Many-body quantum geometric dipole

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Collective excitations of many-body electron systems can carry internal structure, tied to the quantum geometry of the Hilbert space in which they are embedded. This has been shown explicitly for particle-hole-like excitations, which carry a "quantum geometric dipole" (QGD) that is essentially an electric dipole moment associated with the state. We demonstrate in this work that this property can be formulated in a generic way, which does not require wave functions expressed in terms of single particle-hole states. Our formulation exploits the density matrix associated with a branch of excitations that evolves continuously with its momentum **K**, from which one may extract single-particle states allowing a construction of the QGD. We demonstrate the formulation using the single-mode approximation for excited states of two quantum Hall systems: the first for an integrally filled Landau level, and the second for a fractional quantum Hall state at filling factor $\nu = 1/m$, with m an odd integer. In both cases we obtain the same result for the QGD, which can be attributed to the translational invariance assumed of the system. Our study demonstrates that the QGD is an intrinsic property of collective modes which is valid beyond approximations one might make for their wave functions.

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I. INTRODUCTION

In recent years the relevance of quantum geometry to condensed matter physics has become increasingly appreciated. Its realization via Berry phases of single-particle states can be used to understand the quantized Hall effect [1–3], as well as related states with quantized linear response [4–8]. More generally, quantum geometry makes its presence felt in single-particle transport in a variety of electron systems [9–13]. Nonlinear transport can also be understood within quantum geometric frameworks [14–18]. There are also important impacts of the quantum geometry of single-particle states on collective ground states, most prominently ones involving superconductivity or superfluidity [19–36].

The quantum geometry of single-particle states can also impact properties associated with collective modes in a system. For example, for insulating bands with nontrivial Berry curvature, the spectrum of exciton energies deviates from the usual simple hydrogenic form [37,38]. In metallic systems, the single-particle geometry impacts plasmon dynamics [39,40]. For such collective modes, however, quantum geometry impacts the system in a more direct way. In particular, for systems with translational symmetry, collective modes carry a momentum quantum number, and the continuous evolution of the excited states with this momentum label allows other possibilities to characterize the geometry and topology of the Hilbert space in which they are embedded [41,42]. For example, excitons, bound states of a particle and hole, usually in a band structure, carry an intrinsic Berry curvature, distinct from those of the single-particle bands hosting the constituents [41,42].

The fact that collective modes host internal structure without analog in single-particle states suggests that they may have unique quantum geometric properties. Recently, this has been demonstrated for neutral excitations that can be described by particle-hole pairs. Specifically, one finds an intrinsic dipole moment that can be understood in terms of the collective mode wave-function evolution with its momentum. This *quantum geometric dipole* (QGD) is a natural property of excitons [43,44], and a direct consequence for two-dimensional systems is that an applied electric field $\boldsymbol{\mathcal{E}}$ induces exciton drift perpendicular to that field, analogous to the $\boldsymbol{\mathcal{E}}\times\mathbf{B}$ drift one expects for charged particles in a magnetic field **B** [43,45]. An analogous QGD may be present for two-dimensional plasmons, which can be described as particle-hole excitations across a Fermi surface [46]. The QGD for these excitations results in an asymmetry of scattering from circularly symmetric scattering potentials that is present only when the QGD is nonvanishing [47]. In quasi-one-dimensional systems, plasmons may also carry a transverse dipole moment which is closely related to the corresponding two-dimensional QGD in the large wire width limit [48]. Interestingly, such transverse dipole moments can also appear in excitons of quasi-one-dimensional insulators [49].

The two-body forms for wave functions of these types of excitations are convenient, and in many cases accurate, approximations to their true many-body state. Nevertheless, in almost all cases exact wave functions will contain corrections that involve states with more than just a single particle-hole pair. Moreover, some systems, particularly highly correlated ones, may possess excitations with well-defined momenta that are not well described in terms of an effective two-body state. This raises the question of whether the QGD is a well-defined concept which can be applied to more general wave-function forms. In what follows, we demonstrate that indeed it is. We introduce a formalism allowing a computation of the quantum

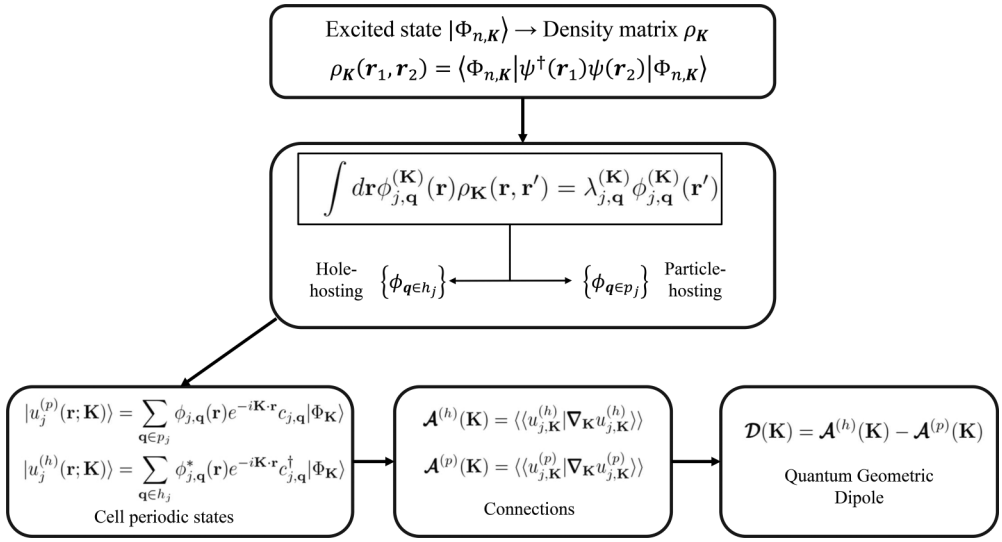

FIG. 1. Basic steps followed to find the QGD for a band n of many-body states $\{\Phi_{n,\mathbf{K}}\}$ with momenta $\mathbf{K}$. One begins by computing a $\mathbf{K}$-dependent density matrix $\rho_{\mathbf{K}}$, whose single-particle eigenstates $\phi_{j,\mathbf{q}}$ are divided into two groups, hole hosting (h_j) and particle hosting (p_j). As continuous functions of $\mathbf{q}$, these are denoted as DM bands. Cell-periodic functions of each type are then constructed for each DM band j, and from these one can define connections associated with holes [$\boldsymbol{\mathcal{A}}^{(h)}(\mathbf{K})$] and with particle [$\boldsymbol{\mathcal{A}}^{(p)}(\mathbf{K})$]. Their discrete difference defines the quantum geometric dipole $\boldsymbol{\mathcal{D}}(\mathbf{K})$. Details of each step are discussed in the main text.

geometric dipole of a many-body state with momentum $\mathbf{K}$ of *generic* form. Briefly, the formalism involves using the density operator associated with the system state for each $\mathbf{K}$ to define a set of single-particle states. These states may be divided into two groups, one which we call *particle hosting*, and the other *hole hosting*, with some flexibility in precisely how states are assigned to each group. With these two sets of states defined, one then defines connections $\mathcal{A}^{(p)}(\mathbf{K})$ and $\mathcal{A}^{(h)}(\mathbf{K})$ associated with particles and holes, respectively, which are analogs of the Berry connection for single-particle band states [50]. Their discrete difference represents the quantum geometric dipole $\boldsymbol{\mathcal{D}}(\mathbf{K})$, and we will show this has the natural interpretation of an internal dipole moment for the many-body state. Figure 1 summarizes the basic steps we use to define the quantum geometric dipole.

To demonstrate the validity of the approach, we consider two explicit examples, in each case computing the QGD of an excitation mode above a quantum Hall state [3,51]. These may be found in two-dimensional electron gas systems immersed in a perpendicular magnetic field $B\hat{z}$. The first of our examples is the low-lying magnetoexciton mode above an integrally filled Landau level. In the strong field limit, this is well described by states involving a single electron in an otherwise empty Landau level, bound to a single hole in an otherwise filled Landau level [52–54]. In this limit, a mode of momentum $\hbar\mathbf{K}$ carries an intrinsic dipole moment $\mathbf{p} = e\mathbf{K}\times\hat{z}\ell^2$, where $\ell = \sqrt{\hbar c/eB}$, with c the speed of light and e the charge of an electron [52–54]. This has been shown to be consistent with the QGD of this neutral mode, for which one finds $\boldsymbol{\mathcal{D}} = \mathbf{K}\times\hat{z}\ell^2$ [43]. Moreover, this value of $\boldsymbol{\mathcal{D}}$ turns out to be precisely what is needed for the exciton equations of motion to be effectively Lorentz invariant [43]. For this study, we adopt a different form for the magnetoexciton wave functions, specifically one generated using the single-mode approximation (SMA) [55–57]. In this case the state is not restricted to a single pair of Landau levels, and so is not limited to strong fields, but it does fall into the paradigm of (a linear combination of) single particle-hole pair states. We find that our many-body approach to the QGD produces exactly the same result as expected in the strong field case ($\boldsymbol{\mathcal{D}} = \mathbf{K}\times\hat{z}\ell^2$), as should be expected since the effective Lorentz invariance in this problem does not require a strong magnetic field.

Our second example involves magnetoplasmon excitations above a partially filled lowest Landau level in a strong magnetic field, with filling factor (defined as the ratio of electron density to magnetic flux density in the electron gas) $\nu = 1/m$, with m an odd integer. For $m = 3$ and 5 such systems are well known to support the fractional quantum Hall effect (FQHE) [3,51,58]. Their ground states are qualitatively well described by Laughlin-Jastrow wave functions [3,51,59,60], with charged excitations of $\pm e/m$. The states may be qualitatively understood in terms of composite fermion theory [61], in which a singular gauge transformation attaches flux quanta to electrons (yielding "composite fermions"), such that at the mean-field level, the ratio of particle density to magnetic flux density is an integer. The state may then be understood in terms of integrally filled Landau levels of composite fermions [61]. This suggests a connection between the magnetoplasmons of the fractionally filled system and the magnetoexcitons of the integrally filled one. With the reduced magnetic flux, the effective magnetic length ℓ^* satisfies $\ell^{*2} = m\ell^2$. These two observations suggest some tension between the reduced charge of the quasiparticles relative to the integer case, which presumably lowers the dipole moment of a collective mode, and the smaller effective magnetic field, which tends to raise it.

The resolution of this tension can be found by direct calculation of the QGD, $\boldsymbol{\mathcal{D}}$. To do this we use approximate wave functions for the magnetoplasmons, generated using the

SMA [56,57]. The computation of $\mathcal{D}$ for these states is more involved than in the integral case, but, as we show below, it may be carried through without further approximation beyond the wave functions themselves. The final result is remarkably simple, and is in fact identical to the result for integrally filled Landau levels. This shows that the two effects discussed above essentially cancel against one another.

While this precise cancellation is at first surprising, it is in fact necessary that it should happen. We show this by demonstrating that *any* state with well-defined momentum $\hbar\mathbf{K}$ which lies fully in the lowest Landau must have an electric dipole moment perpendicular to that momentum, with magnitude $eK\ell^2$, where ℓ is the magnetic length associated with the physical magnetic field. Thus, our many-body approach to the QGD produces a correct result for the FQHE example. This suggests that our formulation indeed gives physically sensible results for states that cannot be described within a simple single particle-hole paradigm. More generally, we see an internal structure associated with the state which can be understood quantum geometrically, without *a priori* assumptions that its wave function has a particular form.

This paper is organized as follows. In Sec. II, we explain in detail our method for computing the QGD of a many-body state of generic form. Section III is devoted to our example of the magnetoexciton excitation above a filled Landau level. In Sec. IV, we present our analysis of the QGD for a fractional quantum Hall state at filling factor $\nu = 1/m$ with m an odd integer, using excited states generated by the single-mode approximation. While the details of this turn out to be involved, the final result is quite simple. We show why this is the case in Sec. V. Section VI summarizes our study, and discusses possible future directions for this work. Our paper also has three Appendixes. Appendix A presents a study of a possible offset term that appears when connecting the quantum geometric dipole to the physical electric dipole moment. We argue that this is essentially a constant term for low-energy excitations, which in many interesting cases vanishes. Appendix B presents some details of the QGD calculation for the magnetoexciton excitation above a filled Landau level, and Appendix C for the fractional quantum Hall magnetoplasmon.

II. MANY-BODY QUANTUM GEOMETRIC DIPOLE

Our formal development of the quantum geometric dipole (QGD) assumes the Hamiltonian commutes with some set of translation operators $T_{\mathbf{a}_i}$, $i = 1, \ldots, D$, where D is the dimensionality of the system, $\mathbf{a}_i$ are primitive lattice vectors, and $[T_{\mathbf{a}_i}, T_{\mathbf{a}_j}] = 0$. In this situation eigenstates of the Hamiltonian $|\Phi_{n,\mathbf{K}}\rangle$ may be labeled by a momentum $\mathbf{K}$ (we herein set $\hbar = 1$), for which $T_{\mathbf{a}_i}|\Phi_{n,\mathbf{K}}\rangle = e^{i\mathbf{K}\cdot\mathbf{a}_i}|\Phi_{n,\mathbf{K}}\rangle$. We assume $\Phi_n(\mathbf{K})$ varies continuously with $\mathbf{K}$, and the index n provides any further quantum numbers needed to specify the state. The states are normalized as $\langle\Phi_{n,\mathbf{K}}|\Phi_{n',\mathbf{K}'}\rangle = \delta(\mathbf{K}-\mathbf{K}')\delta_{n,n'}$. For simplicity, in this work we focus on Hamiltonians and eigenstates of spinless fermions.

For fixed n, from these states we can form as set of density matrices $\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}') = \langle\Phi_{n,\mathbf{K}}|\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')|\Phi_{n,\mathbf{K}}\rangle$, where $\psi(\mathbf{r})$ is an annihilation field operator. (Since we will always work within a fixed n subspace in what follows, for ease of notation we hereon suppress the index n. Thus, the many-body states will be denoted as $|\Phi_{\mathbf{K}}\rangle$.) The density matrices have eigenvalues and eigenfunctions

$$\int d\mathbf{r}\,\phi^{(\mathbf{K})}_{j,\mathbf{q}}(\mathbf{r})\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}') = \lambda^{(\mathbf{K})}_{j,\mathbf{q}}\phi^{(\mathbf{K})}_{j,\mathbf{q}}(\mathbf{r}').$$

In writing this, we have noted that the translation operators $T_{\mathbf{a}_i}$ commute with $\rho_{\mathbf{K}}$, so that the eigenfunctions and eigenvalues of the latter can be labeled by a wave vector $\mathbf{q}$, which in general must be contained within the Brillouin zone associated with the primitive lattice vectors. The index j labels different discrete eigenstates of $\rho_{\mathbf{K}}$ with fixed $\mathbf{q}$. The single-particle eigenfunctions $\phi^{(\mathbf{K})}_{j,\mathbf{q}}$ will be the important quantities in the analysis we describe below. It is interesting to note that an analogous strategy can be used to interrogate the quantum geometry and topology of a many-body *ground* state, by constructing single-particle eigenstates of a Green's function [62]. However, there is no simple relationship between the eigenstates of this operator and those of the excited-state density matrix at the center of our own analysis.

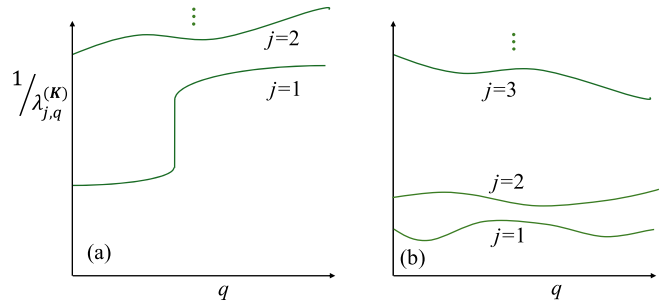

FIG. 2. Qualitative inverse occupations $1/\lambda_{j,\mathbf{q}}$ of single-particle states diagonalizing the density matrix $\rho_{\mathbf{K}}$. (a) Expected form for metal with a single occupied DM band. Discontinuity in $\lambda_{j,\mathbf{q}}$ indicates a Fermi surface. (b) Expected form for an insulator with multiple occupied DM bands. Note that DM bands with very low occupation (small $\lambda_{j,\mathbf{q}}$) float above the frames of these figures.

The eigenstates $\phi^{(\mathbf{K})}_{j,\mathbf{q}}$ are single particle in nature, and their associated eigenvalues $\lambda^{(\mathbf{K})}_{j,\mathbf{q}}$ may be viewed as an effective occupation: $\sum_j\sum_{\mathbf{q}}\lambda^{(\mathbf{K})}_{j,\mathbf{q}}|\phi^{(\mathbf{K})}_{j,\mathbf{q}}(\mathbf{r})|^2$ precisely reproduces the fermion density of the state $|\Phi_{\mathbf{K}}\rangle$, $\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r})$. Figure 2 illustrates possible expected qualitative behaviors for $\lambda^{(\mathbf{K})}_{j,\mathbf{q}}$. We note that as a function of $\mathbf{q}$, the eigenvalues $\lambda_{j,\mathbf{q}}$, and the associated states $|\phi^{(\mathbf{K})}_{j,\mathbf{q}}\rangle$ are organized into bands, analogous to but distinct from energy bands of a single-particle Hamiltonian. Because of the distinction, we denote the bands of density matrix eigenstates as DM bands.

It is interesting to note that these eigenvalues and eigenstates allow us to write the density matrix completely and exactly in terms of single-particle states, as if one is dealing with a system of noninteracting fermions:

$$\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}') = \sum_{j,\mathbf{q}}\phi^{(\mathbf{K})*}_{j,\mathbf{q}}(\mathbf{r})\lambda^{(\mathbf{K})}_{j,\mathbf{q}}\phi^{(\mathbf{K})}_{j,\mathbf{q}}(\mathbf{r}').$$

In this sense, if applied to the ground state rather than an excited state, our procedure bears resemblance to what is done in density-functional theory, where the electron density of an interacting system is written completely in terms of single-particle states [63]. In that case the states are either fully

occupied or unoccupied, so that one models the system as if it is in a *pure* state, and one only reproduces the density. In our procedure, one reproduces the full density matrix, but to do so we need to allow partial occupations, so that the system is modeled by a noninteracting *mixed* state. We note that if our procedure is applied to the system ground state, in the limit of vanishing interactions, all the occupation eigenvalues $\lambda_{j,\mathbf{q}}^{(0)}$ will be either 0 or 1, and the sets of occupied and unoccupied single-particle states will be precisely the same as one expects in a standard band structure.

The key to our formulation of the many-body QGD is a division of the states $\{\phi_{j,\mathbf{q}}^{(\mathbf{K})}\}$ into two groups, one of which we call "hole-hosting states," and the other "particle-hosting states." Qualitatively, these can be understood as analogs of filled and empty states in a band structure. They will allow us to define positively and negatively charged excitations, which, together, facilitate the construction of the QGD. In situations where the state $|\Phi_{\mathbf{K}}\rangle$ is a Slater determinant, the natural choice for the hole-hosting states would be those for which $\lambda_{j,\mathbf{q}}^{(\mathbf{K})} = 1$, while the particle-hosting states would be those for which $\lambda_{j,\mathbf{q}}^{(\mathbf{K})} = 0$. More generally, we will require that the division should be carried out in such a way that gradients of $\{\phi_{j,\mathbf{q}}^{(\mathbf{K})}\}$ with respect to $\mathbf{K}$ are well defined, and such that the total number of hole-hosting states should be equal to the number of fermions in the system. Beyond this, there is some freedom in choosing which group a given $\phi_{j,\mathbf{q}}^{(\mathbf{K})}$ should be placed in, as in principle this does not affect the final result for the QGD. However, when approximations are introduced, some choices will work better than others. Presumably, the best results will be obtained by assigning states with the largest values of $\lambda_{j,\mathbf{q}}^{(\mathbf{K})}$ to the hole-hosting states.

To define the QGD in this setting, we construct second-quantized $\mathbf{K}$-dependent spinor states

$$|u_j(\mathbf{r};\mathbf{K})\rangle = \begin{pmatrix} |u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle \\ |u_j^{(h)}(\mathbf{r};\mathbf{K})\rangle \end{pmatrix}, \tag{1}$$

where

$$|u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle = \sum_{\mathbf{q}\in p_j} \phi_{j,\mathbf{q}}(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} c_{j,\mathbf{q}}^{(\mathbf{K})} |\Phi_{\mathbf{K}}\rangle, \tag{2}$$

$$|u_j^{(h)}(\mathbf{r};\mathbf{K})\rangle = \sum_{\mathbf{q}\in h_j} \phi_{j,\mathbf{q}}^{*}(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} c_{j,\mathbf{q}}^{(\mathbf{K})\dagger} |\Phi_{\mathbf{K}}\rangle. \tag{3}$$

In these expressions, $\sum_{\mathbf{q}\in p_j}$ denotes a sum over particle-hosting states in DM band j, $\sum_{\mathbf{q}\in h_j}$ denotes a sum over hole-hosting states in DM band j, and $c_{j,\mathbf{q}}^{(\mathbf{K})}$ annihilates a particle of momentum $\mathbf{q}$ in DM band j. Note that the construction of $|u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle$ gives it nonzero contributions from particles residing within the manifold of particle-hosting states, so that $\sum_j \langle u_j^{(p)}(\mathbf{r};\mathbf{K})|u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle$ may be interpreted as a density of particles at position $\mathbf{r}$, while $\sum_j \langle u_j^{(h)}(\mathbf{r};\mathbf{K})|u_j^{(h)}(\mathbf{r};\mathbf{K})\rangle$ is the corresponding hole density. Having constructed these states, we endow them with two further properties. The first is their behavior under translations, which we define to be

$$\tilde{T}_{\mathbf{a}}|u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle = \sum_{\mathbf{q}\in p_j} \phi_{j,\mathbf{q}}^{(\mathbf{K})}(\mathbf{r}+\mathbf{a}) e^{-i\mathbf{K}\cdot(\mathbf{r}+\mathbf{a})} T_{\mathbf{a}}\, c_{j,\mathbf{q}}^{(\mathbf{K})} |\Phi_{\mathbf{K}}\rangle,$$

$$\tilde{T}_{\mathbf{a}}|u_j^{(h)}(\mathbf{r};\mathbf{K})\rangle = \sum_{\mathbf{q}\in h_j} \phi_{j,\mathbf{q}}^{(\mathbf{K})*}(\mathbf{r}+\mathbf{a}) e^{-i\mathbf{K}\cdot(\mathbf{r}+\mathbf{a})} T_{\mathbf{a}}\, c_{j,\mathbf{q}}^{(\mathbf{K})\dagger} |\Phi_{\mathbf{K}}\rangle.$$

In these equations, it should be noted that T_a is a translation operator acting on kets representing second-quantized many-body states, shifting all the particle positions by $\mathbf{a}$. Formally, the factors just to the left of T_a in the equations above are complex numbers, specified by the real parameters $\mathbf{r}$ and $\mathbf{K}$. $T_{\mathbf{a}}$ does not by itself act on these. To fully translate $|u_j^{(p,h)}(\mathbf{r};\mathbf{K})\rangle$ the parameter $\mathbf{r}$ must also, separately, be translated. The operator $\tilde{T}_a$ as constructed specifically acts on this set of states, carrying out both the parameter shift and the translation of the many-body ket.

It is not difficult to show $|u_j^{(p,h)}(\mathbf{r};\mathbf{K})\rangle$ is invariant under $\tilde{T}_{\mathbf{a}_i}$, where $\mathbf{a}_i$ is a primitive lattice vector. This is important in that it means we are putting vectors at different $\mathbf{K}$ into a single vector space [i.e., Eqs. (2) and (3) represent affine connections for the states [64]], so that taking derivatives with respect to $\mathbf{K}$ becomes meaningful. The second is an inner product, which we define as

$$\langle\langle u_{j,\mathbf{K}}|u_{j',\mathbf{K}'}\rangle\rangle \equiv \int d\mathbf{r}\langle u_j(\mathbf{r};\mathbf{K})|u_{j'}(\mathbf{r};\mathbf{K}')\rangle. \tag{4}$$

The integration over $\mathbf{r}$ in this expression is over all of real space.

The quantum geometric quantity of interest to us is

$$\begin{aligned}\mathcal{D}(\mathbf{K}) &= i\sum_j \left[\langle\langle u_{j,\mathbf{K}}^{(h)}|\nabla_{\mathbf{K}} u_{j,\mathbf{K}}^{(h)}\rangle\rangle - \langle\langle u_{j,\mathbf{K}}^{(p)}|\nabla_{\mathbf{K}} u_{j,\mathbf{K}}^{(p)}\rangle\rangle\right] \\ &\equiv \mathcal{A}^{(h)}(\mathbf{K}) - \mathcal{A}^{(p)}(\mathbf{K}).\end{aligned} \tag{5}$$

From its form, we see that $\mathcal{D}(\mathbf{K})$ is a measure of how the states $|u_j(\mathbf{r};\mathbf{K})\rangle$ change as one moves through the parameter space $\mathbf{K}$, and in this sense it is a quantum geometric quantity. Physically it is essentially an electric dipole moment associated with the many-body state $|\Phi_{\mathbf{K}}\rangle$, in units where the fermion charge is taken as unity. To see this, define

$$|\psi_{j,\mathbf{K}}(\mathbf{r})\rangle \equiv \begin{pmatrix} |\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\rangle \\ |\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\rangle \end{pmatrix} = e^{i\mathbf{K}\cdot\mathbf{r}} \begin{pmatrix} |u_j^{(p)}(\mathbf{r};\mathbf{K})\rangle \\ |u_j^{(h)}(\mathbf{r};\mathbf{K})\rangle \end{pmatrix},$$

which is a spinor obeying Bloch's theorem [3] constructed from the cell-periodic functions $|u_j^{(p,h)}\rangle$. One then has

$$\begin{aligned}\mathcal{D}(\mathbf{K}) = \sum_j \int d\mathbf{r}\big\{\mathbf{r}\big[\langle\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\rangle - \langle\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\rangle\big] \\ - i\langle\psi_{j,\mathbf{K}}(\mathbf{r})|\sigma_z|\nabla_{\mathbf{K}}\psi_{j,\mathbf{K}}(\mathbf{r})\rangle\big\},\end{aligned} \tag{6}$$

where σ_z is a Pauli matrix. This expression is well defined provided $N_{\mathbf{K}}^{(p)} - N_{\mathbf{K}}^{(h)} = 0$, where

$$N_{\mathbf{K}}^{(p)} = \sum_j \langle\langle\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\rangle\rangle,$$

$$N_{\mathbf{K}}^{(h)} = \sum_j \langle\langle\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\rangle\rangle.$$

In particular, the first integral on the right-hand side of Eq. (6) is independent of the choice of origin of $\mathbf{r}$, and the second term, which can be rewritten as

$$\lim_{\mathbf{K}'\to\mathbf{K}} \nabla_{\mathbf{K}}\big[\big(N_{\mathbf{K}}^{(p)} - N_{\mathbf{K}}^{(h)}\big)\delta(\mathbf{K}-\mathbf{K}')\big],$$

vanishes. Because $\boldsymbol{\mathcal{D}}(\mathbf{K})$ is independent of the origin of coordinates, it is clear that it represents information about the *internal* structure of the excited state.

The quantities $N_{\mathbf{K}}^{(p)}$ and $N_{\mathbf{K}}^{(h)}$ have the interpretations of the average number of particles in the particle-hosting states, and the average number of holes in the hole-hosting states, respectively. The requirement $N_{\mathbf{K}}^{(p)} - N_{\mathbf{K}}^{(h)} = 0$ can be imposed in a simple way. It proceeds by noticing that the state $|\Phi_{\mathbf{K}}\rangle$ can be constructed starting from a Slater determinant in which all the hole-hosting states are filled, and all the particle-hosting states are empty. The excited state $|\Phi_{\mathbf{K}}\rangle$ is in principle a linear combination of states in which different numbers of fermions are excited out of the hole-hosting states into the particle-hosting states. Because the excitation of a particle out of the initial Slater determinant necessarily leaves a hole behind, every term in this linear combination *individually* has equal numbers of particle and holes. Then the averages $N_{\mathbf{K}}^{(p)}$ and $N_{\mathbf{K}}^{(h)}$ must be equal.

Two comments are in order. First, from the above discussion, one sees that if the total number of hole-hosting states is equal to the total number of fermions in the system, we will always meet the condition $N_{\mathbf{K}}^{(p)} - N_{\mathbf{K}}^{(h)} = 0$. In what follows, we enforce $N_{\mathbf{K}}^{(p)} = N_{\mathbf{K}}^{(h)}$ by adopting this condition on the number of hole-hosting states. Second, this construction explains why the precise choice of particle- and hole-hosting states does not affect $\boldsymbol{\mathcal{D}}(\mathbf{K})$: interchanging a pair of states between the two groups changes $\langle\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\rangle$ and $\langle\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})|\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\rangle$ by precisely equal amounts, and $N_{\mathbf{K}}^{(p)} - N_{\mathbf{K}}^{(h)}$ still vanishes.

To complete the demonstration that $\boldsymbol{\mathcal{D}}(\mathbf{K})$ represents a dipole moment, we connect its form back to the density matrix. With some algebra, one may show

$$\begin{aligned}\boldsymbol{\mathcal{D}}(\mathbf{K}) &= \sum_j \int d\mathbf{r}\,\mathbf{r}\left[\left\langle\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\middle|\psi_{j,\mathbf{K}}^{(h)}(\mathbf{r})\right\rangle - \left\langle\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\middle|\psi_{j,\mathbf{K}}^{(p)}(\mathbf{r})\right\rangle\right] \\ &= \left\{\sum_j \sum_{\mathbf{q}\in h_j} \int d\mathbf{r}\,\mathbf{r}\left|\phi_{j,\mathbf{q}}^{(\mathbf{K})}(\mathbf{r})\right|^2\right\} - \mathrm{Tr}(\mathbf{r}\rho_{\mathbf{K}}) \equiv \mathbf{R}_0^{(\mathbf{K})} - \mathrm{Tr}(\mathbf{r}\rho_{\mathbf{K}}). \end{aligned} \tag{7}$$

The last term is the dipole moment of the charge density associated with $|\Phi_{\mathbf{K}}\rangle$. The term $\mathbf{R}_0^{(\mathbf{K})}$ is the dipole moment of the Slater determinant state formed by placing exactly one fermion in each hole-hosting state. In general, this means $\boldsymbol{\mathcal{D}}(\mathbf{K})$ actually represents the *deviation* of the dipole moment from that of a reference state.

Two further comments are in order. First, both the physical dipole moment and that of the reference state depend on the origin of coordinates. However, the QGD itself is independent of this choice, as expected for a quantity that characterizes the internal structure of the states [65]. Second, the reference state dipole moment $\mathbf{R}_0^{(\mathbf{K})}$ may itself be $\mathbf{K}$ dependent. However, in practical situations, it is not. Moreover, if we assume that the set of hole-hosting states at $\mathbf{K} = 0$, $\{\phi_{j,\mathbf{q}}^{(0)}\}$, does not pick some direction in space (as should be possible, for example, of a system with inversion symmetry), then it is natural to have $\mathbf{R}_0^{(\mathbf{K})} = 0$. We discuss the behavior of $\mathbf{R}_0^{(\mathbf{K})}$ in more detail in Appendix A.

Equation (5) is our definition of the quantum geometric dipole, formulated in a such a way that it can be constructed for a general many-body state, without any specific assumptions about its form. To test its validity, we now consider two concrete examples, both for excited states of quantum Hall systems in two dimensions. Our first is the QGD of a magnetoexciton, above a filled Landau level.

III. QGD FOR MAGNETOEXCITONS ABOVE A FILLED LANDAU LEVEL

For two-dimensional electrons in a perpendicular magnetic field, the noninteracting energy spectrum breaks up into Landau levels with energies $\hbar\omega_c(n + \frac{1}{2})$, where $\omega_c = eB/mc$ is the cyclotron frequency, B is the magnetic field, m the electron mass, and n is a non-negative integer. Each of these Landau levels is highly degenerate, with the number of states in a Landau level (LL) being equal to the number of magnetic flux quanta through the two-dimensional system. One way to label the states in a Landau level is to imagine the electrons being subject to an infinitesimal spatially periodic potential, with a single magnetic flux quantum through each unit cell. In this case the Landau level states can be written as eigenstates of the magnetic translation group [3,43]. Each state in a Landau level is then labeled by a unique wave vector.

The electron density ρ_{el} of such a system is parametrized by the filling factor $\nu = 2\pi\ell^2\rho_{el}$, where $\ell = \sqrt{\hbar c/eB}$ is the magnetic length. When ν is integral, the number of electrons is sufficient to fully fill ν Landau levels. In the strong field limit, such a state forms a good approximation to the ground state. Low-energy neutral excitations can be created by exciting an electron out of the highest occupied Landau level into the lowest unoccupied Landau level. By considering linear combinations of such particle-hole states in which each has a fixed momentum difference between the particle and hole, one constructs a wave function for the excitation with well-defined momentum [43] (see Fig. 3). Because the state involves creating a bound-state particle-hole pair across a single-particle energy gap, this excitation can be understood as a magnetoexciton.

An alternative method for constructing these excitations, which is less dependent on taking the strong field limit, is the single-mode approximation (SMA) [56,57]. This involves acting on the ground state with the density operator

$$Q_{\mathbf{K}}^{\dagger} \equiv \sum_{i=1}^{N} e^{i\mathbf{K}\cdot\mathbf{r}_i},$$

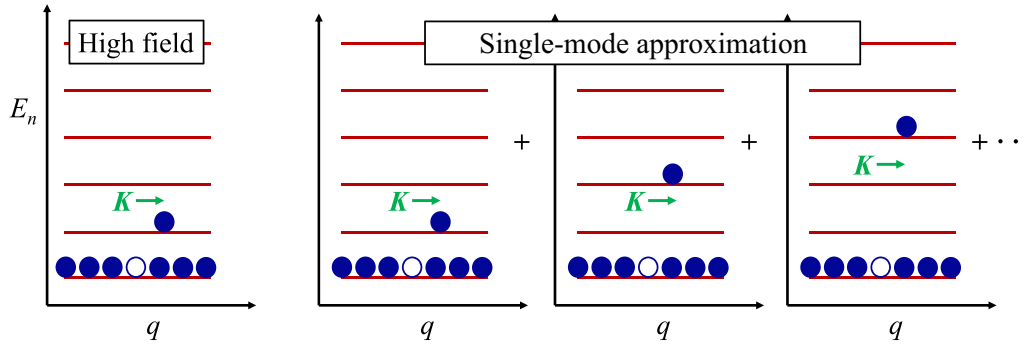

FIG. 3. Qualitative distinction between strong field wave function for magnetoexciton above a filled Landau level (left), and its wave function in the single-mode approximation (SMA) (right). In the former, a particle is excited from the filled level into the lowest-lying empty Landau level, and is given a boost of momentum **K**. In the SMA, the particle is excited into a linear combination of unoccupied Landau levels, but is given the same momentum boost. Note that the actual wave functions involve linear combinations of states with different wave vectors for the hole **q** (not shown).

where $\mathbf{r}_i$ are the positions of the N particles in the system. Such a state has momentum **K**, but also captures the ground-state correlations expected to remain important in a low-energy excitation [55]. In general one need not assume the ground state has the form of a single filled Landau level to apply the SMA; moreover, because of the form of $Q^\dagger_{\mathbf{K}}$, the particle resides in a linear combination of higher Landau levels (Fig. 3). One may show that this approximation produces the correct exact excitation energy $\hbar\omega_c$ in the limit $\mathbf{K} \to 0$, and also saturates the oscillator-strength sum rule [57].

As a first example, we apply our approach to computing the QGD to the magnetoexciton of a single filled Landau level ($\nu = 1$) as described by the SMA.

A. Density matrix $\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}')$

The first step in computing the QGD in this many-body formulation requires we find eigenstates of the **K**-dependent density operator

$$\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}') = \int d^2r_2 d^2r_3 \ldots d^2r_N \times \Phi^*_{\mathbf{K}}(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N)\Phi_{\mathbf{K}}(\mathbf{r}', \mathbf{r}_2, \ldots, \mathbf{r}_N), \quad (8)$$

where $\Phi_{\mathbf{K}}$ is the excited state wave function,

$$\Phi_{\mathbf{K}} = \frac{1}{\sqrt{\mathcal{N}_{\mathbf{K}}}} Q^\dagger_{\mathbf{K}} \Psi_0, \quad (9)$$

and Ψ_0 is the ground state, which we approximate as a filled $n = 0$ Landau level. $\mathcal{N}_{\mathbf{K}}$ normalizes this state. Although our formalism allows for the possibility that the eigenstates of $\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}')$ are **K** dependent, we will see in the thermodynamic limit that there is no such dependence. The eigenstates turn out to be the single-particle Landau level states.

To see this, we exploit the antisymmetry of $\Psi_0(\mathbf{r}_1, \mathbf{r}_2, \ldots)$ to write

$$\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}') = \frac{1}{\mathcal{N}_{\mathbf{K}}} \int d^2r_2 d^2r_3 \ldots d^2r_N \Psi^*_0(\mathbf{r}, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_N) \times \Psi_0(\mathbf{r}', \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_N) \times \left\{ e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')} + (N-1)\left[e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}_2)} + e^{i\mathbf{K}\cdot(\mathbf{r}_2-\mathbf{r}')}\right] + N(N-1)e^{i\mathbf{K}\cdot(\mathbf{r}_2-\mathbf{r}_3)} \right\}. \quad (10)$$

In this approximation, Ψ_0 is a Slater determinant, consisting of $N!$ terms, each a product of N single-particle states, all with different momentum labels **q**, the sum of which should vanish. The integrations over $\mathbf{r}_2 \ldots \mathbf{r}_N$ fix which pairs of terms in Ψ^*_0 and Ψ_0 can yield nonzero values. For each term in the second line of Eq. (10) there are $N!$ nonvanishing terms. Thus, in the large-N limit, the very last term dominates the result.

Because of this, in the thermodynamic limit, the density matrix must have the form

$$\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{q}} \alpha_{\mathbf{q}} \phi^*_{0,\mathbf{q}}(\mathbf{r})\phi_{0,\mathbf{q}}(\mathbf{r}'), \quad (11)$$

where $\alpha_{\mathbf{q}}$ are real numbers, and $\phi_{0,\mathbf{q}}$ are lowest Landau level states. (Note in writing this, we have labeled the states with a wave vector **q**, which is possible if one organizes the states as eigenvectors of magnetic translations. This is discussed in more detail in Sec. IV A.) It immediately follows that any state of the lowest Landau level is an eigenstate of $\rho_{\mathbf{K}}$ with nonzero eigenvalue, and all higher Landau level states are zero eigenvalue states. Since at $\nu = 1$ there are N states in a Landau level, we adopt the lowest Landau level states as our hole-hosting states, and all higher Landau level states as particle-hosting states.

B. Computation of the QGD

To compute the QGD we need explicit expressions for the states defined in Eqs. (2) and (3). The quantities $|u^{(p)}_j(\mathbf{r}; \mathbf{K})\rangle$ and $|u^{(h)}_j(\mathbf{r}; \mathbf{K})\rangle$ involve distinct sets of DM bands (in the case of $|u^{(h)}_j\rangle$ only a single DM band is involved), but because one sums over *all* the states in a DM band, some simplification is possible. In particular, we can choose any form of basis states that fully covers these DM bands to perform the sums. In this application, a particularly simple choice involves eigenstates of the noninteracting Hamiltonian $H = \left(\frac{\hbar}{i}\boldsymbol{\nabla} + \frac{e}{c}\mathbf{A}\right)^2$ with vector potential in the Landau gauge $\mathbf{A} = Bx\hat{\mathbf{y}}$. The single-particle states are then

$$\phi_{n,X}(\mathbf{r}) = \frac{1}{\sqrt{\pi^{1/2}2^n n!\ell L_y}} e^{-iXy/\ell^2} H_n(x - X)e^{-(x-X)^2/2\ell^2},$$

with n the Landau level index, L_y the extent of the system along the $\hat{y}$ direction, H_n a Hermite polynomial, and X the guiding center quantum number labeling states within a Landau level. (The use of guiding center states here rather than the magnetic translation eigenstates described above is possible because we are summing over *all* states in the lowest Landau level. Either set of states forms a basis for a full Landau level; the former is more convenient for the manipulations which follow.) In terms of these states we have

$$\left|u_n^{(p)}(\mathbf{r};\mathbf{K})\right\rangle = \sum_X \phi_{n,X}(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} c_{n,X}|\Phi_{\mathbf{K}}\rangle, \quad \left|u_0^{(h)}(\mathbf{r};\mathbf{K})\right\rangle = \sum_X \phi_{0,X}^*(\mathbf{r}) e^{-i\mathbf{K}\cdot\mathbf{r}} c_{0,X}^\dagger|\Phi_{\mathbf{K}}\rangle,$$

where $c_{n,X}$ annihilates an electron in state $\phi_{n,X}$. Note that, because the sets of particle- and hole-hosting states do not depend on $\mathbf{K}$, we have dropped the $(\mathbf{K})$ superscript on the field operators.

To construct the QGD, we form the quantities

$$\Gamma_n(\mathbf{K}_1,\mathbf{K}_2) \equiv \langle\langle u_{n,\mathbf{K}_1}|u_{n,\mathbf{K}_2}\rangle\rangle = \sum_{X_1,X_2} \left\langle\phi_{n,X_1}\left|e^{i(\mathbf{K}_1-\mathbf{K}_2)\cdot\mathbf{r}}\right|\phi_{n,X_2}\right\rangle \left\langle\Phi_{\mathbf{K}_1}\left|c_{n,X_1}^\dagger c_{n,X_2}\right|\Phi_{\mathbf{K}_2}\right\rangle, \quad n>0 \tag{12}$$

$$= \sum_{X_1,X_2} \left\langle\phi_{0,X_2}\left|e^{i(\mathbf{K}_1-\mathbf{K}_2)\cdot\mathbf{r}}\right|\phi_{n,X_1}\right\rangle \left\langle\Phi_{\mathbf{K}_1}\left|c_{n,X_1} c_{n,X_2}^\dagger\right|\Phi_{\mathbf{K}_2}\right\rangle, \quad n=0. \tag{13}$$

Note that the first factors on the right-hand side of the above equations are single-particle matrix elements, while the second factors are many-body matrix elements. In terms of these, from Eq. (5) we have

$$\mathcal{D}(\mathbf{K}) = -i \lim_{\mathbf{K}_2\to\mathbf{K}_1} \nabla_{\mathbf{K}_2}\left[\sum_{n>0}\Gamma_n(\mathbf{K}_1,\mathbf{K}_2) - \Gamma_0(\mathbf{K}_1,\mathbf{K}_2)\right]. \tag{14}$$

The computations of $\sum_{n>0}\Gamma_n(\mathbf{K}_1,\mathbf{K}_2)$ and $\Gamma_0(\mathbf{K}_1,\mathbf{K}_2)$ are somewhat involved but are in principle straightforward. We present some details for these these in Appendix B. The results are

$$\sum_{n>0}\Gamma_n(\mathbf{K}_1,\mathbf{K}_2) = \frac{g}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}\left\{e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2} - e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2 - K_1^2\ell^2/4 - K_2^2\ell^2/4}\right\} + O(\delta K)^2,$$

$$\Gamma_0(\mathbf{K}_1,\mathbf{K}_2) = \frac{g}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}\left\{1 - e^{i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2 - K_1^2\ell^2/4 - K_2^2\ell^2/4}\right\} + O(\delta K)^2, \tag{15}$$

where $\delta\mathbf{K} \equiv \mathbf{K}_1 - \mathbf{K}_2$ and $g = L_xL_y/2\pi\ell^2$ is the degeneracy of a Landau level, with L_xL_y the system area. The normalization factors are given by [57] $\mathcal{N}_{\mathbf{K}} = \langle\Phi_0|Q_{\mathbf{K}}Q_{\mathbf{K}}^\dagger|\Phi_0\rangle = g(1-e^{-K^2\ell^2/2})$. Finally, from Eq. (14), we obtain

$$\mathcal{D}(\mathbf{K}_1) = \frac{-i}{1-e^{-K_1^2\ell^2/2}} \lim_{\mathbf{K}_2\to\mathbf{K}_1} \nabla_{\mathbf{K}_2}\left\{e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2} + 2\sin[(\hat{z}\cdot\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2]e^{-K_1^2\ell^2/4 - K_2^2\ell^2/4} - 1\right\}$$

$$= \mathbf{K}_1\times\hat{z}\ell^2. \tag{16}$$

This result is identical to that found previously [43] using a different excited-state wave function for the magnetoexciton, in which only a single Landau level was retained for the particle excited out of the filled band. In the strong field limit, the latter yields a lower excitation energy than the SMA wave function we have used here [52–54,57]. The form of the QGD in this context is important because an applied electric field $\boldsymbol{\mathcal{E}}$ couples to the electric dipole moment, causing a drift motion with velocity $\mathbf{v}_D = c(\boldsymbol{\mathcal{E}}\times\mathbf{B})/B^2$, exactly the velocity at which one must move relative to the laboratory frame for the electric field to vanish [43]. This means the system is effectively Lorentz invariant, and that the SMA wave function for the magnetoexciton respects this symmetry.

The QGD found above is identical the to the electric dipole moment of a magnetoexciton in units where the electron charge magnitude $e = 1$ [52–54]. When represented in terms of a pair wave function involving a single particle and a single hole, this may be understood as a guiding center shift between them that must be present when the state carries a nonvanishing momentum $\mathbf{K}$ [66,67]. The analysis above demonstrates that the dipole may alternatively be understood as a manifestation of the quantum geometry inherent to a band of magnetoexciton states.

Finally, we note that the magnetoexciton states above $\nu = 1$ generated by the SMA represent a linear combination of states with a single particle and a single hole, so that the QGD associated with them could have been computed using the methods of Ref. [43]. The present analysis shows that our many-body approach also produces sensible results for such states. We next turn our attention to an example that *cannot* be represented in terms of single particle-hole pair states, and so requires the method developed above to compute the QGD: magnetoplasmons above a fractional quantum Hall state.

IV. QGD FOR MAGNETOPLASMONS ABOVE A LAUGHLIN STATE

We next consider the QGD for a collective excitation above a fractional quantum Hall state, specifically focusing on filling

factors of the form $\nu = 1/m$, with m an odd integer. In a disk geometry, the unnormalized ground state for such a filling is well described by a Laughlin wave function,

$$\Psi_0^{(m)}(\mathbf{r}_1, \mathbf{r}_2, \dots) = \prod_{i<j}(z_i - z_j)^m \prod_k e^{-|z_k|^2/4\ell^2}, \tag{17}$$

where $z_i = x_i - iy_i$ is the particle position in complex notation, and we have assumed the vector potential to be in symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x, 0)$. This wave function has well-defined total angular momentum, and so does *not* have total momentum as a quantum number. While one may write an analog of this state for electrons on a torus [60], allowing for momentum quantum numbers, a fact which we will exploit below, Eq. (17) is more convenient for explicit calculations. We will see that in the thermodynamic limit, the absence of translational invariance can be overcome.

It is well known that the low-energy charged excitations above a Laughlin state carry charge $\pm e/m$ [51,59]. In analogy with the types of excitations discussed for filled Landau levels, it is natural to think of the low-lying neutral excitations as bound pairs of such quasiparticles with opposite charge. This interpretation suggests that the SMA may work well as an approximation for these states. An important caveat, however, is that, in the very strong field limit ($\hbar\omega_c \to \infty$), one expects that higher Landau levels will not be involved in any low-energy excitation. Such excitations then involve motion of electrons within a *single* Landau level. Concretely, the (approximate) wave functions for these excitations take the form [56,57]

$$\Psi_{\mathbf{K}}^{(m)}(\mathbf{r}_1, \mathbf{r}_2, \dots) = \frac{1}{\sqrt{\mathcal{N}_{\mathbf{K}}^{(m)}}} \overline{Q}_{\mathbf{K}}^{\dagger} \Psi_0^{(m)}(\mathbf{r}_1, \mathbf{r}_2, \dots), \tag{18}$$

where $\mathcal{N}_{\mathbf{K}}^{(m)}$ is chosen to normalize the wave function, and

$$\overline{Q}_{\mathbf{K}}^{\dagger} \equiv \sum_{i=1}^{N} P_0 e^{i\mathbf{K}\cdot\mathbf{r}_i} P_0$$

is the same operator as used in the last section, but projected by the operator P_0 into the lowest Landau level (LLL). The effect of acting on the partially filled Landau level with $\overline{Q}_{\mathbf{K}}^{\dagger}$ is to introduce density-wave-like correlations [55–57], and because of this it is natural to think these excitations as magnetoplasmons. The states $\Psi_{\mathbf{K}}^{(m)}$ have been extensively studied [56,57] and are known to have a number of sensible and attractive properties as low-lying collective excitations.

A. Density matrix for $\nu = 1/m$ Laughlin states and its eigenstates

The eigenstates of the density matrices associated with these states can be arrived at following reasoning analogous what we followed for integrally filled Landau levels above. A first observation is that the form of the density matrix,

$$\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}') \equiv \int d^2r_2 d^2r_3 \dots d^2r_N \Psi_{\mathbf{K}}^{(m)*}(\mathbf{r}, \mathbf{r}_2, \mathbf{r}_3, \dots) \times \Psi_{\mathbf{K}}^{(m)}(\mathbf{r}', \mathbf{r}_2, \mathbf{r}_3, \dots),$$

has the property $\int d^2r\, \varphi(\mathbf{r})\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}') = 0$ for *any* state φ that lies outside the lowest Landau level. All such states are then eigenstates of the density matrix, and naturally form part of the particle-hosting states for this system. The zero eigenvalue, however, indicates that they will not contribute to the QGD.

Because the number of states in the LLL exceeds the number of electrons in the system, we need to divide these into hole-hosting and particle-hosting groups. As discussed above, the precise division does not in principle affect the final result, provided all states are eigenstates of $\rho_{\mathbf{K}}$, the number of hole-hosting states is the same as the number of electrons in the system, and states within each group vary in such a way that derivatives with respect to $\mathbf{K}$ are well defined. A convenient way to proceed in this case is to consider placing the system on a torus, for which an analog of the Laughlin wave function, Eq. (17), may be written in terms of elliptic θ functions [60]. Moreover, one may construct the low-lying collective excitations using the SMA [68], in a way directly analogous to that described above for the disk geometry.

The utility of considering the torus geometry is that eigenstates of the Hamiltonian are simultaneously eigenstates of a set of translation operators, such that they have good momentum quantum numbers. In particular, one may consider magnetic translations of the form $T_{a\hat{x}} = \prod_{j=1}^{N} e^{i(p_{j,x}+y_j/2\ell^2)a}$, $T_{a\hat{y}} = \prod_{j=1}^{N} e^{i(p_{j,y}-x_j/2\ell^2)a}$, where $p_{j,x} = \frac{\hbar}{i}\partial_{x_j}$, $p_{j,y} = \frac{\hbar}{i}\partial_{y_j}$ are momentum operators for particle j, for classifying states in the lowest Landau level. This imposes an effective square lattice structure of lattice constant a, and provided there is one magnetic flux quantum per unit cell (i.e., $a^2/2\pi\ell^2 = 1$), $T_{a\hat{x}}$ and $T_{a\hat{y}}$ commute with each other as well as with the Hamiltonian. Imposing periodic boundary conditions such that there are N_c unit cells in the whole system, the eigenstates of the Hamiltonian will have well-defined center-of-mass momentum $\mathbf{K}$ with N_c possible values. In particular, we must have $T_{a\hat{x}}\Psi_{\mathbf{K}} = e^{iK_x a}\Psi_{\mathbf{K}}$, $T_{a\hat{y}}\Psi_{\mathbf{K}} = e^{iK_y a}\Psi_{\mathbf{K}}$.

It follows that the density matrix, viewed as an operator, is invariant under translations, i.e., $T_{\mathbf{a}}\rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}')T_{\mathbf{a}}^{-1} = \rho_{\mathbf{K}}(\mathbf{r}, \mathbf{r}')$, where $\mathbf{a} = a\hat{x}$ or $a\hat{y}$. Its eigenstates $\phi_{0,\mathbf{q}}(\mathbf{r})$ (here the subscript 0 refers to the LLL) are themselves eigenstates of translations, $T_{\mathbf{a}}\phi_{0,\mathbf{q}}(\mathbf{r}) = e^{i\mathbf{q}\cdot\mathbf{a}}\phi_{0,\mathbf{q}}(\mathbf{r})$, with the number of distinct values of $\mathbf{q}$ being the same as the number of states in the LLL. This means that varying $\mathbf{K}$ will not change the eigenstates of the density matrix, and our particle-hosting and hole-hosting state can be chosen in a $\mathbf{K}$-independent way. In the following section, we describe one way in which this can be done that ultimately allows a computation of the QGD.

B. Formal expression for QGD

For concrete calculations, it is preferable to work with the wave functions in Eqs. (17) and (18) than with their counterparts on the torus. However, this introduces a difficulty in that, for any finite-size system, there is an edge which breaks the translational symmetry. Well inside the bulk, we expect that, locally, states for a disk and states for torus will be essentially the same. To take advantage of this, we start with formal steps best defined on the torus, and then carry through concrete calculations for needed correlation functions in the disk geometry, for which the calculations are analytically tractable. We assume the thermodynamic limit has been taken so that the edge does not contribute to these correlation functions.

To compute the QGD, we start with

$$\begin{aligned}\Gamma(\mathbf{K},\mathbf{K}') &= \langle\langle u^{(p)}_{0,\mathbf{K}}|u^{(p)}_{0,\mathbf{K}'}\rangle\rangle - \langle\langle u^{(h)}_{0,\mathbf{K}}|u^{(h)}_{0,\mathbf{K}'}\rangle\rangle \\ &= \sum_{\mathbf{q}_1,\mathbf{q}_1'\in p_0}\langle\phi_{0,\mathbf{q}_1}|e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_1'}\rangle\langle\Psi_{\mathbf{K}}|c^\dagger_{0,\mathbf{q}_1}c_{0,\mathbf{q}_1'}|\Psi_{\mathbf{K}'}\rangle \\ &\quad - \sum_{\mathbf{q}_2,\mathbf{q}_2'\in h_0}\langle\phi_{0,\mathbf{q}_2'}|e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_2}\rangle\langle\Psi_{\mathbf{K}}|c_{0,\mathbf{q}_2}c^\dagger_{0,\mathbf{q}_2'}|\Psi_{\mathbf{K}'}\rangle. \end{aligned} \tag{19}$$

The first and second terms of Γ can be used to form the particle and hole connections $\mathcal{A}^{(p)}$ and $\mathcal{A}^{(h)}$, respectively, by taking gradients of these terms with respect to $\mathbf{K}$. We then have

$$\mathcal{D}(\mathbf{K}) = -\mathcal{A}^{(p)}(\mathbf{K}) + \mathcal{A}^{(h)}(\mathbf{K}) = -i\lim_{\mathbf{K}'\to\mathbf{K}}\nabla_{\mathbf{K}'}\Gamma(\mathbf{K},\mathbf{K}'). \tag{20}$$

Writing $\delta\mathbf{K}\equiv\mathbf{K}-\mathbf{K}'$ allows us to express this as

$$\mathcal{D}(\mathbf{K}) = -i\lim_{\delta\mathbf{K}\to 0}[\Gamma(\mathbf{K},\mathbf{K}-\delta\mathbf{K})-\Gamma(\mathbf{K},\mathbf{K})]/\delta K, \tag{21}$$

and from momentum conservation,

$$\begin{aligned}\Gamma(\mathbf{K},\mathbf{K}-\delta\mathbf{K}) &= \sum_{\mathbf{q}_1\in p_0}\langle\phi_{0,\mathbf{q}_1}|e^{i\delta\mathbf{K}\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_1-\delta\mathbf{K}}\rangle \\ &\quad\times\langle\Psi_{\mathbf{K}}|c^\dagger_{0,\mathbf{q}_1}c_{0,\mathbf{q}_1-\delta\mathbf{K}}|\Psi_{\mathbf{K}-\delta\mathbf{K}}\rangle \\ &\quad - \sum_{\mathbf{q}_2\in h_0}\langle\phi_{0,\mathbf{q}_2+\delta\mathbf{K}}|e^{i\delta\mathbf{K}\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_2}\rangle \\ &\quad\times\langle\Psi_{\mathbf{K}}|c_{0,\mathbf{q}_2}c^\dagger_{0,\mathbf{q}_2+\delta\mathbf{K}}|\Psi_{\mathbf{K}-\delta\mathbf{K}}\rangle. \end{aligned} \tag{22}$$

Equations (21) and (22) raise a subtlety discussed in our initial formulation of the QGD: since $\mathcal{D}$ is expressed as a limit, we need $\Gamma(\mathbf{K},\mathbf{K}-\delta\mathbf{K})$ to behave smoothly as $\delta\mathbf{K}\to 0$. We can in fact guarantee this for both terms in Eq. (22) separately, by a careful division of the states into the particle- and hole-hosting groups, with a corresponding definition of how the $\delta\mathbf{K}\to 0$ limit is taken. To do this, we observe that if $\mathbf{q}$ is among the particle-hosting states, then $\mathbf{q}+\delta\mathbf{K}$ must be as well. Analogous relations should hold for values of $\mathbf{q}$ among the hole-hosting states. One way to guarantee this is to group m neighboring values of $\mathbf{q}$, for example, along a row in the $\hat{x}$ direction, into unit cells, assigning one of these to h_0 and the remainder to p_0. The resulting unit cell has length $2\pi/N_c a$ in the $\hat{y}$ direction, and length $2\pi m/N_c a$ in the $\hat{x}$ direction (see Fig. 4). We then restrict values of $\mathbf{K}$ to discrete points, one associated with each of these unit cells. This guarantees that in Eq. (22), matrix elements are always between states in the same sectors. $\delta\mathbf{K}$ is then taken to be a discrete difference between the locations of nearby unit cells, and the limit $\delta\mathbf{K}\to 0$ is accomplished by taking the thermodynamic limit $N_c\to\infty$.

It is interesting to note that this means the number of momenta allowed to the magnetoplasmons above a given ground state will be $1/m$ of the number of unit cells in the system, which in this construction is the number of magnetic flux quanta passing through it. On a torus, however, there are m different ground states [3,60] upon which one may build a magnetoplasmon in the SMA, so the total number of excited states in a magnetoplasmon band is ultimately equal to the number of flux quanta through the system. We leave it to future work to determine whether this counting remains valid beyond the SMA.

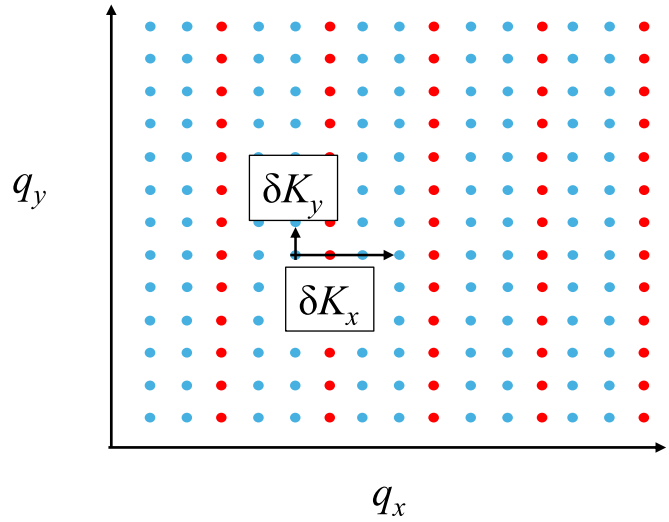

FIG. 4. Illustration of allowed wave vectors for single-particle states for system with periodic boundary conditions. Blue points represent particle-hosting states, red points are hole-hosting. The lattice and the discrete difference vector $\delta\mathbf{K}=(\delta K_x,\delta K_y)$ are constructed so that finite-difference approximations to the derivatives of the single-particle states with respect to $\delta\mathbf{K}$ are always within the particle- or hole-hosting states, so that the limit $\delta\mathbf{K}\to 0$, taken in the thermodynamic limit, behaves smoothly.

Continuing with the calculation, we commute two of the fermion operators to get

$$\begin{aligned}\Gamma(\mathbf{K},\mathbf{K}') &= \sum_{\text{all }\mathbf{q}_1}\langle\phi_{0,\mathbf{q}_1}|e^{i\delta\mathbf{K}\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_1-\delta\mathbf{K}}\rangle\langle\Psi_{\mathbf{K}}|c^\dagger_{0,\mathbf{q}_1}c_{0,\mathbf{q}_1-\delta\mathbf{K}}|\Psi_{\mathbf{K}'}\rangle \\ &\quad - N_h\delta(\mathbf{K}-\mathbf{K}'), \end{aligned} \tag{23}$$

where N_h is the number of hole-hosting states. Since ultimately we will have $\delta\mathbf{K}$ very small, we expand for small δK. Noting $\sum_{\mathbf{q}_1}\langle\Psi_{\mathbf{K}}|c^\dagger_{0,\mathbf{q}_1}c_{0,\mathbf{q}_1}|\Psi_{\mathbf{K}}\rangle=\nu N_c$ is the number of electrons in the system, and that

$$\sum_{\text{all }\mathbf{q}_1}\langle\phi_{0,\mathbf{q}_1}|e^{i\delta\mathbf{K}\cdot\mathbf{r}}|\phi_{0,\mathbf{q}_1-\delta\mathbf{K}}\rangle c^\dagger_{0,\mathbf{q}_1}c_{0,\mathbf{q}_1-\delta\mathbf{K}}\equiv\overline{Q}^\dagger_{\delta\mathbf{K}},$$

we arrive at

$$\begin{aligned}\Gamma(\mathbf{K},\mathbf{K}') &= \sum_{\text{all }\mathbf{q}_1}\left[\langle\Psi_{\mathbf{K}}|\overline{Q}^\dagger_{\delta\mathbf{K}}|\Psi_{\mathbf{K}'}\rangle - N_h\right]\delta(\mathbf{K}-\mathbf{K}') \\ &= (\nu N_c - N_h)\delta(\mathbf{K}-\mathbf{K}') \\ &\quad + \lim_{\mathbf{q}\to 0}\delta\mathbf{K}\cdot\nabla_q\langle\Psi_{\mathbf{K}}|\overline{Q}^\dagger_{\mathbf{q}}|\Psi_{\mathbf{K}-\mathbf{q}}\rangle + O(\delta K^2). \end{aligned} \tag{24}$$

Our construction requires the first term of the last line to vanish, and the $O(\delta K^2)$ correction does not contribute to the QGD. We finally arrive at the expression

$$\mathcal{D}(\mathbf{K}) = -i\lim_{\mathbf{q}\to 0}\nabla_q\langle\Psi_{\mathbf{K}}|\overline{Q}^\dagger_{\mathbf{q}}|\Psi_{\mathbf{K}-\mathbf{q}}\rangle. \tag{25}$$

In arriving at this expression we have not made any specific assumptions about the form of $|\Psi_{\mathbf{K}}\rangle$, except to say that it has a well-defined momentum quantum number, which implies we

are working on a torus. A concrete form of Eq. (25), however, is most easily arrived at by working in the disk geometry. To compute it we will work in the latter, assuming that the thermodynamic limit has been taken, so that edge effects do not enter. As we shall see, this can be written, in the SMA, in terms of ground-state correlation functions.

To this end, it is convenient to write

$$\mathcal{D}(\mathbf{K}) = -i \lim_{\mathbf{q}\to 0} \nabla_q \langle \Psi_{\mathbf{K}+\frac{1}{2}\mathbf{q}} | \overline{Q}^\dagger_{\mathbf{q}} | \Psi_{\mathbf{K}-\frac{1}{2}\mathbf{q}} \rangle \tag{26}$$

$$= -\frac{i}{\mathcal{N}^{(m)}_{\mathbf{K}}} \lim_{\mathbf{q}\to 0} \nabla_q \langle \Psi_0 | \overline{Q^\dagger}_{-\mathbf{K}-\frac{\mathbf{q}}{2}} \overline{Q^\dagger}_{\mathbf{q}} \overline{Q^\dagger}_{\mathbf{K}-\frac{\mathbf{q}}{2}} | \Psi_0 \rangle. \tag{27}$$

In the second line of this, the normalizing factor has not been acted upon by ∇_q because it is even $\mathbf{q}$, so that we may take $\mathbf{q} \to 0$ directly in this term. This is one of the simplifications that follows from rewriting $\mathcal{D}(\mathbf{K})$ in the form shown in Eq. (26).

C. QGD of the magnetoplasmon in the SMA

To carry out the concrete calculation, we use the techniques discussed in Refs. [57,69], to derive, in a straightforward if somewhat involved calculation,

$$\begin{aligned}
&\overline{Q^\dagger}_{-\mathbf{K}-\frac{\mathbf{q}}{2}} \overline{Q^\dagger}_{\mathbf{q}} \overline{Q^\dagger}_{\mathbf{K}-\frac{\mathbf{q}}{2}} \\
&\quad = \overline{Q^\dagger_{-\mathbf{K}-\frac{\mathbf{q}}{2}} Q^\dagger_{\mathbf{q}} Q^\dagger_{\mathbf{K}-\frac{\mathbf{q}}{2}}} + \tfrac{1}{2}(q^* K \ell^2 - q K^* \ell^2) \overline{Q^\dagger_{-\mathbf{K}} Q^\dagger_{\mathbf{K}}} \\
&\qquad - \left(1 - e^{-(K^* + \frac{q^*}{2})(K - \frac{q}{2})\ell^2/2}\right) \overline{Q^\dagger_{-\mathbf{q}} Q^\dagger_{\mathbf{q}}} \\
&\qquad - \overline{Q^\dagger}_0 \left(\tfrac{1}{2} q^* K \ell^2 - \tfrac{1}{2} q K^* \ell^2\right)\left(1 - e^{-|\mathbf{K}|^2\ell^2/2}\right) + O(q^2).
\end{aligned} \tag{28}$$

On the right-hand side of the above expression, wave vectors are written in complex notation (e.g., $q \equiv q_x - i q_y$), and the quantity $\overline{Q^\dagger}_0$ is the same as the particle number νN_c. The derivation of Eq. (28) is provided in Appendix C.

The reason for writing the product of the three projected operators on the left-hand side of Eq. (28) in the form shown on the right-hand side is that, upon taking expectation values, one obtains an explicit expression in terms of ground-state correlation functions. It is not difficult to show that $\langle \Psi_0 | \overline{Q^\dagger_{-\mathbf{K}-\frac{\mathbf{q}}{2}} Q^\dagger_{\mathbf{q}} Q^\dagger_{\mathbf{K}-\frac{\mathbf{q}}{2}}} | \Psi_0 \rangle$ is even in $\mathbf{q}$ for a circularly symmetric state, so this term does not contribute to Eq. (27). The remaining terms involve density-density correlation functions, which may be expressed in terms of pair-correlation functions. In particular,

$$\begin{aligned}
\langle \Psi_0 | \overline{Q^\dagger_{-\mathbf{p}} Q^\dagger_{\mathbf{p}}} | \Psi_0 \rangle &= \int d^2R_1 \int d^2R_2 \sum_{i,j} \langle \Psi_0 | \delta(\mathbf{R}_1 - \mathbf{r}_i)\delta(\mathbf{R}_2 - \mathbf{r}_j) | \Psi_0 \rangle e^{i\mathbf{p}\cdot(\mathbf{R}_1 - \mathbf{R}_2)} \\
&= \sum_i \int d^2R_1 \int d^2R_2 \Bigg\{ \sum_i \langle \Psi_0 | \delta(\mathbf{R}_1 - \mathbf{r}_i) | \Psi_0 \rangle \delta(\mathbf{R}_1 - \mathbf{R}_2) \\
&\qquad + \sum_{i \neq j} \langle \Psi_0 | \delta(\mathbf{R}_1 - \mathbf{r}_i)\delta(\mathbf{R}_2 - \mathbf{r}_j) | \Psi_0 \rangle \Bigg\} e^{i\mathbf{p}\cdot(\mathbf{R}_1 - \mathbf{R}_2)} \\
&= \nu N_c + n_0^2 \int d^2R_1 \int d^2R_2 e^{i\mathbf{p}\cdot(\mathbf{R}_1 - \mathbf{R}_2)} g(\mathbf{R}_1 - \mathbf{R}_2) \\
&= \nu N_c + (2\pi)^2 \nu N_c n_0 \delta(\mathbf{p}) + \nu N_c n_0 \tilde{h}(\mathbf{p}),
\end{aligned} \tag{29}$$

where $n_0 = 2\pi\nu\ell^2$ is the particle density, $g(\mathbf{R})$ is the ground-state pair-correlation function, and $h(\mathbf{p}) = \int d^2R [g(\mathbf{R}) - 1] e^{i\mathbf{p}\cdot\mathbf{R}}$, which has the property $\tilde{h}(0) = -1/n_0$. Note in writing g in terms of the difference in particle positions $\mathbf{R}_1 - \mathbf{R}_2$, we have assumed the infinite-size limit has been taken, so that there is no impact from the system edge and the correlations are translationally invariant.

To compute the QGD we need to consider the expectation value $\langle \Psi_0 | \overline{Q^\dagger}_{-\mathbf{K}-\frac{\mathbf{q}}{2}} \overline{Q^\dagger}_{\mathbf{q}} \overline{Q^\dagger}_{\mathbf{K}-\frac{\mathbf{q}}{2}} | \Psi_0 \rangle$. Some care must be taken in considering the contribution of $\langle \Psi_0 | \overline{Q^\dagger_{-\mathbf{q}} Q^\dagger_{\mathbf{q}}} | \Psi_0 \rangle$ in Eq. (28). For small but finite q, Eq. (29) shows this vanishes as q^2, so that this term does not contribute to the QGD. The absence of any contribution from the $\delta(\mathbf{q})$ term comes about because the thermodynamic limit must be taken *before* taking $\mathbf{q} \to 0$, which is required when we use the disk geometry to compute quantities that are translationally invariant in a system without an edge.

With these considerations, we arrive at

$$\begin{aligned}
&\lim_{\mathbf{q}\to 0} \nabla_q \langle \Psi_0 | \overline{Q^\dagger}_{-\mathbf{K}-\frac{\mathbf{q}}{2}} \overline{Q^\dagger}_{\mathbf{q}} \overline{Q^\dagger}_{\mathbf{K}-\frac{\mathbf{q}}{2}} | \Psi_0 \rangle \\
&\quad = i(\mathbf{K} \times \hat{z}\ell^2) \left\{ \langle \Psi_0 | \overline{Q^\dagger_{-\mathbf{K}} Q^\dagger_{\mathbf{K}}} | \Psi_0 \rangle - \nu N_c \left(1 - e^{-|\mathbf{K}|^2\ell^2/2}\right) \right\} \\
&\quad = i(\mathbf{K} \times \hat{z}\ell^2) \left\{ \nu N_c (1 + n_0 \tilde{h}(\mathbf{K})) - \nu N_c \left(1 - e^{-|\mathbf{K}|^2\ell^2/2}\right) \right\}.
\end{aligned} \tag{30}$$

The normalization $\mathcal{N}^{(m)}_{\mathbf{K}}$ for the excited state is calculated with similar methods [57],

$$\begin{aligned}
\mathcal{N}^{(m)}_{\mathbf{K}} &= \langle \Psi_0 | \overline{Q^\dagger}_{-\mathbf{K}} \overline{Q^\dagger}_{\mathbf{K}} | \Psi_0 \rangle \\
&= \langle \Psi_0 | \overline{Q^\dagger_{-\mathbf{K}} Q^\dagger}_{\mathbf{K}} | \Psi_0 \rangle + \nu N_c \left(e^{-|\mathbf{K}|^2\ell^2/2} - 1\right),
\end{aligned}$$

leading to the final result

$$\mathcal{D}(\mathbf{K}) = (\mathbf{K}\times\hat{z}\ell^2)\frac{\nu N_c(1+n_0\tilde{h}(\mathbf{K}))+\nu N_c(e^{-|\mathbf{K}|^2\ell^2/2}-1)}{\nu N_c(1+n_0\tilde{h}(\mathbf{K})+(e^{-|\mathbf{K}|^2\ell^2/2}-1))} = \mathbf{K}\times\hat{z}\ell^2. \tag{31}$$

We conclude this section with a few observations. First, we have arrived at precisely the same relatively simple result that we found for the inter-Landau-level exciton. The more complicated internal correlations that play a role in intermediate steps of the analysis do not impact the final result. We show below this is a result of the translational invariance of the system. Second, there is a quite different way to formulate the particle- and hole-hosting wave-function spaces than the one we used. This involves composite fermions [61]. In the mean-field approximation, the number of states in a composite fermion Landau level is equal to the number of particles, yielding a natural division between particle- and hole- hosting states. However, to avoid introducing further approximations, it is necessary that the hole-hosting states be written in such a way that they all lie in the LLL of the original electron degrees of freedom. Ultimately one arrives at Eq. (25), but the intermediate steps are more involved than what is presented here [70]. Alternatively, one can adopt a further approximation in which the neutral excited state is simply a single particle-hole pair across two composite fermion Landau levels, with no LLL projection carried out. This turns out to produce [70] a QGD that is m times larger than that of Eq. (31). This reflects the large composite fermion magnetic length, and shows that sacrificing the projection of the state into the electron LLL introduces considerable error.

We now turn to this issue to demonstrate that the simple result found in this section is an outcome of having a translationally invariant system, in a state that has a sharp momentum quantum number, and lies fully in the LLL.

V. DIPOLE MOMENT IN A SINGLE LANDAU LEVEL

The result found for the magnetoplasmon above a Laughlin state is actually generic for any many-body state of a translationally invariant system with well-defined momentum in a single Landau level. In particular, there is a relation between the center-of-mass (c.m.) position and the momentum. To see this, we start with a many-body Hamiltonian in first quantization,

$$H = \sum_j \frac{1}{2m}\left(\mathbf{p}_j + \frac{e}{c}\mathbf{A}(\mathbf{r}_j)\right)^2 + \sum_{i<j} v(\mathbf{r}_i-\mathbf{r}_j).$$

Here it is convenient to work in Landau gauge, $\mathbf{A}(\mathbf{r}) = B(0,x,0)$, $v(\mathbf{r})$ is an interelectron potential, and $\mathbf{p}_j$ is the momentum operator vector of the jth particle. In this gauge, states with a total momentum K in the $\hat{y}$ direction are eigenstates of $\sum_j p_{j,y}$. Consider the Landau level lowering operator for the c.m. degree of freedom,

$$a_{\text{c.m.}} = \frac{1}{\sqrt{2}}\sum_j\left[\ell p_{j,x} - \frac{i}{\ell}(x_j+\ell^2 p_{j,y})\right],$$

which (by design) satisfies

$$[H, a_{\text{c.m.}}] = -\frac{1}{m\ell^2}a_{\text{c.m.}}.$$

If a state Ψ_K with momentum $K\hat{y}$ resides fully in a single Landau level, we must have $\langle\Psi_K|a_{\text{c.m.}}|\Psi_K\rangle = \langle\Psi_K|a^\dagger_{\text{c.m.}}|\Psi_K\rangle = 0$. It follows

$$\langle\Psi_K|\sum_j p_{j,x}|\Psi_K\rangle \equiv \langle\Psi_K|P_x|\Psi_K\rangle = 0, \tag{32}$$

$$\langle\Psi_K|\sum_j p_{j,y} + x_j/\ell^2|\Psi_K\rangle \equiv \langle\Psi_K|P_y + \hat{X}_{CM}/\ell^2|\Psi_K\rangle = 0. \tag{33}$$

This gives the result

$$\langle\Psi_K|\hat{X}_{\text{c.m.}}|\Psi_K\rangle = -\ell^2 K, \tag{34}$$

which is one component of the result we obtained in the previous section. For the other component, there is a subtlety. In this formulation we should think of the system as having periodic boundary conditions in the $\hat{y}$ direction, and having edges in the $\hat{x}$ direction. The state can in principle be compact in the latter dimension, so that the boundaries are very far away from the locations of the electrons. The problem is that $\langle\Psi_K|\hat{Y}_{\text{c.m.}}|\Psi_K\rangle$ is not uniquely defined.

The electron density is uniform along the periodic direction of the cylinder $\hat{y}$, which we take to have size L_y. To uniquely define the y_j coordinate, we fix an origin for coordinates of the positions (x_j, y_j), and also a line at fixed x_j for which y_j jumps by $\pm L_y$ when particle j passes through it. The value of $\langle\Psi_K|\hat{Y}_{\text{c.m.}}|\Psi_K\rangle$ depends on the relative positions of the origin and this cut line. However, whatever values we assign them, $\langle\Psi_K|Y_{\text{c.m.}}|\Psi_K\rangle$ will be the same for any K because $\sum_j y_j$ cannot be localized anywhere on the cylinder if Ψ_K is an eigenstate of $\sum_j p_{y,j}$. We can remove the arbitrariness by choosing some reference state and considering only differences. Then

$$\langle\Psi_K|\sum_j \mathbf{r}_j|\Psi_K\rangle - \langle\Psi_0|\sum_j \mathbf{r}_j|\Psi_0\rangle = -\hat{x}K\ell^2 = -\mathbf{K}\times\hat{z}\ell^2.$$

The overall $-$ sign is present because the shift is in the position of the electrons, which are negatively charged. Note that in our general formulation of the QGD, we also found it to be the *deviation* of the dipole moment from that of a reference state $\mathbf{R}_0^{(K)}$. The latter is not expected to have any $\mathbf{K}$ dependence in the thermodynamic limit for low-energy states, and, with a judicious choice of origin, can be made to vanish.

We see then that the dipole moment for a lowest Landau level state with a momentum quantum number, for a system that is translationally invariant, *generically* takes the simple form found in the last section. By contrast, we expect that had the translational invariance been broken, for example by a periodic potential, one would find deviations from this result. This has been shown to be the case for particle-hole states in Landau levels of Dirac-type Hamiltonians [43].

VI. SUMMARY AND DISCUSSION

In fermion systems that are not ferroelectric, the ground state typically does not carry a static electric dipole moment.

However, there may be dipole *fluctuations* carrying quantum geometric information about the ground state [71,72]. By contrast, neutral excitations of a fermion system with a momentum quantum number carry an internal structure which is geometric in nature, a quantum geometric dipole, or QGD. Previous work has demonstrated that this structure arises naturally for states that are well described by particle-hole wave functions [43,47,48]. In this work, we demonstrated that this concept is not tied to a particular form for the excited-state wave function, and can be defined in a way that can be applied to states $|\Phi_{\mathbf{K}}\rangle$ of any form, provided they are labeled by a continuously varying wave vector $\mathbf{K}$. The formulation exploits the density matrix associated with $|\Phi_{\mathbf{K}}\rangle$, allowing a set of single-particle states to be defined. These states are then divided into two groups, one of which is "particle hosting" and the other "hole hosting," with the number of states in the latter group equal to the number of fermions in the ground state. These collections of states are then exploited to define quantities akin to Berry connections, whose difference is gauge invariant [43], and is essentially the electric dipole moment of the excitation, up to a factor of the charge carried by the fermions.

To demonstrate that this formulation produces sensible results, we considered two concrete examples, both involving excitations of a two-dimensional electron gas in the quantum Hall regime. In the first we considered a magnetoexciton above an integrally filled Landau level ground state. We use the single-mode approximation (SMA) to generate approximate excited-state wave functions as a function of $\mathbf{K}$. The result reproduces the QGD of an inter-Landau-level exciton in the strong magnetic field limit, in which only two Landau levels (one for the hole, one for the electron) are involved in the state. When substituted into semiclassical equations of motion [43], this form is consistent with the Lorentz invariance that leads to drift motion of the exciton in the presence of an in-plane electric field.

In our second example, we considered magnetoplasmon excitations above a Laughlin state of a partially filled Landau level, again using the SMA. In this the details of the calculation were more involved, in particular requiring a careful formulation of the division of states between particle and hole hosting. This was accomplished by labeling single-particle states by wave vectors, allowing a definition of gradients with respect to $\mathbf{K}$ such that the particlelike and holelike connections $\boldsymbol{\mathcal{A}}^{(p)}$ and $\boldsymbol{\mathcal{A}}^{(h)}$ are well defined. The resulting formal expression for the QGD can be explicitly evaluated in terms of correlation functions in the ground state, but the final result does not involve these: one obtains precisely the result found for the magnetoexciton described above. We demonstrated this simple result must emerge due to the combination of continuous translational invariance of the system, so that the state itself can be assigned a momentum along some direction, and the fact that the state lies fully in the lowest Landau level.

It is important to emphasize that the second example involves a state that cannot be written as a linear combination of single particle-hole pair states, demonstrating that to have a QGD, a state need not have this particular form. And while we have applied our formulation to two examples of quantum Hall systems, this formulation is considerably more general than this, and can in principle be applied to any collection of neutral excitations with well-defined momenta, above some many-body ground state, in general dimensionality. The challenge is that one needs explicit wave functions to carry through calculations. It will be interesting to find further examples of states falling outside the single particle-hole paradigm for which the QGD may be computed. One interesting possibility arises for ideal flat bands [73], which have been formulated in the context of twisted van der Waals systems as minimal models that capture the important aspects of the flat bands believed to be present in these systems [74–76]. Among these are models based on Landau levels [77], for which the techniques used in our work could prove useful.

Short of this, one can apply our formulation to approximate wave functions that involve small numbers of particle-hole pairs, for example as corrections to a single particle-hole wave function, to examine their impact on the dipole moment of the excitation. Beyond computations of the QGD for different collective modes, it will be useful to find their equations of motion in applied electric and/or magnetic fields and determine how the QGD enters them [43]. Indeed, the result above for the QGD of the magnetoplasmon above a Laughlin state suggests its equations of motion may be similar to those of a magnetoexciton above the integrally filled Landau level, in which case the former result can again be understood as a consequence of Lorentz invariance in the underlying Hamiltonian, as is the case for the latter. We leave these investigations for future work.

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APPENDIX A: K DEPENDENCE OF $\mathbf{R}_0^{(\mathbf{K})}$

To understand how $\mathbf{R}_0^{(\mathbf{K})}$ varies with $\mathbf{K}$ [cf. Eq. (7)], we examine its behavior for some excitations where we can generate approximate explicit wave functions. As a first example, we consider plasmon excitations in a metal with a single band and a rotationally symmetric Fermi surface. Within the random phase approximation (RPA), the wave function of such a state has the form [46,47]

$$|\Phi_{\mathbf{K}}\rangle = Q_{\mathbf{K}}^{\dagger}|\Phi_0\rangle, \tag{A1}$$

where $|\Phi_0\rangle = \Pi_{q\leqslant k_F} c_{\mathbf{q}}^{\dagger}|0\rangle$ is the approximate ground state, k_F the Fermi wave vector, $|0\rangle$ the vacuum state, and

$$Q_{\mathbf{K}}^{\dagger} = \sum_{|\mathbf{q}|<k_F} \alpha_{\mathbf{q}}(\mathbf{K}) c_{\mathbf{q}+\mathbf{K}}^{\dagger} c_{\mathbf{q}}. \tag{A2}$$

The precise forms for the coefficients $\alpha_{\mathbf{q}}(\mathbf{K})$ may be written, but are unimportant for our present purpose. The $\mathbf{K}$-dependent density matrix in this case is

$$\begin{aligned}\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}') &= \langle \Phi_{n,\mathbf{K}}|\psi^{\dagger}(\mathbf{r})\psi(\mathbf{r}')|\Phi_{n,\mathbf{K}}\rangle \\ &\equiv \sum_{\mathbf{q}_1} \phi_{\mathbf{q}_1}(\mathbf{r})\phi_{\mathbf{q}_1}^{*}(\mathbf{r}')\lambda_{\mathbf{q}_1}^{(\mathbf{K})}.\end{aligned}$$

Here $\phi_{\mathbf{q}}(\mathbf{r})$ is the single-particle state created by the operator $c^\dagger_{\mathbf{q}}$. We see that the eigenstates of $\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}')$ are identical to those of $\rho_{\mathbf{K}=0}(\mathbf{r},\mathbf{r}')$; all that changes is the average occupation of the single-particle states. Moreover, it is not difficult to show that $\alpha_{\mathbf{q}}(\mathbf{K}) \sim 1/\sqrt{N_c}$, where N_c is the number of unit cells in the system, so that $\lambda^{(\mathbf{K})} - \lambda^{(\mathbf{K}=0)} \sim 1/N_c$. Thus, the change in the density matrix with $\mathbf{K}$ is negligibly small in the thermodynamic limit. Independent of this last fact, the natural choice of hole-hosting states here is $\{\phi_{\mathbf{q}}\}$ with $q < k_F$, independent of $\mathbf{K}$. It immediately follows that $\mathbf{R}_0^{(\mathbf{K})}$ is independent of $\mathbf{K}$ for this system. Moreover, it vanishes if the system is inversion symmetric.

As a second example we consider an insulating system whose ground state may be approximated by a single filled band, with quasiparticles which may reside in many higher-energy bands, separated from the occupied band by a gap. Within a Hartree-Fock approximation, each band j hosts a set of single-particle states $\{\phi_{j,\mathbf{q}}\}$, and we assume the $j = 0$ band is completely filled in the Hartree-Fock ground state $|\Phi_0\rangle$. Low-energy neutral collective modes of such systems can typically be written in terms of some linear combination of particle-hole pairs, in the same form as Eq. (A1), but with

$$Q^\dagger_{\mathbf{K}} = \sum_{\mathbf{q}} \sum_{j>0} \alpha_{j,\mathbf{q}}(\mathbf{K}) c^\dagger_{j,\mathbf{q}+\mathbf{K}} c_{0,\mathbf{q}}. \tag{A3}$$

The coefficients $\alpha_{j,\mathbf{q}}(\mathbf{K})$ may be found within the RPA approximation, or within a time-dependent Hartree-Fock approximation. In either case the precise form is again unimportant for our present purpose, except to note that $\alpha_{j,\mathbf{q}}(\mathbf{K}) \sim 1/\sqrt{N_c}$. In this case we have

$$\begin{aligned}\rho_{\mathbf{K}}(\mathbf{r},\mathbf{r}') &= \langle \Phi_{\mathbf{K}}|\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')|\Phi_{\mathbf{K}}\rangle \\ &= \sum_{\mathbf{q}} \sum_{j_1,j_2} \phi^*_{j_1,\mathbf{q}}(\mathbf{r})\phi_{j_2,\mathbf{q}}(\mathbf{r}')\langle \Phi_{\mathbf{K}}|c^\dagger_{j_1,\mathbf{q}} c_{j_2,\mathbf{q}}|\Phi_{\mathbf{K}}\rangle.\end{aligned} \tag{A4}$$

We need to compute

$$\langle \Phi_{\mathbf{K}}|c^\dagger_{j_1,\mathbf{q}} c_{j_2,\mathbf{q}}|\Phi_{\mathbf{K}}\rangle = \sum_{\mathbf{q}_1,\mathbf{q}_2} \sum_{j_3,j_4>0} \alpha^*_{j_3,\mathbf{q}_1}(\mathbf{K})\alpha_{j_4,\mathbf{q}_2}(\mathbf{K})\langle \Phi_0|c^\dagger_{0,\mathbf{q}_1} c_{j_3,\mathbf{q}_1+\mathbf{K}} c^\dagger_{\mathbf{q},j_1} c_{\mathbf{q},j_2} c^\dagger_{j_4,\mathbf{q}_2+\mathbf{K}} c_{0,\mathbf{q}_2}|\Phi_0\rangle. \tag{A5}$$

Taking into account the occupations of the bands, one finds

$$\langle \Phi_0|c^\dagger_{0,\mathbf{q}_1} c_{j_3,\mathbf{q}_1+\mathbf{K}} c^\dagger_{j_1,\mathbf{q}} c_{j_2,\mathbf{q}} c^\dagger_{j_4,\mathbf{q}_2+\mathbf{K}} c_{0,\mathbf{q}_2}|\Phi_0\rangle = \left[\delta_{j_1,0}\delta_{j_2,0}\delta_{\mathbf{q}_1,\mathbf{q}_2} - \delta_{j_1,0}\delta_{j_2,0}\delta_{\mathbf{q},\mathbf{q}_1}\delta_{\mathbf{q},\mathbf{q}_2}\right]\delta_{j_3,j_4} + \delta_{\mathbf{q}_1,\mathbf{q}_2}\delta_{j_1,j_3}\delta_{j_2,j_4}\delta_{\mathbf{q},\mathbf{q}_1+\mathbf{K}}\delta_{\mathbf{q},\mathbf{q}_2+\mathbf{K}}, \tag{A6}$$

so that

$$\langle \Phi_{\mathbf{K}}|c^\dagger_{j_1,\mathbf{q}} c_{j_2,\mathbf{q}}|\Phi_{\mathbf{K}}\rangle = \left[\sum_{j_3} \sum_{\mathbf{q}_1\neq\mathbf{q}} |\alpha_{j_3,\mathbf{q}_1}(\mathbf{K})|^2\right]\delta_{j_1,0}\delta_{j_2,0} + \alpha^*_{j_1,\mathbf{q}-\mathbf{K}}\alpha_{j_2,\mathbf{q}-\mathbf{K}}. \tag{A7}$$

Equations (A4) and (A7)show that the $\mathbf{K} = 0$ wave functions once again diagonalize the density matrix, up to corrections that vanish in the thermodynamic limit.

Equation (A7) points to what is needed for the density matrix to develop $\mathbf{K}$ dependence in the thermodynamic limit: one needs coherences between different bands that actually change with $\mathbf{K}$ within the excited-state band. Such excitations in general require rather high energy because this implies correlations among the electrons that vary strongly away from those of the ground state. In an excitation where the number of particle-hole pairs above the ground state needed to describe the state does not scale as the size of the system, the form of the hole-hosting states does not in practice change with $\mathbf{K}$ in the thermodynamic limit.

APPENDIX B: CALCULATION OF $\sum_{n>0} \Gamma_n(\mathbf{K}_1, \mathbf{K}_2)$ AND $\Gamma_0(\mathbf{K}_1, \mathbf{K}_2)$

In this Appendix, we present some details of the calculations involved in the computation of the QGD for a magnetoexciton state above a single filled Landau level, with the state generated by the single-mode approximation (SMA). In Sec. III B, one finds an expression for the QGD, Eq. (14), that involves the quantities Γ_n defined in Eqs. (12) and (13). These latter quantities involve matrix elements that can be explicitly evaluated. Specifically, one requires

$$\begin{aligned}&\langle \Phi_{\mathbf{K}_1}|c^\dagger_{n,X_1} c_{n,X_2}|\Phi_{\mathbf{K}_2}\rangle \\ &\quad = \frac{1}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}\langle \Phi_0|Q_{\mathbf{K}_1} c^\dagger_{n,X_1} c_{n,X_2} Q^\dagger_{\mathbf{K}_2}|\Phi_0\rangle,\end{aligned}$$

with $Q^\dagger_{\mathbf{K}} = \sum_{n_1,n_2}\sum_{X_1,X_2}\langle \phi_{n_1,X_1}|e^{i\mathbf{K}\cdot\mathbf{r}}|\phi_{n_2,X_2}\rangle c^\dagger_{n_1,X_1} c_{n_2,X_2}$. Noting that, for $n > 0$,

$$\begin{aligned}c_{n,X} Q^\dagger_{\mathbf{K}}|\Phi_0\rangle &= \sum_{X_1,X_2}\sum_{n_1,n_2}\langle \phi_{n_1,X_1}|e^{i\mathbf{K}\cdot\mathbf{r}}|\phi_{n_2,X_2}\rangle c_{n,X} c^\dagger_{n_1,X_1} c_{n_2,X_2}|\Phi_0\rangle \\ &= \sum_{X_1,X_2}\sum_{n_1}\langle \phi_{n_1,X_1}|e^{i\mathbf{K}\cdot\mathbf{r}}|\phi_{0,X_2}\rangle c_{n,X} c^\dagger_{n_1,X_1} c_{0,X_2}|\Phi_0\rangle \\ &= \sum_{X_2}\langle \phi_{n,X}|e^{i\mathbf{K}\cdot\mathbf{r}}|\phi_{0,X_2}\rangle c_{0,X_2}|\Phi_0\rangle,\end{aligned}$$

we arrive at

$$\langle\Phi_0|Q_{\mathbf{K}_1}c^\dagger_{n,X_1}c_{n,X_2}Q^\dagger_{\mathbf{K}_2}|\Phi_0\rangle=\sum_{X_3}\langle\phi_{n,X_2}|e^{i\mathbf{K}_2\cdot\mathbf{r}}|\phi_{0,X_3}\rangle\langle\phi_{0,X_3}|e^{-i\mathbf{K}_1\cdot\mathbf{r}}|\phi_{n,X_1}\rangle. \tag{B1}$$

A similar calculation yields

$$\langle\Phi_0|Q_{\mathbf{K}_1}c_{0,X_1}c^\dagger_{0,X_2}Q^\dagger_{\mathbf{K}_2}|\Phi_0\rangle=\sum_{n_3>0}\sum_{X_3}\langle\phi_{0,X_1}|e^{-i\mathbf{K}_1\cdot\mathbf{r}}|\phi_{n_3,X_3}\rangle\langle\phi_{n_3,X_3}|e^{i\mathbf{K}_2\cdot\mathbf{r}}|\phi_{0,X_2}\rangle. \tag{B2}$$

The quantities of interest can now be written as

$$\sum_{n>0}\Gamma_n(\mathbf{K}_1,\mathbf{K}_2)=\left[\frac{1}{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}\right]^{1/2}\sum_{X_1,X_2,X_3}\sum_{n>0}\langle\phi_{n,X_2}|e^{i\mathbf{K}_2\cdot\mathbf{r}}|\phi_{0,X_3}\rangle\langle\phi_{0,X_3}|e^{-i\mathbf{K}_1\cdot\mathbf{r}}|\phi_{n,X_1}\rangle\langle\phi_{n,X_1}|e^{i(\mathbf{K}_1-\mathbf{K}_2)\cdot\mathbf{r}}|\phi_{n,X_2}\rangle,$$

$$\Gamma_0(\mathbf{K}_1,\mathbf{K}_2)=\left[\frac{1}{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}\right]^{1/2}\sum_{X_1,X_2,X_3}\sum_{n>0}\langle\phi_{0,X_1}|e^{-i\mathbf{K}_1\cdot\mathbf{r}}|\phi_{n,X_3}\rangle\langle\phi_{n,X_3}|e^{i\mathbf{K}_2\cdot\mathbf{r}}|\phi_{n,X_2}\rangle\langle\phi_{0,X_2}|e^{i(\mathbf{K}_1-\mathbf{K}_2)\cdot\mathbf{r}}|\phi_{0,X_1}\rangle.$$

This expression can be evaluated explicitly with use of the matrix element

$$\langle\phi_{n',X'}|e^{i\mathbf{q}\cdot\mathbf{r}}|\phi_{n,X}\rangle=e^{iq_x(X+X')}\delta_{X',X-q_y\ell^2}\left[\frac{n!}{n'!}\right]^{1/2}\left[\frac{(q_y+iq_x)\ell}{\sqrt{2}}\right]^{n'-n}e^{-q^2\ell^2/4}L_n^{n'-n}\left[\frac{q^2\ell^2}{2}\right],$$

where $L_n^{n'-n}$ is an associated Laguerre polynomial, and in writing this we have assumed $n'\geqslant n$. With some algebra, it is possible to show

$$\sum_{X_1,X_2,X_3}\langle\phi_{n,X_2}|e^{i\mathbf{K}_2\cdot\mathbf{r}}|\phi_{0,X_3}\rangle\langle\phi_{0,X_3}|e^{-i\mathbf{K}_1\cdot\mathbf{r}}|\phi_{n,X_1}\rangle\langle\phi_{n,X_1}|e^{i(\mathbf{K}_1-\mathbf{K}_2)\cdot\mathbf{r}}|\phi_{n,X_2}\rangle$$
$$=\frac{g}{n!}e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2}\left[\frac{(K_{2y}+iK_{2x})\ell}{\sqrt{2}}\right]^n\left[\frac{(K_{1y}-iK_{1x})\ell}{\sqrt{2}}\right]^nL_n^0\left[\frac{(\mathbf{K}_1-\mathbf{K}_2)^2\ell^2}{2}\right]e^{-K_1^2\ell^2/4-K_2^2\ell^2/4}e^{-(\mathbf{K}_1-\mathbf{K}_2)^2\ell^2/4},$$

where g is the degeneracy of a Landau level. Noting that $L_n^0(x)\approx 1-nx$ for small x, we can set $L_n^0\left[\frac{(\mathbf{K}_1-\mathbf{K}_2)^2\ell^2}{2}\right]\to 1$ without incurring any error in the final answer since we take only a single gradient and then set $\mathbf{K}_2\to\mathbf{K}_1$. The last Gaussian factor may be dropped for the same reason.

With these results, we can write

$$\sum_{n>0}\Gamma_n(\mathbf{K}_1,\mathbf{K}_2)=\frac{g}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2-K_1^2\ell^2/4-K_2^2\ell^2/4}\left\{\exp\left[\frac{(K_{2y}+iK_{2x})(K_{1y}-iK_{1x})\ell^2}{2}\right]-1\right\}+O(\delta K)^2$$
$$=\frac{g}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}\left\{e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2}-e^{-i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2-K_1^2\ell^2/4-K_2^2\ell^2/4}\right\}+O(\delta K)^2,$$

where $\delta\mathbf{K}\equiv\mathbf{K}_1-\mathbf{K}_2$. A similar calculation yields the result

$$\Gamma_0(\mathbf{K}_1,\mathbf{K}_2)=\frac{g}{\sqrt{\mathcal{N}_{\mathbf{K}_1}\mathcal{N}_{\mathbf{K}_2}}}\left\{1-e^{i\hat{z}\cdot(\mathbf{K}_1\times\mathbf{K}_2)\ell^2/2-K_1^2\ell^2/4-K_2^2\ell^2/4}\right\}+O(\delta K)^2.$$

APPENDIX C: DERIVATION OF EQ. (28)

In Sec. IV C, Eq. (28) is introduced to allow an explicit evaluation of the QGD for a magnetoplasmon excitation above a Laughlin state. In this Appendix, we provide some details of its derivation. The key technique involved is the projection of an operator into the lowest Landau level (LLL). In circular gauge, a state in the LLL has the form $\phi(\mathbf{r})=f(z)e^{-|z|^2/4\ell^2}$, where $z=x-iy$ is a complex representation of a particle position, and f is an analytic function of z. Consider an operator that is a function only of position, for example, a potential energy. This may be written in the form $V(z^*,z)$. Suppose we are interested in the projection of $V\phi$ onto the LLL, which may be written as $\overline{V}\phi$. The prescription for doing this is to make the replacement

$$V(z,z^*)\to\overline{V}\equiv\hat{N}V\left(z,2\ell^2\frac{\partial}{\partial z}\right),$$

where it is understood that when $\overline{V}$ is applied to a LLL state such as ϕ, the derivative operator acts only on $f(z)$ and not the Gaussian part of the wave function. The symbol $\hat{N}$ denotes normal ordering, in which, in a power-law expansion of $V(z,2\ell^2\frac{\partial}{\partial z})$, factors of $2\ell^2\frac{\partial}{\partial z}$ should always be placed to the left of factors of z. This method of implementing LLL projection was introduced in Ref. [69].

As a basic example, consider the density operator

$$Q^\dagger_{\mathbf{K}} \equiv \sum_{i=j}^{N} e^{i\mathbf{K}\cdot\mathbf{r}_j}.$$

This may be rewritten in the form $Q^\dagger_{\mathbf{K}} \equiv \sum_{j=1}^{N} e^{\frac{1}{2}[Kz_j^* + K^* z_j]}$, where $K \equiv K_x - iK_y$ and $z_j = x_j - iy_j$, and the projected operator is formally

$$\overline{Q}^\dagger_{\mathbf{K}} = \sum_{j=1}^{N} e^{K\ell^2 \frac{\partial}{\partial z_j}} e^{\frac{1}{2}K^* z_j}. \tag{C1}$$

In what follows, we will make repeated use of Eq. (C1).

In Eqs. (27) and (28) we encounter a product of three projected density operators $\overline{Q^\dagger}_{-\mathbf{K}-\frac{\mathbf{q}}{2}}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}-\frac{\mathbf{q}}{2}}$, of which we need to take an expectation value with respect to the Laughlin state. As explained in the main text, the result can be written in terms of ground-state correlation functions if we rewrite the expression using terms where we have *first* taken products of density operators, and *then* projected these terms into the LLL. To do this, consider a product of three projected density operators,

$$\begin{aligned}
\overline{Q^\dagger}_{\mathbf{K}_1}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}_2} &= \sum_{ijk} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_j} e^{\frac{1}{2}iq^* z_j} e^{\frac{1}{2}iK_2\bar{z}_k} e^{\frac{1}{2}iK_2^* z_k} \\
&= \sum_{i} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_i} e^{\frac{1}{2}iq^* z_i} e^{\frac{1}{2}iK_2\bar{z}_i} e^{\frac{1}{2}iK_2^* z_i} + \sum_{i\neq k} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_i} e^{\frac{1}{2}iq^* z_i} e^{\frac{1}{2}iK_2\bar{z}_k} e^{\frac{1}{2}iK_2^* z_k} \\
&\quad + \sum_{i\neq j} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_j} e^{\frac{1}{2}iq^* z_j} e^{\frac{1}{2}iK_2\bar{z}_i} e^{\frac{1}{2}iK_2^* z_i} + \sum_{i\neq j} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_j} e^{\frac{1}{2}iq^* z_j} e^{\frac{1}{2}iK_2\bar{z}_j} e^{\frac{1}{2}iK_2^* z_j} \\
&\quad + \sum_{i\neq j\neq k} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_j} e^{\frac{1}{2}iq^* z_j} e^{\frac{1}{2}iK_2\bar{z}_k} e^{\frac{1}{2}iK_2^* z_k},
\end{aligned} \tag{C2}$$

where we have used the notation $\bar{z}_i \equiv 2\ell^2 \frac{\partial}{\partial z_i}$. We then repeatedly use the operator identity $e^A e^B = e^B e^A e^{[A,B]}$, valid when the commutator $[A, B]$ is a c number, to bring this to the form

$$\begin{aligned}
\overline{Q^\dagger}_{\mathbf{K}_1}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}_2} &= \sum_{i} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_i} e^{\frac{1}{2}iq^* z_i} e^{\frac{1}{2}iK_2\bar{z}_i} e^{\frac{1}{2}iK_2^* z_i} \\
&\quad + e^{\frac{1}{2}qK_1^*\ell^2} \sum_{ik} e^{\frac{i}{2}(K_1+q)\bar{z}_i} e^{\frac{i}{2}K_2\bar{z}_k} e^{\frac{i}{2}(K_1^*+q^*)z_i} e^{\frac{i}{2}K_2^* z_k} - e^{\frac{1}{2}qK_1^*\ell^2} \sum_{i} e^{\frac{i}{2}(K_1+q)\bar{z}_i} e^{\frac{i}{2}K_2\bar{z}_i} e^{\frac{i}{2}(K_1^*+q^*)z_i} e^{\frac{i}{2}K_2^* z_i} \\
&\quad + e^{\frac{1}{2}K_1^*K_2\ell^2} \sum_{ij} e^{\frac{i}{2}(K_1+K_2)\bar{z}_i} e^{\frac{i}{2}q\bar{z}_j} e^{\frac{i}{2}(K_1^*+K_2^*)z_i} e^{\frac{i}{2}q^* z_j} - e^{\frac{1}{2}K_1^*K_2\ell^2} \sum_{i} e^{\frac{i}{2}(K_1+K_2)\bar{z}_i} e^{\frac{i}{2}q\bar{z}_i} e^{\frac{i}{2}(K_1^*+K_2^*)z_i} e^{\frac{i}{2}q^* z_i} \\
&\quad + e^{\frac{1}{2}q^*K_2\ell^2} \sum_{ij} e^{\frac{i}{2}(q+K_2)\bar{z}_j} e^{\frac{i}{2}K_1\bar{z}_i} e^{\frac{i}{2}(q^*+K_2^*)z_j} e^{\frac{i}{2}K_1^* z_i} - e^{\frac{1}{2}q^*K_2\ell^2} \sum_{i} e^{\frac{i}{2}(q+K_2)\bar{z}_i} e^{\frac{i}{2}K_1\bar{z}_i} e^{\frac{i}{2}(q^*+K_2^*)z_i} e^{\frac{i}{2}K_1^* z_i} \\
&\quad + \sum_{i\neq j\neq k} e^{\frac{i}{2}K_1\bar{z}_i} e^{\frac{i}{2}q\bar{z}_j} e^{\frac{i}{2}K_2\bar{z}_k} e^{\frac{i}{2}K_2^* z_k} e^{\frac{i}{2}q^* z_j} e^{\frac{i}{2}K_1^* z_i} \\
&= \sum_{i} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_i} e^{\frac{1}{2}iq^* z_i} e^{\frac{1}{2}iK_2\bar{z}_i} e^{\frac{1}{2}iK_2^* z_i} + e^{\frac{1}{2}qK_1^*\ell^2}\big(\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}} Q^\dagger_{\mathbf{K}_2}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\big) \\
&\quad + e^{\frac{1}{2}K_1^*K_2\ell^2}\big(\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{K}_2} Q^\dagger_{\mathbf{q}}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\big) + e^{\frac{1}{2}q^*K_2\ell^2}\big(\overline{Q^\dagger_{\mathbf{K}_1} Q^\dagger_{\mathbf{q}+\mathbf{K}_2}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\big) \\
&\quad + \sum_{i\neq j\neq k} e^{\frac{i}{2}K_1\bar{z}_i} e^{\frac{i}{2}q\bar{z}_j} e^{\frac{i}{2}K_2\bar{z}_k} e^{\frac{i}{2}K_2^* z_k} e^{\frac{i}{2}q^* z_j} e^{\frac{i}{2}K_1^* z_i}.
\end{aligned} \tag{C3}$$

We next focus on the first term in the last equality of Eq. (C3). Carrying out some more commutations brings this to the form

$$\begin{aligned}
\sum_{i} e^{\frac{1}{2}iK_1\bar{z}_i} e^{\frac{1}{2}iK_1^* z_i} e^{\frac{1}{2}iq\bar{z}_i} e^{\frac{1}{2}iq^* z_i} e^{\frac{1}{2}iK_2\bar{z}_i} e^{\frac{1}{2}iK_2^* z_i} &= e^{\frac{1}{2}(K_1^*q + q^*K_2 + K_1^*K_2)\ell^2} \sum_{i} e^{\frac{i}{2}(K_1+q+K_2)\bar{z}_i} e^{\frac{i}{2}\left(K_1^*+q^*+K_2^*\right)z_i} \\
&= e^{\frac{1}{2}(K_1^*q + q^*K_2 + K_1^*K_2)\ell^2} \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}.
\end{aligned} \tag{C4}$$

Finally, we turn to the last term in Eq. (C3). This may be rewritten as

$$\begin{aligned}\sum_{i\neq j\neq k} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_k}e^{\frac{i}{2}K_2^*z_k}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i} &= \sum_{ijk} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_k}e^{\frac{i}{2}K_2^*z_k}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i} - \sum_{i\neq k} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_i}e^{\frac{i}{2}K_2\bar{z}_k}e^{\frac{i}{2}K_2^*z_k}e^{\frac{i}{2}q^*z_i}e^{\frac{i}{2}K_1^*z_i}\\ &\quad - \sum_{i\neq j} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_i}e^{\frac{i}{2}K_2^*z_i}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i} - \sum_{i\neq j} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_j}e^{\frac{i}{2}K_2^*z_j}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i}\\ &\quad - \sum_{i} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_i}e^{\frac{i}{2}K_2\bar{z}_i}e^{\frac{i}{2}K_2^*z_i}e^{\frac{i}{2}q^*z_i}e^{\frac{i}{2}K_1^*z_i}\\ &= \overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}} - \sum_{ik} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_i}e^{\frac{i}{2}K_2\bar{z}_k}e^{\frac{i}{2}K_2^*z_k}e^{\frac{i}{2}q^*z_i}e^{\frac{i}{2}K_1^*z_i} + \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\\ &\quad - \sum_{ij} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_i}e^{\frac{i}{2}K_2^*z_i}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i} + \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\\ &\quad - \sum_{ij} e^{\frac{i}{2}K_1\bar{z}_i}e^{\frac{i}{2}q\bar{z}_j}e^{\frac{i}{2}K_2\bar{z}_j}e^{\frac{i}{2}K_2^*z_j}e^{\frac{i}{2}q^*z_j}e^{\frac{i}{2}K_1^*z_i} + \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}\\ &= \overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} + 2\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} - \overline{Q^\dagger_{\mathbf{K}_1+\mathbf{K}_2}Q^\dagger_{\mathbf{q}}} - \overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}+\mathbf{K}_2}}. \end{aligned} \tag{C5}$$

We can now combine Eqs. (C3), (C4), and (C5) to arrive at the exact relation

$$\begin{aligned}\overline{Q^\dagger}_{\mathbf{K}_1}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}_2} &= \overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} + \left(e^{\frac{1}{2}qK_1^*\ell^2}-1\right)\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} + \left(e^{\frac{1}{2}K_1^*K_2\ell^2}-1\right)\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{K}_2}Q^\dagger_{\mathbf{q}}} + \left(e^{\frac{1}{2}q^*K_2\ell^2}-1\right)\overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}+\mathbf{K}_2}}\\ &\quad + \left[e^{\frac{1}{2}(K_1^*q+K_2q^*+K_1^*K_2)\ell^2} - e^{\frac{1}{2}qK_1^*\ell^2} - e^{\frac{1}{2}K_1^*K_2\ell^2} - e^{\frac{1}{2}q^*K_2\ell^2} + 2\right]\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}}. \end{aligned} \tag{C6}$$

Expanding Eq. (C6) to first nontrivial order in q, one finds

$$\begin{aligned}\overline{Q^\dagger}_{\mathbf{K}_1}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}_2} &= \overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} + \tfrac{1}{2}qK_1^*\ell^2\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}}Q^\dagger_{\mathbf{K}_2}} + \tfrac{1}{2}q^*K_2\ell^2\overline{Q^\dagger_{\mathbf{K}_1}Q^\dagger_{\mathbf{q}+\mathbf{K}_2}}\\ &\quad + \left(e^{\frac{1}{2}K_1^*K_2\ell^2}-1\right)\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{K}_2}Q^\dagger_{\mathbf{q}}} + \left(\tfrac{1}{2}K_1^*q\ell^2 + \tfrac{1}{2}q^*K_2\ell^2\right)\left(e^{\frac{1}{2}K_1^*K_2\ell^2}-1\right)\overline{Q^\dagger_{\mathbf{K}_1+\mathbf{q}+\mathbf{K}_2}} + O(q^2). \end{aligned} \tag{C7}$$

Substituting $\mathbf{K}_1 = -\mathbf{K} - \frac{1}{2}\mathbf{q}$, $\mathbf{K}_2 = \mathbf{K} - \frac{1}{2}\mathbf{q}$ into this yields

$$\begin{aligned}\overline{Q^\dagger}_{-\mathbf{K}-\frac{1}{2}\mathbf{q}}\overline{Q^\dagger}_{\mathbf{q}}\overline{Q^\dagger}_{\mathbf{K}-\frac{1}{2}\mathbf{q}} &= \overline{Q^\dagger_{-\mathbf{K}-\frac{1}{2}\mathbf{q}}Q^\dagger_{\mathbf{q}}Q^\dagger_{\mathbf{K}-\frac{1}{2}\mathbf{q}}} + \tfrac{1}{2}qK^*\ell^2\overline{Q^\dagger_{-\mathbf{K}+\frac{1}{2}\mathbf{q}}Q^\dagger_{\mathbf{K}-\frac{1}{2}\mathbf{q}}} + \tfrac{1}{2}q^*K\ell^2\overline{Q^\dagger_{-\mathbf{K}-\frac{1}{2}\mathbf{q}}Q^\dagger_{\mathbf{K}+\frac{1}{2}\mathbf{q}}}\\ &\quad + \left(e^{-\frac{1}{2}(K^*+\frac{1}{2}q^*)(K+\frac{1}{2}q)\ell^2}-1\right)\overline{Q^\dagger_{-\mathbf{q}}Q^\dagger_{\mathbf{q}}} + \left(-\tfrac{1}{2}K^*q\ell^2 + \tfrac{1}{2}q^*K\ell^2\right)\left(e^{-\frac{1}{2}|K|^2\ell^2}-1\right)\overline{Q^\dagger_0} + O(q^2). \end{aligned} \tag{C8}$$

Finally, setting $\mathbf{q}\to 0$ in the $\overline{Q^\dagger_{-\mathbf{K}+\frac{1}{2}\mathbf{q}}Q^\dagger_{\mathbf{K}-\frac{1}{2}\mathbf{q}}}$ and $\overline{Q^\dagger_{-\mathbf{K}-\frac{1}{2}\mathbf{q}}Q^\dagger_{\mathbf{K}+\frac{1}{2}\mathbf{q}}}$ operators incurs only $O(q^2)$ errors because their coefficient is linear in q and the ground-state expectation value of these operators behaves smoothly as $q\to 0$. With this final change, one arrives at Eq. (28) in the main text.

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