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ORIGINAL PAPER

Generalized Bounded Distortion Property

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Abstract
We prove the nonstationary bounded distortion property for $C^{1+\varepsilon}$ smooth dynamical systems on multidimensional spaces. The results we obtain are motivated by potential application to study of spectral properties of discrete Schrödinger operators with potentials generated by Sturmian sequences.

Keywords Smooth dynamical systems · Non-stationary dynamical systems · Bounded distortion · Discrete Schrödinger operators

1 Introduction

The bounded distortion lemma is a well-known and widely used tool in the theory of smooth dynamical systems. One of its classical forms is the following:

Theorem 1 *Consider a twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with a non-vanishing derivative, such that*

$$\frac{|f''(x)|}{|f'(x)|} < C$$

for some constant C and every $x \in \mathbb{R}$. Assume that there exists an interval $I \subset \mathbb{R}$, such that for some $n \in \mathbb{N}$ we have

$$\sum_{j=0}^{n-1} |f^j(I)| < L$$

for some constant L. Then there exists a constant K that depends only on L and C (and not n), such that for every two subintervals $I_1, I_2 \subset I$,

$$K^{-1}\frac{|I_1|}{|I_2|} < \frac{|f^n(I_1)|}{|f^n(I_2)|} < K\frac{|I_1|}{|I_2|}.$$

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Different forms of this statement are used in [4, 5, 7, 11, 16] to establish various fundamental results.

One of its most famous applications is in the proof of Denjoy's theorem, which states that any diffeomorphism of the circle with an irrational rotation number and derivative of bounded variation is topologically conjugate to an irrational rotation (for details see [4] or [6, theorem 12.1.1]).

Another application of bounded distortion can be seen in the proof of a classical folklore result that asserts existence of a continuous invariant ergodic measure for C^2 expanding maps of the circle. Moreover, if a piecewise differentiable map admits an induced Markov map and satisfies the volume bounded distortion property (also known as geometrical self similarity) and some additional assumptions, then it admits an ergodic invariant absolutely continuous probability measure. For details and other results of a similar nature see [7] and references therein.

Bounded distortion plays an important role in the study of dynamically defined Cantor sets. For example, it is used to prove that the Hausdorff dimension coincides with the box counting dimension (also known as limit capacity) if the generators are in $C^{1+\varepsilon}$ (see [15], or [11, chapter 4, theorem 3] for a modern exposition). Bounded distortion also allows to estimate thickness of a dynamically defined Cantor set, and that estimate is used to prove the celebrated Newhouse's theorem (see [8] and [11] for details).

A special case of bounded distortion for nonstationary sequences of uniformly hyperbolic maps of a plane with the uniform cone condition was studied by J. Palis and J-C. Yoccoz in [12, corollary 3.4]. A generalization of said result that allows unbounded derivatives can be found in [9]. These results have important applications in the theory of SRB measures for surface diffeomorphisms. For more results utilizing bounded distortion technique see [5, 16] and references therein.

We would like to point out that bounded distortion is a phenomenon that is usually observed in systems with $C^{1+\varepsilon}$ regularity (see theorem 2). For example, one could see [10] by H. Ounesli, which proves that within the space of ergodic Lebesgue-preserving C^1 expanding maps of the circle, unbounded distortion is C^1-generic.

Our result is primarily motivated by a potential application to the study of spectral properties of discrete Schrödinger operator with Sturmian potentials. In 1987 A. Sütő (see [13]) found a way to describe the spectrum of the Fibonacci Hamiltonian (which is a special case of a discrete Schrödinger operators with Sturmian potential) using dynamical properties of the trace map. Since then, spectral properties of the Fibonacci Hamiltonian have been extensively studied by many authors (see [3, 14] and references therein). In particular, D. Damanik and A. Gorodetski were able to estimate the thickness and Hausdorff dimension of the spectrum of the Fibonacci Hamiltonian. In their work they used a stationary multidimensional version of the bounded distortion property (see [2, proposition 3.11]). A dynamical description similar to the one discovered by A. Sütő is now available for the spectrum of every discrete Schrödinger operator with Sturmian potential (see [1]). However, the dynamics involved in the general case is nonstationary. In order to generalize the results from [2], the nonstationary bounded distortion property presented in this paper might prove useful.

2 Main Result

In this section we will state and prove the main result of the paper. We start with some definitions.

Definition 1 If γ is a regular C^1 curve, meaning $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{R}^d$ and $\gamma'(t)$ is continuous and does not vanish, then define the **length** of γ as

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t,$$

and define the **maximal angle** of γ as

$$\alpha(\gamma) = \sup_{x,y \in [a,b]} \angle(\gamma'(x), \gamma'(y)).$$

Definition 2 For a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x) = \begin{pmatrix} f^{(1)}(x) \\ \vdots \\ f^{(m)}(x) \end{pmatrix},$$

and a point $x \in \mathbb{R}^n$ define the **total derivative** of f at x as

$$D_x f = \begin{pmatrix} \frac{\partial f^{(1)}}{\partial x_1} & \cdots & \frac{\partial f^{(1)}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^{(m)}}{\partial x_1} & \cdots & \frac{\partial f^{(m)}}{\partial x_n} \end{pmatrix}.$$

Definition 3 For a continuously differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$, define its C^1 **seminorm** as

$$\|f\|_1 = \sup_{x, v \in \mathbb{R}^n, \|v\|=1} \|D_x f \cdot v\|,$$

and for $0 < \varepsilon \leq 1$, define its $C^{1+\varepsilon}$ **seminorm** as

$$\|Df\|_\varepsilon = \sup_{x,y \in \mathbb{R}^n} \frac{\|D_x f - D_y f\|}{\|x - y\|^\varepsilon}.$$

We say that $f \in C^{1+\varepsilon}$ if $\|Df\|_\varepsilon < \infty$.

Remark 1 Notice that the case of $\varepsilon = 1$ corresponds to f with a Lipschitz first derivative.

The following theorem is the main result of the paper:

Theorem 2 *Let $\gamma : [a_0, b_0] \subset \mathbb{R} \to \mathbb{R}^d$ be the natural parameterization of a regular C^1 curve. Fix $0 < \varepsilon \leq 1$ and let $f_i : \mathbb{R}^d \to \mathbb{R}^d$ be $C^{1+\varepsilon}$ maps. Denote $F_i = f_i \circ f_{i-1} \circ \ldots \circ f_1$, where $1 \leq i \leq n$ and let F_0 be the identity map. Assume there exists C such that*

$$\|f_i\|_1, \|f_i^{-1}\|_1, \|Df_i\|_\varepsilon \leq C \tag{1}$$

for all $1 \leq i \leq n$ and assume there exists L and α such that $\sum_{i=0}^{n-1} L(F_i \circ \gamma)^\varepsilon \leq L$ and $\sum_{i=0}^{n-1} \alpha(F_i \circ \gamma) \leq \alpha$. Then there exists a constant K such that for any $x, y \in [a_0, b_0]$,

$$K^{-1} \leq \frac{\|(F_n \circ \gamma)'(x)\|}{\|(F_n \circ \gamma)'(y)\|} \leq K, \tag{2}$$

where K depends only on C, L, and α (and not n).

Moreover, consider two subintervals $[a_1, b_1], [a_2, b_2] \subset [a_0, b_0]$ and related arcs given by $\gamma([a_1, b_1])$ and $\gamma([a_2, b_2])$. Then

$$\frac{|b_1 - a_1|}{|b_2 - a_2|} K^{-2} \leq \frac{L((F_n \circ \gamma)([a_1, b_1]))}{L((F_n \circ \gamma)([a_2, b_2]))} \leq \frac{|b_1 - a_1|}{|b_2 - a_2|} K^2. \tag{3}$$

Proof We start by proving inequality (2). Denote points $x_i = (F_i \circ \gamma)(x),\ y_i = (F_i \circ \gamma)(y)$, and vectors $u_i = (F_i \circ \gamma)'(x),\ v_i = (F_i \circ \gamma)'(y)$ for $0 \le i \le n$. Notice that all $\| f_i^{-1} \|_1$ are finite, so we can say that $\|u_i\|, \|v_i\| \neq 0$ for all $1 \le i \le n$. Now we can deduce the following inequality:

$$\begin{aligned} \left|\log \frac{\|u_n\|}{\|v_n\|}\right| = \left|\log \frac{\|D_{x_{n-1}} f_n \cdot u_{n-1}\|}{\|D_{y_{n-1}} f_n \cdot v_{n-1}\|}\right| \le \\ \le \left|\log \frac{\|D_{x_{n-1}} f_n \cdot u_{n-1}\|}{\|D_{x_{n-1}} f_n \cdot v_{n-1}\|}\right| + \left|\log \frac{\|D_{x_{n-1}} f_n \cdot v_{n-1}\|}{\|D_{y_{n-1}} f_n \cdot v_{n-1}\|}\right|. \end{aligned} \tag{4}$$

We continue by bounding both logarithms, and we will do so with the following lemmas:

Lemma 1 *For every $1 \le i \le n$,*

$$\left|\log \frac{\|D_{x_{i-1}} f_i \cdot u_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot v_{i-1}\|}\right| \le \left|\log \frac{\|u_{i-1}\|}{\|v_{i-1}\|}\right| + C^2 \cdot \alpha(F_{i-1} \circ \gamma).$$

Proof Let $\tilde{u}_{i-1}$ and $\tilde{v}_{i-1}$ notate the normalized vectors u_{i-1} and v_{i-1} respectively. Then

$$\left|\log \frac{\|D_{x_{i-1}} f_i \cdot u_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot v_{i-1}\|}\right| \le \left|\log \frac{\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|}\right| + \left|\log \frac{\|u_{i-1}\|}{\|v_{i-1}\|}\right|.$$

Without loss of generality, assume $\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1}\| \ge \|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|$. If we only focus on the part of the left term that is inside the logarithm, we can rearrange it as

$$\begin{aligned} \frac{\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|} = \frac{\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1} - D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1} + D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|} \le \\ \le \frac{\|D_{x_{i-1}} f_i\| \cdot \|\tilde{u}_{i-1} - \tilde{v}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|} + 1. \end{aligned} \tag{5}$$

Denote $\alpha_{i-1} = \angle(\tilde{u}_{i-1}, \tilde{v}_{i-1})$. We can use the definition of α_{i-1} and the fact that $\sin\left(\frac{\alpha_{i-1}}{2}\right) \le \frac{\alpha_{i-1}}{2}$ to get

$$\|\tilde{u}_{i-1} - \tilde{v}_{i-1}\| = 2 \sin\left(\frac{\alpha_{i-1}}{2}\right) \le \alpha_{i-1} \le \alpha(F_{i-1} \circ \gamma). \tag{6}$$

Note that $\det(D_{x_{i-1}} f_i) \neq 0$ since every $\| f_i^{-1} \|$ is finite, hence its inverse exists. Combining inequalities (5) and (6) gives us

$$\frac{\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|} \le \|D_{x_{i-1}} f_i\| \cdot \|(D_{x_{i-1}} f_i)^{-1}\| \cdot \alpha(F_{i-1} \circ \gamma) + 1 \le C^2 \cdot \alpha(F_{i-1} \circ \gamma) + 1.$$

Since $|\log(1 + x)| \le x$ for all $x \ge 0$ and $|\log x|$ is monotone increasing for $x \ge 1$, we can get our final bound:

$$\left|\log \frac{\|D_{x_{i-1}} f_i \cdot \tilde{u}_{i-1}\|}{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\|}\right| \le \left|\log\left(C^2 \cdot \alpha(F_{i-1} \circ \gamma) + 1\right)\right| \le C^2 \cdot \alpha(F_{i-1} \circ \gamma). \tag{7}$$

□

Lemma 2 *For every $1 \le i \le n$,*

$$\left|\log \frac{\|D_{x_{i-1}} f_i \cdot v_{i-1}\|}{\|D_{y_{i-1}} f_i \cdot v_{i-1}\|}\right| \le C^2 \cdot L(F_{i-1} \circ \gamma)^{\varepsilon}.$$

Proof Once again, let $\tilde{v}_{i-1}$ notate the normalized vector v_{i-1}. Without loss of generality, assume $\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1}\| \geq \|D_{y_{i-1}} f_i \cdot \tilde{v}_{i-1}\|$. Then

$$\frac{\|D_{x_{i-1}} f_i \cdot v_{i-1}\|}{\|D_{y_{i-1}} f_i \cdot v_{i-1}\|} = \frac{\|D_{x_{i-1}} f_i \cdot \tilde{v}_{i-1} - D_{y_{i-1}} f_i \cdot \tilde{v}_{i-1} + D_{y_{i-1}} f_i \cdot \tilde{v}_{i-1}\|}{\|D_{y_{i-1}} f_i \cdot \tilde{v}_{i-1}\|} \leq$$
$$\leq C \cdot \|D_{x_{i-1}} f_i - D_{y_{i-1}} f_i\| + 1,$$

and we can bound the following similar to (7):

$$\left|\log \frac{\|D_{x_{i-1}} f_i \cdot v_{i-1}\|}{\|D_{y_{i-1}} f_i \cdot v_{i-1}\|}\right| \leq C \cdot \|D_{x_{i-1}} f_i - D_{y_{i-1}} f_i\|.$$

Notice that

$$\|x_{i-1} - y_{i-1}\| \leq L(F_{i-1} \circ \gamma),$$

then

$$1 \leq \frac{L(F_{i-1} \circ \gamma)^{\varepsilon}}{\|x_{i-1} - y_{i-1}\|^{\varepsilon}}.$$

It follows that

$$C \cdot \|D_{x_{i-1}} f_i - D_{y_{i-1}} f_i\| \leq C \frac{\|D_{x_{i-1}} f_i - D_{y_{i-1}} f_i\|}{\|x_{i-1} - y_{i-1}\|^{\varepsilon}} \cdot L(F_{i-1} \circ \gamma)^{\varepsilon} \leq C^2 \cdot L(F_{i-1} \circ \gamma)^{\varepsilon}.$$

□

Proof Combining lemma 1 and lemma 2 with Eq. (4) shows

$$\left|\log \frac{\|u_n\|}{\|v_n\|}\right| \leq \left|\log \frac{\|u_{n-1}\|}{\|v_{n-1}\|}\right| + C^2 \cdot \alpha(F_{n-1} \circ \gamma) + C^2 \cdot L(F_{n-1} \circ \gamma)^{\varepsilon}.$$

We can apply this inequality recursively to get

$$\left|\log \frac{\|u_n\|}{\|v_n\|}\right| \leq C^2 \sum_{i=0}^{n-1} \left[\alpha(F_i \circ \gamma) + L(F_i \circ \gamma)^{\varepsilon}\right] \leq C^2(\alpha + L).$$

We finish the proof of (2) by taking $K = e^{C^2(\alpha+L)}$.

Now let us move on to inequality (3). Let $t \in [a_0, b_0]$ be arbitrary. Note that $\|(F_n \circ \gamma)'(t)\| \neq 0$ since $\det(D_{\gamma(t)} f_i) \neq 0$ and $\gamma'(t) \neq 0$. Then by definition of length we have

$$\frac{L((F_n \circ \gamma)([a_1, b_1]))}{\|(F_n \circ \gamma)'(t)\|} = \int_{a_1}^{b_1} \frac{\|(F_n \circ \gamma)'(x)\|}{\|(F_n \circ \gamma)'(t)\|} \, \mathrm{d}x$$

and inequality (2) gives us

$$|b_1 - a_1| K^{-1} \leq \frac{L((F_n \circ \gamma)([a_1, b_1]))}{\|(F_n \circ \gamma)'(t)\|} \leq |b_1 - a_1| K.$$

We can similarly obtain

$$\frac{K^{-1}}{|b_2 - a_2|} \leq \frac{\|(F_n \circ \gamma)'(t)\|}{L((F_n \circ \gamma)([a_2, b_2]))} \leq \frac{K}{|b_2 - a_2|},$$

and multiplying these two inequalities finishes the proof. □

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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