Journal of Functional Analysis 286 (2024) 110366

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Kähler–Einstein metrics and obstruction flatness II: Unit Sphere bundles ☆

Peter Ebenfelt [a,*], Ming Xiao [a], Hang Xu [b]

[a] Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA
[b] School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, Guangdong 519082, China

ARTICLE INFO

Article history:
Received 28 March 2023
Accepted 23 January 2024
Available online 15 February 2024
Communicated by Guido De Philippis

MSC:
32W20
32V15
32Q20

Keywords:
Kähler-Einstein metrics
Obstruction flatness
Unit sphere bundles

ABSTRACT

This paper concerns obstruction flatness of hypersurfaces Σ that arise as unit sphere bundles $S(E)$ of Griffiths negative Hermitian vector bundles (E, h) over Kähler manifolds (M, g). We prove that if the curvature of (E, h) satisfies a splitting condition and (M, g) has constant Ricci eigenvalues, then $S(E)$ is obstruction flat. If, in addition, all these eigenvalues are strictly less than one and (M, g) is complete, then we show that the corresponding ball bundle admits a complete Kähler-Einstein metric.

☆ The first author was supported in part by the NSF grant DMS-1900955 and DMS-2154368. The second author was supported in part by the NSF grants DMS-1800549 and DMS-2045104. The third author was supported in part by the NSFC grant No. 12201040.

* Corresponding author.
E-mail addresses: pebenfelt@ucsd.edu (P. Ebenfelt), m3xiao@ucsd.edu (M. Xiao), xuhang9@mail.sysu.edu.cn (H. Xu).

https://doi.org/10.1016/j.jfa.2024.110366
0022-1236/

1. Introduction

In this paper, the authors continue their investigation of Kähler–Einstein metrics and obstruction flatness in the context of domains in vector bundles. In a recent paper [5] the authors studied obstruction flatness of CR hypersurfaces that arise as the unit circle bundle of a negative Hermitian line bundle over a Kähler manifold. The authors proved, among other results, that for a negative line bundle (L, h) over a complex manifold M, if the Kähler metric g induced by $-\operatorname{Ric}(L, h)$ has constant Ricci eigenvalues, then the unit circle bundle $S(L)$ is obstruction flat. If, in addition, all the Ricci eigenvalues are strictly less than 1, then the disk bundle admits a complete Kähler–Einstein metric. It is natural to consider also the more general case of unit sphere bundles in Hermitian vector bundles of higher rank. The goal of this paper is to find the right conditions on a vector bundle that will guarantee obstruction flatness of the corresponding sphere bundle. This turns out to be more subtle than the line bundle case (cf., e.g., Remark 1.5 below and the penultimate paragraph of this introduction) and we need to pose an additional condition on the curvature, beyond the conditions on the Ricci curvature that we pose in the line bundle case, as will be explained below.

We begin by introducing the notion of obstruction flatness in its classical context. On a smoothly bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, the existence of a complete Kähler–Einstein metric on Ω is governed by the following Dirichlet problem for Fefferman's complex Monge-Ampère equation:

$$\begin{cases} J(u) := (-1)^n \det \begin{pmatrix} u & u_{\overline{z_k}} \\ u_{z_j} & u_{z_j\overline{z_k}} \end{pmatrix} = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with $u > 0$ in Ω. If u is a solution of (1.1), then $-\log u$ is the Kähler potential of a complete Kähler–Einstein metric on Ω with negative Ricci curvature. Fefferman [6] established the existence of an approximate solution $\rho \in C^\infty(\overline{\Omega})$ of (1.1) that only satisfies $J(\rho) = 1 + O(\rho^{n+1})$, and showed that such a ρ is unique modulo $O(\rho^{n+2})$. Such an approximate solution ρ is often referred to as a *Fefferman defining function*. Cheng and Yau [2] then proved the existence and uniqueness of an exact solution $u \in C^\infty(\Omega)$ to (1.1), now called the Cheng–Yau solution. Lee and Melrose [9] showed that the Cheng–Yau solution has the following asymptotic expansion:

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k \left(\rho^{n+1} \log \rho\right)^k, \tag{1.2}$$

where each $\eta_k \in C^\infty(\overline{\Omega})$ and ρ is a Fefferman defining function. The expansion (1.2) shows that, in general, the Cheng–Yau solution u can only be expected to have a finite degree of boundary smoothness; namely, $u \in C^{n+2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$. Graham [8] showed that the obstruction to C^∞ boundary regularity of the Cheng–Yau solution is in fact given by the lowest order obstruction $\eta_1|_{\partial\Omega}$, the restriction of η_1 to the boundary. More precisely, in [8] Graham proved that if $\eta_1|_{\partial\Omega}$ vanishes identically (on $\partial\Omega$), then every η_k vanishes to infinite order on $\partial\Omega$ for all $k \geq 1$. For this reason, $\eta_1|_{\partial\Omega}$ is called the *obstruction function*. Graham also showed [8] that, for any $k \geq 1$, the coefficients η_k mod $O(\rho^{n+1})$ are independent of the choice of Fefferman defining function ρ and are locally determined by the local CR geometry of $\partial\Omega$. As a consequence, the η_k mod $O(\rho^{n+1})$, for $k \geq 1$, are local CR invariants that can be defined on any strongly pseudoconvex CR hypersurface in a complex manifold. In particular, the obstruction function $\mathcal{O} = \eta_1|_{\partial\Omega}$ is a local CR invariant that can be defined on any strongly pseudoconvex CR hypersurface Σ. If Σ is a CR hypersurface for which the obstruction function $\mathcal{O}$ vanishes identically, then Σ is said to be *obstruction flat*. The most basic examples of obstruction flat hypersurfaces are the sphere $\{z \in \mathbb{C}^n : |z| = 1\}$ and, more generally, any CR hypersurface that is locally CR diffeomorphic to an open piece of the sphere; a CR hypersurface that is locally CR diffeomorphic to an open piece of the sphere in a neighborhood of any point is called *spherical*.

As mentioned above, the main aim of this paper is to extend the authors' results in [5] concerning obstruction flatness of unit circle bundles in negative Hermitian line bundles over Kähler manifolds to the more general situation of sphere bundles in Hermitian vector bundles of higher rank. To formulate our results, we first review some standard facts and notions concerning the geometry of Hermitian vector bundles and Kähler manifolds. Let (E, h) be a Hermitian (holomorphic) vector bundle over a complex manifold M. Denote by $\pi : E \to M$ the canonical projection and by $E_z = \pi^{-1}(z)$ the fiber at z. Let $\Theta = \Theta_{E,h}$ be the associated curvature form of the Chern connection of (E, h); thus, Θ is an $\mathrm{End}(E)$-valued $(1, 1)$-form. At each point $z \in M$, the tensor $\Theta_z = \Theta(z)$ can be regarded as a Hermitian bilinear form on $E_z \otimes T^{1,0}_z$. We make the following definition, which will be used in the main results.

Definition 1.1. The curvature Θ *splits* if, at every $z \in M$, there exists a Hermitian form H_z on $T^{1,0}_z M$ such that $\Theta_z = h \cdot H_z$; or, equivalently, for any $e \in E_z$ and any $v \in T^{1,0}_z M$,

$$\Theta_z(e \otimes v, e \otimes v) = h(e, e) \cdot H_z(v, v). \tag{1.3}$$

In addition, we say that the vector bundle (E, h) is *curvature split* if its curvature Θ splits. We remark that when this is the case, H_z is equal to the Ricci curvature of (E, h) at z up to a scaling factor. See the paragraph before (2.1) in §2.

When the Hermitian vector bundle (E, h) is Griffiths negative, i.e., $\sqrt{-1}\Theta_z(e \otimes v, e \otimes v) < 0$ for all non-zero $e \in E_z$, $v \in T^{1,0}_z M$ and $z \in M$, the negative of its Ricci, $-\mathrm{Ric}(E, h)$, induces a Kähler metric g on M. By Lemma 1.2.2 in Mok-Ng [11], the corresponding sphere bundle $S(E) = \{e \in E : |e|_h = 1\}$ is strongly pseudoconvex; here, $|e|_h = \sqrt{h(e, e)}$ denotes the norm of e with respect to the metric h.

Given an n-dimensional Kähler manifold (M, g), let $\mathrm{Ric} = -i\partial\overline{\partial} \log \det(g)$ denote the associated Ricci tensor. The latter naturally induces an endomorphism, the *Ricci* *endomorphism*, of the holomorphic tangent space $T_z^{1,0}M$ given by $\mathrm{Ric}\cdot g^{-1}$ for $z \in M$. The eigenvalues of this endomorphism will be referred to as the *Ricci eigenvalues* of (M,g) and, by design, are functions of $z \in M$. All Ricci eigenvalues are real-valued as both Ric and g are Hermitian tensors. For a fixed z, we label the Ricci eigenvalues such that $\lambda_1(z) \leq \cdots \leq \lambda_n(z)$. Note that the sum of the $\lambda_i(z)$, i.e., the trace of the Ricci endomorphism, gives the scalar curvature at z. The Kähler manifold (M,g) is said to have *constant Ricci eigenvalues*, if each $\lambda_i(z)$, for $1 \leq i \leq n$, is a constant function on M; equivalently, the characteristic polynomial of the Ricci endomorphism, $\mathrm{Ric}\cdot g^{-1} : T_z^{1,0}M \to T_z^{1,0}M$ is the same at every point $z \in M$.

Our first main result is as follows.

Theorem 1.2. *Let (E,h) be a Hermitian holomorphic vector bundle over a complex manifold M. Suppose (E,h) is Griffiths negative and its curvature splits. Let g be the Kähler metric on M induced by $-\mathrm{Ric}(E,h)$. If (M,g) has constant Ricci eigenvalues, then $S(E)$ is obstruction flat.*

Recall that the notion of obstruction flatness originates in the context of complete Kähler–Einstein metrics on domains. It is natural to ask whether a complete Kähler–Einstein metric exists (globally) on the corresponding ball bundle $B(E) = \{e \in E : |e|_h < 1\}$ of (E,h) in the situations we are considering. We shall prove the following result:

Theorem 1.3. *Let $n \geq 1$, $k \geq 1$ and $m = n+k$. Let M be a complex manifold of dimension n and (E,h) a Hermitian holomorphic vector bundle over M of rank k such that (E,h) is Griffiths negative and its curvature splits. Let g be the Kähler metric on M induced by $-\mathrm{Ric}(E,h)$. Assume (M,g) is complete and has constant Ricci eigenvalues. If all the Ricci eigenvalues are strictly less than 1, then the ball bundle $B(E)$ admits a unique complete Kähler–Einstein metric $\widetilde{g}$ with Ricci curvature equal to $-(m+1)$. Moreover, this metric is induced by the following Kähler form*

$$\widetilde{\omega}(w,\overline{w}) := -\frac{1}{m+1}\pi^*(\mathrm{Ric})|_w + \frac{1}{m+1}\pi^*(\omega)|_w - i\partial\overline{\partial}\log\phi(|w|_h), \tag{1.4}$$

where ω and Ric are respectively the Kähler and the Ricci form of (M,g), $\pi : E \to M$ the canonical fiber projection of the vector bundle. Moreover, $\phi : (-1,1) \to \mathbb{R}^+$ is an even real analytic function that depends only on the characteristic polynomial of the Ricci endomorphism of (M,g). (More precisely, ϕ is given by Proposition 4.1 by choosing λ_i's to be the Ricci eigenvalues).

If the Hermitian vector bundle comes from a direct sum of copies of a single Hermitian line bundle, then it is automatically curvature split (cf. Proposition 2.1). Thus we have the following corollary of Theorems 1.2 and 1.3.

Corollary 1.4. *Let (L, h_0) be a negative line bundle over a complex manifold M. Set*

$$(E, h) = (L, h_0) \oplus \cdots \oplus (L, h_0),$$

where there are k copies of (L, h_0) on the right hand side. Let g be the Kähler metric induced by $-\operatorname{Ric}(E, h)$. If (M, g) has constant Ricci eigenvalues, then $S(E)$ is obstruction flat. Furthermore, if in addition (M, g) is complete and all the Ricci eigenvalues are strictly less than 1, then the ball bundle $B(E)$ admits a unique complete Kähler–Einstein metric with Ricci curvature equal to $-(m+1)$, where $m = n+k$ and n is the dimension of M.

Remark 1.5. In the setting of Corollary 1.4, Webster [14] proved that for $n = 1$ and $k \geq 2$, $S(E)$ is spherical if and only if (M, g) has constant Gauss curvature $K = -2/k$. Note that the metric g here is k multiple of the metric used in [14]. This result illustrates the difference between the case where (E, h) is a line bundle ($k = 1$) and the case where it is a vector bundle of rank $k \geq 2$. In the former case provided M is compact, the circle bundle is spherical if and only if K is constant, regardless of its value.

Remark 1.6. Combining Webster's result in Remark 1.5 with Corollary 1.4 yields that, for $n = 1$ and $k \geq 2$, if the Gauss curvature of (M, g) is constant but not equal to -2, then $S(E)$ is obstruction flat but not spherical. For example, taking $M = \mathbb{CP}^1$ and (L, h_0) as the tautological line bundle in Corollary 1.4, we find that $S(E)$ is a compact, obstruction flat and non-spherical CR hypersurface for $k \geq 2$. (More examples of obstruction flat CR hypersurface are provided in §5.)

Remark 1.7. The condition for existence of a complete Kähler–Einstein metric in Theorem 1.3 and Corollary 1.4 is optimal, in the sense that the conclusion fails if some Ricci eigenvalue is greater than or equal to 1. Indeed, the following statement follows from van Coevering's work in [13, Theorem 1.1 and Corollary 1.3]. *Let M be a compact complex manifold of dimension n and (E, h) a Hermitian holomorphic vector bundle over M of rank k. Suppose (E, h) is Griffiths negative. Let g be the Kähler metric on M induced by $-\operatorname{Ric}(E, h)$. Assume (M, g) has constant Ricci eigenvalues. Then all these Ricci eigenvalues are strictly less than 1 if and only if the ball bundle $B(E)$ admits a unique complete Kähler–Einstein metric $\widetilde{g}$ with Ricci curvature equal to $-(n+k+1)$.* We point out, however, that the conditions in van Coevering's work [13] are formulated in terms of negativity of Chern classes: $-c_1(M) - c_1(E) > 0$. In the context of constant Ricci eigenvalues, it can be shown that this condition is equivalent to all Ricci eigenvalues being < 1, as in Theorem 1.3 and Corollary 1.4. To see the equivalence, one can verify that when $-c_1(M) - c_1(E) > 0$, we have

$$\int_M S_k(1 - \lambda_1, \ldots, 1 - \lambda_n)\, \omega^n > 0, \quad k = 1, \ldots, n, \tag{1.5}$$

where S_k denotes the symmetric polynomial of degree k and λ_j the Ricci eigenvalues. In the case where the Ricci eigenvalues are constant, the condition (1.5) clearly implies $\lambda_j < 1$ for $1 \le j \le k$. The implication of the other direction of the equivalence is trivial. The main novelty of Theorem 1.3 is that M need not be compact and, in addition, the explicit formula (1.4) for the Kähler–Einstein metric.

A case of particular interest occurs when the base manifold is a domain in $\mathbb{C}^n$. Recall that, for a smoothly bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ $(n \ge 2)$, the Cheng–Yau solution $u \in C^\infty(\Omega)$ is the unique solution to (1.1) and $-\log u$ is the potential of a complete Kähler–Einstein metric with negative Ricci curvature. More generally, when $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain, it follows from the work of Mok–Yau [12] that Ω admits a unique complete Kähler–Einstein metric with Ricci curvature equal to $-(n+1)$. If we write $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$ and set $u = (\det g_{i\bar{j}})^{-\frac{1}{n+1}}$, then u satisfies (1.1). We will call u the Cheng–Yau–Mok solution for Ω. We have the following corollaries of Theorem 1.3.

Corollary 1.8. *Let $n \ge 1$, $k \ge 1$ and $m = n + k$. Let D be a domain in $\mathbb{C}^n$ and h a positive real analytic function on D such that $\omega := \sqrt{-1}\, k\, \partial\overline{\partial} \log h$ is the Kähler form of a complete Kähler metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$ on D. Assume that (D, g) has constant Ricci eigenvalues and that all eigenvalues are strictly less than 1. Consider the domain*

$$\Omega := \{w = (z, \xi) \in D \times \mathbb{C}^k : |\xi|^2 h(z, \bar{z}) < 1\}, \tag{1.6}$$

and the real hypersurface (which is an open dense subset of the boundary of Ω)

$$\Sigma := \{w = (z, \xi) \in D \times \mathbb{C}^k : |\xi|^2 h(z, \bar{z}) = 1\}. \tag{1.7}$$

Then the Cheng–Yau–Mok solution u of Ω is given by

$$u(w) = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}), \tag{1.8}$$

where $H = h^k$, $G = \det(g_{i\bar{j}})$ and ϕ is as in Theorem 1.3. Moreover, u extends real analytically across the boundary piece Σ.

Corollary 1.9. *Let $n \ge 1$, $k \ge 1$ and $m = n + k$. Let D be a domain in $\mathbb{C}^n$ and h a positive real analytic function on D such that $\omega := \sqrt{-1}\, k\, \partial\overline{\partial} \log h$ is the Kähler form of a complete Kähler metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$ on D. Assume that (D, g) is a homogeneous Kähler manifold (i.e., the group of holomorphic isometries acts transitively on D). Consider the domain Ω and the real hypersurface Σ defined by (1.6) and (1.7). Then the Cheng–Yau–Mok solution u of Ω is given by*

$$u(w) = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}),$$

where $H = h^k$, $G = \det(g_{i\bar{j}})$ and ϕ is as in Theorem 1.3. Moreover, u extends real analytically across the boundary piece Σ.

Finally, by studying the potential rationality of the Cheng–Yau–Mok solution, we obtain a characterization of the unit ball in a class of egg domains in terms of the Bergman-Einstein condition. A well-known conjecture posed by Yau [15] asserts that if the Bergman metric of a bounded pseudoconvex domain is Kähler-Einstein, then the domain must be homogeneous. The following result gives an affirmative answer for a class of egg domains.

Proposition 1.10. *Given $p \in \mathbb{R}^+, n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, we let*

$$E_p := \{(z, \xi) \in \mathbb{C}^n \times \mathbb{C}^k : |z|^2 + |\xi|^{2p} < 1\}.$$

Then the Bergman metric of E_p is Kähler–Einstein if and only if $p = 1$.

Remark 1.11. When $n = 1$ and $k = 1$, the above proposition was proved by Cho [3] (and was known earlier to Fu–Wong [7] if p is an integer). When $k = 1$ and $n \geq 1$, it was proved by the authors in [5].

Although we have established versions of Theorems 1.2 and 1.3 in the case of line bundles in [5], we emphasize that the extension to the case of vector bundles carried out in this paper is far from simple and obvious. To illustrate a basic difference between the two cases, we offer the following observation: If (L, h) is a Hermitian line bundle over a Kähler manifold (M, g), where g is induced by $-\mathrm{Ric}(L, h)$, then under some mild assumptions (e.g., simple-connectedness of M), every holomorphic self-isometry of (M, g) naturally extends to a biholomorphism of (L, h) which preserves the fiber norms. This statement, however, fails dramatically when (L, h) is replaced by a vector bundle (E, h) of higher rank. While we borrow ideas from [5] to construct the desired metric in the proofs of Theorem 1.2 and 1.3, the main difficulty arises in the verification that the metric constructed indeed satisfies the Kähler–Einstein condition (i.e., the complex Monge-Ampère equation). In particular, in the vector bundle case, the calculation of the Ricci curvature of the metric we construct seems very difficult in general. A key observation is that the curvature splitting assumption appears to be the right condition to make the computation tractable. Even under this assumption, however, the calculations in the vector bundle case are significantly more involved than those in the line bundle case (cf. Lemma 3.6 and Lemma 3.7).

The paper is organized as follows. §2 gives some preliminary materials on Hermitian vector bundles, including the Griffiths negativity and the curvature split condition. In §3 and §4, we will respectively prove Theorem 1.2 and Theorem 1.3. In §5, Corollary 1.8, Corollary 1.9 and Proposition 1.10 are established, and some examples of obstruction CR hypersurfaces are constructed.

Acknowledgment. The third author would like to thank Professor Yihong Hao and Professor Liyou Zhang for bringing the paper [13] to his attention and for the helpful discussions. The authors are also grateful to the anonymous referee for the careful reading and many valuable comments.

2. Hermitian vector bundles

Let (E, h) be a Hermitian (holomorphic) vector bundle over a complex manifold M of dimension n. Denote by $\pi : E \to M$ the canonical projection and by $E_z = \pi^{-1}(z)$ the fiber at z. As before, $\Theta = \Theta_{E,h}$ is the curvature form of the Chern connection of (E, h). Set $k = \operatorname{Rank} E$ and let $\{e_\alpha(z)\}_{\alpha=1}^k$ be a basis of E_z. If $\{e_\alpha(z)\}_{\alpha=1}^k$ is orthonormal, then the Ricci curvature of E at z is given by the Hermitian form on $T_z^{1,0}M$:

$$\operatorname{Ric}(E)(v,v) = \sum_{\alpha=1}^{k} \Theta_z(e_\alpha(z) \otimes v, e_\alpha(z) \otimes v).$$

Note that the condition (1.3) in Definition 1.1 implies $\operatorname{Ric}(E)(v,v) = kH_z(v,v)$. Hence the curvature Θ splits if and only if

$$\Theta = \frac{1}{k} h \cdot \operatorname{Ric}(E). \tag{2.1}$$

It will be useful to express the curvature split condition in local coordinates. Let (D, z) with $z = (z_1, \cdots, z_n)$ be a local coordinate chart of M and $\{e_\alpha\}_{\alpha=1}^k$ a local frame of E over D. Then we have

$$\pi^{-1}(D) = \Big\{ \sum_{\alpha=1}^{k} \xi_\alpha e_\alpha(z) : z \in D \text{ and } \xi_\alpha \in \mathbb{C} \text{ for any } 1 \le \alpha \le k \Big\},$$

and $(z, \xi) = (z, \xi_1, \cdots, \xi_k)$ form a local coordinate system on $\pi^{-1}(D)$. With respect to the local coordinates z and the local frame $\{e_\alpha\}$ over D, we can write the curvature tensor Θ into

$$\Theta := \sum_{\alpha,\beta=1}^{k} \sum_{i,j=1}^{n} \Theta_{\alpha\bar\beta i\bar j}\, e_\alpha^* \otimes \overline{e_\beta^*} \otimes dz_i \otimes d\overline{z_j}.$$

In addition, if we denote by $h_{\alpha\bar\beta} := h(e_\alpha, e_\beta)$ and $R_{i\bar j} := \operatorname{Ric}(E)\big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\big)$, then it is well-known that

$$\Theta_{\alpha\bar\beta i\bar j} = -\frac{\partial^2 h_{\alpha\bar\beta}}{\partial z_i \partial \overline{z_j}} + h^{\gamma\bar\delta} \frac{\partial h_{\alpha\bar\delta}}{\partial z_i} \frac{\partial h_{\gamma\bar\beta}}{\partial \overline{z_j}}, \tag{2.2}$$

and the curvature split condition (1.3) becomes

$$\Theta_{\alpha\bar{\beta}i\bar{j}} = \frac{1}{k} h_{\alpha\bar{\beta}} R_{i\bar{j}}. \tag{2.3}$$

For the remainder of this paper, we shall use Greek letters $\alpha, \beta, \gamma \cdots$ to denote indices ranging between 1 and k for the fiber coordinates of E, Roman letters i, j to denote indices ranging between 1 and n for the local coordinates of M, and the Roman letters s, t to denote indices ranging from 1 to $m = n + k$ for local coordinates of the total space E.

The curvature split condition holds true trivially for any Hermitian line bundle. But for Hermitian vector bundles of higher rank this is indeed a strong condition. Nevertheless, we can still construct an abundance of such examples by considering the direct sum of Hermitian line bundles.

Proposition 2.1. *Let $(L_1, h_1), \cdots, (L_k, h_k)$ be Hermitian line bundles over a complex manifold M. Set $(E, h) = (L_1, h_1) \oplus \cdots \oplus (L_k, h_k)$. Then (E, h) is curvature split if and only if $\mathrm{Ric}(L_1, h_1) = \cdots = \mathrm{Ric}(L_k, h_k)$. In particular, if $(L_1, h_1) = \cdots = (L_k, h_k)$, then (E, h) is curvature split.*

Remark 2.2. Let (L_1, h_1) and (L_2, h_2) be two Hermitian line bundles over a complex manifold M. Then $\mathrm{Ric}(L_1, h_1) = \mathrm{Ric}(L_2, h_2)$ implies that L_1 and L_2 are smoothly equivalent, but in general, they are not necessarily biholomorphically equivalent.

Proof of Proposition 2.1. Let (D, z) be a local coordinate chart and e_α a local frame of L_α for $1 \leq k \leq \alpha$ over D. Then $\{e_\alpha\}_{\alpha=1}^k$ forms a local frame of E, in terms of which the metric h becomes the diagonal matrix

$$(h_{\alpha\bar{\beta}}) = \begin{pmatrix} h_1(e_1, e_1) & & \\ & \ddots & \\ & & h_k(e_k, e_k) \end{pmatrix}.$$

By (2.2), when $\alpha \neq \beta$, it follows that $\Theta_{\alpha\bar{\beta}i\bar{j}} = 0$; when $\alpha = \beta$, we have

$$\Theta_{\alpha\bar{\alpha}i\bar{j}} = -\frac{\partial^2 h_{\alpha\bar{\alpha}}}{\partial z_i \partial \overline{z_j}} + h^{\alpha\bar{\alpha}} \frac{\partial h_{\alpha\bar{\alpha}}}{\partial z_i} \frac{\partial h_{\alpha\bar{\alpha}}}{\partial \overline{z_j}} = -h_{\alpha\bar{\alpha}} \frac{\partial^2 \log\big(h_{\alpha\bar{\alpha}}\big)}{\partial z_i \partial \overline{z_j}} = h_{\alpha\bar{\alpha}} \left(\mathrm{Ric}(L_\alpha, h_\alpha)\right)_{i\bar{j}}.$$

As a result,

$$\Theta_{\alpha\bar{\beta}i\bar{j}} = h_{\alpha\bar{\beta}} \left(\mathrm{Ric}(L_\alpha, h_\alpha)\right)_{i\bar{j}}.$$

On the other hand, the curvature split condition (2.3) writes into

$$\Theta_{\alpha\bar{\beta}i\bar{j}} = \frac{1}{k} h_{\alpha\bar{\beta}} \left(\mathrm{Ric}(E, h)\right)_{i\bar{j}} = \frac{1}{k} h_{\alpha\bar{\beta}} \sum_{\gamma=1}^{k} \left(\mathrm{Ric}(L_\gamma, h_\gamma)\right)_{i\bar{j}}.$$

So the result follows by comparing the above two equations. $\square$

We next recall the notion of and some simple facts about Griffiths negativity (and positivity) for a Hermitian vector bundle.

Definition 2.3. *Let (E,h) be a Hermitian vector bundle over a complex manifold. We say (E,h) is Griffiths negative (resp. positive) if for any $z \in M, e \in E_z$ and $v \in T^{1,0}M$ we have*

$$\Theta_z(e \otimes v, e \otimes v) \leq 0 \quad (\textit{resp.} \geq 0),$$

and the equality holds if and only if $e = 0$ or $v = 0$ (i.e., $e \otimes v = 0$). In local coordinates, this means that for any $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, \cdots, \xi_k) \in \mathbb{C}^k$ we have $\Theta_{\alpha\bar{\beta}i\bar{j}}\xi_\alpha\overline{\xi_\beta}v_i\overline{v_j} \leq 0$ (resp. ≥ 0), and the equality holds if and only if $v = 0$ or $\xi = 0$.

Lemma 2.4. *Let (E,h) be a Hermitian vector bundle over a complex manifold. Consider the properties:*

(1) *The Hermitian vector bundle (E,h) is Griffiths negative (resp. positive).*
(2) *The determinant line bundle $L = \det E$ with the determinant metric $\det h$ is negative (resp. positive).*

In general, it holds that (1) implies (2). If the curvature Θ splits, then (1) is equivalent to (2).

Proof. The fact that (1) implies (2) follows directly from the identity

$$\big(\mathrm{Ric}(L)\big)_{i\bar{j}} = \big(\mathrm{Ric}(E)\big)_{i\bar{j}}.$$

When Θ splits, by (2.3) we have $\Theta_{\alpha\bar{\beta}i\bar{j}} = \frac{1}{k}h_{\alpha\bar{\beta}}\big(\mathrm{Ric}(L)\big)_{i\bar{j}}$. Then it is clear that (2) also implies (1) in this case. $\square$

Remark 2.5. Let $(L_1,h_1),\cdots,(L_k,h_k)$ be line bundles over a complex manifold M, and consider the vector bundle $(E,h) := (L_1,h_1) \oplus \cdots \oplus (L_k,h_k)$. Then it follows that $\mathrm{Ric}(\det E, \det h) = \mathrm{Ric}(E,h) = \sum_{j=1}^k \mathrm{Ric}(L_j,h_j)$. By using Proposition 2.1 and Lemma 2.4 we immediately obtain that if (E,h) is curvature split, then (E,h) is Griffiths negative if and only if each (L_j,h_j) for $1 \leq j \leq k$ is negative.

Remark 2.6. Consider the special case when M is a Riemann surface (i.e., $n = 1$). Suppose (E,h) is Griffiths negative vector bundle over M and let g be the metric induced by $-\mathrm{Ric}(E,h)$. Then by (2.3), the vector bundle (E,h) is curvature split if and only if (E,h) is Hermitian-Einstein.

For a Hermitian holomorphic vector bundle (E,h), recall that $S(E) := \{e \in E : |e|_h = 1\}$ denotes the sphere bundle of (E,h). We conclude this section by noting the following fundamental fact; see [11, Lemma 1.2.2] and [13, Proposition 5.3].

Proposition 2.7. *If (E, h) is Griffiths negative, then $S(E)$ is strongly pseudoconvex.*

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. As the obstruction flatness is a local property (cf. [8]), we need to show that for any point $p \in S(E)$, there exists some neighborhood U of p on E such that $S(E) \cap U$ is obstruction flat. By the work of Graham [8] again, it suffices to construct a function $u \in C^\infty(U)$ such that $u = 0$ on $S(E) \cap U$ and $J(u) = 1$ on the pseudoconvex side of $S(E) \cap U$. To do this, we shall establish a series of propositions and lemmas.

Proposition 3.1. *Let $P(y)$ be a monic polynomial in $y \in \mathbb{R}$ of degree $m-1 \geq 1$ and $Q(y)$ a polynomial satisfying $\frac{dQ}{dy} = (m+1)yP(y)$ (thus Q is a monic polynomial of degree $m+1$ and is unique up to a constant). Suppose $\hat{P}$ and $\hat{Q}$ are polynomials defined by*

$$\hat{P}(x) = x^{m-1}P(x^{-1}), \quad \hat{Q}(x) = x^{m+1}Q(x^{-1}).$$

Let $I \subset (0, \infty)$ be an open interval containing $r = 1$ and $Z(r)$ a real analytic function on I satisfying the following conditions:

$$rZ'\hat{P}(Z) + \hat{Q}(Z) = 0 \text{ on } I, \quad Z(1) = 0. \tag{3.1}$$

$$Z'(r) < 0 \text{ and } \hat{P}(Z) > 0 \text{ on } I,\ Z(r) > 0 \text{ on } I_0 := I \cap (0,1), \text{ and } Z'(1) = -1. \tag{3.2}$$

Let $k \in \mathbb{Z}^+$ with $k \leq m-1$ and set $\phi(r) = 2\big(\frac{r^{2k-1}}{-Z'\hat{P}(Z)}\big)^{\frac{1}{m+1}} Z$ on I. Then ϕ is real analytic on I, $\phi(1) = 0$ and $\phi > 0$ on I_0. Moreover, ϕ satisfies that $(m+1)rZ\phi' + (m+1-2kZ)\phi = 0$ on I, and $\phi'(1) = -2$.

Remark 3.2. Since P and Q are monic polynomials, the polynomials $\hat{P}$ and $\hat{Q}$ satisfy $\hat{P}(0) = 1$ and $\hat{Q}(0) = 1$. By elementary ODE theory, (3.1) has a real analytic solution Z which satisfies (3.2) in some open interval I containing 1.

Proof. It is clear that ϕ is real analytic on I, $\phi(1) = 0$, and $\phi > 0$ on I_0 by the definition of ϕ and the assumption of Z. We only need to prove the last assertion in Proposition 3.1. The proof is similar to that of Proposition 2.1 in [5]. We just highlight the following two lemmas and the conclusion will be evident.

Lemma 3.3. *Let Z be as in Proposition 3.1. Then we have*

$$r(Z'Z^{-(m+1)}\hat{P}(Z))' = (m+1)Z'Z^{-(m+2)}\hat{P}(Z) - Z'Z^{-(m+1)}\hat{P}(Z) \text{ on } I_0.$$

Equivalently,

$$r\frac{(Z'Z^{-(m+1)}\hat{P}(Z))'}{Z'Z^{-(m+1)}\hat{P}(Z)} = -1 + (m+1)Z^{-1} \text{ on } I_0.$$

Lemma 3.4. *Let ϕ and Z be as in Proposition 3.1. Then we have*

$$r\frac{(Z'Z^{-(m+1)}\hat{P}(Z))'}{Z'Z^{-(m+1)}\hat{P}(Z)} = (2k-1) - (m+1)r\frac{\phi'}{\phi} \ \textit{on}\ I_0.$$

Lemma 3.3 is identical to Lemma 2.3 in [5]; Lemma 3.4 follows from a similar argument as that of Lemma 2.4 in [5]. Thus, we omit their proofs. Finally we compare (the second equation in) Lemma 3.3 and Lemma 3.4 to obtain $(m+1)rZ\phi' + (m+1-2kZ)\phi = 0$ on I_0. By analyticity, it actually holds on I. Recall $Z(1) = 0, \hat{P}(0) = 1$ and $Z'(1) = -1$. It then follows from the definition of ϕ that $\phi'(1) = -2$. This finishes the proof of Proposition 3.1. $\square$

Let (M, g) and (E, h) be as in Theorem 1.2. Denote by n the complex dimension of M and by k the rank of E. Choose a coordinate chart (D, z) of M with a local frame $\{e_\alpha\}_{\alpha=1}^k$ of E over D. Let $\pi : E \to M$ be the canonical projection. Then we have

$$\pi^{-1}(D) = \Big\{\sum_{\alpha=1}^k \xi_\alpha e_\alpha(z) : (z, \xi) \in D \times \mathbb{C}^k\Big\}.$$

Under this trivialization, the sphere bundle $S(E)$ over D can be written as

$$\Sigma := S(E) \cap \pi^{-1}(D) = \Big\{(z, \xi) \in D \times \mathbb{C}^k : \sum_{\alpha,\beta=1}^k h_{\alpha\bar{\beta}}(z, \bar{z})\xi_\alpha\overline{\xi_\beta} = 1\Big\},$$

where $h_{\alpha\bar{\beta}}(z, \bar{z}) = h(e_\alpha(z), e_\beta(z))$ for $1 \le \alpha, \beta \le k$. In the local coordinates $z = (z_1, \cdots, z_n)$ we write $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$ on D. As g is induced by $-\operatorname{Ric}(E, h)$, we have $g_{i\bar{j}} = \frac{\partial^2 \log H}{\partial z_i \partial \overline{z_j}}$ where $H = \det(h_{\alpha\bar{\beta}})$. Let $G = \det(g_{i\bar{j}})$ on D.

Set $m = n + k$ and denote by I_n the $n \times n$ identity matrix. Given $p \in M$, let $T(y, p)$ be the characteristic polynomial of the linear operator $\frac{2k}{m+1}\operatorname{Ric}\cdot g^{-1} : T_p^{1,0}M \to T_p^{1,0}M$. That is,

$$T(y, p) = \det(yI_n - \frac{2k}{m+1}\mathrm{Ric} \cdot g^{-1}). \tag{3.3}$$

In the local coordinates $z = (z_1, \cdots, z_n)$, writing the Ricci tensor as $\operatorname{Ric} = (R_{i\bar{k}})_{1\le i,k\le n} = -(\frac{\partial^2 \log G}{\partial z_i \partial \overline{z_k}})_{1\le i,k\le n}$, we see that $T(y, p)$ is the determinant of the $n \times n$ matrix $(y\delta_{ij} - \frac{2k}{m+1}R_{i\bar{k}} \cdot g^{j\bar{k}}(p))$.

Now we define

$$P(y, p) := (y - \frac{2k}{m+1})^{k-1}T(y, p). \tag{3.4}$$

By the constant Ricci eigenvalue assumption, $P(y, p)$ does not depend on p. We will therefore just denote it by $P(y)$. It is clear that $P(y)$ is a monic polynomial in y of degree $m-1$. We apply Proposition 3.1 to this polynomial $P(y)$ (with k equal to the rank of E) to obtain polynomials $Q(y), \hat{P}(x), \hat{Q}(x)$, as well as real analytic functions $Z(r)$ and $\phi(r)$ in some interval I containing $r=1$. We let $y(r) = \frac{1}{Z(r)}$ for $r \in I_0$. By Proposition 3.1, $(m+1)rZ\phi' + (m+1-2kZ)\phi = 0$ on I. It then follows that

$$y(r) = \frac{2k\phi(r) - (m+1)r\phi'(r)}{(m+1)\phi(r)} = \frac{2k}{m+1} - r\frac{\phi'}{\phi} \quad \text{on } I_0. \tag{3.5}$$

Theorem 1.2 will follow from the next proposition.

Proposition 3.5. *Let*

$$U = \Big\{ w := (z,\xi) \in D \times \mathbb{C}^k : |w|_h \in I \Big\}, \qquad U_0 = \Big\{ w := (z,\xi) \in D \times \mathbb{C}^k : |w|_h \in I_0 \Big\},$$

where $|w|_h^2 = \sum_{\alpha,\beta=1}^k h_{\alpha\bar{\beta}}(z,\bar{z})\xi_\alpha\overline{\xi_\beta}$. *Set*

$$u(w) = k^{\frac{n}{m+1}}(GH)^{-\frac{1}{m+1}}\phi\big(|w|_h\big) \ \text{for } w \in U. \tag{3.6}$$

Here $G(w)$ is understood as $G(z)$ for $w=(z,\xi)$. Likewise for H. Then u is smooth in U and satisfies

$$J(u) = 1 \ \text{on } U_0, \quad u = 0 \ \text{on } \Sigma.$$

Consequently, Σ is obstruction flat.

Proof. The smoothness of u follows easily from that of ϕ, as well as that of G, H and h. We thus only need to prove the remaining assertions. For that, we first prove the following lemma. Set $X = X(w) = |w|_h$ for $w \in U$, and

$$Y = Y(w) = \frac{2k}{m+1} - X\frac{\phi'(X)}{\phi(X)} \quad \text{for } w \in U_0. \tag{3.7}$$

Then by (3.5), we have $Y = y(r)|_{r=X} = \frac{1}{Z(r)}|_{r=X}$. Note that X and Y are independent of the choice of local coordinates and local frame.

In the following, we will also write the coordinates w of $D\times\mathbb{C}^k$ as $(w_1, \cdots, w_{m-1}, w_m)$. That is, we identify w_i with z_i for $1 \le i \le n$, and $w_{n+\alpha}$ with ξ_α for $1 \le \alpha \le k$. For a sufficiently differentiable function Φ on an open subset of $D \times \mathbb{C}^k$, we write, for $1 \le s,t \le m$, $\Phi_s = \frac{\partial\Phi}{\partial w_s}$, $\Phi_{\bar{t}} = \frac{\partial\Phi}{\partial\overline{w_t}}$, and $\Phi_{s\bar{t}} = \frac{\partial^2\Phi}{\partial w_s\partial\overline{w_t}}$. To simplify the later computations, we introduce the following lemma.

Lemma 3.6. *Let $\pi : (E,h) \to (M,g)$ be a Hermitian vector bundle of rank k over an n-dimensional Hermitian manifold (M,g). Write $m = n+k$. Let Ω be a smooth (m,m)-form on E (regarded as an m-dimensional complex manifold) (resp. an open subset $V \subset E$).*

Then we can define a smooth function Φ *on* E *(resp. on* V*) in the following manner: Pick any local coordinate chart* (D, z) *of* M *and any local frame* $\{e_\alpha\}_{\alpha=1}^k$ *of* E *over* D*. This induces a natural system of coordinates* $w = (z, \xi)$ *for* E *on* $\pi^{-1}(D)$ *with*

$$\pi^{-1}(D) = \Big\{ \sum_{\alpha=1}^{k} \xi_\alpha e_\alpha(z) : (z, \xi) \in D \times \mathbb{C}^k \Big\}.$$

In the above coordinates, we write $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$*,* $h_{\alpha\bar{\beta}} = h(e_\alpha, e_\beta)$ *and* $\Omega = \sigma(z, \xi) dz \wedge d\xi \wedge d\overline{z} \wedge d\overline{\xi}$ *where* $dz = dz_1 \wedge \cdots \wedge dz_n$ *and* $d\xi = d\xi_1 \wedge \cdots \wedge d\xi_k$*. We define the function* $\Phi : \pi^{-1}(D) \to \mathbb{C}$ *by*

$$\Phi(z, \xi) = \frac{\sigma(z, \xi)}{\det\big(g_{i\bar{j}}(z)\big) \cdot \det\big(h_{\alpha\bar{\beta}}(z)\big)}.$$

Then Φ *is independent of the choice of local coordinates and local frame. As a result,* Φ *is a well-defined function on* E *(resp. on* V*).*

Proof. To show Φ is well-defined, we take two local coordinate charts, (D, z) and $(\widetilde{D}, \widetilde{z})$ of M, and two local frames of E, $\{e_\alpha\}_{\alpha=1}^k$ over D and $\{\widetilde{e}_\alpha\}_{\alpha=1}^k$ over $\widetilde{D}$. Then we obtain two induced coordinate systems, $w = (z, \xi)$ on $\pi^{-1}(D)$ and $\widetilde{w} = (\widetilde{z}, \widetilde{\xi})$ on $\pi^{-1}(\widetilde{D})$. On $D \cap \widetilde{D}$ we can write $\widetilde{z} = \varphi(z)$ for some biholomorphism φ and $\widetilde{\xi} = \xi \cdot A(z)$ for some holomorphic map $A : D \cap \widetilde{D} \to \mathrm{Gl}(k, \mathbb{C})$. By the relation $g = \sum_{i,j=1} g_{i\bar{j}}(z) dz_i \otimes d\overline{z_j} = \sum_{i,j=1}^n \widetilde{g}_{i\bar{j}}(\widetilde{z}) d\widetilde{z}_i \otimes d\overline{\widetilde{z}_j}$, we have

$$\det\big(g_{i\bar{j}}(z)\big) = \det\big(\widetilde{g}_{i\bar{j}}(\widetilde{z})\big) \cdot \big|J\varphi(z)\big|^2,$$

where $J(\varphi)$ is the determinant of the Jacobian matrix of φ. Similarly, we also have

$$\det\big(h_{\alpha\bar{\beta}}(z)\big) = \det\big(\widetilde{h}_{\alpha\bar{\beta}}(\widetilde{z})\big) \cdot \big|\det A(z)\big|^2.$$

On the other hand, by writing $\Omega = \sigma(z, \xi) dz \wedge d\xi \wedge d\overline{z} \wedge d\overline{\xi} = \widetilde{\sigma}(\widetilde{z}, \widetilde{\xi}) d\widetilde{z} \wedge d\widetilde{\xi} \wedge d\overline{\widetilde{z}} \wedge d\overline{\widetilde{\xi}}$ on $\pi^{-1}(D) \cap \pi^{-1}(\widetilde{D})$, we get

$$\sigma(z, \xi) = \widetilde{\sigma}(\widetilde{z}, \widetilde{\xi}) \cdot \big|J\varphi(z)\big|^2 \cdot \big|\det A(z)\big|^2.$$

It follows that

$$\frac{\sigma(z, \xi)}{\det\big(g_{i\bar{j}}(z)\big) \cdot \det\big(h_{\alpha\bar{\beta}}(z)\big)} = \frac{\widetilde{\sigma}(\widetilde{z}, \widetilde{\xi})}{\det\big(\widetilde{g}_{i\bar{j}}(\widetilde{z})\big) \cdot \det\big(\widetilde{h}_{\alpha\bar{\beta}}(\widetilde{z})\big)} \quad \text{on } \pi^{-1}(D) \cap \pi^{-1}(\widetilde{D}).$$

So the proof is completed. $\square$

Next we will prove the following lemma on the computation of $\det\big((-\log u)_{s\bar{t}}\big)$.

Lemma 3.7. *Let $u(w)$ be as defined in Proposition 3.5. Write $Y' = \frac{dY}{dX}$, i.e., $Y' = y'(r)|_{r=X}$. Then the following is a well-defined function on U_0:*

$$\Phi(w) = \frac{\det\big((-\log u)_{s\bar{t}}(w)\big)}{\det\big(g_{i\bar{j}}(z)\big)\cdot\det\big(h_{\alpha\bar{\beta}}(z)\big)}.$$

That is, Φ is independent of the choice of local coordinates of M and local frame of E. Moreover,

$$\Phi = \frac{P(Y)Y'}{2^{m+1}k^n X^{2k-1}} \quad \text{on } U_0.$$

Proof. Consider the Hermitian $(1,1)$-form $\widetilde{\omega} = -\sqrt{-1}\partial\bar{\partial}\log u$ on $U_0 \subset E$. In the above local coordinates $w = (z, \xi)$, we can write $\widetilde{\omega} = \sqrt{-1}\sum_{s,t=1}^{m}(-\log u)_{s\bar{t}}dw_s \wedge d\overline{w_t}$. Then $\frac{\widetilde{\omega}^m}{m!}$ is a smooth (m,m)-form on U_0 and

$$\frac{\widetilde{\omega}^m}{m!} = (\sqrt{-1})^m \det\big((-\log u)_{s\bar{t}}\big)dw_1 \wedge d\overline{w_1} \wedge \cdots \wedge dw_m \wedge d\overline{w_m}.$$

By Lemma 3.6, Φ is well-defined on U_0.

To prove the second assertion, we fix a point $p \in E$ and write $q = \pi(p)$. To compute the value of Φ at p, by the first part of the lemma, we can use any local coordinates of M and any local frame of E. In particular, we can choose a local coordinates of M and a local frame of E at q such that the induced coordinates of E near p, which we still denote by $w = (z, \xi)$, satisfy $z(q) = 0, h_{\alpha\bar{\beta}}(q) = \delta_{\alpha\beta}$ and $dh_{\alpha\bar{\beta}}(q) = 0$, where $\delta_{\alpha\beta}$ is the Kronecker delta. Under the above coordinates, the curvature (2.2) at point q simplifies into

$$\Theta_{\alpha\bar{\beta}i\bar{j}}(0) = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z_i \partial \overline{z_j}}(0).$$

We take the logarithm of u (which is defined in (3.6)) and obtain

$$-\log u = -\frac{n}{m+1}\log k + \frac{1}{m+1}\log G + \frac{1}{m+1}\log H - \log\phi(X),$$

where $X = |w|_h = \big(\sum_{\alpha,\beta=1}^{k} h_{\alpha\bar{\beta}}(z,\bar{z})\xi_\alpha\overline{\xi_\beta}\big)^{1/2}$. A straightforward computation yields

$$(-\log u)_{s\bar{t}} = \frac{1}{m+1}\big(\log G\big)_{s\bar{t}} + \frac{1}{m+1}\big(\log H\big)_{s\bar{t}} - \Big(\frac{\phi'}{\phi}X_{s\bar{t}} + \big(\frac{\phi'}{\phi}\big)'X_s X_{\bar{t}}\Big)$$

for any $1 \le s, t \le m$.

Since $X_{s\bar{t}} = X(\log X)_{s\bar{t}} + \frac{1}{X}X_s X_{\bar{t}}$, the above writes into

$$
\begin{aligned}
(-\log u)_{s\bar{t}} = &\frac{1}{m+1}\big(\log G\big)_{s\bar{t}} + \frac{1}{m+1}\big(\log H\big)_{s\bar{t}} - X\big(\frac{\phi'}{\phi}\big)\big(\log X\big)_{s\bar{t}} \\
&- \Big(\frac{1}{X}\big(\frac{\phi'}{\phi}\big) + \big(\frac{\phi'}{\phi}\big)'\Big) X_s X_{\bar{t}}.
\end{aligned}
$$

To continue the computation of this Hessian matrix, we shall divide it into the following three cases.

Case 1. $1 \le s, t \le n$.

For this case, we have $w_s = z_s$ and $w_t = z_t$. We denote $i = s$ and $j = t$ for simplicity. At $w = (0, \xi)$, by the facts $h_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ and $dh_{\alpha\bar{\beta}} = 0$ it follows that $X = |\xi|$ and $X_i = X_{\bar{j}} = 0$. Moreover, we also have

$$
\begin{aligned}
\big(\log X\big)_{i\bar{j}}\big|_{w=(0,\xi)} =&\frac{1}{2}\big(\log X^2\big)_{i\bar{j}}\big|_{w=(0,\xi)} = \frac{1}{2}\frac{(X^2)_{i\bar{j}}}{X^2}\Big|_{w=(0,\xi)} \\
=&\frac{1}{2|\xi|^2}\sum_{\alpha,\beta=1}^{k} \frac{\partial h_{\alpha\bar{\beta}}}{\partial z_i \partial \overline{z_j}}(0)\xi_\alpha \overline{\xi_\beta} = \frac{1}{2|\xi|^2}\sum_{\alpha,\beta=1}^{k}\big(-\Theta_{\alpha\bar{\beta}i\bar{j}}(0)\big)\xi_\alpha\overline{\xi_\beta}.
\end{aligned}
$$

Since the curvature Θ splits and the Kähler metric is induced by $-\operatorname{Ric}(E, h)$ by the assumption of Theorem 1.2, it follows that

$$
\Theta_{\alpha\bar{\beta}i\bar{j}}(0) = \frac{1}{k}h_{\alpha\bar{\beta}}(0) \cdot \big(\operatorname{Ric}(E,h)\big)_{i\bar{j}}(0) = -\frac{1}{k}\delta_{\alpha\beta}\cdot g_{i\bar{j}}(0).
$$

Therefore,

$$
(\log X)_{i\bar{j}}\big|_{w=(0,\xi)} = \frac{1}{2k}g_{i\bar{j}}(0).
$$

Denote the Ricci form $\operatorname{Ric}(g) = \sqrt{-1}R_{i\bar{j}}dz_i \wedge d\overline{z_j}$. Then $(\log G)_{i\bar{j}} = -R_{i\bar{j}}$, $(\log H)_{i\bar{j}} = g_{i\bar{j}}$ and by (3.7) we have

$$
\begin{aligned}
(-\log u)_{s\bar{t}}\big|_{w=(0,\xi)} &= -\frac{1}{m+1}R_{i\bar{j}}(0) + \frac{1}{m+1}g_{i\bar{j}}(0) - \frac{X}{2k}\frac{\phi'}{\phi}g_{i\bar{j}}(0) \\
&= -\frac{1}{m+1}R_{i\bar{j}}(0) + \frac{Y}{2k}g_{i\bar{j}}(0).
\end{aligned}
$$

Case 2. $1 \le s \le n, n+1 \le t \le n+k$ or $n+1 \le s \le n+k, 1 \le t \le n$.

For this case, a similar computation yields $(\log G)_{s\bar{t}}$ and $(\log H)_{s\bar{t}}$ vanish identically and $(\log X)_{s\bar{t}}|_{w=(0,\xi)} = X_s X_{\bar{t}}|_{w=(0,\xi)} = 0$. Therefore, $(-\log u)_{s\bar{t}}|_{w=(0,\xi)} = 0$.

Case 3. $n+1 \le s, t \le n+k$.

For this case, we have $w_s = \xi_{s-n}$ and $w_t = \xi_{t-n}$. We denote $\alpha = s-n$ and $\beta = t-n$ for simplicity. Then a straightforward computation yields that at any $w = (0,\xi) \in U_0$ we have $X = |\xi|$, $X_s = \frac{1}{2}|\xi|^{-1}\overline{\xi_\alpha}$, $\overline{X_t} = \frac{1}{2}|\xi|^{-1}\xi_\beta$ and

$$\big(\log X\big)_{s\bar{t}} = \frac{1}{2X^2}\delta_{\alpha\beta} - \frac{1}{2X^4}\overline{\xi_\alpha}\xi_\beta.$$

As $(\log G)_{s\bar{t}}$ and $(\log H)_{s\bar{t}}$ vanish identically in this case, we have at $w = (0, \xi)$

$$(-\log u)_{s\bar{t}} = -X\big(\frac{\phi'}{\phi}\big)\Big(\frac{1}{2X^2}\delta_{\alpha\beta} - \frac{1}{2X^4}\overline{\xi_\alpha}\xi_\beta\Big) - \Big(\frac{1}{X}\big(\frac{\phi'}{\phi}\big) + \big(\frac{\phi'}{\phi}\big)'\Big)\frac{\overline{\xi_\alpha}\xi_\beta}{4X^2}.$$

By (3.7), we further write it into

$$\begin{aligned}(-\log u)_{s\bar{t}}\,\big|_{w=(0,\xi)} =& \big(Y - \frac{2k}{m+1}\big)\Big(\frac{1}{2X^2}\delta_{\alpha\beta} - \frac{1}{2X^4}\overline{\xi_\alpha}\xi_\beta\Big) + \frac{Y'}{X}\frac{\overline{\xi_\alpha}\xi_\beta}{4X^2}\\ =& \frac{1}{2X^2}\big(Y - \frac{2k}{m+1}\big)\delta_{\alpha\beta} + \big(\frac{XY'}{4} - \frac{Y}{2} + \frac{k}{m+1}\big)\frac{\overline{\xi_\alpha}\xi_\beta}{X^4}.\end{aligned}$$

Combining the above three cases, we see the complex Hessian matrix $\big((-\log u)_{s\bar{t}}\big)$ at $w = (0, \xi)$ is block diagonal. Moreover,

$$\det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le m} = \det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le n}\cdot\det\big((-\log u)_{s\bar{t}}\big)_{n+1\le s,t\le m}. \tag{3.8}$$

Now we need to compute the determinants appearing on the right hand side of the above equation. By the above computation in Case 1, we have at $w = (0, \xi)$

$$\begin{aligned}\det\big((-\log u)_{i\bar{j}}\big)_{1\le i,j\le n} =& \det\Big(-\frac{1}{m+1}R_{i\bar{j}}(0) + \frac{Y}{2k}g_{i\bar{j}}(0)\Big)\\ =& \big(\frac{1}{2k}\big)^n\det\Big(Y\delta_{ij} - \frac{2k}{m+1}R_{i\bar{k}}(0)\,g^{j\bar{k}}(0)\Big)\cdot\det\big(g_{i\bar{j}}(0)\big)\\ =& \big(\frac{1}{2k}\big)^n\,T(Y)\,G,\end{aligned}$$

where the last equality follows from (3.3). For the second determinant on the right hand side of (3.8), by the computation in Case 3, we have at $w = (0, \xi)$,

$$\begin{aligned}\det\big((-\log u)_{s\bar{t}}\big)_{n+1\le s,t\le m} = \det\Big(&\frac{1}{2X^2}\big(Y - \frac{2k}{m+1}\big)\delta_{\alpha\beta}\\ &+ \big(\frac{XY'}{4} - \frac{Y}{2} + \frac{k}{m+1}\big)\frac{\overline{\xi_\alpha}\xi_\beta}{X^4}\Big)_{1\le\alpha,\beta\le k}.\end{aligned}$$

As $X = |\xi|$ at $w = (0, \xi)$, by the matrix determinant lemma we get

$$\begin{aligned}&\det\big((-\log u)_{s\bar{t}}\big)_{n+1\le s,t\le m}\\ &= \Big(\frac{1}{2X^2}\big(Y - \frac{2k}{m+1}\big)\Big)^{k-1}\Big(\frac{1}{2X^2}\big(Y - \frac{2k}{m+1}\big) + \big(\frac{XY'}{4} - \frac{Y}{2} + \frac{k}{m+1}\big)\frac{1}{X^2}\Big)\\ &= \frac{Y'}{2^{k+1}X^{2k-1}}\big(Y - \frac{2k}{m+1}\big)^{k-1}.\end{aligned}$$

We now plug these results back into (3.8) and further use (3.4) to obtain that at $w = (0, \xi)$

$$\det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le m} = \frac{Y'}{2^{m+1}k^n X^{2k-1}} P(Y)\, G.$$

Note that at $w = (0, \xi)$ we have $H = 1$. Therefore,

$$\frac{\det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le m}}{GH} = \frac{P(Y)Y'}{2^{m+1}k^n X^{2k-1}}.$$

This proves Lemma 3.7. $\square$

Now we resume the proof of Proposition 3.5. By the definition of ϕ in Proposition 3.1, for any $r \in I_0$ we have

$$-Z'\hat{P}(Z)Z^{-(m+1)} = 2^{m+1}r^{2k-1}\phi^{-(m+1)}(r)$$

Recall that $y(r) = \frac{1}{Z(r)}$. We thus get

$$y'\hat{P}(y^{-1})y^{m-1} = 2^{m+1}r^{2k-1}\phi^{-(m+1)}(r).$$

By the definition of $\hat{P}$ in Proposition 3.1, we have $P(y) = y^{m-1}\hat{P}(y^{-1})$. Therefore,

$$y'P(y) = 2^{m+1}r^{2k-1}\phi^{-(m+1)}(r).$$

Now we take $r = X = |w|_h$ for $w = (z, \xi) \in U_0$. Then

$$Y'P(Y) = 2^{m+1}X^{2k-1}\phi^{-(m+1)}(X) = 2^{m+1}k^n X^{2k-1}(GH)^{-1}u^{-(m+1)}(X),$$

where the second equality follows from the definition of u. Therefore,

$$\frac{Y'P(Y)}{2^{m+1}k^n X^{2k-1}} = (GH)^{-1}u^{-(m+1)}(X).$$

By Lemma 3.7 we obtain $\det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le m} = u^{-(m+1)}$ on U_0. Therefore, $J(u) = 1$ on U_0. Since $\phi(1) = 0$ as proved in Proposition 3.1, we obtain the boundary condition that $u = 0$ on Σ. The latter part of Proposition 3.5 follows from the first part and Graham's work (see the first paragraph of this section). $\square$

4. Proof of Theorem 1.3

We establish the following proposition before proceeding to the proof of Theorem 1.3.

Proposition 4.1. *Let $n \geq 1$, $k \geq 1$ and $m = n + k$. For given real numbers $\lambda_1 \leq \cdots \leq \lambda_n < 1$, set*

$$P(y) = \big(y - \frac{2k}{m+1}\big)^{k-1} \prod_{i=1}^{n} \big(y - \frac{2k\lambda_i}{m+1}\big).$$

Let $Q(y)$ be the polynomial satisfying $\frac{dQ}{dy} = (m+1)yP(y)$ and $Q(\frac{2k}{m+1}) = 0$ (thus Q is a monic polynomial of degree $m+1$ and is uniquely determined). Suppose $\hat{P}$ and $\hat{Q}$ are polynomials defined by

$$\hat{P}(x) = x^{m-1}P(x^{-1}), \quad \hat{Q}(x) = x^{m+1}Q(x^{-1}).$$

Then the following conclusions hold:

(1) *There exists a unique real analytic function $Z = Z(r)$ on $[-1,1]$ (meaning it extends real analytically to some open interval containing $[-1,1]$) satisfying the following conditions:*

$$rZ'\hat{P}(Z) + \hat{Q}(Z) = 0, \quad Z(1) = 0. \tag{4.1}$$

Moreover, Z is an even function satisfying $Z(0) = \frac{m+1}{2k}$, $Z'(0) = 0$, $Z' < 0$ on $(0,1]$ and $Z'(1) = -1$, $Z''(0) < 0$. Consequently, $Z \in (0, \frac{m+1}{2k})$ on $(-1,0) \cup (0,1)$.

(2) *Let $\phi(r) = 2(\frac{r^{2k-1}}{-Z'\hat{P}(Z)})^{\frac{1}{m+1}} Z$. Then ϕ is real analytic on $[-1,1]$. In addition, ϕ is an even function satisfying $\phi > 0$ on $(-1,1)$ and $\phi(1) = 0$. Moreover, ϕ satisfies $(m+1)rZ\phi' + (m+1-2kZ)\phi = 0$ and $\phi'(1) = -2$.*

Remark 4.2. For the polynomials $P, Q, \hat{P}$ and $\hat{Q}$ defined in Proposition 4.1, note that they satisfy $\hat{P}(0) = \hat{Q}(0) = 1$, $P > 0$ on $(\frac{2k}{m+1}, +\infty)$, $\hat{P} > 0$ on $(0, \frac{m+1}{2k})$ and $\frac{dQ}{dy} > 0$ on $(\frac{2k}{m+1}, \infty)$, $Q > 0$ on $(\frac{2k}{m+1}, \infty)$, $\hat{Q} > 0$ on $(0, \frac{m+1}{2k})$ and $\hat{Q}(\frac{m+1}{2k}) = 0$.

Proof. The proposition was proved in [5, Proposition 2.7] for $k = 1$. We will extend the ideas in [5] to prove for the case $k \geq 2$. Writing

$$\lambda = \frac{m+1}{2k}, \tag{4.2}$$

we can express $\hat{P}$ as follows

$$\hat{P}(x) = \big(1 - \frac{x}{\lambda}\big)^{k-1} \prod_{i=1}^{n} \big(1 - \frac{\lambda_i}{\lambda}x\big) := (x-\lambda)^{k-1}h(x).$$

Here $h(x) = (-1)^{k-1}\lambda^{-(k-1)} \prod_{i=1}^{n}\big(1 - \frac{\lambda_i}{\lambda}x\big)$ and it is a polynomial satisfying $h(\lambda) \neq 0$. Note the polynomial $\hat{Q}$ has a zero of order k at $x = \lambda$. Thus we can write $\hat{Q}(x) = (x-\lambda)^k g(x)$ for some polynomial g satisfying $g(\lambda) \neq 0$.

We introduce the following lemma on the polynomials h and g. For two functions f_1 and f_2, we write $f_1 \sim f_2$ as $x \to \lambda$ if $\lim_{x\to\lambda} \frac{f_1(x)}{f_2(x)} = 1$.

Lemma 4.3. *It holds that $g(\lambda) = -2h(\lambda)$ and $(-1)^{k-1}h(\lambda) > 0$. In particular, $\frac{\hat{Q}(x)}{\hat{P}(x)} \sim -2(x-\lambda)$ as $x \to \lambda$.*

Proof. Note that $(-1)^{k-1}h(\lambda) = \lambda^{-(k-1)} \prod_{i=1}^{n}(1-\lambda_i)$, which is clearly positive as all λ_i's are strictly less than 1. We next prove the first identity. For that, we notice that

$$h(\lambda) = \frac{1}{(k-1)!} \frac{d^{k-1}\hat{P}(x)}{dx^{k-1}}\Big|_{x=\lambda} \quad \text{and} \quad g(\lambda) = \frac{1}{k!}\frac{d^k\hat{Q}(x)}{dx^k}\Big|_{x=\lambda}.$$

Since $\hat{Q}(x) = x^{m+1}Q(\frac{1}{x})$, by the definition of polynomial Q we have

$$\frac{d\hat{Q}(x)}{dx} = (m+1)x^m Q\big(\frac{1}{x}\big) - x^{m-1}Q'\big(\frac{1}{x}\big) = \frac{(m+1)}{x}\hat{Q}(x) - \frac{(m+1)}{x}\hat{P}(x).$$

As $\hat{P}$ and $\hat{Q}$ respectively have a zero of order $(k-1)$ and k at $x = \lambda$, if we further differentiate the above equation $(k-1)$ times and evaluate it at $x = \lambda$, then

$$\frac{d^k\hat{Q}(x)}{dx^k}\Big|_{x=\lambda} = -\frac{(m+1)}{\lambda}\frac{d^{k-1}\hat{P}(x)}{dx^{k-1}}\Big|_{x=\lambda} = -2k!\,h(\lambda).$$

It follows that $g(\lambda) = -2h(\lambda)$. As a result, we get

$$\frac{\hat{Q}(x)}{\hat{P}(x)} = \frac{g(x)}{h(x)}(x-\lambda) \sim -2(x-\lambda) \quad \text{as } x \to \lambda.$$

So the proof is completed. □

Now we are ready to prove part (1) of Proposition 4.1. It follows easily from the assumption and elementary ODE theory that the ODE in (4.1) has a real analytic solution Z in some open interval I containing 1. Since $\hat{P}(0) = 1$ and $\hat{Q}(0) = 1$, we see the ODE in (4.1) implies $Z'(1) = -1$. Set

$$t_0 = \inf\{t \in [0,1) : \text{ on } (t,1] \text{ there exists a real analytic solution } Z \text{ to (4.1) with } Z' < 0\}. \tag{4.3}$$

By definition, $0 \le t_0 < 1$ and on $(t_0, 1]$ there is a real analytic solution Z to (4.1) with $Z' < 0$.

Lemma 4.4. *The number defined in (4.3) satisfies $t_0 = 0$. Consequently, on $(0,1]$ there exists a (unique) real analytic solution Z to (4.1) and it satisfies $Z' < 0$.*

Proof. Seeking a contradiction, we assume $t_0 > 0$. Since Z is decreasing on $(t_0, 1)$, we conclude that $\mu := \lim_{r \to t_0^+} Z(r) > 0$ exists (allowing *a priori* $\mu = +\infty$ as the limit). We note that $\mu \le \lambda$, where λ is the number introduced in (4.2). For, if this were not the case, then since $Z(1) = 0$ there would exist some $t^* \in (t_0, 1)$ such that $Z(t^*) = \lambda$. By Lemma 4.3, at $r = t^*$ the ODE (4.1) gives $Z'(t^*) = 0$, contradicting the fact $Z'(t^*) < 0$. Therefore, we proceed by examining the following two cases.

Case I. Assume $\mu = \lambda$. In this case, by Lemma 4.3 the ODE (4.1) gives

$$-Z'(r) = \frac{\hat{Q}(Z)}{r\hat{P}(Z)} \sim -2\frac{Z-\lambda}{r} \qquad \text{as } r \to t_0^+.$$

Thus, there exists some constants $\delta > 0$ and $C > 0$ such that $-Z'(r) \le C(\lambda - Z)$ for any $r \in (t_0, t_0 + \delta)$. That is, $(\log(\lambda - Z))' \le C$. By taking the integral we obtain

$$\log\big(\lambda - Z(r)\big)\Big|_{r=t_0}^{t} \le C(t - t_0) \qquad \text{for any } t \in (t_0, t_0 + \delta).$$

But this is impossible as the left hand side is $+\infty$ while the right hand side is a finite number.

Case II. Assume $\mu < \lambda$. In this case, $\hat{P}(\mu) > 0$ and thus $\frac{\hat{Q}(Z)}{r\hat{P}(Z)}$ is a smooth function at $(r, Z) = (t_0, \mu)$. Therefore, the following initial value problem has a real analytic solution $\widetilde{Z}$ on some open interval J containing t_0:

$$rZ'\hat{P}(Z) + \hat{Q}(Z) = 0, \quad Z(t_0) = \mu.$$

Shrinking J if necessary, we can assume $\widetilde{Z}' < 0$ on J. By the uniqueness of solutions in the ODE theory, we can glue the previous solution with $\widetilde{Z}$ to obtain a real analytic solution to (4.1), still called Z, on some open interval containing $[t_0, 1]$, which still satisfies $Z' < 0$. This contradicts the definition of t_0.

Since in each case there is a contradiction, we must have $t_0 = 0$ and this proves Lemma 4.4. $\square$

By Lemma 4.4, Z is decreasing on $(0, 1)$ and therefore $\mu = \lim_{r\to 0^+} Z(r) > 0$ exists. By the same reasoning as in the proof of Lemma 4.4, we must have $\mu \le \lambda = \frac{m+1}{2k}$. In fact, it holds that $\mu = \lambda$. Assume $\mu < \lambda$. Note $\hat{P}, \hat{Q} > 0$ on $[0, \mu]$. Since $-Z' = \frac{\hat{Q}}{r\hat{P}}$, we have $-Z' \ge \frac{c}{r}$ on $(0, 1)$ for some positive constant c. This contradicts the fact that Z is bounded on $(0, 1)$. Hence we must have $\mu = \lambda$, i.e., $\lim_{r\to 0^+} Z(r) = \lambda$. Thus, Z is decreasing from λ to 0 on $[0, 1]$.

We write $Z(r) = \lambda + rG(r)$ for some real analytic function G on $(0, 1]$. It is clear that $G < 0$ on $(0, 1]$ as Z is decreasing from λ to 0 on $[0, 1]$. We have the following lemma on G.

Lemma 4.5. *It holds that* $\lim_{r\to 0^+} G(r) = 0$.

Proof. Note

$$\begin{aligned}\hat{P}(Z) =&(Z-\lambda)^{k-1}h(Z) = r^{k-1}G^{k-1}h(Z),\\ \hat{Q}(Z) =&(Z-\lambda)^{k}g(Z) = r^{k}G^{k}g(Z).\end{aligned}$$

By substituting these identities together with $Z(r) = \lambda + rG(r)$ into the ODE in (4.1), we obtain

$$(rG)'h(Z) + Gg(Z) = 0 \quad \text{ on } (0,1).$$

We can rewrite it into

$$\frac{G'}{G} = -\frac{h(Z)+g(Z)}{rh(Z)} \quad \text{ on } (0,1).$$

By Lemma 4.3, $-\frac{h(Z)+g(Z)}{h(Z)} \to 1$ as $r \to 0^+$. Consequently, there exists some constant $\delta > 0$ such that

$$\frac{G'}{G} > \frac{1}{2r} \quad \text{ for any } r \in (0,\delta).$$

As a result, $\frac{-G}{\sqrt{r}}$ is increasing on $(0,\delta)$, which in particular implies that $\frac{-G}{\sqrt{r}}$ is bounded from above on $(0,\delta)$. Therefore, $\lim_{r\to 0^+} G(r) = 0$. □

We may now further write Z as $Z(r) = \lambda + r^2 W(r)$ for some real analytic function W on $(0,1]$. Clearly, $W = rG < 0$ on $(0,1]$, and $W(1) = -\lambda$. In addition, we have the following.

Lemma 4.6. *The limit* $\lim_{r\to 0^+} W(r)$ *exists and it is a negative number.*

Proof. We first note that

$$\begin{aligned}\hat{P}(Z) =&(Z-\lambda)^{k-1}h(Z) = r^{2k-2}W^{k-1}h(Z),\\ \hat{Q}(Z) =&(Z-\lambda)^{k}g(Z) = r^{2k}W^{k}g(Z).\end{aligned}$$

Combining this with $Z = \lambda + r^2 W$ and the ODE in (4.1), we obtain

$$r(r^2W)'h(Z) + (r^2W)g(Z) = 0 \quad \text{ for } r \in (0,1).$$

As h is nonvanishing on $[0,\lambda]$, we can simplify the above equation into

$$W' = -\frac{2h(Z)+g(Z)}{r\,h(Z)}W \quad \text{ for } r \in (0,1). \tag{4.4}$$

Recalling $Z = \lambda + rG$, we have $2h(Z) + g(Z) = 2h(\lambda + rG) + g(\lambda + rG)$. As h and g are both polynomials, $2h(\lambda + rG) + g(\lambda + rG)$ is a polynomial in rG. Moreover, the constant term equals $2h(\lambda) + g(\lambda) = 0$ by Lemma 4.3. By Lemma 4.5, we deduce that

$$f(r) := \frac{2h(\lambda + rG) + g(\lambda + rG)}{rh(\lambda + rG)}$$

extends to a continuous function on $[0, 1]$. Then using (4.4) we obtain

$$\ln(-W(r)) = \ln(-W(1)) + \int_r^1 f(t)dt = \ln \lambda + \int_r^1 f(t)dt \quad \text{for any } r \in (0, 1)$$

Consequently, $\lim_{r\to 0^+} W(r) = -\lambda \exp\left(\int_0^1 f(t)dt\right)$, which is a negative real number. □

Now W naturally extends to a continuous function on $[0, 1]$. Set $a = W(0) = \lim_{r\to 0^+} W(r)$. We have the following lemma.

Lemma 4.7. *There exists a unique real analytic function $T_0(r)$ at $r = 0$ satisfying the following initial value problem:*

$$T' = -\frac{2h(\lambda + r^2T) + g(\lambda + r^2T)}{r\, h(\lambda + r^2T)} T, \quad T(0) = a. \tag{4.5}$$

Moreover, the function is even on $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$.

Remark 4.8. Note $2h(\lambda + r^2T) + g(\lambda + r^2T)$ is a polynomial in r^2T, whose constant term equals $2h(\lambda) + g(\lambda) = 0$ by Lemma 4.3. Therefore $(2h(\lambda + r^2T) + g(\lambda + r^2T))/r$ is a polynomial in r and T.

Proof of Lemma 4.7. By Remark 4.8, the right hand side of the ODE in (4.5) is real analytic in a neighborhood of $(r, T) = (0, a)$. Therefore the existence and uniqueness of the solution, as well as its real analyticity, follow from elementary ODE theory. Note if T_0 is a solution to the initial value problem (4.5), then so is $T_0(-r)$. By uniqueness of the solution, T_0 is an even function. □

Let $T_0 : (-\epsilon, \epsilon) \to \mathbb{R}$ be as in Lemma 4.7 and recall the function W defined before Lemma 4.6. Note T_0 and W are both functions in $C^\omega(0, \epsilon) \cap C[0, \epsilon)$ satisfying the following ODE:

$$T' = -\frac{2h(\lambda + r^2T) + g(\lambda + r^2T)}{rh(\lambda + r^2T)} T \text{ on } (0, \epsilon), \quad T(0) = a. \tag{4.6}$$

By basic ODE theory (cf. the proof of Lemma 2.13 in [5]), it follows that $W = T_0$ on $[0, \epsilon)$.

We now continue the proof of Proposition 4.1. Let λ, T_0 be as above and set $\Psi = \lambda + r^2 T_0$. Then Ψ is a real analytic even function on $(-\epsilon, \epsilon)$. Moreover, $\Psi = Z$ on $[0, \epsilon)$. Therefore we can glue Z with Ψ, and then apply the even extension to obtain a real analytic function on $[-1, 1]$, which we still denote by Z. It is clear that this new function Z still satisfies the ODE in (4.1). Moreover, $Z''(0) = 2W(0) = 2a < 0$. This proves part (1) of Proposition 4.1.

We next prove part (2) of Proposition 4.1. Recall $Z(r) = \lambda + r^2 W(r)$ and $\hat{P}(Z) = (Z-\lambda)^{k-1} h(Z) = r^{2k-2} W^{k-1} h(Z)$. It follows that

$$\frac{r^{2k-1}}{-Z'\hat{P}(Z)} = \frac{r^{2k-1}}{-(2rW + r^2 W')r^{2k-2}W^{k-1}h(Z)} = \frac{1}{-(2W + rW')W^{k-1}h(Z)}.$$

At $r = 0$, $-(2W + rW')W^{k-1}h(Z) = -2W^k(0)\,h(\lambda) > 0$ since $W(0) < 0$ by Lemma 4.6 and $(-1)^{k-1}h(\lambda) > 0$ by Lemma 4.3. Hence $\frac{r^{2k-1}}{-Z'\hat{P}(Z)}$ is real analytic at $r = 0$. By the definition of ϕ and the properties of Z in part (1), ϕ is real analytic and even on $[-1, 1]$. It is also clear that $\phi > 0$ on $(-1, 1)$ and $\phi(1) = 0$. The latter assertion in part (2) can be proved identically as in Proposition 3.1. This finishes the proof of Proposition 4.1. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Choose a coordinate chart (D, z) of M together with a local frame $\{e_\alpha\}_{\alpha=1}^k$ of E over D. Writing $\pi : E \to M$ for the canonical fiber projection, we have

$$\pi^{-1}(D) = \Big\{ \sum_{\alpha=1}^{k} \xi_\alpha e_\alpha(z) : (z, \xi) \in D \times \mathbb{C}^k \Big\}.$$

Under this trivialization, the ball bundle $B(E)$ and the sphere bundle $S(E)$ over D can be expressed as follows:

$$B(E) \cap \pi^{-1}(D) = \Big\{ w = (z, \xi) \in D \times \mathbb{C}^k : \sum_{\alpha,\beta=1}^{k} h_{\alpha\bar{\beta}}(z, \overline{z}) \xi_\alpha \overline{\xi_\beta} < 1 \Big\},$$

$$S(E) \cap \pi^{-1}(D) = \Big\{ w = (z, \xi) \in D \times \mathbb{C}^k : \sum_{\alpha,\beta=1}^{k} h_{\alpha\bar{\beta}}(z, \overline{z}) \xi_\alpha \overline{\xi_\beta} = 1 \Big\}.$$

Here $h_{\alpha\bar{\beta}}(z) = h_{\alpha\bar{\beta}}(z, \overline{z}) = h(e_\alpha, e_\beta)$. In the local coordinates, we write $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\overline{z_j}$. As g is induced by $-\operatorname{Ric}(E, h)$, we have $g_{i\bar{j}} = \frac{\partial^2 \log H}{\partial z_i \partial \overline{z_j}}$ where $H = \det(h_{\alpha\bar{\beta}})$. Let $G(z) = G(z, \overline{z}) = \det(g_{i\bar{j}}) > 0$. We let $\lambda_1 \leq \cdots \leq \lambda_n < 1$ be the Ricci eigenvalues of (M, g) and ϕ be the function resulting from Proposition 4.1. In the local coordinates, the Kähler form $\widetilde{\omega}$ in Theorem 1.3 is given by $\widetilde{\omega} = -i\partial\overline{\partial} \log u$, where $u(w) = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|w|_h)$. Since ϕ is real analytic and even on $[-1, 1]$, u is smooth in a neighborhood of $\overline{B(E)} \cap \pi^{-1}(D)$. Consequently, $\widetilde{\omega}$ is a smooth Kähler form on $\overline{B(E)} \cap \pi^{-1}(D)$. By repeating the proof of Proposition 3.5 (and the smoothness of u), it follows that $u = 0$ on $S(E) \cap \pi^{-1}(D)$ and $J(u) = 1$ on $B(E) \cap \pi^{-1}(D)$.

Since $J(u) = 1$, or equivalently, $\det\big((-\log u)_{s\bar{t}}\big)_{1\le s,t\le m} = u^{-(m+1)}$ in $B(E)\cap\pi^{-1}(D)$, and u is a local defining function of some strongly pseudoconvex piece of the boundary, we conclude that $\widetilde{\omega}$ is positive definite in $B(E) \cap \pi^{-1}(D)$. Also $J(u) = 1$ implies that the metric $\widetilde{g}$ induced by $\widetilde{\omega}$ has constant Ricci curvature $-(m+1)$. Since the coordinate chart D is arbitrarily chosen, $\widetilde{g}$ is a Kähler–Einstein metric in $B(E)$.

It remains to prove that the metric $\widetilde{g}$ is complete on $B(E)$. By the Hopf-Rinow Theorem, it suffices to show $(B(E), \widetilde{g})$ is geodesically complete. Let $\gamma : [0, a) \to B(E)$ be a non-extendible geodesic in $B(E)$ of unit speed with respect to $\widetilde{g}$. We only need to show that $a = +\infty$, that is, γ has infinite length. For that, we first establish the following lemma.

Lemma 4.9. *The metric $\widetilde{g}$ satisfies*

$$\widetilde{g} \ge -\frac{1}{m+1}\pi^*(\mathrm{Ric}) + \frac{1}{m+1}\pi^*(g). \tag{4.7}$$

Consequently, $\widetilde{g} \ge \frac{1-\lambda_n}{m+1}\pi^*(g)$ in $B(E)$.

Proof. Since the validity of (4.7) is independent of the choice of local coordinate chart of $B(E)$, it suffices to establish (4.7) in an arbitrary coordinate chart. Given $p \in B(E)$, recall that in the proof of Lemma 3.7, we have proved that there exist some local coordinates $w = (z, \xi)$, in which $z(\pi(p)) = 0$ and the metric $\widetilde{g}_{s\bar{t}}$ at $w = (0, \xi)$ with $0 < |w|_h < 1$ can be expressed as

$$\widetilde{g}_{s\bar{t}} = \begin{cases} -\dfrac{1}{m+1}R_{i\bar{j}}(0) + \dfrac{Y}{2k}g_{i\bar{j}}(0) & \text{for } 1 \le s,t \le n, \\[2ex] \dfrac{1}{2|\xi|^2}\big(Y - \dfrac{2k}{m+1}\big)\delta_{\alpha\beta} + \big(\dfrac{XY'}{4} - \dfrac{Y}{2} + \dfrac{k}{m+1}\big)\dfrac{\overline{\xi_\alpha}\xi_\beta}{|\xi|^4} & \text{for } n+1 \le s,t \le n+k, \\[2ex] 0 & \text{otherwise}, \end{cases} \tag{4.8}$$

where $i = s, j = t$ when $1 \le s,t \le n$ and $\alpha = s - n, \beta = t - n$ when $n+1 \le s,t \le n+k$. Recall $X = X(w) = (\sum_{\alpha,\beta=1}^n h_{\alpha\bar{\beta}}\xi_\alpha\overline{\xi_\beta})^{1/2}$, $Y = Y(w) = \frac{2k}{m+1} - X\frac{\phi'(X)}{\phi(X)}$ and ϕ is defined in Proposition 4.1. As in §3, by (3.5) we have $Y = y(r)|_{r=X} = \frac{1}{Z}|_{r=X}$. Furthermore, by Proposition 4.1 it follows that $Y(w) \ge \frac{2k}{m+1}$ for any $w \in B(E)$. Note that $(\widetilde{g}_{s\bar{t}})_{1\le s,t\le n+k}$ is a block diagonal matrix. As it is positive definite, the diagonal blocks, $(\widetilde{g}_{st})_{1\le s,t\le n}$ and $(\widetilde{g}_{st})_{n+1\le s,t\le n+k}$, are also individually positive definite. As a result,

$$\widetilde{g} \ge \sum_{i,j=1}^n \widetilde{g}_{i\bar{j}}dz_i \otimes d\overline{z_j} \ge -\frac{1}{m+1}\pi^*(\mathrm{Ric}) + \frac{1}{m+1}\pi^*(g)$$

at any $w = (0, \xi)$ with $0 < |w|_h < 1$.

By continuity, the above actually holds at any $w = (0, \xi)$ with $|w|_h < 1$, and thus (4.7) is proved. Finally, by the assumption on the Ricci eigenvalues, we have $R_{i\bar{j}} \leq \lambda_n g_{i\bar{j}}$ with $\lambda_n < 1$. The latter part of the lemma follows easily. $\square$

With Lemma 4.9, the remaining part of the proof is identical to that of Theorem 1.4 in [5]. We omit the details. $\square$

5. Proofs of corollaries

In this section, we consider the case where the base manifold M is a domain D in $\mathbb{C}^n$, and prove Corollary 1.8 and 1.9, as well as Proposition 1.10. We also exhibit some explicit examples as applications.

We first prove Corollary 1.8.

Proof of Corollary 1.8. Let $L = D \times \mathbb{C}$ be the trivial line bundle over D. By the assumption on h, the Hermitian line bundle (L, h), is negative. Take the Hermitian vector bundle (E, h_E) as $(L, h) \oplus \cdots \oplus (L, h)$, the direct sum of k copies of (L, h). Then the ball bundle $B(E)$ and the sphere bundle $S(E)$ of (E, h_E) are respectively given by

$$
\begin{aligned}
B(E) =& \big\{ w = (z, \xi) \in D \times \mathbb{C}^k : |\xi|^2 h(z, \bar{z}) - 1 < 0 \big\}, \\
S(E) =& \big\{ w = (z, \xi) \in D \times \mathbb{C}^k : |\xi|^2 h(z, \bar{z}) - 1 = 0 \big\}.
\end{aligned}
$$

Note that $B(E)$ coincides with the domain $\Omega \subset \mathbb{C}^m$ (recall $m = n + k$) as defined in (1.6), and $S(E)$ coincides with the hypersurface Σ in $\mathbb{C}^m$ defined by (1.7). In addition, the Hermitian vector bundle (E, h_E) is curvature split by Proposition 2.1 and it is Griffiths negative by Remark 2.5. Moreover, since $\mathrm{Ric}(E, h_E) = k\,\mathrm{Ric}(L, h)$, the Kähler metric induced by $-\mathrm{Ric}(E, h_E)$ is the metric g given in the assumption. By Theorem 1.3, the unique complete Kähler–Einstein metric $\widetilde{g}$ with Ricci curvature $-(m+1)$ is given by (1.4). The explicit formula of the Cheng–Yau–Mok solution u, defined as $(\det(\widetilde{g}_{s\bar{t}}))^{-\frac{1}{m+1}}$, can be seen from the proof of Theorem 1.3. Since the function ϕ is real analytic on $[-1, 1]$ by Proposition 4.1 and G, H are both real analytic on D by the assumption, the Cheng–Yau–Mok solution u extends real analytically across Σ. $\square$

We next prove Corollary 1.9.

Proof of Corollary 1.9. We first note that g is complete since (D, g) is homogeneous. (For the proof of this fact, see for example [10].) Moreover, the homogeneity of (D, g) also implies that g has constant Ricci eigenvalues. The result is now a direct consequence of Corollary 1.8 and the following:

Claim. The Ricci eigenvalues of g are all negative (and thus in particular strictly less than 1).

Proof of Claim. Let g_B be the Bergman metric of D. We denote by $\mathrm{Vol}(g)$ and $\mathrm{Vol}(g_B)$ the volume forms of g and g_B respectively, which are (n,n) forms on D. Set $\Phi := \mathrm{Vol}(g)/\mathrm{Vol}(g_B)$, which is a well-defined function on D. Let $\mathrm{Iso}(g)$ be the group of holomorphic isometries of (Ω, g). Since every biholomorphism also preserves g_B, the group $\mathrm{Iso}(g)$ actually preserves both g and g_B. Thus, Φ is invariant under the action of $\mathrm{Iso}(g)$. As $\mathrm{Iso}(g)$ acts transitively on Ω, Φ is constant on Ω, that is, $\mathrm{Vol}(g)$ and $\mathrm{Vol}(g_B)$ are the same up to some (positive) constant factor. Therefore, g and g_B have the same Ricci form. By the fact $\mathrm{Ric}(g_B) = -g_B$ (see [1] for example), we get $\mathrm{Ric}(g) \cdot g^{-1} = \mathrm{Ric}(g_B) \cdot g^{-1} = -g_B \cdot g^{-1}$. So all the Ricci eigenvalues of g are negative and the proof is completed. $\square$

$\square$

We now present some examples as applications of the above corollaries.

Example 5.1. Let D be a bounded homogeneous domain in $\mathbb{C}^n$. Write $K_D(z,z)$ for its Bergman kernel and g_B for the Bergman metric. Since g_B is biholomorphic invariant, the manifold (Ω, g_B) is homogeneous Kähler. Given $\lambda \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$, we set $h = (K_D)^\lambda$ and consider the domain $\Omega \subset D \times \mathbb{C}^k$ and the hypersurface $\Sigma \subset D \times \mathbb{C}^k$ as defined in (1.6) and (1.7). By Corollary 1.9, the Cheng–Yau–Mok solution of Ω is given by the following with $m = n + k$:

$$u(w) = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}),$$

where $H = (K_D)^{k\lambda}$, $G = k^n \lambda^n G_B$ and G_B is the volume density of g_B. Moreover, the boundary hypersurface Σ is obstruction flat and u extends real analytically across Σ.

Example 5.2. Let D be a bounded domain of holomorphy in $\mathbb{C}^n$. Suppose $g_0 = \big((g_0)_{i\bar{j}}\big)$ is the complete Kähler–Einstein metric with negative Ricci curvature λ_0. (The existence and uniqueness of such a metric is guaranteed by the work of Mok–Yau [12].) Let h be a real analytic function on D such that $(g_0)_{i\bar{j}} = \frac{\partial^2 \log h}{\partial z_i \partial \bar{z}_j}$. (One particular choice of such an h is $\big(\det((g_0)_{i\bar{j}})\big)^{-1/\lambda_0}$ as g_0 is Kähler-Einstein.) For a given $k \in \mathbb{Z}^+$, consider the domain $\Omega \subset D \times \mathbb{C}^k$ and the hypersurface $\Sigma \subset D \times \mathbb{C}^k$ as defined in (1.6) and (1.7). By Corollary 1.8, the Cheng–Yau–Mok solution of Ω is given by the following with $m = n + k$:

$$u = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}),$$

where $H = h^k$ and $G = k^n \det\big((g_0)_{i\bar{j}}\big)$. Moreover, the boundary hypersurface Σ is obstruction flat and u extends real analytically across Σ.

In particular, if we choose $h = 1/u_0$ where u_0 is the Cheng–Yau–Mok solution for the domain D, then $g_0 = \big((\log h)_{i\bar{j}}\big)$ is the complete Kähler metric with Ricci curvature $\lambda_0 = -(n+1)$. A routine computation yields the following expression for the Cheng–Yau–Mok solution of Ω,

$$u = u_0\, \phi(|\xi| h^{\frac{1}{2}}).$$

Example 5.3. Given $l \geq 1$, for each $1 \leq i \leq l$, let D_i be a domain in $\mathbb{C}^{n_i}$, $g^i = \sum_{p,q=1}^{n_i} g^i_{p\bar{q}} dz^i_p \wedge d\overline{z^i_q}$ a complete Kähler–Einstein metric on D_i, and h^i a real analytic function on D_i such that g^i is induced by $\sqrt{-1}\partial\overline{\partial}\log h^i$. Let $D = D_1 \times \cdots \times D_l \subset \mathbb{C}^n$ with $n = n_1 + \cdots + n_l$ and write $z = (z^1, \cdots, z^l)$ for each $z^i \in D_i$. Set $h(z, \bar{z}) = \Pi_{i=1}^l h^i(z^i, \overline{z^i})$. For a fixed $k \in \mathbb{Z}^+$, consider the domain $\Omega \subset D \times \mathbb{C}^k$ and the hypersurface $\Sigma \subset D \times \mathbb{C}^k$ as defined in (1.6) and (1.7). Then the Cheng–Yau–Mok solution of Ω is given by the following with $m = n + k$:

$$u(w) = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}),$$

where $H = h^k$ and $G = k^n \Pi_{i=1}^l \det(g^i_{pq})_{1 \leq p,q \leq n_i}$. Moreover, the boundary hypersurface Σ is obstruction flat and u extends real analytically across Σ.

To conclude the paper, we shall prove Proposition 1.10. Before proceeding to the proof, we first consider the case of the ball bundle over a bounded domain of holomorphy D. As mentioned in Example 5.2, for a given negative real number λ_0, there exists a unique complete Kähler–Einstein metric such that $\mathrm{Ric}(g_0) = \lambda_0 g_0$. As before, L is the trivial line bundle over D and h is a Hermitian metric of L such that g_0 is induced by $-c_1(L, h)$. The Hermitian vector bundle (E, h_E) is the direct sum of k copies of (L, h). By Example 5.2 the Cheng–Yau–Mok solution for $\Omega = B(E)$ as defined in (1.6) is

$$u = k^{\frac{n}{m+1}} (GH)^{-\frac{1}{m+1}} \phi(|\xi| h^{\frac{1}{2}}), \quad \text{with } m = n + k, \tag{5.1}$$

where $H = h^k$, $G = k^n \det((g_0)_{i\bar{j}})$, and the function ϕ is given in Proposition 4.1 with the λ_i's chosen as follows. First, it is clear that $-c_1(E, h_E)$ is $-kc_1(L, h)$, which also induces a Kähler–Einstein metric $g = kg_0$ with negative constant Ricci curvature $\lambda = \lambda_0/k < 0$. Write $\mu = \frac{2k\lambda}{m+1}$ and $\nu = \frac{2k}{m+1}$ and let all λ_i's in Proposition 4.1 be λ. The polynomials $P(y)$ and $\hat{P}(x)$ are then given by

$$P(y) = (y - \nu)^{k-1}(y - \mu)^n, \qquad \hat{P}(x) = (1 - \nu x)^{k-1}(1 - \mu x)^n. \tag{5.2}$$

The polynomial $Q(y)$ from Proposition 4.1 satisfies the following properties:

Lemma 5.4. *The following hold:*

(1) *The polynomial Q is divisible by $(y - \nu)^k$. Moreover, there exists a polynomial $T(y)$ such that $Q(y) = (y - \mu)^{n+1} T(y) + c$, where c is a real number satisfying $c = -(\nu - \mu)^{n+1} T(\nu) = Q(\mu)$.*
(2) *The number $c = 0$ if and only if $\lambda = -\frac{n+1}{k}$ (i.e., $\lambda_0 = -(n + 1)$). In this case, $Q = (y - \mu)^{n+1}(y - \nu)^k$.*

Proof. We first prove part (1). Recall by the definition in Proposition 4.1, Q satisfies

$$\frac{dQ}{dy} = (m+1)yP(y) \quad \text{and} \quad Q(\nu) = 0.$$

It follows immediately that

$$Q(y) = \int_{\nu}^{y} (m+1)tP(t)dt.$$

Note that we can write the integrand function as

$$(m+1)yP(y) = (y-\nu)^{k-1} \sum_{j=0}^{n+1} a_j (y-\nu)^j \quad \text{for some } a_j\text{'s in } \mathbb{R}.$$

We take the integration term by term and obtain

$$Q(y) = \sum_{j=0}^{n+1} \frac{a_j}{k+j} (y-\nu)^{k+j}.$$

Therefore, Q is divisible by $(y-\nu)^k$.

To prove the latter assertion in part (1), note that we can also write

$$(m+1)yP(y) = (y-\mu)^n \sum_{j=0}^{k} b_j (y-\mu)^j \quad \text{for some } b_j\text{'s in } \mathbb{R}.$$

As $Q(y) = \int_{\mu}^{y} (m+1)tP(t)dt + Q(\mu)$, we again take the integration term by term and obtain

$$Q(y) = \sum_{j=0}^{k} \frac{b_j}{n+j+1} (y-\mu)^{n+j+1} + Q(\mu).$$

By setting

$$T(y) = \sum_{j=0}^{k} \frac{b_j}{n+j+1} (y-\mu)^j \quad \text{and} \quad c = Q(\mu),$$

we have $Q(y) = (y-\mu)^{n+1}T(y) + c$. Since $Q(\nu) = 0$, it follows that $c = -(\nu-\mu)^{n+1}T(\nu)$.

Now we prove part (2). It is clear that $c = 0$ if and only if $Q(\mu) = 0$. Since $Q(\nu) = 0$ in the assumption, the former is also equivalent to

$$\int_{\mu}^{\nu} \frac{dQ}{dy} dy = 0, \qquad \text{i.e.,} \quad \int_{\mu}^{\nu} yP(y)dy = 0. \tag{5.3}$$

By writing $y = \frac{\mu}{\mu-\nu}(y-\nu) - \frac{\nu}{\mu-\nu}(y-\mu)$, we have

$$yP(y) = y(y-\nu)^{k-1}(y-\mu)^n = \frac{\mu}{\mu-\nu}(y-\nu)^k(y-\mu)^n - \frac{\nu}{\mu-\nu}(y-\nu)^{k-1}(y-\mu)^{n+1}.$$

Thus, (5.3) is equivalent to

$$\frac{\mu}{\mu-\nu}\int_{\mu}^{\nu}(y-\nu)^k(y-\mu)^n dy = \frac{\nu}{\mu-\nu}\int_{\mu}^{\nu}(y-\nu)^{k-1}(y-\mu)^{n+1}dy.$$

By setting $t = \frac{y-\mu}{\nu-\mu}$, it reduces to

$$\mu\int_0^1 (1-t)^k t^n dt = -\nu\int_0^1 (1-t)^{k-1}t^{n+1}dt. \tag{5.4}$$

Recall the beta function is defined by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$$

and $B(p,q) = \frac{(p-1)!(q-1)!}{(p+q-1)!}$. Therefore, (5.4) writes into

$$\mu\frac{k!n!}{(n+k+1)!} = -\nu\frac{(k-1)!(n+1)!}{(n+k+1)!}.$$

As $\mu = \lambda\nu$, we finally obtain that $c = 0$ is equivalent to $\lambda = -\frac{n+1}{k}$.

In this case, by part (1) we have both $(y-\nu)^k$ and $(y-\mu)^{n+1}$ divide Q. Since Q is a monic polynomial of degree $n+k+1$, it follows that $Q(y) = (y-\nu)^k(y-\mu)^{n+1}$. □

In the special case that P is given by (5.2), we will study the rationality of function Z as defined in Proposition 4.1.

Proposition 5.5. *Let P and $\hat{P}$ be given by (5.2) (and accordingly Q and $\hat{Q}$ are both determined as in Proposition 4.1). Let Z and ϕ be as given in Proposition 4.1. Then the following are equivalent.*

(1) *Z is rational.*
(2) *ϕ^{m+1} is rational.*

(3) $\lambda = -\frac{n+1}{k}$.

Remark 5.6. Moreover, when (1)-(3) in Proposition 5.5 hold, we can see from the proof that $\mu = -\frac{2(n+1)}{m+1}$ and $Z = \frac{1-r^2}{2+\mu-\mu r^2}$. Consequently, $\phi(r) = 1 - r^2$.

Proof of Proposition 5.5. We first prove (1) is equivalent to (2). By Proposition 4.1, we have

$$\frac{1}{Z} = \frac{2k}{m+1} - r\frac{\phi'}{\phi} = \frac{2k}{m+1} - \frac{r}{m+1}\frac{(\phi^{m+1})'}{\phi^{m+1}}.$$

Hence (2) implies (1). Conversely, recall that in Proposition 4.1 the function ϕ is defined by $\phi = 2\Big(\frac{r^{2k-1}}{-Z'\hat{P}(Z)}\Big)^{\frac{1}{m+1}} Z$. Since $\hat{P}$ is a polynomial, the rationality of Z implies that of ϕ^{m+1}.

Now it remains to show (1) is equivalent to (3). We shall first show that (3) implies (1). Suppose $\lambda = -\frac{n+1}{k}$. Then by Lemma 5.4 we get $Q(y) = (y-\mu)^{n+1}(y-\nu)^k$. Thus, $\hat{Q}(x) = (1-\mu x)^{n+1}(1-\nu x)^k$. Recall $\hat{P} = (1-\mu x)^n(1-\nu x)^{k-1}$ as given in (5.2). Since Z satisfies $rZ'\hat{P}(Z) + \hat{Q}(Z) = 0$ for $r \in [-1, 1]$, it follows that

$$\frac{Z'}{(1-\mu Z)(1-\nu Z)} = -\frac{1}{r} \qquad \text{for } r \in (0, 1].$$

As $Z(1) = 0$, by writing $\frac{1}{(1-\mu Z)(1-\nu Z)} = \frac{1}{\mu-\nu}\big(\frac{\mu}{1-\mu Z} - \frac{\nu}{1-\nu Z}\big)$ and integrating the above equation, we obtain

$$\frac{1}{\mu-\nu}\Big(-\ln(1-\mu Z) + \ln(1-\nu Z)\Big) = -\ln r \qquad \text{for } r \in (0, 1].$$

Since $\mu - \nu = \frac{2k\lambda - 2k}{m+1} = -2$, we get $\ln\frac{1-\nu Z}{1-\mu Z} = \ln(r^2)$, and further simplification yields

$$Z = \frac{1-r^2}{\nu - \mu r^2} = \frac{1-r^2}{2+\mu-\mu r^2}.$$

It is clear that Z is a rational function.

Last we check that (1) implies (3). Suppose that Z is rational. Recall $\hat{P}(x) = (1 - \mu x)^n(1-\nu x)^{k-1}$. By Lemma 5.4, we have $Q(y) = (y-\mu)^{n+1}T(y) + c$ where T is some polynomial of degree k and c is some real number. By writing $T(y) = \sum_{j=0}^{k} a_j(y-\mu)^j$ for some $a_j \in \mathbb{R}$, we then have

$$Q(y) = \sum_{j=0}^{k} a_j(y-\mu)^{n+1+j} + c \quad \text{ and } \quad \hat{Q}(x) = \sum_{j=0}^{k} a_j(1-\mu x)^{n+1+j}x^{k-j} + cx^{m+1}.$$

Recall $rZ'\hat{P}(Z) + \hat{Q}(Z) = 0$ for $r \in [-1, 1]$. We divide the equation by $(1-\mu Z)^{m+1}$ to obtain

$$r\frac{Z'(1-\nu Z)^{k-1}}{(1-\mu Z)^{k+1}}+\sum_{j=0}^{k}a_j\frac{Z^{k-j}}{(1-\mu Z)^{k-j}}+c\frac{Z^{m+1}}{(1-\mu Z)^{m+1}}=0 \quad \text{for } r\in[-1,1].$$

Set $\eta(r)=\frac{Z(r)}{1-\mu Z(r)}$ for $r\in[-1,1]$. Then $\eta'=\frac{Z'}{(1-\mu Z)^2}$ and $\frac{1-\nu Z}{1-\mu Z}=1+(\mu-\nu)\eta$. Therefore, we can rewrite the above equation into

$$r\eta'\big(1+(\mu-\nu)\eta\big)^{k-1}+\sum_{j=0}^{k}a_j\eta^{k-j}+c\eta^{m+1}=0. \tag{5.5}$$

As Z is rational, so is η. We can write $\eta=\frac{p}{q}$ for some coprime polynomials p and q. Putting this into (5.5) and multiplying the equation by q^{m+1}, we obtain

$$r(p'q-pq')\big(q+(\mu-\nu)p\big)^{k-1}q^n+\sum_{j=0}^{k}a_jp^{k-j}q^{n+1+j}+cp^{m+1}=0.$$

Assume $c\neq 0$. Then p^{m+1} is divisible by q. As p and q are coprime, q is a constant. Without losing of generality, we can assume $q=1$ and thus η is just the polynomial p. Now that all terms in (5.5) are polynomials, we can count their degrees. Note $c\eta^{m+1}$ is of degree $(m+1)\deg p$ while all other terms on the left hand side of (5.5) are of degree less than or equal to $k\deg p$. Therefore, we have $\deg p=0$, that is, p is a constant. So are the functions η and Z. This is a contradiction as Z is not constant by Proposition 4.1. Hence we must have $c=0$. By Lemma 5.4, (3) holds. So the proof is completed. □

We are now ready to prove Proposition 1.10.

Proof of Proposition 1.10. Let D be the complex n-dimensional unit ball $\{z\in\mathbb{C}^n : |z|<1\}$. We introduce the function $h=(\frac{1}{1-|z|^2})^{1/p}$ and the metric $g_0=(g_0)_{i\bar{j}}=\big((\log h)_{i\bar{j}}\big)$. Note that g_0 is just $\frac{1}{p(n+1)}$ multiple of the Bergman metric of D. Thus g_0 is the complete Kähler–Einstein metric on D with Ricci curvature equal to $\lambda_0=-p(n+1)$. If we take $g=kg_0$, then g is the complete Kähler–Einstein metric on D with Ricci curvature equal to $\lambda=\lambda_0/k$. Recall the domain Ω defined in (1.6), which now becomes

$$\Omega=\big\{(z,\xi)\in D\times\mathbb{C}^k : \big(\frac{1}{1-|z|^2}\big)^{1/p}|\xi|^2<1\big\}.$$

Clearly, we have $\Omega=E_p$. For $m=n+k$, by Example 5.2, the Cheng–Yau–Mok solution for domain Ω is given by

$$u=k^{\frac{n}{m+1}}(GH)^{-\frac{1}{m+1}}\phi(|\xi|h^{\frac{1}{2}}),$$

where $G=\det(g_{i\bar{j}})$ and $H=h^k$. A straightforward computation yields $G=\frac{k^n}{p^n(1-|z|^2)^{n+1}}$, and thus

$$u = p^{\frac{n}{m+1}}(1-|z|^2)^{(n+1+\frac{k}{p})/(m+1)}\phi\big(|\xi|(1-|z|^2)^{-\frac{1}{2p}}\big). \tag{5.6}$$

On the other hand, the Bergman kernel K of E_p was computed by D'Angelo [4]:

$$K\big((z,\xi),(z,\xi)\big) = \sum_{i=0}^{n+1} c_i \frac{(1-|z|^2)^{-(n+1)+\frac{i}{p}}}{\big((1-|z|^2)^{\frac{1}{p}}-|\xi|^2\big)^{k+i}}, \tag{5.7}$$

where c_i are constants depending on i, n, k and p.

To establish Proposition 1.10, we assume the Bergman metric g_{B} of E_p is Kähler-Einstein and first follow the work of Fu–Wong [7] to compute the volume form of g_{B}. Note that a generic boundary point of E_p is smooth and strictly pseudoconvex (indeed spherical). Take an arbitrary strictly pseudoconvex boundary point (z_0, ξ_0). By using Fefferman's expansion for the Bergman kernel near (z_0, ξ_0) and the argument in Cheng–Yau ([2], page 510), we deduce that the Ricci curvature of g_{B} at $(z,\xi) \in E_p$ tends to -1 as (z,ξ) approaches (z_0,ξ_0). Thus the Kähler-Einstein assumption implies that the Ricci curvature of g_{B} is equal to -1. Then by Proposition 1.2 in [7], the determinant of g_{B} equals the Bergman kernel up to a positive constant multiple. On the other hand, the volume form of two complete Kähler–Einstein metrics of negative Ricci curvature, $\det\big((-\log u)_{s\bar{t}}\big)$ and the determinant of g_{B}, on E_p can only differ by a positive constant multiple. As a result, we have $u^{-(m+1)} = cK$ for some constant $c > 0$. Combining this with (5.6) and (5.7), we obtain

$$p^{-n}(1-|z|^2)^{-(n+1+\frac{k}{p})}\phi^{-(m+1)}\big(|\xi|(1-|z|^2)^{-\frac{1}{2p}}\big) = c\sum_{i=0}^{m} c_i \frac{(1-|z|^2)^{-(n+1)+\frac{i}{p}}}{\big((1-|z|^2)^{\frac{1}{p}}-|\xi|^2\big)^{k+i}}.$$

After simplification, this becomes

$$\phi^{-(m+1)}\big(|\xi|(1-|z|^2)^{-\frac{1}{2p}}\big) = c\,p^n \sum_{i=0}^{m} c_i \frac{1}{\big(1-|\xi|^2(1-|z|^2)^{-\frac{1}{p}}\big)^{k+i}}.$$

By setting $r = |\xi|(1-|z|^2)^{-\frac{1}{2p}}$, we observe $\phi^{-(m+1)}(r)$ is equal to $c\,p^n\sum_{i=0}^m c_i\big(1-r^2\big)^{-(k+i)}$. Thus, ϕ^{m+1} is rational. By Proposition 5.5, we get $\lambda = -\frac{n+1}{k}$. Recall that $\lambda = \frac{\lambda_0}{k} = -\frac{p(n+1)}{k}$. So it follows that $p = 1$ and the proof is completed. □

Data availability

No data was used for the research described in the article.

References

[1] Stefan Bergman, The Kernel Function and Conformal Mapping, revised edition, Mathematical Surveys, vol. No. V, American Mathematical Society, Providence, R.I., 1970.

[2] Shiu Yuen Cheng, Shing Tung Yau, On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation, Commun. Pure Appl. Math. 33 (4) (1980) 507–544.
[3] Gunhee Cho, Invariant metrics on the complex ellipsoid, J. Geom. Anal. 31 (2) (2021) 2088–2104.
[4] John P. D'Angelo, An explicit computation of the Bergman kernel function, J. Geom. Anal. 4 (1) (1994) 23–34.
[5] Peter Ebenfelt, Ming Xiao, Hang Xu, Kähler-Einstein metrics and obstruction flatness of circle bundles, J. Math. Pures Appl. 177 (2023) 368–414.
[6] Charles Fefferman, Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, Ann. Math. (2) 103 (2) (1976) 395–416.
[7] Siqi Fu, Bun Wong, On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics, Math. Res. Lett. 4 (5) (1997) 697–703.
[8] C. Robin Graham, Higher asymptotics of the complex Monge-Ampère equation, Compos. Math. 64 (2) (1987) 133–155.
[9] John M. Lee, Richard Melrose, Boundary behaviour of the complex Monge-Ampère equation, Acta Math. 148 (1982) 159–192.
[10] Qi-Keng Lu, On Kähler manifolds with constant curvature, Chin. Math.–Acta 8 (1966) 283–298.
[11] Ngaiming Mok, Sui Chung Ng, Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products, J. Reine Angew. Math. 669 (2012) 47–73.
[12] Ngaiming Mok, Shing-Tung Yau, Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions, in: The Mathematical Heritage of Henri Poincaré, Part 1, Bloomington, Ind., 1980, in: Proc. Sympos. Pure Math., vol. 39, Amer. Math. Soc., Providence, RI, 1983, pp. 41–59.
[13] Craig van Coevering, Kähler-Einstein metrics on strictly pseudoconvex domains, Ann. Glob. Anal. Geom. 42 (3) (2012) 287–315.
[14] S.M. Webster, A remark on the Chern-Moser tensor 28 (2002) 433–435, Special issue for S. S. Chern.
[15] Shing Tung Yau, Problem section, in: Seminar on Differential Geometry, in: Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 669–706.