in Section 2.2. For $U \subset M$, the energy of a section f is

$$E^f[U] = \int_U |\nabla f|^2 \mathrm{dvol}_g. \tag{2.1}$$

Furthermore, for sections f_1, f_2, we define

$$d(f_1(p), f_2(p)) := d(\tilde{f}_1(\tilde{p}), \tilde{f}_2(\tilde{p})) \tag{2.2}$$

where $\tilde{f}_1, \tilde{f}_2$ are the associated ρ-equivariant maps to sections f_1, f_2 respectively.

2.6 Harmonic maps from punctured Riemann surfaces

In this paper, many of the constructions will depend on harmonic maps for punctured Riemann surfaces. Below, we summarize the results of the paper [10], [11].

Let $\bar{\mathcal{R}}$ be a compact Riemann surface and $\mathcal{R} = \bar{\mathcal{R}} \backslash \{p^1, \dots, p^n\}$ a punctured surface. We fix a conformal disk $\mathbb{D}^j \subset \bar{\mathcal{R}}$ centered at each puncture p^j such that $\mathbb{D}^i \cap \mathbb{D}^j = \emptyset$ for $i \neq j$. Furthermore, let $\mathbb{D}^{j*} = \mathbb{D}^j \backslash \{0\}$.

Fix $P_0 \in \tilde{X}$ and a fundamental domain F of $\tilde{\mathcal{R}}$. Let f_0 be the section of the fiber bundle $\tilde{\mathcal{R}} \times_\rho \tilde{X} \to \mathcal{R}$ such that, for any $p \in \mathcal{R} \cap \Pi(F)$, $f_0(p) = [(\tilde{p}, P_0)]$ where $\tilde{p} = \Pi^{-1}(p) \cap F$. (Note that $\Pi(F)$ is of full measure in $\mathcal{R}$.)

For a given section $f: \mathcal{R} \to \mathcal{R} \times_\rho \tilde{X}$, define

$$\delta_j: \mathbb{D}^{j*} \to [0, \infty), \qquad \delta_j(z) = \operatorname*{ess\,inf}_{\{z \in \mathbb{D}^{j*}\}} d(f(z), f_0(z)). \tag{2.3}$$

Recall that $d(f(z), f_0(z))$ is defined by (2.2).

Definition 2.14. We say a section $f: \mathcal{R} \to \tilde{\mathcal{R}} \times_\rho \tilde{X}$ (or its associated equivariant map) has sub-logarithmic growth if for any $j = 1, \dots, n$ and any $\epsilon > 0$

$$\lim_{|z| \to 0} \delta_j(z) + \epsilon \log |z| = -\infty \text{ in } \mathbb{D}^{j*}.$$

By the triangle inequality, this definition is independent of the choice of $P_0 \in \tilde{X}$. We say that f has logarithmic energy growth if near the punctures it satisfies

$$\sum_{j=1}^n \frac{\Delta_{I^j}^2}{2\pi} \log \frac{1}{r} \leq E^f[\mathcal{R}_r] \leq \sum_{j=1}^n \frac{\Delta_{I^j}^2}{2\pi} \log \frac{1}{r} + C \tag{2.4}$$

where $\mathcal{R}_r = \mathcal{R} \backslash \bigcup_{j=1}^n \mathbb{D}_r^j$ and $E^f[\mathcal{R}_r]$ is the energy of f in $\mathcal{R}_r$.

Definition 2.15. For a homomorphism $\rho: 2\pi\mathbb{Z} \simeq \pi_1(\mathbb{S}^1) \to \tilde{X}$, let $\mathbb{R} \times_\rho \tilde{X} \to \mathbb{S}^1$ be the flat $\tilde{X}$-fiber bundle defined by ρ (cf. Subsection 2.5). Define E_ρ to be the infimum of the energies of sections $\mathbb{S}^1 \to \mathbb{R} \times_\rho \tilde{X}$ (cf. (2.1)). If Δ_I is the translation length of the isometry $I := \rho([\mathbb{S}^1])$, then $\mathsf{E}_\rho = \frac{\Delta_I^2}{2\pi}$.

We record our result in [6], [7].

Theorem 2.16. (Existence and Uniqueness of the Dirichlet solution on $\mathbb{D}^*$). *Assume the following:*
- $\rho\colon 2\pi\mathbb{Z} \simeq \pi_1(\mathbb{S}^1) \simeq \pi_1(\bar{\mathbb{D}}^*) \to \mathrm{Isom}(\tilde{X})$ *is a homomorphism*
- $k\colon \bar{\mathbb{D}}^* \to \widetilde{\bar{\mathbb{D}}^*} \times_\rho \tilde{X}$ *is a locally Lipschitz section*
- $I := \rho([\mathbb{S}^1])$ *has exponential decay to its translation length of Property* $(\star)$

Then there exists a harmonic section

$$u\colon \bar{\mathbb{D}}^* \to \widetilde{\bar{\mathbb{D}}^*} \times_\rho \tilde{X} \text{ with } u|_{\mathbb{S}^1} = k|_{\mathbb{S}^1}.$$

Furthermore, there exists a constant $C > 0$ that depends only on E_ρ of Definition 2.15, a, b from Property $(\star)$ and the section k satisfying the following properties:

(i) $E_\rho \log \frac{1}{r} \leq E^u[\mathbb{D}_{r,1}] \leq E_\rho \log \frac{1}{r} + C, \quad 0 < r \leq 1$

(ii) $\left|\frac{\partial u}{\partial r}\right|^2 \leq \frac{C}{r^2(-\log r)}$ *and* $\left(\left|\frac{\partial u}{\partial \theta}\right|^2 - \frac{E_\rho}{2\pi}\right) \leq \frac{C}{-\log r}$ *in* $\mathbb{D}^*_{\frac{1}{2}}$

(iii) *u has sub-logarithmic growth.*

Moreover, u is the only harmonic section satisfying $u|_{\partial\mathbb{D}} = k|_{\partial\mathbb{D}}$ and property (iii).

3 The Poincaré-type metric and its estimates

3.1 Neighborhoods near the divisor

This subsection closely follows [7]. We let $\bar{M}$ be a Kähler surface and Σ be a divisor with normal crossings such that

$$M = \bar{M} \setminus \Sigma.$$

Furthermore, let

$$\Sigma = \bigcup_{j=1}^{L} \Sigma_j$$

where $\{\Sigma_j\}$ is the set of irreducible components of Σ. Let σ_j be the canonical section of $\mathcal{O}(\Sigma_j)$ with zero set Σ_j.

We denote by $\mathbb{D}$ the unit disk in the complex plane and let

$$\mathbb{D}_r := \{z \in \mathbb{D}\colon |z| < r\} \text{ and } \mathbb{D}_{r_1,r_2} := \{z \in \mathbb{D}\colon r_1 < |z| < r_2\}.$$

For clarity, we will also denote the unit disk by $\mathbb{D}_z$ to indicate that $\mathbb{D}$ is being parameterized by the complex variable $z = re^{i\theta}$. We use analogous notation $\mathbb{D}_{z,r}$, $\mathbb{D}_{z,r_1,r_2}$. We also use the notation $\mathbb{S}^1_\theta$ to denote the circle $\mathbb{S}^1$ parameterized by the real variable θ and identify $\mathbb{S}^1_\theta$ as the boundary of $\mathbb{D}_z$ via the map $\theta \mapsto e^{i\theta}$.

To study a neighborhood of the juncture, let $P \in \Sigma_i \cap \Sigma_j$ for some $i, j \in \{1, \dots, L\}$ with $i \neq j$, and let V_P be a neighborhood of P containing no other crossings. Choose holomorphic trivializations e_i (resp. e_j) of $\mathcal{O}(\Sigma_i)$ (resp. $\mathcal{O}(\Sigma_j)$) on V_P and define z^1 (resp. z^2) by setting

$$\sigma_i = z^1 e_i, \ (\text{resp. } \sigma_j = z^2 e_j). \tag{3.1}$$

For each $j = 1, \dots, L$, let h_j be a Hermitian metric on $\mathcal{O}(\Sigma_j)$ such that $|e_j|_{h_j} = 1$ in V_P for any crossing P. Let h be a Hermitian metric on $\bar{M}$, not necessarily Kähler, such that the following holds:

(i) The metric h is the Euclidean metric in a neighborhood V_P of every crossing P, i.e.

$$h|_{V_P} = dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2. \tag{3.2}$$

By rescaling σ_1 and σ_2 if necessary, we can assume without the loss of generality that

$$\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2} \subset V_P. \tag{3.3}$$

(ii) The metric h induces the orthogonal decomposition $T\bar{M}|_{\Sigma_j} = T\Sigma_j \oplus N\Sigma_j$ and under the natural isomorphism

$$N\Sigma_j \simeq \mathcal{O}(\Sigma_j)|_{\Sigma_j}, \tag{3.4}$$

the restriction of h to $N\Sigma_j$ is same as h_j.

For $r \in (0,1]$, we set

$$\mathcal{D}_{j,r} = \{v \in N\Sigma_j: |v|_{h_j} < r\},$$
$$\mathcal{D}^*_{j,r} = \{v \in N\Sigma_j: 0 < |v|_{h_j} < r\},$$
$$\bar{\mathcal{D}}_{j,r} = \{v \in N\Sigma_j: |v|_{h_j} \leq r\}$$
$$\bar{\mathcal{D}}^*_{j,r} = \{v \in\subset N\Sigma_j: 0 < |v|_{h_j} \leq r\},$$
$$\mathcal{D}_r = \bigcup_{j=1}^L \mathcal{D}_{j,r}, \quad \mathcal{D}^*_r = \bigcup_{j=1}^L \mathcal{D}^*_{j,r}, \quad \bar{\mathcal{D}}_r = \bigcup_{j=1}^L \bar{\mathcal{D}}_{j,r}, \quad \bar{\mathcal{D}}^*_r = \bigcup_{j=1}^L \mathcal{D}^*_{j,r}$$
$$\mathcal{D}_{r_1,r_2} = \mathcal{D}_{r_2} \backslash \bar{\mathcal{D}}_{r_1} \text{ for } 0 < r_1 < r_2 \leq 1. \tag{3.5}$$

There exists $r > 0$ such that the restriction of the exponential map

$$\exp: N\Sigma_j \subset T\bar{M}|_{\Sigma_j} \to \bar{M}$$

defines diffeomorphism of $\mathcal{D}_{j,r}$ to a neighborhood of Σ_j in $\bar{M}$. By rescaling σ_j if necessary, we may assume $r > 1$. In particular, we identify (3.5) for sufficiently small $r > 0$ as an open subset of M via the exponential map; i.e.

$$\mathcal{D}_{r_1,r_2} := \bigcup_j \{\exp\ v: v \in N\Sigma_j,\ r_1 < |v|_{h_j} < r_2\} \subset M. \tag{3.6}$$

Denote

$$\bar{\mathcal{D}}_j := \bar{\mathcal{D}}_{j,1} \subset N\Sigma_j.$$

The restriction of $N\Sigma_j \to \Sigma_j$ to $\bar{\mathcal{D}}_j$ defines a disk bundle

$$\pi_j: \bar{\mathcal{D}}_j \to \Sigma_j. \tag{3.7}$$

We also identity $\bar{\mathcal{D}}_j$ as a subset of $\bar{M}$; i.e.

$$\bar{\mathcal{D}}_j \simeq \exp(\bar{\mathcal{D}}_j) \subset \bar{M}. \tag{3.8}$$

We denote by $J_{\bar{M}}$ the holomorphic structure on $\bar{\mathcal{D}}_j$ defined by pulling back the complex structure on $\bar{M}$ via the exponential map.

We now consider a finite collection of sets near the divisor of the following two types:

– A set of type (A) admits a local unitary trivialization

$$\pi_j^{-1}(\Omega) \simeq \Omega \times \bar{\mathbb{D}}_{z^2}, \tag{3.9}$$

of $\pi_j: \bar{\mathcal{D}}_j \to \Sigma_j$ where $\Omega \subset \Sigma_j$ is a contractible open subset of Σ_j containing no crossings. We will use coordinates (z^1, z^2) where z^1 is a holomorphic local coordinate in Ω and z^2 is the standard coordinate of $\bar{\mathbb{D}}$. Although the coordinates (z^1, z^2) are holomorphic with respect to the product complex structure J_{prod} of $\Omega \times \bar{\mathbb{D}}_{z^2}$, they are *not holomorphic* with respect to the complex structure $J_{\bar{M}}$. However, by construction, we have that

$$J_{\bar{M}} = J_{\text{prod}} \quad \text{on} \quad T_{(z^1,0)}\big(\Omega \times \bar{\mathbb{D}}_{z^2}\big).$$

– A set of type (B) is as in (3.3); i.e.

$$\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2} \subset V_P \tag{3.10}$$

where V_P be an open set containing a single crossing $P \in \Sigma_i \cap \Sigma_j$ ($i \neq j$). By the property (i) of the hermitian metric h, (z^1, z^2) are holomorphic coordinates with respect to $J_{\bar{M}}$. Furthermore, with the identification $\bar{\mathbb{D}}_{z^1} \simeq \bar{\mathbb{D}}_{z^1} \times \{0\} \subset \Sigma_1$ (resp. $\bar{\mathbb{D}}_{z^2} \simeq \{0\} \times \bar{\mathbb{D}}_{z^2} \subset \Sigma_2$),

$$\pi_j^{-1}\big(\bar{\mathbb{D}}_{z^1}\big) \simeq \bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2} \quad (\text{resp. } \pi_i^{-1}\big(\bar{\mathbb{D}}_{z^2}\big) \simeq \bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}) \tag{3.11}$$

is a local unitary trivialization of $\pi_j \colon \bar{\mathcal{D}}_j \to \Sigma_j$ (resp. $\pi_i \colon \bar{\mathcal{D}}_i \to \Sigma_i$).

Definition 3.1. We will refer to the coordinates (z^1, z^2) discussed in (A) above as the standard product coordinates on a set $\Omega \times \bar{\mathbb{D}}_{z^2}$ of type (A).

Definition 3.2. We will refer to the coordinates (z^1, z^2) discussed in (B) above as the standard product coordinates and the holomorphic coordinates on a set $\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$ of type (B).

Remark 3.3. Holomorphic coordinates in a set of type (A) are defined later (cf. Definition 3.6).

3.2 Poincaré-type metric

Recall the Poincaré metric

$$g_{\text{poin}} = \text{Re}\left(\frac{\mathrm{d}z \oplus d\bar{z}}{|z|^2(\log|z|^2)^2}\right) \text{ on } \mathbb{D}^*. \tag{3.12}$$

Using the canonical section $\sigma_j \in \mathcal{O}(\Sigma_j)$ and the Hermitian metric h_j on $\mathcal{O}(\Sigma_j)$ given in Section 3.1, we define the Poincare-type metric on M as follows:

Definition 3.4. Let $\bar{\omega}$ be the Kähler form on $\bar{M}$. Scale the metric h_j such that $|\sigma_j|_{h_j} < 1$ and define

$$\omega = \bar{\omega} - \frac{\sqrt{-1}}{2}\sum_{l=1}^{L} \partial\bar{\partial}\log\Big(\log|\sigma_j|_{h_j}^{-2}\Big). \tag{3.13}$$

By scaling $\bar{\omega}$ if necessary, we can assume that ω defines a positive form. We denote by g the Kähler metric on M induced by the Kähler form ω.

Definition 3.5. Fix j and define on $\bar{M}\backslash\bigcup_{i\neq j}\Sigma_i$ the Kähler form

$$\omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\,\log|\sigma_j|_{h_j}^{-2} = \bar{\omega} - \frac{\sqrt{-1}}{2}\sum_{i\neq j}\partial\bar{\partial}\log\,\log|\sigma_i|_{h_i}^{-2}. \tag{3.14}$$

Define g_{Σ_j} to be the restriction to $\Sigma_j \backslash \bigcup_{i\neq j}\Sigma_i$ of the Kähler metric associated to this Kähler form. This is a smooth metric on Σ_j away from the crossings.

Below we derive some estimates for the metric g in a set of type (A) and of type (B) (See (3.9) and (3.10) for definitions of a set of type (A) and (B)). These are an expanded form of the estimates derived by Mochizuki [7].

3.3 Metric estimates in set of type (A)

Let $\pi_j^{-1}(\Omega) \simeq \Omega \times \bar{\mathbb{D}}$ be a set of type (A) with the standard product coordinates (z^1, z^2). (Recall that $\Omega \subset \Sigma_j$ does not intersect Σ_i for $i \neq j$.) We will write

$$z^1 = x + iy \text{ and } z^2 = re^{i\theta}.$$

Fix a local trivialization e of $\mathcal{O}(\Sigma_j)$, holomorphic with respect to the complex structure $J_{\bar{M}}$. Define

$$b:\Omega\times\bar{\mathbb{D}}\to[0,\infty),\quad b=|e|_{h_j}^{-2}. \tag{3.15}$$

With σ_j the canonical section of $\mathcal{O}(\Sigma_j)$ as before, define a function ζ on $\Omega\times\bar{\mathbb{D}}$ by

$$\sigma_j=\zeta e. \tag{3.16}$$

Thus, ζ is holomorphic with respect to $J_{\bar{M}}$.

Definition 3.6. We refer to

$$(z^1,\zeta)$$

as the holomorphic coordinates (with respect to $J_{\bar{M}}$) on a set $\Omega\times\bar{\mathbb{D}}$ of type (A).

Since $z^2=0=\zeta$ on $\Omega\times\{0\}$ and $z^2\neq 0$, $\zeta\neq 0$ on $\Omega\times\bar{\mathbb{D}}^*$,

$$\mathrm{d}\log z^2-\mathrm{d}\log\zeta=\frac{\mathrm{d}z^2}{z^2}-\frac{d\zeta}{\zeta}=O(1)(\mathrm{d}z^1+d\bar{z}^1+\mathrm{d}z^2+d\bar{z}^2).$$

Taking real and imaginary parts,

$$\begin{aligned}\frac{\mathrm{d}r}{r}-\frac{\mathrm{d}s}{s}&=O(1)(\mathrm{d}z^1+\mathrm{d}\bar{z}^1+\mathrm{d}z^2+\mathrm{d}\bar{z}^2)\\ \mathrm{d}\theta-\mathrm{d}\eta&=O(1)(\mathrm{d}z^1+\mathrm{d}\bar{z}^1+\mathrm{d}z^2+\mathrm{d}\bar{z}^2).\end{aligned} \tag{3.17}$$

Let

$$a:\Omega\times\bar{\mathbb{D}}\to\mathbb{C}^*$$

be a smooth function bounded above and bounded away from 0 satisfying

$$a\mathrm{d}\zeta|_{\Omega\times\{0\}}=\mathrm{d}z^2\Big|_{(p,0)},\quad\forall z^1\in\Omega. \tag{3.18}$$

Thus,

$$a\mathrm{d}\zeta=\mathrm{d}z^2(1+O(r))=\frac{\mathrm{d}z^2}{z^2}z^2(1+O(r))=\left(\frac{\mathrm{d}\zeta}{\zeta}+O(1)\right)z^2(1+O(r)).$$

Plugging in $\frac{\partial}{\partial\zeta}$ in the above equation, we obtain

$$a=\frac{z^2}{\zeta}(1+O(r)). \tag{3.19}$$

From this, we immediately obtain

$$\begin{aligned}|a\zeta|&=r(1+O(r))\\ \log|a\zeta|^2&=\log r^2+\log(1+O(r))=\log r^2+O(r).\end{aligned}$$

This implies

$$\log|\sigma_j|_{h_j}^{-2}=\log b|\zeta|^{-2}=\log b-\log r^2+\log|a|^2+O(r)=-\log r^2+A+O(r)$$

where

$$A(z^1)=\log b(z^1,0)+\log|a(z^1,0)|^2.$$

The function a depends on the choice of e and σ whereas the function b depends on the choice of e and h_j. Thus, by scaling σ if necessary, we can assure that b satisfies the following two conditions:

$$-\log|\zeta|^2+\log b>0\text{ on }\Omega\times\bar{\mathbb{D}} \tag{3.20}$$

$$\log b + \log |a|^2 > 0 \text{ on } \Omega \times \{0\}. \tag{3.21}$$

We compute

$$\begin{aligned}
-\frac{\sqrt{-1}}{2}\partial\bar\partial \log \log |\sigma_j|^{-2}_{h_j} &= \frac{\sqrt{-1}}{2}\left(\frac{\partial \log |\sigma_j|^2_{h_j} \wedge \bar\partial \log |\sigma_j|^2_{h_j}}{\left(\log |\sigma_j|^2_{h_j}\right)^2} - \frac{\partial\bar\partial \log |\sigma_j|^2_{h_j}}{\log |\sigma_j|^2_{h_j}}\right) \\
&= \frac{\sqrt{-1}}{2}\left(\frac{(\partial \log \zeta + \partial \log b) \wedge (\bar\partial \log \bar\zeta + \bar\partial \log b)}{\left(\log |\sigma_j|^2_{h_j}\right)^2} - \frac{\partial\bar\partial \log b}{\log |\sigma_j|^2_{h_j}}\right) \\
&= \frac{\sqrt{-1}}{2}\left(\frac{\partial \log \zeta \wedge \bar\partial \log \bar\zeta}{\left(\log |\sigma_j|^2_{h_j}\right)^2} + \frac{\partial \log \zeta \wedge \bar\partial \log b}{\left(\log |\sigma_j|^2_{h_j}\right)^2}\right. \\
&\qquad \left. + \frac{\partial \log b \wedge \bar\partial \log \bar\zeta}{\left(\log |\sigma_j|^2_{h_j}\right)^2} + \frac{\partial \log b \wedge \bar\partial \log b}{\left(\log |\sigma_j|^2_{h_j}\right)^2} - \frac{\partial\bar\partial \log b}{\log |\sigma_j|^2_{h_j}}\right).
\end{aligned} \tag{3.22}$$

Note that because of (3.18) and since b is a smooth function bounded away from 0, we have

$$\begin{aligned}
\frac{\partial \log \zeta \wedge \bar\partial \log \zeta}{\left(\log |\sigma_j|^2_{h_j}\right)^2} &= \frac{dz^2 \wedge d\bar z^2}{r^2(-\log r^2 + A)^2} + \mathrm{Error}_1 + \mathrm{Error}_2 \\
\frac{\partial \log \zeta \wedge \bar\partial \log b}{\left(\log |\sigma_j|^2_{h_j}\right)^2}, \quad \frac{\partial \log b \wedge \bar\partial \log \bar\zeta}{\left(\log |\sigma_j|^2_{h_j}\right)^2} &= \mathrm{Error}_1 + \mathrm{Error}_2 \\
\frac{\partial \log b \wedge \bar\partial \log b}{\left(\log |\sigma_j|^2_{h_j}\right)^2}, \quad \frac{\partial\bar\partial \log b}{\log |\sigma_j|^2_{h_j}} &= \mathrm{Error}_2.
\end{aligned} \tag{3.23}$$

where Error_1 is a form of the type

$$O\left(\frac{1}{r(-\log r^2 + A)^2}\right) dz^1 \wedge d\bar z^2 \text{ or } O\left(\frac{1}{r(-\log r^2 + A)^2}\right) dz^2 \wedge d\bar z^1$$

and Error_2 is a form of the type

$$O\left(\frac{1}{(-\log r^2 + A)^2}\right) dz^1 \wedge d\bar z^1 \text{ or } O\left(\frac{1}{(-\log r^2 + A)^2}\right) dz^2 \wedge d\bar z^2.$$

In coordinate z^1 of $\Omega \subset \Sigma_j$, let the local expression of the metric g_{Σ_j} given by Definition 3.5 be $\lambda\, dz^1 d\bar z^1$. Then it follows from the above estimates that in the coordinates (z^1, z^2) of $\Omega \times \bar{\mathbb{D}}$ and with $r = |z^2|$, the metric expression of g is

$$\begin{pmatrix} g_{1\bar1} & g_{1\bar2} \\ g_{2\bar1} & g_{2\bar2} \end{pmatrix} = \begin{pmatrix} \lambda + O\left(\frac{1}{(-\log r^2 + A)^2}\right) & O\left(\frac{1}{r(-\log r^2 + A)^2}\right) \\ O\left(\frac{1}{r(-\log r^2 + A)^2}\right) & \frac{1}{r^2(-\log r^2 + A)^2} + O\left(\frac{1}{(-\log r^2 + A)^2}\right) \end{pmatrix}. \tag{3.24}$$

Furthermore, the local expression for the inverse g^{-1} is

$$\begin{pmatrix} g^{1\bar1} & g^{1\bar2} \\ g^{2\bar1} & g^{2\bar2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} + O(r^2) & O(r) \\ O(r) & r^2(-\log r^2 + A)^2 + O(r^2) \end{pmatrix}. \tag{3.25}$$

The *product metric* P on $\Omega \times \bar{\mathbb{D}}$ is defined by taking the dominant terms of g. More precisely, let

$$\begin{pmatrix} P_{1\bar{1}} & P_{1\bar{2}} \\ P_{2\bar{1}} & P_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{r^2(-\log r^2 + A)^2} \end{pmatrix}. \tag{3.26}$$

The inverse P^{-1}

$$\begin{pmatrix} P^{1\bar{1}} & P^{1\bar{2}} \\ P^{2\bar{1}} & P^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & r^2(-\log r^2 + A)^2 \end{pmatrix}.$$

Thus,

$$g^{-1} - P^{-1} = \begin{pmatrix} O(r^2) & O(r) \\ O(r) & O(r^2) \end{pmatrix}. \tag{3.27}$$

Comparing the local expression of g and P, we obtain

$$\mathrm{dvol}_g = \mathrm{dvol}_P\left(1 + O\left(\frac{1}{(-\log r^2 + A)^2}\right)\right). \tag{3.28}$$

A straightforward computation gives

$$\begin{aligned} \mathrm{dvol}_P &= \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{-2ir^2(-\log r^2 + A)^2} \\ P^{1\bar{1}}\mathrm{dvol}_P &= g_{\Sigma_j}^{1\bar{1}}\,\mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{-2ir^2(-\log r^2 + A)^2} \\ P^{2\bar{2}}\mathrm{dvol}_P &= \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{-2i}. \end{aligned} \tag{3.29}$$

The metric P (and hence the metric g) is of finite volume over $\Omega \times \mathbb{D}^*$ since g_{Σ_j} is a smooth metric on $\Omega \subset \Sigma_j$ and

$$\begin{aligned} \mathrm{Vol}_P(\Omega \times \mathbb{D}^*) &= \mathrm{Area}(\Omega) \cdot 2\pi \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{r\mathrm{d}r}{r^2(-\log r^2 + A)^2} \\ &= -\mathrm{Area}(\Omega) \cdot 2\pi \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{d(-\log r^2 + A)}{2(-\log r^2 + A)^2} \\ &= -\mathrm{Area}(\Omega) \cdot \pi \lim_{\epsilon \to 0} \frac{1}{-\log r^2 + A}\bigg|_\epsilon^1 < \infty. \end{aligned}$$

Moreover, since

$$2\pi \mathrm{Area}_{g_{\Sigma_j}}(\Omega) \log \frac{r_2}{r_1} = \int_{\Omega \times \mathbb{D}_{r_1,r_2}} \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{r} = \int_{\Omega \times \mathbb{D}_{r_1,r_2}} P^{\theta\theta}\mathrm{dvol}_P,$$

there exists a constant $C > 0$ such that

$$\left| \frac{1}{2\pi} \int_{\Omega \times \mathbb{D}_{r_1,r_2}} g^{\theta\theta}\mathrm{dvol}_g - \mathrm{Area}(\Omega) \log \frac{r_2}{r_1} \right| \leq C, \quad 0 < r_1 < r_2 < 1. \tag{3.30}$$

Lemma 3.7. *The Poincaré type metric g defined by Definition 3.4 satisfies the following: There exists $c > 0$ such that, on the set $\Omega \times \mathbb{D}^*_{\frac{1}{4}}$ away from the crossings with holomorphic coordinates $(z^1, \zeta = re^{i\theta})$,*

$$\frac{1}{c}\,\mathrm{dvol}_g \;\leq\; \rho \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \frac{r\mathrm{d}r \wedge \mathrm{d}\theta}{r^2(-\log r^2)^2} \;\leq\; c\,\mathrm{dvol}_g$$

Proof. This is immediate from the fact that the metric g_{Σ_j} given by Definition 3.5 is smooth combined with (3.28). □

Remark 3.8. The key feature of the metric P is the following: Define Q to be the product metric

$$Q = g_{\Sigma_i} \oplus \mathrm{Re}\Big(\frac{dz^2 d\bar{z}^2}{-2i}\Big) = g_{\Sigma_i} \oplus (dr^2 + r^2 d\theta^2) \text{ on } \Omega \times \mathbb{D}^*.$$

Then

$$P^{2\bar{2}} \mathrm{dvol}_P = Q^{2\bar{2}} \mathrm{dvol}_Q.$$

This is important in Section 5 below where we estimate the energy of v. In particular, we have

$$\begin{aligned}\int_{\Omega\times\mathbb{D}} P^{2\bar{2}} \Big|\frac{\partial v}{\partial z^2}\Big|^2 \mathrm{dvol}_P &= \int_{\Omega\times\mathbb{D}} Q^{2\bar{2}} \Big|\frac{\partial v}{\partial z^2}\Big|^2 \mathrm{dvol}_Q \\ &= \int_{\Omega}\Bigg(\int_{\{z^2\}\times\mathbb{D}} \Big|\frac{\partial v}{\partial z^2}\Big|^2 \frac{dz^2 \wedge d\bar{z}^2}{-2i}\Bigg) \mathrm{dvol}_{g_{\Sigma_j}}.\end{aligned}$$

Note that the inside integral on the right hand side above is exactly the energy of the harmonic map $v|_{\{z^1\}\times\mathbb{D}}$ from the disk.

3.4 Metric estimates in a set of type (B)

First recall that a type (B) set of Section 3.1 is a set $\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$ with

$$\bar{\mathbb{D}}_{z^1} \subset \Sigma_j,\ \bar{\mathbb{D}}_{z^2} \subset \Sigma_i \text{ and } (0,0) \in \Sigma_j \cap \Sigma_i$$

such that the standard product coordinates (z^1, z^2) are also holomorphic coordinates with respect to complex structure $J_{\bar{M}}$. Since $|\sigma_i|_{h_i} = |z^1|$ and $|\sigma_j|_{h_j} = |z^2|$,

$$\begin{aligned}-\frac{\sqrt{-1}}{2}\Big(\partial\bar{\partial} \log\log|\sigma_i|_{h_i}^{-2} + \partial\bar{\partial}\log\log|\sigma_j|_{h_j}^{-2}\Big) &= \frac{\partial \log z^1 \wedge \bar{\partial}\log \bar{z}^1}{(-\log|z^1|^2)^2} \\ &\quad + \frac{\partial \log z^2 \wedge \bar{\partial}\log \bar{z}^2}{(-\log|z^2|^2)^2}.\end{aligned}$$

In the coordinates (z^1, z^2) and with $\rho = |z^1|$ and $r = |z^2|$, the local expression of the metric P associated to the above Kähler form is

$$\begin{pmatrix} P_{1\bar{1}} & P_{1\bar{2}} \\ P_{2\bar{1}} & P_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho^2(\log\rho^2)^2} & 0 \\ 0 & \frac{1}{r^2(\log r^2)^2} \end{pmatrix} \tag{3.31}$$

and the inverse P^{-1} is given by

$$\begin{pmatrix} P^{1\bar{1}} & P^{1\bar{2}} \\ P^{2\bar{1}} & P^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \rho^2(\log\rho^2)^2 & 0 \\ 0 & r^2(\log r^2)^2 \end{pmatrix}.$$

With $\diamond = O(\rho^2(\log\rho^2)^2)$ and $\square = O(r^2(\log r^2)^2)$, the local expression of the metric g and its inverse g^{-1} is

$$\begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\rho^2(\log\rho^2)^2} + O(1) & O(1) \\ O(1) & \frac{1}{r^2(\log r^2)^2} + O(1) \end{pmatrix} \tag{3.32}$$

$$\begin{pmatrix} g^{1\bar{1}} & g^{1\bar{2}} \\ g^{2\bar{1}} & g^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \rho^2(\log\rho^2)^2(1+\diamond+\square) & \diamond\square \\ \diamond\square & r^2(\log r^2)^2(1+\diamond+\square) \end{pmatrix}. \tag{3.33}$$

Thus,

$$g^{-1} - P^{-1} = \begin{pmatrix} \rho^2(\log\rho^2)^2(\diamond+\square) & \diamond\square \\ \diamond\square & r^2(\log r^2)^2(\diamond+\square) \end{pmatrix} \tag{3.34}$$

$$\mathrm{dvol}_g = d\mathrm{vol}_P\big(1 + O\big(\rho^2(\log r^2)^2\big) + O\big(r^2(\log r^2)^2\big)\big). \tag{3.35}$$

A straightforward computation gives

$$\begin{aligned} \mathrm{dvol}_P &= \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i\rho^2(\log\rho^2)^2} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(\log r^2)^2} \\ P^{1\bar{1}}\mathrm{dvol}_P &= \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(\log r^2)^2} \\ P^{2\bar{2}}\mathrm{dvol}_P &= \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i\rho^2(\log\rho^2)^2} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i}. \end{aligned} \tag{3.36}$$

Similarly to (3.30), we also obtain for subsets $\mathbb{D}_{r_1,r_2}\times\Omega$, $\Omega\times\mathbb{D}_{r_1,r_2}$ of $\bar{\mathbb{D}}_{z^1}\times\bar{\mathbb{D}}_{z_2}$

$$\begin{aligned} &\left| \frac{1}{2\pi}\int\limits_{\mathbb{D}_{r_1,r_2}\times\Omega} g^{\phi\phi}\mathrm{dvol}_g - \mathrm{Area}_{g_{\Sigma_i}}(\Omega)\log\frac{r_2}{r_1}\right| \leq C, \\ &\left| \frac{1}{2\pi}\int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} g^{\theta\theta}\mathrm{dvol}_g - \mathrm{Area}_{g_{\Sigma_j}}(\Omega)\log\frac{r_2}{r_1}\right| \leq C. \end{aligned} \tag{3.37}$$

Lemma 3.9. *The Poincaré type metric g of Definition* 3.4 *satisfies the following: There exists $c>0$ such that in neighborhood $\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}$ near a crossing with holomorphic coordinates ($z^1=\rho e^{i\phi}$, $z^2=re^{i\theta}$),*

$$\frac{1}{c}\,\mathrm{d}vol_g \;\leq\; \frac{\rho\mathrm{d}\rho\wedge\mathrm{d}\phi}{\rho^2(-\log\rho^2)^2}\wedge\frac{r\mathrm{d}r\wedge\mathrm{d}\theta}{r^2(-\log r^2)^2} \;\leq\; c\,\mathrm{d}vol_g.$$

Proof. This is immediate from (3.36). □

4 The prototype section

The goal of this section is to construct a *prototype section* with logarithmic energy growth near the divisor. The key is the fiber-wise harmonic sections on the normal bundle of the divisor Σ, the existence of which follows from the Dirichlet problem on the punctured disk (cf. Theorem 2.16).

Recall the sets of type (A) and of type (B) described in (3.9) and (3.10) respectively. In Section 4.1 and Section 4.2, we construct a local prototype section in a set of type (A) and (B) respectively. In Section 4.3, we glue these sections together to define a prototype section near the divisor and extend it to all of M. In summary, we construct a locally Lipschitz global section

$$v\colon M\to\tilde{M}\times_\rho\tilde{X}$$

of logarithmic energy growth near the divisor.

4.1 In a neighborhood away from the junctures

The goal of this subsection is to construct a local prototype section in a set of type (A) and derive some energy estimates. We start with the following:

- $\Omega \times \bar{\mathbb{D}}$ is a set of type (A) with $\Omega \subset \Sigma_j$
- (z^1, z^2) are the standard product coordinates of $\Omega \times \bar{\mathbb{D}}$ (cf. (3.1))
- r, θ are parameters defined by $z^2 = re^{i\theta}$
- $\mathbb{S}^1_\theta$ is the boundary $\partial \bar{\mathbb{D}}_{z^2}$ of $\bar{\mathbb{D}}_{z^2}$
- $\mathbb{R}_\theta \to \mathbb{S}^1$ is the universal cover
- $\widetilde{\bar{\mathbb{D}}^*} \to \bar{\mathbb{D}}^*$ is the universal cover
- $[\mathbb{S}^1_\theta]$ is the element of $\pi_1(\mathbb{S}^1_\theta) \simeq \pi_1(\Omega \times \bar{\mathbb{D}}^*)$ associated to the loop $\mathbb{S}^1_\theta \simeq \{z^1\} \times \mathbb{S}^1_\theta$
- $\rho' : \pi_1(\Omega \times \bar{\mathbb{D}}^*) \to \mathrm{Isom}(\tilde{X})$ is a homomorphism
- $(\Omega \times \widetilde{\bar{\mathbb{D}}^*}) \times_{\rho'} \tilde{X} \to \Omega \times \bar{\mathbb{D}}^*$ is a fiber bundle
- $k : \Omega \times \bar{\mathbb{D}}^* \to (\Omega \times \widetilde{\bar{\mathbb{D}}^*}) \times_{\rho'} \tilde{X}$ is a locally Lipschitz section
- $\mathbb{R}_\theta \times_{\rho'} \tilde{X} \to \mathbb{S}^1_\theta$ is a fiber bundle
- E_j is the infimum of the energies of sections $\mathbb{S}^1_\theta \to \mathbb{R}_\theta \times_{\rho'} \tilde{X}$.

4.1.1 Construction of a prototype section in a set of type (A)

We define

$$v : \Omega \times \bar{\mathbb{D}}^* \to (\Omega \times \widetilde{\bar{\mathbb{D}}^*}) \times_{\rho'} \tilde{X} \tag{4.1}$$

by setting v to be the fiber-wise harmonic section with boundary values given by $k|_{\Omega \times \mathbb{S}^1}$. More precisely, we apply Theorem 2.16 as follows: For each $z^1 \in \Omega$, the restriction

$$v_{z^1} := v|_{\{z^1\} \times \bar{\mathbb{D}}^*} : \bar{\mathbb{D}}^* \approx \{z^1\} \times \bar{\mathbb{D}}^* \to \widetilde{\bar{\mathbb{D}}^*} \times_{\rho'} \tilde{X}$$

is the unique harmonic section with logarithmic energy growth and boundary values

$$v_{z^1}|_{\mathbb{S}^1 \approx \{z^1\} \times \mathbb{S}^1} = k|_{\mathbb{S}^1 \approx \{z^1\} \times \mathbb{S}^1}.$$

4.1.2 Derivative estimates in a set of type (A)

Lemma 4.1. (Derivative estimates in set of type (A)). *There exists a constant C such that*

$$\left| \frac{\partial v}{\partial z^1}(z_0^1, z_0^2) \right| \le C, \quad \forall (z_0^1, z_0^2) \in \Omega \times \bar{\mathbb{D}}^*$$

$$\int\limits_{\{z_0^1\} \times \mathbb{D}_{r,r_0}} \left| \frac{\partial v}{\partial z^2} \right|^2 \frac{dz^2 \wedge d\bar{z}^2}{-2i} \le C + E_j \log \frac{r_0}{r}, \quad \forall z_0^1 \in \Omega$$

where $0 < r < r_0 \le \frac{1}{4}$ *and* $z^2 = re^{i\theta}$.

Proof. Denote the Lipschitz constant of k on $\Omega \times \mathbb{S}^1$ by L. Let $z_0^1, z^1 \in \Omega$. Since $v_{z_0^1}$ and v_{z^1} are harmonic sections, the function $z \mapsto d^2(v_{z_0^1}(z), v_{z^1}(z))$ is subharmonic in $\mathbb{D}^*$ (cf. [14, Remark 2.4.3]).

By Theorem 2.16 (iii) and the triangle inequality,

$$\lim_{z \to 0} d^2(v_{z_0^1}(z), v_{z^1}(z)) + \epsilon \log |z| = -\infty, \quad \forall \epsilon > 0.$$

Thus, $d^2\left(v_{z_0^1}, v_{z^1}\right)$ extends to subharmonic function on $\mathbb{D}$ (cf. [10], [11], Lemma 3.2). The maximum principle implies that for any $z_0^2 = r_0 e^{i\theta_0}$,

$$\begin{aligned} d^2(v_{z_0^1}(r_0 \mathrm{e}^{\mathrm{i}\theta_0}), v_{z^1}(r_0 \mathrm{e}^{\mathrm{i}\theta_0})) &\leq \sup_{\theta\in\mathbb{S}^1} d^2(k(z_0^1, \mathrm{e}^{\mathrm{i}\theta}), k(z^1, \mathrm{e}^{\mathrm{i}\theta})) \\ &\leq L^2 |z_0^1 - z^1|^2. \end{aligned} \tag{4.2}$$

In other words, for every fixed $z_0^2 = r_0 e^{i\theta_0}$, the map $z_1 \mapsto v_{z^1}(z_0^2)$ is Lipschitz which immediately implies the first estimate. The second estimate follows from Theorem 2.16. □

4.2 In a neighborhood of the juncture

The goal of this subsection is to construct a local prototype section in a set of type (B) and derive some derivative estimates. We start with the following:

- $\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$ is a set of type (B) with $\mathbb{D}_{z^1} \subset \Sigma_j$ and $\mathbb{D}_{z^2} \subset \Sigma_i$
- (z^1, z^2) are the standard product (and holomorphic) coordinates
- ρ, ϕ, r, θ are the parameters defined by $z^1 = \rho \mathrm{e}^{\mathrm{i}\phi}$ and $z^2 = re^{\mathrm{i}\theta}$
- $\mathbb{S}^1_\phi$ is the boundary of $\bar{\mathbb{D}}_{z^1}$ and $\mathbb{S}^1_\theta$ is the boundary of $\bar{\mathbb{D}}_{z^2}$
- $\mathbb{R}_\phi \to \mathbb{S}^1_\phi$ and $\mathbb{R}_\theta \to \mathbb{S}^1_\theta$ are the universal covers
- $\widetilde{\bar{\mathbb{D}}^*_{z^1}} \to \bar{\mathbb{D}}^*_{z^1}$ and $\widetilde{\bar{\mathbb{D}}^*_{z^2}} \to \bar{\mathbb{D}}^*_{z^2}$ are the universal covers
- $\mathbb{S}^1_\phi \times \mathbb{S}^1_\theta$ is the boundary of $\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$
- $\mathbb{R}_\phi \times \mathbb{R}_\theta \to \mathbb{S}^1_\phi \times \mathbb{S}^1_\theta$ is the universal covering map
- $\left[\mathbb{S}^1_\phi\right]$ is the element of $\pi_1\left(\bar{\mathbb{D}}^*_{z^1}\right) \simeq \pi_1\left(\mathbb{S}^1_\phi\right)$ generated by $\mathbb{S}^1_\phi$ and $\left[\mathbb{S}^1_\theta\right]$ is the element of $\pi_1\left(\bar{\mathbb{D}}^*_{z^2}\right) \simeq \pi_1(\mathbb{S}^1_\theta)$ generated by $\mathbb{S}^1_\theta$
- $\left[\mathbb{S}^1_\phi\right]$ and $\left[\mathbb{S}^1_\theta\right]$ also are the elements of $\pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}\right)$ generated by $\mathbb{S}^1_\phi \simeq \mathbb{S}^1_\phi \times \{z^2\}$ and $\mathbb{S}^1_\theta \simeq \{z^1\} \times \mathbb{S}^1_\theta$ respectively
- $\pi_1\left(\bar{\mathbb{D}}^*_{z^1}\right) \simeq \pi_1\left(\mathbb{S}^1_\phi\right)$ and $\pi_1\left(\bar{\mathbb{D}}^*_{z^2}\right) \simeq \pi_1(\mathbb{S}^1_\theta)$ are identified as a subgroup of $\pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}\right)$ by the above identification
- $\rho' \colon \pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}\right) \to \mathrm{Isom}(\tilde{X})$ is a homomorphism and $\rho'_k = \rho'|_{\pi_1\left(\bar{\mathbb{D}}^*_{z^k}\right)}$ for $k = 1, 2$
- $\left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho'} \tilde{X} \to \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$ and $\widetilde{\bar{\mathbb{D}}^*_{z^k}} \times_{\rho'_k} \tilde{X} \to \bar{\mathbb{D}}^*_{z^k}$ for $k = 1, 2$ are fiber bundles
- $\mathbb{R}_\phi \times_{\rho'_1} \tilde{X} \to \mathbb{S}^1_\phi$ and $\mathbb{R}_\theta \times_{\rho'_2} \tilde{X} \to \mathbb{S}^1_\theta$ are fiber bundles
- E_i is the infimum of the energies of sections $\mathbb{S}^1_\phi \to \mathbb{R}_\phi \times_{\rho'_1} \tilde{X}$
- E_j is the infimum of the energies of sections $\mathbb{S}^1_\theta \to \mathbb{R}_\theta \times_{\rho'_2} \tilde{X}$
- $k \colon \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho'} \tilde{X}$ is a locally Lipschitz section
- $\tilde{k} \colon \widetilde{\bar{\mathbb{D}}^*_{w^1}} \times \widetilde{\bar{\mathbb{D}}^*_{w^2}} \to \tilde{X}$ is the corresponding ρ'-equivariant map
- κ is the restriction of $\tilde{k}$ to $\mathbb{R}_\phi \times \mathbb{R}_\theta$.

4.2.1 Construction of a prototype section in a set of type (B)

We construct the local section

$$v \colon \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho'} \tilde{X}.$$

In what follows, we will assume that $\mathsf{E}_i, \mathsf{E}_j > 0$ for simplicity. The case when E_i or E_j is equal 0 can be dealt with in the same way.

- By Property (★) and Lemma 2.11, for each $t \in [0, \infty)$, there exist constants $a, b > 0$ (by modifying the constants a, b from Lemma 2.11) and a section

$$h_t \colon \mathbb{S}^1_\phi \times \mathbb{S}^1_\theta \to (\mathbb{R}_\phi \times \mathbb{R}_\theta) \times_{\rho'} \tilde{X}$$

satisfying

$$\left|\frac{\partial h_t}{\partial t}\right|^2 \leq 1, \quad \left|\frac{\partial h_t}{\partial \phi}\right|^2 \leq \frac{\mathsf{E}_i}{2\pi} + be^{-at}, \quad \left|\frac{\partial h_t}{\partial \theta}\right|^2 \leq \frac{\mathsf{E}_j}{2\pi} + be^{-at}. \tag{4.3}$$

- Define the diagonal set

$$D = \left\{(\rho e^{i\phi}, \rho e^{i\theta}) \in \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} : \rho \in (0,1], \quad \phi, \theta \in \mathbb{S}^1\right\}.$$

- Define $v_D : D \to X$ as follows: Fix $(\theta, \phi) \in \mathbb{S}^1 \times \mathbb{S}^1$.
  - For $\rho \in (0, \frac{1}{2}]$, let

$$v_D(\rho e^{i\phi}, \rho e^{i\theta}) = h_{3(-\log\ \rho)^{\frac{1}{3}}}(\phi, \theta).$$

  - For $\rho \in [\frac{1}{2}, 1]$, let the curve

$$\rho \mapsto \gamma_\rho(\phi, \theta) \text{ for } \rho \in \left[\frac{1}{2}, 1\right]$$

be the geodesic between $h_{3(\log\ 2)^{\frac{1}{3}}}(\phi, \theta)$ and $\kappa(\phi, \theta)$. Define

$$v_D(\rho e^{i\phi}, \rho e^{i\theta}) = \gamma_\rho(\phi, \theta)$$

where we use γ_ρ to also denote the section $\gamma_\rho : \mathbb{S}^1_\phi \times \mathbb{S}^1_\theta \to (\mathbb{R}_\phi \times \mathbb{R}_\theta) \times_{\rho'} \tilde{X}$. (Note that ρ' is a representation and ρ is a real number here.)

- Let

$$Z_1 := \left\{(z^1, z^2) \in \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} : |z^1| \geq |z^2|\right\}$$

and

$$\varphi_1 : \bar{\mathbb{D}}^*_{w^1} \times \bar{\mathbb{D}}^*_{w^2} \to Z_1 \subset \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$$

be a homeomorphism defined by (see Figure 1)

$$(z^1, z^2) = \varphi_1(w^1, w^2), \quad z^1 = w^1, \ z^2 = |w^1| w^2.$$

- Define

$$v_1 : \bar{\mathbb{D}}^*_{w^1} \times \bar{\mathbb{D}}^*_{w^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{w^1}} \times \widetilde{\bar{\mathbb{D}}^*_{w^2}}\right) \times_{\rho'} \tilde{X}$$

by setting v_1 to be the fiber-wise harmonic section with boundary values given by $v_D \circ \varphi_1|_{\bar{\mathbb{D}}^*_{w^1} \times \mathbb{S}^1_\theta}$. More precisely, we apply Theorem 2.16 as follows: For each $w^1 \in \bar{\mathbb{D}}^*_{w^1}$, the restriction

$$v_{1,w^1} := v_1\big|_{\{w^1\} \times \bar{\mathbb{D}}^*_{w_2}}$$

is the unique harmonic section with boundary values

$$v_{1,w^1}\big|_{\mathbb{S}^1 \approx \{w^1\} \times \mathbb{S}^1_\theta} = v_D \circ \varphi_1|_{\mathbb{S}^1 \approx \{w^1\} \times \mathbb{S}^1_\theta}.$$

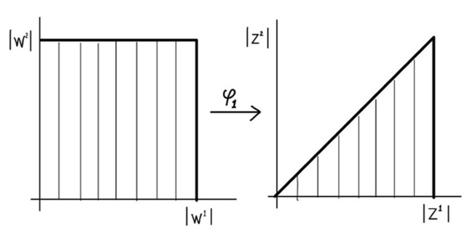

Figure 1: The map φ_1.

– Similarly, let
$$Z_2 := \left\{ (z^1, z^2) \in \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} : |z^1| \le |z^2| \right\}$$
and
$$\varphi_2 : \bar{\mathbb{D}}^*_{w^1} \times \bar{\mathbb{D}}^*_{w^2} \to Z_2 \subset \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$$
be the homeomorphism defined by
$$(z^1, z^2) = \varphi_2(w^1, w^2), \quad z^1 = |w^2| w^1, \ z^2 = w^2.$$
We define
$$v_2 : \bar{\mathbb{D}}^*_{w^1} \times \bar{\mathbb{D}}^*_{w^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{w^1} \times \bar{\mathbb{D}}^*_{w^2}} \right) \times_{\rho'} \tilde{X}$$
by setting v_2 to be the fiber-wise harmonic section with boundary values given by $v_D \circ \varphi_2|_{\mathbb{S}^1_\phi \times \bar{\mathbb{D}}^*_{w^2}}$.

– Let
$$v : \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}} \right) \times_{\rho'} \tilde{X}$$
be the section defined by
$$v = \begin{cases} v_1 \circ \varphi_1^{-1} & \text{on } Z_1, \\ v_2 \circ \varphi_2^{-1} & \text{on } Z_2. \end{cases}$$
Note that v is well defined since $Z_1 \cap Z_2 = D$ and
$$v_1 \circ \varphi_1^{-1}\big|_D = v_D = v_2 \circ \varphi_2^{-1}\big|_D.$$

4.2.2 Derivative estimates in a set of type (B)

On Z_1, since $z^1 = w^1$ and $z^2 = |w^1| w^2$, we have
$$w^1 = z^1 = \rho e^{i\phi}, \quad z^2 = \rho e^{i\theta}, \quad w^2 = \frac{z^2}{|z^1|} = \frac{r}{\rho} e^{i\theta}. \tag{4.4}$$
By (4.3),
$$\left| \frac{\partial h_{3(-\log\rho)^{\frac{1}{3}}}}{\partial \rho}(\theta, \phi) \right|^2 = \left| \frac{\partial h_t}{\partial t}(\theta, \phi) \right|^2 \left| \frac{\partial t}{\partial \rho}(\theta, \phi) \right|^2 \le \frac{1}{\rho^2 (-\log\rho)^{\frac{4}{3}}}.$$
Thus, noting that $w^1 = z^1 = \rho e^{i\phi}$,
$$\left| \frac{\partial (v_D \circ \varphi_1)}{\partial \rho} \right|^2 \le \frac{1}{\rho^2 (-\log\rho)^{\frac{4}{3}}} \text{ on } \mathbb{D}^*_{w^1, \frac{1}{2}} \times \{|w^2| = 1\}.$$
Furthermore, (4.3) implies
$$\left| \frac{\partial (v_D \circ \varphi_1)}{\partial \phi} \right|^2 \le \frac{\mathsf{E}_i}{2\pi} + b e^{-a(-\log\rho)} \le \frac{\mathsf{E}_i}{2\pi} + b\rho^a \ \text{ on } \mathbb{D}^*_{w^1, \frac{1}{2}} \times \{|w^2| = 1\}.$$
Since v_1 is a fiber-wise harmonic section, an argument analogous to the proof of first inequality of Lemma 4.1 (i.e. apply the maximum principle for subharmonic functions $d(u(\rho_1 e^{i\phi}), u(\rho_2 e^{i\phi}))$ and $d(u(\rho e^{i\phi_1}), u(\rho e^{i\phi_2}))$) implies
$$\left| \frac{\partial v_1}{\partial \rho} \right|^2 \le \frac{1}{\rho^2 (-\log\rho)^{\frac{4}{3}}} \ \text{ and } \left| \frac{\partial v_1}{\partial \phi} \right|^2 \le \frac{\mathsf{E}_i}{2\pi} + b\rho^a \quad \text{in } \mathbb{D}^*_{w^1, \frac{1}{2}} \times \mathbb{D}^*_{w^2}.$$
Thus,
$$\left| \frac{\partial v_1}{\partial w^1} \right|^2 \le \frac{1}{\rho^2 (-\log\rho)^{\frac{4}{3}}} + \frac{\mathsf{E}_i}{2\pi\rho^2} + b\rho^{a-2} \quad \text{in } \mathbb{D}^*_{w^1, \frac{1}{2}} \times \mathbb{D}^*_{w^1}. \tag{4.5}$$

Since κ is a Lipschitz map, $v_D \circ \varphi_1$ is a Lipschitz section for $\rho \geq \frac{1}{2}$. Thus,

$$\left|\frac{\partial v_1}{\partial w^1}\right|^2 \leq C \quad \text{in } \mathbb{D}^*_{w^1,\frac{1}{2},1} \times \mathbb{D}^*_{w^2}. \tag{4.6}$$

Furthermore, using the harmonicity of v_1 restricted to the slice $\{w_0^1\} \times \bar{\mathbb{D}}^*_{w^2}$, we have by Theorem 2.16 that

$$\mathsf{E}_j \log \frac{r_0}{r} \leq \int\limits_{\{w_0^1\}\times\mathbb{D}_{r,r_0}} \left|\frac{\partial v_1}{\partial w^2}\right|^2 \frac{dw^2 \wedge d\bar{w}^2}{-2i} \leq C + \mathsf{E}_j \log \frac{r_0}{r} \tag{4.7}$$

for $0 < r < r_0 \leq \frac{1}{4}$.

Lemma 4.2. (Derivative estimates in a set of type (B) away from the juncture).
For $\Omega := \mathbb{D}_{\frac{1}{4},1}$, there exists a constant C such that the following estimates hold:

$$\left|\frac{\partial v}{\partial z^1}(z_0^1, z_0^2)\right| \leq C, \quad \forall (z_0^1, z_0^2) \in \Omega \times \mathbb{D}^*$$

$$E_j \log \frac{r_0}{r} \leq \int\limits_{\{z_0^1\}\times\mathbb{D}_{r,r_0}} \left|\frac{\partial v}{\partial z^2}\right|^2 \frac{dz^2 \wedge d\bar{z}^2}{-2i} \leq C + E_j \log \frac{r_0}{r}$$

where $0 < r < r_0 \leq \frac{1}{4}$, $z_0^1 \in \Omega$ *and* $z^2 = re^{i\theta}$.

Proof. Since $\mathbb{D}_{\frac{1}{4},1} \times \mathbb{D}^*_{\frac{1}{4}} \subset Z_1$, the estimates follow by applying the change of variables (4.4) to estimates (4.5), (4.6), (4.7) and noting that $\rho > \frac{1}{4}$ in Ω. □

Lemma 4.3. (Derivative estimates in a set of type (B) near the juncture).
For v restricted to $\mathbb{D}^*_{\frac{1}{4}} \times \mathbb{D}^*_{\frac{1}{4}}$, there exists a constant C such that the following estimates hold:

$$E_j \log \frac{r_0}{r} \leq \int\limits_{\{z_0^1\}\times\mathbb{D}_{r,r_0}} \left|\frac{\partial v}{\partial z^2}\right|^2 \frac{dz^2 \wedge d\bar{z}^2}{-2i} \leq C + E_j \log \frac{r_0}{r},$$

$$E_i \log \frac{r_0}{r} \leq \int\limits_{\mathbb{D}_{r,r_0}\times\{z_0^2\}} \left|\frac{\partial v}{\partial z^1}\right|^2 \frac{dz^1 \wedge d\bar{z}^1}{-2i} \leq C + E_i \log \frac{r_0}{r}$$

for $z_0^1, z_0^2 \in \mathbb{D}^*_{\frac{1}{4}}$ *and* $0 < r < r_0 \leq \frac{1}{4}$.

Proof. We only prove the first estimate, the second being similar. First, note that the lower bound follows from the definition of E_j. Next, we estimate the upper bound. For $z_0^1 \in \mathbb{D}^*_{\frac{1}{4}}$, we have the inclusion $\{z_0^1\} \times \mathbb{D}_{r,r_0} \subset Z_1$ whenever $r_0 \leq |z_0^1|$. Thus, the change of variables $w^2 \mapsto z^2 = |w^1|w^2$ in (4.7) yields

$$\int\limits_{\{z_0^1\}\times\mathbb{D}_{r,r_0}} \left|\frac{\partial v}{\partial z^2}\right|^2 \frac{dz^2 \wedge d\bar{z}^2}{-2i} \leq C + \mathsf{E}_j \log \frac{r_0}{r}$$

for $z_0^1 \in \mathbb{D}^*_{\frac{1}{4}}$ and $0 < r < r_0 \leq |z_0^1|$.

If $r < |z_0^1| < r_0$, we break up the integral into two integrals since the estimate for $\left|\frac{\partial v}{\partial z^2}\right|^2$ is different in Z_2 than in Z_1. Indeed, by an analogous argument to the proof of (4.5), we have

$$\left|\frac{\partial v_2}{\partial w^2}\right|^2 \leq \frac{1}{r^2(-\log r)^{\frac{4}{3}}} + \frac{\mathsf{E}_j}{2\pi r^2} + r^{a-2}$$

After a change of variables $z^1 = |w^2|w^1, z^2 = w^2$,

$$\left|\frac{\partial v}{\partial z^2}\right|^2 \leq \frac{1}{r^2(-\log r)^{\frac{4}{3}}} + \frac{\mathsf{E}_j}{2\pi r^2} + r^{a-2} \quad \text{in } Z_2. \tag{4.8}$$

Thus, we have

$$\begin{aligned}
\int\limits_{\{z_0^1\}\times\mathbb{D}_{r,r_0}} \left|\frac{\partial v}{\partial z^2}\right|^2 \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i} &\leq \int_0^{2\pi}\left(\int_r^{|z_0^1|}\left|\frac{\partial v}{\partial z^2}\right|^2 r\mathrm{d}r + \int_{|z_0^1|}^{r_0}\left|\frac{\partial v}{\partial z^2}\right|^2 r\mathrm{d}r\right)\mathrm{d}\theta \\
&\leq \left(C + \mathsf{E}_j \log\frac{|z_0^1|}{r}\right) + \int_0^{2\pi}\int_{|z_0^1|}^{r_0}\left(\frac{1}{r^2(-\log r)^{\frac{4}{3}}} + \frac{\mathsf{E}_j}{2\pi r^2} + r^{a-2}\right) r\mathrm{d}r\mathrm{d}\theta \\
&\leq C + \mathsf{E}_j \log\frac{r_0}{r}.
\end{aligned}$$

Finally, if $|z_0^1| < r < r_0$, we only use the estimate (4.8). We omit the details. □

4.3 Gluing the maps

Given the homomorphism $\rho\colon \pi_1(M) \to \mathrm{Isom}(\tilde{X})$ of Theorem 1, we will construct a prototype section $v\colon M \to \tilde{M}\times_\rho \tilde{X}$.

Let $P = \{0\} \times \{0\} \in \Sigma_j \cap \Sigma_i$ and $U = \bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$ be a set of type (B) with

$$\bar{\mathbb{D}}_{z^1} \simeq \bar{\mathbb{D}}_{z^1} \times \{0\} \subset \Sigma_j \quad\text{and}\quad \bar{\mathbb{D}}_{z^2} \simeq \{0\} \times \bar{\mathbb{D}}_{z^2} \subset \Sigma_i.$$

The identification of the product space $\bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$ as a subset of M is simultaneously induced by the local trivializations of the disk bundles $\pi_j\colon \bar{D}_j \to \Sigma_j$ and $\pi_i\colon \bar{D}_i \to \Sigma_i$ via

$$\pi_j^{-1}(\bar{\mathbb{D}}_{z^1}) \simeq \bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2} \quad\text{and}\quad \pi_i^{-1}(\bar{\mathbb{D}}_{z^2}) \simeq \bar{\mathbb{D}}_{z^1} \times \bar{\mathbb{D}}_{z^2}$$

(cf. (3.11)).

Recall the following items associated with the set $U := \mathbb{D}_{z^1} \times \mathbb{D}_{z^2}$.

- $\left[\mathbb{S}^1_{\theta^k}\right]$ is the element of $\pi_1\left(\bar{\mathbb{D}}^*_{z^k}\right)$ generated by $\mathbb{S}^1_{\theta^k}$ for $k = 1, 2$
- $\left[\mathbb{S}^1_{\theta^1}\right]$ and $\left[\mathbb{S}^1_{\theta^2}\right]$ also are the elements of $\pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}\right)$ generated by $\mathbb{S}^1_{\theta^1} \simeq \mathbb{S}^1_{\theta^1} \times \{z^2\}$ and $\mathbb{S}^1_{\theta^1} \simeq \{z^1\} \times \mathbb{S}^1_{\theta^2}$ respectively
- $\pi_1\left(\bar{\mathbb{D}}^*_{z^k}\right)$ is identified with its image in $\pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}\right)$ for $k = 1, 2$ as a subgroup
- $\rho'\colon \pi_1\left(\bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^1}\right) \to \mathrm{Isom}(\tilde{X})$ is defined as $\rho \circ \imath_*$ where $\imath_*$ is the induced map by the inclusion and $\rho'_k = \rho'|_{\pi_1\left(\bar{\mathbb{D}}^*_{z^k}\right)}$ for $k = 1, 2$
- $\left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right)\times_{\rho'} \tilde{X} \to \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$ and $\widetilde{\bar{\mathbb{D}}^*_{z^k}}\times_{\rho'_k} \tilde{X} \to \bar{\mathbb{D}}^*_{z^k}$ for $k = 1, 2$ are fiber bundles.

Next, let $V = \Omega \times \bar{\mathbb{D}}_{z_2}$ be a set of type (A) with

$$\Omega \simeq \Omega \times \{0\} \subset \Sigma_j.$$

The identification $V \simeq \Omega \times \bar{\mathbb{D}}^*_{z^2}$ is induced by the local trivialization

$$\pi_j^{-1}(\Omega) \simeq \Omega \times \bar{\mathbb{D}}_{z^2}$$

of the bundle $\pi_j\colon \bar{\mathcal{D}}_j \to \Sigma_j$ (cf. (3.9)). If $U \cap V \neq \emptyset$ (and hence $\Omega \cap \bar{\mathbb{D}}_{z^1} \neq \emptyset$), then the transition function of the disk bundle $\pi_j\colon \bar{\mathcal{D}}_j \to \Sigma_j$ defines a smooth map

$$\tau\colon \Omega \cap \bar{\mathbb{D}}_{z^1} \to U(1).$$

By [14, Proposition 2.6.1], there exists a locally Lipschitz section $k\colon M \to \tilde{M} \times_\rho \tilde{X}$ of the fiber bundle $\tilde{M} \times_\rho \tilde{X} \to M$. Let k_U be the lift to $\left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho'} \tilde{X}$ of the restriction of k to $U^* := \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2}$ and let

$$v_U\colon U^* := \bar{\mathbb{D}}^*_{z^1} \times \bar{\mathbb{D}}^*_{z^2} \to \left(\widetilde{\bar{\mathbb{D}}^*_{z^1}} \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho'} \tilde{X}$$

be the local prototype section defined in Section 4.2. The composition of v_U and the quotient map $\tilde{M} \times_{\rho'} \tilde{X} \to \tilde{M} \times_\rho \tilde{X}$ defines a section of $\tilde{M} \times_\rho \tilde{X} \to U$ which we call again v_U.

Also let k_V be the lift to $\left(\Omega \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho_2'} \tilde{X}$ of the restriction of k to $V^* = \Omega \times \bar{\mathbb{D}}^*_{z^2}$ and let

$$v_V\colon V^* := \Omega \times \bar{\mathbb{D}}^*_{z^2} \to \left(\Omega \times \widetilde{\bar{\mathbb{D}}^*_{z^2}}\right) \times_{\rho_2'} \tilde{X}$$

be the local prototype section defined in Section 4.1.1. The composition of v_V and the quotient map $\tilde{M} \times_{\rho_2'} \tilde{X} \to \tilde{M} \times_\rho \tilde{X}$ defines a section of $\tilde{M} \times_\rho \tilde{X} \to V$ which we call again v_V.

We claim that we can glue these local sections together to define v in $U \cup V$. To do so, we have to show the following.

Lemma 4.4. *If U and V are sets of type (B) and (A) respectively, then the sections v_U and v_V agree on $U^* \cap V^*$.*

Proof. For $p \in \bar{\mathbb{D}}_{z^1} \cap \Omega$, let $v_{U,p}, k_{U,p}$ be the restrictions of v_U, k_U respectively to $\{p\} \times \bar{\mathbb{D}}^*_{z^2}$ and $v_{V,p}, k_{V,p}$ be the restrictions of v_V, k_V respectively to $\{p\} \times \bar{\mathbb{D}}^*_{z^2}$. We claim that the harmonic sections $v_{U,p}$ and $v_{V,p}$ are related by the transition relation

$$v_{U,p} = v_{V,p} \circ \tau(p).$$

Indeed, since multiplication by $\tau(p) \in U(1)$ is a conformal map, $v_{V,p} \circ \tau(p)$ is harmonic on $\bar{\mathbb{D}}^*$ with boundary values $k_{U,p} = k_{V,p} \circ \tau(p)\big|_{\partial\mathbb{D} \simeq \{p\} \times \mathbb{S}^1_\theta}$. The assertion follows from the uniqueness of harmonic maps (cf. Theorem 2.16). ☐

Similar construction holds for two sets of type (A).

Lemma 4.5. *If U and V are both of type (A) and $U \cap V \neq \emptyset$, then v_U and v_V agree on $U^* \cap V^*$.*

Proof. Apply the same argument as Lemma 4.4. ☐

Let $\mathcal{U}$ be a finite open cover of $\bar{\mathcal{D}}$ by sets of type (A) and of type (B). Let $\mathcal{U}_A \subset \mathcal{U}$ be the collection of sets of type (A) and $\mathcal{U}_B \subset \mathcal{U}$ be the collections of sets of type (B). Without the loss of generality, we can assume $\mathcal{U}_B$ is a collection of disjoint sets. Set

$$v = \begin{cases} v_U & \text{in } U \cap \mathcal{D}^*_{\frac{1}{4}} \text{ where } U \in \mathcal{U}_B \\ v_V & \text{in } \left(V \cap \mathcal{D}^*_{\frac{1}{4}}\right) \setminus \bigcup\limits_{U \in \mathcal{U}_B} U \text{ where } V \in \mathcal{U}_A \end{cases} \tag{4.9}$$

and extend to the rest of M as a well-defined, locally Lipschitz global section of $\tilde{M} \times_\rho \tilde{X} \to M$.

Definition 4.6. The map

$$v: M \to \tilde{M} \times_\rho \tilde{X} \tag{4.10}$$

constructed above is called the prototype section. The corresponding ρ-equivariant map $\tilde{v}: \tilde{M} \to \tilde{X}$ is called the prototype map.

5 Energy estimates of the prototype section

The goal of this section is to obtain the energy estimates of the prototype section

$$v: M \to \tilde{M} \times_\rho \tilde{X}$$

of Definition 4.6 with respect to the Poincaré-type metric g given by Definition 3.4. Throughout this section, we use C to denote constants that are independent of the distance to the divisor. (Note that C may change from line to line.)

We consider the following three types of sets intersecting the divisor $\Sigma \subset \bar{M}$:

(i) $\Omega \times \bar{\mathbb{D}}_{\frac{1}{4}} \subset \Omega \times \bar{\mathbb{D}}$, a subset in a set of type (A)

(ii) $\Omega \times \bar{\mathbb{D}}_{\frac{1}{4}} := \mathbb{D}_{\frac{1}{4},1} \times \bar{\mathbb{D}}_{\frac{1}{4}} \subset \bar{\mathbb{D}} \times \bar{\mathbb{D}}$, a subset in a set of type (B) away from the crossing (cf. Figure 2)

(iii) $\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}} \subset \bar{\mathbb{D}} \times \bar{\mathbb{D}}$, a subset in a set of type (B) at the crossing (cf. Figure 2)

A neighborhood of Σ can be covered by a *finite collection of sets* of the above type. In order to estimate the energy of v, we will compute its energy in each such set.

5.1 Energy in a set of type (A)

In this subsection we will use the following notation in addition to the one used in Section 4.1.

- $\Omega \times \mathbb{D}_{r_1,r_2}$ is the subset of $\Omega \times \bar{\mathbb{D}}$ with $0 < r_1 < |z^2| < r_2 < \frac{1}{4}$.
- g_{Σ_j} is the smooth metric on Ω as defined in Definition 3.5
- $\mathrm{Area}_{g_{\Sigma_j}}$ is the area with respect to g_{Σ_j}
- P is the product metric on $\Omega \times \bar{\mathbb{D}}^*$ defined by (3.26).

Note that

$$\mathcal{D}_{r_1,r_2} \cap (\Omega \times \bar{\mathbb{D}}) = \Omega \times \mathbb{D}_{r_1,r_2} \quad \text{(cf. (3.6))}.$$

The strategy for estimating the energy of the prototype section v in $\Omega \times \bar{\mathbb{D}}$ will be to first compute the energy of v with respect to the product metric P (cf. (3.26)). Since the metrics P and g are close (cf. (3.27)), this will give us the estimate of the energy of v with respect to g.

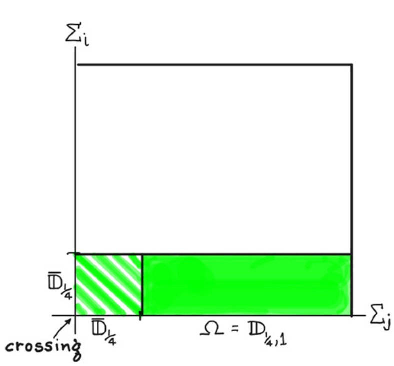

Figure 2: Subsets of a set of type (B).

Lemma 5.1. *For a subset $\Omega \times \bar{\mathbb{D}}$ of a set of type (A), there exists a constant $C > 0$ such that the energy with respect to the metric P of the prototype section v satisfies*

$$0 \leq {}^P E^v[\Omega \times \mathbb{D}_{r_1,r_2}] - E_j Area_{g_{\Sigma_j}}(\Omega) \log \frac{r_2}{r_1} \leq C, \qquad 0 < r_1 < r_2 \leq \frac{1}{4}.$$

Proof. By (3.29),

$$\begin{aligned}{}^P E^v[\Omega \times \mathbb{D}_{r_1,r_2}] &= \int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} P^{1\bar{1}}\left|\frac{\partial v}{\partial z^1}\right|^2 + P^{2\bar{2}}\left|\frac{\partial v}{\partial z^2}\right|^2 \mathrm{dvol}_P \\ &= \int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} P^{1\bar{1}}\left|\frac{\partial v}{\partial z^1}\right|^2 \mathrm{dvol}_P + \int\limits_{\Omega}\left(\int\limits_{\mathbb{D}_{r_1,r_2}}\left|\frac{\partial v}{\partial z^2}\right|^2 \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i}\right)\mathrm{dvol}_{g_{\Sigma_j}},\end{aligned}$$

hence the inequality on the right follows from Lemma 4.1 (cf. Remark 3.8).

By the definition of E_j,

$$\begin{aligned}\mathsf{E}_j \mathrm{Area}_{g_{\Sigma_j}}(\Omega) \log\frac{r_2}{r_1} &= \int\limits_{\Omega}\int\limits_{r_1}^{r_2} \mathsf{E}_j \frac{\mathrm{d}r}{r}\mathrm{dvol}_{g_{\Sigma_j}} \\ &\leq \int\limits_{\Omega}\left(\int\limits_{\mathbb{D}_{r_1,r_2}}\left|\frac{\partial v}{\partial \theta}\right|^2 \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{r}\right)\mathrm{dvol}_{g_{\Sigma_j}} \\ &\leq \int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} P^{1\bar{1}}\left|\frac{\partial v}{\partial z^1}\right|^2 + P^{2\bar{2}}\left|\frac{\partial v}{\partial z^2}\right|^2 \mathrm{dvol}_P \\ &= {}^P E^v[\Omega \times \mathbb{D}_{r_1,r_2}],\end{aligned}$$

which proves the inequality on the left. □

Lemma 5.2. *For a subset $\Omega \times \bar{\mathbb{D}}$ of a set type (A), there exists a constant $C > 0$ such that the prototype section v satisfies*

$$\left|{}^g E^v[\Omega \times \mathbb{D}_{r_1,r_2}] - E_j Area_{g_{\Sigma_j}}(\Omega) \log \frac{r_2}{r_1}\right| \leq C, \quad 0 < r_1 < r_2 \leq \frac{1}{4}.$$

Proof. By Lemma 5.1, it suffices to show that the difference of the energy of v with respect to g and with respect to P is bounded; i.e.

$$\left|{}^g E^v[\Omega \times \mathbb{D}_{r_1,r_2}] - {}^P E^v[\Omega \times \mathbb{D}_{r_1,r_2}]\right| \leq C.$$

We obtain the above estimate with the help of Lemma 5.15 found in the Appendix to this chapter. Therefore we need to first show that the assumption (5.3) of Lemma 5.15 is satisfied; in other words, we need an estimate of the integral of $\frac{1}{r^2}\left|\frac{\partial v}{\partial \theta}\right|^2$. Below, we will derive the estimate (5.3) by bounding the z^1-energy and r-energy of v and then subtracting those from the full energy estimate of Lemma 5.1.

First, to bound the z^1-energy of v, we use estimate $\left|\frac{\partial v}{\partial z^1}\right|^2 \leq C$ of (4.1) to see that

$$\int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} \left|\frac{\partial v}{\partial z^1}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(\log r^2 + A)^2} \leq C.$$

Second, from [6, 7, Proof of Lemma 3.4], we have

$$\int_{\Omega\times\mathbb{D}_{r_1,r_2}} \left|\frac{\partial v}{\partial r}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i} \leq C.$$

Thus, Lemma 5.1 and the identities for $P^{1\bar{1}}\mathrm{dvol}_P$ and $P^{2\bar{2}}\mathrm{dvol}_P$ given by (3.29) imply the following integral estimate on $\frac{1}{r^2}\left|\frac{\partial v}{\partial\theta}\right|^2$:

$$\int_{\Omega\times\mathbb{D}_{r_1,r_2}} \frac{1}{r^2}\left|\frac{\partial v}{\partial \theta}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i} - \mathsf{E}_j \mathrm{Area}_{g_{\Sigma_j}}(\Omega)\log\frac{r_2}{r_1} \leq C.$$

We set $r_2=\frac{1}{4}$ and let $r_1\to 0$ above to obtain

$$\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \left(\left|\frac{\partial v}{\partial \theta}\right|^2 - \mathsf{E}_j\right)\mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{r} \leq C.$$

Noting that g_{Σ_j} is a smooth metric, the assumption (5.3) of Lemma 5.15 is satisfied. Consequently (noting that A appears in the metric expression of P is a bounded function),

$$\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \left|\frac{\partial v}{\partial \theta}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{r(\log r^2+A)^2} \leq C' \tag{5.1}$$

where the constant C' depends only on C.

We use the above estimate to compute the difference between ${}^gE^v[\Omega\times\mathbb{D}_{r_1,r_2}]$ and ${}^PE^v[\Omega\times\mathbb{D}_{r_1,r_2}]$. The trickiest to bound include the following two terms for which we use the estimate (5.1):

- $\left|\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} P^{\theta\theta}\left|\frac{\partial v}{\partial\theta}\right|^2(\mathrm{dvol}_g - \mathrm{dvol}_P)\right|$

  To bound this term, note that since by (3.28)

$$\mathrm{dvol}_P - \mathrm{dvol}_g = O\left(\frac{1}{(-\log r^2+A)^2}\right)\mathrm{dvol}_P = O(1)\mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(-\log r^2+A)^4}$$

and

$$P^{2\bar{2}} = r^2(\log r^2+A)^2.$$

and thus

$$P^{2\bar{2}}(\mathrm{dvol}_P - \mathrm{dvol}_g) = O(1)\mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2i(-\log r^2+A)^2}.$$

Thus,

$$P^{\theta\theta}\big(\mathrm{dvol}_P - \mathrm{dvol}_g\big) = \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(-\log r^2+A)^2} = \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{r(-\log r^2+A)^2}$$

which in turn implies

$$\begin{aligned}\left|\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} P^{\theta\theta}\left|\frac{\partial v}{\partial\theta}\right|^2(\mathrm{dvol}_g - \mathrm{dvol}_P)\right| &\leq C\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \left|\frac{\partial v}{\partial\theta}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{r(-\log r^2+A)^2}\\ &\leq CC'.\end{aligned}$$

- $\left|\int_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \big(g^{\theta\theta} - P^{\theta\theta}\big)\left|\frac{\partial v}{\partial\theta}\right|^2\mathrm{dvol}_P\right|$

  To bound this term, note that since $g^{2\bar{2}} - P^{2\bar{2}} = O(r^2)$, we have

$$g^{\theta\theta} - P^{\theta\theta} = O(1) \text{ and } \mathrm{dvol}_P = \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}z^2\wedge \mathrm{d}\bar{z}^2}{-2ir^2(\log r^2+A)^2}$$

by (3.29). Combining the above, we obtain

$$\left(g^{\theta\theta} - P^{\theta\theta}\right)\mathrm{dvol}_P = O(1)\mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{r(-\log r^2 + A)^2}.$$

Thus

$$\left| \int\limits_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \left(g^{\theta\theta} - P^{\theta\theta}\right)\left|\frac{\partial v}{\partial\theta}\right|^2 \mathrm{dvol}_P \right| \le C \int\limits_{\Omega\times\mathbb{D}_{\frac{1}{4}}} \left|\frac{\partial v}{\partial\theta}\right|^2 \mathrm{dvol}_{g_{\Sigma_j}} \wedge \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{r(-\log r^2 + A)^2} \le CC'.$$

The other terms of $\left|{}^gE^v[\Omega\times\mathbb{D}_{r_1,r_2}] - {}^PE^v[\Omega\times\mathbb{D}_{r_1,r_2}]\right|$ are also bounded by similar computations. We omit the details. □

Lemma 5.3. *For a subset $\Omega\times\bar{\mathbb{D}}$ of a set type (A), there exists a constant $C > 0$ such that the prototype section v satisfies*

$${}^gE^v[\Omega\times\mathbb{D}_{r_1,r_2}] \le {}^gE^f[\Omega\times\mathbb{D}_{r_1,r_2}] + C, \quad 0 < r_1 < r_2 \le \frac{1}{4}$$

for any locally Lipschitz section $f\colon \Omega\times\mathbb{D}_{r_1,r_2} \to \tilde{M}\times_\rho\tilde{X}$.

Proof. Since E_j is the infimum of the energies of sections $\mathbb{S}^1_\theta \to \mathbb{R}_\theta\times_{\rho_2'}\tilde{X}$,

$$\begin{aligned}
\mathsf{E}_j\mathrm{Area}_{g_{\Sigma_j}}(\Omega)\log\frac{r_2}{r_1} &\le \int_0^{2\pi}\left|\frac{\partial f}{\partial\theta}\right|^2\mathrm{d}\theta\int_\Omega \mathrm{dvol}_{g_{\Sigma_j}}\int_{r_1}^{r_2}\frac{\mathrm{d}r}{r}\\
&= \int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}}\left|\frac{\partial f}{\partial\theta}\right|^2\mathrm{dvol}_{g_{\Sigma_j}}\wedge\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r}\\
&\le \int\limits_{\Omega\times\mathbb{D}_{r_1,r_2}} P^{1\bar{1}}\left|\frac{\partial f}{\partial z^1}\right|^2 + P^{2\bar{2}}\left|\frac{\partial f}{\partial z^2}\right|^2 \mathrm{dvol}_P\\
&={}^P E^f[\Omega\times\mathbb{D}_{r_1,r_2}].
\end{aligned} \tag{5.2}$$

Thus, the desired estimate with g replaced by P follows from combining the above estimate with Lemma 5.1. Thus, we are left to show that

$$\left|{}^gE^f[\Omega\times\mathbb{D}_{r_1,r_2}] - {}^PE^f[\Omega\times\mathbb{D}_{r_1,r_2}]\right| \le C.$$

To do so, note that if the inequality

$$\int\limits_{\Omega\times\bar{\mathbb{D}}_{\frac{1}{4}}}\left(\left|\frac{\partial f}{\partial\theta}\right|^2 - \frac{\mathsf{E}_j}{2\pi}\right)\mathrm{dvol}_{g_{\Sigma_j}}\wedge\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r} < \infty$$

does not hold, then we are done since the desired estimate holds by Lemma 5.2. Hence, we can assume the above inequality and apply Lemma 5.15 to conclude

$$\int\limits_{\Omega\times\mathbb{D}_{\frac{1}{4}}}\left|\frac{\partial f}{\partial\theta}\right|^2\mathrm{dvol}_{g_{\Sigma_j}}\wedge\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r(\log r^2 + A)^2} \le C'$$

where the constant C' depends only on C. The rest of the proof is exactly as in the proof of Lemma 5.1. □

5.2 Energy in a set of type (B) away from the crossing

In this subsection we will use the following notation in addition to the one used in Section 4.2.

- $\Omega := \mathbb{D}_{\frac{1}{4},1}$
- $\Omega \times \mathbb{D}_{r_1,r_2}$ is the subset of $\Omega \times \bar{\mathbb{D}} = \mathbb{D}_{\frac{1}{4},1} \times \bar{\mathbb{D}}$ with $0 < r_1 < |z^2| < r_2 < \frac{1}{4}$.
- g_{Σ_j} is the smooth metric on Ω as in Definition 3.5
- $\text{Area}_{g_{\Sigma_j}}$ is the area with respect to g_{Σ_j}
- P is the product metric on $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ defined by (3.31).

Since $\Omega := \mathbb{D}_{\frac{1}{4},1} \subset \mathbb{D}$, the points of $\Omega \times \mathbb{D}_{r_1,r_2}$ are uniformly away from the crossing. In particular, since

$$\frac{1}{4} < \rho \text{ in } \Omega \times \mathbb{D}_{r_1,r_2},$$

the metric expressions of g in a set of type (A) and of type (B) (cf. (3.24) and (3.32) respectively) show that g restricted to $\Omega \times \mathbb{D}_{r_1,r_2}$ in set type (B) has the same asymptotic behavior as $r \to 0$ as g in a set of type (A). Thus, the procedure for obtaining energy estimates of v will be analogous to that in the previous subsection. Note that

$$\mathcal{D}_{r_1,r_2} \cap (\Omega \times \bar{\mathbb{D}})) = \Omega \times \mathbb{D}_{r_1,r_2} \text{ for } 0 < r_1 < r_2 < \frac{1}{4} \text{ (cf. (3.6)).}$$

Lemma 5.4. *For a subset $\Omega \times \mathbb{D}_{r_1,r_2}$ of a set of type (B) (with $\Omega = \mathbb{D}_{\frac{1}{4},1}$), there exists a constant $C > 0$ such that the prototype section v satisfies*

$${}^{P}E^{v}[\Omega \times \mathbb{D}_{r_1,r_2}] \le E_j Area_{g_{\Sigma_j}}(\Omega) \log \frac{r_2}{r_1} + C, \qquad 0 < r_1 < r_2 \le \frac{1}{4}.$$

Proof. Follow the proof of Lemma 5.1 but by replacing (3.29) by (3.36) and Lemma 4.1 by Lemma 4.2. (Note that $\frac{1}{4} < \rho$, i.e. ρ is bounded away from 0, so the expressions in (3.29) and (3.36) are comparable.) □

Lemma 5.5. *For a subset $\Omega \times \mathbb{D}_{r_1,r_2}$ of a set of type (B) (with $\Omega = \mathbb{D}_{\frac{1}{4},1}$), there exists a constant $C > 0$ such that the prototype section v satisfies*

$$\left| {}^{g}E^{v}[\Omega \times \mathbb{D}_{r_1,r_2}] - E_j Area_{g_{\Sigma_j}}(\Omega) \log \frac{r_2}{r_1} \right| \le C, \quad 0 < r_1 < r_2 \le \frac{1}{4}.$$

Proof. Follow the proof of Lemma 5.2 using Lemma 5.4 instead of Lemma 5.1. □

Lemma 5.6. *For a subset $\Omega \times \mathbb{D}_{r_1,r_2}$ of a set of type (B) (with $\Omega = \mathbb{D}_{\frac{1}{4},1}$), there exists a constant $C > 0$ such that the prototype section v satisfies*

$${}^{g}E^{v}[\Omega \times \mathbb{D}_{r_1,r_2}] \le {}^{g}E^{f}[\Omega \times \mathbb{D}_{r_1,r_2}] + C, \quad 0 < r_1 < r_2 \le \frac{1}{4}$$

for any locally Lipschitz section $f : \Omega \times \mathbb{D}_{r_1,r_2} \to \tilde{M} \times_\rho \tilde{X}$.

Proof. Follow the proof of Lemma 5.3 using Lemma 5.5 instead of Lemma 5.2. □

5.3 Energy in a set of type (B) at the crossing

In this subsection we will use the following notation in addition the one used in Section 4.2.

- $\mathbb{D}_{r_1,r_2} \times \mathbb{D}_{r_1,r_2}$ is the subset of $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ with $0 < r_1 < |z^k| < r_2 < \frac{1}{4}$ for $k = 1, 2$.
- $g_{\Sigma_j}, g_{\Sigma_i}$ are as in Definition 3.5
- $\text{Area}_{g_{\Sigma_j}}$ and $\text{Area}_{g_{\Sigma_i}}$ are the areas with respect to g_{Σ_i}
- P is the product metric on $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ defined by (3.31).

The goal is to estimate the energy of v in the set

$$U_{r_1,r_2} = \left(\mathbb{D}_{r_1,\frac{1}{4}} \times \mathbb{D}_{r_1,\frac{1}{4}}\right) \setminus \left(\mathbb{D}_{r_2,\frac{1}{4}} \times \mathbb{D}_{r_2,\frac{1}{4}}\right),$$

pictured in Figure 3. The procedure for doing so involves an extra step compared to the procedure in the previous two subsections. Namely, we will first derive an expression for the energy with respect to the product metric P in the box $\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}$ pictured in Figure 4 (cf. Lemma 5.9 below). Then we take the difference of the energy with respect to P contained in $\mathbb{D}_{r_1,\frac{1}{4}} \times \mathbb{D}_{r_1,\frac{1}{4}}$ and in $\mathbb{D}_{r_2,\frac{1}{4}} \times \mathbb{D}_{r_2,\frac{1}{4}}$ to bound the energy in U_{r_1,r_2} (cf. Lemma 5.10 below). Finally, since the difference between the metric g and P is small, we obtain a bound for the energy with respect to g in U_{r_1,r_2} (cf. Lemma 5.11 below).

Definition 5.7. We define $P_{\Sigma_i}, P_{\Sigma_j}$ to be the restriction of P (cf. (3.31)) to Σ_i, Σ_j respectively.

Remark 5.8. Note that $\mathrm{Area}_{P_{\Sigma_i}}(\mathbb{D}_{r,r'}) = \mathrm{Area}_{P_{\Sigma_j}}(\mathbb{D}_{r,r'})$ for $0 < r < r' \leq \frac{1}{4}$ by the symmetry of P.

Lemma 5.9. *In a subset $\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}$ of a set $\mathbb{D}^* \times \mathbb{D}^*$ of type (B), the prototype section v satisfies*

$$\left| {}^{P}E^{v}\left[\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}\right] - \left(E_j \mathrm{Area}_{P_{\Sigma_j}}\left(\mathbb{D}_{r,\frac{1}{4}}\right) \log \frac{1}{4r} + E_i \mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}_{r,\frac{1}{4}}\right) \log \frac{1}{4r}\right) \right| \leq C$$

for $0 < r < \frac{1}{4}$.

Proof. By Lemma 4.3,

$$\mathsf{E}_i \log \frac{r_0}{r} \leq \int\limits_{\mathbb{D}_{r,r_0} \times \{z_0^2\}} \left|\frac{\partial v}{\partial z^1}\right|^2 \frac{dz^1 \wedge d\bar{z}^1}{-2i} \leq C + \mathsf{E}_i \log \frac{r_0}{r}$$

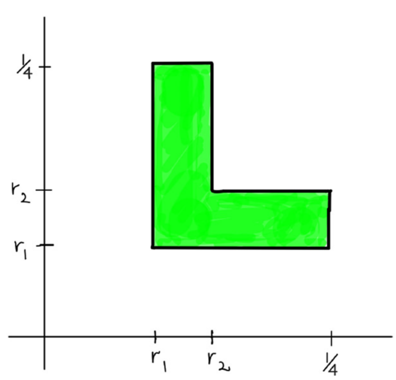

Figure 3: The region U_{r_1,r_2}.

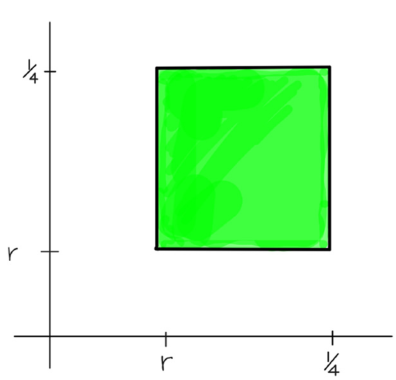

Figure 4: The region $\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}$.

for $z_0^2 \in \mathbb{D}^*_{\frac{1}{4}}$ and $0 < r < r_0$. By (3.36),

$$\int\limits_{\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}} P^{1\bar{1}} \left| \frac{\partial v}{\partial z^1} \right|^2 \mathrm{dvol}_P = \int\limits_{\mathbb{D}_{r,\frac{1}{4}}} \left(\int\limits_{\mathbb{D}_{r,\frac{1}{4}}} \left| \frac{\partial v}{\partial z^1} \right|^2 \frac{\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1}{-2i} \right) \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{-2ir^2(\log r^2)^2}.$$

Combining the above, we obtain

$$\begin{aligned} \mathsf{E}_i \mathrm{Area}_{P_{\Sigma_i}} \left(\mathbb{D}_{r,\frac{1}{4}} \right) \log \frac{1}{4r} &\leq \int\limits_{\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}} P^{1\bar{1}} \left| \frac{\partial v}{\partial z^1} \right|^2 \mathrm{dvol}_P \\ &\leq \mathsf{E}_i \mathrm{Area}_{P_{\Sigma_i}} \left(\mathbb{D}_{r,\frac{1}{4}} \right) \log \frac{1}{4r} + C \end{aligned}$$

and similarly

$$\begin{aligned} \mathsf{E}_j \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r,\frac{1}{4}} \right) \log \frac{1}{4r} &\leq \int\limits_{\mathbb{D}_{r,\frac{1}{4}} \times \mathbb{D}_{r,\frac{1}{4}}} P^{2\bar{2}} \left| \frac{\partial v}{\partial z^2} \right|^2 \mathrm{dvol}_P \\ &\leq \mathsf{E}_j \mathrm{Area}_{P_{\Sigma_i}} \left(\mathbb{D}_{r,\frac{1}{4}} \right) \log \frac{1}{4r} + C. \end{aligned}$$

□

Lemma 5.10. *In a subset $\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}$ of a set of type (B), there exists a constant $C > 0$ such that in any subset (cf.* (3.6)*)*

$$U_{r_1,r_2} = \mathcal{D}_{r_1,r_2} \cap \left(\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}} \right), \quad 0 < r_1 < r_2 \leq \frac{1}{4},$$

the prototype section v satisfies

$$0 \leq {}^P E^v[U_{r_1,r_2}] - E_j Area_{P_{\Sigma_j}} \left(\mathbb{D}^*_{\frac{1}{4}} \right) \log \frac{r_2}{r_1} - E_i Area_{P_{\Sigma_i}} \left(\mathbb{D}^*_{\frac{1}{4}} \right) \log \frac{r_2}{r_1} \leq C.$$

Proof. By a straightforward computation,

$$\begin{aligned} &\mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{1}{4r_1} - \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_2,\frac{1}{4}} \right) \log \frac{1}{4r_2} \\ &\quad = \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{1}{4r_1} - \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{1}{4r_2} + \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{1}{4r_2} - \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_2,\frac{1}{4}} \right) \log \frac{1}{4r_2} \\ &\quad = \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{r_2}{r_1} + \left(\mathrm{Area}_{P_{\Sigma_j}} (\mathbb{D}_{r_1,r_2}) \right) \log \frac{1}{4r_2}. \end{aligned}$$

The second term is bounded by

$$\begin{aligned} 0 \leq \left(\mathrm{Area}_{P_{\Sigma_j}} (\mathbb{D}_{r_1,r_2}) \right) \log \frac{1}{4r_2} &= \left(\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r(\log r^2)^2} \right) \log \frac{1}{4r_2} \\ &= \left(\frac{1}{\log r_2} - \frac{1}{\log r_1} \right) \log 4r_2 \leq 1. \end{aligned}$$

Thus, combining the above equality and the inequality and then multiplying by E_j, we obtain

$$\mathsf{E}_j \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_1,\frac{1}{4}} \right) \log \frac{1}{4r_1} - \mathsf{E}_j \mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}_{r_2,\frac{1}{4}} \right) \log \frac{1}{4r_2} \leq \mathsf{E}_j \left(\mathrm{Area}_{P_{\Sigma_j}} \left(\mathbb{D}^*_{\frac{1}{4}} \right) \log \frac{r_2}{r_1} + 1 \right).$$

Similarly,

$$\mathsf{E}_i \mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}_{r_1,\frac{1}{4}}\right) \log \frac{1}{4r_1} - \mathsf{E}_i \mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}_{r_2,\frac{1}{4}}\right) \log \frac{1}{4r_2} \le \mathsf{E}_i \left(\mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}^*_{\frac{1}{4}}\right) \log \frac{r_2}{r_1} + 1\right).$$

Furthermore, Lemma 5.9 implies

$$\begin{aligned}
{}^P E^v[U_{r_1,r_2}] &= {}^P E^v\left[\mathbb{D}_{r_1,\frac{1}{4}} \times \mathbb{D}_{r_1,\frac{1}{4}}\right] - {}^P E^v\left[\mathbb{D}_{r_2,\frac{1}{4}} \times \mathbb{D}_{r_2,\frac{1}{4}}\right] \\
&\le \mathsf{E}_j \mathrm{Area}_{P_{\Sigma_j}}\left(\mathbb{D}_{r_1,\frac{1}{4}}\right) \log \frac{1}{4r_1} + \mathsf{E}_i \mathrm{Area}_{P_{\Sigma_j}}\left(\mathbb{D}_{r_1,\frac{1}{4}}\right) \log \frac{1}{4r_1} \\
&\quad - \mathsf{E}_j \mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}_{r_2,\frac{1}{4}}\right) \log \frac{1}{4r_2} - \mathsf{E}_i \mathrm{Area}_{P_{\Sigma_i}}\left(\mathbb{D}_{r_2,\frac{1}{4}}\right) \log \frac{1}{4r_2} + C.
\end{aligned}$$

Thus, the desired estimate follows from the fact that $\mathrm{Area}_{P_{\Sigma_i}}(\mathbb{D}_{r,\frac{1}{4}}) = \mathrm{Area}_{P_{\Sigma_j}}(\mathbb{D}_{r,\frac{1}{4}})$ (cf. Remark 5.8). □

Lemma 5.11. *In a subset $\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}$ of a set $\bar{\mathbb{D}}^* \times \bar{\mathbb{D}}^*$ of type (B), there exists a constant $C > 0$ such that in any subset (cf.* (3.6)*)*

$$U_{r_1,r_2} = \mathcal{D}_{r_1,r_2} \cap \left(\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}\right), \quad 0 < r_1 < r_2 \le \frac{1}{4},$$

the prototype section v satisfies

$$\left| {}^g E^v[U_{r_1,r_2}] - E_j \mathrm{Area}_{g_{\Sigma_j}}\left(\mathbb{D}^*_{\frac{1}{4}}\right) \log \frac{r_2}{r_1} - E_i \mathrm{Area}_{g_{\Sigma_i}}\left(\mathbb{D}^*_{\frac{1}{4}}\right) \log \frac{r_2}{r_1} \right| \le C.$$

Proof. Follows from the metric estimate (3.34) and Lemma 5.10. □

Lemma 5.12. *In a subset $\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}$ of a set $\bar{\mathbb{D}}^* \times \bar{\mathbb{D}}^*$ of type (B), there exists a constant $C > 0$ such that in any subset (cf.* (3.6)*)*

$$U_{r_1,r_2} = \mathcal{D}_{r_1,r_2} \cap \left(\bar{\mathbb{D}}_{\frac{1}{4}} \times \bar{\mathbb{D}}_{\frac{1}{4}}\right), \quad 0 < r_1 < r_2 \le \frac{1}{4},$$

the prototype section v satisfies

$${}^g E^v[U_{r_1,r_2}] \le {}^g E^f[U_{r_1,r_2}] + C$$

for any locally Lipschitz section $f\colon U_{r_1,r_2} \to \tilde{M} \times_\rho \tilde{X}$.

Proof. Follow the proof of Lemma 5.3 using Lemma 5.11 instead of Lemma 5.2. □

5.4 Energy estimates for the prototype section near the divisor

Combining the results of the previous three subsections, we obtain the following estimate in an open set $\mathcal{D}_{r_1,r_2}$ (cf. (3.6)) near the divisor.

Proposition 5.13. *There exists a constant $C > 0$ such that the prototype section v of Definition* 4.6 *satisfies*

$$\left| {}^g E^v[\mathcal{D}_{r_1,r_2}] - \sum_{j=1}^{L} E_j \mathrm{Area}_{g_{\Sigma_j}}(\Sigma_j) \log \frac{r_2}{r_1} \right| < C, \quad 0 < r_1 < r_2 \le \frac{1}{4}.$$

Proof. Since we can cover a neighborhood of Σ by a finite collection of sets of type (A) and type (B), the estimate follows from Lemma 5.2, Lemma 5.5 and Lemma 5.11. □

Proposition 5.14. *The section v is almost minimizing in M in the following sense: There exists a constant $C > 0$ such that*

$$^{g}E^{v}[\mathcal{D}_{r_1,r_2}] \leq {}^{g}E^{f}[\mathcal{D}_{r_1,r_2}] + C, \quad 0 < r_1 < r_2 \leq \frac{1}{4}$$

for any section $f\colon M \to \tilde{M}\times_\rho \tilde{X}$.

Proof. Since we can cover a neighborhood of Σ by a finite collection of sets of type (A) and type (B), the estimate, the estimate follows from Lemma 5.3, Lemma 5.6 and Lemma 5.12. □

5.5 Appendix

We conclude this chapter with the following calculus result which was used in the derivation of the energy estimates.

Lemma 5.15. *Let* $\Omega \times \mathbb{D}^*_{\frac{1}{4}}$ *be a subset of* $\Omega \times \bar{\mathbb{D}}$ *of type (A) or* $\Omega \times \bar{\mathbb{D}}^* := \mathbb{D}_{\frac{1}{4},1} \times \bar{\mathbb{D}}$ *of type (B)*$_1$ *with standard product coordinates* $(z^1, z^2 = re^{i\theta})$. *If a locally Lipschitz map* f *defined on* $\Omega \times \mathbb{D}^*_{\frac{1}{4}}$ *satisfies*

$$\int\limits_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial f}{\partial \theta}\right|^2 - c\right) dz^1 \wedge d\bar{z}^1 \wedge \frac{dz^2 \wedge d\bar{z}^2}{r^2} \leq C \tag{5.3}$$

where

$$\int\limits_{\{z^1\}\times\partial\mathbb{D}_r} \left|\frac{\partial f}{\partial \theta}\right|^2 d\theta \geq c \geq 0,$$

then

$$\int\limits_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial f}{\partial \theta}\right|^2 dz^1 \wedge d\bar{z}^1 \wedge \frac{dz^2 \wedge d\bar{z}^2}{r^2(-\log r^2)^2} \leq C'$$

where C' *is a constant depending only on* C *and* c.

Proof. We start with the following claim: For any function $\psi\colon [0, \frac{1}{4}] \to \mathbb{R}$ satisfying $\psi(r) \geq c$,

$$\int\limits_0^{\frac{1}{4}} \psi(r) \frac{dr}{r(\log r^2)^2} \leq c \log 2 + \int\limits_0^{\frac{1}{4}} (\psi(r) - c)\frac{dr}{r}. \tag{5.4}$$

To prove (5.4), first note that $2^{-i-1} \leq r \leq 2^{-i}$ implies

$$\frac{1}{(\log r^2)^2} \leq \frac{1}{(\log 2^{-2i})^2} = \frac{1}{4(\log 2)^2}\frac{1}{i^2}.$$

Furthermore, by the assumption that $\psi(r) \geq c \geq 0$,

$$\int\limits_{2^{-i-1}}^{2^{-i}} \psi(r)\frac{dr}{r} = c\int\limits_{2^{-i-1}}^{2^{-i}} \frac{dr}{r} + \int\limits_{2^{-i-1}}^{2^{-i}} (\psi(r) - c)\frac{dr}{r} \leq c \log 2 + \int\limits_0^{\frac{1}{4}} (\psi(r) - c)\frac{dr}{r}.$$

The above two inequalities imply

$$
\begin{aligned}
\int_0^{\frac{1}{4}} \frac{\psi(r)}{r(\log r^2)^2}\,\mathrm{d}r &= \sum_{i=2}^{\infty} \int_{2^{-i-1}}^{2^{-i}} \frac{\psi(r)}{r(\log r^2)^2}\,\mathrm{d}r \\
&\leq \frac{1}{4(\log 2)^2} \sum_{i=2}^{\infty} \frac{1}{i^2} \int_{2^{-i-1}}^{2^{-i}} \psi(r)\frac{\mathrm{d}r}{r} \\
&= \frac{1}{4(\log 2)^2} \sum_{i=2}^{\infty} \frac{1}{i^2} \cdot \left(c \log 2 + \int_0^{\frac{1}{4}} \big(\psi(r) - c\big)\frac{\mathrm{d}r}{r} \right) \\
&\leq c \log 2 + \int_0^{\frac{1}{4}} \big(\psi(r) - c\big)\frac{\mathrm{d}r}{r}
\end{aligned}
$$

which proves (5.4).

Let

$$
\psi(r) := \int_{\{z^1\}\times\partial\mathbb{D}_r} \left|\frac{\partial f}{\partial\theta}\right|^2 \mathrm{d}\theta.
$$

Since $\psi(r) \geq c$, we have by (5.4) that

$$
\int_0^{\frac{1}{4}} \left(\int_{\{z^1\}\times\partial\mathbb{D}_r} \left|\frac{\partial f}{\partial\theta}\right|^2 \mathrm{d}\theta \right) \frac{\mathrm{d}r}{r(\log r^2)^2} \leq c \log 2 + \int_0^{\frac{1}{4}} \left(\int_{\{z^1\}\times\partial\mathbb{D}_r} \left|\frac{\partial f}{\partial\theta}\right|^2 \mathrm{d}\theta - c \right) \frac{\mathrm{d}r}{r}
$$

for a.e. $z^1 \in \Omega$. Thus,

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{D}_{z^2,\frac{1}{4}}} \left|\frac{\partial f}{\partial\theta}\right|^2 \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} \wedge \frac{r\mathrm{d}r\wedge\mathrm{d}\theta}{r^2(\log r^2)^2} \\
&\quad= \int_\Omega \int_0^{\frac{1}{4}} \left(\int_{\{z^1\}\times\partial\mathbb{D}_r} \left|\frac{\partial f}{\partial\theta}\right|^2 \mathrm{d}\theta \right) \frac{\mathrm{d}r}{r(\log r^2)^2} \wedge \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} \\
&\quad\leq \int_\Omega \left(c \log 2 + \int_0^{\frac{1}{4}} \left(\int_{\{z^1\}\times\mathbb{D}_{z^2,\frac{1}{4}}} \left|\frac{\partial f}{\partial\theta}\right|^2 \mathrm{d}\theta - c \right) \frac{\mathrm{d}r}{r} \right) \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} \\
&\quad= c \log 2 \cdot \int_\Omega \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} + \int_{\Omega\times\bar{\mathbb{D}}_{\frac{1}{4}}} \left(\left|\frac{\partial f}{\partial\theta}\right|^2 - c \right) \frac{\mathrm{d}z^1\wedge \mathrm{d}\bar{z}^1}{-2i} \wedge \frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r}.
\end{aligned}
$$

□

6 Harmonic maps of possibly infinite energy

The goal of this section is to prove Theorem 1, the existence of a harmonic map of logarithmic energy growth. In Section 6.1, we show the existence of a harmonic map with the help of the prototype map. In Section 6.2, we record the energy growth estimates for this map.

Throughout this section, we use C to denote constants that are independent of the parameter r. Note that C may change from line to line.

6.1 Proof of existence, Theorem 1

Proof. For $r \in (0, \frac{1}{4}]$, let $M_r = M \backslash \mathcal{D}_r$ (see Figure 5).

Next, let $v: M \to \tilde{M} \times_\rho \tilde{X}$ be the prototype section of Definition 4.6 and let

$$u_r: M_r \to \tilde{M} \times_\rho \tilde{X}$$

be the energy minimizer among all sections that agree with v on ∂M_r for each $r \in (0, r_1]$. The existence of such a section u_r follows from the proof of [14, Theorem 2.7.2].

Since

$$\begin{aligned}
{}^gE^{u_r}[\mathcal{D}_{r,r_1}] + {}^gE^{u_r}[M_{r_1}] &= {}^gE^{u_r}[M_r] \\
&\leq {}^gE^{v}[M_r] \quad (\text{since } u_r \text{ is minimizing in } M_r) \\
&= {}^gE^{v}[\mathcal{D}_{r,r_1}] + {}^gE^{v}[M_{r_1}] \\
&\leq {}^gE^{u_r}[\mathcal{D}_{r,r_1}] + C + {}^gE^{v}[M_{r_1}] \quad (\text{by Proposition 5.14}),
\end{aligned}$$

we have that

$$ {}^gE^{u_r}[M_{r_1}] \leq {}^gE^{v}[M_{r_1}] + C. \tag{6.1}$$

The right hand side of the inequality (6.1) is independent of the parameter r; i.e. once we fix $r_1 \in (0, \frac{1}{4}]$, the quantity ${}^gE^{u_r}[M_{r_1}]$ is uniformly bounded for all $r \in (0, r_1]$. This implies a uniform Lipschitz bound, say L, of u_r for $r \in (0, r_1]$ in M_{2r_1} (cf. [14, Theorem 2.4.6]).

Let $\tilde{u}_r$ and $\tilde{v}$ be the ρ-equivariant maps corresponding to sections u_r and v. Thus,

$$d(\tilde{u}_r(\lambda(p)), \tilde{u}_r(p)) \leq L d_{\tilde{M}}(\lambda(p), p), \quad p \in M_{2r_1}, \ \lambda \in \Lambda, \quad r \in (0, r_1]$$

where Λ is the finite set of generators used in the definition of *proper*, Definition 2.7. If we let

$$c = L \max\{d_{\tilde{M}}(\lambda(p), p) : \lambda \in \Lambda, \ p \in \overline{M_{2r_1}}\},$$

then by equivariance

$$d(\rho(\lambda)\tilde{u}_r(p), \tilde{u}_r(p))) \leq c, \quad p \in M_{2r_1}, \ \lambda \in \Lambda, \quad r \in (0, r_1].$$

In other words, $\delta(\tilde{u}_r(p)) \leq L$ for all $p \in M_{2r_1}$ and $r \in (0, r_1]$. By the properness of ρ, there exists $P_0 \in \tilde{X}$ and $R_0 > 0$ such that

$$\{\tilde{u}_r(p) : \ p \in M_{2r_1}, \ r \in (0, r_1]\} \subset B_{R_0}(P_0).$$

Thus, following the proof of [15, Theorem 2.1.3], taking a compact exhaustion and applying the usual diagonalization argument, there exists a subsequence of $\tilde{u}_r$ that converges locally uniformly to a ρ-equivariant harmonic map $\tilde{u}: \tilde{M} \to \tilde{X}$. Let $u: M \to \tilde{M} \times_\rho \tilde{X}$ be the corresponding harmonic section. □

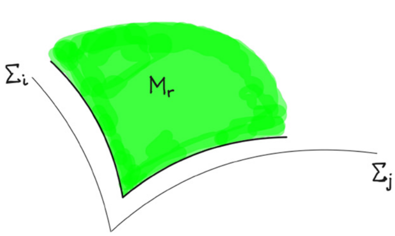

Figure 5: The region $M_r \subset M$.

6.2 Energy estimates for the harmonic section

Lemma 6.1. *For the harmonic section $u\colon M \to \tilde{M} \times_\rho \tilde{X}$ of Theorem 1 and the prototype section of Definition 4.6, we have*

$$^{g}E^{u}[M_{r_1}] \leq {}^{g}E^{v}[M_{r_1}] + C, \quad \forall r_1 \in (0, \tfrac{1}{4}].$$

Proof. Follows from (6.1) and the lower semicontinuity of energy (cf. [14, Lemma 1.6.1]). □

Lemma 6.2. *If $v\colon M \to \tilde{M} \times_\rho \tilde{X}$ is the prototype section of Definition 4.6 and $u\colon M \to \tilde{M} \times_\rho \tilde{X}$ is the harmonic section of Theorem 1, there exists a constant $C > 0$ such that*

$$\left| {}^{g}E^{u}[\mathcal{D}_{r_1,r_2}] - {}^{g}E^{v}[\mathcal{D}_{r_1,r_2}] \right| \leq C, \quad 0 < r_1 < r_2 \leq \frac{1}{4}.$$

Proof. From the fact that $\mathcal{D}_{r_1,\frac{1}{4}} \subset \mathcal{D}_{r_1,\frac{1}{4}} \cup M_{\frac{1}{4}} = M_{r_1}$, Lemma 6.1, and the lower semicontinuity of energy (cf. [14, Theorem 1.6.1]), we obtain

$$^{g}E^{u}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] \leq {}^{g}E^{u}[M_{r_1}] \leq {}^{g}E^{v}[M_{r_1}] + C = {}^{g}E^{v}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] + {}^{g}E^{v}\left[M_{\frac{1}{4}}\right] + C.$$

Proposition 5.14 implies

$$^{g}E^{v}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] \leq {}^{g}E^{u}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] + C.$$

Combining the above two inequalities we obtain

$$\left| {}^{g}E^{u}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] - {}^{g}E^{v}\left[\mathcal{D}_{r_1,\frac{1}{4}}\right] \right| \leq {}^{g}E^{v}\left[M_{\frac{1}{4}}\right] + C$$

and similarly

$$\left| {}^{g}E^{u}\left[\mathcal{D}_{r_2,\frac{1}{4}}\right] - {}^{g}E^{v}\left[\mathcal{D}_{r_2,\frac{1}{4}}\right] \right| \leq {}^{g}E^{v}\left[M_{\frac{1}{4}}\right] + C.$$

The desired estimate follows from the above two inequalities. □

Lemma 6.3. *If $u\colon M \to \tilde{M} \times_\rho \tilde{X}$ is the harmonic section of Theorem 1, then there exists $C > 0$ such that*

$$\left| {}^{g}E^{u}[\mathcal{D}_{r_1,r_2}] - \sum_{j=1}^{L} E_j Area_{g_{\Sigma_j}}(\Sigma_j) \log \frac{r_2}{r_1} \right| \leq C, \quad 0 < r_1 < r_2 \leq \frac{1}{4}.$$

Proof. The estimate follows from Proposition 5.13 and Lemma 6.2. □

Lemma 6.4. *If $u\colon M \to \tilde{M} \times_\rho \tilde{X}$ is the harmonic section of Theorem 1, then we have the following estimates in the subset $\Omega \times \mathbb{D}^*_{\frac{1}{4}}$ of a set $\Omega \times \bar{\mathbb{D}}$ of type (A) or the subset $\Omega \times \bar{\mathbb{D}}^* := \mathbb{D}_{\frac{1}{4},1} \times \bar{\mathbb{D}}^*$ of a set $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ of type (B):*

$$\int_{\Omega \times \bar{\mathbb{D}}^*_{\frac{1}{4}}} \left| \frac{\partial u}{\partial z^1} \right|^2 dz^1 \wedge d\bar{z}^1 \wedge \frac{dz^2 \wedge d\bar{z}^2}{r^2(-\log r^2)^2} < \infty$$

$$\int_{\Omega \times \bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left| \frac{\partial u}{\partial z^2} \right|^2 - \frac{E_j}{8\pi r^2} \right) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 < \infty$$

$$\int_{\Omega \times \bar{\mathbb{D}}^*_{\frac{1}{4}}} \left| \frac{\partial u}{\partial r} \right|^2 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial\theta}\right|^2 - \frac{E_j}{2\pi}\right) dz^1\wedge d\bar{z}^1\wedge \frac{dz^2\wedge d\bar{z}^2}{r^2} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial\theta}\right|^2 dz^1\wedge d\bar{z}^1\wedge \frac{dz^2\wedge d\bar{z}^2}{r^2(-\log r^2)^2} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial z^2}\right|^2 dz^1\wedge d\bar{z}^1\wedge \frac{dz^2\wedge d\bar{z}^2}{(-\log r^2)^2} < \infty$$

where $(z^1, z^2 = re^{i\theta})$ *are the standard product coordinates on* $\Omega\times\bar{\mathbb{D}}$.

Proof. All the estimates except for the last two follow immediately from Lemma 6.3. The last two follow from the other estimates and Lemma 5.15. □

Recall that the standard product coordinates (z^1, z^2) on a set $\Omega\times\bar{\mathbb{D}}$ of type (A) are not necessarily the holomorphic coordinates (z^1,ζ) of Definition 3.6. We will now reframe the statements of Lemma 6.4 in terms of the holomorphic coordinates on the set of type (A). We first need some estimates that compares ζ to z^2.

Lemma 6.5. *If* $(z^1, z^2 = re^{i\theta})$ *and* $(z^1, \zeta = se^{i\eta})$ *are the standard product coordinates and holomorphic coordinates respectively on a set* $\Omega\times\bar{\mathbb{D}}$ *of type (A), then*

$$\begin{aligned} &r = as + O(r^2) & & \\ &\frac{\partial z^2}{\partial\zeta} = a(1+O(r)) & &\frac{\partial \bar{z}^2}{\partial\zeta} = O(r) \\ &\frac{\partial r}{\partial s} = |\alpha|(1+O(r)) & &\frac{\partial \theta}{\partial s} = O(1) \\ &\frac{\partial r}{\partial \eta} = O(r^2) & &\frac{\partial \theta}{\partial \eta} = O(1) \end{aligned}$$

where a is a smooth function both bounded above and bounded away from 0 (cf. (3.18) *and* (3.19)*).*

Proof. Since

$$z^2 = a\zeta + O(r^2)$$

by (3.19), the first estimate follows immediately. Furthermore, differentiating the above with respect to ζ, we obtain the next two estimates. The last four estimates are obtained by evaluating the differential forms of (3.17) on the vector fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial\eta}$ and using the fact that $|\frac{\partial}{\partial s}| = O(1)$, $|\frac{\partial}{\partial \eta}| = O(r^2)$. □

Theorem 6.6. *If* $u\colon M\to\tilde{M}\times_\rho\tilde{X}$ *is the harmonic section of Theorem* 1*, then we have the following estimates in the set* $\Omega\times\mathbb{D}^*_{\frac{1}{4}}$ *away from a crossing (i.e. a subset of a set* $\Omega\times\bar{\mathbb{D}}$ *of type (A) or a subset* $\Omega\times\bar{\mathbb{D}}^* := \mathbb{D}_{\frac{1}{4},1}\times\bar{\mathbb{D}}^*$ *of a set* $\bar{\mathbb{D}}\times\bar{\mathbb{D}}$ *of type (B)),*

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial z^1}\right|^2 dz^1\wedge d\bar{z}^1\wedge \frac{d\zeta\wedge d\bar{\zeta}}{s^2(-\log s^2)^2} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial\zeta}\right|^2 - \frac{E_j}{8\pi s^2}\right) dz^1\wedge d\bar{z}^1\wedge d\zeta\wedge d\bar{\zeta} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial \zeta}\right|^2 \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{(-\log s^2)^2} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial s}\right|^2 \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial \eta}\right|^2 - \frac{E_j}{2\pi}\right) \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{s^2} < \infty$$

$$\int_{\Omega\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial \eta}\right|^2 \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{s^2(-\log s^2)^2} < \infty$$

where $(z^1, \zeta = s e^{\mathrm{i}\eta})$ *are the holomorphic coordinates on* $\Omega \times \bar{\mathbb{D}}$ *(cf. Definition* 3.6*).*

Proof. By Lemma 6.5, we have

$$\begin{aligned}
s^2(\log s^2)^2 &= r^2(\log r^2)O(1),\\
\frac{1}{s^2} &= \frac{|a|}{r^2}(1+O(r)),\\
\frac{\partial u}{\partial \zeta} &= \frac{\partial u}{\partial z^2}\frac{\partial z^2}{\partial \zeta} + \frac{\partial u}{\partial \bar{z}^2}\frac{\partial \bar{z}^2}{\partial \zeta} = \frac{\partial u}{\partial z^2} a(1+O(r)) + \frac{\partial u}{\partial \bar{z}^2} O(r),\\
\frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial \eta} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial \eta} = \frac{\partial u}{\partial r} O(r^2) + \frac{\partial u}{\partial \theta} O(1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\frac{\partial u}{\partial \zeta}\right|^2 - \frac{\mathsf{E}_j}{8\pi s^2} &= |a|^2\left(\left|\frac{\partial u}{\partial z^2}\right|^2 - \frac{\mathsf{E}_j}{8\pi r^2}\right)(1+O(r))\\
\left|\frac{\partial u}{\partial \eta}\right|^2 &= \left|\frac{\partial u}{\partial \theta}\right|^2 O(1) + \left|\frac{\partial u}{\partial r}\right|^2 O(r^2).
\end{aligned}$$

Thus, the second, third and fourth estimates now follow from Lemma 6.4. The first estimate is a restatement of the first estimate of Lemma 6.4. □

Theorem 6.7. *If* $u\colon M \to \tilde{M}\times_\rho \tilde{X}$ *is the harmonic section of Theorem* 1*, then we have the following estimates in the set* $\bar{\mathbb{D}}^*_{\frac{1}{4}} \times \bar{\mathbb{D}}^*_{\frac{1}{4}}$ *at a crossing (i.e. a subset of a set* $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ *of type (B)),*

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial z^1}\right|^2 - \frac{E_i}{8\pi\rho^2}\right) \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{r^2(-\log r^2)^2} < \infty$$

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial z^2}\right|^2 - \frac{E_j}{8\pi r^2}\right) \frac{\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1}{\rho^2(-\log \rho^2)^2} \wedge \mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2 < \infty$$

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial \rho}\right|^2 \mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \frac{\mathrm{d}z^2 \wedge \mathrm{d}\bar{z}^2}{r^2(-\log r^2)^2} < \infty$$

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left|\frac{\partial u}{\partial r}\right|^2 \frac{dz^1\wedge d\bar{z}^1}{\rho^2(-\log\rho^2)^2}\wedge dz^2\wedge d\bar{z}^2 < \infty$$

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial \phi}\right|^2 - \frac{E_j}{2\pi}\right)\frac{dz^1\wedge d\bar{z}^1}{\rho^2}\wedge\frac{dz^2\wedge d\bar{z}^2}{r^2(-\log r^2)^2} < \infty$$

$$\int_{\bar{\mathbb{D}}^*_{\frac{1}{4}}\times\bar{\mathbb{D}}^*_{\frac{1}{4}}} \left(\left|\frac{\partial u}{\partial \theta}\right|^2 - \frac{E_i}{2\pi}\right)\frac{dz^1\wedge d\bar{z}^1}{\rho^2(-\log\rho^2)^2}\wedge\frac{dz^2\wedge d\bar{z}^2}{r^2} < \infty$$

where $(z^1 = \rho e^{i\phi}, z^2 = re^{i\theta})$ *are the holomorphic coordinates on* $\bar{\mathbb{D}}\times\bar{\mathbb{D}}$ *(cf. Definition* 3.2*).*

Proof. The standard product coordinates of a set of type (B) are also the holomorphic coordinates. Thus, these estimates follow immediately from Lemma 5.9 and (3.35). □

7 Generalization to higher dimensions

The construction of harmonic maps from quasi-compact Kähler surfaces generalizes to quasi-compact Kähler manifolds of arbitrary dimension. Indeed, let $M = \bar{M}\backslash\Sigma$ be a n-dimensional quasi-compact Kähler manifold where Σ is a normal crossing divisor. Then, every point x in an irreducible component Σ_j of Σ has a neighborhood U which can be written in holomorphic coordinates as $U = \mathbb{D}^{n-k}\times\mathbb{D}^{*k}$. A neighborhood $\mathbb{D}^{n-1}\times\mathbb{D}^*$ (resp. $\mathbb{D}^{n-2}\times\mathbb{D}^{*2}$) is analogous to a neighborhood of type (A) (resp. type (B)) defined above. Thus, the prototype map is defined analogously in those neighborhoods. We can also define the prototype map in other neighborhoods using an inductive argument. Once the prototype map is constructed, the existence of harmonic maps follows as in the two-dimensional case.

In our upcoming papers, we will only use the two-dimensional case. More precisely, we combine Theorem 1 with an inductive argument due to Mochizuki (cf. [7]) to deduce the existence of pluriharmonic maps in any dimension in the quasi-projective case. This is why we gave the details only for Kähler surface domains.

Acknowledgments: The authors would like to thank Y. Deng, T. Mochizuki and Y. Siu for illuminating discussions.
Research ethics: Not applicable.
Author contributions: The authors have accepted responsibility for the entire content of this manuscript and approved its submission.
Competing interests: The authors state no conflict of interest.
Research funding: Georgios Daskalopoulos supported in part by NSF DMS-2105226 and a Simons collaboration grant, Chikako Mese supported in part by NSF DMS-2304697.
Data availability: Not applicable.

References

[1] J. Lohkamp, "An existence theorem for harmonic maps," *Manuscripta Math.*, vol. 67, no. 1, pp. 21–23, 1990.
[2] M. Wolf, "Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli spaces," *J. Diff. Geom.*, vol. 33, no. 2, pp. 487–539, 1991.
[3] S. Gupta and M. Wolf, "Quadratic differentials, half-plane structures, and harmonic maps to trees," *Comm. Math. Helv.*, vol. 91, no. 2, pp. 317–356, 2016.
[4] J. Jost and K. Zuo, "Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasi-projective varieties," *J. Diff. Geom.*, vol. 47, no. 3, pp. 469–503, 1997.

[5] K. Zuo, "Representations of fundamental groups of algebraic varieties," in *Lecture Notes in Mathematics*, vol. 1708, Berlin, Heidelberg, Springer, 1999.
[6] D. Brotbek, G. Daskalopoulos, Y. Deng and C. Mese. "Pluriharmonic maps into buildings and symmetric differentials." arXiv:2206.11835.
[7] T. Mochizuki, "Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules," *Memoirs AMS*, vol. 185, no. 870, p. 0, 2007.
[8] S. Donaldson, "Twisted harmonic maps and the self-duality equations," *Proc. London Math. Soc.*, vol. 55, no. 1, pp. 127—131, 1987.
[9] K. Corlette, "Archimedian superrigidity and hyperbolic geometry," *Ann. Math.*, vol. 135, no. 1, pp. 165—182, 1990.
[10] G. Daskalopoulos and C. Mese, "Notes on harmonic maps." Preprint arXiv:2301.04190.
[11] G. Daskalopoulos and C. Mese, "Infinite energy harmonic maps from Riemann surfaces to CAT(0) spaces," *J. Geom. Anal*.
[12] M. Cornalba and P. Griffiths, "Analytic cycles and vector bundles on noncompact algebraic varieties," *Invent. Math.*, vol. 28, no. 1, pp. 1—106, 1975.
[13] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Berlin, Springer-Verlag, 1999.
[14] N. Korevaar and R. Schoen, "Sobolev spaces and harmonic maps for metric space targets," *Comm. Anal. Geom.*, vol. 1, no. 4, pp. 561—659, 1993.
[15] N. Korevaar and R. Schoen, "Global existence theorem for harmonic maps to non-locally compact spaces," *Comm. Anal. Geom.*, vol. 5, no. 2, pp. 333—387, 1997.