



The replica-symmetric free energy for Ising spin glasses with orthogonally invariant couplings

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Abstract

We study a variant of the Sherrington–Kirkpatrick (S–K) spin glass model with external field, where the random symmetric couplings matrix does not consist of i.i.d. entries but is instead orthogonally invariant in law. For sufficiently high temperature, we prove a replica-symmetric formula for the first-order limit of the model free energy. Our analysis is an adaptation of a conditional second-moment-method argument previously introduced by Bolthausen for studying the high-temperature regime of the S–K model, where one conditions on the iterates of an Approximate Message Passing (AMP) algorithm for solving the TAP equations for the model magnetization. We apply this method using a memory-free version of AMP that is tailored to the orthogonally invariant structure of the model couplings.

Keywords Mean-field spin glasses · Free probability · AMP algorithms

Mathematics Subject Classification 60F10 · 60K35

1 Introduction

We study a probability model on the hypercube $\sigma \in \{+1, -1\}^n$ given by

$$P(\sigma) = \frac{1}{Z} \exp \left(\frac{\beta}{2} \sigma^\top J \sigma + h^\top \sigma \right). \quad (1.1)$$

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Here $h \in \mathbb{R}^n$ is a deterministic vector, $J \in \mathbb{R}^{n \times n}$ is a random symmetric matrix which we will assume satisfies the orthogonal invariance in law

$$J \stackrel{L}{=} O^\top J O \text{ for any orthogonal matrix } O \in \mathbb{R}^{n \times n},$$

and Z is the partition function

$$Z = \sum_{\sigma \in \{+1, -1\}^n} \exp\left(\frac{\beta}{2} \sigma^\top J \sigma + h^\top \sigma\right).$$

We will refer to J as the couplings matrix, h as the external field, and β as the inverse temperature.

The specific example of the Sherrington–Kirkpatrick (S–K) model [49], where J has i.i.d. Gaussian entries above the diagonal, is well-studied and known to exhibit rich phenomena. At high temperatures, $P(\sigma)$ is “replica-symmetric”, the large- n limit of the free energy is described by a simple replica-symmetric formula [1, 8, 49], and the magnetization $m = \sum_{\sigma \in \{+1, -1\}^n} \sigma \cdot P(\sigma)$ satisfies in this limit the Thouless–Anderson–Palmer (TAP) mean-field equations [13, 55, 56]. At low temperatures, the limit free energy is described more generally by Parisi’s variational formula [25, 43, 44, 54]. The solution of the variational problem may be understood as corresponding to an ultrametric tree structure for $P(\sigma)$, and the TAP equations describe the conditional means of the “pure states” in this ultrametric tree [35–37]. This picture has been formalized and proven rigorously for certain mixed p -spin analogues of the S–K model in [3, 42].

Here, we are interested in the more general setting of (1.1) where J is orthogonally invariant, but can have arbitrary spectral distribution and dependent entries. Examples include the random orthogonal model (ROM) [34] where J has all eigenvalues equal to $\{+1, -1\}$, and the Gaussian Hopfield model [27] where $J = G^\top G$ and G is a rectangular Gaussian matrix. In the physics literature, the replica-symmetric and 1-RSB free energies for the ROM were computed by Marinari et al. in [34], and extended to models with general orthogonally invariant couplings in [14]. Parisi and Potters derived in [45] the TAP mean-field equations for the ROM, using a perturbative expansion approach of [23, 46] and a conjectured resummation of the terms of this expansion. Oppor and Winther provided in [41] an alternative derivation of the TAP equations using the cavity method and a system of self-consistent equations for the cavity fields (which we review in Appendix E), and verified also via a replica calculation that the TAP free energy evaluated at the model magnetization coincides with the free energy given by the replica-symmetric formula. At present, few rigorous mathematical results are known for models with general orthogonally invariant couplings matrices J .

Our work is in large part motivated by a renewed interest in these types of mean-field models in information theory, statistics, and machine learning [4, 18, 21, 24, 30, 31, 33, 47, 48, 50–53, 57], where orthogonally invariant matrices may serve as more robust models of regression and sensing designs or more accurate models of noise in data applications. Indeed, following the initial posting of this work, several dynamical universality results were obtained in [19, 20, 58] showing that the mean-field dynamical

ics of Approximate Message Passing (AMP) and other first-order iterative algorithms applied to orthogonally invariant matrices are universal across broad classes of matrices with delocalized eigenvectors. In many of these applications, replica predictions for the model free energy are conjectured, but not rigorously known. Maillard et al. studied in [32] a class of computational algorithms in the context of such orthogonally invariant models, extending the diagrammatic expansion method of [23, 45, 46] to describe the connections between these algorithms and the predicted mean field theory; the authors of [32] highlighted the mathematical verification of these predictions as an open question.

We study in this work the specific model (1.1), and prove a replica-symmetric formula for the first-order limit of its free energy in a sufficiently high temperature regime. This extends previous work of [6], which showed such a result in the absence of an external field ($h = 0$). Similar to the S–K model [1], the $h = 0$ setting is special in that the quenched free energy $n^{-1} \mathbb{E} \log Z$ coincides asymptotically with the annealed free energy $n^{-1} \log \mathbb{E} Z$, as was verified in [6] using the second moment method. This no longer holds when $h \neq 0$, and our proof applies instead a conditional version of this idea developed in [8] for the S–K model, where one establishes that the quenched and annealed free energies coincide upon conditioning on an appropriately chosen sigma-field that is informative about the random magnetization. This method was refined for the S–K model in [11] to cover a large and explicit part of the high-temperature regime, and is also related to analyses of [9, 16] for the Ising perceptron model.

Our construction of the conditioning sigma-field relies on recent developments on iterative algorithms for solving the TAP equations for the model (1.1). We summarize these developments and the proof strategy in Sect. 1.2 below, after presenting our main result. Following the initial posting of this work, our analyses have been extended in [22] to show also the validity of the TAP equations for the magnetization under a similar high-temperature assumption, and in [29] to obtain analogous results in a statistical linear model with orthogonally invariant regression design.

1.1 Model and main result

Consider the Gibbs distribution (1.1) on the binary hypercube, under the following assumptions for the couplings matrix J and external field h .

Assumption 1.1 Let $J = O^\top D O$ be the eigen-decomposition of J .

- (a) $O \sim \text{Haar}(\mathbb{O}(n))$ is a random Haar-distributed orthogonal matrix.
- (b) $D = \text{diag}(d_1, \dots, d_n)$ is a deterministic diagonal matrix of eigenvalues, whose empirical distribution converges weakly to a limit law

$$\frac{1}{n} \sum_{i=1}^n \delta_{d_i} \rightarrow \mu_D$$

as $n \rightarrow \infty$. This law μ_D has strictly positive variance and a compact support $\text{supp}(\mu_D)$. Furthermore,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \max(d_1, \dots, d_n) \\ &= d_+ \triangleq \max(x : x \in \text{supp}(\mu_D)), \quad \liminf_{n \rightarrow \infty} \min(d_1, \dots, d_n) > -\infty. \end{aligned}$$

- (c) $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ is a deterministic vector, whose empirical distribution of entries converges weakly to a limit law

$$\frac{1}{n} \sum_{i=1}^n \delta_{h_i} \rightarrow \mu_H$$

as $n \rightarrow \infty$. For every $p \geq 1$, the law μ_H has finite p^{th} moment, and $n^{-1} \sum_{i=1}^n h_i^p \rightarrow \mathbb{E}_{H \sim \mu_H}[H^p]$.¹

We remark that our results apply also to models with random (D, h) independent of O which satisfy these conditions almost surely as $n \rightarrow \infty$, by applying the results conditionally on (D, h) .

We are interested in the asymptotic free energy

$$\Psi = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z. \quad (1.2)$$

For sufficiently small $\beta > 0$, we prove that this limit exists almost surely and is given by the following replica-symmetric formula: Denote the Cauchy- and R-transforms of μ_D by

$$G(z) = \int \frac{1}{z-x} \mu_D(dx), \quad R(z) = G^{-1}(z) - \frac{1}{z}.$$

We define $G(z)$ for real arguments $z \in (d_+, \infty)$. The function $G : (d_+, \infty) \rightarrow (0, G(d_+))$ is strictly decreasing, where we denote

$$G(d_+) \triangleq \lim_{z \downarrow d_+} G(z) \in (0, \infty].$$

We define $R(z)$ for real arguments $z \in (0, G(d_+))$, where G^{-1} is the functional inverse of G over the domain (d_+, ∞) .

Proposition 1.2 *Under Assumption 1.1, for some $\beta_0 = \beta_0(\mu_D) > 0$ and all $\beta \in (0, \beta_0)$, there is a unique solution $q_* \in [0, 1)$ to the fixed-point equation*

$$q_* = \mathbb{E}[\tanh(H + \sigma_* \mathbb{G})^2], \quad \sigma_*^2 = \beta^2 q_* R'(\beta(1 - q_*)) \quad (1.3)$$

where the expectation is over independent random variables $\mathbb{G} \sim \mathcal{N}(0, 1)$ and $H \sim \mu_H$.

¹ This moment condition for h is used to apply the AMP state evolution analysis of [21] to deduce Theorem 2.2, and is not used in the rest of the argument.

Let q_* , σ_*^2 and G , H be as above. Then the replica-symmetric prediction for the free energy Ψ is (see e.g. [41, Eq. (56)])

$$\begin{aligned} \Psi_{RS} = & \mathbb{E} \left[\log 2 \cosh(H + \sigma_* G) \right] + \frac{\beta q_*}{2} R(\beta(1 - q_*)) \\ & - \frac{\beta^2 q_*(1 - q_*)}{2} R'(\beta(1 - q_*)) + \frac{1}{2} \int_0^{1-q_*} \beta R(\beta z) dz. \end{aligned} \quad (1.4)$$

The correctness of this prediction for sufficiently high temperature is justified by the next theorem, which is the main result of the paper.

Theorem 1.3 *Suppose Assumption 1.1 holds. Then for some $\beta_0 = \beta_0(\mu_D) > 0$ depending only on μ_D , and for any fixed $\beta \in (0, \beta_0)$, almost surely*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z = \Psi_{RS}.$$

We mention that for the spherical counterpart of the Ising model (1.1), the free energy can be computed directly and also agrees with its replica-symmetric prediction. We carry out this computation in Appendix D by applying one of the technical results in Sect. 2.4.

1.2 Overview of the proof

We will adopt the conditional second moment method of [8], and show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}] \approx \Psi_{RS}, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z^2 \mid \mathcal{G}] \approx 2\Psi_{RS} \quad (1.5)$$

for an appropriately chosen sigma-field $\mathcal{G}$. Together with classical concentration-of-measure results for Haar measure over the orthogonal group, this will be enough to show Theorem 1.3.

We define $\mathcal{G} = \mathcal{G}_t$ as the sigma-field generated by a fixed number t of iterations of an AMP algorithm designed to solve the TAP mean-field equations described in [41, 45]—see (2.11) below. Such an algorithm was introduced for the S–K model in [7] and for applications in compressed sensing in [5, 17]. For the Ising model (1.1) with orthogonally invariant couplings, a general class of AMP-type procedures was described in [40], including a “single-step memory” algorithm for solving the TAP equations that reduces to the one of [7] in the S–K setting. Our analyses will rely on a rigorous characterization of the state evolution of such algorithms obtained in [21].

The specific algorithm we use to construct $\mathcal{G}_t$ is not of the single-step memory form of [7, 40], but rather an alternative “memory-free” form introduced by Çakmak and Oppor in [12], applying the general procedure in [40] with a resolvent of J instead of the couplings matrix J itself. This memory-free algorithm is related to a class of Vector/Orthogonal AMP algorithms developed for compressed sensing applications in [31, 47, 51], and may be derived also from the Expectation Propagation framework

of Minka [38]. For our purposes, use of this algorithm leads to two important simplifications: First, its state evolution has a simple description when J is non-Gaussian, whereas that of alternative iterative procedures may have a complicated dependence on the spectral free cumulants of J . Second, the analysis of [21] reveals that iterates of this algorithm have an asymptotic freeness property with respect to J , which we describe below in Proposition 2.4. Both simplifications are important in enabling our computations of the conditional moments in (1.5).

The details of our strategy for showing (1.5) are somewhat different from those presented in [8], and we proceed in two high-level steps: We first apply the AMP state evolution and a large deviations argument to give exact expressions for the large- n limits of the conditional moments (1.5) in terms of low-dimensional variational problems. This leverages and extends some results of Guionnet and Maïda [26] that relate exponential integrals over the orthogonal group to the R-transform of J . We then analyze these variational problems, proving upper and lower bounds for their values that are tight for Ψ_{RS} and $2\Psi_{\text{RS}}$ in the limit as the number of algorithm iterations $t \rightarrow \infty$. The assumption of small β (i.e. sufficiently high temperature) is used in a crucial way in the upper bounds, to show a global concavity property of these variational problems.

The remainder of the paper is organized as follows: In Sect. 2, we collect the general ingredients of the proof, including a more detailed description of the AMP algorithm and its state evolution, and the evaluations of the required exponential integrals over the orthogonal group. In Sects. 3 and 4, we analyze the conditional first moment $\mathbb{E}[Z \mid \mathcal{G}_t]$ and second moment $\mathbb{E}[Z^2 \mid \mathcal{G}_t]$ respectively, leading to the proof of Theorem 1.3 in Sect. 5.

Notation

$\mathbb{O}(n)$ and $\mathbb{SO}(n)$ are the orthogonal and special orthogonal groups of $n \times n$ matrices. $\text{Haar}(\cdot)$ denotes the Haar-measure on these groups.

$\|\cdot\|$ is the ℓ_2 -norm for vectors and $\ell_2 \rightarrow \ell_2$ operator norm for matrices; we may write the latter as $\|\cdot\|_{\text{op}}$ in situations where this is unclear. $\|\cdot\|_{\text{F}}$ is the Frobenius norm for matrices. We use the convention that for scalar values $x_1, \dots, x_k$, $(x_1, \dots, x_k) \in \mathbb{R}^k$ denotes the *column* vector containing these values. We write $\triangleq$ for a definition or assignment. We reserve the sans-serif font G, H, X, Y for scalar random variables.

2 Preliminaries

2.1 Centering and rescaling

Adding a multiple of the identity to J shifts the free energy Ψ and Ψ_{RS} by the same additive constant. Thus, we may assume without loss of generality that

$$\int x \mu_D(dx) = 0. \quad (2.1)$$

Since μ_D has positive variance by Assumption 1.1(b), we may also assume without loss of generality that

$$\int x^2 \mu_D(dx) = 1, \quad (2.2)$$

by rescaling $J = O^\top D O$ and incorporating this scaling into β .

For most of the proof, it will be notationally convenient to absorb the parameter β into the couplings matrix J , after this centering and rescaling. We define

$$\bar{J} = \beta J, \quad \bar{D} = \text{diag}(\bar{d}_1, \dots, \bar{d}_n) = \beta D, \quad \mu_{\bar{D}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{\bar{d}_i}, \quad \bar{d}_+ = \beta d_+. \quad (2.3)$$

Thus $\mu_{\bar{D}}$ is the rescaling of the limit spectral law μ_D , and $\bar{d}_+ = \max(x : x \in \text{supp}(\mu_{\bar{D}}))$ is its maximum point of support.

We denote the Cauchy- and R-transforms of $\mu_{\bar{D}}$ by

$$\bar{G}(z) = \int \frac{1}{z-x} \mu_{\bar{D}}(dx), \quad \bar{R}(z) = \bar{G}^{-1}(z) - \frac{1}{z}, \quad (2.4)$$

where $\bar{G}(z)$ is defined on $(\bar{d}_+, \infty)$, and $\bar{R}(z)$ on $(0, \bar{G}(\bar{d}_+))$. These are related to the Cauchy- and R-transforms of μ_D by

$$\bar{G}(z) = \frac{1}{\beta} G\left(\frac{z}{\beta}\right), \quad \bar{R}(z) = \beta R(\beta z). \quad (2.5)$$

Let $\{\kappa_k\}_{k \geq 1}$ be the free cumulants of the law μ_D . Since κ_1 and κ_2 correspond to the mean and variance of μ_D (cf. [39, Examples 11.6]), (2.1) and (2.2) imply that $\kappa_1 = 0$ and $\kappa_2 = 1$. Writing $\|\mu_D\|_\infty = \max(|x| : x \in \text{supp}(\mu_D))$, we have

$$|\kappa_k| \leq (16\|\mu_D\|_\infty)^k \quad (2.6)$$

for all $k \geq 1$, and the R-transform admits the convergent series expansion for small z given by

$$R(z) = \sum_{k \geq 1} \kappa_k z^{k-1}$$

(cf. [39, Notation 12.6, Proposition 13.15]). The free cumulants of $\mu_{\bar{D}}$ are then $\bar{\kappa}_k = \beta^k \kappa_k$, satisfying $\bar{\kappa}_1 = 0$, $\bar{\kappa}_2 = \beta^2$, and $|\bar{\kappa}_k| \leq (16\|\mu_D\|_\infty \beta)^k$ for $k \geq 3$. The R-

transform of $\mu_{\bar{D}}$ for small z is

$$\bar{R}(z) = \sum_{k \geq 1} \bar{k}_k z^{k-1}. \quad (2.7)$$

The Gibbs distribution and partition function in (1.1) may be written in this rescaled notation as

$$P(\sigma) = \frac{1}{Z} \exp\left(\frac{1}{2} \sigma^\top \bar{J} \sigma + h^\top \sigma\right), \quad Z = \sum_{\sigma \in \{+1, -1\}^n} \exp\left(\frac{1}{2} \sigma^\top \bar{J} \sigma + h^\top \sigma\right). \quad (2.8)$$

The fixed-point equation (1.3) for q_* is written in terms of $\bar{R}(z)$ as

$$q_* = \mathbb{E}[\tanh(H + \sigma_* G)^2], \quad \sigma_*^2 = q_* \bar{R}'(1 - q_*) \quad (2.9)$$

and the replica-symmetric free energy (1.4) is

$$\begin{aligned} \Psi_{\text{RS}} = & \mathbb{E}\left[\log 2 \cosh(H + \sigma_* G)\right] + \frac{q_*}{2} \bar{R}(1 - q_*) - \frac{q_*(1 - q_*)}{2} \bar{R}'(1 - q_*) \\ & + \frac{1}{2} \int_0^{1-q_*} \bar{R}(z) dz. \end{aligned} \quad (2.10)$$

2.2 AMP for solving the TAP equations

Denote by

$$m = \sum_{\sigma \in \{+1, -1\}^n} \sigma \cdot P(\sigma) \in (-1, 1)^n$$

the magnetization vector of the Gibbs distribution (2.8). It is predicted that for sufficiently small $\beta > 0$, this vector m approximately satisfies the TAP mean-field equations [41, 45]

$$m = \tanh\left(h + \bar{J}m - \bar{R}(1 - q_*)m\right). \quad (2.11)$$

Here and below, $\tanh(\cdot)$ is applied coordinatewise. For the S-K model where $\bar{J}$ is Gaussian, we have $\bar{R}(x) = \beta^2 x$, and this coincides with the classical TAP equations of [56]. We provide a brief review of the cavity-method derivation of these TAP equations from [41] in Appendix E.

Our proof of Theorem 1.3 will compute the first and second moments of the partition function Z conditioned on a sigma-field generated by an iterative AMP algorithm for solving the TAP equations. We consider the following algorithm from [12] having

“memory-free” dynamics: Define

$$\lambda_* = \bar{G}^{-1}(1 - q_*) = \bar{R}(1 - q_*) + \frac{1}{1 - q_*} \quad (2.12)$$

so that $\bar{G}(\lambda_*) = 1 - q_*$. This is well-defined for any $\beta \in (0, G(d_+))$, since $1 - q_* \leq 1 < \bar{G}(\bar{d}_+) = G(d_+)/\beta$. Consider the matrix

$$\Gamma = \frac{1}{1 - q_*}(\lambda_* I - \bar{J})^{-1} - I,$$

which admits the eigen-decomposition

$$\Gamma = O^\top \Lambda O, \quad \Lambda = \frac{1}{1 - q_*}(\lambda_* I - \bar{D})^{-1} - I. \quad (2.13)$$

In particular, Γ is also orthogonally invariant in law. Let $y^0 \in \mathbb{R}^n$ be an initialization of the AMP algorithm with entries

$$y_1^0, \dots, y_n^0 \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_*^2), \quad (2.14)$$

where σ_*^2 is defined in (2.9). Then the AMP algorithm is given by the iterations

$$x^t = \frac{1}{1 - q_*} \tanh(h + y^{t-1}) - y^{t-1}, \quad (2.15)$$

$$y^t = \Gamma x^t. \quad (2.16)$$

An approximate solution of the TAP equations (2.11) is obtained from the iterates of this algorithm as $m^t = (1 - q_*)(x^t + y^{t-1}) = \tanh(h + y^{t-1})$. For any fixed point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ of this algorithm, it is easily checked that $m = (1 - q_*)(x + y) = \tanh(h + y)$ exactly satisfies (2.11).

Applying the diagonalization $\Gamma = O^\top \Lambda O$ in (2.13), let us write the AMP iterations in an expanded form

$$x^t = \frac{1}{1 - q_*} \tanh(h + y^{t-1}) - y^{t-1}, \quad (2.17)$$

$$s^t = O x^t, \quad (2.18)$$

$$y^t = O^\top \Lambda s^t. \quad (2.19)$$

For each fixed $t \geq 1$, we define the sigma-field (in the probability space of O)

$$\mathcal{G}_t = \mathcal{G}(y^0, x^1, s^1, y^1, \dots, x^t, s^t, y^t) \quad (2.20)$$

generated by all iterates of (2.17–2.19) up to y^t . The proof of Theorem 1.3 will compute the first and second moments of Z conditioned on $\mathcal{G}_t$.

A key property of this algorithm is that the scalar function $f(h, y) = (1 - q_*)^{-1} \tanh(h + y) - y$ applied entrywise in (2.17) is divergence-free in y , in the sense

$$\mathbb{E}[\partial_y f(\mathbf{H}, \sigma_* \mathbf{G})] = 0 \quad (2.21)$$

for independent random variables $\mathbf{H} \sim \mu_H$ and $\mathbf{G} \sim \mathcal{N}(0, 1)$, which follows from the definition of q_* in (1.3). This substantially simplifies the state evolution that describes the AMP iterates x^t, y^t —discussed in the next section—when $\bar{J}$ is a non-Gaussian orthogonally invariant couplings matrix.

2.3 State evolution for AMP

The state evolution for general AMP algorithms of this form was described in [12, 21, 40]. We first review the specialization of these results to the specific algorithm (2.17–2.19). Proofs are deferred to Appendix A.

Define

$$\kappa_* = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Tr } \Gamma^2, \quad \delta_* = \sigma_*^2 / \kappa_*. \quad (2.22)$$

These quantities are given more explicitly as follows.

Proposition 2.1 *We have*

$$\kappa_* = \frac{1}{1 - (1 - q_*)^2 \bar{R}'(1 - q_*)} - 1 \quad (2.23)$$

and

$$\delta_* = \frac{q_*}{(1 - q_*)^2} - \sigma_*^2 = \mathbb{E} \left[\left(\frac{1}{1 - q_*} \tanh(\mathbf{H} + \sigma_* \mathbf{G}) - \sigma_* \mathbf{G} \right)^2 \right] \quad (2.24)$$

for independent random variables $\mathbf{H} \sim \mu_H$ and $\mathbf{G} \sim \mathcal{N}(0, 1)$.

Under Assumption 1.1, let $\mathbf{H} \sim \mu_H$ and $Y_0 \sim \mathcal{N}(0, \sigma_*^2)$ be independent of each other. Then, iteratively for each $s = 1, \dots, t$, set

$$\mathbf{X}_s = \frac{1}{1 - q_*} \tanh(\mathbf{H} + \mathbf{Y}_{s-1}) - \mathbf{Y}_{s-1}, \quad (2.25)$$

$$\Delta_s = \mathbb{E}[(\mathbf{X}_1, \dots, \mathbf{X}_s)(\mathbf{X}_1, \dots, \mathbf{X}_s)^\top], \quad (2.26)$$

and draw Y_s independently of $(\mathbf{H}, Y_0)$ so that $(Y_1, \dots, Y_s) \sim \mathcal{N}(0, \kappa_* \Delta_s)$. This defines a joint law for the variables $(\mathbf{H}, Y_0, Y_1, \dots, Y_t, \mathbf{X}_1, \dots, \mathbf{X}_t)$, for any $t \geq 1$.

Theorem 2.2 Fix any $t \geq 1$, and let $Y_t = (y^1, \dots, y^t) \in \mathbb{R}^{n \times t}$ and $X_t = (x^1, \dots, x^t) \in \mathbb{R}^{n \times t}$ collect the iterates of (2.17–2.19), starting from the initialization (2.14). Then, under Assumption 1.1, almost surely as $n \rightarrow \infty$, the empirical distribution of rows of (h, y^0, Y_t, X_t) satisfies the convergence

$$\frac{1}{n} \sum_{i=1}^n \delta_{(h_i, y_i^0, y_i^1, \dots, y_i^t, x_i^1, \dots, x_i^t)} \rightarrow (H, Y_0, Y_1, \dots, Y_t, X_1, \dots, X_t) \quad (2.27)$$

weakly and in p^{th} moment for each fixed order $p \geq 1$.

Furthermore, Δ_t is non-singular, and almost surely as $n \rightarrow \infty$,

$$n^{-1} X_t^\top X_t \rightarrow \mathbb{E}[(X_1, \dots, X_t)(X_1, \dots, X_t)^\top] = \Delta_t, \quad (2.28)$$

$$n^{-1} Y_t^\top Y_t \rightarrow \mathbb{E}[(Y_1, \dots, Y_t)(Y_1, \dots, Y_t)^\top] = \kappa_* \Delta_t, \quad (2.29)$$

$$n^{-1} X_t^\top Y_t \rightarrow \mathbb{E}[(X_1, \dots, X_t)(Y_1, \dots, Y_t)^\top] = 0. \quad (2.30)$$

By definition, the second-moment matrix Δ_t in Theorem 2.2 is the upper-left $t \times t$ submatrix of Δ_{t+1} . Thus it is unambiguous to write the entries of these matrices as

$$\Delta_t = (\delta_{ss'})_{1 \leq s, s' \leq t}.$$

For our purposes, we will require only the following property of the entries of Δ_t .

Proposition 2.3 In the setting of Theorem 2.2, for some $\beta_0 = \beta_0(\mu_D) > 0$ and all $\beta \in (0, \beta_0)$, we have

$$\delta_{tt} = \delta_* \text{ and } \kappa_* \delta_{tt} = \sigma_*^2 \text{ for all } t \geq 1, \quad \lim_{\min(s,t) \rightarrow \infty} \delta_{st} = \delta_*, \quad \lim_{\min(s,t) \rightarrow \infty} \kappa_* \delta_{st} = \sigma_*^2.$$

Thus the algorithm (2.17–2.19) is convergent for sufficiently small β ,² in the sense

$$\begin{aligned} \lim_{\min(s,t) \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \|x^t - x^s\|^2 \right) &= \lim_{\min(s,t) \rightarrow \infty} (\delta_{ss} + \delta_{tt} - 2\delta_{st}) = 0, \\ \lim_{\min(s,t) \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \|y^t - y^s\|^2 \right) &= \lim_{\min(s,t) \rightarrow \infty} \kappa_* (\delta_{ss} + \delta_{tt} - 2\delta_{st}) = 0. \end{aligned}$$

Defining $S_t = (s^1, \dots, s^t) = O X_t$, where the second equality holds by (2.18), the convergence (2.28) implies that

$$n^{-1} X_t^\top X_t = n^{-1} S_t^\top S_t \rightarrow \Delta_t.$$

A second important property of the memory-free dynamics (2.17–2.19) is the following more general statement.

² It is shown in [12] that this convergence in fact holds in the entirety of a high-temperature region defined by an Almeida-Thouless type condition for stability of the replica-symmetric phase, which depends on μ_D and μ_H .

Proposition 2.4 *In the setting of Theorem 2.2, fix any $t \geq 1$, and let $X_t = (x^1, \dots, x^t) \in \mathbb{R}^{n \times t}$ and $S_t = (s^1, \dots, s^t) \in \mathbb{R}^{n \times t}$ collect the iterates of (2.17–2.18). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function which is continuous and bounded in a neighborhood of $\text{supp}(\mu_{\bar{D}})$, and define $f(\bar{J})$ by the functional calculus. Then almost surely as $n \rightarrow \infty$,*

$$n^{-1} X_t^\top f(\bar{J}) X_t = n^{-1} S_t^\top f(\bar{D}) S_t \rightarrow \Delta_t \cdot \int f(x) \mu_{\bar{D}}(dx).$$

Informally, this states that for large n ,

$$n^{-1} X_t^\top f(\bar{J}) X_t \approx n^{-1} X_t^\top X_t \cdot n^{-1} \text{Tr} f(\bar{J}).$$

Thus, in a certain sense, the AMP iterates X_t are “free” of the couplings matrix $\bar{J}$, despite being dependent on $\bar{J}$. This result is a consequence of the divergence-free property (2.21), and it follows from the state evolution analysis in [21, Lemma A.4(b)]. We provide a proof in Appendix A.

Finally, we record here the leading-order behaviors of the above constants q_* , σ_*^2 , λ_* , κ_* , δ_* for small β .

Proposition 2.5 *Under Assumption 1.1, let $O(f(\beta, z))$ denote a quantity having magnitude at most $C \cdot f(\beta, z)$, for some constants C , $\beta_0 > 0$ depending only on μ_D and for all $\beta \in (0, \beta_0)$ and $z \in (0, 1)$. Then*

$$\bar{R}(z) = \beta^2 z (1 + O(\beta z)), \quad \bar{R}'(z) = \beta^2 (1 + O(\beta z)), \quad \bar{R}''(z) = O(\beta^3) \quad (2.31)$$

and

$$\begin{aligned} q_* &= \mathbb{E}[\tanh(H)^2] + O(\beta^2), \quad \sigma_*^2 = \beta^2 q_* + O(\beta^3), \\ \lambda_* &= \frac{1}{1 - q_*} + \beta^2 (1 - q_*) (1 + O(\beta(1 - q_*))) \\ \kappa_* &= \beta^2 (1 - q_*)^2 (1 + O(\beta(1 - q_*))), \quad \delta_* = \frac{q_*}{(1 - q_*)^2} + O(\beta^2). \end{aligned}$$

2.4 Conditioning and large deviations for Haar-orthogonal matrices

We collect here several results on the conditioning of Haar-orthogonal matrices, and large deviations for integrals over the orthogonal group.

Proposition 2.6 (Lemma 4 of [47]) *Let $A, B \in \mathbb{R}^{n \times k}$ be deterministic matrices of rank k , such that $A = QB$ for some orthogonal matrix $Q \in \mathbb{O}(n)$. Let $V_{A^\perp}, V_{B^\perp} \in \mathbb{R}^{n \times (n-k)}$ be matrices with orthonormal columns spanning the orthogonal complements of the column spans of A and B , respectively. Let $O \sim \text{Haar}(\mathbb{O}(n))$. Then the law of O conditioned on the event $A = OB$ is given by*

$$O|_{A=OB} \stackrel{L}{=} V_{A^\perp} \tilde{O} V_{B^\perp}^\top + A(A^\top A)^{-1} B^\top = V_{A^\perp} \tilde{O} V_{B^\perp}^\top + A(B^\top B)^{-1} B^\top,$$

where $\tilde{O} \sim \text{Haar}(\mathbb{O}(n-k))$.

Proposition 2.7 *Let $O \sim \text{Haar}(\mathbb{O}(n))$. Let $D \in \mathbb{R}^{n \times n}$ be a deterministic symmetric matrix whose eigenvalue distribution satisfies Assumption 1.1(b) as $n \rightarrow \infty$. Let μ_D be its limit eigenvalue distribution, let $d_+ = \max(x : x \in \text{supp}(\mu_D))$, and let $G(z)$ be the Cauchy transform of μ_D . Fix any constants $C, \varepsilon > 0$, and define the domain*

$$\Omega_n = \left\{ (a, b) \in \mathbb{R}^n \times \mathbb{R}^n : 0 < \frac{\|a\|^2}{n} \leq G(d_+ + \varepsilon) - \varepsilon, \frac{\|b\|^2}{n} \leq C \right\}.$$

Then

$$\lim_{n \rightarrow \infty} \sup_{(a,b) \in \Omega_n} \left| \frac{1}{n} \log \mathbb{E} \left[\exp \left(b^\top O a + \frac{a^\top O^\top D O a}{2} \right) \right] - \frac{1}{2} E_n(a, b) \right| = 0, \quad (2.32)$$

where

$$E_n(a, b) = \inf_{\gamma \geq d_+ + \varepsilon} \left\{ \frac{\gamma \|a\|^2}{n} + \frac{b^\top (\gamma I - D)^{-1} b}{n} - \frac{1}{n} \log \det(\gamma I - D) - \left(1 + \log \frac{\|a\|^2}{n} \right) \right\}. \quad (2.33)$$

Proposition 2.8 *Let O, D, μ_D, d_+ , and $G(z)$ be as in Proposition 2.7. Fix any constants $C, \varepsilon > 0$, and define the domains*

$$\mathcal{D}_\varepsilon = \left\{ (\gamma, v, \rho) \in \mathbb{R}^3 : \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \geq (d_+ + \varepsilon) I_{2 \times 2} \right\}, \quad (2.34)$$

$$\Omega_n = \left\{ (a, b, c, d) \in (\mathbb{R}^n)^4 : 0 < \frac{1}{n} \begin{pmatrix} \|a\|^2 & a^\top c \\ a^\top c & \|c\|^2 \end{pmatrix} \preceq \left(G(d_+ + \varepsilon) - \varepsilon \right) I_{2 \times 2}, \frac{\|b\|^2}{n}, \frac{\|d\|^2}{n} \leq C \right\}.$$

Then

$$\lim_{n \rightarrow \infty} \sup_{(a,b,c,d) \in \Omega_n} \left| \frac{1}{n} \log \mathbb{E} \left[\exp \left(b^\top O a + d^\top O c + \frac{a^\top O^\top D O a}{2} + \frac{c^\top O^\top D O c}{2} \right) \right] - \frac{1}{2} E_n(a, b, c, d) \right| = 0 \quad (2.35)$$

where

$$E_n(a, b, c, d) = \inf_{(\gamma, v, \rho) \in \mathcal{D}_\varepsilon} \left\{ \frac{1}{n} \operatorname{Tr} \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \begin{pmatrix} \|a\|^2 & a^\top c \\ a^\top c & \|c\|^2 \end{pmatrix} + \frac{1}{n} \begin{pmatrix} b \\ d \end{pmatrix}^\top \begin{pmatrix} \gamma I - D & vI \\ vI & \rho I - D \end{pmatrix}^{-1} \begin{pmatrix} b \\ d \end{pmatrix} \right. \\ \left. - \frac{1}{n} \log \det \begin{pmatrix} \gamma I - D & vI \\ vI & \rho I - D \end{pmatrix} - \left(2 + \log \det \frac{1}{n} \begin{pmatrix} \|a\|^2 & a^\top c \\ a^\top c & \|c\|^2 \end{pmatrix} \right) \right\}. \quad (2.36)$$

When $b = d = 0$, the expectations evaluated in Propositions 2.7 and 2.8 are finite-rank HCIZ integrals over the orthogonal group, and such results were obtained in [26, Theorems 2 and 7]. The above propositions extend these results to $b, d \neq 0$, and also establish the approximations in a more uniform sense. We note that the content of Proposition 2.7 for $b \neq 0$ is essentially the calculation of the limit free energy in the spherical analogue of the model (1.1) with external field, and we discuss this in Appendix D.

For $b = d = 0$, asymptotic versions of the infima in Propositions 2.7 and 2.8 may be explicitly evaluated, and we record these evaluations here.

Proposition 2.9 *Let μ_D be a compactly supported probability distribution on $\mathbb{R}$. Let $G(z)$ and $R(z)$ be the Cauchy- and R -transforms of μ_D , and let $d_+ = \max\{x : x \in \operatorname{supp}(\mu_D)\}$.*

(a) *Suppose that $\alpha \in (0, G(d_+))$. Then*

$$\inf_{\gamma > d_+} \gamma \alpha - \int \log(\gamma - x) \mu_D(dx) - (1 + \log \alpha) = \int_0^\alpha R(z) dz$$

and the infimum is achieved at $\gamma = G^{-1}(\alpha) = R(\alpha) + 1/\alpha$.

(b) *Suppose that $A \in \mathbb{R}^{2 \times 2}$ is symmetric and satisfies $0 \prec A \prec G(d_+)I$. Define $f(A) \in \mathbb{R}^{2 \times 2}$ for any function $f : (0, G(d_+)) \rightarrow \mathbb{R}$ by the functional calculus. Let*

$$\mathcal{D}_+ = \left\{ (\gamma, v, \rho) \in \mathbb{R}^3 : \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \succ d_+ I_{2 \times 2} \right\}.$$

Then

$$\inf_{(\gamma, v, \rho) \in \mathcal{D}_+} \operatorname{Tr} \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} A - \int \log \det \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix} \mu_D(dx) - (2 + \log \det A) \\ = \operatorname{Tr} f(A),$$

where $f(\alpha) = \int_0^\alpha R(z) dz$. The infimum is achieved at $\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} = G^{-1}(A) = R(A) + A^{-1}$.

We prove Propositions 2.7, 2.8, and 2.9 in Appendix B, building on the large-deviations arguments of [26].

3 Conditional first moment

Let Z be the partition function in (2.8), and let $\mathcal{G}_t$ be the sigma-field defined by (2.20). We show in this section the following result.

Lemma 3.1 *In the setting of Theorem 1.3,*

$$\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}_t] = \Psi_{\text{RS}},$$

where the inner limit as $n \rightarrow \infty$ exists almost surely for each fixed t .

3.1 Derivation of the variational formula

For scalar arguments $\gamma > \bar{d}_+$ and $u, U \in \mathbb{R}$, and vector arguments $v, w, V, W \in \mathbb{R}^t$ with $\|v\|^2 + \|w\|^2 < 1$, we define the function

$$\begin{aligned} \Phi_{1,t}(u, v, w; \gamma, U, V, W) &= \mathbb{E} \left[\log 2 \cosh \left(U \cdot H + V^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \right] \\ &\quad - u \cdot U - v^\top V - w^\top W + u + \bar{R}(1 - q_*) \kappa_*^{-1/2} v^\top w + \frac{\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}}{2} \|w\|^2 \\ &\quad + \frac{1}{2} \mathcal{F}(\gamma) \|v - \kappa_*^{-1/2} w\|^2 + \frac{1}{2} \mathcal{H}(\gamma, 1 - \|v\|^2 - \|w\|^2) \end{aligned} \quad (3.1)$$

and the variational formula

$$\Psi_{1,t} = \sup_{\substack{u \in \mathbb{R} \\ v, w \in \mathbb{R}^t: \|v\|^2 + \|w\|^2 < 1}} \inf_{\gamma > \bar{d}_+} \inf_{U \in \mathbb{R}, V, W \in \mathbb{R}^t} \Phi_{1,t}(u, v, w; \gamma, U, V, W). \quad (3.2)$$

Here, the random variables $(H, Y_1, \dots, Y_t, X_1, \dots, X_t)$ and the positive-definite matrix Δ_t are as described in Theorem 2.2, and the functions $\mathcal{F}$ and $\mathcal{H}$ are given by

$$\mathcal{F}(\gamma) \triangleq \mathcal{F}_{22}(\gamma) - \mathcal{F}_{12}(\gamma)^\top \mathcal{F}_{11}(\gamma)^{-1} \mathcal{F}_{12}(\gamma), \quad (3.3)$$

$$\mathcal{H}(\gamma, \alpha) \triangleq \gamma \alpha - \int \log(\gamma - x) \mu_{\bar{D}}(dx) - (1 + \log \alpha) \quad (3.4)$$

where we set

$$\lambda(x) \triangleq \frac{1}{(1 - q_*)(\lambda_* - x)} - 1, \quad \theta(x) \triangleq x + \frac{\bar{R}(1 - q_*)}{\kappa_*} \left(1 - \frac{1}{(1 - q_*)(\lambda_* - x)} \right), \quad (3.5)$$

and

$$\mathcal{F}_{11}(\gamma) \triangleq \int \frac{1}{\gamma - x} \begin{pmatrix} 1 & \lambda(x) \\ \lambda(x) & \lambda(x)^2 \end{pmatrix} \mu_{\bar{D}}(dx) \in \mathbb{R}^{2 \times 2} \quad (3.6)$$

$$\mathcal{F}_{12}(\gamma) \triangleq \int \frac{1}{\gamma - x} \begin{pmatrix} \theta(x) \\ \lambda(x)\theta(x) \end{pmatrix} \mu_{\bar{D}}(dx) \in \mathbb{R}^2 \quad (3.7)$$

$$\mathcal{F}_{22}(\gamma) \triangleq \int \frac{1}{\gamma - x} \theta(x)^2 \mu_{\bar{D}}(dx). \quad (3.8)$$

Note that under Assumption 1.1(b), $\mu_{\bar{D}}$ is supported on at least two points, and $\lambda_* > \bar{d}_+$ by definition so $x \mapsto \lambda(x)$ is one-to-one on $\text{supp}(\mu_{\bar{D}})$. As a result, $\mathcal{F}_{11}(\gamma)$ is strictly positive-definite and invertible for $\gamma > \bar{d}_+$ and thus $\mathcal{F}(\gamma)$ is well-defined.

Lemma 3.2 *In the setting of Theorem 1.3, for any fixed $t \geq 1$, almost surely*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}_t] = \Psi_{1,t}.$$

Proof Recall the $n \times t$ matrices $X_t = (x^1, \dots, x^t)$, $Y_t = (y^1, \dots, y^t)$, and $S_t = (s^1, \dots, s^t)$ which collect the AMP iterates. We fix t and write $\mathcal{G}, X, Y, S, \Delta$ for $\mathcal{G}_t, X_t, Y_t, S_t, \Delta_t$. From the definition of Z in (2.8),

$$\begin{aligned} \mathbb{E}[Z \mid \mathcal{G}] &= \sum_{\sigma \in \{+1, -1\}^n} \exp \left(h^\top \sigma + \frac{n}{2} \cdot f_n(\sigma) \right), \\ f_n(\sigma) &\triangleq \frac{2}{n} \log \mathbb{E} \left[\exp \left(\frac{1}{2} \sigma^\top O^\top \bar{D} O \sigma \right) \mid \mathcal{G} \right]. \end{aligned} \quad (3.9)$$

The function $f_n(\sigma)$ is well-defined for any $\sigma \in \mathbb{R}^n$. We first approximate $f_n(\sigma)$ over the sphere where $\|\sigma\|^2 = n$.

Conditional law of O . Theorem 2.2 guarantees that Δ is non-singular. The assumption of positive variance in (2.2) and the definitions of Γ and κ_* in (2.13) and (2.22) ensure that $\kappa_* > 0$. Then applying (2.28–2.30), almost surely for all large n , $n^{-1}(X, Y)^\top(X, Y) \in \mathbb{R}^{2t \times 2t}$ is also non-singular and $(X, Y) \in \mathbb{R}^{n \times 2t}$ has full column rank $2t$. Furthermore, we have the bounds

$$\limsup_{n \rightarrow \infty} n^{-1/2} \|X\| < \infty, \quad \limsup_{n \rightarrow \infty} n^{-1/2} \|Y\| < \infty, \quad \limsup_{n \rightarrow \infty} n^{-1/2} \|S\| < \infty, \quad (3.10)$$

which follow from $\|n^{-1}X^\top X\| = \|n^{-1}S^\top S\| \rightarrow \|\Delta\|$ and $\|n^{-1}Y^\top Y\| \rightarrow \|\kappa_* \Delta\|$.

Conditional on $\mathcal{G}$, the law of O is that of a Haar-orthogonal matrix conditioned on the event

$$(S, \Lambda S) = O(X, Y).$$

By Proposition 2.6, we may represent this conditional law of O as

$$O|_{\mathcal{G}} \stackrel{L}{=} V_{(S, \Lambda S)^\perp} \tilde{O} V_{(X, Y)^\perp}^\top + (S, \Lambda S) \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix}^{-1} (X, Y)^\top,$$

where $V_{(X,Y)^\perp}, V_{(S,\Lambda S)^\perp} \in \mathbb{R}^{n \times (n-2t)}$ have orthonormal columns orthogonal to the column spans of $(X, Y) \in \mathbb{R}^{n \times 2t}$ and $(S, \Lambda S) \in \mathbb{R}^{n \times 2t}$ respectively, and $\tilde{O} \sim \text{Haar}(\mathbb{O}(n - 2t))$ is an independent Haar-orthogonal matrix. Let us write as shorthand

$$V = V_{(S,\Lambda S)^\perp}.$$

For any vector $\sigma \in \mathbb{R}^n$, let us denote

$$\sigma_\perp = V_{(X,Y)^\perp}^\top \sigma \in \mathbb{R}^{n-2t}, \quad \sigma_\parallel = (S, \Lambda S) \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix}^{-1} (X, Y)^\top \sigma \in \mathbb{R}^n. \quad (3.11)$$

This yields the equality in conditional law $O\sigma|_{\mathcal{G}} \stackrel{L}{=} V\tilde{O}\sigma_\perp + \sigma_\parallel$, so (3.9) reduces to

$$f_n(\sigma) = \frac{1}{n} \sigma_\parallel^\top \bar{D} \sigma_\parallel + \frac{2}{n} \log \mathbb{E} \left[\exp \left(\frac{1}{2} \sigma_\perp^\top \tilde{O}^\top V^\top \bar{D} V \tilde{O} \sigma_\perp + \sigma_\parallel^\top \bar{D} V \tilde{O} \sigma_\perp \right) \right]. \quad (3.12)$$

Expectation over $\tilde{O}$. We first restrict to the domain

$$U_n = \{\sigma \in \mathbb{R}^n : \|\sigma\|^2 = n, \sigma_\perp \neq 0\}$$

and evaluate the expectation over $\tilde{O} \sim \text{Haar}(\mathbb{O}(n - 2t))$ using Proposition 2.7. Throughout the proof, we write $r_n(\sigma)$ to indicate any σ -dependent scalar, vector, or matrix remainder term with dimension independent of n , satisfying the uniform convergence almost surely

$$\lim_{n \rightarrow \infty} \sup_{\sigma \in U_n} \|r_n(\sigma)\| = 0, \quad (3.13)$$

and changing from instance to instance. We check the conditions of Proposition 2.7:

- The matrix $V = V_{(S,\Lambda S)^\perp}$ has $n - 2t$ orthonormal columns, where t is independent of n . Then by Assumption 1.1(b) and Weyl eigenvalue interlacing, as $n \rightarrow \infty$, the empirical eigenvalue distribution of $V^\top \bar{D} V$ has the same weak limit $\mu_{\bar{D}}$ as that of $\bar{D}$. Furthermore, from the conditions on $\max(d_1, \dots, d_n)$ and $\min(d_1, \dots, d_n)$ in Assumption 1.1(b), the largest eigenvalue of $V^\top \bar{D} V$ also converges to $\bar{d}_+$, and the smallest eigenvalue remains bounded away from $-\infty$.
- Take $a = \sigma_\perp$ in Proposition 2.7. Applying (2.5), we have $\bar{G}(\bar{d}_+) = \beta^{-1} G(d_+)$, where $G(d_+) \in (0, \infty]$ depends only on μ_D . Then for some $\beta_0 = \beta_0(\mu_D) > 0$, any $\beta \in (0, \beta_0)$, and any sufficiently small constant $\varepsilon > 0$, we have

$$\bar{G}(\bar{d}_+ + \varepsilon) - \varepsilon > 1 \quad (3.14)$$

so that $\|\sigma_\perp\|^2/n \leq \|\sigma\|^2/n = 1 < \bar{G}(\bar{d}_+ + \varepsilon) - \varepsilon$.

- Take $b = V^\top \bar{D}\sigma_\parallel$ in Proposition 2.7. Observe that $(S, \Lambda S) = O(X, Y)$, so $\sigma_\parallel = O\Pi_{X,Y}\sigma$ where $\Pi_{X,Y} = I - V_{(X,Y)^\perp} V_{(X,Y)^\perp}^\top \in \mathbb{R}^{n \times n}$ is the orthogonal projection onto the column span of (X, Y) . Then $\|V^\top \bar{D}\sigma_\parallel\|^2/n \leq \|\bar{D}\|^2 \cdot \|\sigma_\parallel\|^2/n \leq \|\bar{D}\|^2 \cdot \|\sigma\|^2/n = \|\bar{D}\|^2$.

Thus Proposition 2.7 (applied with dimension $n - 2t$) yields uniformly over $\sigma \in U_n$

$$f_n(\sigma) = \frac{1}{n} \sigma_\parallel^\top \bar{D}\sigma_\parallel + E_n(\sigma) + r_n(\sigma) \quad (3.15)$$

where

$$E_n(\sigma) = \inf_{\gamma \geq \bar{d}_+ + \varepsilon} \left\{ \frac{\gamma \|\sigma_\perp\|^2}{n} + \frac{\sigma_\parallel^\top \bar{D}V(\gamma I - V^\top \bar{D}V)^{-1} V^\top \bar{D}\sigma_\parallel}{n} - \frac{1}{n} \log \det(\gamma I - V^\top \bar{D}V) - \left(1 + \log \frac{\|\sigma_\perp\|^2}{n} \right) \right\}. \quad (3.16)$$

Approximation by v, w . For $\sigma \in U_n$, define the low-dimensional linear functionals

$$u(\sigma) = \frac{1}{n} h^\top \sigma, \quad \begin{pmatrix} v(\sigma) \\ w(\sigma) \end{pmatrix} = \left[\frac{1}{n} \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix} \right]^{-1/2} \cdot \frac{1}{n} (X, Y)^\top \sigma \quad (3.17)$$

where $u(\sigma) \in \mathbb{R}$ and $v(\sigma), w(\sigma) \in \mathbb{R}^t$. Note that

$$\|v(\sigma)\|^2 + \|w(\sigma)\|^2 = \frac{1}{n} \|\Pi_{(X,Y)} \sigma\|^2 = 1 - \frac{\|\sigma_\perp\|^2}{n} < 1. \quad (3.18)$$

Let us approximate the terms of (3.15) by functions of $v(\sigma)$ and $w(\sigma)$.

We begin with $\sigma_\parallel^\top \bar{D}\sigma_\parallel/n$: Applying (2.28–2.30) to (3.11),

$$\begin{aligned} \sigma_\parallel &= (S, \Lambda S) \left[\frac{1}{n} \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix} \right]^{-1/2} \begin{pmatrix} v(\sigma) \\ w(\sigma) \end{pmatrix} \\ &= S \cdot \Delta^{-1/2} v(\sigma) + \Lambda S \cdot (\kappa_* \Delta)^{-1/2} w(\sigma) + (S, \Lambda S) \cdot r_n(\sigma). \end{aligned} \quad (3.19)$$

From the definition of λ_* in (2.12) and the definition of the Cauchy-transform in (2.4), as $n \rightarrow \infty$,

$$\begin{aligned} n^{-1} \operatorname{Tr} \bar{D}(\lambda_* I - \bar{D})^{-1} &= n^{-1} \operatorname{Tr} \left[\lambda_*(\lambda_* I - \bar{D})^{-1} - I \right] \rightarrow \lambda_* \bar{G}(\lambda_*) - 1 \\ &= \lambda_*(1 - q_*) - 1. \end{aligned} \quad (3.20)$$

Differentiating the R-transform in (2.4),

$$\bar{R}'(z) = \frac{1}{\bar{G}'(\bar{G}^{-1}(z))} + \frac{1}{z^2}.$$

Then applying the form of κ_* in (2.23), also

$$\begin{aligned} n^{-1} \operatorname{Tr} \bar{D}(\lambda_* I - \bar{D})^{-2} &= n^{-1} \operatorname{Tr} \left[\lambda_* (\lambda_* I - \bar{D})^{-2} - (\lambda_* I - \bar{D})^{-1} \right] \\ &\rightarrow -\lambda_* \bar{G}'(\lambda_*) - \bar{G}(\lambda_*) = \lambda_* (\kappa_* + 1)(1 - q_*)^2 - (1 - q_*). \end{aligned} \quad (3.21)$$

Let us write as a shorthand

$$a_* \triangleq \bar{R}(1 - q_*) = \lambda_* - \frac{1}{1 - q_*}. \quad (3.22)$$

Then in view of the definition of Γ in (2.13), applying (2.1), (3.20), and (3.21) yields

$$n^{-1} \operatorname{Tr} \bar{D} \rightarrow 0, \quad n^{-1} \operatorname{Tr} \bar{D} \Lambda \rightarrow a_*, \quad n^{-1} \operatorname{Tr} \bar{D} \Lambda^2 \rightarrow \lambda_* \kappa_* - a_*.$$

So Proposition 2.4 yields almost surely

$$\frac{1}{n} (S, \Lambda S)^\top \bar{D} (S, \Lambda S) \rightarrow \begin{pmatrix} 0 & a_* \Delta \\ a_* \Delta & (\lambda_* \kappa_* - a_*) \Delta \end{pmatrix}. \quad (3.23)$$

Combining this with (3.19), we obtain for the first term of (3.15) that

$$\frac{\sigma_\parallel^\top \bar{D} \sigma_\parallel}{n} = \frac{2a_*}{\kappa_*^{1/2}} v(\sigma)^\top w(\sigma) + \left(\lambda_* - \frac{a_*}{\kappa_*} \right) \|w(\sigma)\|^2 + r_n(\sigma). \quad (3.24)$$

Next, we approximate $E_n(\sigma)$ in (3.16) by approximating each term inside the infimum uniformly over $\gamma \geq \bar{d}_+ + \varepsilon$ and $\sigma \in U_n$. Note that for all large n , all eigenvalues of $V^\top \bar{D} V$ are contained in a compact interval $\mathcal{K} \subset (-\infty, \bar{d}_+ + \varepsilon/2)$ that is disjoint from $[\bar{d}_+ + \varepsilon, \infty)$. Fixing $\gamma \geq \bar{d}_+ + \varepsilon$, the function $x \mapsto \log(\gamma - x)$ is bounded and continuous on $\mathcal{K}$, so by weak convergence in Assumption 1.1(b),

$$\frac{1}{n} \log \det(\gamma I - V^\top \bar{D} V) = \int \log(\gamma - x) \mu_{\bar{D}}(dx) + r_n(\gamma)$$

where $r_n(\gamma) \rightarrow 0$ as $n \rightarrow \infty$. The function $\gamma \mapsto n^{-1} \log \det(\gamma I - V^\top \bar{D} V)$ on the left side is uniformly Lipschitz over $\gamma \geq \bar{d}_+ + \varepsilon$ for all large n , so by Arzelà-Ascoli, in fact $r_n(\gamma) \rightarrow 0$ uniformly in γ over any compact subset $\mathcal{K}' \subset [\bar{d}_+ + \varepsilon, \infty)$. For any $\delta > 0$, we may take a sufficiently large such compact subset $\mathcal{K}'_\delta$ and bound

$$\left| \log(\gamma - x) - \log \gamma \right| \leq |x| \cdot \frac{1}{|\gamma| - |x|} < \delta \text{ for all } x \in \mathcal{K}, \gamma \in [\bar{d}_+ + \varepsilon, \infty) \setminus \mathcal{K}'_\delta.$$

Then also $|r_n(\gamma)| < 2\delta$ for all $\gamma \in [\bar{d}_+ + \varepsilon, \infty) \setminus \mathcal{K}'_\delta$, implying that the convergence $r_n(\gamma) \rightarrow 0$ is uniform over all $\gamma \geq \bar{d}_+ + \varepsilon$. Then, applying also $\|\sigma_\perp\|^2/n = 1 -$

$\|v(\sigma)\|^2 - \|w(\sigma)\|^2$ from (3.18) and recalling the function $\mathcal{H}$ defined in (3.4), we obtain

$$\begin{aligned} & \frac{\gamma \|\sigma_{\perp}\|^2}{n} - \frac{1}{n} \log \det(\gamma I - V^{\top} \bar{D} V) - \left(1 + \log \frac{\|\sigma_{\perp}\|^2}{n}\right) \\ &= \mathcal{H}(\gamma, 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2) + r_n(\gamma). \end{aligned} \quad (3.25)$$

To analyze the remaining second term of $E_n(\sigma)$ in (3.16), let us introduce

$$W = (S, \Lambda S) \begin{pmatrix} S^{\top} S & S^{\top} \Lambda S \\ S^{\top} \Lambda S & S^{\top} \Lambda^2 S \end{pmatrix}^{-1/2} = (S, \Lambda S) \begin{pmatrix} X^{\top} X & X^{\top} Y \\ Y^{\top} X & Y^{\top} Y \end{pmatrix}^{-1/2} \in \mathbb{R}^{n \times 2t} \quad (3.26)$$

whose columns are the orthogonalization of $(S, \Lambda S)$. Then the columns of (V, W) form a full orthonormal basis for $\mathbb{R}^n$. We write $\Pi = V V^{\top} = I - W W^{\top}$ as the projection orthogonal to $(S, \Lambda S)$. Applying (3.19), (3.23) and (2.28–2.30), observe that

$$\begin{aligned} & \begin{pmatrix} X^{\top} X & X^{\top} Y \\ Y^{\top} X & Y^{\top} Y \end{pmatrix}^{-1} (S, \Lambda S)^{\top} \bar{D} \sigma_{\parallel} \\ &= \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & (\kappa_* \Delta)^{-1} \end{pmatrix} \begin{pmatrix} 0 & a_* \Delta \\ a_* \Delta & (\lambda_* \kappa_* - a_*) \Delta \end{pmatrix} \begin{pmatrix} \Delta^{-1/2} v(\sigma) \\ (\kappa_* \Delta)^{-1/2} w(\sigma) \end{pmatrix} + r_n(\sigma) \\ &= \begin{pmatrix} 0 & a_* I \\ a_* \kappa_*^{-1} I & (\lambda_* - a_* \kappa_*^{-1}) I \end{pmatrix} \begin{pmatrix} \Delta^{-1/2} v(\sigma) \\ (\kappa_* \Delta)^{-1/2} w(\sigma) \end{pmatrix} + r_n(\sigma). \end{aligned}$$

Then applying

$$\Pi = I - W W^{\top} = I - (S, \Lambda S) \begin{pmatrix} X^{\top} X & X^{\top} Y \\ Y^{\top} X & Y^{\top} Y \end{pmatrix}^{-1} (S, \Lambda S)^{\top},$$

we obtain

$$\begin{aligned} \Pi \bar{D} \sigma_{\parallel} &= (\bar{D} S - a_* \kappa_*^{-1} \Lambda S) \cdot \Delta^{-1/2} v(\sigma) \\ &\quad + (\bar{D} \Lambda S - a_* S - (\lambda_* - a_* \kappa_*^{-1}) \Lambda S) \cdot (\kappa_* \Delta)^{-1/2} w(\sigma) \\ &\quad + (\bar{D} S, \bar{D} \Lambda S, S, \Lambda S) r_n(\sigma). \end{aligned}$$

Substituting

$$\Lambda = \frac{1}{1 - q_*} (\lambda_* I - \bar{D})^{-1} - I, \quad \bar{D} \Lambda = \frac{1}{1 - q_*} (\lambda_* (\lambda_* I - \bar{D})^{-1} - I) - \bar{D}$$

and applying the identity (3.22) and some algebraic simplification,

$$\Pi \bar{D} \sigma_{\parallel} = \bar{D} S \cdot (\Delta^{-1/2} v(\sigma) - (\kappa_* \Delta)^{-1/2} w(\sigma)) + (\bar{D} S, \bar{D} \Lambda S, S, \Lambda S) r_n(\sigma)$$

(3.27)

where $\tilde{D}$ is the diagonal matrix

$$\tilde{D} \triangleq \bar{D} + \frac{a_*}{\kappa_*} \left(I - \frac{1}{1 - q_*} (\lambda_* I - \bar{D})^{-1} \right). \quad (3.28)$$

Now let us apply $(V, W)^\top (V, W) = I$ to write

$$\begin{aligned} \begin{pmatrix} \gamma I - V^\top \bar{D} V & -V^\top \bar{D} W \\ -W^\top \bar{D} V & \gamma I - W^\top \bar{D} W \end{pmatrix} &= \begin{pmatrix} V^\top \\ W^\top \end{pmatrix} (\gamma I - \bar{D}) \begin{pmatrix} V & W \end{pmatrix} \\ &= \left[\begin{pmatrix} V^\top \\ W^\top \end{pmatrix} (\gamma I - \bar{D})^{-1} \begin{pmatrix} V & W \end{pmatrix} \right]^{-1} \\ &= \begin{pmatrix} V^\top (\gamma I - \bar{D})^{-1} V & V^\top (\gamma I - \bar{D})^{-1} W \\ W^\top (\gamma I - \bar{D})^{-1} V & W^\top (\gamma I - \bar{D})^{-1} W \end{pmatrix}^{-1}. \end{aligned}$$

Equating the upper-left blocks and applying the Schur-complement formula to the right side yields

$$\begin{aligned} \gamma I - V^\top \bar{D} V &= \left[V^\top (\gamma I - \bar{D})^{-1} V - V^\top (\gamma I - \bar{D})^{-1} W (W^\top (\gamma I - \bar{D})^{-1} W)^{-1} W^\top (\gamma I - \bar{D})^{-1} V \right]^{-1}. \end{aligned}$$

Thus, recalling $\Pi = V V^\top$, the second term of (3.16) is

$$\begin{aligned} &\frac{1}{n} \sigma_\parallel^\top \bar{D} V (\gamma I - V^\top \bar{D} V)^{-1} V^\top \bar{D} \sigma_\parallel \\ &= \frac{1}{n} \sigma_\parallel^\top \bar{D} \Pi (\gamma I - \bar{D})^{-1} \Pi \bar{D} \sigma_\parallel \\ &\quad - \frac{1}{n} \sigma_\parallel^\top \bar{D} \Pi (\gamma I - \bar{D})^{-1} W (W^\top (\gamma I - \bar{D})^{-1} W)^{-1} W^\top (\gamma I - \bar{D})^{-1} \Pi \bar{D} \sigma_\parallel. \end{aligned} \quad (3.29)$$

We apply (3.27) and Proposition 2.4 to approximate these two terms: By Proposition 2.4, we have almost surely

$$\frac{1}{n} S^\top \tilde{D} (\gamma I - \bar{D})^{-1} \tilde{D} S \rightarrow \mathcal{F}_{22}(\gamma) \cdot \Delta$$

for each fixed $\gamma \geq \bar{d}_+ + \varepsilon$, where $\mathcal{F}_{22}(\gamma)$ is as defined in (3.8) and $\tilde{D}$ in (3.28). Applying (3.10), the left side is a $t \times t$ matrix that is entrywise uniformly Lipschitz as a function of $\gamma \geq \bar{d}_+ + \varepsilon$ for all large n . So this convergence is again uniform in γ over any compact subset $\mathcal{K}' \subset [\bar{d}_+ + \varepsilon, \infty)$ by Arzelà-Ascoli. For any $\delta > 0$, we may take a sufficiently large such subset $\mathcal{K}'_\delta$ so that the left side is entrywise bounded

by δ for all γ outside $\mathcal{K}'_\delta$. In all, we conclude the above convergence is uniform over all $\gamma \geq \bar{d}_+ + \varepsilon$. Since

$$\begin{aligned} & \frac{1}{n} S^\top \tilde{D}(\gamma I - \bar{D})^{-1} (\bar{D}S, \bar{D}\Lambda S, S, \Lambda S), \\ & \frac{1}{n} (\bar{D}S, \bar{D}\Lambda S, S, \Lambda S)^\top (\gamma I - \bar{D})^{-1} (\bar{D}S, \bar{D}\Lambda S, S, \Lambda S) \end{aligned}$$

are also uniformly bounded over $\gamma \geq \bar{d}_+ + \varepsilon$ for all large n , this combined with (3.27) shows for the first term of (3.29) that

$$\frac{1}{n} \sigma_\parallel^\top \bar{D} \Pi (\gamma I - \bar{D})^{-1} \Pi \bar{D} \sigma_\parallel = \mathcal{F}_{22}(\gamma) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 + r_n(\sigma, \gamma) \quad (3.30)$$

where $r_n(\sigma, \gamma) \rightarrow 0$ uniformly over $\gamma \geq \bar{d}_+ + \varepsilon$ and $\sigma \in U_n$ as $n \rightarrow \infty$.

For the second term of (3.29), recalling $\Lambda = \frac{1}{1-q_*}(\lambda_* I - \bar{D})^{-1} - I$ from (2.13) and again applying Proposition 2.4, we have

$$\frac{1}{n} (S, \Lambda S)^\top (\gamma I - \bar{D})^{-1} (S, \Lambda S) \rightarrow \mathcal{F}_{11}(\gamma) \otimes \Delta \in \mathbb{R}^{2l \times 2l}$$

where

$$\mathcal{F}_{11}(\gamma) = \lim_{n \rightarrow \infty} \begin{pmatrix} \frac{1}{n} \text{Tr}(\gamma I - \bar{D})^{-1} & \frac{1}{n} \text{Tr}(\gamma I - \bar{D})^{-1} \Lambda \\ \frac{1}{n} \text{Tr} \Lambda (\gamma I - \bar{D})^{-1} & \frac{1}{n} \text{Tr} \Lambda (\gamma I - \bar{D})^{-1} \Lambda \end{pmatrix},$$

and this coincides with the matrix defined in (3.6). Then, recalling the form of W from (3.26),

$$\begin{aligned} W^\top (\gamma I - \bar{D})^{-1} W & \rightarrow \begin{pmatrix} \Delta & 0 \\ 0 & \kappa_* \Delta \end{pmatrix}^{-1/2} [\mathcal{F}_{11}(\gamma) \otimes \Delta] \begin{pmatrix} \Delta & 0 \\ 0 & \kappa_* \Delta \end{pmatrix}^{-1/2} \\ & = \left[\begin{pmatrix} 1 & 0 \\ 0 & \kappa_*^{-1/2} \end{pmatrix} \mathcal{F}_{11}(\gamma) \begin{pmatrix} 1 & 0 \\ 0 & \kappa_*^{-1/2} \end{pmatrix} \right] \otimes I. \end{aligned}$$

Similarly, for $\mathcal{F}_{12}(\gamma)$ as defined in (3.7),

$$\begin{aligned} \frac{1}{\sqrt{n}} W^\top (\gamma I - \bar{D})^{-1} \tilde{D} S & \rightarrow \begin{pmatrix} \Delta & 0 \\ 0 & \kappa_* \Delta \end{pmatrix}^{-1/2} [\mathcal{F}_{12}(\gamma) \otimes \Delta] \\ & = \left[\begin{pmatrix} 1 & 0 \\ 0 & \kappa_*^{-1/2} \end{pmatrix} \mathcal{F}_{12}(\gamma) \right] \otimes \Delta^{1/2}. \end{aligned}$$

Thus

$$\frac{1}{n} S^\top \tilde{D} (\gamma I - \bar{D})^{-1} W (W^\top (\gamma I - \bar{D})^{-1} W)^{-1} W^\top (\gamma I - \bar{D})^{-1} \tilde{D} S$$

$$\rightarrow \mathcal{F}_{12}(\gamma)^\top \mathcal{F}_{11}(\gamma)^{-1} \mathcal{F}_{12}(\gamma) \cdot \Delta.$$

Applying the bounds $\|(W^\top(\gamma I - \bar{D})^{-1}W)^{-1}\| \leq \gamma - \bar{d}_-$ and $\|W^\top(\gamma I - \bar{D})^{-1}\| \leq \frac{1}{\gamma - \bar{d}_+}$, we may check that the left side is again uniformly Lipschitz over $\gamma \geq \bar{d}_+ + \varepsilon$ and, for any $\delta > 0$, is bounded in magnitude by δ when γ lies outside a compact subset $\mathcal{K}'_\delta \subset [\bar{d}_+ + \varepsilon, \infty)$. Thus this convergence is again uniform over $\gamma \geq \bar{d}_+ + \varepsilon$. Then, combining with (3.27) and applying the same argument as leading to (3.30), we have for the second term of (3.29) that

$$\begin{aligned} & \frac{1}{n} \sigma_\parallel^\top \bar{D} \Pi (\gamma I - \bar{D})^{-1} W (W^\top (\gamma I - \bar{D})^{-1} W)^{-1} W^\top (\gamma I - \bar{D})^{-1} \Pi \bar{D} \sigma_\parallel \\ &= \mathcal{F}_{12}(\gamma)^\top \mathcal{F}_{11}(\gamma)^{-1} \mathcal{F}_{12}(\gamma) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 + r_n(\sigma, \gamma) \end{aligned}$$

where $r_n(\sigma, \gamma) \rightarrow 0$ uniformly over $\gamma \geq \bar{d}_+ + \varepsilon$ and $\sigma \in U_n$. Defining $\mathcal{F} = \mathcal{F}_{22} - \mathcal{F}_{12}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12}$ as in (3.3), this shows that almost surely as $n \rightarrow \infty$, the second term of (3.16) satisfies

$$\frac{1}{n} \sigma_\parallel^\top \bar{D} V (\gamma I - V^\top \bar{D} V)^{-1} V^\top \bar{D} \sigma_\parallel = \mathcal{F}(\gamma) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 + r_n(\sigma, \gamma). \quad (3.31)$$

Observe that this also implies

$$\mathcal{F}(\gamma) \text{ is non-increasing and convex over } \gamma > \bar{d}_+. \quad (3.32)$$

Indeed, fixing any $\gamma > \bar{d}_+$, let us take ε above small enough such that $\gamma \geq \bar{d}_+ + \varepsilon$. For each n , let us take $\sigma \in U_n$ such that $\|v(\sigma)\|^2 \rightarrow 1$ and $\|w(\sigma)\|^2 \rightarrow 0$ as $n \rightarrow \infty$. (For example, we may choose $\sigma = \sqrt{n}(x + \delta_n r)/\|x + \delta_n r\|$ where x is the first column of X , r is a unit vector orthogonal to the column span of (X, Y) , and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.) Then as $n \rightarrow \infty$, the right side of (3.31) converges to $\mathcal{F}(\gamma)$. The left side is non-increasing and convex at γ for each finite n , so the same properties hold for the limit $\mathcal{F}(\gamma)$.

Combining (3.24), (3.25), and (3.31) and applying this to (3.15), we obtain the approximation for $\sigma \in U_n$

$$\begin{aligned} f_n(\sigma) = & \inf_{\gamma \geq \bar{d}_+ + \varepsilon} \left(\frac{2a_*}{\kappa_*^{1/2}} v(\sigma)^\top w(\sigma) + \left(\lambda_* - \frac{a_*}{\kappa_*} \right) \|w(\sigma)\|^2 \right. \\ & + \mathcal{F}(\gamma) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 \\ & \left. + \mathcal{H}(\gamma, 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2) \right) + r_n(\sigma), \end{aligned}$$

where $r_n(\sigma) \rightarrow 0$ uniformly over $\sigma \in U_n$. Observe that for any fixed $\sigma \in U_n$, we have $\|v(\sigma)\|^2 + \|w(\sigma)\|^2 < 1$ strictly, so the argument to this infimum is a well-defined

and convex function of $\gamma \in (\bar{d}_+, \infty)$. Its derivative in γ is

$$\mathcal{F}'(\gamma) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 + 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2 - \bar{G}(\gamma).$$

For any $\gamma \in (\bar{d}_+, \bar{d}_+ + \varepsilon]$, $\mathcal{F}'(\gamma) \leq 0$ as shown in (3.32), and $1 < \bar{G}(\bar{d}_+ + \varepsilon) - \varepsilon$ as previously argued in (3.14), so $\bar{G}(\gamma) > 1 + \varepsilon$. Thus this derivative is negative for $\gamma \in (\bar{d}_+, \bar{d}_+ + \varepsilon]$, so it is equivalent to write this infimum over the range $\gamma > \bar{d}_+$, i.e.

$$f_n(\sigma) = f(v(\sigma), w(\sigma)) + r_n(\sigma) \quad (3.33)$$

where the function f on the domain $\mathcal{V} \triangleq \{(v, w) : \|v\|^2 + \|w\|^2 < 1\}$ is defined by

$$\begin{aligned} f(v, w) &\triangleq \inf_{\gamma > \bar{d}_+} \frac{2a_*}{\kappa_*^{1/2}} v^\top w \\ &+ \left(\lambda_* - \frac{a_*}{\kappa_*} \right) \|w\|^2 + \mathcal{F}(\gamma) \cdot \|v - \kappa_*^{-1/2} w\|^2 + \mathcal{H}(\gamma, 1 - \|v\|^2 - \|w\|^2). \end{aligned} \quad (3.34)$$

Finally, observe that $f_n(\sigma)$ is continuous on the sphere $\{\sigma \in \mathbb{R}^n : \|\sigma\|^2 = n\}$, and the function $\sigma \mapsto (v(\sigma), w(\sigma))$ is continuous, relatively open, and maps the dense subset U_n of this sphere to $\mathcal{V}$ for every n . By Proposition C.1 in Appendix C, $f(v, w)$ admits a continuous extension³ to the closure $\bar{\mathcal{V}} = \{(v, w) : \|v\|^2 + \|w\|^2 \leq 1\}$, and (3.33) holds uniformly over all σ on this sphere. Thus, we have shown the almost sure uniform convergence

$$\lim_{n \rightarrow \infty} \sup_{\sigma \in \mathbb{R}^n : \|\sigma\|^2 = n} |f_n(\sigma) - f(v(\sigma), w(\sigma))| = 0. \quad (3.35)$$

Large deviations analysis. We conclude the proof by applying Varadhan's Lemma and the Gärtner-Ellis Theorem: Consider now the discrete uniform law $\sigma \sim \text{Unif}(\{+1, -1\}^n)$ and write $\langle \cdot \rangle$ for the expectation over this law. For arguments $U \in \mathbb{R}$ and $V, W \in \mathbb{R}^t$, define the limiting cumulant generating function

$$\begin{aligned} \lambda(U, V, W) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left[n(U \cdot u(\sigma) + V^\top v(\sigma) + W^\top w(\sigma)) \right] \right\rangle \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left[U \cdot h^\top \sigma + (V^\top, W^\top) \left[\frac{1}{n} \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix} \right]^{-1/2} \begin{pmatrix} X^\top \sigma \\ Y^\top \sigma \end{pmatrix} \right] \right\rangle \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left[U \cdot h^\top \sigma + V^\top \Delta^{-1/2} X^\top \sigma + W^\top (\kappa_* \Delta)^{-1/2} Y^\top \sigma + n \cdot r_n(\sigma) \right] \right\rangle. \end{aligned}$$

³ Here, it is not hard to show that this extension to $\|v\|^2 + \|w\|^2 = 1$ is given explicitly by $f(v, w) = \frac{2a_*}{\kappa_*^{1/2}} v^\top w + (\lambda_* - \frac{a_*}{\kappa_*}) \|w\|^2$, but this explicit form is not needed for proving the end result in (3.2).

Here $r_n(\sigma)$ is a remainder term satisfying $r_n(\sigma) \rightarrow 0$ uniformly over $\sigma \in \{+1, -1\}^n$ for any fixed arguments U, V, W , and hence is negligible in the large- n limit. Evaluating the average over σ using $\langle e^{a\sigma_i} \rangle = \cosh a$, and writing $h_i \in \mathbb{R}$ and $x_i, y_i \in \mathbb{R}^t$ for the entries of h and rows of X, Y , we obtain

$$\lambda(U, V, W) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \cosh \left(U \cdot h_i + V^\top \Delta^{-1/2} x_i + \kappa_*^{-1/2} W^\top \Delta^{-1/2} y_i \right)$$

Then the weak convergence in law (2.27) from the AMP state evolution of Theorem 2.2 shows that this limit indeed exists almost surely, and is given by

$$\begin{aligned} \lambda(U, V, W) \\ = \mathbb{E} \left[\log \cosh \left(U \cdot H + V^\top \Delta^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W^\top \Delta^{-1/2} (Y_1, \dots, Y_t) \right) \right]. \end{aligned}$$

Note that the function $\lambda(U, V, W)$ is finite and differentiable at all $(U, V, W) \in \mathbb{R}^{2t+1}$. Then, denoting by

$$\lambda^*(u, v, w) = \sup_{U \in \mathbb{R}, V, W \in \mathbb{R}^t} U \cdot u + V^\top v + W^\top w - \lambda(U, V, W) \quad (3.36)$$

its Fenchel-Legendre dual, the Gärtner-Ellis Theorem shows that $(u(\sigma), v(\sigma), w(\sigma))$ satisfies a large deviations principle with good rate function $\lambda^*(u, v, w)$ [15, Theorem 2.3.6].

The function $(u, v, w) \mapsto u + f(v, w)/2$ is continuous over $\{u \in \mathbb{R}, v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 \leq 1\}$. Here $f(v, w)$ must be bounded over the compact set $\{v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 \leq 1\}$, and for any $c > 0$ we have the exponential integrability

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \langle e^{cnu(\sigma)} \rangle &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \langle e^{c \cdot h^\top \sigma} \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \cosh(ch_i) \\ &= \mathbb{E}[\log \cosh(cH)] < \infty. \end{aligned}$$

Then by (3.9), (3.35), and Varadhan's lemma [15, Theorem 4.3.1],

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}] &= \log 2 + \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left(n \cdot \left[u(\sigma) + \frac{1}{2} f(v(\sigma), w(\sigma)) \right] \right) \right\rangle \\ &= \sup_{\substack{u \in \mathbb{R} \\ v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 \leq 1}} \log 2 + u + \frac{f(v, w)}{2} - \lambda^*(u, v, w). \end{aligned}$$

The domain $\|v\|^2 + \|w\|^2 \leq 1$ in this supremum may now be restricted to $\|v\|^2 + \|w\|^2 < 1$, by continuity of $f(v, w)$ and lower-semicontinuity of the rate function $\lambda^*(u, v, w)$. Substituting the forms of f and λ^* from (3.34) and (3.36) concludes the proof. $\square$

3.2 Analysis of the variational formula

Denote by $\partial_u \Phi_{1,t} \in \mathbb{R}$, $\partial_v \Phi_{1,t} \in \mathbb{R}^t$, etc. the partial derivatives of the function $\Phi_{1,t}$ in each argument. We now consider an approximate stationary point of (3.2), given by

$$\begin{aligned} u_* &= \mathbb{E}[H \cdot \tanh(H + \sigma_* G)], \quad v_* = (1 - q_*) \Delta_t^{1/2} e_t, \\ w_* &= \kappa_*^{1/2} (1 - q_*) \Delta_t^{1/2} e_t \\ \gamma_* &= \bar{G}^{-1} (1 - q_*) = \bar{R} (1 - q_*) + (1 - q_*)^{-1}, \quad U_* = 1, \quad V_* = 0, \\ W_* &= \kappa_*^{1/2} \Delta_t^{1/2} e_t \end{aligned}$$

where $e_t = (0, \dots, 0, 1)$ is the t^{th} standard basis vector in $\mathbb{R}^t$. We check in two steps that first, this is approximately stationary for the optimization in (3.2) and yields the desired value Ψ_{RS} in (2.10), and second, that it is approximately the global solution to (3.2) when $\beta > 0$ is sufficiently small.

For these steps, we require the following properties of $\mathcal{F}(\gamma)$ defined in (3.3).

Lemma 3.3 (a) $\mathcal{F}(\gamma)$ is monotonically decreasing and convex over $\gamma > \bar{d}_+$.
(b) Fix any $\delta > 0$, open neighborhood $U \subset \mathbb{R}$, and twice differentiable function $\gamma : U \rightarrow (\bar{d}_+ + \delta, \infty)$. Then for some constants $C, \beta_0 > 0$ depending only on μ_D and δ , any $s \in U$, and all $\beta \in (0, \beta_0)$,

$$\begin{aligned} |\mathcal{F}(\gamma(s))| &\leq C\beta^4 (1 - q_*)^2 \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |(\gamma(s) - x)^{-1}| \\ |\partial_s \mathcal{F}(\gamma(s))| &\leq C\beta^4 (1 - q_*)^2 \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |\partial_s (\gamma(s) - x)^{-1}| \\ \partial_s^2 \mathcal{F}(\gamma(s)) &\leq C\beta^4 (1 - q_*)^2 \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |\partial_s^2 (\gamma(s) - x)^{-1}|. \end{aligned}$$

Proof Part (a) was verified in (3.32).

For part (b), we use the notation $O(f(\beta))$ as in Proposition 2.5, and allow the constant in this notation to depend also on δ throughout the proof. We have $x = O(\beta)$ uniformly over $x \in \text{supp}(\mu_{\bar{D}})$. Applying this and Proposition 2.5,

$$\begin{aligned} (1 - q_*)(\lambda_* - x) &= 1 - (1 - q_*)x + O(\beta^2(1 - q_*)^2), \quad \bar{R}(1 - q_*)\kappa_*^{-1} \\ &= \frac{1}{1 - q_*} + O(\beta). \end{aligned}$$

Then for $\lambda(x)$ and $\theta(x)$ defined in (3.5), uniformly over $x \in \text{supp}(\mu_{\bar{D}})$, we have

$$\lambda(x) = O(\beta(1 - q_*)), \quad \theta(x) = O(\beta^2(1 - q_*)).$$

Abbreviate $\lambda \equiv \beta(1 - q_*)$ and $\theta \equiv \beta^2(1 - q_*)$. Then differentiating $\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{22}$ in γ , this implies for $k = 0, 1, 2$,

$$\partial_s^k \mathcal{F}_{11}(\gamma(s)) = O\left(\begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix}\right) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |\partial_s^k(\gamma(s) - x)^{-1}| \quad (3.37)$$

$$\partial_s^k \mathcal{F}_{12}(\gamma(s)) = O\left(\begin{pmatrix} \theta \\ \lambda\theta \end{pmatrix}\right) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |\partial_s^k(\gamma(s) - x)^{-1}| \quad (3.38)$$

$$\partial_s^k \mathcal{F}_{22}(\gamma(s)) = O(\theta^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |\partial_s^k(\gamma(s) - x)^{-1}|. \quad (3.39)$$

Here and below, $O(\cdot)$ for a matrix or vector is in the sense of entrywise comparison.

To bound $\mathcal{F}_{11}(\gamma)^{-1}$ appearing in $\mathcal{F}(\gamma)$, we first bound $\det \mathcal{F}_{11}(\gamma)$ from below in terms of the variance of μ_D as follows: Let D_1, D_2 be independently drawn from μ_D . Let $X_i = \frac{1}{\gamma - \beta D_i}$ and $Y_i = \frac{1}{(1 - q_*)(\lambda_* - \beta D_i)} - 1$ for $i = 1, 2$, where X_i is positive for $\gamma > \bar{d}_+$. Let $\bar{d}_- = \beta d_-$ denote the minimum point of support of $\mu_{\bar{D}}$. Then

$$\begin{aligned} \det \mathcal{F}_{11}(\gamma) &= \mathbb{E}[X_1] \mathbb{E}[X_1 Y_1^2] - (\mathbb{E}[X_1 Y_1])^2 = \frac{1}{2} \mathbb{E}[X_1 X_2 (Y_1 - Y_2)^2] \\ &\geq \frac{\beta^2}{2(1 - q_*)^2 (\gamma - \bar{d}_-)^2 (\lambda_* - \bar{d}_-)^2} \mathbb{E}[(D_1 - D_2)^2] \\ &= \frac{\beta^2}{(1 - q_*)^2 (\gamma - \bar{d}_-)^2 (\lambda_* - \bar{d}_-)^2} \text{Var}(D_1). \end{aligned}$$

Using the small- β expansion of λ_* in Proposition 2.5, we have $(1 - q_*)(\lambda_* - \bar{d}_-) = 1 + O(\beta)$. Then for any $\gamma \geq \bar{d}_+ + \delta$ and some constant $c = c(\mu_D, \delta) > 0$, we have $\det \mathcal{F}_{11}(\gamma) \geq c\beta^2/\gamma^2$. Applying the explicit 2×2 matrix inverse of $\mathcal{F}_{11}$, and combining this with (3.37) for $k = 0$ and the bound $|(\gamma(s) - x)^{-1}| = O(1/\gamma(s))$,

$$\mathcal{F}_{11}(\gamma(s))^{-1} = O\left(\beta^{-2} \gamma(s) \begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix}\right). \quad (3.40)$$

Then applying (3.40), (3.38) and (3.39) for $k = 0$, and $|(\gamma(s) - x)^{-1}| = O(1/\gamma(s))$ again,

$$\begin{aligned} \mathcal{F}(\gamma(s)) &= \mathcal{F}_{22} - \mathcal{F}_{12}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12} \\ &= O(\theta^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |(\gamma(s) - x)^{-1}| + O(\theta^2 \lambda^2 / \beta^2) \cdot \gamma(s) \left(\sup_{x \in \text{supp}(\mu_{\bar{D}})} |(\gamma(s) - x)^{-1}| \right)^2 \\ &= O(\beta^4 (1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} |(\gamma(s) - x)^{-1}|. \end{aligned}$$

Here, the second equality uses $\theta^2 = \beta^4(1 - q_*)^2$ and $\lambda^2 = \beta^2(1 - q_*)^2 = O(\beta^2)$. This is the desired bound for $|\mathcal{F}(\gamma(s))|$.

For the derivative, let us write as shorthand $\mathcal{F}'_{11} = \partial_s \mathcal{F}_{11}(\gamma(s))$ and similarly for the other terms. Now differentiating $\mathcal{F}_{11}^{-1}$ and applying (3.40) and (3.37) with $k = 1$,

$$[\mathcal{F}_{11}^{-1}]' = -\mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} = O\left(\beta^{-4} \begin{pmatrix} \lambda^4 & \lambda^3 \\ \lambda^3 & \lambda^2 \end{pmatrix}\right) \cdot \gamma(s)^2 \sup_{x \in \text{supp}(\mu_{\tilde{D}})} |\partial_s(\gamma(s) - x)^{-1}|.$$

Then applying also (3.38) and (3.39) with $k \in \{0, 1\}$ and $|(\gamma(s) - x)^{-1}| \leq O(1/\gamma(s))$,

$$\begin{aligned} \partial_s \mathcal{F}(\gamma(s)) &= \mathcal{F}'_{22} - 2\mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12} - \mathcal{F}_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12} \\ &= O\left(\theta^2 \left(1 + \frac{\lambda^2}{\beta^2} + \frac{\lambda^4}{\beta^4}\right)\right) \cdot \sup_{x \in \text{supp}(\mu_{\tilde{D}})} |\partial_s(\gamma(s) - x)^{-1}|. \end{aligned}$$

The desired bound for $|\partial_s \mathcal{F}(\gamma(s))|$ follows again from $\theta^2 = \beta^4(1 - q_*)^2$ and $\lambda^2 = O(\beta^2)$.

Finally, differentiating $\mathcal{F}_{11}^{-1}$ again and applying (3.40) and (3.37) with $k = 2$,

$$\begin{aligned} [\mathcal{F}_{11}^{-1}]'' &= -\mathcal{F}_{11}^{-1} \mathcal{F}_{11}'' \mathcal{F}_{11}^{-1} + 2\mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \\ &= O\left(\beta^{-4} \begin{pmatrix} \lambda^4 & \lambda^3 \\ \lambda^3 & \lambda^2 \end{pmatrix}\right) \cdot \gamma(s)^2 \sup_{x \in \text{supp}(\mu_{\tilde{D}})} |\partial_s^2(\gamma(s) - x)^{-1}| \\ &\quad + 2\mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1}. \end{aligned}$$

Then applying (3.40), (3.38) and (3.39) with $k \in \{0, 2\}$,

$$\begin{aligned} \partial_s^2 \mathcal{F}(\gamma(s)) &= \mathcal{F}_{22}'' - 2\mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12} - \mathcal{F}_{12}{}^\top [\mathcal{F}_{11}^{-1}]'' \mathcal{F}_{12} - 2\mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12} \\ &\quad - 4\mathcal{F}'_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12} \\ &= O\left(\theta^2 \left(1 + \frac{\lambda^2}{\beta^2} + \frac{\lambda^4}{\beta^4}\right)\right) \cdot \sup_{x \in \text{supp}(\mu_{\tilde{D}})} |\partial_s^2(\gamma(s) - x)^{-1}| \\ &\quad - 2\left(\mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} + \mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12}\right. \\ &\quad \left.+ 2\mathcal{F}'_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12}\right). \end{aligned} \tag{3.41}$$

Note that for the second term above,

$$\begin{aligned} &\mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} + \mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12} + 2\mathcal{F}'_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12} \\ &= \left[\mathcal{F}'_{12} - \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12}\right]{}^\top \mathcal{F}_{11}^{-1} \left[\mathcal{F}'_{12} - \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12}\right] \geq 0 \end{aligned}$$

where this inequality holds because $\mathcal{F}_{11}^{-1} \succ 0$. Then, applying again $\theta^2 = \beta^4(1 - q_*)^2$ and $\lambda^2 = O(\beta^2)$, we obtain the desired upper bound for $\partial_s^2 \mathcal{F}(\gamma(s))$. $\square$

Lemma 3.4 For all $t \geq 1$ and each $\iota \in \{u, v, w, \gamma, U, W\}$,

$$\Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) = \Psi_{\text{RS}}, \quad \partial_t \Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) = 0.$$

Furthermore, for $\iota = V$,

$$\lim_{t \rightarrow \infty} \|\partial_V \Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*)\| = 0.$$

Proof Let $\delta_{tt} = e_t^\top \Delta_t e_t$ be the lower-right entry of Δ_t , and recall that $\delta_{tt} = \delta_*$ by Proposition 2.3. Then the function $\Phi_{1,t}$ in (3.1) evaluated at $u_*, v_*, w_*, \gamma_*, U_*, V_*, W_*$ is

$$\begin{aligned} \Phi_{1,t} &= \mathbb{E}[\log 2 \cosh(H + Y_t)] - \kappa_*(1 - q_*)\delta_* + \bar{R}(1 - q_*) \cdot (1 - q_*)^2 \delta_* \\ &\quad + \frac{\lambda_* \kappa_* - \bar{R}(1 - q_*)}{2} (1 - q_*)^2 \delta_* \\ &\quad + \frac{1}{2} \mathcal{H}(\bar{G}^{-1}(1 - q_*), 1 - \|v_*\|^2 - \|w_*\|^2). \end{aligned}$$

For the first term, by Theorem 2.2 and Proposition 2.3, $Y_t \sim \mathcal{N}(0, \kappa_* \delta_{tt})$ where $\kappa_* \delta_{tt} = \sigma_*^2$, so

$$\mathbb{E}[\log 2 \cosh(H + Y_t)] = \mathbb{E}[\log 2 \cosh(H + \sigma_* G)], \quad G \sim \mathcal{N}(0, 1).$$

For the second term, applying $\delta_* \kappa_* = \sigma_*^2 = q_* \bar{R}'(1 - q_*)$ by (2.22) and (2.9),

$$-\kappa_*(1 - q_*)\delta_* = -q_*(1 - q_*)\bar{R}'(1 - q_*).$$

For the third and fourth terms, applying also $\lambda_* = \bar{R}(1 - q_*) + (1 - q_*)^{-1}$ from (2.12) and $\delta_* = q_*(1 - q_*)^{-2} - \sigma_*^2$ from (2.24),

$$\begin{aligned} &\bar{R}(1 - q_*) \cdot (1 - q_*)^2 \delta_* + \frac{\lambda_* \kappa_* - \bar{R}(1 - q_*)}{2} (1 - q_*)^2 \delta_*, \\ &= \frac{\bar{R}(1 - q_*) \cdot (1 - q_*)^2}{2} \left(\frac{q_*}{(1 - q_*)^2} - \sigma_*^2 \right) + \left(\bar{R}(1 - q_*) + \frac{1}{1 - q_*} \right) \frac{(1 - q_*)^2}{2} \sigma_*^2, \\ &= \frac{q_*}{2} \bar{R}(1 - q_*) + \frac{q_*(1 - q_*)}{2} \bar{R}'(1 - q_*). \end{aligned}$$

For the last term, observe that $1 - \|v_*\|^2 - \|w_*\|^2 = 1 - (1 + \kappa_*)(1 - q_*)^2 \delta_*$. Applying $\delta_* = \sigma_*^2 / \kappa_* = q_* \bar{R}'(1 - q_*) / \kappa_*$ and the form of κ_* in (2.23), this is

$$1 - \|v_*\|^2 - \|w_*\|^2 = 1 - (1 + \kappa_*^{-1})(1 - q_*)^2 q_* \bar{R}'(1 - q_*) = 1 - q_*.$$

For some $\beta_0 = \beta_0(\mu_D) > 0$ and all $\beta \in (0, \beta_0)$, we have $1 - q_* \leq 1 < \bar{G}(\bar{d}_+) = \beta^{-1}G(d_+)$. Then by Proposition 2.9(a),

$$\mathcal{H}(\bar{G}^{-1}(1 - q_*), 1 - q_*) = \int_0^{1-q_*} \bar{R}(z) dz.$$

Combining all of the above yields

$$\begin{aligned} \Phi_{1,t} &= \mathbb{E}[\log 2 \cosh(H + \sigma_* G)] + \frac{q_*}{2} \bar{R}(1 - q_*) - \frac{q_*(1 - q_*)}{2} \bar{R}'(1 - q_*) \\ &\quad + \frac{1}{2} \int_0^{1-q_*} \bar{R}(z) dz = \Psi_{RS}. \end{aligned}$$

To check the stationary conditions, first by the form of $\mathcal{H}$ in (3.4), we have $\partial_\gamma \mathcal{H}(\gamma, \alpha) = \alpha - \bar{G}(\gamma)$ and $\partial_\alpha \mathcal{H}(\gamma, \alpha) = \gamma - 1/\alpha$. Since $\gamma_* = \bar{G}^{-1}(1 - q_*) = \bar{R}(1 - q_*) + (1 - q_*)^{-1}$ and $\|v_*\|^2 + \|w_*\|^2 = q_*$, we have

$$\partial_\gamma \mathcal{H}(\gamma_*, 1 - q_*) = 0, \quad \partial_\alpha \mathcal{H}(\gamma_*, 1 - q_*) = \gamma_* - \frac{1}{1 - q_*} = \bar{R}(1 - q_*). \quad (3.42)$$

Then evaluating at $u_*, v_*, w_*, \gamma_*, U_*, V_*, W_*$ where $v_* = \kappa_*^{-1/2} w_*$, $V_* = 0$, and $W_* = (1 - q_*)^{-1} w_*$,

$$\partial_u \Phi_{1,t} = -U_* + 1 = 0, \quad (3.43)$$

$$\partial_v \Phi_{1,t} = -V_* + \bar{R}(1 - q_*) \kappa_*^{-1/2} w_* - \bar{R}(1 - q_*) v_* = 0, \quad (3.44)$$

$$\partial_w \Phi_{1,t} = -W_* + \bar{R}(1 - q_*) \kappa_*^{-1/2} v_* + (\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}) w_* - \bar{R}(1 - q_*) w_* = 0, \quad (3.45)$$

$$\partial_\gamma \Phi_{1,t} = \frac{1}{2} \partial_\gamma \mathcal{H}(\gamma_*, 1 - q_*) = 0. \quad (3.46)$$

The third line above applies again $\lambda_* = \bar{R}(1 - q_*) + (1 - q_*)^{-1}$.

For the derivatives in U, V, W , observe that the derivative of $\log 2 \cosh x$ is $\tanh x$, and

$$\tanh(H + Y_t) = (1 - q_*)(X_{t+1} + Y_t)$$

by the definition of the AMP state evolution (2.25). Hence

$$\partial_U \Phi_{1,t} = \mathbb{E}[H \cdot (1 - q_*)(X_{t+1} + Y_t)] - u_*, \quad (3.47)$$

$$\partial_V \Phi_{1,t} = \mathbb{E}[\Delta_t^{-1/2} (X_1, \dots, X_t) \cdot (1 - q_*)(X_{t+1} + Y_t)] - v_*, \quad (3.48)$$

$$\partial_W \Phi_{1,t} = \mathbb{E}[\kappa_*^{-1/2} \Delta_t^{-1/2} (Y_1, \dots, Y_t) \cdot (1 - q_*)(X_{t+1} + Y_t)] - w_*. \quad (3.49)$$

From the joint law of $H, Y_1, \dots, Y_t, X_1, \dots, X_{t+1}$ described in Theorem 2.2, we have $\mathbb{E}[H \cdot Y_t] = 0$ and $\mathbb{E}[(1 - q_*)H \cdot X_{t+1}] = \mathbb{E}[H \cdot \tanh(H + \sigma_* G)]$, so $\partial_U \Phi_{1,t} = 0$. We have $\mathbb{E}[(Y_1, \dots, Y_t) \cdot Y_t] = \kappa_* \Delta_t e_t$ and $\mathbb{E}[(Y_1, \dots, Y_t) \cdot X_{t+1}] = 0$, so also $\partial_W \Phi_{1,t} = 0$. Finally, writing a block decomposition of Δ_{t+1} as

$$\Delta_{t+1} = \begin{pmatrix} \Delta_t & \delta_t \\ \delta_t^\top & \delta_* \end{pmatrix}, \quad \delta_t = (\delta_{1,t+1}, \dots, \delta_{t,t+1}), \quad (3.50)$$

we have $\mathbb{E}[(X_1, \dots, X_t) \cdot X_{t+1}] = \delta_t$ and $\mathbb{E}[(X_1, \dots, X_t) \cdot Y_t] = 0$. Thus

$$\partial_V \Phi_{1,t} = (1 - q_*) \Delta_t^{-1/2} \delta_t - v_* = (1 - q_*) \left[\Delta_t^{-1/2} \delta_t - \Delta_t^{1/2} e_t \right], \quad (3.51)$$

so (recalling that $\delta_{t+1,t+1} = \delta_{tt} = \delta_*$)

$$\begin{aligned} (1 - q_*)^{-2} \|\partial_V \Phi_{1,t}\|^2 &= \delta_t^\top \Delta_t^{-1} \delta_t - 2e_t^\top \delta_t + \delta_* \\ &= \left(\delta_t^\top \Delta_t^{-1} \delta_t - \delta_{t+1,t+1} \right) - 2(\delta_{t,t+1} - \delta_*). \end{aligned}$$

By Proposition 2.3, $\lim_{t \rightarrow \infty} \delta_{t,t+1} = \delta_*$. By (2.28) applied at $t + 1$,

$$\delta_{t+1,t+1} - \delta_t^\top \Delta_t^{-1} \delta_t = \inf_{\alpha \in \mathbb{R}^t} \mathbb{E} \left[\left(X_{t+1} - \alpha^\top (X_1, \dots, X_t) \right)^2 \right],$$

where the infimum is attained at the least-squares coefficients

$$\alpha = \mathbb{E} \left[(X_1, \dots, X_t) (X_1, \dots, X_t)^\top \right]^{-1} \mathbb{E} \left[(X_1, \dots, X_t) \cdot X_{t+1} \right] = \Delta_t^{-1} \delta_t.$$

Then

$$0 \leq \delta_{t+1,t+1} - \delta_t^\top \Delta_t^{-1} \delta_t \leq \mathbb{E}[(X_{t+1} - X_t)^2] = 2\delta_* - 2\delta_{t,t+1},$$

so also $\lim_{t \rightarrow \infty} \delta_{t+1,t+1} - \delta_t^\top \Delta_t^{-1} \delta_t = 0$. Thus $\lim_{t \rightarrow \infty} \|\partial_V \Phi_{1,t}\| = 0$. $\square$

Lemma 3.5 For a constant $\beta_0 = \beta_0(\mu_D) > 0$ and any $\beta \in (0, \beta_0)$,

$$\lim_{t \rightarrow \infty} \Psi_{1,t} = \Psi_{\text{RS}}.$$

Proof We will establish separately

$$\liminf_{t \rightarrow \infty} \Psi_{1,t} \geq \Psi_{\text{RS}}, \quad (3.52)$$

$$\limsup_{t \rightarrow \infty} \Psi_{1,t} \leq \Psi_{\text{RS}}. \quad (3.53)$$

We write $o_t(1)$ for any scalar, vector, or matrix error (with dimension depending on t) that satisfies $\lim_{t \rightarrow \infty} \|o_t(1)\| = 0$, where $\|\cdot\|$ is the Euclidean norm for

vectors and operator norm for matrices. Note that $\Psi_{1,t}$ takes the max-min form $\Psi_{1,t} = \sup_{u,v,w} \inf_{\gamma,U,V,W} \Phi_{1,t}$. Here the supremum and the infimum cannot be interchanged due to the non-concavity in the (u, v, w) parameter. Our strategy is as follows:

- For the lower bound, we specialize the outer supremum to a fixed choice of (u, v, w) near (u_*, v_*, w_*) and minimize the resulting (convex) function over (γ, U, V, W) . This minimizer is shown to be approximately $(\gamma_*, U_*, V_*, W_*)$.
- For the upper bound, we specialize the inner infimum to a choice of (γ, U, V, W) depending on (u, v, w) in such a way that the resulting function is globally concave for sufficiently small β . This concave function is then shown to be approximately maximized at (u_*, v_*, w_*) .

To show the lower bound (3.52), we specialize $\Phi_{1,t}$ to $(u, v, w) = (u_*, \tilde{v}_*, w_*)$ where

$$\tilde{v}_* = v_* + (1 - q_*)[\Delta_t^{-1/2}\delta_t - \Delta_t^{1/2}e_t] = v_* + o_t(1),$$

and δ_t was defined in (3.50). Here the second equality has been verified in the preceding proof of Lemma 3.4. As defined in (3.1), $\Phi_{1,t}(u_*, \tilde{v}_*, w_*; \gamma, U, V, W)$ decomposes as $X(U, V, W) + Y(\gamma)$, where X and Y are both convex functions; specifically, $Y(\gamma) = \frac{1}{2}\mathcal{F}(\gamma)\|\tilde{v}_* - v_*\|^2 + \frac{1}{2}\mathcal{H}(\gamma, 1 - \|\tilde{v}_*\|^2 - \|w_*\|^2)$ which is convex applying Lemma 3.3(a). Then

$$\begin{aligned} \Psi_{1,t} &\geq \inf_{\gamma > \bar{d}_+} \inf_{U \in \mathbb{R}, V, W \in \mathbb{R}^t} \Phi_{1,t}(u_*, \tilde{v}_*, w_*; \gamma, U, V, W) \\ &= \inf_{U \in \mathbb{R}, V, W \in \mathbb{R}^t} X(U, V, W) + \inf_{\gamma > \bar{d}_+} Y(\gamma). \end{aligned}$$

In view of (3.47), (3.49), and (3.51), note that $\tilde{v}_*$ is chosen so that (U_*, V_*, W_*) is now an exact stationary point of X . Hence by the convexity of X , $\inf_{U \in \mathbb{R}, V, W \in \mathbb{R}^t} X(U, V, W) = X(U_*, V_*, W_*)$. For the infimum over γ , recall from (3.46) that $\partial_\gamma \mathcal{H}(\gamma_*, 1 - \|v_*\|^2 - \|w_*\|^2) = 1 - \|v_*\|^2 - \|w_*\|^2 - \bar{G}(\gamma_*) = 0$. Since $\|\tilde{v}_* - v_*\| = o_t(1)$, we have that $Y'(\gamma_*) = \frac{1}{2}\mathcal{F}'(\gamma_*)\|\tilde{v}_* - v_*\|^2 + \frac{1}{2}(\|v_*\|^2 - \|\tilde{v}_*\|^2) = o_t(1)$. Furthermore, $Y''(\gamma) = \frac{1}{2}\mathcal{F}''(\gamma)\|\tilde{v}_* - v_*\|^2 + \frac{1}{2}\bar{G}'(\gamma)$. Thus there exist some constants $c, \delta > 0$ independent of t , such that $Y''(\gamma) \geq c$ whenever $|\gamma - \gamma_*| < \delta$. Applying Proposition C.2, we conclude that $\inf_{\gamma > \bar{d}_+} Y(\gamma) \geq Y(\gamma_*) + o_t(1)$. Combining these two bounds,

$$\begin{aligned} \Psi_{1,t} &\geq X(U_*, V_*, W_*) + Y(\gamma_*) + o_t(1) \\ &= \Phi_{1,t}(u_*, \tilde{v}_*, w_*; \gamma_*, U_*, V_*, W_*) + o_t(1) = \Psi_{\text{RS}} + o_t(1). \end{aligned}$$

Here the last step follows from $\|\partial_v \Phi_{1,t}(u_*, v, w_*; \gamma_*, U_*, V_*, W_*)\| \leq C$ for all $\|v - v_*\| \leq \delta$, where C, δ are constants independent of t . This shows the lower bound (3.52).

For the upper bound (3.53), we now specialize $\Phi_{1,t}$ to

$$\gamma = \gamma(v, w) = \bar{G}^{-1}(1 - \|v\|^2 - \|w\|^2),$$

$$U = U_* = 1, \quad V = V(v) = \beta^{1/2}(v - v_*), \quad W = W(w) = \beta^{1/2}(w - w_*) + W_*.$$

Here $\gamma(v, w)$ is well-defined for any (v, w) such that $\|v\|^2 + \|w\|^2 < 1$, since $\bar{G}(\bar{d}_+) > 1$ in view of (3.14). Note that at $(v, w) = (v_*, w_*)$, this gives $(\gamma(v_*, w_*), U, V(v_*), W(w_*)) = (\gamma_*, U_*, V_*, W_*)$. Furthermore,

$$\Psi_{1,t} \leq \sup_{\substack{u \in \mathbb{R} \\ v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 < 1}} \Phi_{1,t}(u, v, w; \gamma(v, w), 1, V(v), W(w)).$$

Due to the choice $U = 1$, the function on the right no longer depends on u . We denote it by

$$\tilde{\Phi}_{1,t}(v, w) = \Phi_{1,t}(u, v, w; \gamma(v, w), 1, V(v), W(w)) = \text{I} + \text{II} + \text{III} + \text{IV}$$

where

$$\begin{aligned} \text{I} &= \mathbb{E} \left[\log 2 \cosh \left(H + V(v)^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W(w)^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \right] \\ \text{II} &= -v^\top V(v) - w^\top W(w) + \bar{R}(1 - q_*) \kappa_*^{-1/2} v^\top w + \frac{\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}}{2} \|w\|^2 \\ \text{III} &= \frac{1}{2} \mathcal{F}(\gamma(v, w)) \|v - \kappa_*^{-1/2} w\|^2 \\ \text{IV} &= \frac{1}{2} \mathcal{H}(\gamma(v, w), 1 - \|v\|^2 - \|w\|^2). \end{aligned}$$

We claim that for some $\beta_0 = \beta_0(\mu_D) > 0$ and all $\beta \in (0, \beta_0)$, this function $\tilde{\Phi}_{1,t}(v, w)$ is concave over the domain $\{v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 < 1\}$. To show this claim, we analyze the Hessian of each term I, II, III, IV using the small- β approximations of Proposition 2.5—the desired concavity will arise from the first two terms of II. We write $O(\beta^k)$ for a scalar, vector, or matrix whose (Euclidean or operator) norm is at most $C\beta^k$ uniformly over $\{v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 < 1\}$, for a constant $C = C(\mu_D) > 0$ depending only on μ_D .

For I, we have

$$\nabla_{v,w}^2 \text{I} = \beta \cdot \mathbb{E} \left[Z_t Z_t^\top \cdot \left(1 - \tanh^2 \left(H + V(v)^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W(w)^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \right) \right]$$

where

$$Z_t \triangleq \left(\Delta_t^{-1/2} (X_1, \dots, X_t), \kappa_*^{-1/2} \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \in \mathbb{R}^{2t}. \quad (3.54)$$

Then $0 \leq \nabla_{v,w}^2 \text{I} \leq \beta \mathbb{E}[Z_t Z_t^\top] = \beta I_{2t \times 2t}$, the last equality applying (2.28–2.30).

For II, observe that Proposition 2.5 implies

$$\bar{R}(1 - q_*) \kappa_*^{-1/2} = O(\beta), \quad \lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1} = O(\beta).$$

Then $\nabla_{v,w}^2 \Pi = -2\beta^{1/2} I_{2t \times 2t} + O(\beta)$.

For III, consider any scalar linear parametrization

$$(v(s), w(s))_{s \in \mathbb{R}} = (v, w) + s \cdot (v', w')$$

where $\|(v', w')\| = 1$. Write as shorthand

$$A(s) \triangleq 1 - \|v(s)\|^2 - \|w(s)\|^2, \quad B(s) \triangleq \|v(s) - \kappa_*^{-1/2} w(s)\|^2.$$

Applying $\|v\|, \|w\|, \|v'\|, \|w'\| \leq 1$, it is easily checked that

$$|A(s)|, |\partial_s A(s)|, |\partial_s^2 A(s)| = O(1) \text{ at } s = 0. \quad (3.55)$$

Applying also $\kappa_*^{-1} = O(\beta^{-2}(1 - q_*)^{-2})$ by Proposition 2.5, we have

$$|B(s)|, |\partial_s B(s)|, |\partial_s^2 B(s)| = O(\beta^{-2}(1 - q_*)^{-2}) \text{ at } s = 0. \quad (3.56)$$

Now write also as shorthand

$$\mathcal{F}(s) \triangleq \mathcal{F}(\gamma(v(s), w(s))) = \mathcal{F}(\bar{G}^{-1}(A(s))).$$

Then

$$\begin{aligned} & (v', w')^\top \cdot \nabla_{v,w}^2 \text{III} \cdot (v', w') \\ &= \partial_s^2 \text{III} \Big|_{s=0} = \frac{1}{2} \left(\partial_s^2 \mathcal{F}(s) \cdot B(s) + 2\partial_s \mathcal{F}(s) \cdot \partial_s B(s) + \mathcal{F}(s) \cdot \partial_s^2 B(s) \right) \Big|_{s=0}. \end{aligned} \quad (3.57)$$

Observe that since $\|v\|^2 + \|w\|^2 < 1$, we have $A(s) \in (0, 1]$ at $s = 0$. Let $\bar{d}_- = \beta d_-$ be the smallest point of support of $\mu_{\bar{D}}$. For any $x > \bar{d}_+$, since $\bar{G}(x) \geq 1/(x - \bar{d}_-)$ and $\bar{G}$ is decreasing, we have $x \leq \bar{G}^{-1}(1/(x - \bar{d}_-))$. Thus

$$\bar{G}^{-1}(A(s)) \geq \bar{G}^{-1}(1) \geq 1 + \bar{d}_- > \bar{d}_+ + 0.1,$$

where the last inequality holds for all sufficiently small β . Then Lemma 3.3(b) implies

$$|\mathcal{F}(s)| \leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left| (\bar{G}^{-1}(A(s)) - x)^{-1} \right|, \quad (3.58)$$

$$|\partial_s \mathcal{F}(s)| \leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left| \partial_s (\bar{G}^{-1}(A(s)) - x)^{-1} \right|, \quad (3.59)$$

$$\partial_s^2 \mathcal{F}(s) \leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left| \partial_s^2 (\bar{G}^{-1}(A(s)) - x)^{-1} \right| \quad (3.60)$$

where this third inequality (3.60) is a one-sided bound without absolute value on the left side. Here $\bar{G}^{-1}(A(s)) = \bar{R}(A(s)) + A(s)^{-1}$, where $A(s) \in (0, 1]$ at $s = 0$. To

further bound (3.58–3.60),⁴ we may apply the series expansion for $\bar{R}(z)$ from (2.7), recalling $\bar{\kappa}_1 = 0$, to write

$$\begin{aligned} (\bar{G}^{-1}(z) - x)^{-1} &= (\bar{R}(z) + z^{-1} - x)^{-1} = z \left(1 - xz + \sum_{k \geq 2} \bar{\kappa}_k z^k \right)^{-1} \\ &= z \cdot \sum_{j \geq 0} \left(xz - \sum_{k \geq 2} \bar{\kappa}_k z^k \right)^j \\ &\triangleq \sum_{k \geq 0} c_k(x) z^{k+1}. \end{aligned} \quad (3.61)$$

Applying $|x| \leq C\beta$ and $|\bar{\kappa}_k| \leq (C\beta)^k$ for a constant $C = C(\mu_D) > 0$ and all k , we have

$$|c_k(x)| \leq 2^{k-1} \cdot (C\beta)^k,$$

where 2^{k-1} is the number of ordered partitions of k into positive integers. Then for sufficiently small $\beta_0(\mu_D) > 0$ and any $\beta \in (0, \beta_0)$ and $z \in (0, 1]$, all summations of (3.61) are absolutely convergent, and the right side is an analytic power series for the function $(\bar{G}^{-1}(z) - x)^{-1}$ on the left. The derivatives in z may be computed term-by-term, to yield

$$\left| (\bar{G}^{-1}(z) - x)^{-1} \right|, \left| \partial_z (\bar{G}^{-1}(z) - x)^{-1} \right|, \left| \partial_z^2 (\bar{G}^{-1}(z) - x)^{-1} \right| = O(1).$$

Combining with (3.55) and applying this to (3.58–3.60) using the chain rule, we obtain that at $s = 0$, $|\mathcal{F}(s)|, |\partial_s \mathcal{F}(s)| = O(\beta^4(1 - q_*)^2)$ and $\partial_s^2 \mathcal{F}(s) \leq C\beta^4(1 - q_*)^2$, for a constant $C = C(\mu_D) > 0$. Note that $B(s) \geq 0$, so this last inequality implies also $\partial_s^2 \mathcal{F}(s) \cdot B(s) \leq C\beta^4(1 - q_*)^2 \cdot B(s)$. Then combining with (3.56) and applying this to (3.57), we obtain the upper bound $\nabla_{v,w}^2 \text{III} < C'\beta^2$ for a constant $C' = C'(\mu_D) > 0$.

Finally, for IV, observe that by Proposition 2.9(a),

$$\text{IV} = \frac{1}{2} \mathcal{H}(\gamma(v, w), 1 - \|v\|^2 - \|w\|^2) = \frac{1}{2} \int_0^{1 - \|v\|^2 - \|w\|^2} \bar{R}(z) dz.$$

Writing as shorthand $f(s) = \int_0^{A(s)} \bar{R}(z) dz$ with $A(s) = 1 - \|v(s)\|^2 - \|w(s)\|^2$ previously defined, we have similarly

$$(v', w')^\top \nabla_{v,w}^2 \text{IV} \cdot (v', w') = \partial_s^2 \text{IV} \Big|_{s=0} = \frac{1}{2} \partial_s^2 f(s) \Big|_{s=0}.$$

⁴ One may apply more explicit bounds for $\bar{G}^{-1}$ and its derivatives here, such as $|\bar{G}^{-1}(z) - \frac{1}{z}| \leq \beta \|D\|_\infty$, but the current argument allows an easier generalization to the second moment computation (cf. Lemma 4.5).

Applying again (3.55) and the bounds $\bar{R}(z), \bar{R}'(z) = O(\beta^2)$ over $z \in (0, 1)$ from Proposition 2.5, we obtain $\nabla_{v,w}^2 \text{IV} = O(\beta^2)$. Combining I–IV, we conclude

$$\nabla_{v,w}^2 \tilde{\Phi}_{1,t}(v, w) < -2\beta^{1/2} I_{2t \times 2t} + O(\beta).$$

Then for some sufficiently small $\beta_0 = \beta_0(\mu_D) > 0$, all $\beta \in (0, \beta_0)$, and any t , we have

$$\nabla_{v,w}^2 \tilde{\Phi}_{1,t}(v, w) < -\beta^{1/2} I_{2t \times 2t} \quad (3.62)$$

over the whole domain $\{v, w \in \mathbb{R}^t : \|v\|^2 + \|w\|^2 < 1\}$. In particular, $\tilde{\Phi}_{1,t}$ is concave as claimed.

Finally, we argue that $(v, w) = (v_*, w_*)$ is an approximate maximizer for $\tilde{\Phi}_{1,t}(v, w)$. Indeed,

$$\partial_v \tilde{\Phi}_{1,t}(v_*, w_*) = \partial_v \Phi_{1,t} + \partial_\gamma \Phi_{1,t} \cdot \partial_v \gamma(v_*, w_*) + \partial_V \Phi_{1,t} \cdot \partial_v V(v_*)$$

where the derivatives of $\Phi_{1,t}$ are evaluated at $(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*)$. Applying Lemma 3.4, we have $\partial_v \tilde{\Phi}_{1,t}(v_*, w_*) = o_t(1)$. Similarly, $\partial_w \tilde{\Phi}_{1,t}(v_*, w_*) = 0$. In view of (3.62), applying Proposition C.2 yields $\sup_{\|v\|^2 + \|w\|^2 < 1} \tilde{\Phi}_{1,t}(v, w) = \tilde{\Phi}_{1,t}(v_*, w_*) + o_t(1)$. Thus

$$\begin{aligned} \Psi_{1,t} &\leq \tilde{\Phi}_{1,t}(v_*, w_*) + o_t(1) = \Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) + o_t(1) \\ &= \Psi_{\text{RS}} + o_t(1), \end{aligned}$$

which is the desired (3.53). $\square$

Lemma 3.1 follows immediately from Lemmas 3.2 and 3.5.

4 Conditional second moment

We now provide a similar computation for the conditional second moment.

Lemma 4.1 *In the setting of Theorem 1.3,*

$$\lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z^2 \mid \mathcal{G}_t] = 2\Psi_{\text{RS}}$$

where the inner limit as $n \rightarrow \infty$ exists almost surely for each fixed t .

4.1 Derivation of the variational formula

Define the domain

$$\mathcal{D}_+ = \left\{ (\gamma, v, \rho) \in \mathbb{R}^3 : \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \succ \bar{d}_+ \cdot I_{2 \times 2} \right\}. \quad (4.1)$$

For scalar arguments $(\gamma, v, \rho) \in \mathcal{D}_+$ and $u, k, U, K, P \in \mathbb{R}$ and $p \in [-1, 1]$, and vector arguments $v, w, \ell, m, V, W, L, M \in \mathbb{R}^t$ satisfying

$$A(p, v, w, \ell, m) \triangleq \begin{pmatrix} 1 - \|v\|^2 - \|w\|^2 & p - v^\top \ell - w^\top m \\ p - v^\top \ell - w^\top m & 1 - \|\ell\|^2 - \|m\|^2 \end{pmatrix} \succ 0, \quad (4.2)$$

we define

$$\begin{aligned} \Phi_{2,t}(u, v, w, k, \ell, m, p; \gamma, v, \rho, U, V, W, K, L, M, P) \\ = \mathbb{E} \left[\mathcal{L} \left(P, U \cdot H + V^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t), \right. \right. \\ \left. \left. K \cdot H + L^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} M^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \right] \\ - u \cdot U - k \cdot K - v^\top V - w^\top W - \ell^\top L - m^\top M - p \cdot P \\ + u + k + \bar{R}(1 - q_*) \kappa_*^{-1/2} (v^\top w + \ell^\top m) + \frac{\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}}{2} (\|w\|^2 + \|m\|^2) \\ + \frac{1}{2} \text{Tr} \mathcal{F}(\gamma, v, \rho) \cdot B(v, w, \ell, m) \\ + \frac{1}{2} \mathcal{H}(\gamma, v, \rho; 1 - \|v\|^2 - \|w\|^2, p - v^\top \ell - w^\top m, 1 - \|\ell\|^2 - \|m\|^2). \end{aligned} \quad (4.3)$$

Here, $\mathcal{L}$ is a multivariate analogue of $\log 2 \cosh$ defined as

$$\mathcal{L}(x, y, z) = \log[e^{x+y+z} + e^{x-y-z} + e^{-x+y-z} + e^{-x-y+z}], \quad (4.4)$$

$\mathcal{F}(\gamma, v, \rho)$ denotes the univariate function $\mathcal{F}$ from (3.3) applied spectrally to $(\begin{smallmatrix} \gamma & v \\ v & \rho \end{smallmatrix})$ via the functional calculus, B is the 2×2 -matrix-valued function

$$B(v, w, \ell, m) = \begin{pmatrix} \|v - \kappa_*^{-1/2} w\|^2 & (v - \kappa_*^{-1/2} w)^\top (\ell - \kappa_*^{-1/2} m) \\ (v - \kappa_*^{-1/2} w)^\top (\ell - \kappa_*^{-1/2} m) & \|\ell - \kappa_*^{-1/2} m\|^2 \end{pmatrix}, \quad (4.5)$$

and $\mathcal{H}$ is the scalar-valued function

$$\begin{aligned} \mathcal{H}(\gamma, v, \rho; a, b, c) = \text{Tr} \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} - \int \log \det \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix} \mu_{\bar{D}}(dx) \\ - \left(2 + \log \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right). \end{aligned} \quad (4.6)$$

Define the variational formula

$$\begin{aligned} \Psi_{2,t} = \sup_{\substack{u, k \in \mathbb{R}, p \in [-1, 1] \\ v, w, \ell, m \in \mathbb{R}^t: A(p, v, w, \ell, m) \succ 0}} \inf_{(\gamma, v, \rho) \in \mathcal{D}_+} \inf_{\substack{U, K, P \in \mathbb{R} \\ V, W, L, M \in \mathbb{R}^t}} \Phi_{2,t}(u, v, w, k, \ell, m, p; \\ \gamma, v, \rho, U, V, W, K, L, M, P). \end{aligned} \quad (4.7)$$

Lemma 4.2 *In the setting of Theorem 1.3, for any fixed $t \geq 1$, almost surely*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z^2 \mid \mathcal{G}_t] = \Psi_{2,t}.$$

Proof The proof is analogous to that of Lemma 3.2, and we will omit details where the arguments are the same. We again fix t and write $\mathcal{G}, X, Y, S, \Delta$ for $\mathcal{G}_t, X_t, Y_t, S_t, \Delta_t$. We have

$$\mathbb{E}[Z^2 \mid \mathcal{G}] = \sum_{\sigma, \tau \in \{+1, -1\}^n} \exp\left(h^\top \sigma + h^\top \tau + \frac{n}{2} \cdot f_n(\sigma, \tau)\right),$$

where we define

$$f_n(\sigma, \tau) = \frac{2}{n} \log \mathbb{E} \left[\exp \left(\frac{1}{2} \sigma^\top O^\top \bar{D} O \sigma + \frac{1}{2} \tau^\top O^\top \bar{D} O \tau \right) \mid \mathcal{G} \right].$$

We will approximate this function $f_n(\sigma, \tau)$ on the spheres $\|\sigma\|^2 = n$ and $\|\tau\|^2 = n$.

Conditional law of O . Recall the shorthand $V = V_{(S, \Lambda S)^\perp}$ and $\sigma_\perp, \sigma_\parallel$ from (3.11), and define similarly

$$\tau_\perp = V_{(X, Y)^\perp}^\top \tau, \quad \tau_\parallel = (S, \Lambda S) \left(\begin{matrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{matrix} \right)^{-1} (X, Y)^\top \tau.$$

Then similarly to (3.12), an application of Proposition 2.6 yields

$$\begin{aligned} f_n(\sigma, \tau) &= \frac{1}{n} \sigma_\parallel^\top \bar{D} \sigma_\parallel + \frac{1}{n} \tau_\parallel^\top \bar{D} \tau_\parallel \\ &\quad + \frac{2}{n} \log \mathbb{E} \left[\exp \left(\frac{1}{2} \sigma_\perp^\top \tilde{O}^\top V^\top \bar{D} V \tilde{O} \sigma_\perp \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \tau_\perp^\top \tilde{O}^\top V^\top \bar{D} V \tilde{O} \tau_\perp + \sigma_\parallel^\top \bar{D} V \tilde{O} \sigma_\perp + \tau_\parallel^\top \bar{D} V \tilde{O} \tau_\perp \right) \right]. \end{aligned}$$

Expectation over $\tilde{O}$. We first restrict to the domain

$$U_n = \left\{ (\sigma, \tau) \in \mathbb{R}^n \times \mathbb{R}^n : \|\sigma\|^2 = n, \|\tau\|^2 = n, \right. \\ \left. \sigma_\perp \text{ and } \tau_\perp \text{ are (non-zero and) linearly independent} \right\}.$$

In particular, σ and τ must be different on this domain. We evaluate the expectation over $\tilde{O}$ using Proposition 2.8: Taking $a = \sigma_\perp, c = \tau_\perp, b = V^\top \bar{D} \sigma_\parallel, d = V^\top \bar{D} \tau_\parallel$, and defining Ω_n by some constants $\varepsilon, C > 0$ depending only on μ_D and β , for sufficiently small β and all large n , we have $(a, b, c, d) \in \Omega_n$. Then

$$f_n(\sigma, \tau) = \frac{1}{n} \sigma_\parallel^\top \bar{D} \sigma_\parallel + \frac{1}{n} \tau_\parallel^\top \bar{D} \tau_\parallel + E_n(\sigma, \tau) + r_n(\sigma, \tau) \quad (4.8)$$

where

$$\begin{aligned}
 E_n(\sigma, \tau) = & \inf_{(\gamma, v, \rho) \in \mathcal{D}_\varepsilon} \left\{ \frac{1}{n} \text{Tr} \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \begin{pmatrix} \|\sigma_\perp\|^2 & \sigma_\perp^\top \tau_\perp \\ \sigma_\perp^\top \tau_\perp & \|\tau_\perp\|^2 \end{pmatrix} \right. \\
 & + \frac{1}{n} \begin{pmatrix} V^\top \bar{D} \sigma_\parallel \\ V^\top \bar{D} \tau_\parallel \end{pmatrix}^\top \begin{pmatrix} \gamma I - V^\top \bar{D} V & v I \\ v I & \rho I - V^\top \bar{D} V \end{pmatrix}^{-1} \begin{pmatrix} V^\top \bar{D} \sigma_\parallel \\ V^\top \bar{D} \tau_\parallel \end{pmatrix} \\
 & - \frac{1}{n} \log \det \begin{pmatrix} \gamma I - V^\top \bar{D} V & v I \\ v I & \rho I - V^\top \bar{D} V \end{pmatrix} \\
 & \left. - \left(2 + \log \det \frac{1}{n} \begin{pmatrix} \|\sigma_\perp\|^2 & \sigma_\perp^\top \tau_\perp \\ \sigma_\perp^\top \tau_\perp & \|\tau_\perp\|^2 \end{pmatrix} \right) \right\}
 \end{aligned}$$

and

$$\mathcal{D}_\varepsilon = \left\{ (\gamma, v, \rho) \in \mathbb{R}^3 : \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \succeq (\bar{d}_+ + \varepsilon) \cdot I \right\}. \quad (4.9)$$

We use $r_n(\sigma, \tau)$ to denote any remainder satisfying

$$\lim_{n \rightarrow \infty} \sup_{(\sigma, \tau) \in U_n} \|r_n(\sigma, \tau)\| \rightarrow 0$$

almost surely, and changing from instance to instance.

Approximation by v, w, ℓ, m, p . Define the functionals

$$\begin{aligned}
 u(\sigma) &= \frac{1}{n} h^\top \sigma, \quad \begin{pmatrix} v(\sigma) \\ w(\sigma) \end{pmatrix} = \left[\frac{1}{n} \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix} \right]^{-1/2} \cdot \frac{1}{n} (X, Y)^\top \sigma \\
 k(\tau) &= \frac{1}{n} h^\top \tau, \quad \begin{pmatrix} \ell(\tau) \\ m(\tau) \end{pmatrix} = \left[\frac{1}{n} \begin{pmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{pmatrix} \right]^{-1/2} \cdot \frac{1}{n} (X, Y)^\top \tau, \quad p(\sigma, \tau) = \frac{1}{n} \sigma^\top \tau.
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{\|\sigma_\perp\|^2}{n} &= 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2, & \frac{\|\tau_\perp\|^2}{n} &= 1 - \|\ell(\tau)\|^2 - \|m(\tau)\|^2, \\
 \frac{\sigma_\perp^\top \tau_\perp}{n} &= p(\sigma, \tau) - v(\sigma)^\top \ell(\tau) - w(\sigma)^\top m(\tau),
 \end{aligned}$$

and

$$\begin{aligned}
 \sigma_\parallel &= S \cdot \Delta^{-1/2} v(\sigma) + \Lambda S \cdot (\kappa_* \Delta)^{-1/2} w(\sigma) + (S, \Lambda S) \cdot r_n(\sigma, \tau) \\
 \tau_\parallel &= S \cdot \Delta^{-1/2} \ell(\tau) + \Lambda S \cdot (\kappa_* \Delta)^{-1/2} m(\tau) + (S, \Lambda S) \cdot r_n(\sigma, \tau).
 \end{aligned}$$

We approximate the terms of (4.8) using the low-dimensional parameters v, w, ℓ, m, p : Setting $a_* = \bar{R}(1 - q_*)$ and following arguments similar to (3.24),

$$\begin{aligned} \frac{\sigma_{\parallel}^{\top} \bar{D} \sigma_{\parallel}}{n} + \frac{\tau_{\parallel}^{\top} \bar{D} \tau_{\parallel}}{n} &= \frac{2a_*}{\kappa_*^{1/2}} \left(v(\sigma)^{\top} w(\sigma) + \ell(\tau)^{\top} m(\tau) \right) \\ &\quad + \left(\lambda_* - \frac{a_*}{\kappa_*} \right) \left(\|w(\sigma)\|^2 + \|m(\tau)\|^2 \right) + r_n(\sigma, \tau). \end{aligned}$$

For approximating $E_n(\sigma, \tau)$, we will refer to the eigen-decomposition

$$\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} = (y_1 \ y_2) \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} y_1^{\top} \\ y_2^{\top} \end{pmatrix} \quad (4.10)$$

for $(\gamma, v, \rho) \in \mathcal{D}_{\varepsilon}$. Here $\alpha_1, \alpha_2 \geq \bar{d}_+ + \varepsilon$ are the eigenvalues, and $y_1 \in \mathbb{R}^2$ and $y_2 \in \mathbb{R}^2$ are the two corresponding eigenvectors. We write also

$$V^{\top} \bar{D} V = V'^{\top} \bar{D}' V' = V'^{\top} \text{diag}(\bar{d}'_1, \dots, \bar{d}'_{n-2t}) V' \quad (4.11)$$

as the eigendecomposition of $V^{\top} \bar{D} V \in \mathbb{R}^{(n-2t) \times (n-2t)}$, where $V' \in \mathbb{R}^{(n-2t) \times (n-2t)}$ is orthogonal and $\bar{d}'_i$ are the eigenvalues. Then as $n \rightarrow \infty$,

$$\begin{aligned} \frac{1}{n} \log \det \begin{pmatrix} \gamma I - V^{\top} \bar{D} V & v I \\ v I & \rho I - V^{\top} \bar{D} V \end{pmatrix} &= \frac{1}{n} \sum_{i=1}^{n-2t} \log \det \begin{pmatrix} \gamma - \bar{d}'_i & v \\ v & \rho - \bar{d}'_i \end{pmatrix} \\ &\rightarrow \int \log \det \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix} \mu_{\bar{D}}(dx). \end{aligned}$$

This convergence is uniform over $(\gamma, v, \rho) \in \mathcal{D}_{\varepsilon}$, because the left side is

$$\frac{1}{n} \sum_{i=1}^{n-2t} \log(\alpha_1 - \bar{d}'_i) + \frac{1}{n} \sum_{i=1}^{n-2t} \log(\alpha_2 - \bar{d}'_i),$$

and the uniform convergence of each sum over $\alpha_1, \alpha_2 \geq \bar{d}_+ + \varepsilon$ was verified in the first-moment calculation of Lemma 3.1. Thus, for any $(\sigma, \tau) \in U_n$,

$$\begin{aligned} &\frac{1}{n} \text{Tr} \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \begin{pmatrix} \|\sigma_{\perp}\|^2 & \sigma_{\perp}^{\top} \tau_{\perp} \\ \sigma_{\perp}^{\top} \tau_{\perp} & \|\tau_{\perp}\|^2 \end{pmatrix} - \frac{1}{n} \log \det \begin{pmatrix} \gamma I - V^{\top} \bar{D} V & v I \\ v I & \rho I - V^{\top} \bar{D} V \end{pmatrix} \\ &\quad - \left(2 + \log \det \frac{1}{n} \begin{pmatrix} \|\sigma_{\perp}\|^2 & \sigma_{\perp}^{\top} \tau_{\perp} \\ \sigma_{\perp}^{\top} \tau_{\perp} & \|\tau_{\perp}\|^2 \end{pmatrix} \right) \\ &= \mathcal{H} \left(\gamma, v, \rho; 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2, p(\sigma, \tau) \right. \\ &\quad \left. - v(\sigma)^{\top} \ell(\tau) - w(\sigma)^{\top} m(\tau), 1 - \|\ell(\tau)\|^2 - \|m(\tau)\|^2 \right) \\ &\quad + r_n(\gamma, v, \rho) \end{aligned}$$

where $r_n(\gamma, \nu, \rho) \rightarrow 0$ uniformly over $(\gamma, \nu, \rho) \in \mathcal{D}_\varepsilon$.

For the remaining second term of $E_n(\sigma, \tau)$, let us write

$$\begin{pmatrix} \gamma I - V^\top \bar{D} V & \nu I \\ \nu I & \rho I - V^\top \bar{D} V \end{pmatrix}^{-1} = \begin{pmatrix} V' & 0 \\ 0 & V' \end{pmatrix}^\top \begin{pmatrix} \gamma I - \bar{D}' & \nu I \\ \nu I & \rho I - \bar{D}' \end{pmatrix}^{-1} \begin{pmatrix} V' & 0 \\ 0 & V' \end{pmatrix}. \quad (4.12)$$

We may invert the matrix on the right by inverting separately the non-zero 2×2 blocks,

$$\begin{pmatrix} \gamma - \bar{d}'_i & \nu \\ \nu & \rho - \bar{d}'_i \end{pmatrix}^{-1} = \frac{1}{\alpha_1 - \bar{d}'_i} y_1 y_1^\top + \frac{1}{\alpha_2 - \bar{d}'_i} y_2 y_2^\top.$$

Then for each $j, k \in \{1, 2\}$, the (j, k) block of (4.12) is

$$\begin{aligned} \left(\gamma I - V^\top \bar{D} V \quad \nu I \right)_{jk}^{-1} &= y_{1j} y_{1k} V'^\top \operatorname{diag} \left(\frac{1}{\alpha_1 - \bar{d}'_i} \right) V' \\ &\quad + y_{2j} y_{2k} V'^\top \operatorname{diag} \left(\frac{1}{\alpha_2 - \bar{d}'_i} \right) V' \\ &= y_{1j} y_{1k} (\alpha_1 I - V^\top \bar{D} V)^{-1} \\ &\quad + y_{2j} y_{2k} (\alpha_2 I - V^\top \bar{D} V)^{-1}. \end{aligned}$$

Let us consider first $j = k = 1$. Then by (3.31) from the first-moment calculation of Lemma 3.1,

$$\begin{aligned} &y_{11}^2 \cdot \frac{\sigma_\parallel^\top \bar{D} V (\alpha_1 I - V^\top \bar{D} V)^{-1} V^\top \bar{D} \sigma_\parallel}{n} + y_{21}^2 \cdot \frac{\sigma_\parallel^\top \bar{D} V (\alpha_2 I - V^\top \bar{D} V)^{-1} V^\top \bar{D} \sigma_\parallel}{n} \\ &= \left(y_{11}^2 \mathcal{F}(\alpha_1) + y_{21}^2 \mathcal{F}(\alpha_2) \right) \cdot \|v(\sigma) - \kappa_*^{-1/2} w(\sigma)\|^2 + r_n(\sigma, \alpha_1, \alpha_2, y_{11}, y_{21}) \end{aligned}$$

where $r_n(\sigma, \alpha_1, \alpha_2, y_{11}, y_{21}) \rightarrow 0$ uniformly over $\alpha_1, \alpha_2 \geq \bar{d}_+ + \varepsilon, y_{11}, y_{21} \in [-1, 1]$, and $\{\sigma : \|\sigma\|^2 = n, \sigma_\perp \neq 0\}$. Similarly, for the other blocks $j, k \in \{1, 2\}$,

$$\begin{aligned}
& y_{11}y_{12} \cdot \frac{\sigma_{\parallel}^{\top} \bar{D}V(\alpha_1 - V^{\top} \bar{D}V)^{-1} V^{\top} \bar{D}\tau_{\parallel}}{n} \\
& + y_{21}y_{22} \cdot \frac{\sigma_{\parallel}^{\top} \bar{D}V(\alpha_2 - V^{\top} \bar{D}V)^{-1} V^{\top} \bar{D}\tau_{\parallel}}{n} \\
& = \left(y_{11}y_{12}\mathcal{F}(\alpha_1) + y_{21}y_{22}\mathcal{F}(\alpha_2) \right) \cdot (v(\sigma) - \kappa_*^{-1/2}w(\sigma))^{\top} (\ell(\tau) - \kappa_*^{-1/2}m(\tau)) \\
& + r_n(\sigma, \tau, \alpha_1, \alpha_2, y_1, y_2), \\
& y_{12}^2 \cdot \frac{\tau_{\parallel}^{\top} \bar{D}V(\alpha_1 - V^{\top} \bar{D}V)^{-1} V^{\top} \bar{D}\tau_{\parallel}}{n} + y_{22}^2 \cdot \frac{\tau_{\parallel}^{\top} \bar{D}V(\alpha_2 - V^{\top} \bar{D}V)^{-1} V^{\top} \bar{D}\tau_{\parallel}}{n} \\
& = \left(y_{12}^2\mathcal{F}(\alpha_1) + y_{22}^2\mathcal{F}(\alpha_2) \right) \cdot \|\ell(\tau) - \kappa_*^{-1/2}m(\tau)\|^2 + r_n(\tau, \alpha_1, \alpha_2, y_{12}, y_{22})
\end{aligned}$$

where these remainders converge to 0 uniformly over $(\sigma, \tau) \in U_n$, $\alpha_1, \alpha_2 \geq \bar{d}_+ + \varepsilon$, and $y_{11}, y_{12}, y_{21}, y_{22} \in [-1, 1]$. Combining these statements, we have for the second term of $E_n(\sigma, \tau)$ that

$$\begin{aligned}
& \frac{1}{n} \begin{pmatrix} V^{\top} \bar{D}\sigma_{\parallel} \\ V^{\top} \bar{D}\tau_{\parallel} \end{pmatrix}^{\top} \begin{pmatrix} \gamma I - V^{\top} \bar{D}V & vI \\ vI & \rho I - V^{\top} \bar{D}V \end{pmatrix}^{-1} \begin{pmatrix} V^{\top} \bar{D}\sigma_{\parallel} \\ V^{\top} \bar{D}\tau_{\parallel} \end{pmatrix} \\
& = \text{Tr} \begin{pmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{pmatrix} \begin{pmatrix} \mathcal{F}(\alpha_1) & \\ & \mathcal{F}(\alpha_2) \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \cdot B(v(\sigma), w(\sigma), \ell(\tau), m(\tau)) \\
& + r_n(\sigma, \tau, \gamma, v, \rho) \\
& = \text{Tr} \mathcal{F}(\gamma, v, \rho) \cdot B(v(\sigma), w(\sigma), \ell(\tau), m(\tau)) + r_n(\sigma, \tau, \gamma, v, \rho) \quad (4.13)
\end{aligned}$$

where $\mathcal{F}(\gamma, v, \rho)$ is the function $\mathcal{F}$ applied to $\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix}$ spectrally, and $r_n(\sigma, \tau, \gamma, v, \rho) \rightarrow 0$ uniformly over $(\sigma, \tau) \in U_n$ and $(\gamma, v, \rho) \in \mathcal{D}_{\varepsilon}$.

Observe that this also implies, for any fixed vector $z \in \mathbb{R}^2$, with respect to the positive-definite ordering for $\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix}$,

$$z^{\top} \mathcal{F}(\gamma, v, \rho) z \text{ is non-increasing and convex over } (\gamma, v, \rho) \in \mathcal{D}_+. \quad (4.14)$$

Indeed, it suffices to show this for unit vectors $z = (z_1, z_2) \in \mathbb{R}^2$. Fixing any $(\gamma, v, \rho) \in \mathcal{D}_+$, we may take ε above small enough such that $(\gamma, v, \rho) \in \mathcal{D}_{\varepsilon}$. For each n , we may then take $(\sigma, \tau) \in U_n$ such that $\|v(\sigma)\|^2 \rightarrow z_1^2$, $\|\ell(\tau)\|^2 \rightarrow z_2^2$, $v(\sigma)^{\top} \ell(\tau) \rightarrow z_1 z_2$, $\|w(\sigma)\|^2 \rightarrow 0$, and $\|m(\tau)\|^2 \rightarrow 0$. (For example, we may choose

$$\sigma = \sqrt{n} \frac{z_1 x + (\sqrt{1 - z_1^2} + \delta_n) r_1}{\|z_1 x + (\sqrt{1 - z_1^2} + \delta_n) r_1\|}, \quad \tau = \sqrt{n} \frac{z_2 x + (\sqrt{1 - z_2^2} + \delta_n) r_2}{\|z_2 x + (\sqrt{1 - z_2^2} + \delta_n) r_2\|}$$

where x is the first column of X , r_1 and r_2 are vectors with $\|r_1\| = \|r_2\| = \|x\|$ that are orthogonal to each other and to the column span of (X, Y) , and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.) Then as $n \rightarrow \infty$, the right side of (4.13) converges to

$\text{Tr } \mathcal{F}(\gamma, \nu, \rho) \cdot \begin{pmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{pmatrix} = z^\top \mathcal{F}(\gamma, \nu, \rho) z$. The left side is non-increasing with respect to the positive-definite ordering and convex at (γ, ν, ρ) , so the same properties hold for the limit $z^\top \mathcal{F}(\gamma, \nu, \rho) z$, showing (4.14).

Combining the above, we obtain the uniform approximation over $(\sigma, \tau) \in U_n$

$$\begin{aligned} f_n(\sigma, \tau) = & \inf_{(\gamma, \nu, \rho) \in \mathcal{D}_\varepsilon} \left(\frac{2a_*}{\kappa_*^{1/2}} (v(\sigma)^\top w(\sigma) + \ell(\tau)^\top m(\tau)) \right. \\ & + \left(\lambda_* - \frac{a_*}{\kappa_*} \right) (\|w(\sigma)\|^2 + \|m(\tau)\|^2) \\ & + \text{Tr } \mathcal{F}(\gamma, \nu, \rho) \cdot B(v(\sigma), w(\sigma), \ell(\tau), m(\tau)) \\ & + \mathcal{H}(\gamma, \nu, \rho; 1 - \|v(\sigma)\|^2 - \|w(\sigma)\|^2, \\ & p(\sigma, \tau) - v(\sigma)^\top \ell(\tau) - w(\sigma)^\top m(\tau), 1 - \|\ell(\tau)\|^2 - \|m(\tau)\|^2) \Big) \\ & + r_n(\sigma, \tau). \end{aligned} \quad (4.15)$$

We now show that, for small β and ε , the above infimum over $\mathcal{D}_\varepsilon$ is the same as that over the large domain $\mathcal{D}_+$ in (4.1). Indeed, for any fixed $(\sigma, \tau) \in U_n$, denote by $S(\gamma, \nu, \rho)$ the quantity inside this infimum. Recall the eigendecomposition $\begin{pmatrix} \gamma & \nu \\ \nu & \rho \end{pmatrix} = \alpha_1 y_1 y_1^\top + \alpha_2 y_2 y_2^\top$ in (4.10). For any $(\gamma, \nu, \rho) \in \mathcal{D}_+ \setminus \mathcal{D}_\varepsilon$, we compare $S(\gamma, \nu, \rho)$ with $S(\gamma', \nu', \rho')$, where $\begin{pmatrix} \gamma' & \nu' \\ \nu' & \rho' \end{pmatrix} = \max\{\alpha_1, \bar{d}_+ + \varepsilon\} y_1 y_1^\top + \max\{\alpha_2, \bar{d}_+ + \varepsilon\} y_2 y_2^\top$ and $(\gamma', \nu', \rho') \in \mathcal{D}_\varepsilon$. Note first that since $B(v, w, \ell, m) \geq 0$, (4.14) implies

$$\begin{aligned} & \text{Tr } \mathcal{F}(\gamma', \nu', \rho') \cdot B(v(\sigma), w(\sigma), \ell(\tau), m(\tau)) \\ & \leq \text{Tr } \mathcal{F}(\gamma, \nu, \rho) \cdot B(v(\sigma), w(\sigma), \ell(\tau), m(\tau)). \end{aligned}$$

Next, let Δ denote the matrix derivative of the term $\mathcal{H}(\gamma, \nu, \rho; \cdot)$ of (4.15),

$$\Delta \triangleq \begin{pmatrix} \partial_\gamma \mathcal{H}(\gamma, \nu, \rho, \cdot) & \frac{1}{2} \partial_\nu \mathcal{H}(\gamma, \nu, \rho, \cdot) \\ \frac{1}{2} \partial_\nu \mathcal{H}(\gamma, \nu, \rho, \cdot) & \partial_\rho \mathcal{H}(\gamma, \nu, \rho, \cdot) \end{pmatrix}$$

which has the explicit form

$$\Delta = A(p(\sigma, \tau), v(\sigma), w(\sigma), \ell(\tau), m(\tau)) - \int \begin{pmatrix} \gamma - x & \nu \\ \nu & \rho - x \end{pmatrix}^{-1} \mu_{\bar{D}}(dx).$$

Since $(\gamma, \nu, \rho) \in \mathcal{D}_+ \setminus \mathcal{D}_\varepsilon$, there is at least one eigenvalue, say α_1 , which is less than $\bar{d}_+ + \varepsilon$. Then by the monotonicity of $\bar{G}$, for the corresponding eigenvector y_1 , we have

$$y_1^\top \left(\int \begin{pmatrix} \gamma - x & \nu \\ \nu & \rho - x \end{pmatrix}^{-1} \mu_{\bar{D}}(dx) \right) y_1 = \bar{G}(\alpha_1) \geq \bar{G}(\bar{d}_+ + \varepsilon).$$

So

$$\begin{aligned} \text{Tr} [\Delta \cdot y_1 y_1^\top] &\leq \text{Tr} [A(p(\sigma, \tau), v(\sigma), w(\sigma), \ell(\tau), m(\tau)) \cdot y_1 y_1^\top] - \bar{G}(\bar{d}_+ + \varepsilon) \\ &\leq 4 - \bar{G}(\bar{d}_+ + \varepsilon) < 0 \end{aligned}$$

where the second inequality is by Cauchy-Schwarz and the fact that all entries of A are in $[-2, 2]$, and the last inequality holds for $\beta \in (0, \beta_0)$ and sufficiently small $\beta_0 = \beta_0(\mu_D) > 0$ and all sufficiently small ε . Integrating this bound from α_1 to $\bar{d}_+ + \varepsilon$, and also from α_2 to $\bar{d}_+ + \varepsilon$ if $\alpha_2 < \bar{d}_+ + \varepsilon$, we obtain $\mathcal{H}(\gamma', v', \rho'; \cdot) < \mathcal{H}(\gamma, v, \rho; \cdot)$. Combining the above, $S(\gamma', v', \rho') < S(\gamma, v, \rho)$. This shows that $\inf_{\mathcal{D}_\varepsilon} S(\gamma, v, \rho) = \inf_{\mathcal{D}_+} S(\gamma, v, \rho)$.

Finally, observe that $(\sigma, \tau) \mapsto (p(\sigma, \tau), v(\sigma), w(\sigma), \ell(\tau), m(\tau))$ is continuous, relatively open, and maps U_n onto the fixed domain

$$\mathcal{V} \triangleq \{p \in [-1, 1], v, w, \ell, m \in \mathbb{R}^t : A(p, v, w, \ell, m) > 0\}, \quad (4.16)$$

where $A(p, v, w, \ell, m)$ is as defined in (4.2). Then, applying Proposition C.1 as in the proof of Lemma 3.2 to extend the uniform approximation from U_n to its closure $\{\sigma, \tau \in \mathbb{R}^n : \|\sigma\|^2 = \|\tau\|^2 = n\}$, we obtain

$$\lim_{n \rightarrow \infty} \sup_{\sigma, \tau \in \mathbb{R}^n : \|\sigma\|^2 = \|\tau\|^2 = n} |f_n(\sigma, \tau) - f(p(\sigma, \tau), v(\sigma), w(\sigma), \ell(\tau), m(\tau))| = 0$$

where we define for $(p, v, w, \ell, m) \in \mathcal{V}$ the function

$$\begin{aligned} f(p, v, w, \ell, m) &\triangleq \inf_{(\gamma, v, \rho) \in \mathcal{D}_+} \frac{2a_*}{\kappa_*^{1/2}} (v^\top w + \ell^\top m) \\ &\quad + \left(\lambda_* - \frac{a_*}{\kappa_*} \right) (\|w\|^2 + \|m\|^2) + \text{Tr } \mathcal{F}(\gamma, v, \rho) \times \\ &\quad B(v, w, \ell, m) + \mathcal{H}(\gamma, v, \rho; 1 - \|v\|^2 - \|w\|^2, p - v^\top \ell \\ &\quad - w^\top m, 1 - \|\ell\|^2 - \|m\|^2), \end{aligned}$$

and extend this definition by continuity to the closure $\bar{\mathcal{V}}$.

Large deviations analysis. Finally, writing $\langle \cdot \rangle$ for the expectation over the independent discrete uniform laws $\sigma \sim \text{Unif}(\{+1, -1\}^n)$ and $\tau \sim \text{Unif}(\{+1, -1\}^n)$, we may define the limiting cumulant generating function

$$\begin{aligned} &\lambda(U, V, W, K, L, M, P) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left[n(U \cdot u(\sigma) + V^\top v(\sigma) + W^\top w(\sigma) + K \cdot k(\tau) \right. \right. \\ &\quad \left. \left. + L^\top \ell(\tau) + M^\top m(\tau) + P \cdot p(\sigma, \tau) \right] \right\rangle \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\langle \exp \left[U \cdot h^\top \sigma + V^\top \Delta^{-1/2} X^\top \sigma + W^\top (\kappa_* \Delta)^{-1/2} Y^\top \sigma \right. \right. \\ \left. \left. + K \cdot h^\top \tau + L^\top \Delta^{-1/2} X^\top \tau + M^\top (\kappa_* \Delta)^{-1/2} Y^\top \tau + P \cdot \sigma^\top \tau + n \cdot r_n(\sigma, \tau) \right] \right\rangle$$

where $r_n(\sigma, \tau) \rightarrow 0$ uniformly over $\sigma, \tau \in \{+1, -1\}^n$. Evaluating the average over (σ, τ) using

$$\langle e^{x\sigma_i \tau_i + y\sigma_i + z\tau_i} \rangle = e^{\mathcal{L}(x, y, z)} / 4, \quad (4.17)$$

where $\mathcal{L}(x, y, z)$ is as defined in (4.4), and applying the AMP convergence (2.27), this limit exists and is given by

$$\lambda(U, V, W, K, L, M, P) \\ = \mathbb{E} \left[\mathcal{L} \left(P, U \cdot H + V^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t), \right. \right. \\ \left. \left. K \cdot H + L^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} M^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \right) \right] - \log 4.$$

The proof is then concluded by the same argument as in the first-moment calculation of Lemma 3.1, using the Gärtner-Ellis Theorem and Varadhan's Lemma. $\square$

4.2 Analysis of the variational formula

We now consider the approximate stationary point of (4.7) given by

$$u_* = k_* = \mathbb{E}[H \cdot \tanh(H + \sigma_* G)], \quad v_* = \ell_* = (1 - q_*) \Delta_t^{1/2} e_t, \\ w_* = m_* = \kappa_*^{1/2} (1 - q_*) \Delta_t^{1/2} e_t, \\ \gamma_* = \rho_* = \bar{G}^{-1}(1 - q_*), \quad \nu_* = 0, \quad U_* = K_* = 1, \quad V_* = L_* = 0, \\ W_* = M_* = \kappa_*^{1/2} \Delta_t^{1/2} e_t, \quad p_* = q_*, \quad P_* = 0.$$

We write $\Phi_{2,t}(u_*, \dots, P_*)$ for the evaluation of $\Phi_{2,t}$ at this point. We again verify in two steps that this approximately solves (4.7) for $\beta > 0$ sufficiently small.

For these steps, we require the following properties of $\mathcal{F}(\gamma, \nu, \rho)$ analogous to Lemma 3.3.

- Lemma 4.3** (a) For any fixed vector $z \in \mathbb{R}^2$, $z^\top \mathcal{F}(\gamma, \nu, \rho) z$ is non-increasing (with respect to the positive-definite ordering) and convex over $(\gamma, \nu, \rho) \in \mathcal{D}_+$.
 (b) Fix any $\delta > 0$, open neighborhood $U \subset \mathbb{R}$, and twice differentiable function $(\gamma, \nu, \rho) : U \rightarrow \mathcal{D}_\delta$ where $\mathcal{D}_\delta$ is as defined in (4.9). Then for some constants $C, \beta_0 > 0$ depending only on μ_D and δ , any $s \in U$, and all $\beta \in (0, \beta_0)$,

$$\|\mathcal{F}(\gamma(s), \nu(s), \rho(s))\| \leq C\beta^4(1 - q_*)^2 \sup_{x \in \text{supp}(\mu_D)} \left\| \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\| \\ \|\partial_s \mathcal{F}(\gamma(s), \nu(s), \rho(s))\| \leq C\beta^4(1 - q_*)^2 \sup_{x \in \text{supp}(\mu_D)} \left\| \partial_s \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\|$$

$$\partial_s^2 \mathcal{F}(\gamma(s), v(s), \rho(s)) \leq C\beta^4(1 - q_*)^2 \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left\| \partial_s^2 \left(\begin{pmatrix} \gamma(s) & v(s) \\ v(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\| \cdot I_{2 \times 2}.$$

Proof Part (a) was verified in (4.14).

For part (b), as in Lemma 3.3, let us write $O(f(\beta))$ for a quantity bounded in magnitude by $C|f(\beta)|$ for a constant $C = C(\mu_D, \delta) > 0$, and interpret this entrywise for vectors and matrices. We again diagonalize

$$\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} = (y_1 \ y_2) \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \begin{pmatrix} y_1^\top \\ y_2^\top \end{pmatrix},$$

where (y_1, y_2) are the two unit eigenvectors. Then by definition,

$$\mathcal{F}(\gamma, v, \rho) = y_1 y_1^\top \cdot \mathcal{F}(\alpha_1) + y_2 y_2^\top \cdot \mathcal{F}(\alpha_2)$$

where $\mathcal{F}(\alpha)$ is the univariate function defined in (3.3). Then $\|\mathcal{F}(\gamma, v, \rho)\| = \max(|\mathcal{F}(\alpha_1)|, |\mathcal{F}(\alpha_2)|)$, and also $\left\| \left(\begin{pmatrix} \gamma(s) & v(s) \\ v(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\| = \max(|\alpha_1 - x|^{-1}, |\alpha_2 - x|^{-1})$, so the bound for $\|\mathcal{F}(\gamma(s), v(s), \rho(s))\|$ follows directly from Lemma 3.3.

To bound the derivatives, let us write $\mathcal{F}(\gamma, v, \rho)$ in a more explicit form that parallels (3.3):

$$\mathcal{F}(\gamma, v, \rho) = \mathcal{F}_{22}(\gamma, v, \rho) - \mathcal{F}_{12}(\gamma, v, \rho)^\top \mathcal{F}_{11}(\gamma, v, \rho)^{-1} \mathcal{F}_{12}(\gamma, v, \rho) \quad (4.18)$$

where

$$\mathcal{F}_{11}(\gamma, v, \rho) = \int \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix}^{-1} \otimes \begin{pmatrix} 1 & \lambda(x) \\ \lambda(x) & \lambda(x)^2 \end{pmatrix} \mu_{\bar{D}}(dx) \in \mathbb{R}^{4 \times 4}, \quad (4.19)$$

$$\mathcal{F}_{12}(\gamma, v, \rho) = \int \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix}^{-1} \otimes \begin{pmatrix} \theta(x) \\ \lambda(x)\theta(x) \end{pmatrix} \mu_{\bar{D}}(dx) \in \mathbb{R}^{4 \times 2}, \quad (4.20)$$

$$\mathcal{F}_{22}(\gamma, v, \rho) = \int \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix}^{-1} \theta(x)^2 \mu_{\bar{D}}(dx) \in \mathbb{R}^{2 \times 2}. \quad (4.21)$$

and $\lambda(x)$ and $\theta(x)$ were defined in (3.5). To verify this form, recall the univariate $\mathcal{F}_{11}(\gamma)$, $\mathcal{F}_{12}(\gamma)$, $\mathcal{F}_{22}(\gamma)$ defined in (3.6)–(3.8) and observe that

$$\begin{aligned} \mathcal{F}_{11}(\gamma, v, \rho) &= \int \left(y_1 y_1^\top \otimes \frac{1}{\alpha_1 - x} \begin{pmatrix} 1 & \lambda(x) \\ \lambda(x) & \lambda(x)^2 \end{pmatrix} \right. \\ &\quad \left. + y_2 y_2^\top \otimes \frac{1}{\alpha_2 - x} \begin{pmatrix} 1 & \lambda(x) \\ \lambda(x) & \lambda(x)^2 \end{pmatrix} \right) \mu_{\bar{D}}(dx) \\ &= y_1 y_1^\top \otimes \mathcal{F}_{11}(\alpha_1) + y_2 y_2^\top \otimes \mathcal{F}_{11}(\alpha_2). \end{aligned}$$

Then, using $y_1 y_1^\top \cdot y_2 y_2^\top = y_2 y_2^\top \cdot y_1 y_1^\top = 0$,

$$\mathcal{F}_{11}(\gamma, \nu, \rho)^{-1} = y_1 y_1^\top \otimes \mathcal{F}_{11}(\alpha_1)^{-1} + y_2 y_2^\top \otimes \mathcal{F}_{11}(\alpha_2)^{-1}. \quad (4.22)$$

Similarly

$$\mathcal{F}_{12}(\gamma, \nu, \rho) = y_1 y_1^\top \otimes \mathcal{F}_{12}(\alpha_1) + y_2 y_2^\top \otimes \mathcal{F}_{12}(\alpha_2), \quad (4.23)$$

$$\mathcal{F}_{22}(\gamma, \nu, \rho) = y_1 y_1^\top \otimes \mathcal{F}_{22}(\alpha_1) + y_2 y_2^\top \otimes \mathcal{F}_{22}(\alpha_2). \quad (4.24)$$

Combining these yields the identity (4.18).

As in the proof of Lemma 3.3, we use again the abbreviations $\lambda \equiv \beta(1 - q_*)$ and $\theta \equiv \beta^2(1 - q_*)$. Then, from the forms (4.19–4.21), for $k = 1, 2$,

$$\partial_s^k \mathcal{F}_{11}(\gamma(s)) = O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \lambda \\ \lambda & \lambda^2 \end{pmatrix}\right) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left\| \partial_s^k \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\| \quad (4.25)$$

$$\partial_s^k \mathcal{F}_{12}(\gamma(s)) = O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} \theta \\ \lambda\theta \end{pmatrix}\right) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left\| \partial_s^k \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\| \quad (4.26)$$

$$\partial_s^k \mathcal{F}_{22}(\gamma(s)) = O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \theta^2\right) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left\| \partial_s^k \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\|. \quad (4.27)$$

Writing $\mathcal{F}'_{11} = \partial_s \mathcal{F}_{11}(\gamma(s), \nu(s), \rho(s))$ and similarly for the other terms,

$$\mathcal{F}' = \mathcal{F}'_{22} - \mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12} - \mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12} + \mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12}.$$

Taking the product of (4.22) and (4.23),

$$\begin{aligned} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} &= y_1 y_1^\top \otimes [\mathcal{F}_{11}(\alpha_1)^{-1} \mathcal{F}_{12}(\alpha_1)] + y_2 y_2^\top \otimes [\mathcal{F}_{11}(\alpha_2)^{-1} \mathcal{F}_{12}(\alpha_2)] \\ &= O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \beta^{-2} \begin{pmatrix} \lambda^2 \theta \\ \lambda \theta \end{pmatrix}\right), \end{aligned} \quad (4.28)$$

where the second equality applies (3.40) and (3.38) from Lemma 3.3. Then, applying also (4.25–4.27) for $k = 1$, and $\lambda^2 = O(\beta^2)$ and $\theta^2 = \beta^4(1 - q_*)^2$, we obtain

$$\|\mathcal{F}'\| = O(\beta^4(1 - q_*)^2) \cdot \left\| \partial_s \left(\begin{pmatrix} \gamma(s) & \nu(s) \\ \nu(s) & \rho(s) \end{pmatrix} - xI \right)^{-1} \right\|$$

which is the desired bound for $\|\partial_s \mathcal{F}(\gamma(s), \nu(s), \rho(s))\|$.

For the second derivative, similar to (3.41), we have

$$\begin{aligned} \mathcal{F}'' &= \mathcal{F}''_{22} - \mathcal{F}''_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}_{12} - \mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}''_{12} + \mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}''_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} \\ &\quad - 2\left(\mathcal{F}_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} + \mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12} + \mathcal{F}'_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12} \right. \\ &\quad \left. + \mathcal{F}_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}'_{12}\right). \end{aligned}$$

Bounding the terms on the first line using (4.28) and (4.25–4.27) for $k = 2$, and applying for the second line

$$\begin{aligned} & \mathcal{F}_{12}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} + \mathcal{F}'_{12}{}^\top \mathcal{F}_{11}^{-1} \mathcal{F}'_{12} + \mathcal{F}'_{12}{}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}_{12} + \mathcal{F}_{12}^\top [\mathcal{F}_{11}^{-1}]' \mathcal{F}'_{12} \\ &= \left[\mathcal{F}'_{12} - \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} \right]^\top \mathcal{F}_{11}^{-1} \left[\mathcal{F}'_{12} - \mathcal{F}'_{11} \mathcal{F}_{11}^{-1} \mathcal{F}_{12} \right] \geq 0, \end{aligned}$$

we obtain

$$\mathcal{F}'' \leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\bar{D}})} \left\| \partial_s^2 \left(\binom{\gamma(s)}{\nu(s)} \binom{\nu(s)}{\rho(s)} - xI \right)^{-1} \right\|$$

which is the desired upper bound for $\partial_s^2 \mathcal{F}(\gamma(s), \nu(s), \rho(s))$. $\square$

Lemma 4.4 For each $\iota \in \{u, v, w, k, \ell, m, p, \gamma, \nu, \rho, U, W, K, M, P\}$, we have

$$\Phi_{2,\iota}(u_*, \dots, P_*) = 2\Psi_{\text{RS}}, \quad \partial_\iota \Phi_{2,\iota}(u_*, \dots, P_*) = 0.$$

For $\iota \in \{V, L\}$, we have

$$\lim_{t \rightarrow \infty} \|\partial_\iota \Phi_{2,\iota}(u_*, \dots, P_*)\| = 0.$$

Proof At $P_* = 0$, we have $\mathcal{L}(0, y, z) = \log(e^y + e^{-y})(e^z + e^{-z}) = \log 2 \cosh y + \log 2 \cosh z$. Recalling the definition of $\mathcal{F}(\gamma, \nu, \rho)$ by the spectral calculus, at $\nu_* = 0$, we have $\mathcal{F}(\gamma, 0, \rho) = \text{diag}(\mathcal{F}(\gamma), \mathcal{F}(\rho))$, where $\mathcal{F}(\cdot)$ is the function defined in (3.3). Hence

$$\text{Tr } \mathcal{F}(\gamma, 0, \rho) \cdot B(v, w, \ell, m) = \mathcal{F}(\gamma) \cdot \|v - \kappa_*^{-1/2} w\|^2 + \mathcal{F}(\rho) \cdot \|\ell - \kappa_*^{-1/2} m\|^2.$$

At the above $\nu_*, \ell_*, w_*, m_*, p_*$, from the computation in Lemma 3.4, we also have

$$\begin{aligned} 1 - \|v_*\|^2 - \|w_*\|^2 &= 1 - (1 + \kappa_*)(1 - q_*)^2 \delta_* = 1 - q_*, \\ 1 - \|\ell_*\|^2 - \|m_*\|^2 &= 1 - (1 + \kappa_*)(1 - q_*)^2 \delta_* = 1 - q_*, \\ p_* - v_*^\top \ell_* - w_*^\top m_* &= q_* - (1 + \kappa_*)(1 - q_*)^2 \delta_* = 0, \end{aligned}$$

and

$$\mathcal{H}(\gamma, 0, \rho; 1 - q_*, 0, 1 - q_*) = \mathcal{H}(\gamma, 1 - q_*) + \mathcal{H}(\rho, 1 - q_*)$$

where $\mathcal{H}(\cdot, \cdot)$ on the right is the function (3.4). Thus,

$$\begin{aligned} \Phi_{2,\iota}(u_*, \dots, P_*) &= \Phi_{1,\iota}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) \\ &\quad + \Phi_{1,\iota}(k_*, \ell_*, m_*; \rho_*, K_*, L_*, M_*) = 2\Psi_{\text{RS}}, \end{aligned}$$

the second equality applying Lemma 3.4. Also, in view of (4.17),

$$\begin{aligned}\partial_x \mathcal{L}(x, y, z)|_{x=0} &= \tanh(y) \tanh(z), \quad \partial_y \mathcal{L}(x, y, z)|_{x=0} \\ &= \tanh(y), \quad \partial_z \mathcal{L}(x, y, z)|_{x=0} = \tanh(z).\end{aligned}\quad (4.29)$$

Furthermore,

$$\begin{aligned}\partial_\gamma \mathcal{F}(\gamma, v, \rho)|_{v=0} &= \partial_\gamma \mathcal{F}(\gamma) e_1 e_1^\top, \quad \partial_\rho \mathcal{F}(\gamma, v, \rho)|_{v=0} = \partial_\rho \mathcal{F}(\rho) e_2 e_2^\top, \\ \partial_\gamma \mathcal{H}(\gamma, v, \rho; a, b, c)|_{v=0} &= \partial_\gamma \mathcal{H}(\gamma, a), \quad \partial_\rho \mathcal{H}(\gamma, v, \rho; a, b, c)|_{v=0} = \partial_\rho \mathcal{H}(\rho, c).\end{aligned}$$

Using these identities and applying Lemma 3.4, we obtain

$$\begin{aligned}\partial_t \Phi_{2,t}(u_*, \dots, P_*) &= \partial_t \Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) = 0 \text{ for } t \in \{u, \gamma, U, W\} \\ \partial_V \Phi_{2,t}(u_*, \dots, P_*) &= \partial_V \Phi_{1,t}(u_*, v_*, w_*; \gamma_*, U_*, V_*, W_*) = o_t(1), \\ \partial_t \Phi_{2,t}(u_*, \dots, P_*) &= \partial_t \Phi_{1,t}(k_*, \ell_*, m_*; \rho_*, K_*, L_*, M_*) = 0 \text{ for } t \in \{k, \rho, K, M\} \\ \partial_L \Phi_{2,t}(u_*, \dots, P_*) &= \partial_L \Phi_{1,t}(k_*, \ell_*, m_*; \rho_*, K_*, L_*, M_*) = o_t(1),\end{aligned}$$

where $o_t(1)$ denotes a length- t vector satisfying $\lim_{t \rightarrow \infty} \|o_t(1)\| = 0$.

It remains to check the derivatives in $\iota \in \{v, w, \ell, m, p, v, P\}$. Since $v_* = \kappa_*^{-1/2} w_*$ and $\ell_* = \kappa_*^{-1/2} m_*$, we have $B(v_*, w_*, \ell_*, m_*) = 0$ and $\partial_t B(v_*, w_*, \ell_*, m_*) = 0$ for each $\iota \in \{v, w, \ell, m\}$. Writing $a_* = c_* = 1 - q_*$ and $b_* = 0$, we have

$$\partial_b \mathcal{H}(\gamma_*, v_*, \rho_*; a_*, b_*, c_*) = \partial_v \mathcal{H}(\gamma_*, v_*, \rho_*; a_*, b_*, c_*) = 0$$

by the identities $v_* = b_* = 0$ and

$$\partial_y \log \det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \Big|_{y=0} = 0. \quad (4.30)$$

Then it follows directly that

$$\partial_p \Phi_{2,t}(u_*, \dots, P_*) = 0, \quad \partial_v \Phi_{2,t}(u_*, \dots, P_*) = 0.$$

Furthermore,

$$\begin{aligned}\partial_a \mathcal{H}(\gamma_*, v_*, \rho_*; a_*, b_*, c_*) &= \partial_a \mathcal{H}(\gamma_*, a_*) = \bar{R}(1 - q_*), \\ \partial_c \mathcal{H}(\gamma_*, v_*, \rho_*; a_*, b_*, c_*) &= \partial_c \mathcal{H}(\rho_*, c_*) = \bar{R}(1 - q_*),\end{aligned}$$

where the latter two equalities follow from (3.42). Applying also (3.44–3.45) and the identity $\lambda_* = \bar{R}(1 - q_*) + (1 - q_*)^{-1}$, we have

$$\begin{aligned}\partial_v \Phi_{2,t}(u_*, \dots, P_*) &= -V_* + \bar{R}(1 - q_*) \kappa_*^{-1/2} w_* - \bar{R}(1 - q_*) v_* = 0, \\ \partial_w \Phi_{2,t}(u_*, \dots, P_*) &= -W_* + \bar{R}(1 - q_*) \kappa_*^{-1/2} v_* + (\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}) w_*\end{aligned}$$

$$-\bar{R}(1 - q_*)w_* = 0;$$

similarly $\partial_\ell \Phi_{2,t}(u_*, \dots, P_*) = 0$ and $\partial_m \Phi_{2,t}(u_*, \dots, P_*) = 0$.

Finally, for the derivative in P , applying (4.29), together with

$$\begin{aligned} U_* \cdot H + V_*^\top \Delta_t^{-1/2}(X_1, \dots, X_t) + \kappa_*^{-1/2} W_*^\top \Delta_t^{-1/2}(Y_1, \dots, Y_t) \\ = K_* \cdot H + L_*^\top \Delta_t^{-1/2}(X_1, \dots, X_t) + \kappa_*^{-1/2} M_*^\top \Delta_t^{-1/2}(Y_1, \dots, Y_t) = H + Y_t, \end{aligned}$$

$p_* = q_*$, and the definition of q_* from (2.9), we obtain

$$\partial_P \Phi_{2,t}(u_*, \dots, P_*) = \mathbb{E}[\tanh(H + Y_t)^2] - p_* = \mathbb{E}[\tanh(H + \sigma_* G)^2] - q_* = 0.$$

□

Lemma 4.5 For a constant $\beta_0 = \beta_0(\mu_D) > 0$ and any $\beta \in (0, \beta_0)$,

$$\lim_{t \rightarrow \infty} \Psi_{2,t} = 2\Psi_{\text{RS}}.$$

Proof The proof is analogous to that of Lemma 3.5. We establish separately

$$\liminf_{t \rightarrow \infty} \Psi_{2,t} \geq 2\Psi_{\text{RS}}, \quad (4.31)$$

$$\limsup_{t \rightarrow \infty} \Psi_{2,t} \leq 2\Psi_{\text{RS}}. \quad (4.32)$$

Recall the max-min form of $\Psi_{2,t}$ in (4.7). For the lower bound (4.31), we specialize the outer supremum of $\Phi_{2,t}$ to $(u, v, w, k, \ell, m, p) = (u_*, \tilde{v}_*, w_*, k_*, \tilde{\ell}_*, m_*, p_*)$ where

$$\begin{aligned} \tilde{v}_* &= v_* + (1 - q_*)[\Delta_t^{-1/2}\delta_t - \Delta_t^{1/2}e_t] = v_* + o_t(1) \\ \tilde{\ell}_* &= \ell_* + (1 - q_*)[\Delta_t^{-1/2}\delta_t - \Delta_t^{1/2}e_t] = \ell_* + o_t(1) \end{aligned} \quad (4.33)$$

and $\delta_t = (\delta_{1,t+1}, \dots, \delta_{t,t+1})$ is as defined in the proof of Lemma 3.5. Note that

$$\Phi_{2,t}(u_*, \tilde{v}_*, w_*, k_*, \tilde{\ell}_*, m_*, p_*; \gamma, \dots, P) = X(U, V, W, K, L, M, P) + Y(\gamma, v, \rho),$$

where both X and Y are convex functions. (Convexity of Y holds by Lemma 4.3(a).) Then

$$\Psi_{2,t} \geq \inf_{U, K, P \in \mathbb{R}, V, W, L, M \in \mathbb{R}^t} X(U, V, W, K, L, M, P) + \inf_{(\gamma, \rho, v) \in \mathcal{D}_+} Y(\gamma, v, \rho). \quad (4.34)$$

Under the above definitions of $\tilde{v}_*$ and $\tilde{\ell}_*$, the point $(U_*, V_*, W_*, K_*, L_*, M_*, P_*)$ is an exact stationary point of X hence its minimizer. For the minimum of Y , note that $Y(\gamma, v, \rho) = \frac{1}{2} \text{Tr } \mathcal{F}(\gamma, v, \rho) \tilde{B} + \frac{1}{2} \mathcal{H}(\gamma, v, \rho; \tilde{a}_*, \tilde{b}_*, \tilde{c}_*)$, where we denote $\tilde{B} =$

$B(\tilde{v}_*, w_*, \tilde{\ell}_*, m_*) = o_t(1)$, $\tilde{a}_* = \tilde{c}_* = 1 - \|\tilde{v}_*\|^2 - \|w_*\|^2 = 1 - \|\tilde{\ell}_*\|^2 - \|m_*\|^2$, and $\tilde{b}_* = p_* - \tilde{v}_*^\top \tilde{\ell}_* - w_*^\top m_* = q_* - \|\tilde{v}_*\|^2 - \|w_*\|^2$. Recalling the identity $\bar{G}(\gamma_*) = 1 - q_* = 1 - \|v_*\|^2 - \|w_*\|^2$, we have for each $\iota \in \{\gamma, v, \rho\}$,

$$\partial_t \mathcal{H}(\gamma_*, v_*, \rho_*; \tilde{a}_*, \tilde{b}_*, \tilde{c}_*) = \|v_*\|^2 - \|\tilde{v}_*\|^2 = o_t(1).$$

Therefore $\nabla Y(\gamma^*, v^*, \rho^*) = o_t(1)$. Furthermore, there exist some constants $c, \delta > 0$ independent of t , such that $\nabla Y(\gamma, v, \rho) \geq cI$ whenever $\|(\gamma, v, \rho) - (\gamma_*, v_*, \rho_*)\| \leq \delta$. Applying Proposition C.2 yields $\inf_{(\gamma, v, \rho) \in \mathcal{D}_+} Y(\gamma, v, \rho) \geq Y(\gamma_*, v_*, \rho_*) + o_t(1)$. Note that $\|\nabla_{v, \ell} \Phi_{2,t}(u_*, v, w_*, k_*, \ell, m_*; \gamma_*, \dots, P_*)\| \leq C$ for all $\|v - v_*\| \leq \delta$ and $\|\ell - \ell_*\| \leq \delta$, where C, δ are constants independent of t . In view of (4.33) and (4.34), we then have

$$\Psi_{2,t} \geq \Phi_{2,t}(u_*, \dots, P_*) + o_t(1) = 2\Psi_{\text{RS}} + o_t(1),$$

implying the lower bound (4.31).

For the upper bound (4.32), let $A \in \mathbb{R}^{2 \times 2}$ be a symmetric matrix satisfying $0 < A < \bar{G}(\bar{d}_+)I$. We define by the spectral calculus

$$\begin{pmatrix} \gamma(A) & v(A) \\ v(A) & \rho(A) \end{pmatrix} = \bar{G}^{-1}(A) = \bar{R}(A) + A^{-1} \succ \bar{d}_+ I. \quad (4.35)$$

We then specialize the inner infimum of $\Phi_{2,t}$ to

$$\begin{aligned} (\gamma, v, \rho) &= (\gamma(A(p, v, w, \ell, m)), v(A(p, v, w, \ell, m)), \rho(A(p, v, w, \ell, m))), \\ U &= U_* = 1, \quad K = K_* = 1, \quad V = V(v) = \beta^{1/2}(v - v_*), \\ L &= L(\ell) = \beta^{1/2}(\ell - \ell_*), \\ W &= W(w) = \beta^{1/2}(w - w_*) + W_*, \quad M = M(m) = \beta^{1/2}(m - m_*) + M_*, \\ P &= P(p) = \beta^{1/2}(p - p_*), \end{aligned}$$

where $A(p, v, w, \ell, m)$ is in (4.2). Note that the above (γ, v, ρ) is well defined for any (p, v, w, ℓ, m) in the domain $\mathcal{V}$ defined in (4.16), provided that $\beta < G(d_+)/2$, in which case $\bar{G}(\bar{d}_+) > 2$. Indeed, since $p \in [-1, 1]$, we have $A(p, v, w, \ell, m) \leq \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \leq 2I < \bar{G}(\bar{d}_+)I$.

At $(p, v, w, \ell, m) = (p_*, v_*, w_*, \ell_*, m_*)$, this specialization gives $(\gamma, v, \rho) = (\gamma_*, v_*, \rho_*)$, because $A(p_*, v_*, w_*, \ell_*, m_*) = (1 - q_*)I_{2 \times 2}$, $(V, L, W, M) = (V_*, L_*, W_*, M_*)$, and $P = P_* = 0$. Now write the function $\Phi_{2,t}$ under this specialization (which no longer depends on u or k , thanks to the choice of U and K) as

$$\tilde{\Phi}_{2,t}(p, v, w, \ell, m) = \text{I} + \text{II} + \text{III} + \text{IV}$$

where

$$\text{I} = \mathbb{E} \left[\mathcal{L} \left(P(p), H + V(v)^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} W(w)^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t), \right. \right.$$

$$\begin{aligned}
& \mathbf{H} + L(\ell)^\top \Delta_t^{-1/2} (X_1, \dots, X_t) + \kappa_*^{-1/2} M(m)^\top \Delta_t^{-1/2} (Y_1, \dots, Y_t) \Big] \\
\text{II} = & -v^\top V(v) - w^\top W(w) - \ell^\top L(\ell) - m^\top M(m) - p \cdot P(p) \\
& + \bar{R}(1 - q_*) \kappa_*^{-1/2} (v^\top w + \ell^\top m) + \frac{\lambda_* - \bar{R}(1 - q_*) \kappa_*^{-1}}{2} (\|w\|^2 + \|m\|^2) \\
\text{III} = & \frac{1}{2} \text{Tr} \mathcal{F} \Big(\gamma(A(p, v, w, \ell, m)), v(A(p, v, w, \ell, m)), \rho(A(p, v, w, \ell, m)) \Big) \cdot B(v, w, \ell, m) \\
\text{IV} = & \frac{1}{2} \mathcal{H} \Big(\gamma(A(p, v, w, \ell, m)), v(A(p, v, w, \ell, m)), \rho(A(p, v, w, \ell, m)) \Big); \\
& 1 - \|v\|^2 - \|w\|^2, p - v^\top \ell - w^\top m, 1 - \|\ell\|^2 - \|m\|^2 \Big).
\end{aligned}$$

Recalling the definition of $\Psi_{2,t}$ in (4.7) and the domain $\mathcal{V}$ in (4.16), we have

$$\Psi_{2,t} \leq \sup_{(p,v,w,\ell,m) \in \mathcal{V}} \tilde{\Phi}_{2,t}(p, v, w, \ell, m). \quad (4.36)$$

Note that $\mathcal{V}$ is a convex set, since $A(p, v, w, \ell, m) \succ 0$ is equivalent to $\begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \prec \begin{pmatrix} v & \ell \\ w & m \end{pmatrix}^\top \begin{pmatrix} v & \ell \\ w & m \end{pmatrix}$.

For all $\beta \in (0, \beta_0)$ and sufficiently small $\beta_0 = \beta_0(\mu_D) > 0$, we claim that $\tilde{\Phi}_{2,t}$ is globally concave on $\mathcal{V}$. As in Lemma 3.5, we write $O(\beta^k)$ for a scalar, vector, or matrix of norm at most $C\beta^k$, uniformly over $\mathcal{V}$, for a constant $C = C(\mu_D) > 0$ depending only on μ_D .

For I, recall from (4.17) that $\mathcal{L}(x, y, z) = \log(4\langle e^{x\sigma\tau + y\sigma + z\tau} \rangle)$, where $\langle \cdot \rangle$ is the mean with respect to σ and τ independent and uniform on $\{+1, -1\}$. Thus its Hessian coincides with the covariance matrix of the random vector $r = (\sigma\tau, \sigma, \tau)$

$$\nabla^2 \mathcal{L}(x, y, z) = \langle rr^\top \rangle' - \langle r \rangle' \langle r \rangle'^\top,$$

under the tilted distribution of (σ, τ) defined by $\langle f(\sigma, \tau) \rangle' = \frac{\langle f(\sigma, \tau) e^{x\sigma\tau + y\sigma + z\tau} \rangle}{\langle e^{x\sigma\tau + y\sigma + z\tau} \rangle}$. So

$$0 \leq \nabla^2 \mathcal{L}(x, y, z) \leq 3I. \quad (4.37)$$

Recall the random vector $Z_t \in \mathbb{R}^{2t}$ defined in (3.54), which satisfies $\mathbb{E}[Z_t Z_t^\top] = I$. For any unit vector $q = (a, b, c) \in \mathbb{R}^{4t+1}$ where $a \in \mathbb{R}$ and $b, c \in \mathbb{R}^{2t}$, define $\eta = (a, b^\top Z_t, c^\top Z_t) \in \mathbb{R}^3$. Then $q^\top (\nabla_{p,v,w,\ell,m}^2 \mathbf{I}) q = \beta \cdot \mathbb{E}[\eta^\top (\nabla^2 \mathcal{L}) \eta]$, where $\nabla^2 \mathcal{L}$ is evaluated in the same point as in the definition of I. Applying (4.37), we have $0 \leq q^\top (\nabla_{p,v,w,\ell,m}^2 \mathbf{I}) q \leq 3\beta \cdot \mathbb{E}[\|\eta\|^2] = 3\beta$, and thus $0 \leq \nabla_{p,v,w,\ell,m}^2 \mathbf{I} \leq 3\beta \cdot I$.

For II, by the same arguments as in Lemma 3.5, we have $\nabla_{p,v,w,\ell,m}^2 \text{II} = -2\beta^{1/2} I + O(\beta)$.

For III, consider any scalar linear parametrization

$$(p(s), v(s), w(s), \ell(s), m(s))_{s \in \mathbb{R}} = (p, v, w, \ell, m) + s \cdot (p', v', w', \ell', m')$$

where $\|(p', v', w', \ell', m')\| = 1$. Write as shorthand the following 2×2 matrices

$$A(s) = A(p(s), v(s), w(s), \ell(s), m(s)), \quad B(s) = B(v(s), w(s), \ell(s), m(s)),$$

and $\mathcal{F}(s) = \mathcal{F}(\gamma(A(s)), \nu(A(s)), \rho(A(s)))$. As in Lemma 3.5, it is easily checked from the definitions (4.2) and (4.5) and the bound $\kappa_* = O(\beta^{-2}(1 - q_*)^{-2})$ in Proposition 2.5 that at $s = 0$, we have

$$\begin{aligned} \|A(s)\|, \|\partial_s A(s)\|, \|\partial_s^2 A(s)\| &= O(1), \quad \|B(s)\|, \|\partial_s B(s)\|, \|\partial_s^2 B(s)\| \\ &= O(\beta^{-2}(1 - q_*)^{-2}). \end{aligned} \quad (4.38)$$

We may write

$$\begin{aligned} &(p', v', w', \ell', m')^\top \cdot \partial_{p,v,w,\ell,m}^2 \text{III} \cdot (p', v', w', \ell', m') \\ &= \partial_s^2 \text{III} \Big|_{s=0} = \frac{1}{2} \text{Tr} \left(\partial_s^2 \mathcal{F}(s) \cdot B(s) + 2\partial_s \mathcal{F}(s) \cdot \partial_s B(s) + \mathcal{F}(s) \cdot \partial_s^2 B(s) \right) \Big|_{s=0}. \end{aligned}$$

Applying (4.35) and Lemma 4.3, we have analogous to Lemma 3.5 that

$$\begin{aligned} \left| \text{Tr} \mathcal{F}(s) \cdot \partial_s^2 B(s) \right| &\leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\tilde{D}})} \left\| \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\| \cdot \left\| \partial_s^2 B(s) \right\| \\ \left| \text{Tr} \partial_s \mathcal{F}(s) \cdot \partial_s B(s) \right| &\leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\tilde{D}})} \left\| \partial_s \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\| \cdot \left\| \partial_s B(s) \right\| \\ \text{Tr} \partial_s^2 \mathcal{F}(s) \cdot B(s) &\leq O(\beta^4(1 - q_*)^2) \cdot \sup_{x \in \text{supp}(\mu_{\tilde{D}})} \left\| \partial_s^2 \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\| \cdot \left\| B(s) \right\| \end{aligned}$$

where the last inequality above applies $B(s) \geq 0$ and holds without absolute value on the left side. Applying the series expansion (3.61), for some $\beta_0 = \beta_0(\mu_D) > 0$ and all $\beta \in (0, \beta_0)$, we have

$$\left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} = \sum_{k \geq 0} c_k(x) A(s)^{k+1}$$

as a convergent matrix series. Then, differentiating in s term-by-term,

$$\begin{aligned} &\left\| \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\|, \left\| \partial_s \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\|, \left\| \partial_s^2 \left(\tilde{G}^{-1}(A(s)) - xI \right)^{-1} \right\| \\ &= O(1), \end{aligned}$$

so $\nabla_{p,v,w,\ell,m}^2 \text{III} < C\beta^2$ for a constant $C = C(\mu_D) > 0$.

For IV, observe that by Proposition 2.9(b), we have

$$\text{IV} = \frac{1}{2} \text{Tr} f(A(p, v, w, \ell, m)), \quad f(\alpha) \triangleq \int_0^\alpha \bar{R}(z) dz.$$

Then similarly

$$(p', v', w', \ell', m')^\top \nabla_{p,v,w,\ell,m}^2 \text{IV} \cdot (p', v', w', \ell', m') = \partial_s^2 \text{IV} \Big|_{s=0}$$

$$= \frac{1}{2} \operatorname{Tr} \partial_s^2 f(A(s)) \Big|_{s=0}.$$

For all $\beta \in (0, \beta_0)$, we may integrate (2.7) term by term to write $f(A(s))$ as the convergent matrix series

$$f(A(s)) = \sum_{k \geq 2} \frac{\bar{\kappa}_k}{k} A(s)^k,$$

where $|\bar{\kappa}_k| \leq (C\beta)^k$ for some $C = C(\mu_D)$. Differentiating in s at $s = 0$ and using (4.38), we have for some constant C' independent of t ,

$$\left\| \partial_s^2 f(A(s)) \Big|_{s=0} \right\| \leq \sum_{k \geq 2} (C'\beta)^k = O(\beta^2).$$

Then also $\nabla_{p,v,w,\ell,m}^2 \mathbf{IV} = O(\beta^2)$.

Combining the above, $\nabla_{p,v,w,\ell,m}^2 \tilde{\Phi}_{2,t} < -2\beta^{1/2} I_{(4t+1) \times (4t+1)} + O(\beta)$. Applying Lemma 4.4, we have that $\nabla_t \tilde{\Phi}_{2,t}(p_*, v_*, w_*, \ell_*, m_*) = 0$ for $\iota = p, w, m$ and $o_t(1)$ for $\iota = v, \ell$. Thus, recalling that $\mathcal{V}$ is convex and applying Proposition C.2,

$$\begin{aligned} \sup_{(p,v,w,\ell,m) \in \mathcal{V}} \tilde{\Phi}_{2,t}(v, w) &= \tilde{\Phi}_{2,t}(p_*, v_*, w_*, \ell_*, m_*) + o_t(1) \\ &= \Phi_{2,t}(u_*, \dots, P_*) + o_t(1) = 2\Psi_{\text{RS}} + o_t(1). \end{aligned}$$

Then $\Psi_{2,t} \leq 2\Psi_{\text{RS}} + o_t(1)$ in view of (4.36), proving the upper bound (4.32). $\square$

Lemma 4.1 follows immediately from Lemmas 4.2 and 4.5.

5 Proof of Theorem 1.3

Finally, using Lemmas 3.1 and 4.1, we conclude the proof of Theorem 1.3.

Proof We first show concentration of $n^{-1} \log Z$ around its mean: Writing $\sigma^\top J \sigma = \operatorname{Tr} \sigma \sigma^\top O^\top D O$ and viewing $Z = Z(O)$ as a function of $O \in \mathbb{R}^{n \times n}$, we have

$$\partial_O \log Z(O) = \frac{1}{Z} \sum_{\sigma \in \{+1, -1\}^n} \beta \sigma \sigma^\top O^\top D \cdot \exp \left(\frac{\beta}{2} \sigma^\top J \sigma + h^\top \sigma \right).$$

Then the Frobenius norm of this derivative (for any $O \in \mathbb{R}^{n \times n}$) is bounded as

$$\begin{aligned} \|\partial_O \log Z(O)\|_{\text{F}} &\leq \max_{\sigma \in \{+1, -1\}^n} \|\beta \sigma \sigma^\top O^\top D\|_{\text{F}} \\ &= \sqrt{n\beta^2 \cdot \sigma^\top O^\top D^2 O \sigma} \leq n\beta \|D\|_{\text{op}} \|O\|_{\text{op}}. \end{aligned}$$

So for any $O, O' \in \mathbb{O}(n)$, integrating along a linear path from O to O' in $\mathbb{R}^{n \times n}$,

$$|\log Z(O) - \log Z(O')| \leq \|O - O'\|_F \cdot n\beta \|D\|_{\text{op}}.$$

We apply Gromov's concentration inequality in the form of [2, Corollary 4.4.28]: Let $Q \sim \mathbb{SO}(n)$ and $O \sim \mathbb{O}(n)$ be independent. Then for any $\varepsilon > 0$,

$$\mathbb{P} \left[\left| \frac{1}{n} \log Z(O) - \frac{1}{n} \mathbb{E}[\log Z(OQ) \mid O] \right| > \varepsilon \right] \leq 2 \exp \left(-\frac{\left(\frac{n}{4} - \frac{1}{2}\right) \varepsilon^2}{2\beta^2 \|D\|_{\text{op}}^2} \right). \quad (5.1)$$

For any diagonal sign matrix P with diagonal entries $\{+1, -1\}$, note that $O^\top D O = O^\top P^\top D P O$, so that $Z(O) = Z(PO)$. Then for any fixed $O \in \mathbb{O}(n)$, the conditional expectation $\mathbb{E}[\log Z(OQ) \mid O]$ over $Q \sim \text{Haar}(\mathbb{SO}(n))$ coincides with that over $Q \sim \text{Haar}(\mathbb{O}(n))$, which in turn equals the unconditional expectation $\mathbb{E}[\log Z(O)]$ over $O \sim \text{Haar}(\mathbb{O}(n))$ by the invariance of the Haar measure. Thus under Assumption 1.1(b), for any $\varepsilon > 0$ and a constant $c = c(\varepsilon, \beta, \mu_D) > 0$,

$$\mathbb{P} \left[\left| \frac{1}{n} \log Z - \frac{1}{n} \mathbb{E} \log Z \right| \leq \varepsilon \right] \geq 1 - e^{-cn}. \quad (5.2)$$

The remainder of the argument is the same as in [8], but for convenience we reproduce it here. Fix any $\varepsilon > 0$. First observe that by Lemma 3.1, for a large enough iteration $t = t(\varepsilon)$, almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}_t] \leq \Psi_{\text{RS}} + \varepsilon.$$

Since

$$\begin{aligned} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}_t] &\leq \log 2 + \max_{\sigma \in \{+1, -1\}^n} \frac{1}{n} \left(\frac{\beta}{2} \sigma^\top J \sigma + h^\top \sigma \right) \\ &\leq \log 2 + \frac{\beta}{2} \|D\|_{\text{op}} + \frac{1}{n} \sum_{i=1}^n |h_i|, \end{aligned}$$

and the right side has a constant upper bound under Assumption 1.1, this and Jensen's inequality yields

$$\frac{1}{n} \mathbb{E} \log Z \leq \mathbb{E} \frac{1}{n} \log \mathbb{E}[Z \mid \mathcal{G}_t] \leq \Psi_{\text{RS}} + 2\varepsilon \text{ for all large } n.$$

For the complementary lower bound, for any $t \geq 1$, Markov's inequality gives

$$\begin{aligned} \mathbb{P} \left[\frac{1}{n} \log Z \geq \Psi_{\text{RS}} - \varepsilon \right] &= \mathbb{E} \left[\mathbb{P} \left[\frac{1}{n} \log Z \geq \Psi_{\text{RS}} - \varepsilon \mid \mathcal{G}_t \right] \right] \\ &\geq \mathbb{P} \left[\mathbb{P} \left[\frac{1}{n} \log Z \geq \Psi_{\text{RS}} - \varepsilon \mid \mathcal{G}_t \right] \geq e^{-cn/2} \right] \cdot e^{-cn/2}, \end{aligned}$$

where we take $c > 0$ to be the constant in (5.2) for this ε . Taking $t = t(\varepsilon)$ large enough and applying Lemma 3.1 again, almost surely

$$\Psi_{\text{RS}} - \varepsilon \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\mathbb{E}[Z | \mathcal{G}_t]}{2}.$$

Then applying also the Paley-Zygmund inequality and Lemma 4.1, for $t = t(\varepsilon, c)$ large enough, almost surely for all large n ,

$$\begin{aligned} \mathbb{P} \left[\frac{1}{n} \log Z \geq \Psi_{\text{RS}} - \varepsilon \middle| \mathcal{G}_t \right] &\geq \mathbb{P} \left[\frac{1}{n} \log Z \geq \frac{1}{n} \log \frac{\mathbb{E}[Z | \mathcal{G}_t]}{2} \middle| \mathcal{G}_t \right] \\ &= \mathbb{P} \left[Z \geq \frac{\mathbb{E}[Z | \mathcal{G}_t]}{2} \middle| \mathcal{G}_t \right] \geq \frac{\mathbb{E}[Z | \mathcal{G}_t]^2}{4\mathbb{E}[Z^2 | \mathcal{G}_t]} \geq e^{-cn/2}. \end{aligned}$$

Then for all large n ,

$$\mathbb{P} \left[\frac{1}{n} \log Z \geq \Psi_{\text{RS}} - \varepsilon \right] \geq 0.99e^{-cn/2} > e^{-cn}. \quad (5.3)$$

Together (5.2) and (5.3) imply

$$\frac{1}{n} \mathbb{E} \log Z \geq \Psi_{\text{RS}} - 2\varepsilon \text{ for all large } n.$$

Thus $n^{-1} \mathbb{E} \log Z \rightarrow \Psi_{\text{RS}}$, and applying again the concentration (5.2) finishes the proof of almost sure convergence by Borel–Cantelli. $\square$

Appendix A. Analysis of AMP

We prove Propositions 1.2, 2.3, and 2.5, followed by Theorem 2.2 and Propositions 2.3 and 2.4.

Proof of Proposition 1.2 Recall $\bar{R}(z) = \beta R(\beta z)$ from (2.5). We note that the statement (2.31) in Proposition 2.5 immediately follows from the series expansion (2.7) for $\bar{R}(z)$, where $\bar{\kappa}_1 = 0$, $\bar{\kappa}_2 = \beta^2$, and $\bar{\kappa}_k = O(\beta^k)$.

Set $t(x) = \tanh(x)^2$. The fixed-point equation (1.3) is equivalently given in (2.9) by $f(q_*) = q_*$, where $f : [0, 1] \rightarrow [0, 1]$ is the function

$$f(q) = \mathbb{E} \left[t \left(H + \sqrt{q \bar{R}'(1-q)} \cdot G \right) \right].$$

Applying Gaussian integration by parts,

$$f'(q) = \mathbb{E} \left[t' \left(H + \sqrt{q \bar{R}'(1-q)} \cdot G \right) \cdot \frac{-q \bar{R}''(1-q) + \bar{R}'(1-q)}{2\sqrt{q \bar{R}'(1-q)}} \cdot G \right]$$

$$= \mathbb{E} \left[t'' \left(H + \sqrt{q \bar{R}'(1-q)} \cdot G \right) \cdot \frac{-q \bar{R}''(1-q) + \bar{R}'(1-q)}{2} \right].$$

We have $|t''(x)| \leq 2$. By (2.31), we have $|\bar{R}'(1-q)| \leq C\beta^2$ and $|\bar{R}''(1-q)| \leq C\beta^3$ for all $q \in [0, 1]$, $\beta \in (0, \beta_0)$, and some constants $C, \beta_0 > 0$ depending only on μ_D . So $|f'(q)| < 1$ for any such β and sufficiently small β_0 . Then $f : [0, 1] \rightarrow [0, 1]$ is contractive and has a unique fixed point $q_* \in [0, 1)$. $\square$

Proof of Proposition 2.1 Note that by Assumption 1.1(b),

$$\bar{G}(z) = \lim_{n \rightarrow \infty} n^{-1} \text{Tr}(zI - \bar{J})^{-1}, \quad -\bar{G}'(z) = \lim_{n \rightarrow \infty} n^{-1} \text{Tr}(zI - \bar{J})^{-2}.$$

Recall, by definition of λ_* in (2.12), that $\bar{G}(\lambda_*) = 1 - q_*$. Then by the definitions of κ_* and Γ in (2.22) and (2.13),

$$\begin{aligned} \kappa_* &= \lim_{n \rightarrow \infty} \text{Tr} \left(\frac{1}{1 - q_*} (\lambda_* I - \bar{D})^{-1} - I \right)^2 \\ &= \frac{1}{(1 - q_*)^2} \left(-\bar{G}'(\lambda_*) \right) - \frac{2}{1 - q_*} \bar{G}(\lambda_*) + 1 = -\frac{1}{(1 - q_*)^2} \bar{G}'(\lambda_*) - 1. \end{aligned}$$

We have $\bar{R}(z) = \bar{G}^{-1}(z) - 1/z$, so that $\bar{G}'(z) = 1/[(\bar{G}^{-1})'(\bar{G}(z))] = 1/[\bar{R}'(\bar{G}(z)) - 1/\bar{G}(z)^2]$. Then

$$\kappa_* = -\frac{1}{(1 - q_*)^2} \cdot \frac{1}{\bar{R}'(1 - q_*) - (1 - q_*)^{-2}} - 1 = \frac{1}{1 - (1 - q_*)^2 \bar{R}'(1 - q_*)} - 1.$$

Substituting $\bar{R}'(1 - q_*) = \sigma_*^2/q_*$ from the definition of σ_*^2 in (2.9), this yields

$$\delta_* = \frac{\sigma_*^2}{\kappa_*} = \frac{\sigma_*^2[1 - (1 - q_*)^2 \sigma_*^2/q_*]}{(1 - q_*)^2 \sigma_*^2/q_*} = \frac{q_*}{(1 - q_*)^2} - \sigma_*^2.$$

The second equality of (2.24) may be checked by expanding the square on the right side, and applying the definition of q_* in (2.9) and Gaussian integration by parts. $\square$

Proof of Proposition 2.1 As already argued, the statement (2.31) follows from (2.7). This implies $\sigma_*^2 = O(\beta^2)$ by its definition in (2.9). Setting $t(x) = \tanh(x)^2$, we have

$$q_* = \mathbb{E}[t(H) + t'(H) \cdot \sigma_* G + t''(H') \cdot (\sigma_*^2 G^2/2)]$$

for some random variable H' between H and $H + \sigma_* G$. Here $|t''(x)| \leq 2$ and $\mathbb{E}[t'(H) \cdot G] = 0$, so $q_* = \mathbb{E}[\tanh(H)^2] + O(\beta^2)$. The remaining statements follow immediately from (2.31) and the forms of $\sigma_*^2, \lambda_*, \kappa_*, \delta_*$ in (2.9), (2.12), (2.23), and (2.24). $\square$

Proof of Theorem 2.2 The AMP algorithm (2.15–2.16) is a particular instance of the more general algorithm studied in [21, Eqs. (4.2–4.3)], whose state evolution is obtained in [21, Theorem 4.3]. We apply this result with the notational identifications

$\mathbf{u}_t \leftrightarrow x^t$, $\mathbf{z}_t \leftrightarrow y^t$, $\mathbf{W} \leftrightarrow \Gamma$, $\Lambda \leftrightarrow \Lambda$, $\mathbf{E} \leftrightarrow h$, $(Z_1, \dots, Z_t, E) \leftrightarrow (Y_1, \dots, Y_t, H)$, and

$$u_{t+1}(Z_1, \dots, Z_t, E) \leftrightarrow f(H, Y_t) \triangleq (1 - q_*)^{-1} \tanh(H + Y_t) - Y_t.$$

Applying the property (2.21) for this function f , the matrix Φ_t of [21, Eq. (4.4)] satisfies

$$\lim_{n \rightarrow \infty} \Phi_t = 0.$$

Furthermore, by the definitions of λ_* and κ_* in (2.12) and (2.22),

$$\frac{1}{n} \text{Tr } \Lambda \rightarrow 0, \quad \frac{1}{n} \text{Tr } \Lambda^2 \rightarrow \kappa_*,$$

so that the second free cumulant of the empirical spectral distribution of Λ converges to κ_* . Then the matrices $\Theta_t^{(j)}$, $\mathbf{B}_t$, and Σ_t of [21, Eqs. (4.5) and (4.7)] satisfy

$$\lim_{n \rightarrow \infty} \Theta_t^{(j)} = \begin{cases} \Delta_t & \text{if } j = 0 \\ 0 & \text{otherwise,} \end{cases} \quad \lim_{n \rightarrow \infty} \mathbf{B}_t = 0, \quad \lim_{n \rightarrow \infty} \Sigma_t = \kappa_* \Delta_t, \quad (\text{A.1})$$

where we define

$$\Delta_t = \lim_{n \rightarrow \infty} n^{-1} X_t^\top X_t$$

provided that this limit exists. Thus, (2.15–2.16) is a special case of the general AMP algorithm of [21, Section 4], replacing the debiasing coefficients b_{ts} therein by their large- n limits $b_{ts}^\infty = 0$.

From the initialization $y^0 \sim \mathcal{N}(0, \sigma_*^2 I)$, [21, Proposition E.1] ensures that the empirical distribution of rows of (h, y^0) converges almost surely in the Wasserstein space W_p to (H, Y_0) , for every $p \geq 1$. Since f is Lipschitz, the distribution of entries of $x^1 = f(h, y^0)$ then converges in W_p to X_1 . By definition, $\lambda_* > \max(x : x \in \text{supp}(\mu_{\tilde{D}}))$, so Assumption 1.1(b) implies that the empirical eigenvalue distribution of Λ also converges in W_p to a compactly supported limit. The remaining conditions of [21, Assumption 4.2] are easily checked from Assumption 1.1. Thus, [21, Theorem 4.3] shows the distributional convergence (2.27) in W_p , for any fixed $p \geq 1$. In particular, the above matrix Δ_t is well-defined and non-singular for every $t \geq 1$, and coincides with the definition of (2.26). Thus (2.28) holds.

The limit (2.29) then immediately follows from the distributional convergence (2.27) and the specification of the law $(Y_1, \dots, Y_t) \sim \mathcal{N}(0, \kappa_* \Delta_t)$. The limit (2.30) follows from writing each X_s as a function of Y_{s-1} according to (2.25), and applying the divergence-free condition (2.21) and Gaussian integration by parts for a multivariate Gaussian vector—see [21, Proposition E.5]. $\square$

Proof of Proposition 2.3 Since $Y_0 \sim \mathcal{N}(0, \sigma_*^2)$, we have $\delta_{11} = \mathbb{E}[X_1^2] = \delta_*$ by (2.25) and the second equality of (2.24). Then $\kappa_* \delta_{11} = \sigma_*^2$ by definition of δ_* in (2.22), so

$Y_1 \sim \mathcal{N}(0, \sigma_*^2)$ by the characterization of its law in Theorem 2.2. The statements $\delta_{tt} = \delta_*$ and $\kappa_* \delta_{tt} = \sigma_*^2$ then hold for all $t \geq 1$ by induction.

To show the convergence $\delta_{st} \rightarrow \delta_*$ as $\min(s, t) \rightarrow \infty$, let us set $\delta_{0t} = \delta_{t0} = 0$ for all $t \geq 0$. We first show that $0 \leq \delta_{st} \leq \delta_*$ for all s, t . Observe that

$$\delta_{s+1,t+1} = \mathbb{E}[X_{s+1}X_{t+1}] = \mathbb{E}[f(H, Y_s)f(H, Y_t)],$$

where $f(h, y) = (1 - q_*)^{-1} \tanh(h + y) - y$. By induction on $\min(s, t)$, it suffices to show that $\delta_{st} \in [0, \delta_*]$ implies that $\delta_{s+1,t+1} \in [0, \delta_*]$. Represent the bivariate Gaussian law of (Y_s, Y_t) as

$$(Y_s, Y_t) = \left(\sqrt{\kappa_* \delta_{st}} G + \sqrt{\sigma_*^2 - \kappa_* \delta_{st}} G', \sqrt{\kappa_* \delta_{st}} G + \sqrt{\sigma_*^2 - \kappa_* \delta_{st}} G'' \right),$$

where G, G', G'' are independent $\mathcal{N}(0, 1)$ variables. Note that this representation holds also when $s = 0$ and/or $t = 0$, because Y_0 is independent of Y_t for $t \neq 0$. Then $\delta_{s+1,t+1} = g(\delta_{st})$, where $g(\delta)$ is the map defined on $[0, \delta_*]$ by

$$g(\delta) \triangleq \mathbb{E} \left[f \left(H, \sqrt{\kappa_* \delta} \cdot G + \sqrt{\sigma_*^2 - \kappa_* \delta} \cdot G' \right) f \left(H, \sqrt{\kappa_* \delta} \cdot G + \sqrt{\sigma_*^2 - \kappa_* \delta} \cdot G'' \right) \right].$$

Denote $Y' = \sqrt{\kappa_* \delta} \cdot G + \sqrt{\sigma_*^2 - \kappa_* \delta} \cdot G'$, and define Y'' similarly with G'' in place of G' . By Cauchy-Schwarz, $|g(\delta)| \leq \mathbb{E}[f(H, Y')^2] = \delta_*$ by (2.24). At $\delta = \delta_*$, we have $Y' = Y'' = \sigma_* G$ and hence $g(\delta_*) = \mathbb{E}[f(H, Y')^2] = \delta_*$. Furthermore, taking the expectation first over G' and G'' , for any $\delta \in [0, \delta_*]$ we have

$$g(\delta) = \mathbb{E} \left[\mathbb{E}[f(H, Y') \mid H, G]^2 \right] \in [0, \delta_*]$$

as claimed.

Next, applying symmetry with respect to (Y', Y'') and Gaussian integration by parts,

$$\begin{aligned} g'(\delta) &= 2\mathbb{E} \left[\partial_y f(H, Y') \cdot f(H, Y'') \cdot \left(\frac{\kappa_*}{2\sqrt{\kappa_* \delta}} G - \frac{\kappa_*}{2\sqrt{\sigma_*^2 - \kappa_* \delta}} G' \right) \right] \\ &= 2\mathbb{E} \left[\partial_y^2 f(H, Y') \cdot f(H, Y'') \cdot \frac{\kappa_*}{2} \right. \\ &\quad \left. - \partial_y^2 f(H, Y') \cdot f(H, Y'') \cdot \frac{\kappa_*}{2} + \partial_y f(H, Y') \cdot \partial_y f(H, Y'') \cdot \frac{\kappa_*}{2} \right] \\ &= \kappa_* \mathbb{E} [\partial_y f(H, Y') \partial_y f(H, Y'')]. \end{aligned}$$

Here $|\partial_y f(h, y)| \leq 2/(1 - q_*)$. Then, applying $\kappa_* = O(\beta^2(1 - q_*)^2)$ by Proposition 2.5, we have $|g'(\delta)| \leq 1/2$ for any $\beta \in (0, \beta_0)$ and some constant $\beta_0 > 0$ depending only on μ_D . So $g : [0, \delta_*] \rightarrow [0, \delta_*]$ is contractive, and δ_* is the unique fixed point. We then have

$$|\delta_{st} - \delta_*| \leq (1/2)^{\min(s,t)} |\delta_{s-\min(s,t), t-\min(s,t)} - \delta_*| = (1/2)^{\min(s,t)} \delta_* \leq (1/2)^{\min(s,t)},$$

so $\lim_{\min(s,t) \rightarrow \infty} \delta_{st} = \delta_*$ as desired. Finally, $\lim_{\min(s,t) \rightarrow \infty} \kappa_* \delta_{st} \rightarrow \sigma_*^2$ follows from $\sigma_*^2 = \kappa_* \delta_*$. $\square$

Proof of Proposition 2.4 Since $\bar{J} = O^\top \bar{D} O$, we have $f(\bar{J}) = O^\top f(\bar{D}) O$ by the functional calculus. Then applying $S_t = O X_t$ yields $n^{-1} X_t^\top f(\bar{J}) X_t = n^{-1} S_t^\top f(\bar{D}) S_t$.

Let Λ be as defined in (2.13). Applying [21, Lemma A.4(b)] with the notational identification $\mathbf{r}_t \leftrightarrow s^t$, for each fixed integer $k \geq 0$, almost surely

$$\lim_{n \rightarrow \infty} n^{-1} S_t^\top \Lambda^k S_t = \mathbf{L}_t^{(k, \infty)}.$$

This limit matrix $\mathbf{L}_t^{(k, \infty)}$ is defined by [21, Eq. (A.6) and Lemma A.1]. Under the divergence-free condition (2.21), applying (A.1), we have simply

$$\mathbf{L}_t^{(k, \infty)} = m_k \cdot \Delta_t, \quad m_k = \lim_{n \rightarrow \infty} n^{-1} \text{Tr } \Lambda^k = \int \left(\frac{1}{1 - q_*} (\lambda_* - x)^{-1} - 1 \right)^k \mu_{\bar{D}}(dx).$$

Define the increasing map $g : (-\infty, \lambda_*) \rightarrow (-1, \infty)$ by

$$g(x) = \frac{1}{1 - q_*} (\lambda_* - x)^{-1} - 1,$$

so that $\Lambda = g(\bar{D})$. Then, for any fixed polynomial $p : \mathbb{R} \rightarrow \mathbb{R}$, this shows

$$\lim_{n \rightarrow \infty} n^{-1} S_t^\top p(\Lambda) S_t = \Delta_t \cdot \int p(g(x)) \mu_{\bar{D}}(dx).$$

We apply Weierstrass polynomial approximation to extend the above to general continuous functions: Let $g^{-1} : (-1, \infty) \rightarrow (-\infty, \lambda_*)$ be the functional inverse of g . Then, for any $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous and bounded on a neighborhood of $\text{supp}(\mu_{\bar{D}})$, the function $f \circ g^{-1}$ is continuous and bounded on some compact neighborhood $\mathcal{K}$ of $g(\text{supp}(\mu_{\bar{D}}))$. Applying the Weierstrass approximation, for any $\varepsilon > 0$, there is a polynomial p for which

$$\max_{x \in \mathcal{K}} |p(x) - f \circ g^{-1}(x)| < \varepsilon.$$

Then

$$\begin{aligned} & \left\| \lim_{n \rightarrow \infty} n^{-1} S_t^\top f(\bar{D}) S_t - \Delta_t \cdot \int f(x) \mu_{\bar{D}}(dx) \right\| \\ &= \left\| \lim_{n \rightarrow \infty} n^{-1} S_t^\top (f \circ g^{-1}(\Lambda)) S_t - \Delta_t \cdot \int f \circ g^{-1}(g(x)) \mu_{\bar{D}}(dx) \right\| \\ &\leq \limsup_{n \rightarrow \infty} \varepsilon \cdot n^{-1} \|S_t\|^2 + \varepsilon \cdot \|\Delta_t\| \leq \varepsilon \cdot \text{Tr } \Delta_t + \varepsilon \cdot \|\Delta_t\|. \end{aligned}$$

This holds for any $\varepsilon > 0$, so

$$\lim_{n \rightarrow \infty} n^{-1} S_t^\top f(\bar{D}) S_t = \Delta_t \cdot \int f(x) \mu_{\bar{D}}(dx).$$

□

Appendix B. Large deviations for integrals over the orthogonal group

5.1 B.1 Proof of Proposition 2.7

By applying a transformation $D \mapsto QDQ^\top$ and $b \mapsto Qb$ for an orthogonal matrix Q , we may assume without loss of generality that $D = \text{diag}(d_1, \dots, d_n)$ is diagonal. Let $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{d_i}$, $d_{n,+} = \max d_i$, $d_{n,-} = \min d_i$, and $\|D\|_{\text{op}} = \max |d_i|$. Let

$$G_n(\gamma) = \frac{1}{n} \text{Tr}(\gamma I - D)^{-1} = \frac{1}{n} \sum_{i=1}^n \frac{1}{\gamma - d_i}. \quad (\text{B.1})$$

Lemma B.1 *In the setting of Proposition 2.7, there exists $n_0 > 0$ such that for any $n \geq n_0$ and any $(a, b) \in \Omega_n$, the following holds: Set $\alpha = \|a\|^2/n$ and*

$$F_n(\gamma) = G_n(\gamma) + \frac{b^\top (\gamma I - D)^{-2} b}{n}.$$

Then the equation

$$F_n(\gamma) = \alpha \quad (\text{B.2})$$

has a unique solution $\gamma_n^ \in (d_+ + \varepsilon, \infty)$, and $|\gamma_n^* - \alpha^{-1}| \leq C + \|D\|_{\text{op}}$.*

Proof By Assumption 1.1(b), $\mu_n \rightarrow \mu_D$ weakly, $d_{n,+} \rightarrow d_+$ as $n \rightarrow \infty$, and $\|D\|_{\text{op}}$ is bounded. Then $G_n(\gamma)$ converges to $G(\gamma)$ pointwise for each $\gamma > d_+$. So for some $n_0 > 0$ and all $n \geq n_0$, we have $d_{n,+} < d_+ + \varepsilon$ and $G_n(d_+ + \varepsilon) > G(d_+ + \varepsilon) - \varepsilon$. Then

$$F_n(d_+ + \varepsilon) \geq G_n(d_+ + \varepsilon) > G(d_+ + \varepsilon) - \varepsilon \geq \alpha.$$

Since $F_n(\gamma) \rightarrow 0$ monotonically as $\gamma \rightarrow \infty$, this shows (B.2) has a unique solution $\gamma_n^* > d_+ + \varepsilon$. Next, since $\|b\|^2 \leq Cn$, for any $\gamma > d_{n,+}$ we have $b^\top (\gamma I - D)^{-2} b/n \leq C/(\gamma - d_{n,+})^2$, and hence

$$\frac{1}{\gamma - d_{n,-}} \leq F_n(\gamma) \leq \frac{1}{\gamma - d_{n,+}} + \frac{C}{(\gamma - d_{n,+})^2}.$$

Applying this at $\gamma = \gamma_n^*$ and $F_n(\gamma) = \alpha$, and rearranging,

$$d_{n,-} + \frac{1}{\alpha} \leq \gamma_n^* \leq d_{n,+} + \frac{1}{\alpha'},$$

where $\alpha' = \frac{\sqrt{1+4C\alpha}-1}{2C}$. We conclude the proof by noting that $\frac{1}{\alpha'} - \frac{1}{\alpha} = \frac{2C}{\sqrt{1+4C\alpha}+1} \in [0, C]$. $\square$

Proof of Proposition 2.7 We now bound the expectation in (2.32) for any $(a, b) \in \Omega_n$. Let $g \sim \mathcal{N}(0, I_n)$ be a standard Gaussian vector. Then $\frac{g}{\|g\|}$ is uniformly distributed over the sphere and $Oa \stackrel{L}{=} \frac{g\|a\|}{\|g\|}$. Then

$$\mathbb{E} \left[\exp \left(b^\top Oa + \frac{a^\top O^\top D O a}{2} \right) \right] = \mathbb{E} \left[\exp \left(\frac{\|a\|}{\|g\|} b^\top g + \frac{\|a\|^2}{2\|g\|^2} g^\top D g \right) \right].$$

Let $\mathcal{E} = \{g : \|\|g\|^2/n - 1\| \leq \delta\}$ for some small δ to be specified. Since $\|g\|^2 \sim \chi_n^2$, by the χ^2 -tail bound (see e.g. [28, Lemma 1]), we have for all $\delta \in (0, 1)$,

$$\mathbb{P}[g \in \mathcal{E}] \geq 1 - 2e^{-\delta^2 n/16}. \quad (\text{B.3})$$

By the independence of $\|g\|$ and $\frac{g}{\|g\|}$, we have

$$1 \leq \frac{\mathbb{E} \left[\exp \left(\frac{\|a\|}{\|g\|} b^\top g + \frac{\|a\|^2}{2\|g\|^2} g^\top D g \right) \right]}{\mathbb{E} \left[\exp \left(\frac{\|a\|}{\|g\|} b^\top g + \frac{\|a\|^2}{2\|g\|^2} g^\top D g \right) \mathbf{1}_{g \in \mathcal{E}} \right]} = \frac{1}{\mathbb{P}[g \in \mathcal{E}]} \leq \frac{1}{1 - 2e^{-\delta^2 n/16}}.$$

Set $\alpha = \|a\|^2/n$, and fix $v \in \mathbb{R}$ such that $v > d_{n,+} - \frac{1}{\alpha}$. Then

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{\|a\|}{\|g\|} b^\top g + \frac{\|a\|^2}{2\|g\|^2} g^\top D g \right) \mathbf{1}_{g \in \mathcal{E}} \right] \\ & \leq \mathbb{E} \left[\exp \left(\sqrt{\alpha} b^\top g + \frac{\alpha}{2} g^\top D g + \frac{\alpha v}{2} (n - \|g\|^2) \right) \mathbf{1}_{g \in \mathcal{E}} \right] \\ & \quad \times \underbrace{\exp \left(\delta \|a\| \|b\| + \frac{1}{2} \delta \|a\|^2 \|D\|_{\text{op}} + \frac{\alpha}{2} \delta |v| n \right)}_{\triangleq \tau} \\ & \leq \prod_{i=1}^n \mathbb{E} \left[\exp \left(\sqrt{\alpha} b_i g_i - \frac{\alpha(v - d_i)}{2} g_i^2 \right) \right] \exp \left(\frac{\alpha v n}{2} + \tau \right) \\ & = \prod_{i=1}^n \frac{1}{\sqrt{1 + \alpha(v - d_i)}} \exp \left(\frac{\alpha b_i^2}{2(1 + \alpha(v - d_i))} \right) \exp \left(\frac{\alpha v n}{2} + \tau \right) \\ & = \exp \left(\sum_{i=1}^n \frac{\alpha v}{2} + \frac{b_i^2}{2(\frac{1}{\alpha} + v - d_i)} - \frac{1}{2} \log(1 + \alpha(v - d_i)) \right) \exp(\tau) \end{aligned}$$

$$= \exp \left\{ \frac{n}{2} \left(\alpha v + \frac{1}{n} b^\top \left((\alpha^{-1} + v) I - D \right)^{-1} b - \frac{1}{n} \log \det(I + \alpha(vI - D)) \right) \right\} \exp(\tau).$$

Next we minimize the leading term over $v > d_{n,+} - \frac{1}{\alpha}$. Write $v = \gamma - \frac{1}{\alpha}$. Since the exponent is convex in v , for all large n the minimum is achieved at $v_n^* = \gamma_n^* - \frac{1}{\alpha}$, where γ_n^* is previously defined as the unique solution on $(d_+ + \varepsilon, \infty)$ to (B.2), and this minimum is exactly $E_n(a, b)$ defined in (2.33). By Lemma B.1, we have $|v_n^*| \leq C + \|D\|_{\text{op}}$. Choosing $\delta = n^{-1/4}$ yields $\tau \leq C_1 n^{3/4}$ for some constant C_1 depending on $\varepsilon, C, \|D\|_{\text{op}}$ and $G(d_+ + \varepsilon)$ only. This proves

$$\mathbb{E} \left[\exp \left(b^\top Oa + \frac{a^\top O^\top D O a}{2} \right) \right] \leq \frac{1}{1 - 2e^{-\sqrt{n}/16}} \exp \left(\frac{n}{2} E_n(a, b) + C_1 n^{3/4} \right)$$

For the lower bound,

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{\|a\|}{\|g\|} b^\top g + \frac{\|a\|^2}{2\|g\|^2} g^\top D g \right) \mathbf{1}_{g \in \mathcal{E}} \right] \\ & \geq \mathbb{E} \left[\exp \left(\sqrt{\alpha} b^\top g + \frac{\alpha}{2} g^\top D g + \frac{\alpha v}{2} (n - \|g\|^2) \right) \mathbf{1}_{g \in \mathcal{E}} \right] \exp(-\tau) \\ & = \mathbb{E} \left[\prod_{i=1}^n \exp \left(\sqrt{\alpha} b_i g_i - \frac{\alpha(v - d_i)}{2} g_i^2 \right) \mathbf{1}_{g \in \mathcal{E}} \right] \exp \left(\frac{\alpha v n}{2} - \tau \right) \\ & = \exp \left\{ \frac{n}{2} \left(\alpha v + \frac{1}{n} b^\top \left((\alpha^{-1} + v) I - D \right)^{-1} b - \frac{1}{n} \log \det(I + \alpha(vI - D)) \right) \right\} \mathbb{P}[\tilde{g} \in \mathcal{E}] \exp(-\tau) \end{aligned}$$

where the last step follows from a change of measure from g to $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_n)$, whose coordinates are drawn independently as $g_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, with

$$\mu_i = \frac{\sqrt{\alpha} b_i}{1 + \alpha(v - d_i)}, \quad \sigma_i^2 = \frac{1}{1 + \alpha(v - d_i)}.$$

Note that

$$\begin{aligned} \mathbb{E}[\|\tilde{g}\|^2] &= \sum_{i=1}^n (\mu_i^2 + \sigma_i^2) = \frac{1}{\alpha} \sum_{i=1}^n \frac{b_i^2}{(v + 1/\alpha - d_i)^2} + \frac{1}{v + 1/\alpha - d_i} \\ &= \frac{n}{\alpha} F_n \left(v + \frac{1}{\alpha} \right) \end{aligned}$$

where F_n is as defined in Lemma B.1. As before, choose $v = v_n^* = \gamma_n^* - \frac{1}{\alpha}$, where γ_n^* is the solution to (B.2). Then we have $\mathbb{E}[\|\tilde{g}\|^2] = n$. Moreover,

$$\begin{aligned} \text{Var}(\|\tilde{g}\|^2) &= \sum_{i=1}^n (2\sigma_i^4 + 4\mu_i^2 \sigma_i^2) = \sum_{i=1}^n \frac{2}{(1 + \alpha(v - d_i))^2} + \frac{4\alpha b_i^2}{(1 + \alpha(v - d_i))^3} \\ &= \frac{1}{\alpha^2} \left(\sum_{i=1}^n \frac{2}{(\gamma_n^* - d_i)^2} + \sum_{i=1}^n \frac{4b_i^2}{(\gamma_n^* - d_i)^3} \right). \end{aligned}$$

If $\frac{1}{\alpha} \leq 4(C + \|D\|_{\text{op}})$, we may apply $\gamma_n^* > d_+ + \varepsilon$ in Lemma B.1 and $\|b\|^2 \leq Cn$ to obtain $\text{Var}(\|\tilde{g}\|^2) \leq \frac{n}{\alpha^2} \left(\frac{2}{\varepsilon^2} + \frac{4C}{\varepsilon^3} \right)$. If $\frac{1}{\alpha} \geq 4(C + \|D\|_{\text{op}})$, then we apply $\gamma_n^* \geq \frac{1}{\alpha} - C - \|D\|_{\text{op}}$ from Lemma B.1 to obtain $\gamma_n^* - d_i \geq \frac{1}{\alpha} - 2(C + \|D\|_{\text{op}}) \geq \frac{1}{2\alpha}$ and hence $\text{Var}(\|\tilde{g}\|^2) \leq n(8 + 32C\alpha)$. In both cases, we conclude that

$$\text{Var}(\|\tilde{g}\|^2) \leq C_2 n$$

for some constant C_2 depending on $(C, \|D\|_{\text{op}}, \varepsilon)$. By Chebyshev's inequality,

$$\mathbb{P}[\tilde{g} \notin \mathcal{E}] \leq \frac{\text{Var}(\|\tilde{g}\|^2)}{(\delta n)^2} \leq \frac{C_2}{\sqrt{n}}.$$

This shows

$$\mathbb{E} \left[\exp \left(b^\top Oa + \frac{a^\top O^\top D O a}{2} \right) \right] \geq \frac{1}{1 - \frac{C_2}{\sqrt{n}}} \exp \left(\frac{n}{2} E_n(a, b) - C_1 n^{3/4} \right).$$

Combining these upper and lower bounds completes the proof. $\square$

B.2 Proof of Proposition 2.8

Let $s_1, \dots, s_n \in \mathbb{R}^2$ be the rows of $(b, d) \in \mathbb{R}^{n \times 2}$. We again assume without loss of generality that $D = \text{diag}(d_1, \dots, d_n)$ is diagonal, and write $\mu_n, d_{n,+}, d_{n,-}, \|D\|_{\text{op}}$ as in the preceding section. (Here d_i are the diagonal entries of D , not the entries of the vector d .)

Define

$$M \triangleq \begin{pmatrix} \|a\|^2 & \frac{a^\top c}{n} \\ \frac{a^\top c}{n} & \frac{\|c\|^2}{n} \end{pmatrix}. \quad (\text{B.4})$$

Define

$$\mathcal{F}_n(\Lambda) \triangleq \text{Tr } \Lambda M + \frac{1}{n} \sum_{i=1}^n \left(s_i^\top (\Lambda - d_i I)^{-1} s_i - \log \det(\Lambda - d_i I) \right) - 2 - \log \det(M) \quad (\text{B.5})$$

so that E_n defined in (2.36) is given by

$$E_n(a, b, c, d) = \inf_{\Lambda \succeq (d_+ + \varepsilon) I} \mathcal{F}_n(\Lambda).$$

We have the following lemma that parallels Lemma B.1:

Lemma B.2 *Under the assumption of Proposition 2.8, there exists n_0 such that for all $n \geq n_0$ and all $(a, b, c, d) \in \Omega_n$,*

$$\inf_{\Lambda \geq (d_+ + \varepsilon)I} \mathcal{F}_n(\Lambda)$$

is achieved at a unique minimizer Λ^ such that $\Lambda^* \succ (d_+ + \varepsilon)I$ and $\|\Lambda^* - M^{-1}\| \leq 2C + \|D\|_{\text{op}}$. Furthermore, Λ^* satisfies the equation*

$$F_n(\Lambda) = M \quad (\text{B.6})$$

where

$$F_n(\Lambda) \triangleq \frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1} + \frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1} s_i s_i^\top (\Lambda - d_i I)^{-1}. \quad (\text{B.7})$$

Proof of Lemma B.2 Let n_0 be sufficiently large such that $d_{n,+} < d_+ + \varepsilon$ and $G_n(d_+ + \varepsilon) > G(d_+ + \varepsilon) - \varepsilon$, where G and G_n are the Cauchy transform of μ_D and its empirical version, defined in (2.4) and (B.1). Write the gradient $\nabla \mathcal{F}_n \triangleq \begin{pmatrix} \partial_{11} \mathcal{F}_n & \frac{1}{2} \partial_{12} \mathcal{F}_n \\ \frac{1}{2} \partial_{12} \mathcal{F}_n & \partial_{22} \mathcal{F}_n \end{pmatrix}$ as a 2×2 symmetric matrix. Then one can verify that

$$\begin{aligned} \nabla \mathcal{F}_n(\Lambda) &= M - \underbrace{\frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1}}_{\triangleq G_n(\Lambda)} - \underbrace{\frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1} s_i s_i^\top (\Lambda - d_i I)^{-1}}_{\triangleq g_n(\Lambda)} \\ &= M - F_n(\Lambda). \end{aligned}$$

We first claim that $\inf_{\Lambda \geq (d_+ + \varepsilon)I} \mathcal{F}_n(\Lambda)$ is attained at a unique minimizer Λ^* satisfying $RI \succ \Lambda^* \succ (d_+ + \varepsilon)I$, for some $R > 0$ depending only on M, μ_D, ε . To this end, suppose Λ has an eigenvalue $\lambda \geq R$ with unit-norm eigenvector u . Then

$$\begin{aligned} u^\top \nabla \mathcal{F}_n(\Lambda) u &= u^\top M u - \frac{1}{n} \sum_{i=1}^n (\lambda - d_i)^{-1} - \frac{1}{n} \sum_{i=1}^n (\lambda - d_i)^{-2} (s_i^\top u)^2 \\ &\geq \lambda_{\min}(M) - (R - d_{n,+})^{-1} - 2C(R - d_{n,+})^{-2}, \end{aligned}$$

where the last inequality follows from Cauchy-Schwarz and the assumption that $\frac{1}{n} \sum \|s_i\|^2 = \frac{1}{n} (\|b\|^2 + \|d\|^2) \leq 2C$. Since $\lambda_{\min}(M) > 0$ by assumption and $d_{n,+} < d_+ + \varepsilon$, for sufficiently large R depending only on M, μ_D, ε , we have $u^\top \nabla \mathcal{F}_n(\Lambda) u > 0$, and hence $\mathcal{F}_n(\Lambda - \delta u u^\top) < \mathcal{F}_n(\Lambda)$ for sufficiently small δ . Now suppose that Λ has an eigenvalue equal to $d_+ + \varepsilon$ with unit-norm eigenvector u . Then

$$u^\top \nabla \mathcal{F}_n(\Lambda) u \leq \lambda_{\max}(M) - \frac{1}{n} \sum_{i=1}^n (d_+ + \varepsilon - d_i)^{-1}$$

$$\leq G(d_+ + \varepsilon) - \varepsilon - G_n(d_+ + \varepsilon) < 0,$$

where we used the assumption that $M \preceq (G(d_+ + \varepsilon) - \varepsilon)I$ and $G_n(d_+ + \varepsilon) > G(d_+ + \varepsilon) - \varepsilon$. Thus $\mathcal{F}_n(\Lambda + \delta uu^\top) < \mathcal{F}_n(\Lambda)$ for sufficiently small δ . In view of the strict convexity of $\mathcal{F}_n$, this verifies our claim. Furthermore, the unique minimizer Λ^* must be a critical point of $\mathcal{F}_n$, satisfying the gradient equation (B.6).

Finally, we show that $\|\Lambda^* - M^{-1}\| \leq 2C + \|D\|_{\text{op}}$ by showing that

$$\Lambda^* \succeq M^{-1} + d_{n,-}I, \quad (\text{B.8})$$

$$\Lambda^* \preceq M^{-1} + (d_{n,+} + 2C)I. \quad (\text{B.9})$$

Since $g_n(\Lambda) \geq 0$, (B.8) simply follows from

$$M = F_n(\Lambda^*) \geq G_n(\Lambda^*) \geq (\Lambda^* - d_{n,-}I)^{-1}.$$

To show (B.9), note that for any $x \in \mathbb{R}^n$, by Cauchy-Schwarz and the bound $\frac{1}{n} \sum \|s_i\|^2 \leq 2C$, we have

$$x^\top g_n(\Lambda)x = \frac{1}{n} \sum_{i=1}^n (s_i^\top (\Lambda - d_i I)^{-1} x)^2 \leq 2C x^\top (\Lambda - d_{n,+} I)^{-2} x.$$

In other words, $g_n(\Lambda) \leq 2C(\Lambda - d_{n,+} I)^{-2}$. Writing $Y = \Lambda^* - d_{n,+} I$, this shows

$$M = F_n(\Lambda^*) \preceq Y^{-1} + 2CY^{-2}.$$

Then

$$\begin{aligned} M^{-1} &\succeq (Y^{-1} + 2CY^{-2})^{-1} = (Y^{-1/2}(I + 2CY^{-1})Y^{-1/2})^{-1} \\ &= Y^{1/2}(I + 2CY^{-1})^{-1}Y^{1/2} \\ &\succeq Y^{1/2}(I - 2CY^{-1})Y^{1/2} = Y - 2CI, \end{aligned}$$

where the second line applies $(I + X)^{-1} \succeq I - X$. Then $Y \preceq M^{-1} + 2CI$, which implies (B.9). $\square$

Proof of Proposition 2.8 Let $g_1, g_2 \sim \mathcal{N}(0, I_n)$ be independent standard Gaussian vectors. Let g'_1, g'_2 be their Gram-Schmidt orthogonalized versions

$$g'_1 = \frac{g_1}{\|g_1\|}, \quad g'_2 = \frac{1}{\sin \theta} \left(\frac{g_2}{\|g_2\|} - \cos \theta \frac{g_1}{\|g_1\|} \right)$$

where $\cos \theta = \frac{g_1^\top g_2}{\|g_1\| \|g_2\|}$ and $\theta \in [0, \pi]$. Let

$$x_1 = \|a\|g'_1, \quad x_2 = \|c\|(g'_1 \cos \phi + g'_2 \sin \phi)$$

where $\cos \phi = \frac{a^\top c}{\|a\|\|c\|}$ and $\phi \in [0, \pi]$. Then $(Oa, Oc) \stackrel{L}{=} (x_1, x_2)$ and

$$\begin{aligned} & \mathbb{E} \left[\exp \left(b^\top Oa + d^\top Oc + \frac{a^\top O^\top DOa}{2} + \frac{c^\top O^\top DOc}{2} \right) \right] \\ &= \mathbb{E} \left[\exp \left(b^\top x_1 + d^\top x_2 + \frac{x_1^\top Dx_1}{2} + \frac{x_2^\top Dx_2}{2} \right) \right] \end{aligned}$$

Define the event

$$\mathcal{E} = \left\{ (g_1, g_2) : \|\|g_i\|^2/n - 1\| \leq \delta \text{ for } i = 1, 2, \text{ and } |\cos \theta| \leq \delta \right\} \quad (\text{B.10})$$

for some small $\delta \in (0, \frac{1}{2})$ to be specified. Note that $\mathbb{E}[\exp(\lambda g_1^\top g_2)] = (1 - \lambda^2)^{-n/2}$ for all $|\lambda| < 1$. Thus for $\lambda \in (0, 1)$, $\log \mathbb{E}[\exp(\lambda g_1^\top g_2)] = -\frac{n}{2} \log(1 - \lambda^2) \leq \frac{n\lambda^2}{2(1-\lambda)}$. By [10, Theorem 2.3], we have $\mathbb{P}[|g_1^\top g_2| \geq \sqrt{2nt} + t] \leq 2e^{-t}$. Taking $t = \frac{\delta^2 n}{32}$ and using (B.3), we conclude that

$$\mathbb{P}[(g_1, g_2) \in \mathcal{E}] \geq 1 - 6e^{-\delta^2 n/32}.$$

Crucially, (g'_1, g'_2) and $(\|g_1\|, \|g_2\|, \cos \theta)$ are independent. Since the event $\{(g_1, g_2) \in \mathcal{E}\}$ is measurable with respect to the latter, it is also independent of (g'_1, g'_2) . Thus

$$\begin{aligned} 1 &\leq \frac{\mathbb{E} \left[\exp \left(b^\top x_1 + d^\top x_2 + \frac{x_1^\top Dx_1}{2} + \frac{x_2^\top Dx_2}{2} \right) \right]}{\mathbb{E} \left[\exp \left(b^\top x_1 + d^\top x_2 + \frac{x_1^\top Dx_1}{2} + \frac{x_2^\top Dx_2}{2} \right) \mathbf{1}_{(g_1, g_2) \in \mathcal{E}} \right]} \\ &= \frac{1}{\mathbb{P}[(g_1, g_2) \in \mathcal{E}]} \leq \frac{1}{1 - 6e^{-\delta^2 n/32}}. \end{aligned}$$

Define

$$\xi \triangleq \frac{\|a\|}{\sqrt{n}} g_1, \quad \zeta \triangleq \|c\| \left(\cos \phi \frac{g_1}{\sqrt{n}} + \sin \phi \frac{g_2}{\sqrt{n}} \right),$$

which satisfy $(\xi_i, \zeta_i) \stackrel{iid}{\sim} \mathcal{N}(0, M)$, with M defined in (B.4). On the event $\mathcal{E}$, for an absolute constant $C' > 0$, we have the approximations

$$|b^\top x_1 - b^\top \xi| \leq C' \delta \|a\| \|b\|, \quad |d^\top x_2 - d^\top \zeta| \leq C' \delta \|c\| \|d\|, \quad (\text{B.11})$$

$$|x_1^\top Dx_1 - \xi^\top D\xi| \leq C' \delta \|D\|_{\text{op}} \|a\|^2, \quad |x_2^\top Dx_2 - \zeta^\top D\zeta| \leq C' \delta \|D\|_{\text{op}} \|c\|^2, \quad (\text{B.12})$$

$$\|a\|^2 - \|\xi\|^2 \leq C' \delta \|a\|^2, \quad \|c\|^2 - \|\zeta\|^2 \leq C' \delta \|c\|^2, \quad |a^\top c - \xi^\top \zeta| \leq C' \delta \|a\| \|c\|. \quad (\text{B.13})$$

Fix any $(\gamma, \rho, \nu) \in \mathcal{D}_\varepsilon$ such that $\Lambda = \begin{pmatrix} \gamma & \nu \\ \nu & \rho \end{pmatrix} \succeq (d_+ + \varepsilon)I_{2 \times 2}$. Let $\Lambda' = \begin{pmatrix} \gamma' & \nu' \\ \nu' & \rho' \end{pmatrix} \triangleq \Lambda - M^{-1}$. Define

$$\tau \triangleq C' \delta \left(\|a\| \|b\| + \|c\| \|d\| + \|D\|_{\text{op}} \|a\|^2 + \|D\|_{\text{op}} \|c\|^2 + \frac{|\gamma'|}{2} \|a\|^2 + \frac{|\rho'|}{2} \|c\|^2 + |\nu'| \|a\| \|c\| \right).$$

By the assumption of $(a, b, c, d) \in \Omega_n$, we have

$$\tau \leq C_0 \delta n (1 + \|\Lambda'\|) \quad (\text{B.14})$$

for some C_0 depending on $G(d_+ + \varepsilon)$, C' , and the constant C defining Ω_n .

Recall that $s_1, \dots, s_n \in \mathbb{R}^2$ are the rows of $(b, d) \in \mathbb{R}^{n \times 2}$, and write $z_1, \dots, z_n \in \mathbb{R}^2$ for the rows of $(g_1, g_2) \in \mathbb{R}^{n \times 2}$. Then $z_i \stackrel{iid}{\sim} \mathcal{N}(0, I_2)$ and $(\xi_i, \zeta_i) = T z_i$ for a matrix T satisfying $T T^\top = M$. Define

$$\mu_i \triangleq T^{-1}(\Lambda - d_i I)^{-1} s_i, \quad \Sigma_i \triangleq T^{-1}(\Lambda - d_i I)^{-1} (T^{-1})^\top$$

so that $\det \Sigma_i = \det(M)^{-1} \det(\Lambda - d_i I)^{-1}$. Since $\Lambda \succeq (d_+ + \varepsilon)I$, each Σ_i is well-defined and positive definite. By (B.11)–(B.13), for some error term r_n that satisfies $|r_n| \leq \tau$, we have

$$\begin{aligned} & \mathbb{E} \left[\exp \left(b^\top x_1 + d^\top x_2 + \frac{x_1^\top D x_1}{2} + \frac{x_2^\top D x_2}{2} \right) \mathbf{1}_{(g_1, g_2) \in \mathcal{E}} \right] \\ &= \mathbb{E} \left[\exp \left(b^\top \xi + d^\top \zeta + \frac{\xi^\top D \xi}{2} + \frac{\zeta^\top D \zeta}{2} + \frac{\gamma'}{2} (\|a\|^2 - \|\xi\|^2) \right. \right. \\ &\quad \left. \left. + \frac{\rho'}{2} (\|c\|^2 - \|\zeta\|^2) + \nu' (a^\top c - \xi^\top \zeta) + r_n \right) \mathbf{1}_{(g_1, g_2) \in \mathcal{E}} \right] \\ &= \exp \left(\frac{n}{2} \text{Tr} \Lambda' M + r_n \right) \int \mathbf{1}_{(g_1, g_2) \in \mathcal{E}} \prod_{i=1}^n \exp \left(s_i^\top T z_i - \frac{1}{2} z_i^\top T^\top (\Lambda' - d_i I) T z_i \right) \frac{1}{2\pi} \exp \left(-\frac{1}{2} \|z_i\|^2 \right) \\ &= \exp \left(\frac{n}{2} (\text{Tr} \Lambda M - 2) + r_n \right) \int \mathbf{1}_{(g_1, g_2) \in \mathcal{E}} \prod_{i=1}^n \frac{1}{2\pi} \exp \left(-\frac{1}{2} (z_i - \mu_i)^\top \Sigma_i^{-1} (z_i - \mu_i) + \frac{1}{2} \mu_i^\top \Sigma_i^{-1} \mu_i \right) \\ &= \exp \left\{ \frac{n}{2} (\text{Tr} \Lambda M - 2 - \log \det M) + \frac{1}{2} \sum_{i=1}^n \left(-\log \det(\Lambda - d_i I) + s_i^\top (\Lambda - d_i I)^{-1} s_i \right) + r_n \right\} \\ &\quad \times \mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \in \mathcal{E}] \\ &= \exp \left\{ \frac{n}{2} \mathcal{F}_n(\Lambda) + r_n \right\} \mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \in \mathcal{E}], \end{aligned}$$

where $(\tilde{g}_1, \tilde{g}_2)$ consists of independent pairs $(\tilde{g}_{i1}, \tilde{g}_{i2}) \stackrel{\text{ind}}{\sim} \mathcal{N}(\mu_i, \Sigma_i)$.

Now choose $\delta = n^{-1/4}$ and $\Lambda = \Lambda^*$ as in Lemma B.2. Then $\mathcal{F}_n(\Lambda^*) = \inf_{\Lambda \succeq (d_+ + \varepsilon)I} \mathcal{F}_n(\Lambda) = E_n(a, b, c, d)$. By Lemma B.2, $\|\Lambda'\| = \|\Lambda^* - M^{-1}\| \leq 2C + \|D\|_{\text{op}}$. By (B.14), we have $\tau \leq C_1 n^{3/4}$, which yields the desired upper bound

in (2.35). For the lower bound, we analyze $\mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \in \mathcal{E}]$ by a union bound:

$$\begin{aligned} \mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \notin \mathcal{E}] &\leq \mathbb{P}\left[\left|\frac{1}{n} \sum_{i=1}^n \tilde{g}_{i1}^2 - 1\right| \geq \delta\right] \\ &\quad + \mathbb{P}\left[\left|\frac{1}{n} \sum_{i=1}^n \tilde{g}_{i2}^2 - 1\right| \geq \delta\right] + \mathbb{P}\left[\left|\frac{1}{n} \sum_{i=1}^n \tilde{g}_{i1} \tilde{g}_{i2}\right| \geq \frac{\delta}{2}\right]. \end{aligned} \quad (\text{B.15})$$

Furthermore, the gradient equation (B.6) reads

$$T T^\top = M = \frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1} + \frac{1}{n} \sum_{i=1}^n (\Lambda - d_i I)^{-1} s_i s_i^\top (\Lambda - d_i I)^{-1}.$$

Thus at $\Lambda = \Lambda^*$ which satisfies this equation, we have $n^{-1} \sum_{i=1}^n (\mu_i \mu_i^\top + \Sigma_i) = I_2$, i.e.,

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{g}_{i1}^2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{g}_{i2}^2 = 1, \quad \frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{g}_{i1} \tilde{g}_{i2} = 0.$$

Note that $\text{Var}(\tilde{g}_{i1}^2) = 4\mu_{i1}^2 \Sigma_{i,11} + 2\Sigma_{i,11}^2$, $\text{Var}(\tilde{g}_{i2}^2) = 4\mu_{i2}^2 \Sigma_{i,22} + 2\Sigma_{i,22}^2$, and $\text{Var}(\tilde{g}_{i1} \tilde{g}_{i2}) = \mu_{i1}^2 \Sigma_{i,22} + \mu_{i2}^2 \Sigma_{i,11} + 2\mu_{i1} \mu_{i2} \Sigma_{i,12} + \Sigma_{i,11} \Sigma_{i,22} + \Sigma_{i,12}^2$. Applying $\|T T^\top\| = \|M\| \leq G(d_+ + \varepsilon)$, we have $\|\mu_i\|^2 = s_i^\top T \Sigma_i^2 T^\top s_i \leq G(d_+ + \varepsilon) \|\Sigma_i\|^2 \|s_i\|^2$. Then applying Chebyshev's inequality to (B.15), we have

$$\mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \notin \mathcal{E}] \leq \frac{100(1 + G(d_+ + \varepsilon))}{n\delta^2} \sum_{i=1}^n (\|s_i\|^2 \text{Tr}(\Sigma_i)^3 + \text{Tr}(\Sigma_i)^2).$$

Let $M = \sum_{j=1}^2 \alpha_j u_j u_j^\top$ be its eigenvalue decomposition, and let $T = \sum_{j=1}^2 \sqrt{\alpha_j} u_j v_j^\top$ be the associated singular value decomposition of T . Then

$$\text{Tr}(\Sigma_i) = \sum_{j=1}^2 v_j^\top \Sigma_i v_j = \sum_{j=1}^2 \frac{1}{\alpha_j} u_j^\top (\Lambda^* - d_i I)^{-1} u_j.$$

Recall from Lemma B.2 that $\Lambda^* \succeq (d_+ + \varepsilon)I$ and $\Lambda^* \succeq M^{-1} - C_2 I$, where $C_2 = 2C + \|D\|_{\text{op}}$. Thus $u_j^\top (\Lambda^* - d_i I)^{-1} u_j \leq \frac{1}{\varepsilon}$ always, and $u_j^\top (\Lambda^* - d_i I)^{-1} u_j \leq \frac{\alpha_j}{1 - (C_2 + d_i)\alpha_j}$ provided $\alpha_j < \frac{1}{C_2 + d_i}$. Overall, we have $\text{Tr}(\Sigma_i) \leq C_3$ for some C_3 depending on $(C, \|D\|_{\text{op}}, \varepsilon)$. Consequently, $\mathbb{P}[(\tilde{g}_1, \tilde{g}_2) \notin \mathcal{E}] \leq C_4/\sqrt{n}$, for some constant C_4 depending on $(C, \|D\|_{\text{op}}, \varepsilon, G(d_+ + \varepsilon))$. This completes the required lower estimate for (2.35). $\square$

B.3 Proof of Proposition 2.9

Proof For part (a), write $\mathcal{H}(\gamma, \alpha)$ for the function inside the infimum. This is strictly convex over $\gamma > d_+$, and its derivative is $\partial_\gamma \mathcal{H}(\gamma, \alpha) = \alpha - G(\gamma)$. For $\alpha \in (0, G(d_+))$, this derivative vanishes at $\gamma = G^{-1}(\alpha)$, so $\gamma = G^{-1}(\alpha)$ must be the minimizer by convexity. At this minimizer, writing $G^{-1}(\alpha) = R(\alpha) + \alpha^{-1}$ and combining the logarithmic terms,

$$\mathcal{H}(G^{-1}(\alpha), \alpha) = \alpha R(\alpha) - \int \log(\alpha R(\alpha) + 1 - \alpha x) \mu_D(dx).$$

This evaluates to 0 at $\alpha = 0$. Its derivative in α is

$$\begin{aligned} & R(\alpha) + \alpha R'(\alpha) - \frac{1}{\alpha} \int \frac{R(\alpha) + \alpha R'(\alpha) - x}{R(\alpha) + \alpha^{-1} - x} \mu_D(dx) \\ &= R(\alpha) + \alpha R'(\alpha) - \alpha^{-1} + \alpha^{-1} \int \frac{\alpha^{-1} - \alpha R'(\alpha)}{R(\alpha) + \alpha^{-1} - x} \mu_D(dx) \\ &= R(\alpha) + \alpha R'(\alpha) - \alpha^{-1} + \alpha^{-1} \left(\alpha^{-1} - \alpha R'(\alpha) \right) \cdot G \left(G^{-1}(\alpha) \right) = R(\alpha). \end{aligned}$$

Hence $\inf_{\gamma > d_+} \mathcal{H}(\gamma, \alpha) = \mathcal{H}(G^{-1}(\alpha), \alpha) = \int_0^\alpha R(z) dz$.

For part (b), applying the orthogonal transformations

$$\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} \mapsto Q^\top \begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} Q, \quad A \mapsto Q^\top A Q$$

for any orthogonal matrix $Q \in \mathbb{O}(2)$ preserves both the value of the objective and the optimization domain $\mathcal{D}_+$. Thus we may assume without loss of generality that $A = \text{diag}(\alpha_1, \alpha_2)$ is diagonal. In this case, the function to be minimized is

$$\gamma \alpha_1 - (1 + \log \alpha_1) + \rho \alpha_2 - (1 + \log \alpha_2) - \int \log \det \begin{pmatrix} \gamma - x & v \\ v & \rho - x \end{pmatrix} \mu_D(dx).$$

This is strictly convex over $(\gamma, v, \rho) \in \mathcal{D}_+$, and its gradient is 0 at $(\gamma, v, \rho) = (G^{-1}(\alpha_1), 0, G^{-1}(\alpha_2))$ by (4.30) and part (a). Thus the minimizer is

$$\begin{pmatrix} \gamma & v \\ v & \rho \end{pmatrix} = G^{-1}(A),$$

and the value is $\int_0^{\alpha_1} R(z) dz + \int_0^{\alpha_2} R(z) dz = \text{Tr } f(A)$ also by part (a). $\square$

Appendix C. Auxiliary results

Proposition C.1 *Let S, T be two fixed metric spaces. For each $n \geq 1$, let K_n be a compact metric space, $f_n : K_n \rightarrow S$ a continuous map, and $v_n : K_n \rightarrow T$ a map that*

is both continuous and relatively open.⁵ For each $n \geq 1$, let U_n be a dense subset of K_n such that

- For some fixed subset $V \subset T$, we have $v_n(U_n) = V$ for every n , and
- There exists a function $f : V \rightarrow S$ such that $f_n(x) - f(v_n(x)) \rightarrow 0$ as $n \rightarrow \infty$, uniformly over $x \in U_n$.

Then $v_n(K_n) = \bar{V}$ (the closure of V in T) for every n , this function f is continuous on V and extends continuously to $\bar{V}$, and $f_n(x) - f(v_n(x)) \rightarrow 0$ uniformly also over $x \in K_n$.

Proof Since K_n is compact and v_n is continuous, $v_n(K_n)$ is also compact, so $v_n(K_n) \supseteq \bar{V}$. The reverse inclusion $v_n(K_n) \subseteq \bar{V}$ is immediate by continuity, so $v_n(K_n) = \bar{V}$.

For $x \in K_n$ and $v \in T$, denote $B_\eta(x) = \{x' \in K_n : \|x - x'\| < \eta\}$ and $B_\delta(v) = \{v' \in T : \|v - v'\| < \delta\}$. To check that f is continuous on V and extends continuously to $\bar{V}$, it suffices to show that for any $\varepsilon > 0$ and any $v \in \bar{V}$, there exists $\delta > 0$ for which

$$\|f(v') - f(v'')\| < \varepsilon \text{ for all } v', v'' \in B_\delta(v) \cap V. \quad (\text{C.1})$$

Fix any such ε , v , and let $n = n(\varepsilon)$ be large enough so that $\|f_n(x) - f(v_n(x))\| < \varepsilon/3$ for all $x \in U_n$. For this n , let $x_n \in K_n$ be a point where $v_n(x_n) = v$. By continuity of f_n , there exists $\eta = \eta(n) > 0$ sufficiently small such that $\|f_n(x') - f_n(x'')\| < \varepsilon/3$ for all $x', x'' \in B_\eta(x_n)$. Then

$$\|f(v_n(x')) - f(v_n(x''))\| < \varepsilon \text{ for all } x', x'' \in B_\eta(x_n) \cap U_n. \quad (\text{C.2})$$

Since $v_n(x_n) = v$ and v_n is relatively open, for some $\delta = \delta(n) > 0$, the image $v_n(B_\eta(x_n))$ must contain $B_\delta(v) \cap \bar{V}$. Then $v_n(B_\eta(x_n) \cap U_n) \supseteq B_\delta(v) \cap V$, so (C.2) implies (C.1) as desired.

Finally, since U_n is dense in K_n and f_n , f , and v_n are continuous,

$$\begin{aligned} \sup_{x \in K_n} |f_n(x) - f(v_n(x))| &= \sup_{x \in K_n} \left(\lim_{\substack{x' \rightarrow x \\ x' \in U_n}} |f_n(x') - f(v_n(x'))| \right) \\ &\leq \sup_{x' \in U_n} |f_n(x') - f(v_n(x'))|, \end{aligned}$$

so the uniform convergence $|f_n(x) - f(v_n(x))| \rightarrow 0$ over $x \in K_n$ follows from that over $x \in U_n$. $\square$

Proposition C.2 Let $D \subset \mathbb{R}^d$ be a convex set and $f : D \rightarrow \mathbb{R}$ be convex and twice differentiable. Given $x_* \in D$ such that $B(x_*, \delta) = \{x : \|x - x_*\| < \delta\} \subset D$, suppose $\|\nabla f(x_*)\| \leq \varepsilon$ and $\nabla^2 f(x) \geq cI$ for all $x \in B(x_*, \delta)$, where $c\delta > 4\varepsilon$. Then

$$\inf_{x \in D} f(x) \geq f(x_*) - \frac{4\varepsilon^2}{c}.$$

⁵ That is, $v_n(U_n)$ is open in $v_n(K_n)$ for any open subset $U_n \subset K_n$.

Proof For each $x \in B(x_*, \delta)$, we have $f(x) \geq f(x_*) + (\nabla f(x_*))^\top (x - x_*) + \frac{c}{2} \|x - x_*\|^2$. So $f(x) > f(x_*)$ for all $\|x - x_*\| \geq 4\varepsilon/c$. Therefore, the local minimum $\min_{\|x - x_*\| \leq 4\varepsilon/c} f(x)$ is achieved at some $\tilde{x}$ such that $\|\tilde{x} - x_*\| < 4\varepsilon/c$ and hence $\nabla f(\tilde{x}) = 0$. By convexity of f , $\tilde{x}$ is also the global minimizer so $\inf_{x \in D} f(x) = f(\tilde{x})$. Finally, $f(\tilde{x}) \geq f(x_*) + (\nabla f(x_*))^\top (x - x_*) \geq f(x_*) - 4\varepsilon^2/c$. $\square$

Appendix D. Spherical model

Consider the spherical counterpart of the Ising model (1.1), with partition function

$$Z_{\text{sphere}} \triangleq \int_{S^{n-1}(\sqrt{n})} \pi(d\sigma) \exp\left(\frac{\beta}{2} \sigma^\top J \sigma + h^\top \sigma\right),$$

where $J = O^\top D O$ and π is the uniform distribution on $S^{n-1}(\sqrt{n})$, the n -sphere of radius $\sqrt{n}$. The replica-symmetric prediction of the limit free energy is

$$\Psi_{\text{RS,sphere}} = \frac{1}{2} \inf_{\gamma > \bar{d}_+} \left\{ \gamma + \mathbb{E}[\mathbf{H}^2] \cdot \bar{G}(\gamma) - \int \log(\gamma - x) \mu_{\bar{D}}(x) - 1 \right\}, \quad (\text{D.1})$$

where the rescaled notations $\bar{d}_+$, $\bar{G}$, $\mu_{\bar{D}}$ were defined in (2.3). The following theorem justifies this formula.

Theorem D.1 Under Assumption 1.1, for any fixed $\beta \in (0, G(d_+))$, almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_{\text{sphere}} = \Psi_{\text{RS,sphere}}.$$

A derivation of this result in the special case of $h = 0$ is given in [32, Section 2.1].⁶ We prove Theorem D.1 using Proposition 2.7, which we have stated under the assumption $\beta < G(d_+)$. Dropping this assumption requires removing the upper-bound condition on $\|a\|$ in Proposition 2.7; such an extension was obtained in [26, Theorem 6] for $b = 0$.

Proof of Theorem D.1 We express the uniform distribution of $\sigma \in S^{n-1}(\sqrt{n})$ as $\sigma = Qa$, where $a \in S^{n-1}(\sqrt{n})$ is any fixed vector on the sphere, and $Q \sim \text{Haar}(\mathbb{O}(n))$ is independent of J . By the given condition $\beta < G(d_+)$, we have $\|a\|^2/n = 1 < \bar{G}(\bar{d}_+) = G(d_+)/\beta$. Thus there exists $\varepsilon > 0$ for which $\|a\|^2/n = 1 < \bar{G}(\bar{d}_+ + \varepsilon) - \varepsilon$. Setting $b = h$ and applying Proposition 2.7 to evaluate the expectation over Q (conditional on J), we obtain

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} \log Z - f(\bar{J}) \right| = 0, \quad f(\bar{J}) \triangleq \frac{1}{2} \inf_{\gamma \geq \bar{d}_+ + \varepsilon} f(\bar{J}, \gamma)$$

⁶ [32, Eq. (14)] studies the unnormalized surface area measure on the sphere, and hence has an extra additive term of $\frac{1}{2} \log(2\pi e)$.

where

$$\begin{aligned} f(\bar{J}, \gamma) &\triangleq \gamma + \frac{h^\top (\gamma I - \bar{J})^{-1} h}{n} - \frac{1}{n} \log \det(\gamma I - \bar{J}) - 1 \\ &= \gamma + \frac{(Oh)^\top (\gamma I - \bar{D})^{-1} (Oh)}{n} - \frac{1}{n} \log \det(\gamma I - \bar{D}) - 1. \end{aligned} \quad (\text{D.2})$$

For any $\gamma \geq \bar{d} + \varepsilon$ and all large n , note that $f(\bar{J}, \gamma) \geq \gamma - \frac{1}{n} \log \det(\gamma I - \bar{D}) - 1$, where the right side diverges as $\gamma \rightarrow \infty$. Thus there exists some constant $\Gamma > 0$ independent of $\bar{J}$ and n such that

$$f(\bar{J}) = \frac{1}{2} \inf_{\gamma \in [\bar{d}_+ + \varepsilon, \Gamma]} f(\bar{J}, \gamma). \quad (\text{D.3})$$

Writing $\Psi_{\text{RS}, \text{sphere}} = \frac{1}{2} \inf_{\gamma > \bar{d}_+} \Psi(\gamma)$ where $\Psi(\gamma)$ is the function in (D.1), by the same reasoning, this infimum may be restricted to $\gamma \leq \Gamma$. For $\gamma \in (\bar{d}_+, \bar{d}_+ + \varepsilon)$, we have $\Psi'(\gamma) = 1 + \mathbb{E}[\mathbf{H}^2] \cdot \bar{G}'(\gamma) - \bar{G}(\gamma) \leq 1 - \bar{G}(\gamma) < 0$, and hence the infimum may also be restricted to $\gamma \geq \bar{d}_+ + \varepsilon$. So

$$\Psi_{\text{RS}, \text{sphere}} = \frac{1}{2} \inf_{\gamma \in [\bar{d}_+ + \varepsilon, \Gamma]} \Psi(\gamma). \quad (\text{D.4})$$

Finally, we check the convergence of $f(\bar{J}, \gamma)$ to $\Psi(\gamma)$. Note that $\mathbb{E}[(Oh)^\top (\gamma I - \bar{D})^{-1} (Oh)] = \sum_{i=1}^n \mathbb{E}[(Oh)_i^2] / (\gamma - \bar{d}_i)$, where $\mathbb{E}[(Oh)_i^2] = \|h\|^2 \mathbb{E}[O_{1i}^2] = \frac{\|h\|^2}{n}$ by symmetry. Thus, applying Assumption 1.1(b) and (c),

$$\mathbb{E}[f(\bar{J}, \gamma)] = \gamma + \frac{\|h\|^2}{n^2} \sum_{i=1}^n \frac{1}{\gamma - \bar{d}_i} - \frac{1}{n} \log \det(\gamma I - \bar{D}) - 1 \xrightarrow{n \rightarrow \infty} \Psi(\gamma). \quad (\text{D.5})$$

Next we argue that $f(\bar{J}, \gamma)$ concentrates, similar to the proof of Theorem 1.3. Viewing $f(\bar{J}, \gamma)$ as a function of O via (D.2), we may compute its derivative

$$\partial_O f(\bar{J}, \gamma) = \frac{2}{n} h h^\top O^\top (\gamma I - \bar{D})^{-1}.$$

Thus for large enough n and any $\gamma \geq d_+ + \varepsilon$, we have $\|\partial_O f(\bar{J}, \gamma)\|_F \leq \frac{4\|h\|^2}{n\varepsilon} \|O\|_{\text{op}}$. By Assumption 1.1(c), for all sufficiently large n , $\frac{1}{n} \|h\|^2 \leq 2\mathbb{E}[\mathbf{H}^2]$ and hence $O \mapsto f(\bar{J}, \gamma)$ is L -Lipschitz on $\mathbb{O}(n)$ with $L = \frac{8\mathbb{E}[\mathbf{H}^2]}{\varepsilon}$. Then by the same argument that leads to (5.1) and (5.2), we have for each $\gamma \geq \bar{d}_+ + \varepsilon$,

$$\mathbb{P}[|f(\bar{J}, \gamma) - \mathbb{E}[f(\bar{J}, \gamma)]| \geq \delta] \leq 2 \exp \left\{ -\frac{(\frac{n}{4} - \frac{1}{2})\delta^2}{2L^2} \right\}. \quad (\text{D.6})$$

Furthermore, $|\partial_\gamma f(\bar{J}, \gamma)| \leq 1 + \frac{\|h\|^2}{n\varepsilon} + \frac{1}{\varepsilon}$. Thus for all $O \in \mathbb{O}(n)$ and all sufficiently large n , $\gamma \mapsto f(\bar{J}, \gamma)$ is L' -Lipschitz with $L' = 1 + (2\mathbb{E}[H^2] + 1)/\varepsilon$ on $[\bar{d}_+ + \varepsilon, \Gamma]$. The same Lipschitz continuity holds for $\Psi(\gamma)$. Combining (D.5) and (D.6), and applying Borel-Cantelli and a union bound over a sufficiently fine grid of values $\gamma \in [\bar{d}_+ + \varepsilon, \Gamma]$, we obtain the almost-sure convergence $f(\bar{J}, \gamma) \rightarrow \Psi(\gamma)$ uniformly over $\gamma \in [\bar{d}_+ + \varepsilon, \Gamma]$. Then by (D.3) and (D.4), also $f(\bar{J}) \rightarrow \Psi_{\text{RS, sphere}}$, completing the proof. $\square$

Appendix E. Cavity-method derivation of the TAP equations

We provide a brief review of the heuristic approach in [41] for deriving the TAP equations (2.11). Let

$$m_i = \langle \sigma_i \rangle, \quad \chi_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \quad (\text{E.1})$$

where $\langle \cdot \rangle$ is the expectation under the law $P(\sigma)$ in our model of interest (2.8). Define the cavity field $\theta_i = \sum_{j \neq i} \bar{J}_{ij} \sigma_j$. Then the single-spin marginals of $P(\sigma)$ are

$$P(\sigma_i) = \frac{1}{Z_i} \langle e^{\sigma_i(h_i + \theta_i)} \rangle_{\setminus i}, \quad Z_i = \sum_{\sigma_i \in \{\pm 1\}} \langle e^{\sigma_i(h_i + \theta_i)} \rangle_{\setminus i}$$

where $\langle \cdot \rangle_{\setminus i}$ denotes the expectation over $\{\sigma_j\}_{j \neq i}$ (defining θ_i) in the cavity system with the spin σ_i removed. We have the exact identities

$$m_i = \partial_{h_i} \log Z_i, \quad \chi_{ij} = \partial_{h_j} m_i. \quad (\text{E.2})$$

Approximating the law of the cavity field θ_i under $\langle \cdot \rangle_{\setminus i}$ by a Gaussian law $\mathcal{N}(\mu_i, v_i)$, one obtains from the Gaussian moment-generating-function

$$P(\sigma_i) \approx \frac{1}{Z_i} e^{\sigma_i(h_i + \mu_i) + v_i/2}, \quad m_i \approx \tanh(h_i + \mu_i), \quad Z_i \approx \sum_{\sigma_i \in \{\pm 1\}} e^{\sigma_i(h_i + \mu_i) + v_i/2}. \quad (\text{E.3})$$

Furthermore, the law of θ_i under $\langle \cdot \rangle$ in the original model is then approximately a two-component Gaussian mixture $P(\theta_i) \propto \sum_{\sigma_i \in \{\pm 1\}} e^{\sigma_i(h_i + \theta_i) - (\theta_i - \mu_i)^2/2v_i} \propto \sum_{\sigma_i \in \{\pm 1\}} e^{\sigma_i(h_i + \mu_i)} e^{-(\theta_i - \mu_i - \sigma_i v_i)^2/2v_i}$, from which one obtains $\langle \theta_i \rangle \approx \mu_i + v_i \tanh(h_i + \mu_i) \approx \mu_i + v_i m_i$. Equating this with $\langle \theta_i \rangle = \sum_{k \neq i} \bar{J}_{ik} m_k$ from the definition of θ_i gives

$$\mu_i \approx \sum_k \bar{J}_{ik} m_k - v_i m_i. \quad (\text{E.4})$$

Finally, the approach of [41] is to derive an equation for the cavity field variances $\{v_i\}$ by implicit differentiation in h_j , assuming $\partial_{h_j} v_i \approx 0$. Then, differentiating (E.4)

and applying (E.2),

$$\partial_{h_j} \mu_i \approx \sum_k \bar{J}_{ik} \chi_{kj} - v_i \chi_{ij}. \quad (\text{E.5})$$

Differentiating the first equality of (E.2) using the approximation for Z_i in (E.3),

$$\chi_{ij} = \partial_{h_j} m_i \approx \partial_{h_j} \frac{\sum_{\sigma_i} \sigma_i e^{\sigma_i(h_i + \mu_i) + v_i/2}}{\sum_{\sigma_i} e^{\sigma_i(h_i + \mu_i) + v_i/2}} \approx \chi_{ii} \left(\mathbf{1}\{i = j\} + \partial_{h_j} \mu_i \right). \quad (\text{E.6})$$

Combining (E.5) and (E.6), and denoting $\chi = (\chi_{ij})_{i,j=1}^N$, $X = \text{diag}(\chi_{ii})_{i=1}^N$, and $V = \text{diag}(v_i)_{i=1}^N$, one obtains $\chi \approx X(I + \bar{J}\chi - V\chi)$, hence $\chi \approx (V + X^{-1} - \bar{J})^{-1}$. Taking the trace and assuming further by the symmetries of the model that $\chi_{ii} \approx 1 - q_*$ and $v_i \approx v$ for some $v > 0$ and all $i = 1, \dots, N$, this gives $1 - q_* \approx \bar{G}(v + \frac{1}{1-q_*})$, i.e. $v \approx \bar{R}(1 - q_*)$, where $\frac{1}{N} \text{Tr}(zI - \bar{J}) \rightarrow \bar{G}(z)$ as $N \rightarrow \infty$ and $\bar{G}, \bar{R}$ are the Cauchy and R transforms of the limiting law $\bar{\mu}_{\bar{D}}$ defined in (2.4). Substituting this into (E.3–E.4) yields $m \approx \tanh(h + \mu) \approx \tanh(h + \bar{J}m - \bar{R}(1 - q_*)m)$ which are the TAP equations (2.11).

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