Stationary states of boundary-driven quantum systems: Some exact results

Eric A. Carlen,[1] David A. Huse,[2] and Joel L. Lebowitz[3]
[1]Department of Mathematics, Rutgers University, Piscataway, New Jersey 08854-8019, USA
[2]Department of Physics, Princeton University, Princeton, New Jersey 08544, USA
[3]Departments of Mathematics and Physics, Rutgers University, Piscataway, New Jersey 08854-8019, USA

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We study finite-dimensional open quantum systems in contact with macroscopic equilibrium systems at their boundaries such that the system density matrix ρ evolves via a Lindbladian, $\dot{\rho} = -i[H,\rho] + \mathcal{D}\rho$. Here H is the Hamiltonian of the system and $\mathcal{D}$ is the dissipator. We consider the case where the system consists of two parts, the "boundary" A and the "bulk" B, and $\mathcal{D}$ acts only on A, so $\mathcal{D} = \mathcal{D}_A \otimes \mathcal{I}_B$, where $\mathcal{I}_B$ is the identity superoperator on part B. Let $\mathcal{D}_A$ be ergodic, so $\mathcal{D}_A\pi_A = 0$ only for one unique density matrix π_A. We show that any stationary density matrix $\bar{\rho}$ on the full system which commutes with H must be of the product form $\bar{\rho} = \pi_A \otimes \rho_B$ for some ρ_B. This rules out finding any $\mathcal{D}_A$ that has the Gibbs measure $\rho_\beta = e^{-\beta H}/Z(\beta)$ as a stationary state with $\beta \neq 0$, unless there is no interaction between parts A and B. We give criteria for the uniqueness of the stationary state $\bar{\rho}$ for systems with interactions between A and B. Related results for nonergodic cases are also discussed.

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I. INTRODUCTION

There is much current interest, theoretical and experimental, in open quantum systems coupled at their boundaries to macroscopic equilibrium systems; see [1,2] and references therein. These are often many-body open quantum systems for which the macroscopic equilibrium systems act as reservoirs, with energy and/or particles exchanged between the reservoirs and the boundaries of the open system, and transported through the system. Since the quantities of interest are the time-evolution and stationary states of the open quantum system, its interaction with the reservoirs is commonly idealized by saying that they cause the system to evolve under the influence of a stochastic quantum process [3]. This leads to a Markovian master equation for the system's density matrix $\rho(t)$. The requirement of complete positivity of the evolution then restricts the form of this equation to a Lindbladian form,

$$\frac{\partial \rho}{\partial t} = -i[H,\rho] + \mathcal{D}\rho, \tag{1}$$

where H is the Hamiltonian of the isolated system, and $\mathcal{D}\rho$ is the "dissipation" caused by its interactions with the reservoirs. By a theorem of Lindblad [4] and Gorini, Kossakowski, and Sudarshan [5], the generator $\mathcal{D}$ (a superoperator) has the form

$$\mathcal{D}\rho = -i[K,\rho] + \sum_{\alpha=1}^{n}\left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\right), \tag{2}$$

where K is a self-adjoint operator, often called the Lamb shift Hamiltonian [3], because it can be seen as adjusting the system's energy levels due to interaction with the reservoirs. K could be combined with H, but for our purposes, it is convenient to keep the effect of the reservoirs on the dynamics clearly separated from the dynamics of the isolated system.

As in [1,2,6] and the references therein, the Hilbert space of the system has the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_B$ corresponds to the "bulk" of the system and $\mathcal{H}_A$ corresponds to the part coupled to the reservoirs (the boundary). The Lindbladian generators we consider have the form

$$\mathcal{D} = \mathcal{D}_A \otimes \mathcal{I}_B, \tag{3}$$

where the superoperator $\mathcal{D}_A$ acts only on operators on subsystem A, and $\mathcal{I}_B$ denotes the identity superoperator on subsystem B. Then with $\mathbf{1}_B$ denoting the identity operator on A, this means that the jump operators in (2) all can be written as

$$L_\alpha = L_{A,\alpha} \otimes \mathbf{1}_B, \tag{4}$$

in terms of $L_{A,\alpha}$ that act only on $\mathcal{H}_A$, and similarly the Lamb shift Hamiltonian can be written as

$$K = K_A \otimes \mathbf{1}_B. \tag{5}$$

We shall not discuss here the derivation of K and the "jump operators" $\{L_\alpha\}$ from the interactions with the reservoirs. There are various ways that such generators arise in physical models. The most commonly discussed is the weak coupling limit studied by Davies [7]. An alternative is the singular coupling limit studied in Gorini and Kossakowski [8]. (See [9] for a treatment of both of these limits limits in a common framework.) The weak coupling limit leads to a special class of Lindbladian generators for which the Gibbs state of the system Hamiltonian (at a temperature and chemical potential set by the reservoir) is always a steady state. In general, in the weak coupling limit, even if the coupling to the heat bath is local, acting only on $\mathcal{H}_A$, the jump operators will not have the local form (4). However, the singular coupling limit is more flexible, and the parameters can be adjusted to produce any desired Lindbladian [8].

As in [2], we focus instead on the relations between the properties of $\mathcal{D}_A \otimes \mathcal{I}_B$ and H and the stationary state(s) $\bar{\rho}$ of (1). These stationary states are the nonequilibrium analog of the equilibrium Gibbs states, and they determine the properties of the system in the steady state. However, when the dimension $\mathcal{H}_B$ is large, it is impossible to exactly solve for steady states in all but the simplest settings. Even the question of existence and uniqueness is not entirely trivial, but our goal is to go beyond this and to deduce properties of the steady states from the specified dynamics.

It follows from general results that there is always at least one stationary state $\bar{\rho}$. (See, e.g., [10] and Sec. IV of this paper.) The question of uniqueness of such steady states is closely connected with the existence of a positive definite steady state [11,12] and is discussed below in Sec. V.

As discussed in [1,2] and references therein, there are two desiderata for the dissipator $\mathcal{D}$: (i) we would like $\mathcal{D}$ to act only on the "boundary" degrees of freedom of the system, as do the reservoirs in certain situations of interest, and (ii) we would like the stationary state $\bar{\rho}$ to be unique and to be that of thermal equilibrium at a finite temperature $1/\beta$ set by the reservoir when the system is interacting with only one such reservoir, i.e., $\bar{\rho} = \frac{1}{Z(\beta)}\exp(-\beta H)$, or $\bar{\rho} = \frac{1}{Z(\beta,\mu)}\exp[-\beta(H-\mu N)]$ if the reservoir also sets a chemical potential for total particle (or excitation) number N. Note that for quantum systems this N is an operator. These two desired properties are readily realizable for classical systems [13] but seem incompatible for quantum systems in the particular cases investigated in [1,2]. Here we prove a "no go" theorem showing that this incompatibility is indeed the case quite generally. However, infinite temperature ($\beta \to 0$) steady states of this form with $\beta\mu$ finite do occur (and are unique) for some such models when $[H, N] = 0$, as we show.

II. LOCAL LINDBLADIANS

As explained in the introduction, we consider a general setup where the system of interest can be divided into two parts, A and B. The Lindbladian dissipator will couple only to part A, which we can consider to be the boundary of the system, while part B is the bulk of the system. For example, if our system is a finite spin chain, part A could be the first m spins at one or both ends of the chain (e.g., m could be one or two spins), while part B is all the remaining spins. This type of "local coupling" has been discussed in some detail for various systems, such as the well-known XXZ or XYZ spin chain models with $\mathcal{D}$ acting on the spins at the ends of the chain [1,2,6,12]. A particular such spin chain example is discussed below.

The Hilbert space $\mathcal{H}$ of the full system is the direct product of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, each assumed to be of finite dimension more than one. (We are particularly interested in cases where B, although finite, is large.) The Hamiltonian of the system is assumed to be finite and traceless. Then defining $H_A = \mathrm{tr}_B\{H\}$ and $H_B = \mathrm{tr}_A\{H\}$, H can be written as

$$H = H_A \otimes \mathbf{1}_B + H_{AB} + \mathbf{1}_A \otimes H_B, \tag{6}$$

which defines H_{AB}, the part of the Hamiltonian producing interaction between subsystems A and B. Then automatically, $\mathrm{Tr}_A\{H_{AB}\} = \mathrm{Tr}_B\{H_{AB}\} = 0$. H_{AB} then describes the dynamics of separately isolated A and B systems.

As explained in the introduction, the dissipators $\mathcal{D}$ with which we are concerned have the form (3); that is, $\mathcal{D} = \mathcal{D}_A \otimes \mathcal{I}_B$. We assume further (for the moment) that $\mathcal{D}_A$ is ergodic on subsystem A, i.e., that there exists a unique density matrix π_A on $\mathcal{H}_A$ such that

$$\mathcal{D}_A\pi_A = 0. \tag{7}$$

(We also discuss the case where $\mathcal{D}_A$ is nonergodic in the Appendix.)

An example of such a $\mathcal{D}$ is given by

$$\mathcal{D}\rho = \frac{1}{\tau}[\pi_A \otimes \mathrm{tr}_A\{\rho(t)\} - \rho(t)]. \tag{8}$$

The solution of $\frac{\partial}{\partial t}\rho = \mathcal{D}\rho$ with initial state ρ_0 is

$$\rho(t) = (1 - e^{-t/\tau})\pi_A \otimes \mathrm{tr}_A\{\rho_0\} + e^{-t/\tau}\rho_0, \tag{9}$$

so that the dynamics described by this $\mathcal{D}$ is replacement of $\rho(t)$ by $\pi_A \otimes \mathrm{tr}_A\{\rho(t)\}$ at rate $1/\tau$. [This dissipator can be written in the form (2) and (3), using the jump operators $L_{i,j} = \pi_A^{1/2}|i\rangle\langle j| \otimes \mathbf{1}_B$ where $|j\rangle$ runs over an orthonormal basis of $\mathcal{H}_A$.]

The question we now investigate is: Given a Lindbladian dissipator of the form (3) that acts only on part A (the boundary) of our system, what can we say about a stationary density matrix of the full system, $\bar{\rho}$? We first restrict our attention to cases where there is a steady state $\bar{\rho}$ that is a "generalized Gibbs state," meaning that it commutes with the Hamiltonian H: $[\bar{\rho}, H] = 0$. This ensures that $\bar{\rho}$ is also a stationary state of the system if we set $\mathcal{D} = 0$. It includes the standard Gibbs state $\rho_\beta = e^{-\beta H}/Z(\beta)$ as a special case. We will show later that in some cases such a stationary state is unique. Our main theorems and a corollary are the following.

Theorem 1. Let $\mathcal{D}$ have the form (3). We further assume that $\mathcal{D}_A$ is ergodic, so on $\mathcal{H}_A$ there is a unique density matrix π_A satisfying $\mathcal{D}_A\pi_A = 0$. Let $\overline{\rho}$ be a steady-state solution of (1).

Assume that $\bar{\rho}$ commutes with H. Then there exists a density matrix ρ_B on $\mathcal{H}_B$ such that

$$\overline{\rho} = \pi_A \otimes \rho_B. \tag{10}$$

In particular, such a steady state $\overline{\rho}$ always satisfies $\mathrm{tr}_B\{\overline{\rho}\} = \pi_A$. If we further assume that $\bar{\rho}$ is positive definite, then this implies that

$$[\pi_A, H_A] = 0, \quad [\rho_B, H_B] = 0, \quad \text{and} \quad [\bar{\rho}, H_{AB}] = 0. \tag{11}$$

Conversely if $[\bar{\rho}, H] \neq 0$, then $\bar{\rho} \neq \pi_A \otimes \rho_B$ for any ρ_B.

Corollary 1. Under the same assumptions made in Theorem 1, suppose that for some finite $\beta > 0$, the Gibbs state

$$\rho_\beta = \frac{1}{Z(\beta)}e^{-\beta H} \tag{12}$$

is a steady-state solution of (1). Then necessarily $H_{AB} = 0$, and for any density matrix ρ_B on $\mathcal{H}_B$ such that $[\rho_B, H_B] = 0$, $\pi_A \otimes \rho_B$ is a steady-state solution of (1). The dissipator relaxes A to its unique steady state π_A. However, due to $H_{AB} = 0$, subsystem B remains isolated and autonomous so has many possible steady states.

We next turn to uniqueness and positivity. As already noted, the uniqueness of steady states is closely connected with the existence of positive definite steady states, as we recall below. In our setting, in which the Lindbladian acting at the boundary has a unique positive definite steady state, a simple algebraic condition on the Hamiltonian H is necessary and sufficient for any steady state $\bar{\rho}$ that commutes with H to be the unique steady state, and in this case $\bar{\rho}$ is necessarily positive definite.

Theorem 2. Let $\mathcal{D}$ be a Lindbladian dissipator of the form (2) and (3) and suppose that $\mathcal{D}_A$ is ergodic with a positive definite steady state π_A on $\mathcal{H}_A$. Suppose that $\bar{\rho}$ is a steady state for (1) that commutes with H.

Then $\bar{\rho}$ is the unique steady state of (1) if and only if the only traceless self-adjoint operator X_B acting on $\mathcal{H}_B$ such that $[H, \mathbf{1}_A \otimes X_B] = 0$ is $X_B = 0$, and moreover, in this case $\bar{\rho}$ is positive definite.

Furthermore, if there is a traceless operator X_B on $\mathcal{H}_B$ so that $[H, \mathbf{1}_A \otimes X_B] = 0$ and thus the steady state is not unique, this also implies $[H_B, X_B] = 0$ and $[H_{AB}, \mathbf{1}_A \otimes X_B] = 0$.

The proofs will follow below, after the following discussion.

There are many examples, one of which we discuss below, for which both (10) and (11) hold. These include cases for which

$$\bar{\rho} = e^V = e^{V_A} \otimes e^{V_B}, \tag{13}$$

where $V = \lambda N + a\mathbf{1}$, where $N = N_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes N_B$ is the number operator for the number of particles in the system, which is conserved by H but is not conserved by $\mathcal{D}$. $\mathbf{1} = \mathbf{1}_A \otimes \mathbf{1}_B$ is the identity operator on the full system, $V_A = \lambda N_A + a\mathbf{1}_A$ and $V_B = \lambda N_B + a\mathbf{1}_B$. These "particles" can be the z component of the magnetization for the spin chain examples [1,2]. In [14] they are the excitations in a system of oscillators. In these cases we have $\bar{\rho} \sim e^{\lambda N}$, which is the equilibrium Gibbs ensemble in the limit of infinite temperature, where the reduced chemical potential $\lambda = \beta\mu$ remains finite in the limit. In order for $\bar{\rho}$ to be the unique steady state in these cases, we need V_B to be unique, which requires that $H_{AB} \neq 0$ and that the only operator acting on B that commutes with H is the identity, as in Theorem 2. Thus, $[H_{AB}, \mathbf{1}_A \otimes V_B] \neq 0$: H_{AB} moves particles between A and B.

We define the inner product between two operators V and W that act on some Hilbert space $\mathcal{H}$ as $\langle V|W\rangle = \mathrm{tr}\{V^\dagger W\}$, which is known as the Hilbert-Schmidt inner product. This makes the space of operators on $\mathcal{H}$ into a Hilbert space denoted $\widehat{\mathcal{H}}$. Operators on $\widehat{\mathcal{H}}$ are superoperators. If $\mathcal{D}$ is an operator on $\widehat{\mathcal{H}}$, we write $\mathcal{D}^\dagger$ to denote its adjoint with respect to the Hilbert-Schmidt inner product. If ρ is a density matrix on $\mathcal{H}$, then ρ is a vector in $\widehat{\mathcal{H}}$, and it will be convenient to use the Dirac notation such as $\mathcal{D}|\rho\rangle = |\mathcal{D}\rho\rangle$.

Proof of Theorem 1. Suppose that $\bar{\rho}$ is a steady state of (1). Then since $[H, \bar{\rho}] = 0$, $\mathcal{D}\bar{\rho} = 0$. Since $\mathcal{D}_A$ is ergodic, the nullspace of $\mathcal{D}_A$ is spanned by π_A. Since $\langle \rho_A|\mathcal{D}_A^\dagger\mathcal{D}_A\rho_A\rangle = \langle \mathcal{D}_A\rho_A|\mathcal{D}_A\rho_A\rangle$ for any ρ_A, π_A also spans the nullspace of $\mathcal{D}_A^\dagger\mathcal{D}_A$. The operator $\mathcal{D}_A^\dagger\mathcal{D}$ is positive semidefinnite, and its nullspace is spanned by π. Let d_A be the dimension of $\mathcal{H}_A$, and let

$$\mathcal{D}_A^\dagger\mathcal{D}_A = \sum_{j=1}^{d_A^2} \lambda_j |X_j\rangle\langle X_j|$$

be an eigenfunction expansion of the superoperator $\mathcal{D}_A$ in which X_1 is a multiple of π, so that $\lambda_1 = 0$, but $\lambda_j > 0$ for $j \geqslant 2$. Then $\{X_1, \ldots, X_{d_A^2}\}$ is a complete orthonormal basis, with respect to the Hilbert-Schmidt inner product, of the operators acting on $\mathcal{H}_A$, consisting of eigenvectors of $\mathcal{D}_A^\dagger\mathcal{D}_A$.

Then the startionary state $\bar{\rho}$ has the expansion

$$\bar{\rho} = \sum_{j=1}^{d_A^2} X_j \otimes W_j, \tag{14}$$

where each W_j acts on $\mathcal{H}_B$. Since $\mathcal{D}\bar{\rho} = 0$ and $\mathcal{D} = \mathcal{D}_A \otimes \mathcal{I}_B$ we have

$$0 = \sum_{j=2}^{d_A^2} (\mathcal{D}_A X_j) \otimes W_j. \tag{15}$$

For $j > 1$, define $Y_j := \lambda_j^{-1}\mathcal{D}_A X_j$. Note that

$$\begin{aligned}\mathrm{tr}_A[Y_j^\dagger(\mathcal{D}_A X_k)] &= \langle \lambda_j^{-1}\mathcal{D}_A X_j, \mathcal{D}_A X_k\rangle \\ &= \lambda_j^{-1}\langle \mathcal{D}_A^\dagger\mathcal{D}_A X_j, X_k\rangle = \delta_{j,k}.\end{aligned}$$

Therefore, for each $k > 1$,

$$0 = \sum_{j=2}^{M} \mathrm{tr}_A[(Y_k^\dagger \otimes \mathbf{1}_B)((\mathcal{D}_A X_j) \otimes W_j)] = W_k. \tag{16}$$

The conclusion is that for some normalization constant c,

$$\bar{\rho} = c\pi_A \otimes W_1, \tag{17}$$

where $\mathrm{tr}_B[cW_1] = 1$ and $cW_1 \geqslant 0$. Defining $\rho_B := cW_1$, we see that $\bar{\rho} = \pi_A \otimes \rho_B$, which proves (10). Note that the assumption that $\bar{\rho}$ (and thus also π_A) is positive definite was not used yet, so this part of the theorem [unlike (11)] is also true even if $\bar{\rho}$ has null eigenvectors, as can occur when $\mathcal{D}$ is approximating a zero-temperature bath.

On the other hand, if $\bar{\rho} = \pi_A \otimes \rho_B$ then using (7) the r.h.s. of (1) is just $-i[H, \bar{\rho}]$ which would have to be zero if $\bar{\rho}$ is stationary.

Now, to prove (11), we assume that $\bar{\rho}$ is positive definite, so we can define $V_A = \ln \pi_A$ and $V_B = \ln \rho_B$. Since H commutes with $\bar{\rho} = e^{V_A} \otimes e^{V_B}$, it commutes with $\ln \bar{\rho} = V_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes V_B = V$. Therefore

$$\begin{aligned}0 &= [V_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes V_B, H_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes H_B + H_{AB}] \\ &= [V_A, H_A] \otimes \mathbf{1}_B + \mathbf{1}_A \otimes [V_B, H_B] \\ &\quad + [V_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes V_B, H_{AB}].\end{aligned} \tag{18}$$

Apply the partial trace tr_B to each term on the r.h.s. First,

$$\mathrm{tr}_B\{[V_A, H_A] \otimes \mathbf{1}_B\} = [V_A, H_A]\dim(\mathcal{H}_B). \tag{19}$$

We claim that tr_B of all other terms in (18) are zero so that $[V_A, H_A] = 0$. To see this, $\mathrm{tr}_B\{\mathbf{1}_A \otimes [V_B, H_B]\} = 0$ since it is the trace of a commutator on $\mathcal{H}_B$. Next, $\mathrm{tr}_B\{[V_A \otimes \mathbf{1}_B, H_{AB}]\} = [V_A, \mathrm{tr}_B\{H_{AB}\}] = 0$ by our convention that $\mathrm{tr}_B\{H_{AB}\} = 0$. Finally, $\mathrm{tr}_B\{[\mathbf{1}_A \otimes V_B, H_{AB}]\} = 0$ by the partial cyclicity of the partial trace; that is, $\mathrm{tr}_B\{V_B H_{AB}\} = \mathrm{tr}_B\{H_{AB}V_B\}$. This proves that $[V_A, H_A] = 0$, and the same reasoning using instead tr_A shows that $[V_B, H_B] = 0$. Then (18) simplifies to $[V, H_{AB}] = 0$. For each of these vanishing commutators, we then use the fact that $[C, D] = 0$ implies $[e^C, D] = 0$ for any two operators C, D, to prove (11). ■

Proof of Corollary 1. By (12), ρ_β is of the form (10) if and only if $H_{AB} = 0$, given that $\mathrm{tr}_A\{H_{AB}\} = 0$ and $\mathrm{tr}_B\{H_{AB}\} = 0$. ■

Before proving Theorem 2 we recall a theorem of Frigerio [11] that we will use.

Theorem 3 (Frigerio's Theorem). Suppose that the equation $\frac{d\rho}{dt} = \mathcal{D}\rho$ for density matrices on $\mathcal{H}$ with $\mathcal{D}$ given by (2) has at least one positive definite steady state. Then there is a unique steady-state density matrix $\bar{\rho}$ with $\mathcal{D}\bar{\rho} = 0$ if and only if K and $\{L_1, \dots, L_n\}$ are such that any operator X on $\mathcal{H}$ that satisfies

$$[K, X] = 0 \quad \text{and for all} \quad \alpha, \quad [L_\alpha, X] = [L_\alpha^\dagger, X] = 0 \tag{20}$$

is a multiple of the identity. This is equivalent to saying that $\{K, L_\alpha, L_\alpha^\dagger\}$ generate all operators on $\mathcal{H}$.

Note that Eq. (1) has the form considered in Frigerio's Theorem if we simply replace K by $(K + H)$, so that Frigerio's Theorem may also be applied to Eq. (1).

Proof of Theorem 2. Let A be an operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $[\hat{K} \otimes \mathbf{1}_B, A] = 0$ and for all α: $[\hat{L}_\alpha \otimes \mathbf{1}_B, A] = [\hat{L}_\alpha^\dagger \otimes \mathbf{1}_B, A] = 0$, where $L_{A,\alpha}$ and K_A are defined in (4) and (5). Expand $A = \sum_\gamma W_\gamma \otimes E_\gamma$ where the W_γ are operators on $\mathcal{H}_A$ and the E_γ are an orthonormal basis for operators on $\mathcal{H}_B$. Then $[\hat{L}_\alpha \otimes \mathbf{1}_B, A] = 0$ becomes

$$\sum_\gamma [\hat{L}_\alpha, W_\gamma] \otimes E_\gamma = 0, \tag{21}$$

and since the E_γ are orthonormal, $[\hat{L}_\alpha, W_\gamma] = 0$ for each α, γ. A similar argument shows that $[\hat{L}_\alpha^\dagger, W_\gamma] = 0$ for each α, γ, and that $[\hat{K}, W_\gamma] = 0$ for each γ.

Since $\mathcal{D}_A$ is ergodic on $\mathcal{H}_A$, by Theorem 3, the only operators on $\mathcal{H}_A$ that commute with $\hat{K}$, $\hat{L}_\alpha$, and $\hat{L}_\alpha^\dagger$ for all α are multiples of the identity. Hence each W_γ is of the form $W_\gamma = c_\gamma \mathbf{1}_A$ for some constant c_γ. It follows that

$$A = \mathbf{1}_A \otimes X \quad \text{where} \quad X = \sum_\gamma c_\gamma E_\gamma. \tag{22}$$

Therefore, the only operators A on $\mathcal{H}_A \otimes \mathcal{H}_B$ that satisfy (20) of Theorem 3 are operators of the form $\mathbf{1}_A \otimes X$ such that $[H, \mathbf{1}_A \otimes X] = 0$.

Now suppose that the only operators of the form $\mathbf{1}_A \otimes X$ such that $[H, \mathbf{1}_A \otimes X] = 0$ are multiples of the identity. Let $\widetilde{\rho}$ denote a steady state that has maximal support, which exists by Theorem 4. If $\widetilde{\rho}$ is positive definite, then by Frigerio's Theorem, $\widetilde{\rho}$ is the unique steady state, and so $\bar{\rho} = \widetilde{\rho}$, which is positive definite.

On the other hand, if $\tilde{\rho}$ is not positive definite, then neither is any other steady state, including our steady state $\bar{\rho}$ that commutes with H. We claim that in this case, there would exist self-adjoint operators X on $\mathcal{H}_B$ other than multiples of the identity such that $[H, \mathbf{1}_A \otimes X] = 0$. Hence under our assumption on operators satisfying $[H, \mathbf{1}_A \otimes X] = 0$, $\tilde{\rho}$ must be positive definite, and must be the unique steady state by Frigerio's Theorem, and hence equals $\bar{\rho}$.

To see this, note that by Theorem 1, $\bar{\rho}$ has the form $\bar{\rho} = \pi_A \otimes \rho_B$. Since we assumed that π_A is positive definite, the projector P onto the null space of $\bar{\rho}$ has the form $\mathbf{1}_A \otimes P_B$ where P_B is the projector onto the null space of ρ_B. By hypothesis, $[H, \bar{\rho}] = 0$. Then since all of the spectral projections of $\bar{\rho}$ are polynomials in $\bar{\rho}$, $\mathbf{1}_A \otimes P_B$ is can be written as a polynomial, and hence it commutes with H. But then $X_B := P_B - c\mathbf{1}_B$, where c is chosen to make X_B traceless, is a nonzero traceless self-adjoint operator such that $\mathbf{1}_A \otimes X_B$ commutes with H, and therefore, if $\bar{\rho}$ is a degenerate (i.e., not positive definite) steady state commuting with H, it is not the unique steady state.

To simplify the condition on solutions of $[H, \mathbf{1}_A \otimes X] = 0$, observe, that since H is self adjoint, $[H, \mathbf{1}_A \otimes X] = 0$ if and only if $[H, \mathbf{1}_A \otimes X^\dagger] = 0$, and hence it suffices to consider self-adjoint X. Finally since $\mathbf{1}_A \otimes X$ commutes with H if and only if $\mathbf{1}_A \otimes (X - \mathrm{tr}[X]\mathbf{1}_B)$ commutes with H, we may freely assume X to be traceless. Thus, the steady state is unique if and only if the only traceless self-adjoint operator X on $\mathcal{H}_B$ such that $[H, \mathbf{1}_A \otimes X] = 0$ is $X = 0$.

Now suppose that the stationary state is not unique, so that there exists a nontrivial operator X on $\mathcal{H}_B$ such that $[H, \mathbf{1}_A \otimes X] = 0$. Then $[H_B + H_{AB}, \mathbf{1}_A \otimes X] = 0$, and since $\mathrm{tr}_A[H_{AB}] = 0$,

$$0 = \mathrm{tr}_A[H_{AB}, \mathbf{1}_A \otimes X] = [H_B, X], \tag{23}$$

from which the rest follows. ■

III. SPIN CHAIN EXAMPLE

The boundary-driven *XX* (or *XY*) spin model on a chain of ℓ sites for which the dissipator is of the form (8) is exactly solvable, and the unique $\bar{\rho}$ is of the form (13). This model, and close relatives of it, are also presented in [1,6,15] and references therein to illustrate various theorems discussed in those papers. In this section we discuss this model as an illustration of Theorem 2 for a $\mathcal{D}_A$ of the form (8).

After the Jordan-Wigner (JW) transformation its Hamiltonian has the form [cf. Eq. (15) in [1]]

$$H = \sum_{j=1}^{\ell-1} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j), \tag{24}$$

where $a_j, a_j^\dagger$ are the usual annihilation and creation operators of the JW fermions at site j. As is well known, the particle number operator

$$N = \sum_{j=1}^{\ell} a_j^\dagger a_j \tag{25}$$

commutes with H.

Let A be the first site of this chain, $j = 1$, while B is all the remaining sites. Fix $\beta > 0$ and define

$$\pi_A := \frac{1}{1 + e^{-\beta}}(e^{-\beta}|1\rangle\langle 1|_A + |0\rangle\langle 0|_A) = \frac{1}{1 + e^{-\beta}} e^{-\beta a_1^\dagger a_1}. \tag{26}$$

Let $\mathcal{D} = \mathcal{D}_A \otimes \mathcal{I}_B$ and let $\mathcal{D}_A$ be the dissipator defined as in (8) by

$$\mathcal{D}_A \rho = \epsilon[\pi_A \otimes \mathrm{tr}_A[\rho(t)] - \rho(t)] \tag{27}$$

in terms of π_A as in (26). Let $\ell \geqslant 2$, and let H be the Hamiltonian defined in (24). Define $\bar{\rho}$ to be the ℓ-fold tensor product state

$$\bar{\rho} := \left(\frac{1}{1+e^{-\beta}}(e^{-\beta}|1\rangle\langle 1| + |0\rangle\langle 0|)\right)^{\otimes \ell} \tag{28}$$

acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ so that $\bar{\rho} = (\frac{1}{1+e^{-\beta}})^{\ell} e^{-\beta N}$.

Note that $\bar{\rho}$ has the form $\pi_A \otimes \rho_B$, so that $\mathcal{D}\bar{\rho} = 0$. Moreover, since H commutes with N, and since $\bar{\rho}$ is a function of N, $[H, \bar{\rho}] = 0$. Therefore $\bar{\rho}$ is a steady state of (1). Since $\bar{\rho}$ is positive definite, one could apply Frigerio's Theorem to prove that $\bar{\rho}$ is the unique steady state—there are many ways to treat this simple model. However, the work is especially simple using Theorem 2 since we need only concern ourselves with H and not the operators L_α and $L_\alpha^\dagger$ in the Lindblad description of $\mathcal{D}_A$.

Proof that $\bar{\rho}$ is the unique steady state via Theorem 2: Since $\{n, a, a^\dagger, \mathbf{1} - n\}$ is an orthonormal basis for operators on $\mathcal{H}_A$, we may expand

$$H = K_{1,1} \otimes n + K_{1,0} \otimes a + K_{0,1} \otimes a^\dagger + K_{0,0} \otimes (\mathbf{1} - n) \tag{29}$$

and then write H in the block matrix form

$$H = \begin{bmatrix} K_{1,1} & K_{1,0} \\ K_{0,1} & K_{0,0} \end{bmatrix} \tag{30}$$

with operators $K_{i,j}$ on $\mathcal{H}_B$.

We will proceed by induction on ℓ. For $\ell = 2$, (30) reduces to $H = [\begin{smallmatrix} 0 & a \\ a^\dagger & 0 \end{smallmatrix}]$. Likewise, the block form of $\mathbf{1} \otimes X$ is $\mathbf{1} \otimes X = [\begin{smallmatrix} X & 0 \\ 0 & X \end{smallmatrix}]$. Then $[H, \mathbf{1} \otimes X] = 0$ becomes $[\begin{smallmatrix} 0 & [X, a] \\ [X, a^\dagger] & 0 \end{smallmatrix}]$, which reduces to $[a, X] = 0$ and $[a^\dagger, X] = 0$. Any operator that commutes with both a and $a^\dagger$ also commutes with n and $\mathbf{1} - n$, and hence with everything. Therefore, any such operator X is a multiple of the identity. Since $\mathrm{tr}[X] = 0$, $X = 0$. This proves uniqueness for $N = 2$.

For $N > 2$, let X be self-adjoint on $\mathcal{H}_B$ and such that $[\mathbf{1} \otimes X, H] = 0$. We claim that then X has the form $X = \mathbf{1} \otimes Y$ corresponding to the decomposition $\mathcal{H}_B = \mathcal{H} \otimes \mathcal{H}^{\otimes N-2}$.

To see this, again write H in the block form (30) with operators on $\mathcal{H}_B$ as entries

$$H = \begin{bmatrix} K & a \otimes \mathbf{1} \\ a^\dagger \otimes \mathbf{1} & K \end{bmatrix}, \tag{31}$$

where $a \otimes \mathbf{1}$, $a^\dagger \otimes \mathbf{1}$ act on $\mathcal{H}_B$ through its identification with $\mathcal{H} \otimes \mathcal{H}^{\otimes N-2}$, and where $K := \sum_{j=2}^{N-1} H_{j,j+1}$. Then $[H, \mathbf{1} \otimes X] = 0$ is equivalent to

$$[K, X] = 0\ , \quad [a \otimes \mathbf{1}, X] = 0, \quad \text{and} \quad [a^\dagger \otimes \mathbf{1}, X] = 0. \tag{32}$$

Now let $\{E_1, \dots, E_{2^{N-1}}\}$ be an orthonormal basis of operators on $\mathcal{H}^{\otimes N-2}$. Then X has a unique expansion $X = \sum_{j=1}^{2^{N-1}} W_j \otimes E_j$ where each W_j is an operator on $\mathcal{H}$. Then $0 = [a \otimes \mathbf{1}, X] = \sum_{j=1}^{2^{N-1}} [a, W_j] \otimes E_j$ and $0 = [a^\dagger \otimes \mathbf{1}, X] = \sum_{j=1}^{2^{N-1}} [a^\dagger, W_j] \otimes E_j$. It follows that for each j $[a, W_j] = [a^\dagger, W_j] = 0$, and then $W_j = c_j \mathbf{1}$ for some constant c_j. Therefore $X = \sum_{j=1}^{2^{N-1}} \mathbf{1} \otimes c_j E_j = \mathbf{1} \otimes Y$ where $Y = \sum_{j=1}^{2^{N-1}} c_j E_j$.

Now make the inductive assumption that this has been proved for $N \leqslant M$; we shall show it is then true for $N = M + 1$.

Let X be traceless and self-adjoint on $\mathcal{H}_B = \mathcal{H}^{\otimes M}$, and suppose that $\mathbf{1} \otimes X$ commutes with $H = \sum_{j=1}^{M} H_{j,j+1}$. By what we proved just above, $X = \mathbf{1} \otimes Y$, where Y is traceless and self-adjoint on the last $M - 1$ factors of $\mathcal{H}$ in $\mathcal{H}_B$. Then $\mathbf{1} \otimes X = \mathbf{1} \otimes \mathbf{1} \otimes Y$, which evidently commutes with H_{12}. Therefore $[\mathbf{1} \otimes X, H] = 0$ becomes

$$[\mathbf{1} \otimes Y, H'] = 0 \quad \text{where} \quad H' = \sum_{j=2}^{M} H_{j,j+1}. \tag{33}$$

By the inductive hypothesis, $Y = 0$. ■

Remark 1. Note that the form $\bar{\rho} = \pi_A \otimes \rho_B$ of the unique steady state is independent of the parameter ϵ, and this proves analytically that, as a function of ϵ, the steady state does not converge to the Gibbs state as ϵ converges to zero, an issue discussed in [2].

IV. EXISTENCE OF STEADY STATES WITH MAXIMAL SUPPORT

We give a simple proof of the existence of a stationary state of (1) which yields some additional information that is used here. Many proofs of existence of steady states invoke fixed point theorems, e.g., the Markov-Kakutani Fixed Point Theorem in [10] in a general infinite-dimensional setting, and the Brower Fixed Point Theorem in [15] in a finite-dimensional setting. The mean ergodic theorem provides a more constructive approach and additional information.

Theorem 4. For a d-dimensional Hilbert space $\mathcal{H}$, Eq. (1) has at least one steady-state solution. Moreover, there exists a steady-state solution $\bar{\rho}$ that has maximal support in the sense that if ρ is any steady-state solution, then

$$\rho \leqslant d\bar{\rho}. \tag{34}$$

Proof. Let $\mathcal{L}\rho := -i[H, \rho] + \mathcal{D}\rho$ as in (1). Then each $e^{t\mathcal{L}}$, $t > 0$ is completely positive and trace preserving. As a consequence, by a mean ergodic theorem of Lance [16], for any operator A on $\mathcal{H}$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{t\mathcal{L}}(A)\, dt := \mathcal{P}_{\mathcal{L}}(A) \tag{35}$$

exists. (In our finite-dimensional setting, all topologies are equivalent, so the sense of convergence is immaterial.) It is clear from the definition that for all t, $e^{t\mathcal{L}}\mathcal{P}_{\mathcal{L}}(A) = \mathcal{P}_{\mathcal{L}}(A)$. Furthermore, since $\mathcal{P}_{\mathcal{L}}$ preserves positivity and traces, if A is any density matrix, then $\mathcal{P}_{\mathcal{L}}(A)$ is a density matrix. This proves existence.

Next, define the density matrix ρ_0 by $\rho_0 := \frac{1}{d}\mathbf{1}_{\mathcal{H}}$, and define

$$\overline{\rho} := \mathcal{P}_{\mathcal{L}}(\rho_0). \tag{36}$$

Then $\bar{\rho}$ is a steady state. Now let ρ be any other steady state. Since $\rho \leqslant \mathbf{1}_{\mathcal{H}}$, $\rho \leqslant d\rho_0$, and then for each t, $\rho = e^{t\mathcal{L}}\rho \leqslant de^{t\mathcal{L}}\rho_0$ so that (34) is satisfied. ■

We remark that in our finite-dimensional setting, the theorem of Lance has an elementary proof using the Jordan canonical form and a well-known contractive property of trace preserving completely positive operators.

V. UNIQUENESS

Frigerio's theorem (4) [11] gives a general if and only if result for uniqueness of the stationary solution $\bar{\rho}$ of (1) once we know the existence of a positive definite $\bar{\rho}$. The latter requirement is essential, as pointed out in [15]. An avoidance of this requirement is given by Yoshida [12,15] who proved that a sufficient condition for uniqueness of $\bar{\rho}$ is that the Lindbladian $\mathcal{L}$ is such that all operators in $\mathcal{H}$ are linear combinations of products of the operators in the set $\{H - \frac{i}{2}\sum_\alpha L_\alpha^\dagger L_\alpha, L_\alpha\}$ (all α). This set generally contains fewer operators than the set used by Frigerio.

Theorem (3) gives necessary and sufficient conditions for uniqueness for the case when the dissipator $\mathcal{D}$ has the form (3) and $\mathcal{D}_A$ is ergodic. We do not require the *a priori* existence of a positive definite $\bar{\rho}$ but find the conditions for uniqueness and strict positivity of a $\bar{\rho}$ of the form $\pi_A \otimes \rho_B$ which commutes with H. Our conditions also ensure that when there exists a unique $\bar{\rho}$ then it is positive definite.

VI. DISCUSSION

The Gibbs measure, $\rho_\beta = e^{-\beta H}/Z(\beta)$, is the standard thermal equilibrium state of a system with Hamiltonian H at inverse temperature β. For an open quantum system coupled to a Lindbladian dissipator that acts only on part A of the system (the system is otherwise fully isolated), one might have naively thought that one could choose a dissipator acting on part A that produces the Gibbs measure as the resulting exact steady state of the full system. In this paper we have shown that this is not possible when $\beta H_{AB} \neq 0$, where H_{AB} is the part of the Hamiltonian that is the interactions between part A of the system and the rest of the system (part B). Why is this not possible? An informal justification of our result is: For $\beta H_{AB} \neq 0$ the Gibbs measure contains specific detailed correlations between parts A and B of the system. The dissipator acts only on A without being able to use any "information" about part B, so it cannot produce these correct correlations between A and B. On the contrary, it necessarily will disrupt those correlations by dissipating part A in a way that is independent of the state of part B.

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APPENDIX: NONERGODIC $\mathcal{D}_A$

In this Appendix, we explain how quantum ergodic decompositions may be used to extend Theorem 1 to the case in which $\mathcal{D}_A$ is not assumed to be ergodic

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and let $\mathcal{L}$ be the generator of a quantum dynamical semigroup $e^{t\mathcal{L}}$ on operators on $\mathcal{H}$ so that each $e^{t\mathcal{L}}$ is completely positive and trace preserving. Then $\mathcal{L}^\dagger$ is the generator of a quantum Markov semigroup $(e^{t\mathcal{L}})^\dagger := e^{t\mathcal{L}^\dagger}$. That is, for each t, $e^{t\mathcal{L}^\dagger}$ is completely positive with the property that $e^{t\mathcal{L}^\dagger}\mathcal{I}_\mathcal{H} = \mathcal{I}_\mathcal{H}$. Because of this last property, $\mathcal{L}^\dagger\mathcal{I}_\mathcal{H} = 0$.

Let $\mathcal{C}$ denote the null space of $\mathcal{L}^\dagger$. Suppose that there exists at least one positive definite steady state; that is, at least one positive definite density matrix such that $\mathcal{L}\rho = 0$. Then Frigerio's Theorem [11] says that $\mathcal{C}$ is not just a vector space of operators on $\mathcal{H}$; it is also closed under multiplcation and taking Hermitian adjoints, and evidently it contains $\mathcal{I}_\mathcal{H}$. This makes it a a von Neumann algebra. Let $\mathcal{Z}$ denote $\mathcal{C} \cap \mathcal{C}'$ where $\mathcal{C}'$ is the commutant of $\mathcal{C}$. This is a commutative von Neumann algebra callled the *center* of $\mathcal{C}$.

Every commutative von Neumann algebra on a finite-dimensional Hilbert space $\mathcal{H}$ has the following simple structure (see, e.g., [17]): There is a set $\{P_1, \ldots, P_m\}$ of mutually orthogonal projections summing to $\mathcal{I}_\mathcal{H}$ whose complex span is the algebra.

The projectors $\{P_1, \ldots, P_m\}$ provide the basis for an *ergodic decomposition* of $e^{t\mathcal{L}^\dagger}$. Let $\mathcal{H}_j$ denote the range of P_j so that

$$\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_j. \tag{A1}$$

The following theorem is proved in [17,18]: Each of the Hilbert spaces $H^{(j)}$ has a factorization $\mathcal{H}^{(j)} = \mathcal{K}_\ell^{(j)} \otimes \mathcal{K}_r^{(j)}$, determined by the generator $\mathcal{L}$, where either of these factors may be, but neither need be, one-dimensional. There is a set of m density matrices on the "right" factors $\mathcal{K}_r^{(j)}$, $\{\omega_1, \ldots, \omega_m\}$ such that a density matrix ρ on $\mathcal{H}$ satisfies $\mathcal{L}\rho = 0$ if and only if it has the form

$$\rho = \sum_{j=1}^{m} p_j \rho_j \otimes \omega_j, \tag{A2}$$

where each ρ_j is *any* density matrix on $\mathcal{K}_1^{(j)}$ and the p_j' are probabilities.

The ergodic case is that in which $m = 1$ and $\mathcal{K}_\ell^{(1)}$ is one dimensional so that $\mathcal{H} = \mathcal{K}_r^{(1)}$ and then ω_1 is the unique steady state.

If we relax the assumption that $\mathcal{D}_A$ is ergodic with a positive definite steady state to only the assumption that $\mathcal{D}_A$ has at least one positive definite steady state, so that every steady state for $\mathcal{D}_A$ has the form (A2), then the method of proof of Theorem 1 can be used to prove that every steady state $\bar{\rho}$ of (1) that commutes with H has an expansion of the form (A2) where now ρ_j is a density matrix on $\mathcal{K}_r^{(j)} \otimes \mathcal{H}_B$: In this nonergodic case, the steady states that commute with H are a direct sum of components that again factor as tensor products. Finally, if $\mathcal{D}_A$ does not have any positive definite steady state, let $\bar{\rho}_A$ be a steady state of maximal support, as in Theorem 4, and let $\mathcal{K}_A$ be the subspace of $\mathcal{H}_A$ that supports $\bar{\rho}_A$. (That is, $\mathcal{K}_A$ is the orthogonal complement of the null spaces of $\bar{\rho}_A$.) Let P_A be the orthogonal projection onto $\mathcal{K}_A$. Then for any operator X on $\mathcal{H}_A$

$$e^{t\mathcal{L}}(P_A X P_A) = P_A e^{t\mathcal{L}}(P_A X P_A) P_A, \tag{A3}$$

so that the Lindbladian evolution may be restricted to operators on $\mathcal{K}_A$, and then it has a positive definite steady state (but a different Lindbladian description in terms of operators L_α now acting on $\mathcal{K}_A$ instead of $\mathcal{H}_A$). The above consideration apply to this reduced system, in which a "transient part" has been discarded. The transient part is irrelevant as far the the structure of steady states is concerned.

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