Subsystem CSS codes, a tighter stabilizer-to-CSS mapping, and Goursat's Lemma

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The CSS code construction is a powerful framework used to express features of a quantum code in terms of a pair of underlying classical codes. Its subsystem extension allows for similar expressions, but the general case has not been fully explored. Extending previous work of Aly, Klappenecker, and Sarvepalli [quant-ph/0610153], we determine subsystem CSS code parameters, express codewords, and develop a Steane-type decoder using only data from the two underlying classical codes. Generalizing a result of Kovalev and Pryadko [Phys. Rev. A 88 012311 (2013)], we show that any subsystem stabilizer code can be "doubled" to yield a subsystem CSS code with twice the number of physical, logical, and gauge qudits and up to twice the code distance. This mapping preserves locality and is tighter than the Majorana-based mapping of Bravyi, Terhal, and Leemhuis [New J. Phys. 12 083039 (2010)]. Using Goursat's Lemma, we show that every subsystem stabilizer code can be constructed from two nested subsystem CSS codes satisfying certain constraints, and we characterize subsystem stabilizer codes based on the nested codes' properties.

1 Introduction and summary of results

An extensive theory of quantum error-correcting codes has been developed to protect quantum computers from noise [2, 54, 66] ᴱᶜ. Qualitatively, a quantum error-correcting code describes how to "hide" quantum information within a protected subsystem of a quantum system, such that noise within the quantum system can be detected and removed from the protected subsystem. The class of quantum error-correcting codes called *stabilizer codes* [25, 39] ᴱᶜ remains the most promising route to a working and robust quantum computer, in part due to extra structure that is useful for promptly detecting

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and correcting errors.

We are interested in the special class of *subsystem stabilizer codes* [53, 63] ᴱᶜ, which correspond to the nontrivial normal subgroups of a Pauli group. While subsystem stabilizer codes can be derived from subspace stabilizer codes by using only a subset of the original logical qudits to store quantum information, such codes can exhibit advantageous and/or inherently different properties from subspace stabilizer codes such as lower weights of check operators [20], new fault-tolerant protocols [7], the association with more general types of anyon theories in two dimensions [35], as well as single-shot error correction in three dimensions [16, 56]. The extra data used to define subsystem stabilizer codes yields a richer code structure, warranting its own investigation.

A large part of our work studies *subsystem CSS codes* [4, 5] ᴱᶜ, where the corresponding normal subgroup admits a generating set consisting of X-type and Z-type Pauli operators. Subsystem CSS codes generalize subspace CSS codes [24, 69, 71] ᴱᶜ, of which there are many useful examples [13, 52], recently culminating in the first asymptotically good quantum low-density parity-check codes [31, 37, 62]. Subspace CSS codes have rich connections to classical coding theory [26, 48, 70] and homology theory [14, 18, 22], and they admit a recovery procedure that utilizes classical linear codes to independently correct X-type and Z-type Pauli errors [24, 61]. Thus, it is interesting to investigate the generalization of subspace CSS codes to the subsystem setting. Indeed, many subsystem stabilizer codes are CSS [9, 10, 27], so a general theory of subsystem CSS codes would apply to a broad class of examples developed to date.

To study subsystem CSS codes, we adopt a linear-algebraic perspective that streamlines previous proofs and distills the essential mathematical ideas [43]. This places subsystem CSS codes on the same footing as subspace CSS codes, allowing us to generalize many important features of subspace CSS codes to the subsystem setting.

Our main tool is the symplectic representation, which

allows us to view a subsystem stabilizer code as a subspace of a vector space by considering Pauli groups modulo phase factors. In this representation, the structure and parameters of a subsystem stabilizer code are encoded in a "tower" of subspaces, which can then be manipulated using tools from linear algebra [6, 59, 73] to yield several general results.

Our results include a mapping from subsystem stabilizer codes to subsystem CSS codes, a fleshing out of the connection between subsystem CSS codes and classical coding theory, a general recovery procedure for subsystem CSS codes, and a characterization of subsystem stabilizer codes based on Goursat's Lemma from group theory.

In Section 2, we review the theory of Pauli groups and subsystem stabilizer codes, and we explain how to translate these objects to the linear algebraic framework.

In Section 3, we review the subspace CSS construction and its generalization to the subsystem case. While the subspace CSS construction utilizes two classical linear codes [49] 🔒 under certain constraints, the subsystem CSS construction utilizes two classical linear codes with no constraints. We express the logical and gauge codewords in terms of the two underlying classical linear codes.

In Section 4, we show that the performance of subsystem CSS codes is comparable to that of subsystem stabilizer codes. More precisely, we show that every modular-qudit subsystem stabilizer code 🔒 can be "doubled" to yield a subsystem CSS code with twice the number of physical, logical, and gauge qudits and up to twice the code distance. This generalizes the mapping for subspace stabilizer codes obtained in [55, Theorem 1].

In Section 5, we present an error-correction procedure for subsystem CSS codes that generalizes the well-known Steane procedure for subspace CSS codes (see, e.g., [24, 40, 61]). While specific instances of this procedure have been presented previously [8, 17, 20, 23, 38, 47, 56], we are unaware of a presentation of the general case that uses only information from the underlying classical linear codes. Our error-correction procedure for subsystem CSS codes decodes up to arbitrary actions on the gauge qudits, and the corresponding classical step in our error-correction procedure decodes up to prescribed "redundant" subcodes obtained from the underlying classical codes. We explore an alternative decoder that generalizes the recovery procedure for the Bacon–Shor code [9] in Appendix C.

In Section 6, we show that every subsystem stabilizer code can be constructed from two subsystem CSS codes satisfying certain constraints, much as every subsystem CSS code can be constructed from two classical codes.

We investigate the structure of a given subsystem stabilizer code in terms of its two underlying subsystem CSS codes, and we define two code families that can be viewed as generalizations of subsystem CSS codes from this perspective (see Fig. 2).

Finally, we provide concluding remarks in Section 7. Most proofs have been deferred to Appendix A for readability.

2 Preliminaries

In Section 2.1, we review the construction of Pauli groups. In Section 2.2, we explain how to interpret Pauli groups as vector spaces, and we review the related theory of bilinear forms on vector spaces. In Section 2.3, we review the construction of subsystem stabilizer codes. Finally, in Section 2.4, we explain how to interpret subsystem stabilizer codes as subspaces of vector spaces, and we define the properties of subsystem stabilizer codes in the vector space setting.

2.1 Pauli groups

Consider a quantum system with n prime qudits, each of dimension p. The Hilbert space of this system is given by the quantization of $G \coloneqq \mathbb{F}_p^n$, i.e.,

$$\mathbb{H} \coloneqq \langle \{|g\rangle \mid g \in G\} \rangle \cong (\mathbb{C}^p)^{\otimes n}. \tag{1}$$

Acting on this space is a special class of linear operators called *Pauli operators*, denoted by X^a, Z^b for any $a, b \in G$. For any basis vector $|g\rangle \in \mathbb{H}$, the Pauli operators act as

$$X^a |g\rangle \coloneqq |a + g\rangle \quad \text{and} \tag{2}$$

$$Z^b |g\rangle \coloneqq e^{\frac{2\pi i}{p} b \cdot g} |g\rangle, \tag{3}$$

where $\cdot$ is the usual dot product on G. One verifies the relations

$$X^a Z^b X^c Z^d = e^{\frac{2\pi i}{p} b \cdot c} X^{a+c} Z^{b+d} \quad \text{and} \tag{4}$$

$$X^a Z^b X^c Z^d = e^{\frac{2\pi i}{p} (b \cdot c - a \cdot d)} X^c Z^d X^a Z^b \tag{5}$$

among the Pauli operators. We define the *weight* of a Pauli operator to be the number of qudits on which it acts nontrivially. The Pauli operators generate the *Pauli group*, which is given explicitly by

$$\begin{aligned} \mathcal{P} &\coloneqq \langle \{X^a, Z^b \mid a, b \in G\} \rangle \\ &= \left\{ e^{\frac{2\pi i}{p} \kappa} X^a Z^b \,\middle|\, \kappa \in \mathbb{F}_p, a, b \in G \right\}. \end{aligned} \tag{6}$$

We remark that in other works, the phase factors included in the Pauli group may differ depending on whether p is odd or even [28, 35, 60]. This will not matter much for our work, since we will henceforth consider Pauli groups modulo phase factors.

2.2 Pauli groups as vector spaces

In light of Eqs. (4) and (6), one sees that the Pauli group modulo phases is essentially equivalent to the vector space $G \times G$. Formally, letting

$$\Phi := \left\{ e^{\frac{2\pi i}{p} \kappa} \,\middle|\, \kappa \in \mathbb{F}_p \right\} \tag{7}$$

denote the collection of phases in the Pauli group, one verifies that the map

$$\begin{aligned} \mathcal{P}/\Phi &\to G \times G \\ \Phi X^a Z^b &\mapsto (a, b) \end{aligned} \tag{8}$$

is a vector space isomorphism. For most of this work, it will be convenient to view the Pauli group modulo phases as the vector space $G \times G$. We review the necessary theory below [43, 73].

A function $\xi : G \times G \to \mathbb{F}_p$ is called *bilinear* if

$$\begin{aligned} \xi(a + \lambda b, c) &= \xi(a, c) + \lambda \xi(b, c) \text{ and} \\ \xi(a, b + \lambda c) &= \xi(a, b) + \lambda \xi(a, c), \end{aligned} \tag{9}$$

symmetric if

$$\xi(a, b) = \xi(b, a), \tag{10}$$

antisymmetric if

$$\xi(a, b) = -\xi(b, a), \tag{11}$$

and *nondegenerate* if

$$\xi(a, G) = 0 \implies a = 0. \tag{12}$$

We call ξ a *form* if it is bilinear and nondegenerate. Henceforth, ξ will denote a symmetric or antisymmetric form, θ will denote a symmetric form, and ω will denote an antisymmetric form. Every subspace $H \leq G$ has a *complement with respect to ξ*

$$H^\xi := \{a \in G \,|\, \xi(a, H) = 0\} \leq G. \tag{13}$$

One verifies that

$$\dim H + \dim H^\xi = \dim G \text{ and} \tag{14}$$

$$\left(H^\xi\right)^\xi = H. \tag{15}$$

Moreover, if $K \leq G$ is another subspace, then

$$(H + K)^\xi = H^\xi \cap K^\xi \text{ and} \tag{16}$$

$$(H \cap K)^\xi = H^\xi + K^\xi. \tag{17}$$

A symmetric form θ on G passes naturally to a symmetric form

$$((a, b), (c, d)) \mapsto \theta(a, c) + \theta(b, d) \tag{18}$$

on $G \times G$, which by abuse of notation we also call θ (note that for the Pauli group, θ is the dot product). In addition, the original symmetric form θ on G induces the antisymmetric form

$$\omega((a, b), (c, d)) := \theta(b, c) - \theta(a, d) \tag{19}$$

on $G \times G$ (note that for the Pauli group, ω encodes the commutation of the Pauli operators as in Eq. (5)). Direct products behave well with respect to these induced forms. Indeed, if $H_X \times H_Z \leq G \times G$, then

$$(H_X \times H_Z)^\theta = H_X^\theta \times H_Z^\theta \text{ and} \tag{20}$$

$$(H_X \times H_Z)^\omega = H_Z^\theta \times H_X^\theta. \tag{21}$$

For any $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in G$, we call

$$\mathrm{wt}(a) := |\{j \in \{1, \ldots, n\} \,|\, a_j \neq 0\}| \tag{22}$$

the *weight* of a, and we call

$$\mathrm{swt}(a, b) := |\{j \in \{1, \ldots, n\} \,|\, (a_j, b_j) \neq (0, 0)\}| \tag{23}$$

the *symplectic weight* of (a, b). For any nontrivial $S \subseteq G$ and $T \subseteq G \times G$, we write

$$\min \mathrm{wt}(S) := \min_{a \in S \setminus \{0\}} \mathrm{wt}(a) \text{ and} \tag{24}$$

$$\min \mathrm{swt}(T) := \min_{(a, b) \in T \setminus \{0\}} \mathrm{swt}(a, b). \tag{25}$$

Note that the weight of a Pauli operator $X^a Z^b$ is equal to the symplectic weight of the corresponding vector (a, b).

Finally, for any algebraic objects G, H, we write

$$\begin{array}{c} G \\ \Big| \, l \\ H \end{array} \tag{26}$$

to indicate that $H \leq G$ and, if G, H are vector spaces, that $\dim G/H = l$.

2.3 Subsystem stabilizer codes

A subsystem stabilizer code can be interpreted as a subspace stabilizer code with some of its logical qudits relegated to gauge qudits. This necessitates the use of a "gauge group" to determine the gauge qudits.

In the usual formalism for subsystem stabilizer codes [28, 63], one begins with an abelian "stabilizer group" $\mathcal{S} \leq \mathcal{P}$ whose intersection with Φ is trivial. Then, one considers the centralizer of $\mathcal{S}$, denoted $\mathcal{C}(\mathcal{S})$, which is the subgroup of all Pauli operators that commute with everything in $\mathcal{S}$. Finally, one selects a "gauge group"

$\mathcal{G} \leq \mathcal{C}(\mathcal{S})$ whose center is $\langle \Phi, \mathcal{S} \rangle$. These relationships can be summarized in the tower of normal subgroups

$$
\begin{array}{c}
\mathcal{P} \\
| \\
\mathcal{C}(\mathcal{S}) \\
| \\
\mathcal{G} \\
| \\
\langle \Phi, \mathcal{S} \rangle \\
| \\
\Phi
\end{array}
\tag{27}
$$

which encodes the parameters of the subsystem stabilizer code. Indeed, the number of logical (resp. gauge) qudits in $\mathcal{G}$ is given by half the size of any minimal generating set for $\mathcal{C}(\mathcal{S})/\mathcal{G}$ (resp. $\mathcal{G}/\langle \Phi, \mathcal{S} \rangle$). Moreover, the distance of $\mathcal{G}$ is defined to be the minimum weight of a Pauli operator in $\mathcal{C}(\mathcal{S}) \setminus \mathcal{G}$.

Observe that every tower as in Eq. (27) determines a gauge group $\Phi \leq \mathcal{G} \leq \mathcal{P}$. Conversely, every gauge group $\Phi \leq \mathcal{G} \leq \mathcal{P}$ determines a tower as in Eq. (27). Thus, in the usual formalism, a *subsystem stabilizer code* is defined to be a subgroup $\Phi \leq \mathcal{G} \leq \mathcal{P}$. Moreover, one calls $\mathcal{G}$ a *subsystem CSS code* if $\mathcal{G}$ admits a generating set that consists only of X-type or Z-type Pauli operators.

Any subsystem stabilizer code $\mathcal{G}$ generates a von Neumann algebra $\mathrm{Alg}\,\mathcal{G}$ of linear operators on the physical Hilbert space $\mathbb{H}$. This algebra induces an orthogonal decomposition of $\mathbb{H}$ as [28, 44, 63]

$$
\mathbb{H} \cong \bigoplus_{\tau \mathcal{C}(\mathcal{S}) \in \mathcal{P}/\mathcal{C}(\mathcal{S})} \tau(\mathbb{L} \otimes \mathbb{G}),
\tag{28}
$$

where $\mathbb{L} \otimes \mathbb{G}$ is the subspace of $\mathbb{H}$ fixed by the stabilizer group $\mathcal{S}$. With respect to this decomposition, the algebra $\mathrm{Alg}\,\mathcal{G}$ takes the form

$$
\mathrm{Alg}\,\mathcal{G} \cong \bigoplus_{\tau \mathcal{C}(\mathcal{S}) \in \mathcal{P}/\mathcal{C}(\mathcal{S})} \tau(\mathbb{1}_{\mathbb{L}} \otimes \mathrm{End}(\mathbb{G}))\tau^{-1},
\tag{29}
$$

where $\mathrm{End}(\mathbb{G})$ denotes the collection of linear operators on $\mathbb{G}$. We call $\mathcal{G}$ a *subspace stabilizer code* if the gauge space $\mathbb{G}$ is one-dimensional.

2.4 Subsystem stabilizer codes as subspaces of vector spaces

In this work, we find it convenient to consider the tower in Eq. (27) modulo phase factors, since this allows us to apply tools from linear algebra to analyze subsystem stabilizer codes. To begin, recall that $\mathcal{P}/\Phi \cong G \times G$, so there exists some subspace $H \leq G \times G$ corresponding

to the gauge group modulo phases, $\mathcal{G}/\Phi \cong H$. Now, let $\mathcal{C}(\mathcal{G})$ denote the centralizer of $\mathcal{G}$. Then by Eq. (5), we have $\mathcal{C}(\mathcal{G})/\Phi \cong H^\omega$. Since $\langle \Phi, \mathcal{S} \rangle$ is the center of $\mathcal{G}$ and $\mathcal{C}(\mathcal{S})$ is the centralizer of $\langle \Phi, \mathcal{S} \rangle$, we have

$$
\begin{array}{ccc}
\mathcal{P}/\Phi & \cong & G \times G \\
| & & | \\
\mathcal{C}(\mathcal{S})/\Phi & \cong & H + H^\omega \\
| & & | \\
\mathcal{G}/\Phi & \cong & H \\
| & & | \\
\langle \Phi, \mathcal{S} \rangle/\Phi & \cong & H \cap H^\omega \\
| & & | \\
\Phi/\Phi & \cong & 0
\end{array}
\tag{30}
$$

We claim that this tower still encodes the parameters of the original subsystem stabilizer code. To see this, note that by the third and fourth isomorphism theorems [33], we have

$$
\mathcal{C}(\mathcal{S})/\mathcal{G} \cong (H + H^\omega)/H \text{ and} \tag{31}
$$
$$
\mathcal{G}/\langle \Phi, \mathcal{S} \rangle \cong H/(H \cap H^\omega), \tag{32}
$$

so the number of logical and gauge qudits can still be detected in Eq. (30). Moreover, note that the weight of a Pauli operator is phase invariant, so the code distance can still be detected in Eq. (30). This motivates the following proposition, which describes the parameters of a subsystem stabilizer code in the vector space setting (see also [53, Theorem 5]).

Proposition 1. *Let ω be an antisymmetric form on $G \times G$, let $H \leq G \times G$, and let $n := \dim G$. Then*

$$
\begin{array}{c}
G \times G \\
\left| \, n - k - r \right. \\
H + H^\omega \\
\left| \, 2k \right. \\
H \\
\left| \, 2r \right. \\
H \cap H^\omega \\
\left| \, n - k - r \right. \\
0
\end{array}
\tag{33}
$$

for some $k, r \in \mathbb{Z}$. Moreover, we can write

$$
d = \min \mathrm{swt}\left((H + H^\omega) \setminus H\right) \tag{34}
$$

for some $d \in \mathbb{Z}$.

Proof. See Appendix A.1. $\qquad\square$

This discussion leads to an alternative definition of subsystem stabilizer codes in the vector space setting.

Definition 1. A *subsystem stabilizer code* is a subspace $H \leq G \times G$. We call H a *subsystem CSS code* if H is a direct product of subspaces of G. With n, k, r, d as in Proposition 1, we say that H encodes k *logical qudits* and r *gauge qudits* into n *physical qudits*, and that H has *distance* d. To summarize these parameters, we call H an $[[n, k, r, d]]$ code.

3   Subsystem CSS codes

In this section, we summarize what is known about subsystem CSS codes, and we fill in some details that, to our knowledge, have been missing in the general case.

Subsystem CSS codes are constructed from two classical linear codes C_1 and C_2, which are subspaces of the vector space $G = \mathbb{F}_p^n$. The subsystem CSS construction does not require any relations between the two classical codes. Our construction is the same as that in Ref. [5, Corollary 4] via the association $C_1 \leftrightarrow H_X$ and $C_2 \leftrightarrow H_Z$; we define the gauge group to be the direct product of the two classical codes, i.e.,

$$H := H_X \times H_Z. \tag{35}$$

Restricting to the subspace case requires additionally that the gauge group and stabilizer group coincide up to phases, which forces the gauge group to be abelian. This imposes the relations

$$H_X \leq H_Z^\theta \quad \text{and} \quad H_Z \leq H_X^\theta, \tag{36}$$

which are the well-known relations between the two classical codes in the subspace CSS construction.

In general, H_X does not have to be a subspace of H_Z^θ, and the overlap between the two spaces controls the parameters of the subsystem CSS code. For example, suppose that we fix the dimensions of H_X and H_Z, but we choose H_Z such that H_Z^θ has less overlap with H_X. Then the resulting subsystem CSS code encodes more logical and gauge qudits into the same number of physical qudits. This is demonstrated in the following proposition, which expresses the tower in Eq. (33) and the code parameters in Proposition 1 in terms of H_X and H_Z.

Proposition 2. *Let $H = H_X \times H_Z \leq G \times G$ be an*

$[[n, k, r, d]]$ *subsystem CSS code. Then*

$$
\begin{array}{ccccc}
G \times G & = & G & \times & G \\
\mid & & \mid & & \mid \\
H + H^\omega & = & H_X + H_Z^\theta & \times & H_Z + H_X^\theta \\
\;\;\mid 2k & & \;\mid k & & \;\mid k \\
H & = & H_X & \times & H_Z \\
\;\;\mid 2r & & \;\mid r & & \;\mid r \\
H \cap H^\omega & = & H_X \cap H_Z^\theta & \times & H_Z \cap H_X^\theta \\
\mid & & \mid & & \mid \\
0 & = & 0 & \times & 0
\end{array}
\tag{37}
$$

Moreover, let

$$d^{H_X} := \min \operatorname{wt} \left(\left(H_X + H_Z^\theta \right) \setminus H_X \right) \tag{38}$$

be the minimum weight of a non-gauge X-type logical operator, and let

$$d^{H_Z} := \min \operatorname{wt} \left(\left(H_Z + H_X^\theta \right) \setminus H_Z \right) \tag{39}$$

be the minimum weight of a non-gauge Z-type logical operator. Then

$$d = \min \left\{ d^{H_X}, d^{H_Z} \right\}. \tag{40}$$

Proof. See Appendix A.2. □

The above tower (Eq. (37)) is summarized in the fourth column of Table 1, while the fifth column shows the consequences of the subspace restriction (Eq. (36)). Each subspace in these columns is a direct product, where the second factor is obtained from the first factor by switching the letters X and Z. This illustrates the $X \leftrightarrow Z$ symmetry of the CSS construction.

Recall that the subspace CSS construction yields a simple expression for a basis of logical codewords in terms of cosets of the underlying classical codes [24, 40, 61]. Indeed, the logical X-type Paulis commute and generate the entire codespace (up to scalars) when acting on the all-zero logical state, so their labels — cosets of H_X in H_Z^θ — form a complete set of quantum numbers with which we can label a basis of logical codewords. Since a subsystem code can be interpreted as a subspace code with some of its logical qudits relegated to gauge qudits, we can extend this construction to subsystem CSS codes.

In the subsystem case, the logical and gauge qudits are labeled by elements of their own subspaces, $(H_X + H_Z^\theta)/H_X$ and $H_X/(H_X \cap H_Z^\theta)$. A basis for the codespace is given by the collection of vectors of the form

$$|l, g\rangle := \frac{1}{\sqrt{\left| H_X \cap H_Z^\theta \right|}} \sum_{s \in H_X \cap H_Z^\theta} |l + g + s\rangle, \tag{41}$$

Name	Subgroup of $\mathcal{P}$	Subspace of $G \times G$		
		Subsystem Stabilizer	Subsystem CSS	Subspace CSS
Centralizer	$\mathcal{C}(\mathcal{S})/\Phi$	$H + H^\omega$	$H_X + H_Z^\theta \times H_Z + H_X^\theta$	$H_Z^\theta \times H_X^\theta$
Gauge	$\mathcal{G}/\Phi$	H	$H_X \times H_Z$	$H_X \times H_Z$
Stabilizer	$\langle\Phi,\mathcal{S}\rangle/\Phi$	$H \cap H^\omega$	$H_X \cap H_Z^\theta \times H_Z \cap H_X^\theta$	$H_X \times H_Z$
Logical Paulis	$\mathcal{C}(\mathcal{S})/\mathcal{G}$	$(H + H^\omega)/H$	$(H_X + H_Z^\theta)/H_X \times (H_Z + H_X^\theta)/H_Z$	$H_Z^\theta/H_X \times H_X^\theta/H_Z$
Gauge Paulis	$\mathcal{G}/\langle\Phi,\mathcal{S}\rangle$	$H/(H \cap H^\omega)$	$H_X/(H_X \cap H_Z^\theta) \times H_Z/(H_Z \cap H_X^\theta)$	0
Gauge-preserving	$\mathcal{C}(\mathcal{G})/\Phi$	H^ω	$H_Z^\theta \times H_X^\theta$	$H_Z^\theta \times H_X^\theta$
Bare logical Paulis	$\mathcal{C}(\mathcal{G})/\langle\Phi,\mathcal{S}\rangle$	$H^\omega/H \cap H^\omega$	$H_Z^\theta/\left(H_X \cap H_Z^\theta\right) \times H_X^\theta/\left(H_Z \cap H_X^\theta\right)$	$H_Z^\theta/H_X \times H_X^\theta/H_Z$

Table 1: Dictionary between the names (first column), Pauli group representations (second column), and vector space representations of seven components of a subsystem stabilizer code (third column), subsystem CSS code (fourth column), and subspace CSS code (fifth column) (see also [50, Table 2]). For a subsystem stabilizer code, these components are derived from one object — the gauge group H — up to the phases of the stabilizer group elements. For a subsystem CSS code, these components are derived from two objects — H_X and H_Z — which are two unrelated classical linear codes. Imposing the relations in Eq. (36) yields the subspace CSS construction.

where l and g represent elements in $(H_X + H_Z^\theta)/H_X$ and $H_X/(H_X \cap H_Z^\theta)$, respectively (see Appendix D). This reduces to the subspace CSS construction upon imposing Eq. (36).

4 Stabilizer-to-CSS mapping

In this section, we show that every subsystem stabilizer code can be used to construct a subsystem CSS code with comparable parameters (see [55, Theorem 1] for the analogous result for subspace stabilizer codes). Our construction is simple to express in the notation of Sections 2.1 and 2.3. Namely, if $\mathcal{G}$ is a subsystem stabilizer code generated by $\left\{ X^{a_j} Z^{b_j} \right\}_{j=1}^{2r}$, then the corresponding subsystem CSS code $\Delta(\mathcal{G})$ is generated by $\left\{ X^{a_j} \otimes X^{b_j}, Z^{b_j} \otimes Z^{-a_j} \right\}_{j=1}^{2r}$ (see Examples 1, 2, and 3). However, to prove that the codes $\mathcal{G}$ and $\Delta(\mathcal{G})$ do indeed have comparable parameters, it is convenient to work in the vector space formalism introduced in Sections 2.2 and 2.4. Thus, we define our mapping Δ in the vector space setting, and we exhibit its key properties in the following lemma.

Lemma 1. *For any $(a, b) \in G \times G$, define*

$$\Psi(a, b) := (b, -a). \tag{42}$$

For any $H \leq G \times G$, define

$$\Delta(H) := H \times \Psi(H) \leq (G \times G) \times (G \times G). \tag{43}$$

Then for any $H, K \leq G \times G$, we have

$$\Delta(H + K) = \Delta(H) + \Delta(K), \tag{44a}$$
$$\Delta(H \cap K) = \Delta(H) \cap \Delta(K), \tag{44b}$$
$$\Delta(H^\omega) = \Delta(H)^\omega, \text{ and} \tag{44c}$$
$$\dim \Delta(H) = 2 \dim H. \tag{44d}$$

That is, Δ is a lattice embedding that respects ω-complement and doubles dimension.

Proof. To verify Eq. (44c), observe that for any $(a, b), (c, d) \in G \times G$, we have

$$\theta(\Psi(a, b), \Psi(c, d)) = \theta((a, b), (c, d)), \tag{45a}$$
$$\omega(\Psi(a, b), \Psi(c, d)) = \omega((a, b), (c, d)), \tag{45b}$$
$$\theta(\Psi(a, b), (c, d)) = \omega((a, b), (c, d)), \text{ and} \tag{45c}$$
$$\omega((a, b), \Psi(c, d)) = \theta((a, b), (c, d)). \tag{45d}$$

Then for any subspace $H \leq G \times G$, we have

$$\Psi(H^\theta) = \Psi(H)^\theta, \tag{46a}$$
$$\Psi(H^\omega) = \Psi(H)^\omega, \tag{46b}$$
$$H^\omega = \Psi(H)^\theta, \text{ and} \tag{46c}$$
$$H^\theta = \Psi(H)^\omega. \tag{46d}$$

Thus, we have

$$\begin{aligned} \Delta(H^\omega) &= H^\omega \times \Psi(H^\omega) \\ &= \Psi(H)^\theta \times H^\theta \\ &= (H \times \Psi(H))^\omega \\ &= \Delta(H)^\omega, \end{aligned} \tag{47}$$

where the second line holds by Eq. (46), and the third line holds by Eq. (21). To verify Eq. (44a), observe that

$$\begin{aligned} \Delta(H) + \Delta(K) &= H \times \Psi(H) + K \times \Psi(K) \\ &= (H + K) \times (\Psi(H) + \Psi(K)) \\ &= (H + K) \times \Psi(H + K) \\ &= \Delta(H + K). \end{aligned} \tag{48}$$

To verify Eq. (44b), observe that

$$\begin{aligned} \Delta(H \cap K) &= \Delta((H^\omega + K^\omega)^\omega) \\ &= (\Delta(H)^\omega + \Delta(K)^\omega)^\omega \\ &= \Delta(H) \cap \Delta(K). \end{aligned} \tag{49}$$

Finally, to verify Eq. (44d), note that Ψ is an isomorphism on $G \times G$, and take the dimension of the right-hand side of Eq. (43). □

We are now ready to state and prove the main result of this section.

Theorem 1. *Let $H \leq G \times G$ be an $[[n, k, r, d]]$ subsystem stabilizer code. Then $\Delta(H)$ is a $[[2n, 2k, 2r, d']]$ subsystem CSS code, where $d \leq d' \leq 2d$. Moreover, if H admits a collection of generators with symplectic weight at most w, then $\Delta(H)$ admits a collection of generators with symplectic weight at most $2w$.*

Proof. Apply Δ to the tower

$$
\begin{array}{c}
G \times G \\
\left|\, n - k - r \right. \\
H + H^\omega \\
\left|\, 2k \right. \\
H \\
\left|\, 2r \right. \\
H \cap H^\omega \\
\left|\, n - k - r \right. \\
0
\end{array}
\qquad (50)
$$

to obtain the tower

$$
\begin{array}{c}
(G \times G) \times (G \times G) \\
\left|\, 2n - 2k - 2r \right. \\
\Delta(H) + \Delta(H)^\omega \\
\left|\, 4k \right. \\
\Delta(H) \\
\left|\, 4r \right. \\
\Delta(H) \cap \Delta(H)^\omega \\
\left|\, 2n - 2k - 2r \right. \\
0
\end{array}
\qquad (51)
$$

which shows that $\Delta(H)$ is a $[[2n, 2k, 2r, d']]$ subsystem CSS code.

To bound d', first note that by Proposition 2 along with Eq. (46), it follows that the tower in Eq. (51) is equal to the tower

$$
\begin{array}{ccc}
G \times G & \times & \Psi(G \times G) \\
| & & | \\
H + H^\omega & \times & \Psi(H + H^\omega) \\
| & & | \\
H & \times & \Psi(H) \\
| & & | \\
H \cap H^\omega & \times & \Psi(H \cap H^\omega) \\
| & & | \\
0 & \times & \Psi(0)
\end{array}
\qquad (52)
$$

Since the map Ψ is weight-preserving, it follows from Proposition 2 that

$$
d' = \min \mathrm{wt}\left((H + H^\omega) \setminus H \right). \qquad (53)
$$

Now, for any $x \in G \times G$, we have

$$
\mathrm{swt}(x) \leq \mathrm{wt}(x) \leq 2\,\mathrm{swt}(x). \qquad (54)
$$

Minimizing over $(H + H^\omega) \setminus H$, we find that

$$
d \leq d' \leq 2d, \qquad (55)
$$

as needed.

Finally, if T is a generating set for H, then $(T \times 0) \cup (0 \times \Psi(T))$ is a generating set for $\Delta(H)$, and for any $t \in T$, we have

$$
\mathrm{swt}(t, 0) \leq 2\,\mathrm{swt}(t) \text{ and} \qquad (56)
$$
$$
\mathrm{swt}(0, \Psi(t)) \leq 2\,\mathrm{swt}(\Psi(t)) = 2\,\mathrm{swt}(t), \qquad (57)
$$

and we are done. □

Intuitively, our mapping in Eq. (43) embeds two copies of the subsystem stabilizer code H into the subsystem CSS code $\Delta(H)$. One copy $H \times 0$ consists entirely of X-type Pauli operators, while the other (isomorphic) copy $0 \times \Psi(H)$ consists entirely of Z-type Pauli operators. This is shown explicitly in Eq. (52), which explains why Δ can be called a "doubling" mapping.

We remark that Theorem 1 can be generalized to subsystem stabilizer codes over modular qudits (see Appendix B). Now, we illustrate our construction with three examples.

Example 1 (Five-qubit code)**.** Consider the $[[5, 1, 0, 3]]$ code [57] 🔗, which is the smallest subspace stabilizer code that can correct any single-qubit Pauli error. Note that the $[[5, 1, 0, 3]]$ code does not admit a CSS representation under single-qubit Clifford rotations. The stabilizer group is

$$
\mathcal{G} = \langle ZXXZI, IZXXZ, ZIZXX, XZIZX \rangle, \qquad (58)
$$

and the logical Pauli operators are

$$\bar{X} = XXXXX \text{ and} \tag{59}$$
$$\bar{Z} = ZZZZZ.$$

Applying the mapping in Lemma 1 and Theorem 1, the corresponding subspace CSS code has ten physical qubits with stabilizer group

$$\Delta(\mathcal{G}) = \left\langle \begin{array}{cccc} XIIXI & IXIIX & XIXII & IXIXI \\ IXXII, & IIXXI, & IIIXX, & XIIIX, \\ IZZII & IIZZI & IIIZZ & ZIIIZ \\ ZIIZI, & IZIIZ, & ZIZII, & IZIZI \end{array} \right\rangle, \tag{60}$$

where we arrange the ten physical qubits in two rows of five for readability.

Example 2 (Double semion code)**.** The double semion code is a topological subspace stabilizer code [34] ⬚ encoding one logical qubit. To define the code, we place one 4-dimensional qudit on each edge of a square lattice with periodic boundary conditions. Then, we specify the gauge group, which is generated by the four Pauli operators

$$
\begin{array}{ccccc}
\begin{array}{c} \ulcorner Z \urcorner \\ Y^\dagger \quad Z^\dagger \\ -X^\dagger \!\!\dashv\!\!\vdash\!\! \tilde{Y} \lrcorner \\ X^\dagger \\ | \end{array}
&
\begin{array}{c} \ulcorner Z^2 \urcorner \\ Z^2 \quad Z^2 \\ \llcorner Z^2 \lrcorner \end{array}
&
&
\begin{array}{c} | \\ X^2 \\ -Z^2 \lrcorner \end{array}
&
\begin{array}{c} \ulcorner X^2 - \\ Z^2 \\ | \end{array}
\end{array}
\tag{61}
$$

per unit cell, where

$$Y := e^{i\pi/4} X^\dagger Z^\dagger \text{ and} \tag{62}$$
$$\tilde{Y} := e^{i\pi/4} Z^\dagger X. \tag{63}$$

Note that the double semion code is not a subspace CSS code. In fact, one can argue that there does not exist a shallow depth Clifford circuit transforming the double semion code into a subspace CSS code. Indeed, if such a circuit exists, then one can define a commuting projector Hamiltonian which is stoquastic [^1], and whose ground state lies in the double semion topological phase. However, it is known that the double semion phase has an intrinsic sign problem [45, 67].

Although the double semion code is not Clifford equivalent to a subspace CSS code, one can apply the mapping in Lemma 1 and Theorem 1 to construct a subspace CSS code with parameters comparable to that of the double semion code. The resulting subspace CSS

[^1]: the X stabilizers in Eq. (61) only contain positive off-diagonal entries, while the Z stabilizers are diagonal and can therefore be shifted by a constant such that all coefficients are positive

code has two physical qudits per edge and eight stabilizer generators per unit cell given by

$$
\begin{array}{cccc}
\begin{array}{c} \ulcorner IX \urcorner \\ XX \quad IX^\dagger \\ -X^\dagger\!\!\dashv\!\!\vdash\!\!XX^\ddagger\!\!\dashv \\ X^\dagger I \\ | \end{array}
&
\begin{array}{c} \ulcorner IX^2 \urcorner \\ IX^2 \quad IX^2 \\ \llcorner IX^2 \lrcorner \end{array}
&
&
\begin{array}{c} | \\ X^2 I \\ -IX^2 \lrcorner \end{array}
\begin{array}{c} \ulcorner X^2 I - \\ IX^2 \\ | \end{array}
\\[2em]
\begin{array}{c} \ulcorner ZI \urcorner \\ ZZ^\dagger \quad Z^\dagger I \\ -IZ\!\!\dashv\!\!\vdash\!\!Z^\dagger Z^\ddagger\!\!\dashv \\ IZ \\ | \end{array}
&
\begin{array}{c} \ulcorner Z^2 I \urcorner \\ Z^2 I \quad Z^2 I \\ \llcorner Z^2 I \lrcorner \end{array}
&
&
\begin{array}{c} | \\ IZ^2 \\ -Z^2 I \lrcorner \end{array}
\begin{array}{c} \ulcorner IZ^2 - \\ Z^2 I \\ | \end{array}
\end{array}
\tag{64}
$$

Here, we use the shorthand $X^a X^b$ to mean $X^a \otimes X^b$ (as opposed to operator multiplication) and similarly for the Z-type operators.

It would be interesting to determine the phase of matter corresponding to the subspace CSS code above. This code cannot be two copies of the double semion code due to the sign problem, but we conjecture that this code is equivalent to two copies of a $\mathbb{Z}_2$ toric code. We leave further investigation of this point to future work.

Example 3 ($\mathbb{Z}_N^{(1)}$ subsystem code)**.** The $\mathbb{Z}_N^{(1)}$ subsystem code is a topological subsystem stabilizer code based on the qudit generalization of the Kitaev honeycomb model introduced in [11, 35] ⬚. It encodes one logical qudit, which is of dimension N if N is odd and $\frac{N}{2}$ if N is even. To define the code, we place one N-dimensional qudit on each vertex of a honeycomb lattice. The gauge group is generated by the two-body checks

$$\tag{65}$$

where

$$Y := XZ. \tag{66}$$

Since each vertex hosts three checks of XX, YY, and ZZ type, there does not exist a shallow depth Clifford circuit transforming the $\mathbb{Z}_N^{(1)}$ subsystem code into a subsystem CSS code. However, using the mapping in Lemma 1 and Theorem 1, we can construct a subsystem CSS code with comparable parameters. This code

is generated by the six checks

$$
\begin{array}{c}
\quad\quad\quad\quad\quad IX \\
XI \diagdown \quad\quad XX \diagup \quad\bigg| \\
\quad\ XI \quad XX \quad IX
\end{array}
\tag{67}
$$

$$
\begin{array}{c}
\quad\quad\quad\quad\quad ZI \\
IZ^\dagger \diagdown \quad ZZ^\dagger \diagup \quad\bigg| \\
\quad\ IZ^\dagger \quad ZZ^\dagger \quad ZI
\end{array}
$$

per unit cell.

The proof of Theorem 1 suggests a general strategy for constructing subsystem CSS codes from subsystem stabilizer codes. Indeed, any function Δ satisfying Eq. (44) yields a mapping of codes that doubles the number of physical, logical, and gauge qudits. Moreover, by modifying Eq. (44d) appropriately, one can allow Δ to map to subspaces of $G^m \times G^m$ for any $m \in \mathbb{N}$, thus producing $[[mn, mk, mr, d']]$ subsystem CSS codes as output. We hope that the proof of Theorem 1 and the related linear algebraic ideas will inspire new mappings from subsystem stabilizer codes to subsystem CSS codes.

We remark that our Theorem 1 is a direct generalization of Theorem 1 in [55]. In addition, results analogous to our Theorem 1 have been obtained previously [19, 42, 58]. The mappings in these works take $[[n, k, 0, d]]$ subspace stabilizer codes to $[[4n, 2k, 0, d']]$ subspace CSS codes. These mappings take subspace stabilizer codes to Majorana fermion codes to subspace CSS codes, which increases the complexity of the proof (a mapping from $[[n, k, r, d]]$ subsystem stabilizer codes to $[[4n, 2k, 2r, 2d]]$ subsystem CSS codes was obtained in [21, Theorem 5.7], but the proof remains rather long despite avoiding the intermediate Majorana fermion codes). Nonetheless, the mappings in [19, 42, 58] can be formulated as maps analogous to Δ in Lemma 1. With some modifications, it is possible to demonstrate the validity of the mappings in [19, 42, 58] by following the general proof structure in Lemma 1 and Theorem 1.

5 Recovery procedure for subsystem CSS codes

Having shown that subsystem CSS codes achieve performance comparable to that of subsystem stabilizer codes, we now study subsystem CSS codes in greater detail. In this section, we present a recovery procedure to correct an unknown Pauli error affecting the physical Hilbert space of a subsystem CSS code. Our recovery procedure generalizes the usual Steane-type decoder for subspace stabilizer codes [24, 40, 61] to the subsystem setting.

To begin, let $H = H_X \times H_Z \leq G \times G$ be a subsystem CSS code. Let $\mathbb{L} \otimes \mathbb{G} \leq \mathbb{H}$ be the codespace of H, i.e., the fixed space of the stabilizer group of H. Suppose that a code state

$$
|\psi\rangle = |\psi_\mathbb{L}\rangle \otimes |\psi_\mathbb{G}\rangle \in \mathbb{L} \otimes \mathbb{G}
\tag{68}
$$

is corrupted by a Pauli error $X^a Z^b \in \mathcal{P}$, where $(a, b) \in G \times G$. Our recovery procedure identifies the error (a, b) up to some unknown gauge term in $H_X \times H_Z$.

The first step in our recovery procedure is purely quantum: in Section 5.1, we use gauge generator measurements to determine the syndrome of the Pauli error. The second step in our recovery procedure is purely classical: in Section 5.2, we use two classical linear codes to identify the Pauli error up to gauge terms. Our discussion focuses on correcting the X-component of the error; correcting the Z-component is similar.

5.1 Quantum step: syndrome measurement

We determine the syndrome of the Pauli error by measuring gauge generators. Usually, the syndrome of a Pauli error is defined to be its commutation with each of the stabilizer generators. In the linear algebraic framework, the syndrome of a Pauli error is defined (for subsystem CSS codes) as follows.

Definition 2. Let $H = H_X \times H_Z \leq G \times G$ be a subsystem CSS code. Let $(a, b) \in G \times G$. Let

$$
\begin{aligned}
\mathrm{Syn}_X : G &\to G / \big(H_X + H_Z^\theta \big) \\
a &\mapsto a + \big(H_X + H_Z^\theta \big)
\end{aligned}
\tag{69}
$$

and

$$
\begin{aligned}
\mathrm{Syn}_Z : G &\to G / \big(H_Z + H_X^\theta \big) \\
b &\mapsto b + \big(H_Z + H_X^\theta \big)
\end{aligned}
\tag{70}
$$

be the canonical projections. We call $\mathrm{Syn}_X(a)$ the *syndrome* of a and similarly for b.

For intuition, note that the space $H_X + H_Z^\theta$ is precisely the space of all X-type logical operators, i.e., the space of all X-type Pauli operators that commute with all the Z-type stabilizers in $H_Z \cap H_X^\theta$. Thus, the syndrome of an X-type error a is precisely the set of all X-type Pauli operators with the same Z-type stabilizer commutations as a.

Now, our goal is to use gauge generator measurements to determine the syndrome of a. To begin, fix bases for the spaces

$$
\begin{array}{ll}
H_X = \langle h_X \rangle & H_Z = \langle h_Z \rangle \\
H_X \cap H_Z^\theta = \langle s_X \rangle & H_Z \cap H_X^\theta = \langle s_Z \rangle
\end{array} .
\tag{71}
$$

Observe that for each basis vector s_Z in $H_Z \cap H_X^\theta$, we have

$$s_Z = \sum_{h_Z} c_{h_Z} h_Z \tag{72}$$

for some $c_{h_Z} \in \mathbb{F}$, so

$$Z^{s_Z} = \prod_{h_Z} \left(Z^{h_Z} \right)^{c_{h_Z}} . \tag{73}$$

That is, every Z-type stabilizer generator Z^{s_Z} can be written as a product of Z-type gauge generators Z^{h_Z}. This simple observation suggests that one can determine the syndrome of the error a, i.e., the commutation of X^a with the stabilizer generators Z^{s_Z}, by measuring the gauge generators Z^{h_Z}. To see this formally, note that the stabilizer generators X^{s_X}, Z^{s_Z} and the Z-type gauge generators Z^{h_Z} all commute, so there exists an orthonormal basis $\{|e_i\rangle\}_i$ for $\mathbb{H}$ in which the X^{s_X}, Z^{s_Z}, and Z^{h_Z} are all diagonal. Write the corrupted state in this basis, i.e.,

$$X^a Z^b |\psi\rangle = \sum_i \psi_i |e_i\rangle , \tag{74}$$

where the $\psi_i \in \mathbb{C} \setminus 0$. Since $|\psi\rangle \in \mathbb{L} \otimes \mathbb{G}$, by applying Z^{s_Z} to both sides above, we find that

$$Z^{s_Z} |e_i\rangle = e^{\frac{2\pi i}{p} \theta(s_Z, a)} |e_i\rangle \tag{75}$$

for each $|e_i\rangle$ in Eq. (74). Similarly, applying X^{s_X} to both sides in Eq. (74), we find that

$$X^{s_X} |e_i\rangle = e^{-\frac{2\pi i}{p} \theta(s_X, b)} |e_i\rangle \tag{76}$$

for each $|e_i\rangle$ in Eq. (74). Now, since the $|e_i\rangle$ are also eigenstates of the Z^{h_Z}, measuring any Z^{h_Z} projects the corrupted state $X^a Z^b |\psi\rangle$ onto a subspace spanned by some subset of the $|e_i\rangle$ in Eq. (74). Thus, by Eq. (73), if we measure each gauge generator Z^{h_Z} and multiply each corresponding measurement outcome c_{h_Z} times, we obtain the value $e^{\frac{2\pi i}{p} \theta(s_Z, a)}$, from which we can determine $\theta(s_Z, a)$. Repeating this for each s_Z, we can determine the image of a under the linear map $a \mapsto \sum_{s_Z} \theta(s_Z, a) s_Z$ on G. Since this map has kernel $\left(H_Z \cap H_X^\theta \right)^\theta = H_X + H_Z^\theta$, we can determine $a + \left(H_X + H_Z^\theta \right) = \mathrm{Syn}_X(a)$, as desired.

We claim that this procedure does not disturb the logical component of the corrupted state. Intuitively, this is because the measurement of gauge generators can only modify the gauge component of the corrupted state. To see this formally, recall that the gauge generators Z^{h_Z} are of the form

$$Z^{h_Z} \cong \bigoplus_\tau \tau \left(\mathbb{1}_\mathbb{L} \otimes U_\mathbb{G}^\tau \right) \tau^{-1} \tag{77}$$

for some unitaries $U_\mathbb{G}^\tau \in \mathrm{End}(\mathbb{G})$. Since any projection onto an eigenspace of Z^{h_Z} can be written as a polynomial in Z^{h_Z}, any measurement of Z^{h_Z} effects a projection of the form

$$P \cong \bigoplus_\tau \tau \left(\mathbb{1}_\mathbb{L} \otimes P_\mathbb{G}^\tau \right) \tau^{-1} \tag{78}$$

on $X^a Z^b (|\psi_\mathbb{L}\rangle \otimes |\psi_\mathbb{G}\rangle)$, where the $P_\mathbb{G}^\tau \in \mathrm{End}(\mathbb{G})$ are some projections. Thus, after measuring the gauge generators Z^{h_Z}, the corrupted state is of the form $X^a Z^b (|\psi_\mathbb{L}\rangle \otimes |\psi_\mathbb{G}'\rangle)$ for some gauge state $|\psi_\mathbb{G}'\rangle \in \mathbb{G}$.

Finally, since the state $|\psi_\mathbb{L}\rangle \otimes |\psi_\mathbb{G}'\rangle$ still belongs to the codespace $\mathbb{L} \otimes \mathbb{G}$, one can apply an analogous procedure to determine $\mathrm{Syn}_Z(b)$ by measuring the X-type gauge generators X^{h_X}, and after this procedure, the state of the system is $X^a Z^b (|\psi_\mathbb{L}\rangle \otimes |\psi_\mathbb{G}''\rangle)$ for some $|\psi_\mathbb{G}''\rangle \in \mathbb{G}$. This concludes the first step in our recover procedure.

We remark that our syndrome measurement procedure uses homogeneous scheduling of gauge generator measurements. That is, in our procedure, all Z-type gauge generators are measured, and subsequently all X-type gauge generators are measured. However, other measurement schedules might be relevant for fault-tolerant implementations of our recovery procedure; this is related to the notion of "gauge fixing" [47, 68]. Indeed, the special case of our recovery procedure for subspace CSS codes (i.e., the Steane recovery procedure) is fault-tolerant [40]. Future work could investigate potential fault-tolerant implementations of our generalized recovery procedure for subsystem CSS codes.

5.2 Classical step: error recovery

In the first step of our recovery procedure, we determined the syndrome $\mathrm{Syn}_X(a)$ of the error a. That is, we determined the image of a under the map

$$\mathrm{Syn}_X : G \to G / \left(H_X + H_Z^\theta \right). \tag{79}$$

In the second step of our recovery procedure, we use Syn_X as the parity check of the classical linear code $H_X + H_Z^\theta$ to identify the error a up to gauge terms in H_X. We emphasize here a key point: the syndrome map used in the first step of our recovery procedure to determine the commutation of the error with the stabilizers is *precisely the same as* the parity check of the classical linear code used in the second step of our recovery procedure to determine the error up to gauge terms. Thus, in our recovery procedure, the quantum and classical syndromes coincide, which further illustrates the connection between subsystem CSS codes and classical linear codes.

To begin, we introduce a general notion of a classical linear code capable of identifying an error up to some prescribed "redundant" subcode.

Definition 3. Let N be a vector space over $\mathbb{F}_p$. A *classical linear code* is a subspace $K \leq N$. A *parity check* for K is a linear map $F : N \to M$ such that $\ker F = K$. A *redundant subcode* for K is a subspace $R \leq K$. We say that K has *logical dimension* $k := \dim K$ and *physical dimension* $n := \dim N$. The *distance of K with respect to R* is

$$d^R := \min \text{wt}(K \setminus R). \tag{80}$$

To summarize these parameters, we call K an $\left[n, k, d^R\right]$ code.

Now, suppose that we only wish to identify an error $a \in N$ up to some redundant subcode R. That is, suppose that we only wish to recover the coset $a + R$ from the syndrome Fa of the error. As shown in the following proposition, this is possible when $a + R$ contains some vector of sufficiently small weight relative to the code distance d^R.

Proposition 3. *Let K be a classical linear code with parity check $F : N \to M$ and redundant subcode R. Let $a \in N$ be unknown, but suppose that Fa is known. Suppose that $\min \text{wt}(a + R) < \frac{d^R}{2}$. Then we can determine $a + R$, i.e., we can determine $a + k$ for some unknown redundant vector $k \in R$.*

Proof. See Appendix A.3. $\qquad \square$

The special case of Proposition 3 with trivial redundant subcode is well known [49]. We state it as a corollary below for a sanity check.

Corollary 1. *Let K be a classical linear code with parity check $F : N \to M$. Let $a \in N$ be unknown, but suppose that Fa is known. Suppose that $\text{wt}(a) < \frac{\min \text{wt} K}{2}$. Then we can determine a.*

Let us return to the second step of our recovery procedure. Suppose that $H = H_X \times H_Z$ is an $[[n, k, r, d]]$ subsystem CSS code. Write $k_X := \dim\left(H_X + H_Z^\theta\right)$ and $k_Z := \dim\left(H_Z + H_X^\theta\right)$ (note that $k_X + k_Z = n + k + r$ by Proposition 1). Let d^{H_X}, d^{H_Z} be as in Proposition 2. One verifies from the definitions that $H_X + H_Z^\theta$ with parity check Syn_X and redundant subcode H_X is an $[n, k_X, d^{H_X}]$ classical linear code, and $H_Z + H_X^\theta$ with parity check Syn_Z and redundant subcode H_Z is an $[n, k_Z, d^{H_Z}]$ classical linear code.

Now, suppose that the error a satisfies $\min \text{wt}(a + H_X) < \frac{d^{H_X}}{2}$. Then by Proposition 3, we can recover the error class $a + H_X$ from the syndrome $\text{Syn}_X(a)$, which was determined previously in the first step of our recovery procedure. To correct the X-type error, we can pick any $a + h_X \in a + H_X$ and apply $\left(X^{a+h_X}\right)^{-1}$ to the corrupted code state $X^a Z^b(|\psi_{\mathbb{L}}\rangle \otimes |\psi_{\mathbb{G}}''\rangle)$, which yields the state $Z^b(|\psi_{\mathbb{L}}\rangle \otimes |\psi_{\mathbb{G}}'''\rangle)$. Assuming that the error b

satisfies $\min \text{wt}(b + H_Z) < \frac{d^{H_Z}}{2}$, we can apply a similar procedure to correct the Z-type error and restore the state $|\psi_{\mathbb{L}}\rangle \otimes |\psi_{\mathbb{G}}''''\rangle$. This concludes the second step in our recovery procedure.

We remark that the second step in our recovery procedure performs as well as can be expected, given that the distance of the parent subsystem CSS code is $d = \min\{d^{H_X}, d^{H_Z}\}$ (as shown in Proposition 2). Indeed, if the original Pauli error $X^a Z^b$ satisfies $\text{swt}(a, b) < \frac{d}{2}$, which is the usual correctability condition for subsystem stabilizer codes, then a and b satisfy $\text{wt}(a) < \frac{d^{H_X}}{2}$ and $\text{wt}(b) < \frac{d^{H_Z}}{2}$, so by the above discussion, the second step in our recovery procedure succeeds.

One might note that the second step in our recovery procedure identifies a little more than just the coset $a + H_X$. In fact, as indicated in the proof of Proposition 3, our recovery procedure always identifies a representative $a + h_X$ of $a + H_X$ with weight less than $\frac{d^{H_X}}{2}$ (provided that such a representative exists). One might ask whether a weaker recovery procedure exists, which only recovers the coset $a + H_X$ and nothing else. We explore this idea further in Appendix C.

6 Structure of subsystem stabilizer codes

After a detailed study of subsystem CSS codes, we now turn our attention to the structure of general subsystem stabilizer codes. Our main observation is that every subsystem stabilizer code is associated with two subsystem CSS codes and an isomorphism that determine the structure of the subsystem stabilizer code. In Section 6.1, we formalize this observation using Goursat's Lemma. Then, in Section 6.2, we investigate the structure of subsystem stabilizer codes by associating two subsystem CSS codes to each subsystem stabilizer code in Eq. (33), just as we associated two classical linear codes to each subsystem CSS code in Eq. (37). Finally, in Section 6.3, we apply this theory to introduce two generalizations of subsystem CSS codes.

6.1 Goursat data of the gauge group

We can quantify the degree of "non-CSS-ness" of any subsystem stabilizer code $H \leq G \times G$ by sandwiching it between two subsystem CSS codes,

$$N \leq H \leq E, \tag{81}$$

where the *internal CSS code N* is the largest subsystem CSS code contained in H, and the *external CSS code E* is the smallest subsystem CSS code containing H.

The internal CSS code consists of all elements in the gauge group that are purely X-type or purely Z-type,

i.e., all X^a and Z^b where $X^a, Z^b \in \mathcal{G}$. In particular, the internal CSS code equals H iff H is itself a subsystem CSS code, and the internal CSS code is trivial iff H has no pure X or pure Z elements.

The external CSS code is constructed by independently peeling off the X- and Z-components of each element in the gauge group. That is, the external CSS code contains all X^a and Z^b such that $X^a Z^b \in \mathcal{G}$. In particular, the external CSS code equals H iff H is itself a subsystem CSS code, and the external CSS code always contains H.

If H is not a subsystem CSS code, the inclusions

$$N < H < E \qquad (82)$$

are strict. In particular, the internal code is a strict subspace of the external code, since there are elements of E that cannot be obtained by taking products of the pure gauge group elements in N. The relative size of the internal and external codes corresponds to the number of non-CSS elements in the gauge group. This claim is made precise using Goursat's Lemma.

Goursat's Lemma associates a gauge group with a set of *Goursat data* — the internal code $N = N_X \times N_Z$, the external code $E = E_X \times E_Z$, and an isomorphism $\phi : E_X/N_X \to E_Z/N_Z$ that associates the X- and Z-components of each gauge group element $(e_X, e_Z) \in H$ up to pure X, Z elements:

$$e_X + N_X \underset{\phi^{-1}}{\overset{\phi}{\rightleftharpoons}} e_Z + N_Z. \qquad (83)$$

In other words, for each gauge group element (e_X, e_Z), the isomorphism ϕ associates the non-pure component of e_X with the non-pure component of its partner e_Z. The number of such independent pairs quantifies the "non-CSS-ness" of the code: if E_X/N_X has one element, then H is CSS, but if E_X/N_X is large, then H is highly "non-CSS". This is all stated formally below.

Lemma 2 (Goursat's Lemma). *Let G be a vector space.*

(1) Let $H \leq G \times G$. Define

$$E_X := \{e_X \in G \mid \exists e_Z \in G, (e_X, e_Z) \in H\}, \quad (84a)$$
$$E_Z := \{e_Z \in G \mid \exists e_X \in G, (e_X, e_Z) \in H\}, \quad (84b)$$
$$N_X := \{n_X \in G \mid (n_X, 0) \in H\}, \ and \quad (84c)$$
$$N_Z := \{n_Z \in G \mid (0, n_Z) \in H\}. \quad (84d)$$

For each $e_X \in E_X$, define

$$\phi(e_X + N_X) := e_Z + N_Z, \qquad (85)$$

where $(e_X, e_Z) \in H$. Then

$$\begin{array}{c} G \times G \\ | \\ E_X \times E_Z \\ | \\ N_X \times N_Z \end{array} \qquad (86)$$

and

$$E_X/N_X \xrightarrow{\phi} E_Z/N_Z. \qquad (87)$$

(2) Conversely, let

$$\begin{array}{c} G \times G \\ | \\ E_X \times E_Z \\ | \\ N_X \times N_Z \end{array} \qquad (88)$$

and

$$E_X/N_X \xrightarrow{\phi} E_Z/N_Z. \qquad (89)$$

Define

$$\begin{aligned} H := \{(e_X, e_Z) \in E_X \times E_Z \mid \\ \phi(e_X + N_X) = e_Z + N_Z\}. \end{aligned} \qquad (90)$$

Then $H \leq G \times G$.

(3) The constructions in (1) and (2) are inverses.

Proof. This is [59, Theorem 3]. See also [6, Theorem 4]. $\qquad\qquad\square$

We emphasize that item (3) above states that the correspondence between H and its Goursat data is bijective. Thus, we can write

$$\begin{array}{c} G \times G \\ | \\ H \end{array} \longleftrightarrow \begin{array}{c} G \times G \\ | \\ E_X \times E_Z \\ | \\ N_X \times N_Z \end{array} , \ \phi \qquad (91)$$

to indicate that the subsystem stabilizer code on the left-hand side corresponds via Lemma 2 to the Goursat data on the right-hand side. This allows us to study the structure of a subsystem stabilizer code H by studying the corresponding internal code N, external code E, and isomorphism ϕ.

Goursat's Lemma implicitly states that the subsystem stabilizer code H consists of "blocks" (i.e., cosets) of $N_X \times N_Z$ inside $E_X \times E_Z$. The isomorphism ϕ essentially pairs pure X elements e_X with pure Z elements e_Z to obtain the representatives (e_X, e_Z) of these cosets comprising H. We illustrate this in Fig. 1 and state it formally below.

Proposition 4. *Suppose that Eq. (91) holds. Fix a basis*

$$\left\{ e_X^j + N_X \right\}_j \qquad (92)$$

for E_X/N_X, and consider the basis

$$\left\{ \phi\left(e_X^j + N_X \right) =: e_Z^j + N_Z \right\}_j \qquad (93)$$

for E_Z/N_Z. Then

$$H = \left\{ \sum_j c_j(e_X^j, e_Z^j) + (n_X, n_Z) \in E_X \times E_Z \;\middle|\; \right.$$
$$\left. c_j \in \mathbb{F}_p, (n_X, n_Z) \in N_X \times N_Z \right\}. \tag{94}$$

Moreover, H is a subsystem CSS code iff $E_X = N_X$, which occurs iff $E_Z = N_Z$.

Proof. See Appendix A.4. $\qquad\square$
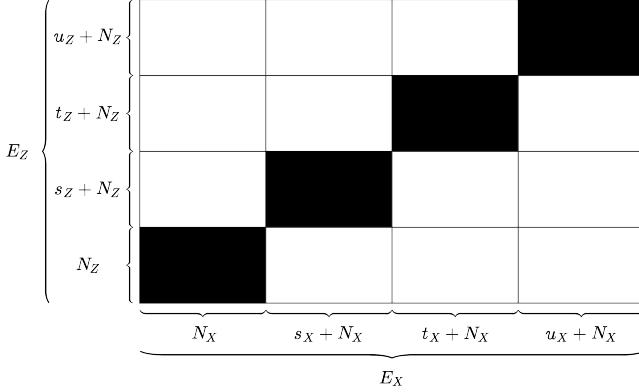
Figure 1: Illustration of Lemma 2 and Proposition 4 with $|E_X/N_X| = 4 = |E_Z/N_Z|$. Any subspace $H \leq G \times G$ consists of "blocks" — i.e., cosets — (dark shaded rectangles) of a direct product $N_X \times N_Z$ (bottom left dark shaded rectangle) contained inside a direct product $E_X \times E_Z$ (large rectangle). Shaded regions belong to H, while unshaded regions do not. Here, $\phi(v_X + N_X) = v_Z + N_Z$ for $v = s, t, u$.

As indicated in Eq. (94), the isomorphism ϕ specifies which X-type operators e_X^j and Z-type operators e_Z^j are "paired together" to form generators of H. We illustrate this in the following example.

Example 4 (5-qubit code)**.** The five-qubit code [57] is a stabilizer code, so its gauge group is the join of its stabilizer group and phases, $\mathcal{G} = \langle \mathcal{S}, \Phi \rangle$.

Inspecting the stabilizer generators of this code in Eq. (58), we see that there are no elements of the gauge group that are purely X-type or Z-type. Thus, the internal code $N_X \times N_Z$ is trivial.

On the other hand, the external code $E_X \times E_Z$ is generated by splitting the gauge group into its X-part and Z-part. That is, with a slight abuse of notation, we have

$$E_X = \langle IXXII, IIXXI, IIIXX, XIIIX \rangle \text{ and} \tag{95}$$
$$E_Z = \langle ZIIZI, IZIIZ, ZIZII, IZIZI \rangle. \tag{96}$$

The isomorphism ϕ is constructed from combinations of $e_X \in E_X$ and $e_Z \in E_Z$ such that $e_X e_Z \in \mathcal{G}$. Thus,

again with a slight abuse of notation, the action of ϕ is

$$IXXII \mapsto ZIIZI, \qquad IIXXI \mapsto IZIIZ, \tag{97}$$
$$IIIXX \mapsto ZIZII, \qquad XIIIX \mapsto IZIZI. \tag{98}$$

An alternative set of generators for the external code is

$$E_X = \langle XXIII, IXXII, IIXXI, IIIXX \rangle \text{ and} \tag{99}$$
$$E_Z = \langle ZZIII, IZZII, IIZZI, IIIZZ \rangle. \tag{100}$$

The external code can be gauged fixed to obtain the $[[5,1,0,3]]$ code. However, the external code has distance one, since $XIIII$ and $ZIIII$ are dressed logicals.

To aid in computer implementations, we note that the Goursat data of a subsystem stabilizer code H can be expressed explicitly in terms of its generators. This is useful for explicitly computing the Goursat data of subsystem stabilizer codes.

Proposition 5. *Suppose that*

$$H = \langle (e_X^1, e_Z^1), \ldots, (e_X^l, e_Z^l) \rangle \leq G \times G. \tag{101}$$

Define the $\dim G \times l$ matrices

$$\pi_X := \begin{pmatrix} | & & | \\ e_X^1 & \ldots & e_X^l \\ | & & | \end{pmatrix} \tag{102}$$

and

$$\pi_Z := \begin{pmatrix} | & & | \\ e_Z^1 & \ldots & e_Z^l \\ | & & | \end{pmatrix}. \tag{103}$$

Then

$$\begin{array}{ccc}
 & G \times G & \\
G \times G & | & \\
| & \longleftrightarrow & \pi_X(\mathbb{F}_p^l) \times \pi_Z(\mathbb{F}_p^l) \qquad , \quad \phi \\
H & | & \\
 & \pi_X(\ker \pi_Z) \times \pi_Z(\ker \pi_X) &
\end{array}$$
$$\tag{104}$$

for some isomorphism ϕ. Moreover, H is a subsystem CSS code iff

$$\ker \pi_X + \ker \pi_Z = \mathbb{F}_p^l. \tag{105}$$

Proof. See Appendix A.5. $\qquad\square$

Note that Eq. (105) provides a characterization of subsystem CSS codes that can be readily checked given the generators of H. Indeed, to check whether a given subsystem stabilizer code is a subsystem CSS code, one can compute spanning sets for $\ker \pi_X$ and $\ker \pi_Z$, take their union to form a spanning set for $\ker \pi_X + \ker \pi_Z$, and compare the dimension of that spanning set with l. However, this procedure is more complicated than simply computing the reduced row echelon form of the generator matrix of H, so Eq. (105) may be most interesting as a theoretical characterization of subsystem CSS codes.

6.2 Goursat data of the stabilizer group

The stabilizer group $H \cap H^\omega$ of a subsystem stabilizer code H can be described in terms of the Goursat data of H. More broadly, our goal in this section is to determine the Goursat data of each subsystem stabilizer code in Eq. (33) in terms of the Goursat data of H, just as we determined the classical linear codes comprising each subsystem CSS code in Eq. (37) in terms of the classical linear codes comprising H.

We first derive the Goursat data of the gauge-preserving group H^ω in terms of that of H.

Proposition 6. *Suppose that*

$$
\begin{array}{c}
 \quad G \times G \\
G \times G \qquad \quad | \\
| \qquad \longleftrightarrow \quad E_X \times E_Z \quad , \quad \phi \ . \qquad (106) \\
H \qquad \qquad\quad | \\
\quad N_X \times N_Z
\end{array}
$$

Then there exist isomorphisms ϕ_X, ϕ_Z in the diagram

$$
N_Z^\theta / E_Z^\theta \xrightarrow{\phi_Z} E_Z/N_Z \xrightarrow{\phi^{-1}} E_X/N_X \xleftarrow{\phi_X} N_X^\theta / E_X^\theta, \quad (107)
$$

such that with

$$
\phi^\omega := \phi_X^{-1} \phi^{-1} \phi_Z, \qquad (108)
$$

we have

$$
\begin{array}{c}
\quad G \times G \\
G \times G \qquad\quad | \\
| \qquad \longleftrightarrow \quad N_Z^\theta \times N_X^\theta \quad , \quad \phi^\omega \qquad (109) \\
H^\omega \qquad\qquad\quad | \\
\quad E_Z^\theta \times E_X^\theta
\end{array}
$$

Proof. See Appendix A.6. □

Intuitively, Proposition 6 states that if a subsystem stabilizer code H corresponds to the subsystem CSS codes $N \leq H \leq E$, then the complement H^ω corresponds to the subsystem CSS codes $E^\omega \leq H^\omega \leq N^\omega$ (c.f. Eq. (21)).

Next, we derive the Goursat data of an intersection of codes $H \cap \tilde{H}$ in terms of the Goursat data of the original codes H and $\tilde{H}$. This result is slightly more general than is needed to determine the Goursat data of the stabilizer $H \cap H^\omega$, since $\tilde{H}$ does not have to equal H^ω.

Proposition 7. *Suppose that*

$$
\begin{array}{c}
\quad G \times G \\
G \times G \qquad\quad | \\
| \qquad \longleftrightarrow \quad E_X \times E_Z \quad , \quad \phi \qquad (110) \\
H \qquad\qquad\quad | \\
\quad N_X \times N_Z
\end{array}
$$

and

$$
\begin{array}{c}
\quad G \times G \\
G \times G \qquad\quad | \\
| \qquad \longleftrightarrow \quad \tilde{E}_X \times \tilde{E}_Z \quad , \quad \tilde{\phi} \ . \qquad (111) \\
\tilde{H} \qquad\qquad\quad | \\
\quad \tilde{N}_X \times \tilde{N}_Z
\end{array}
$$

Then there exist subspaces

$$
\begin{array}{ccc}
E_X \cap \tilde{E}_X & \left(E_X/N_X\right) \times \left(\tilde{E}_X/\tilde{N}_X\right) & \left(E_Z/N_Z\right) \times \left(\tilde{E}_Z/\tilde{N}_Z\right) & E_Z \cap \tilde{E}_Z \\
| & | & | & | \\
T \quad , & \mathcal{U} & \mathcal{V} \quad , & W \qquad (112) \\
| & | & | & | \\
N_X \cap \tilde{N}_X & 0 & 0 & N_Z \cap \tilde{N}_Z
\end{array}
$$

and isomorphisms ϕ_T, ϕ_W in the diagram

$$
T \big/ \big(N_X \cap \tilde{N}_X\big) \xrightarrow{\phi_T} \mathcal{U} \xrightarrow{\phi \times \tilde{\phi}} \mathcal{V} \xleftarrow{\phi_W} W \big/ \big(N_Z \cap \tilde{N}_Z\big) \qquad (113)
$$

such that with

$$
\phi^\cap := \phi_W^{-1} \big(\phi \times \tilde{\phi}\big) \phi_T, \qquad (114)
$$

we have

$$
\begin{array}{c}
\qquad\quad G \times G \\
G \times G \qquad\qquad\qquad | \\
| \qquad \longleftrightarrow \qquad\quad T \times W \qquad\qquad , \quad \phi^\cap \ . \\
H \cap \tilde{H} \qquad\qquad\qquad\qquad | \\
\quad \big(N_X \cap \tilde{N}_X\big) \times \big(N_Z \cap \tilde{N}_Z\big)
\end{array}
$$

$$(115)$$

Proof. See Appendix A.7. □

Intuitively, Proposition 7 states that the internal code of the intersection of H and $\tilde{H}$ is exactly the intersection of the internal codes of H and $\tilde{H}$. However, as indicated in the first and fourth columns of Eq. (112), the external code of the intersection of H and $\tilde{H}$ need only be contained in (but not necessarily equal to) the intersection of the external codes of H and $\tilde{H}$.

Propositions 6 and 7 are sufficient to determine the Goursat data of each subsystem stabilizer code in Eq. (33) in terms of the Goursat data of H. However, these expressions for subsystem stabilizer codes are not as explicit as the expressions for subsystem CSS codes in Eq. (37). Nonetheless, Propositions 6 and 7 might be useful for some applications, such as a characterizaztion of subsystem stabilizer codes.

6.3 Characterizing subsystem stabilizer codes using the Goursat data of the stabilizer group

Here, we characterize subsystem stabilizer codes according to the Goursat data of their stabilizer groups (see Fig. 2 for a summary).
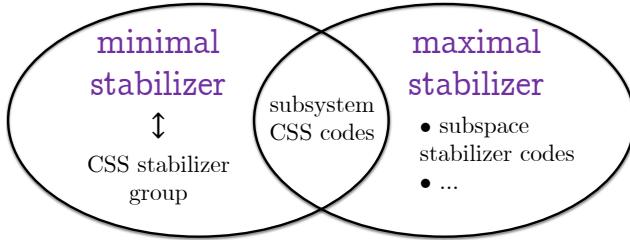
Figure 2: Venn diagram depicting features of subsystem stabilizer codes with maximal and minimal stabilizer groups, per Definition 4. A code has minimal stabilizer iff its stabilizer group is CSS. A code has minimal and maximal stabilizer iff its gauge group is CSS. Every subspace stabilizer code has maximal stabilizer, but there are codes with maximal stabilizer that are not subspace stabilizer codes.

Recall that a subsystem stabilizer code H can equivalently be described by its Goursat data — the internal CSS code N, the external CSS code E, and the isomorphism ϕ. Here, we are interested in the number of blocks in the stabilizer group $H \cap H^\omega$ of H (see Figure 1). To motivate this inquiry, recall that the stabilizer group has one block iff it is CSS, and the stabilizer group has many blocks iff it is "far from CSS". As we will show, the number of blocks in the stabilizer group is bounded above by a function of N and E. This upper bound is independent of the particular isomorphism ϕ, which determines the true number of blocks in the stabilizer group of H. Thus, for fixed internal and external codes N and E, we can define two extreme classes of codes which, in some sense, achieve the largest and the smallest number of blocks admitted by N and E.

Definition 4. Let

$$
\begin{matrix} & & G \times G \\ G \times G & & | \\ | & \longleftrightarrow & E_X \times E_Z \\ H & & | \\ & & N_X \times N_Z \end{matrix} \quad , \quad \phi \ . \tag{116}
$$

By Propositions 6 and 7, we have

$$
\begin{matrix} & & \left(E_X \cap N_Z^\theta\right) \times \left(E_Z \cap N_X^\theta\right) \\ G \times G & & | \\ | & \longleftrightarrow & T \times W \\ H \cap H^\omega & & | \\ & & \left(N_X \cap E_Z^\theta\right) \times \left(N_Z \cap E_X^\theta\right) \end{matrix} \quad , \quad \phi^\cap \tag{117}
$$

for some subspace $T \times W$ (interpreted as the external code of the stabilizer) and some isomorphism $\phi^\cap$. We say that H has *maximal stabilizer* if

$$
T \times W = \left(E_X \cap N_Z^\theta\right) \times \left(E_Z \cap N_X^\theta\right), \tag{118}
$$

i.e., if $T \times W$ is as large as possible. We say that H has *minimal stabilizer* if

$$
T \times W = \left(N_X \cap E_Z^\theta\right) \times \left(N_Z \cap E_X^\theta\right), \tag{119}
$$

i.e., if $T \times W$ is as small as possible.

It is in the following sense that subsystem stabilizer codes with maximal or minimal stabilizer generalize subsystem CSS codes.

Proposition 8. *A subsystem stabilizer code has both maximal and minimal stabilizer iff it is a subsystem CSS code.*

Proof. See Appendix A.8. □

Since subsystem CSS codes have many nice properties, as exhibited in this work and others, it may be interesting to study these two generalizations of subsystem CSS codes. In particular, the class of codes with minimal stabilizer is precisely the class of codes with CSS stabilizer, so it seems particularly natural to investigate whether codes with minimal stabilizer retain some useful properties enjoyed by subsystem CSS codes. In addition, the class of codes with maximal stabilizer is precisely the class of codes where the external code of the intersection of H and H^ω is equal to the intersection of the external codes of H and H^ω (see the discussion following Proposition 7), so this class of codes may also exhibit some interesting properties.

We remark that codes with minimal stabilizer have multiple characterizations.

Corollary 2. *The following are equivalent:*

(1) H has minimal stabilizer

(2) $H + H^\omega$ *is a direct product*

(3) $H \cap H^\omega$ *is a direct product*

(4) $H + H^\omega = (E_X + N_Z^\theta) \times (E_Z + N_X^\theta)$

(5) $H \cap H^\omega = (N_X \cap E_Z^\theta) \times (N_Z \cap E_X^\theta)$

Proof. By definition, (1) and (3) are equivalent. By Eq. (21), (2) and (3) are equivalent, and (4) and (5) are equivalent. By Propositions 6 and 7, (3) and (5) are equivalent. $\square$

This corollary yields a simple algorithm to test whether a subsystem stabilizer code has minimal stabilizer. Indeed, given a spanning set for H, one can compute a spanning set for $H + H^\omega$ and apply Eq. (105) (or row reduction) to test if $H + H^\omega$ is a direct product, i.e., if H has minimal stabilizer. We leave a more detailed investigation of codes with maximal and minimal stabilizer for future research.

7 Conclusion

In this work, we study subsystem stabilizer codes in a linear algebraic framework by viewing Pauli groups as vector spaces and subsystem stabilizer codes as subspaces via the symplectic representation. This perspective allows us to construct a doubling map that converts any modular-qudit subsystem stabilizer code into a subsystem CSS code with comparable parameters (see Theorem 1 and Appendix B).

The linear algebraic perspective also elucidates the correspondence between a subsystem CSS code and a pair of classical linear codes, yielding an explicit basis of codewords (see Section 3), a generalized Steane-type recovery procedure (see Section 5), and other results (see Appendices C and D).

Finally, the linear-algebraic perspective enables a characterization of subsystem stabilizer codes via Goursat's Lemma, which yields two generalizations of subsystem CSS codes (see Section 6).

We conclude with questions and outlook for future investigations.

1. Is our "doubling" map from $[[n, k, r, d]]$ subsystem stabilizer codes to $[[2n, 2k, 2r, d \le d' \le 2d]]$ subsystem CSS codes in Theorem 1 tight in terms of the required number of physical qudits?

2. Which subsystem stabilizer codes yield "doubled" subsystem CSS codes with *exactly* twice the code distance?

3. Does our recovery procedure in Section 5 interact nicely with the "doubled" subsystem CSS code?

4. Keeping in mind that Steane error correction is fault-tolerant for subspace codes, can our subsystem syndrome extraction procedure in Section 5.1 be made fault-tolerant? Fault-tolerant Steane error correction for subspace CSS codes relies on the codes' transversal implementations of inter-block logical CNOT gates and check operator measurements. A subsystem CSS code can be viewed as a subspace CSS code with some logical qubits relegated as gauge qubits. Thus, subsystem CSS codes inherit the ingredients required to make Steane error correction fault-tolerant [41]. Proving fault tolerance of Steane error correction for subsystem CSS codes may be possible with, e.g., the exRec formalism [3], and we leave this as an exciting question for future work.

5. Does the Goursat-based characterization in Section 6.1 simplify significantly for subspace stabilizer codes?

6. How does the Goursat data of a code behave under Clifford deformations?

7. Is there a concise characterization of subsystem stabilizer codes with maximal (minimal) stabilizer, as introduced in Definition 4?

8. Do subsystem stabilizer codes with maximal (minimal) stabilizer have nice structural properties, analogous to those described in Proposition 2 for subsystem CSS codes?

9. Our doubling map holds for all modular-qudit subsystem stabilizer codes ⬚ — can we generalize our other results to modular-qudit codes or operator algebra stabilizer codes [28, 53] ⬚? For a generalization to modular-qudit codes, it is helpful that the theory of classical linear codes, subsystem stabilizer codes, and Goursat's Lemma all generalize to rings of the form $(\mathbb{Z}/m\mathbb{Z})^n$ [35, 59, 65], but many of our proofs in Sections 5 and 6 rely on the vector space structure of the underlying Pauli group. In particular, the lack of a basis for modular-qudit Pauli groups may complicate the existing proofs in this work. For a generalization to operator algebra stabilizer codes, one may need to add to our perspective of subsystem stabilizer codes as subspaces of vector spaces (Section 2.4), since this perspective ignores phases and thus is insensitive to the particular factor of the logical algebra used to encode information (see Eqs. (28) and (29)). Since operator algebra codes encode information in more than one such factor in general, one must keep track of these factors when generalizing our results to operator algebra codes. One may also investigate how

to transition from the Schrödinger picture, in which this paper is expressed, to the Heisenberg picture, in which the theory of operator algebra codes is usually expressed.

10. Using existing (non-CSS) Floquet codes [12, 30, 32, 36, 46, 72, 75] 🖻, new CSS Floquet codes [1, 15, 29, 51, 74] can be constructed using our mapping. Such CSS codes could be more amenable to the analysis of fault-tolerance using ZX calculus [12, 15, 74].

Acknowledgments

A Proofs of Propositions

A.1 Proof of Proposition 1

Write

$$\overline{H} := (H + H^{\omega})/H \text{ and} \tag{120}$$

$$\underline{H} := H/(H \cap H^{\omega}). \tag{121}$$

One verifies that the map

$$(a + (H \cap H^{\omega}), b + (H \cap H^{\omega})) \mapsto \omega(a, b) \tag{122}$$

is an antisymmetric form on $\underline{H}$. Since every vector space with an antisymmetric form has even dimension [73], we have $\dim \underline{H} = 2r$ for some $r \in \mathbb{Z}$. Moreover, observe that

$$\begin{aligned}
\dim \overline{H} + \dim \underline{H} &= \dim H + H^{\omega} - \dim H \cap H^{\omega} \\
&= \dim G \times G - 2 \dim H \cap H^{\omega},
\end{aligned} \tag{123}$$

so $\dim \overline{H} = 2k$ for some $k \in \mathbb{Z}$. Next, note that

$$\dim G \times G - \dim (H + H^{\omega}) = \dim (H \cap H^{\omega}). \tag{124}$$

Since the dimensions in Eq. (33) must sum to $\dim G \times G = 2n$, we find that $\dim H \cap H^{\omega} = n - k - r$, as needed. The final statement follows by defining d appropriately.

A.2 Proof of Proposition 2

First, note that Eq. (37) follows directly from Eq. (21). Next, observe that

$$\begin{aligned}
\dim \left(H_X + H_Z^{\theta}\right)/H_X &= \dim H_X^{\theta}/\left(H_Z \cap H_X^{\theta}\right) \\
&= \dim \left(H_Z + H_X^{\theta}\right)/H_Z
\end{aligned} \tag{125}$$

and

$$\begin{aligned}
\dim H_X/\left(H_X \cap H_Z^{\theta}\right) &= \dim \left(H_Z + H_X^{\theta}\right)/H_X^{\theta} \\
&= \dim H_Z/\left(H_Z \cap H_X^{\theta}\right),
\end{aligned} \tag{126}$$

so the claim about dimensions follows. Finally, recall that

$$d := \min \operatorname{swt}\left((H + H^{\omega}) \setminus H\right). \tag{127}$$

Fix $(a, b) \in (H + H^{\omega}) \setminus H$ with $\operatorname{swt}(a, b) = d$. Then $a \in \left(H_X + H_Z^{\theta}\right)$ and $b \in \left(H_Z + H_X^{\theta}\right)$, but either $a \notin H_X$ or $b \notin H_Z$. If $a \notin H_X$, then

$$d = \operatorname{swt}(a, b) \geq \operatorname{swt}(a, 0) = \operatorname{wt}(a) \geq d^{H_X}. \tag{128}$$

If instead $b \notin H_Z$, a similar argument shows that $d \geq d^{H_Z}$. Conversely, suppose that $a \in \left(H_X + H_Z^{\theta}\right) \setminus H_X$ satisfies $\operatorname{wt}(a) = \min\left\{d^{H_X}, d^{H_Z}\right\}$. Then

$$d \leq \operatorname{swt}(a, 0) = \operatorname{wt}(a) = \min\left\{d^{H_X}, d^{H_Z}\right\}. \tag{129}$$

If instead $b \in \left(H_Z + H_X^{\theta}\right) \setminus H_Z$ satisfies $\operatorname{wt}(b) = \min\left\{d^{H_X}, d^{H_Z}\right\}$, a similar argument shows that $d \leq \min\left\{d^{H_X}, d^{H_Z}\right\}$.

A.3 Proof of Proposition 3

Since we know Fa, we can determine $a + K$ by the first isomorphism theorem. By assumption, there exists $r \in R$ such that $\operatorname{wt}(a + r) < \frac{d^R}{2}$. Thus, we can identify some $a + k \in a + K$ with $\operatorname{wt}(a + k) < \frac{d^R}{2}$, where $k \in K$ is unknown. By the triangle inequality, we have

$$\begin{aligned}
\operatorname{wt}(r - k) &= \operatorname{wt}((a + r) - (a + k)) \\
&\leq \operatorname{wt}(a + r) + \operatorname{wt}(a + k) \\
&< d^R \\
&= \min \operatorname{wt}(K \setminus R).
\end{aligned} \tag{130}$$

Thus, since $r - k \in K$, we must have $r - k \in R$, so $k \in R$.

A.4 Proof of Proposition 4

On the one hand, observe that

$$\begin{aligned}
\phi\left(\sum_j c_j e_X^j + n_X + N_X\right) &= \sum_j c_j \phi\left(e_X^j + N_X\right) \\
&= \sum_j c_j \left(e_Z^j + N_Z\right) \\
&= \sum_j c_j e_Z^j + n_Z + N_Z.
\end{aligned} \tag{131}$$

Conversely, suppose that $(e_X, e_Z) \in E_X \times E_Z$ satisfies $\phi(e_X + N_X) = e_Z + N_Z$. Write

$$e_X + N_X = \sum_j c_j \left(e_X^j + N_X\right) \tag{132}$$

and

$$e_Z + N_Z = \sum_j d_j \left(e_Z^j + N_Z\right) \tag{133}$$

for some $c_j, d_j \in \mathbb{F}_p$. Then

$$\sum_j c_j \left(e_Z^j + N_Z\right) = \sum_j d_j \left(e_Z^j + N_Z\right), \tag{134}$$

so $c_j = d_j$ for all j. Then

$$e_X = \sum_j c_j e_X^j + n_X \tag{135}$$

and

$$e_Z = \sum_j c_j e_Z^j + n_Z \tag{136}$$

for some $n_X \in N_X$ and $n_Z \in N_Z$, so Eq. (94) holds.

A.5 Proof of Proposition 5

One verifies Eq. (104) from the definitions in Eq. (84). To prove Eq. (105), we need the following technical lemma.

Lemma 3. *Let $W \leq V \leq U$, and let $\pi : U \to X$ be any linear map. Then $\pi(W) = \pi(V)$ iff $(V \cap \ker \pi) + W = V$.*

Proof. Suppose that $\pi(W) = \pi(V)$. Fix $v \in V$. Then there exists $w \in W$ such that $\pi(v) = \pi(w)$. Then $v = (v-w)+w$, where $v-w \in V \cap \ker \pi$. Conversely, suppose that $(V \cap \ker \pi) + W = V$. Fix $v \in V$. Then there exist $k \in V \cap \ker \pi$ and $w \in W$ such that $v = k + w$. Then $\pi(v) = \pi(w)$. Thus, $\pi(V) \subseteq \pi(W)$, as needed. $\square$

Now, recall that H is a direct product iff $\pi_X(\ker \pi_Z) = \pi_X(\mathbb{F}_p^l)$. Applying the above lemma with $U = V = \mathbb{F}_p^l$, $W = \ker \pi_Z$, and $\pi = \pi_X$, we obtain Eq. (105).

A.6 Proof of Proposition 6

Fix a basis $\left\{ e_X^j + N_X \right\}_j$ for E_X/N_X. Define the function

$$\tilde{\phi}_X : N_X^\theta \to E_X/N_X$$
$$n_X^\theta \mapsto \sum_j \theta\left(e_X^j, n_X^\theta\right)\left(e_X^j + N_X\right). \quad (137)$$

Since $n_X^\theta \in N_X^\theta$, the function $\tilde{\phi}_X$ is independent of the choice of representatives e_X^j for the cosets $e_X^j + N_X$. Since θ is bilinear, $\tilde{\phi}_X$ is linear. Now, if $n_X^\theta \in E_X^\theta$, then $\tilde{\phi}_X\left(n_X^\theta\right) = N_X$. Conversely, suppose that $\tilde{\phi}_X\left(n_X^\theta\right) = N_X$. Since the $e_X^j + N_X$ are linearly independent, we have $\theta\left(e_X^j, n_X^\theta\right) = 0$ for all j. Since every $e_X \in E_X$ can be written as

$$e_X = \sum_j c_j e_X^j + n_X \quad (138)$$

for some $c_j \in \mathbb{F}_p$ and $n_X \in N_X$, we find that $n_X^\theta \in E_X^\theta$. Thus, $\ker \tilde{\phi}_X = E_X^\theta$. Then $\tilde{\phi}_X$ induces the injection

$$\phi_X : N_X^\theta/E_X^\theta \to E_X/N_X$$
$$n_X^\theta + E_X^\theta \mapsto \tilde{\phi}_X\left(n_X^\theta\right). \quad (139)$$

In fact, since $\dim N_X^\theta/E_X^\theta = \dim E_X/N_X$, the map ϕ_X is an isomorphism.

Similarly, given the basis

$$\left\{ \phi\left(e_X^j + N_X\right) =: e_Z^j + N_Z \right\}_j \quad (140)$$

for E_Z/N_Z, there exists an isomorphism $\phi_Z : N_Z^\theta/E_Z^\theta \to E_Z/N_Z$, where

$$\phi_Z\left(n_Z^\theta + E_Z^\theta\right) := \sum_j \theta\left(e_Z^j, n_Z^\theta\right)\left(e_Z^j + N_Z\right). \quad (141)$$

Now, define $\phi^\omega := \phi_X^{-1}\phi^{-1}\phi_Z$. Fix $\left(n_Z^\theta, n_X^\theta\right) \in N_Z^\theta \times N_X^\theta$. By definition, we have

$$\phi^\omega\left(n_Z^\theta + E_Z^\theta\right) = n_X^\theta + E_X^\theta \quad (142)$$

iff

$$\phi_Z\left(n_Z^\theta + E_Z^\theta\right) = \phi\left(\phi_X\left(n_X^\theta + E_X^\theta\right)\right). \quad (143)$$

By Eqs. (139) and (141), this is equivalent to

$$\sum_j \theta\left(e_Z^j, n_Z^\theta\right)\left(e_Z^j + N_Z\right)$$
$$= \phi\left(\sum_j \theta\left(e_X^j, n_X^\theta\right)\left(e_X^j + N_X\right)\right), \quad (144)$$

which is equivalent to

$$\sum_j \theta\left(e_Z^j, n_Z^\theta\right)\phi\left(e_X^j + N_X\right)$$
$$= \sum_j \theta\left(e_X^j, n_X^\theta\right)\phi\left(e_X^j + N_X\right). \quad (145)$$

Since the $\phi\left(e_X^j + N_X\right)$ are linearly independent, this is equivalent to

$$\forall j, \, \theta\left(e_Z^j, n_Z^\theta\right) = \theta\left(e_X^j, n_X^\theta\right), \quad (146)$$

which by Eq. (94) is equivalent to

$$\forall (e_X, e_Z) \in H, \, \theta\left(e_Z, n_Z^\theta\right) = \theta\left(e_X, n_X^\theta\right). \quad (147)$$

Rewriting, this occurs iff

$$\forall (e_X, e_Z) \in H, \, \omega\left((e_X, e_Z), \left(n_Z^\theta, n_X^\theta\right)\right) = 0, \quad (148)$$

which is equivalent to $\left(n_Z^\theta, n_X^\theta\right) \in H^\omega$. Now, since $N_X \times N_Z \leq H$, we have $H^\omega \leq N_Z^\theta \times N_X^\theta$. Thus, we find

$$H^\omega = \left\{ \left(n_Z^\theta, n_X^\theta\right) \in N_Z^\theta \times N_X^\theta \, \middle| \, \phi^\omega\left(n_Z^\theta + E_Z^\theta\right) = n_X^\theta + E_X^\theta \right\}, \quad (149)$$

as needed.

A.7 Proof of Proposition 7

Consider the injections ϕ_T, ϕ_W defined in the diagram

$$\begin{array}{ccccccc}
\left(E_X \cap \tilde{E_X}\right)/\left(N_X \cap \tilde{N}_X\right) & \overset{\phi_T}{\hookrightarrow} & \left(E_X/N_X\right) \times \left(\tilde{E_X}/\tilde{N}_X\right) & \overset{\phi \times \tilde{\phi}}{\sim} & \left(E_Z/N_Z\right) \times \left(\tilde{E_Z}/\tilde{N}_Z\right) & \overset{\phi_W}{\hookleftarrow} & \left(E_Z \cap \tilde{E_Z}\right)/\left(N_Z \cap \tilde{N}_Z\right) \\
a + \left(N_X \cap \tilde{N}_X\right) & \mapsto & \left(a + N_X, a + \tilde{N}_X\right) & & \left(b + N_Z, b + \tilde{N}_Z\right) & \hookleftarrow & b + \left(N_Z \cap \tilde{N}_Z\right)
\end{array}$$
$$(150)$$

Moving "left to right" in the above diagram, define

$$\mathcal{W} \coloneqq \phi_W^{-1}\left(\phi \times \tilde{\phi}\right)\phi_T\left(\left(E_X \cap \tilde{E_X}\right)/\left(N_X \cap \tilde{N}_X\right)\right). \tag{151}$$

Moving "right to left" in the above diagram, define

$$\mathcal{V} \coloneqq \phi_W(\mathcal{W}), \tag{152}$$
$$\mathcal{U} \coloneqq \left(\phi \times \tilde{\phi}\right)^{-1}(\mathcal{V}), \tag{153}$$
$$\mathcal{T} \coloneqq \phi_T^{-1}(\mathcal{U}). \tag{154}$$

From the definitions, it is straightforward to verify that the appropriate restrictions of $\phi_T, \phi \times \tilde{\phi}, \phi_W$ satisfy

$$\mathcal{T} \overset{\phi_T}{\sim} \mathcal{U} \overset{\phi \times \tilde{\phi}}{\sim} \mathcal{V} \overset{\phi_W}{\sim} \mathcal{W}. \tag{155}$$

By the lattice isomorphism theorem, there exist subspaces

$$\begin{array}{ccc}
E_X \cap \tilde{E_X} & & E_Z \cap \tilde{E_Z} \\
| & & | \\
T & \text{and} & W \\
| & & | \\
N_X \cap \tilde{N}_X & & N_Z \cap \tilde{N}_Z
\end{array} \tag{156}$$

such that

$$\begin{aligned}
\mathcal{T} &= T/\left(N_X \cap \tilde{N}_X\right) \text{ and} \\
\mathcal{W} &= W/\left(N_Z \cap \tilde{N}_Z\right).
\end{aligned} \tag{157}$$

Now, fix $(a, b) \in T \times W$. Then with

$$\phi^\cap \coloneqq \phi_W^{-1}\left(\phi \times \tilde{\phi}\right)\phi_T, \tag{158}$$

we have

$$\phi^\cap\left(a + \left(N_X \cap \tilde{N}_X\right)\right) = b + \left(N_Z \cap \tilde{N}_Z\right) \tag{159}$$

iff

$$\left(\phi \times \tilde{\phi}\right)\phi_T\left(a + \left(N_X \cap \tilde{N}_X\right)\right) = \phi_W\left(b + \left(N_Z \cap \tilde{N}_Z\right)\right), \tag{160}$$

which occurs iff

$$\left(\phi \times \tilde{\phi}\right)\left(a + N_X, a + \tilde{N}_X\right) = \left(b + N_Z, b + \tilde{N}_Z\right). \tag{161}$$

Reading coordinate-wise, this occurs iff

$$\begin{aligned}
\phi\left(a + N_X\right) &= b + N_Z \text{ and} \\
\tilde{\phi}\left(a + \tilde{N}_X\right) &= b + \tilde{N}_Z,
\end{aligned} \tag{162}$$

which occurs iff

$$(a, b) \in H \text{ and } (a, b) \in \tilde{H}. \tag{163}$$

Moreover, if $(a, b) \in H \cap \tilde{H} \subseteq E_X \cap \tilde{E_X}$, then by the same computation above, we find

$$\begin{aligned}
\left(\phi \times \tilde{\phi}\right)&\left(\phi_T\left(a + \left(N_X \cap \tilde{N}_X\right)\right)\right) \\
&= \phi_W\left(b + \left(N_Z \cap \tilde{N}_Z\right)\right).
\end{aligned} \tag{164}$$

That is, $b + \left(N_Z \cap \tilde{N}_Z\right) \in \mathcal{W}$, so $a + \left(N_X \cap \tilde{N}_X\right) \in \mathcal{T}$. Then $(a, b) \in T \times W$, so $H \cap \tilde{H} \subseteq T \times W$. Thus, we have

$$\begin{aligned}
H \cap \tilde{H} = \Big\{ (a, b) \in T \times W \,\Big|\, &\phi^\cap\left(a + \left(N_X \cap \tilde{N}_X\right)\right) \\
&= b + \left(N_Z \cap \tilde{N}_Z\right) \Big\},
\end{aligned} \tag{165}$$

as needed.

A.8 Proof of Proposition 8

Suppose that H has both maximal and minimal stabilizer. Taking orthogonal complements in the right-hand side factor, we have

$$N_X \cap E_Z^\theta = E_X \cap N_Z^\theta \text{ and} \tag{166}$$
$$E_X + N_Z^\theta = N_X + E_Z^\theta. \tag{167}$$

On the one hand,

$$\begin{aligned}
\dim E_X + \dim N_Z^\theta &= \dim\left(E_X + N_Z^\theta\right) + \dim\left(E_X \cap N_Z^\theta\right) \\
&= \dim\left(N_X + E_Z^\theta\right) + \dim\left(N_X \cap E_Z^\theta\right) \\
&= \dim N_X + \dim E_Z^\theta.
\end{aligned} \tag{168}$$

On the other hand,

$$\dim E_X \geq \dim N_X \text{ and} \tag{169}$$
$$\dim N_Z^\theta \geq \dim E_Z^\theta. \tag{170}$$

Thus, we have

$$E_X = N_X \text{ and } E_Z = N_Z, \tag{171}$$

as needed. The converse is clear — assume Eq. (171) and simplify the right-hand side of Eq. (117).

B Generalizing Theorem 1 to subsystem stabilizer codes over modular qudits

In this appendix, we briefly describe how to generalize Theorem 1 to subsystem stabilizer codes over modular qudits. The material in the preliminaries, Section 2, generalizes as follows [28, 35, 73].

- The prime p is now an arbitrary integer m, so G is now the finite abelian group $(\mathbb{Z}/m\mathbb{Z})^n$ with standard generating set $\{(1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$

- The Pauli group modulo phases is still isomorphic to $G \times G$

- Eq. (14) now reads as $|H|\left|H^\xi\right| = |G|$, where the absolute value denotes group order

- A subsystem stabilizer code is now a subgroup $H \leq G \times G$. We call H a subsystem CSS code if H is a direct product of subgroups of G

- We no longer keep track of the number of logical and gauge qudits in a subsystem stabilizer code. Rather, we keep track of the *logical Pauli group* $\overline{H} := (H + H^\omega)/H$ and the *gauge Pauli group* $\underline{H} := H/(H \cap H^\omega)$ — see Eqs. (31) and (32). The logical (resp. gauge) Pauli group is isomorphic to the group of Pauli operators on the logical (resp. gauge) subsystem $\mathbb{L}$ (resp. $\mathbb{G}$) [35]

Having established the necessary background, we now sketch the generalization of Theorem 1.

- Eq. (44d) in Lemma 1 now reads as $|\Delta(H)| = |H|^2$, and the analogue of Lemma 1 holds with essentially the same proof

- Theorem 1 now reads as follows, where the proof is similar

Theorem 2. *Let $H \leq G \times G$ be a subsystem stabilizer code with logical Pauli group $\overline{H}$, gauge Pauli group $\underline{H}$, and distance d. Then $\Delta(H)$ is a subsystem CSS code with logical Pauli group isomorphic to $\overline{H} \times \overline{H}$, gauge Pauli group isomorphic to $\underline{H} \times \underline{H}$, and distance d', where $d \leq d' \leq 2d$. Moreover, if H admits a collection of generators with symplectic weight at most w, then $\Delta(H)$ admits a collection of generators with symplectic weight at most $2w$.*

C An alternative classical step for error recovery

In this appendix, we explore an alternative to the classical step for error recovery described in Section 5.2. To begin, consider the commutative diagram of canonical quotients

$$\begin{array}{c}
G \\
\end{array} \quad \xrightarrow{\text{Syn}_X} \quad G/(H_X + H_Z^\theta) \; . \qquad (172)$$

In Section 5.2, we used the parity check Syn_X of the classical linear code $H_X + H_Z^\theta$ to recover an error class $a + H_X$ from its syndrome $a + (H_X + H_Z^\theta)$. Here, we investigate the utility of Par_X for the same task — this line of inquiry is motivated by the Bacon–Shor decoder [9], which utilizes Par_X for error recovery (see Example 5). We will need the following notation.

Let $\sigma \subseteq G$ denote the standard basis for G. Let σ_0 be any subset of σ such that $\overline{\sigma_0}$ is a basis for G/H_X, where the over-line denotes quotient by H_X. For any $a + H_X \in G/H_X$, we write $\text{wt}_{\overline{\sigma_0}}(a + H_X)$ to denote the weight of $a + H_X$ in the basis $\overline{\sigma_0}$, i.e., the number of nonzero coordinates of $a + H_X$ in the basis $\overline{\sigma_0}$.

Now, recall that by Proposition 3, Syn_X can recover the error class $a + H_X$ from its syndrome $a + (H_X + H_Z^\theta)$ provided that

$$\min \text{wt}(a + H_X) < \frac{d^{H_X}}{2}. \qquad (173)$$

Similarly, Par_X can recover the error $a + H_X$ from its syndrome $a + (H_X + H_Z^\theta)$ provided that

$$\text{wt}_{\overline{\sigma_0}}(a + H_X) < \frac{d^{\text{Par}_X}}{2}, \qquad (174)$$

where

$$d^{\text{Par}_X} := \min \text{wt}_{\overline{\sigma_0}}\left((H_X + H_Z^\theta)/H_X\right) \qquad (175)$$

is the distance of Par_X in the basis $\overline{\sigma_0}$. Our first observation is that the notions of weight and distance in Eqs. (173) and (174) are related.

Proposition 9. *We have*

$$\min \text{wt}(a + H_X) \leq \text{wt}_{\overline{\sigma_0}}(a + H_X) \qquad (176)$$

for all $a \in G$. Moreover, we have

$$d^{H_X} \leq d^{\text{Par}_X}. \qquad (177)$$

Proof. First, note that for any $a \in G$, we have

$$a + H_X = \sum_{\alpha \in \sigma_0} a_\alpha (\alpha + H_X) \qquad (178)$$

for some $a_\alpha \in \mathbb{F}_p$, so

$$a + h_X = \sum_{\alpha \in \sigma_0} a_\alpha \alpha \qquad (179)$$

for some $h_X \in H_X$. Thus,

$$\min \text{wt}(a + H_X) \leq \text{wt}(a + h_X) = \text{wt}_{\overline{\sigma_0}}(a + H_X). \qquad (180)$$

Now, suppose in addition that

$$a + H_X \in (H_X + H_Z^\theta)/H_X \qquad (181)$$

is nonzero and satisfies

$$\mathrm{wt}_{\overline{\sigma_0}}(a + H_X) = d^{\mathrm{Par}_X}. \tag{182}$$

Then $a + h_X \in (H_X + H_Z^\theta) \setminus H_X$, so

$$d^{H_X} \leq \mathrm{wt}(a + h_X) = \mathrm{wt}_{\overline{\sigma_0}}(a + H_X) = d^{\mathrm{Par}_X}. \tag{183}$$

$\square$

Unfortunately, this proposition does not immediately reveal whether Par_X performs better or worse than Syn_X. Indeed, the distance of Par_X is nominally lower bounded by the distance of Syn_X, but the notion of error weight relevant for Par_X is also lower bounded by that for Syn_X. Thus, it is not immediately clear whether one code performs better than the other. To shed light on this issue, we investigate the special case in which the notions of error weight for Par_X and Syn_X coincide.

Definition 5. We say that $H_X \leq G$ *respects weight* if there exists $\sigma_0 \subseteq \sigma$ such that $\overline{\sigma_0}$ is a basis for G/H_X satisfying

$$\min \mathrm{wt}(a + H_X) = \mathrm{wt}_{\overline{\sigma_0}}(a + H_X) \tag{184}$$

for all $a \in G$.

Note that by definition, H_X respects weight iff the notions of weight for Syn_X and Par_X in Eqs. (173) and (174) coincide. In fact, if H_X respects weight, then the distances of Syn_X and Par_X also coincide. To see this, note that

$$\begin{aligned} d^{\mathrm{Par}_X} &= \min_{a \in (H_X + H_Z^\theta) \setminus H_X} \mathrm{wt}_{\overline{\sigma_0}}(a + H_X) \\ &\leq \min_{a \in (H_X + H_Z^\theta) \setminus H_X} \mathrm{wt}(a) \\ &= d^{H_X}, \end{aligned} \tag{185}$$

so by Proposition 9, we have $d^{H_X} = d^{\mathrm{Par}_X}$.

Interestingly, it turns out that there are not very many subspaces $H_X \leq G$ that respect weight.

Proposition 10. *A subspace $H_X \leq G$ respects weight iff H_X admits a basis whose elements have weight at most 2.*

Proof. ($\Longrightarrow$) Fix $\beta \in \sigma_0^c := \sigma \setminus \sigma_0$ and write

$$\beta + H_X = \sum_{\alpha \in \sigma_0} a_{\alpha\beta}(\alpha + H_X), \tag{186}$$

where the $a_{\alpha\beta} \in \mathbb{F}_p$. If more than one $a_{\alpha\beta}$ is nonzero, then $\mathrm{wt}_{\overline{\sigma_0}}(\beta + H_X) > 1$ but $\mathrm{wt}(\beta) = 1$, a contradiction. Thus, at most one $a_{\alpha\beta}$ is nonzero, so $\beta = b_\beta \alpha_\beta + \gamma_\beta$ for some $b_\beta \in \mathbb{F}_p$, $\alpha_\beta \in \sigma_0$ and $\gamma_\beta \in H_X$. We claim

that $\Gamma := \{\gamma_\beta \mid \beta \in \sigma_0^c\} \subseteq H_X$ is linearly independent. Indeed, suppose that

$$\sum_{\beta \in \sigma_0^c} c_\beta \gamma_\beta = 0 \tag{187}$$

for some $c_\beta \in \mathbb{F}_p$. Then

$$\sum_{\beta \in \sigma_0^c} c_\beta \beta = \sum_{\beta \in \sigma_0^c} c_\beta b_\beta \alpha_\beta, \tag{188}$$

which by linear independence of σ implies that $c_\beta = 0$ for all $\beta \in \sigma_0^c$. Thus, Γ has exactly $|\sigma_0^c| = \dim H_X$ elements, so Γ is a basis for H_X whose elements have weight at most 2.

($\Longleftarrow$) Let M be the $\dim H_X \times \dim G$ matrix whose rows are the basis vectors of H_X. Let M' be the reduced row echelon form of M. Note that the row space of M' is H_X. Let σ_0 be the subset of σ corresponding to the non-pivot columns of M', so σ_0^c corresponds to the pivot columns of M'. Since each row of M has weight at most 2, it follows that for each pivot $\beta \in \sigma_0^c$, there is at most one other nonzero entry (which lies to the right of β and is not a pivot) in the corresponding row of M'. Call that entry $-b_\beta \alpha_\beta$, where $b_\beta \in \mathbb{F}_p$ and $\alpha_\beta \in \sigma_0$. Now, fix any $a \in G$, and write

$$a = \sum_{\alpha \in \sigma_0} a_\alpha \alpha + \sum_{\beta \in \sigma_0^c} a_\beta \beta, \tag{189}$$

where the $a_\alpha, a_\beta \in \mathbb{F}_p$. Then

$$\begin{aligned} a + H_X &= \sum_{\alpha \in \sigma_0} a_\alpha (\alpha + H_X) + \sum_{\beta \in \sigma_0^c} a_\beta (\beta + H_X) \\ &= \sum_{\alpha \in \sigma_0} a_\alpha (\alpha + H_X) + \sum_{\beta \in \sigma_0^c} a_\beta (b_\beta \alpha_\beta + H_X). \end{aligned} \tag{190}$$

Thus, the weight of $a + H_X$ is at most the number of nonzero a_α plus the number of nonzero a_β. That is,

$$\mathrm{wt}_{\overline{\sigma_0}}(a + H_X) \leq \mathrm{wt}(a). \tag{191}$$

Since this holds for any $a \in G$, we conclude that H_X respects weight. $\square$

In the following example, we illustrate the usage of Par_X to correct errors affecting a subsystem CSS code whose underlying classical linear codes respect weight.

Example 5 (Bacon–Shor code). The physical qubits of a Bacon–Shor code [9] are located at the vertices of an $l \times l$ grid, so we set $G = \mathbb{F}_2^{l^2}$. For concreteness, let $l = 4$. Every pair of row-adjacent sites $(i, j), (i, j + 1)$ on the grid contributes an X-type gauge generator $X^{(i,j)} \otimes X^{(i,j+1)}$, and every pair of column-adjacent sites $(i, j), (i + 1, j)$ on the grid contributes a Z-type gauge

generator $Z^{(i,j)} \otimes Z^{(i+1,j)}$. Thus, the gauge group is the direct product of

$H_X :=$

$$\left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\cdots,$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle \tag{192}$$

and

$H_Z :=$

$$\left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\cdots,$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle. \tag{193}$$

Note that H_X and H_Z respect weight by Proposition 10. Now, the complement H_X^θ corresponds to the collection of all Z-type Pauli operators which commute with all the X-type Pauli operators in H_X (and similarly for H_Z^θ). Explicitly, we have

$$H_X^\theta = \left\langle \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\rangle \tag{194}$$

and

$$H_Z^\theta = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle. \tag{195}$$

Thus, we find that

$$H_X + H_Z^\theta = \left\langle H_X, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle, \tag{196}$$

$$H_Z + H_X^\theta = \left\langle H_Z, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle, \tag{197}$$

$H_X \cap H_Z^\theta =$

$$\left\langle \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right\rangle, \tag{198}$$

and

$H_Z \cap H_X^\theta =$

$$\left\langle \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\rangle. \tag{199}$$

Moreover, we have

$$G/H_X = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + H_X, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + H_X, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + H_X, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + H_X \right\rangle \tag{200}$$

and

$$G/\left(H_X + H_Z^\theta\right) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \left(H_X + H_Z^\theta\right), \right.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \left(H_X + H_Z^\theta\right),$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \left(H_X + H_Z^\theta\right) \right\rangle. \tag{201}$$

Now, in the bases of Eqs. (200) and (201), the parity check Par_X is represented by the $(l-1) \times l$ matrix

$$\mathrm{Par}_X = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{202}$$

Abstractly, the kernel of Par_X is

$$\left(H_X + H_Z^\theta\right)/H_X = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + H_X \right\rangle, \tag{203}$$

which in the basis of Eq. (200) is

$$\ker \mathrm{Par}_X = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle. \qquad (204)$$

Thus, for the Bacon–Shor code, Par_X is the parity check of the l-fold repetition code, which coincides with the l-fold repetition code described in the original work [9]. To correct an X-type error $a \in G$, we first determine its syndrome $\mathrm{Syn}_X(a)$ by measuring the Z-type gauge generators in Eq. (193), as described in Section 5.1. We then express $\mathrm{Syn}_X(a) = a + (H_X + H_Z^\theta)$ in the basis of Eq. (201). Next, by viewing Par_X as the matrix in Eq. (202), we compute $(a + H_X) + \ker \mathrm{Par}_X$ in the basis of Eq. (200). Now, suppose that the original X-type error a has odd weight in less than $\frac{l}{2}$ rows. Then $a + H_X$ has weight less than $\frac{l}{2}$ in the basis of Eq. (200). Thus, since the distance of Par_X is l, we can determine $a + H_X$ in the basis of Eq. (200). This allows us to correct the X-type error a up to gauge terms.

D Parameterizing the codespace of a subsystem CSS code

In this appendix, we parameterize the codespace of a subsystem CSS code $H = H_X \times H_Z$ in terms of the underlying classical codes H_X and H_Z. The main result is as follows.

Proposition 11. *Let $H = H_X \times H_Z$ be a subsystem CSS code. Write*

$$\left(H_X + H_Z^\theta\right)/H_X = \{l_i + H_X\}_i \ and \qquad (205)$$

$$H_X/\left(H_X \cap H_Z^\theta\right) = \left\{g_j + H_X \cap H_Z^\theta\right\}_j, \qquad (206)$$

where all elements on the right-hand side are distinct. For any $a \in G$, write

$$\left|a + H_X \cap H_Z^\theta\right\rangle = \sum_{s \in H_X \cap H_Z^\theta} \left|a + s\right\rangle. \qquad (207)$$

Then the codespace of H is

$$\mathbb{L} \otimes \mathbb{G} = \mathrm{Span}\left\{\left|l_i + g_j + H_X \cap H_Z^\theta\right\rangle\right\}_{i,j}. \qquad (208)$$

Proof. Say that H has parameters $[[n, k, r, d]]$. From the standard theory of stabilizer codes, the codespace $\mathbb{L} \otimes \mathbb{G}$ has dimension $p^k p^r$ [64]. Moreover, one verifies that every state of the form

$$\left|l_i + g_j + H_X \cap H_Z^\theta\right\rangle \qquad (209)$$

is fixed by every stabilizer of H and so belongs to the codespace $\mathbb{L} \otimes \mathbb{G}$. Thus, we must show that the set

$$\left\{\left|l_i + g_j + H_X \cap H_Z^\theta\right\rangle\right\}_{i,j} \qquad (210)$$

contains exactly $p^k p^r$ distinct elements. To see this, note that by Proposition 2, the index i ranges over an index set of size p^k, and the index j ranges over an index set of size p^r. Moreover, suppose that

$$\left|l_i + g_j + H_X \cap H_Z^\theta\right\rangle = \left|l_{i'} + g_{j'} + H_X \cap H_Z^\theta\right\rangle. \quad (211)$$

Then

$$l_i + g_j + H_X \cap H_Z^\theta = l_{i'} + g_{j'} + H_X \cap H_Z^\theta, \qquad (212)$$

which implies

$$l_i + H_X = l_{i'} + H_X. \qquad (213)$$

Since the $l_i + H_X$ are distinct by assumption, we must have $l_i = l_i'$, so $i = i'$. This implies that

$$g_j + H_X \cap H_Z^\theta = g_{j'} + H_X \cap H_Z^\theta, \qquad (214)$$

so we conclude that $g_j = g_{j'}$, so $j = j'$. $\qquad \square$

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