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RESEARCH ARTICLE

Continuous in time bubble decomposition for the harmonic map heat flow

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Abstract
We consider the harmonic map heat flow for maps $\mathbb{R}^2 \to \mathbb{S}^2$. It is known that solutions to the initial value problem exhibit bubbling along a well-chosen sequence of times. We prove that every sequence of times admits a subsequence along which bubbling occurs. This is deduced as a corollary of our main theorem, which shows that the solution approaches the family of multi-bubble configurations in continuous time.

Contents

1. Introduction

1.1. Setting of the problem

Consider the harmonic map heat flow (HMHF) for maps $u : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ – that is, the gradient flow of the Dirichlet energy

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, \mathrm{d}x, \tag{1.1}$$

for the L^2 inner product. The initial value problem for the HMHF is given by

$$\begin{aligned} \partial_t u &= \Delta u + u\,|\nabla u|^2 \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.2}$$

We consider initial data in the energy class

$$\mathcal{E} := H^1(\mathbb{R}^2; \mathbb{S}^2) := \{u_0 \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \mid |u_0(x)|^2 = 1 \text{ for almost every } x \in \mathbb{R}^2\}. \tag{1.3}$$

The HMHF was proved to be well-posed in $\mathcal{E}$ by Struwe [28], and we can associate to each initial data $u_0 \in \mathcal{E}$ a maximal time of existence $T_+ = T_+(u_0) \in (0, \infty]$ and unique solution $u(t) \in \mathcal{E}$, which is regular for $t \in (0, T_+)$. The maximal time T_+ is characterized as the first time at which energy concentrates at a point in space; see Lemma 2.7. Of fundamental importance is the energy identity

$$E(u(t_2)) + \int_{t_1}^{t_2} \|\mathcal{T}(u(t))\|_{L^2(\mathbb{R}^2)}^2 \, \mathrm{d}t = E(u(t_1)), \tag{1.4}$$

which holds for any $0 \le t_1 < t_2 < T_+$ (see [28, Lemma 3.4]), and where $\mathcal{T}(u) := \Delta u + u\,|\nabla u|^2$, which is called the *tension* of u.

The HMHF for maps between Riemannian manifolds was introduced by Eells and Sampson [12]. Though we do not do this here, when studying the HMHF for maps $\mathbb{R}^2 \to \mathbb{S}^2$, it is natural to further restrict the class of initial data by intersecting the space $\mathcal{E}$ with the set of continuous maps u_0 that tend to a fixed vector on $\mathbb{S}^2$ at ∞ (i.e., such that there exists $u_\infty \in \mathbb{S}^2$ so that $\lim_{x\to\infty} |u_0(x) - u_\infty| = 0$). By assigning to the point at ∞ the vector u_∞, u_0 induces a continuous map $\widetilde{u}_0 : \mathbb{S}^2 \to \mathbb{S}^2$, and we can define the topological degree of u_0 to be the degree of $\widetilde{u}_0$. One can show that this condition is preserved by the flow – that is, the solution $u(t,x)$ satisfies $\lim_{x\to\infty} |u(t,x) - u_\infty| = 0$ for all $0 \le t < T_+$. Under this restriction, the solution $u(t,x)$ gives a continuous deformation of the initial data $u_0(x)$ within its homotopy class, which was one of the motivations mentioned in [12].

Harmonic maps $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ have vanishing tension and give stationary solutions to (1.2). They are formal critical points of the energy (1.1) and satisfy the PDE

$$\Delta\omega + \omega|\nabla\omega|^2 = 0. \tag{1.5}$$

It is a well-known general property of harmonic maps in two dimensions that they are conformal (up to change of orientation) and minimize the energy in their homotopy class; [10, 11, 19]. The energy of a harmonic map ω is given by $E(\omega) = 4\pi|\deg(\omega)|$. Weak solutions – that is, $\omega \in \mathcal{E}$ for which (1.5) holds in the weak sense – are smooth by a result of Hélein [15]; see Theorem 2.1 in Section 2.1.

An influential series of works by Struwe [28], Qing [23], Ding-Tian [9], Wang [37], Qing-Tian [24], Lin-Wang [20] and Topping [31] showed that solutions $u(t)$ to (1.2) admit a bubble decomposition along a well-chosen sequence of times $t_n \to T_+$; see also the book by Lin-Wang [21]. In these works, the bubbling time sequence $t_n \to T_+$ and corresponding sequence of maps $u(t_n)$ become a Palais-Smale sequence after rescaling. Indeed, in the case when $T_+ = \infty$, it follows from (1.4) that

$$\int_0^\infty \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t < \infty,$$

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so there exists a sequence $t_n \to \infty$ so that

$$\lim_{n\to\infty} \sqrt{t_n}\|\mathcal{T}(u(t_n))\|_{L^2} = 0.$$

By similar logic, in the case of finite time blow-up ($T_+ < \infty$), there is a sequence $t_n \to T_+$ so that

$$\lim_{n\to\infty} \sqrt{T_+ - t_n}\|\mathcal{T}(u(t_n))\|_{L^2} = 0.$$

In other words, after the rescaling $u_n(x) := u(t_n, \sqrt{t_n}x)$ (or $u_n(x) = u(t_n, \sqrt{T_+ - t_n}x)$), the u_n are Palais-Smale sequences for the energy functional because $\sup_n E(u_n) < \infty$ and

$$DE(u_n) = -\mathcal{T}(u_n) \to 0$$

in L^2. Elliptic bubbling analysis (see, for example, [2, 23, 28]) is then used to extract bubbles up to the scale $\sqrt{t_n}$ (or $\sqrt{T_+ - t_n}$ in the case $T_+ < \infty$). The main result of this paper (see Theorem 1.1 below) is distinct from this classical literature in that we show bubbling occurs along every time sequence (after passing to a suitable subsequence) without the aid of a Palais-Smale sequence in the sense described above. However, the works [21, 24] show L^∞ convergence, including in the neck regions between the bubbles, whereas here we control only the energy in the neck regions; we do not address the question of L^∞ convergence on the neck regions.

1.2. Statement of the results

The goal of this paper is to give asymptotic descriptions of solutions $u(t)$ to (1.2) with initial data $u_0 \in \mathcal{E}$. Our first main result is that every sequence of times tending to the maximal time admits a subsequence $t_n \to T_+$ along which $u(t_n)$ admits a decomposition into a finite superposition of rescaled and translated harmonic maps.

We use the notation $D(y, \rho) \subset \mathbb{R}^2$ to denote the open disc of radius $\rho > 0$ centered at the point $y \in \mathbb{R}^2$.

Theorem 1.1 (Bubble decomposition along any time sequence). *Let $u(t)$ be the unique solution to (1.2) associated to initial data $u_0 \in \mathcal{E}$. Let $T_+ = T_+(u_0) \in (0, \infty]$ denote the maximal time of existence.*

(Finite time blow-up) *Suppose $T_+ < \infty$. There exist a finite energy map $u^* : \mathbb{R}^2 \to \mathbb{S}^2$, an integer $L \geq 1$ and points $\{x^\ell\}_{\ell=1}^L \subset \mathbb{R}^2$ with the following properties.*

Let $t_n \to T_+$ be any time sequence. After passing to a subsequence, which we still denote by t_n, we can associate to each $\ell \in \{1, \dots, L\}$ an integer $M^{(\ell)}$, sequences $a_{j,n}^{(\ell)} \in \mathbb{R}^2$ and $\lambda_{j,n}^{(\ell)} \in (0, \infty)$ for each $j \in \{1, \dots, M^{(\ell)}\}$, with $a_{j,n}^{(\ell)} \to x^\ell$, $\frac{\lambda_{j,n}^{(\ell)}}{\sqrt{T_+ - t_n}} \to 0$ as $n \to \infty$, and nontrivial harmonic maps $\omega_1^{(\ell)}, \dots, \omega_{M^{(\ell)}}^{(\ell)}$ so that

$$\lim_{n\to\infty} \left(\frac{\lambda_{j,n}^{(\ell)}}{\lambda_{k,n}^{(\ell)}} + \frac{\lambda_{k,n}^{(\ell)}}{\lambda_{j,n}^{(\ell)}} + \frac{|a_{j,n}^{(\ell)} - a_{k,n}^{(\ell)}|}{\lambda_{j,n}^{(\ell)}} \right) = \infty \quad \text{for all } j \neq k, \tag{1.6}$$

and

$$\lim_{n\to\infty} E\Big(u(t_n) - u^* - \sum_{\ell=1}^{L} \sum_{j=1}^{M^{(\ell)}} \Big(\omega_j^{(\ell)}\Big(\frac{\cdot - a_{j,n}^{(\ell)}}{\lambda_{j,n}^{(\ell)}} \Big) - \omega_j^{(\ell)}(\infty) \Big) \Big) = 0, \tag{1.7}$$

where $\omega_j^{(\ell)}(\infty) := \lim_{|x|\to\infty} \omega_j^{(\ell)}(x) \in \mathbb{S}^2$.

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Moreover, there exists a sequence $r_n \to \infty$ with the following property. Fix any $\ell \in \{1, \dots, L\}$. For each $j \in \{1, \dots, M^{(\ell)}\}$, there exists $0 \le K_j^{(\ell)} < M^{(\ell)}$ many discs $D(x_{j,k,n}, \mu_{j,k,n}) \subset D(a_{j,n}^{(\ell)}, r_n \lambda_{j,n}^{(\ell)})$ such that for each $k \in \{1, \dots, K_j^{(\ell)}\}$,

$$\lim_{n\to\infty} \Big(\frac{\mu_{j,k,n}}{\lambda_{j,n}^{(\ell)}} + \frac{\mu_{j,k,n}}{\operatorname{dist}(x_{j,k,n}, \partial D(a_{j,n}^{(\ell)}, r_n \lambda_{j,n}^{(\ell)}))} \Big) = 0, \tag{1.8}$$

and so that

$$\lim_{n\to\infty} \Big\| u(t_n) - \omega_j^{(\ell)} \Big(\frac{\cdot - a_{j,n}^{(\ell)}}{\lambda_{j,n}^{(\ell)}} \Big) \Big\|_{L^\infty(D_{j,n}^*)} = 0, \tag{1.9}$$

where $D_{j,n}^* = D(a_{j,n}^{(\ell)}, r_n \lambda_{j,n}^{(\ell)}) \setminus \bigcup_{k=1}^{K_j^{(\ell)}} D(x_{j,k,n}, \mu_{j,k,n})$.

Finally, there exist constants $\omega_\infty^{(1)}, \dots, \omega_\infty^{(L)} \in \mathbb{S}^2$ and sequences $\xi_n, \nu_n \to 0$ so that for each $\ell \in \{1, \dots, L\}$,

$$\lim_{n\to\infty} \Big(\| u(t_n) - \omega_\infty^{(\ell)} \|_{L^\infty(D(x^\ell, \nu_n) \setminus D(x^\ell, \xi_n))} + \frac{\xi_n}{\sqrt{T_+ - t_n}} + \frac{\sqrt{T_+ - t_n}}{\nu_n} \Big) = 0. \tag{1.10}$$

(Global solution) *Suppose $T_+ = \infty$. Let $t_n \to \infty$ be any time sequence. After passing to a subsequence, which we still denote by t_n, we can find an integer $M \ge 0$, sequences $a_{j,n} \in \mathbb{R}^2$ and $\lambda_{j,n} \in (0, \infty)$ for each $j \in \{1, \dots, M\}$, with $\lim_{n\to\infty} \frac{|a_{j,n}| + \lambda_{j,n}}{\sqrt{t_n}} = 0$, and nontrivial harmonic maps $\omega_1, \dots, \omega_M$, so that*

$$\lim_{n\to\infty} \Big(\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} + \frac{|a_{j,n} - a_{k,n}|}{\lambda_{j,n}} \Big) = \infty \quad \text{for all } j \neq k, \tag{1.11}$$

and

$$\lim_{n\to\infty} E\Big(u(t_n) - \omega_\infty - \sum_{j=1}^{M} \Big(\omega_j \Big(\frac{\cdot - a_{j,n}}{\lambda_{j,n}} \Big) - \omega_j(\infty) \Big) \Big) = 0, \tag{1.12}$$

where $\omega_j(\infty) := \lim_{|x|\to\infty} \omega_j(x) \in \mathbb{S}^2$.

Moreover, there exists a sequence $r_n \to \infty$ with the following property. For each $j \in \{1, \dots, M\}$, there exists $0 \le K_j < M$ many discs $D(x_{j,k,n}, \mu_{j,k,n}) \subset D(a_{j,n}, r_n \lambda_{j,n})$ such that for each $k \in \{1, \dots, K_j\}$,

$$\lim_{n\to\infty} \Big(\frac{\mu_{j,k,n}}{\lambda_{j,n}} + \frac{\mu_{j,k,n}}{\operatorname{dist}(x_{j,k,n}, \partial D(a_{j,n}, r_n \lambda_{j,n}))} \Big) = 0, \tag{1.13}$$

and so that

$$\lim_{n\to\infty} \Big\| u(t_n) - \omega_j^{(\ell)} \Big(\frac{\cdot - a_{j,n}^{(\ell)}}{\lambda_{j,n}^{(\ell)}} \Big) \Big\|_{L^\infty(D_{j,n}^*)} = 0, \tag{1.14}$$

where $D_{j,n}^* = D(a_{j,n}, r_n \lambda_{j,n}) \setminus \bigcup_{k=1}^{K_j} D(x_{j,k,n}, \mu_{j,k,n})$.

Finally, there exists a constant $\omega_\infty \in \mathbb{S}^2$ and sequences $\xi_n, \nu_n \in (0, \infty)$ so that

$$\lim_{n\to\infty} \Big(\| u(t_n) - \omega_\infty \|_{L^\infty(D(x^\ell, \nu_n) \setminus D(x^\ell, \xi_n))} + \frac{\xi_n}{\sqrt{t_n}} + \frac{\sqrt{t_n}}{\nu_n} \Big) = 0. \tag{1.15}$$

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Remark 1.2. One can also study the two-dimensional HMHF for more general domains and targets – that is, for maps $u : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ is a 2-dimensional closed, orientable Riemannian manifold (or $\mathbb{R}^2$) and $\mathcal{N}$ is a closed n-dimensional sub-manifold of $\mathbb{R}^N$ for some N – as in this, case the bubbling theory of [9, 20, 23, 24, 28] is understood. But we do not pursue this here. Moreover, the choice of $\mathbb{R}^2$ as the domain is for convenience, as we could have instead considered maps $u : \mathbb{S}^2 \to \mathbb{S}^2$.

We deduce Theorem 1.1 as a consequence of a more refined result, where we show that every smooth solution $u(t)$ converges, continuously in time, to the family of multi-bubble configurations, locally about any point in space. To state this result, we first define a notion of *scale* and *center* of a nontrivial harmonic map.

Definition 1.3 (Scale of a harmonic map). To each non-constant harmonic map $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ and each $\gamma_0 \in (0, 2\pi)$, we associate a scale $\lambda(\omega; \gamma_0)$ defined by

$$\lambda(\omega;\gamma_0) := \inf\{\lambda \in (0,\infty) \mid \text{there exists } a \in \mathbb{R}^2 \text{ such that } E(\omega; D(a,\lambda)) \geq E(\omega) - \gamma_0\}. \tag{1.16}$$

Definition 1.4 (Center of a harmonic map). Given the scale of a harmonic map ω as above, we define the associated center of ω by fixing a choice of $a = a(\omega; \gamma_0) \in \mathbb{R}^2$ so that

$$E(\omega; D(a(\omega;\gamma_0), \lambda(\omega;\gamma_0))) \geq E(\omega) - \gamma_0. \tag{1.17}$$

We prove in Lemma 2.3 that these notions are well-defined and transform naturally under the rescaling and translation of a harmonic map. Indeed, the scale $\lambda(\omega; \gamma_0)$ is a uniquely defined, strictly positive number. Regarding a choice of center, equality occurs in (1.17). However, $a(\omega; \gamma_0)$ is defined only up to a distance of $2\lambda(\omega; \gamma_0)$. Given a harmonic map $\omega(x)$, translating by $b \in \mathbb{R}^2$ and rescaling by $\mu \in (0, \infty)$, we obtain $\omega_{b,\mu}(x) := \omega\big(\frac{x-b}{\mu}\big)$. Then $\lambda(\omega_{b,\mu}) = \lambda(\omega)\mu$ and $|a(\omega_{b,\mu}) - a(\omega)\mu - b| \leq 2\lambda(\omega)\mu$.

Definition 1.5 (Multi-bubble configuration). Let $M \in \{0, 1, 2, \dots\}$. We define an M-bubble configuration to be a superposition

$$\mathcal{Q}(\omega, \omega_1, \dots, \omega_M; x) = \omega + \sum_{j=1}^{M} (\omega_j(x) - \omega_j(\infty)), \tag{1.18}$$

where $\omega \in \mathbb{S}^2$ is a constant, and each $\omega_j : \mathbb{R}^2 \to \mathbb{S}^2$ is a smooth non-constant harmonic map, and $\omega_j(\infty) := \lim_{|x|\to\infty} \omega_j(x)$. We include constant maps as $M = 0$.

We will occasionally use boldface notation, $\boldsymbol{\omega} := (\omega, \omega_1, \dots, \omega_M)$, for finite sequences of harmonic maps with $\omega \in \mathbb{S}^2$ a constant harmonic map and $\omega_1, \dots, \omega_M$ non-constant, and we reserve the arrow notation for vectors (finite sequences) in other contexts. With this notation, we will often express multi-bubbles as $\mathcal{Q}(\boldsymbol{\omega}) := \mathcal{Q}(\omega, \omega_1, \dots, \omega_M)$. We reserve the character $\mathfrak{h}$ to denote an infinite sequence of non-constant harmonic maps (i.e., $\mathfrak{h} := \{\omega_n\}_{n=1}^{\infty}$, where each ω_n is a harmonic map).

Definition 1.6 (Localized distance to a multi-bubble configuration). Let $\xi, \rho, \nu \in (0, \infty)$, with $\xi \leq \rho \leq \nu$, $y \in \mathbb{R}^2$, $u : D(y, \nu) \to \mathbb{S}^2$ and $\gamma_0 \in (0, 2\pi)$ as in Definition 1.3. Let $M \in \{0, 1, 2, \dots\}$, $\omega \in \mathbb{S}^2$ a constant, and let $\omega_1, \dots, \omega_M$ be non-constant harmonic maps with centers $a(\omega_j) \in D(y, \xi)$ for each $j \in \{1, \dots, M\}$ and scales $\lambda(\omega_j) \in (0, \infty)$. Let $\mathcal{Q}(\boldsymbol{\omega})$ be the associated multi-bubble configuration. Let $\vec{\nu} = (\nu, \nu_1, \dots, \nu_M) \in (0, \infty)^{M+1}$ be such that $D(a(\omega_j), \nu_j) \subset D(y, \xi)$ for each $j \in \{1, \dots, M\}$. Let $\vec{\xi} = (\xi, \xi_1, \dots, \xi_M) \in (0, \infty)^{M+1}$ be such that $\xi_j < \lambda(\omega_j)$ for each $j \in \{1, \dots, M\}$. Denote by $\mathcal{I}_j := \{k \neq j \mid D(a(\omega_k), \xi_j) \subset D(a(\omega_j), \nu_j)\}$, and let

$$D_j^* := D(a(\omega_j), \nu_j) \setminus \bigcup_{k \in \mathcal{I}_j} D(a(\omega_k), \xi_j). \tag{1.19}$$

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Define

$$\mathbf{d}_{\gamma_0}(u, \mathcal{Q}(\omega); D(y,\rho); \vec{\nu}, \vec{\xi}) := E\big(u - \mathcal{Q}(\omega); D(y,\rho)\big) + \sum_j \|u - \omega_j\|_{L^\infty(D_j^*)}$$
$$+ \|u - \omega\|_{L^\infty(D(y,\nu)\setminus D(y,\xi))} + E(u; D(y,\nu)\setminus D(y,\xi)) + \frac{\xi}{\rho} + \frac{\rho}{\nu}$$
$$+ \sum_{j\neq k}\left(\frac{\lambda(\omega_j)}{\lambda(\omega_k)} + \frac{\lambda(\omega_k)}{\lambda(\omega_j)} + \frac{|a(\omega_j) - a(\omega_k)|}{\lambda(\omega_j)}\right)^{-1}$$
$$+ \sum_j \left(\frac{\lambda(\omega_j)}{\operatorname{dist}(a(\omega_j), \partial D(y,\xi))} + \frac{\lambda(\omega_j)}{\nu_j} + \frac{\xi_j}{\lambda(\omega_j)}\right)$$
$$+ \sum_j \sum_{k\in\mathcal{I}_j} \frac{\xi_j}{\operatorname{dist}(a(\omega_k), \partial D(a(\omega_j),\nu_j))}. \tag{1.20}$$

We define a localized distance function to the family of all multi-bubble configurations as follows.

Definition 1.7 (Localized multi-bubble proximity function). Let $y \in \mathbb{R}^2$, $\rho \in (0,\infty)$, $u : D(y,\rho) \to \mathbb{S}^2$, and let $\gamma_0 \in (0, 2\pi)$ as in Definition 1.3. We define

$$\boldsymbol{\delta}_{\gamma_0}(u; D(y,\rho)) := \min_{M\in\{0,1,2,\dots\}} \inf_{\omega,\vec{\nu},\vec{\xi}} \mathbf{d}_{\gamma_0}(u, \mathcal{Q}(\omega); D(y,\rho); \vec{\nu}, \vec{\xi}) \tag{1.21}$$

where the infimum above is taken over all possible M-bubble configurations $\mathcal{Q}(\omega)$ and over all admissible $\vec{\nu} = (\nu, \nu_1, \dots, \nu_M) \in (0,\infty)^{M+1}$, $\vec{\xi} = (\xi, \xi_1, \dots, \xi_M) \in (0,\infty)^{M+1}$ in the sense of Definition 1.6. Since γ_0 will eventually be fixed, we will often suppress the dependence of $\mathbf{d}_{\gamma_0}$ and $\boldsymbol{\delta}_{\gamma_0}$ on γ_0 and just write $\mathbf{d}$, $\boldsymbol{\delta}$.

We prove the following theorem.

Theorem 1.8 (Convergence to multi-bubbles in continuous time). *Let $u(t)$ be the unique solution to (1.2) associated to initial data $u_0 \in \mathcal{E}$. Let $T_+ = T_+(u_0) \in (0,\infty]$ denote the maximal time of existence. There exists $\gamma_0 = \gamma_0(E(u_0)) > 0$ as in Definition 1.3 sufficiently small so that the following conclusions hold.*

(Finite time blow-up) Suppose $T_+ < \infty$. For every $y \in \mathbb{R}^2$,

$$\lim_{t\to T_+} \boldsymbol{\delta}_{\gamma_0}\big(u(t); D(y, \sqrt{T_+ - t})\big) = 0. \tag{1.22}$$

Moreover, let $t_n \to T_+$ be any sequence and let $D(y_n, \rho_n)$ be any sequence of discs such that $D(y_n, R_n\rho_n) \subset D(y, \sqrt{T_+ - t})$ for some sequence $R_n \to \infty$. Suppose α_n, β_n are sequences with $\alpha_n \to 0$, $\beta_n \to \infty$, $\lim_{n\to\infty} \beta_n R_n^{-1} = 0$, and

$$\lim_{n\to\infty} E\big(u(t_n); D(y_n, \beta_n\rho_n) \setminus D(y_n, \alpha_n\rho_n)\big) = 0. \tag{1.23}$$

Then,

$$\lim_{n\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t_n); D(y_n, \rho_n)\big) = 0. \tag{1.24}$$

(Global solution) *Suppose $T_+ = \infty$. For every $y \in \mathbb{R}^2$,*

$$\lim_{t\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t); D(y, \sqrt{t})\big) = 0. \tag{1.25}$$

Moreover, let $t_n \to \infty$ be any sequence and let $D(y_n, \nu_n)$ any sequence of discs such that $D(y_n, R_n\nu_n) \subset D(y, \sqrt{t_n})$ for some sequence $R_n \to \infty$. Suppose α_n, β_n are sequences with $\alpha_n \to 0$, $\beta_n \to \infty$, $\lim_{n\to\infty} \beta_n R_n^{-1} = 0$ and

$$\lim_{n\to\infty} E\big(u(t_n); D(y_n, \beta_n\rho_n) \setminus D(y_n, \alpha_n\rho_n)\big) = 0. \tag{1.26}$$

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Then,

$$\lim_{n\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t_n); D(y_n, \rho_n)\big) = 0. \tag{1.27}$$

Remark 1.9. Theorem 1.8 can be viewed as partial progress toward the following questions, which arise naturally from the classical sequential bubbling results [9, 20, 23, 24, 28, 31, 37]:

- Can the harmonic maps (bubbles) appearing in Theorem 1.1 be taken independently of the time sequence? For example, Theorem 1.8 does not fix even the number of bubbles in the decomposition, but rather proves convergence in continuous time to the entire family of multi-bubble configurations.
- In particular, can the decomposition in Theorem 1.1 be taken in continuous time – that is, does $u(t)$ converge in the energy space to u^* plus a superposition of a fixed collection of harmonic maps that are continuously modulated by a finite number of parameters independently of the degree (for example, via the underlying symmetries such as scaling, spatial translations, and rotations)?

Topping [30, 33] made important progress on these and related questions in the case of a global-in-time solution ($T_+ = \infty$), showing the uniqueness of the locations of the bubbling points and that $u(t)$ converges weakly to a unique harmonic map as $t \to \infty$, all under restrictions on the configurations of bubbles appearing in the sequential decomposition. His assumption, roughly, is that all of the *concentrating* bubbles have the same orientation. Here, we do not make any assumptions on the orientations of the bubbles, but our results in the global-in-time case are of a different nature, and we do not recover Topping's conclusions.

Topping answered the questions above in the negative for the HMHF for maps from $\mathbb{S}^2$ into certain target manifolds; see [31].

The first two authors answered the questions above in the affirmative in the case that the target is $\mathbb{S}^2$ and the initial data for (1.2) is k-equivariant; see [16].

Remark 1.10. One can view Theorem 1.8 as a statement about the nonexistence of bubble collisions (asymptotically in time) that destroy multi-bubble structure. Here a bubble collision on a disc $D(y,\rho)$ is defined via the growth of the function $\boldsymbol{\delta}(u(t); D(y,\rho))$ (i.e., $u(t)$ starts close to, but then moves away from the family of multi-bubble configurations on some time interval). Roughly speaking, Theorem 1.8 reduces the questions in Remark 1.9 to an analysis of the dynamics of solutions close to the manifold of multi-bubble configurations. Note that [1, 8, 36] suggest proximity to multi-bubbles cannot be achieved exclusively from energy considerations.

Remark 1.11. There are solutions to the HMHF that develop a bubbling singularity in finite time, the first being the examples of Coron and Ghidaglia [4] (in dimensions ≥ 3) and Chang, Ding and Ye [3] in two dimensions. Guan, Gustafson and Tsai [13] and Gustafson, Nakanishi and Tsai [14] showed that k-equivariant harmonic maps are asymptotically stable for perturbations within their equivariance classes when $k \geq 3$, and thus, there is no finite time blow-up for energies close to the harmonic map in that setting. For $k = 2$, [14] gave examples of solutions exhibiting infinite time blow-up and eternal oscillations, and recently, Wei, Zhang and Zhou [38] constructed such examples in the case $k = 1$. When $k = 1$, the ground state harmonic map is unstable, as Topping [32] proved that there are solutions blowing up in finite time with any initial energy that is slightly above the ground state. Raphaël and Schweyer constructed a stable equivariant blow-up regime for $k = 1$ in [25] and then equivariant blow-up solutions with different rates in [26]. Davila, Del Pino and Wei [6] constructed examples of solutions simultaneously concentrating a single copy of the ground state harmonic map at distinct points in space. See also the work of Topping on reverse bubbling [35] and on the existence of bubble towers [34], and the recent work of Del Pino, Musso and Wei [7] for a construction of bubble towers with an arbitrary number of bubbles in the case of the critical semi-linear heat equation.

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1.3. Summary of the proof

We give an informal description of the proof of Theorem 1.8 and then we discuss how to deduce Theorem 1.1 from it.

To fix ideas, we consider a solution blowing up at a finite time $T_+ < \infty$. Theorem 1.8 is proved by contradicting the finiteness of the integral

$$\int_0^{T_+} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t < \infty \tag{1.28}$$

via a collision analysis in the event that the theorem fails. The collision analysis hinges on the notion of a *minimal collision energy* and the corresponding *collision (time) intervals* that accompany it. These are defined as follows (see Section 3.1). We let K be the smallest integer so that there exist time sequences $\sigma_n, \tau_n \to T_+$, a sequence of discs $D(y_n, \rho_n) \subset \mathbb{R}^2$, a number $\eta > 0$ and a sequence $\epsilon_n \to 0$ so that $\boldsymbol{\delta}(u(\sigma_n); D(y_n, \rho_n)) \le \epsilon_n$, $\boldsymbol{\delta}(u(\tau_n); D(y_n, \rho_n)) \ge \eta$, and $E(u(\sigma_n); D(y_n, \rho_n)) \to 4K\pi$ as $n \to \infty$. To ensure that K is well-defined and ≥ 1 in the event that theorem fails (see Lemma 3.3), we also require that $|[\sigma_n, \tau_n]| \le \epsilon_n \rho_n^2$. We emphasize that the quantization of the energy of harmonic maps $\mathbb{R}^2 \to \mathbb{S}^2$ is used to define K as above. Roughly speaking, the intervals $I_n := [\sigma_n, \tau_n]$ have the property that u is close to a multi-bubble configuration on the left endpoint $t = \sigma_n$ (which we call *bubbling times*) and far from every multi-bubble at the right endpoint $t = \tau_n$ (which we call *ejection times*).

The minimality of K is used crucially to relate the lengths of the collision intervals $|I_n|$ to the largest scale of the bubbles involved in the collision (i.e., those bubbles that concentrate within the discs $D(y_n, \rho_n)$). We call this largest scale $\lambda_{\max,n}$, and the key Lemma 3.4 shows (roughly) that every sequence of collision intervals I_n has subintervals J_n of length at least

$$|J_n| \gtrsim \lambda_{\max,n}^2, \tag{1.29}$$

on which $u(t)$ bounded away from the multi-bubble family (i.e,. $\boldsymbol{\delta}(u(t); D(y_n, \rho_n)) \ge \epsilon > 0$ for all $t \in J_n$, for some $\epsilon > 0$). The intuition behind this is the following. Suppose there were a sequence of intervals $J_n = [s_n, t_n] \subset I_n$ for which the s_n's are bubbling times and the t_n's are ejection times, but $|J_n| \ll \lambda_{\max,n}^2$. This leads to a contradiction of the minimality of K because the time-interval J_n is too short relative to the scales of the largest bubbles ($\lambda_{\max,n}$) for them to become involved in a collision; thus, collisions are captured on smaller discs $D(\widetilde{y}_n, \widetilde{\rho}_n) \subset D(y_n, \rho_n)$ with $\widetilde{\rho}_n \ll \lambda_{\max,n}$, and these carry strictly less energy than $4\pi K$; see the proof of Lemma 3.4.

The fact that $u(t)$ is at least distance $\epsilon > 0$ away from the multi-bubble family on J_n can be combined with the classical localized elliptic bubbling lemma described in Section 1.1 (the Compactness Lemma 2.15) to show that on the interval J_n, the tension satisfies

$$\inf_{t \in J_n} \lambda_{\max,n}^2 \|\mathcal{T}(u(t))\|_{L^2}^2 \gtrsim 1. \tag{1.30}$$

The main point here is that the Compactness Lemma 2.15 says that $u(t)$ bubbles at scale $\lambda_{\max,n}$ along any sequence of times $\widetilde{s}_n$ for which $\lim_{n\to\infty} \lambda_{\max,n}^2 \|\mathcal{T}(u(\widetilde{s}_n))\|_{L^2}^2 = 0$, which is impossible. At this point, we have contradicted (1.28) since the previous two displayed equations combine to give

$$\sum_n \int_{J_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t \gtrsim \sum_n |J_n| \, \lambda_{\max,n}^{-2} \gtrsim \sum_n 1 = \infty. \tag{1.31}$$

The idea of a (minimal) collision energy and associated collision time intervals are related to analogous concepts in the first two authors' work on the soliton resolution conjecture for nonlinear waves and on continuous bubbling for the k-equivariant HMHF; see [16, 17, 18]. Unlike in these earlier works, we do not use modulation analysis in this paper.

Theorem 1.8 and the Compactness Lemma 2.15 are the main ingredients in the proof of Theorem 1.1. Again, focusing on the finite time blow-up case, it is well-known (see Lemma 2.13) that energy does

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not concentrate at or outside the self-similar scale $\sqrt{T_+ - t}$, so it suffices to examine the behavior of $u(t)$ restricted to discs $D(y, \sqrt{T_+ - t})$, for $y \in \mathbb{R}^2$ a point where energy concentrates. Let $t_n \to T_+$ be any time sequence. By Theorem 1.8, and after passing to a subsequence, there exists an integer $\widetilde{M} \geq 1$ and a sequence of M-bubble configurations $\mathbf{\Omega}_n = (\Omega_n, \Omega_{1,n}, \ldots, \Omega_{\widetilde{M},n})$ and sequences $\vec{\xi}_n, \vec{\nu}_n$ as in Definition 1.6 so that

$$\mathbf{d}(u(t_n), \mathbf{\Omega}_n; D(y, \sqrt{T_+ - t}); \vec{\xi}_n, \vec{\nu}_n) \to 0 \text{ as } n \to \infty. \tag{1.32}$$

However, the decomposition in Theorem 1.1 involves a *fixed* collection of finitely many harmonic maps, $\omega_1, \ldots, \omega_M$ (i.e., a collection independent of n). To find such a collection from the $\Omega_{j,n}$ we apply the Compactness Lemma 2.15 to each $\Omega_{j,n}$, obtaining a fixed collection of bubbles $\{\omega_{j,k}\}_{k=1}^{L_j}$ for each $j \in \{1, \ldots, \widetilde{M}\}$. A delicate point is that the scales and centers $(b_{j,k,n}, \mu_{j,k,n})$ associated to the harmonic maps $\omega_{j,k}$ given by the Compactness Lemma 2.15 may not satisfy (1.6) for distinct j. But this potential pitfall is remedied by the refined information in Theorem 1.8, which says $u(t)$ approaches the multi-bubble family at every smaller scale $\rho_n \leq \sqrt{T_+ - t_n}$ (excluding of course the precise scales of the bubbles themselves).

In Section 2, we give background information on harmonic maps and the harmonic map heat flow. Much of Section 2 is classical, except perhaps the notions of scale and center of harmonic maps and Lemma 2.12, which involves the propagation of localized L^∞ estimates for solutions to (1.2), which we did not find a reference for in the literature. Section 3 contains the proofs of the main theorems.

1.4. Notational conventions

Constants are denoted C, C_0, C_1, c, c_0, c_1. We write $A \lesssim B$ if $A \leq CB$ and $A \gtrsim B$ if $A \geq cB$. Given sequences A_n, B_n, we write $A_n \ll B_n$ if $\lim_{n\to\infty} A_n/B_n = 0$.

For any sets X, Y, Z, we identify $Z^{X\times Y}$ with $(Z^Y)^X$, which means that if $\phi : X \times Y \to Z$ is a function, then for any $x \in X$, we can view $\phi(x)$ as a function $Y \to Z$ given by $(\phi(x))(y) := \phi(x, y)$.

2. Preliminaries

2.1. Properties of harmonic maps

We use a few well-known features of finite energy harmonic maps $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ – namely, their smoothness, the invariance of harmonicity and the energy under conformal transformations of the domain, and the fact that the energy is quantized.

Theorem 2.1. [15, Theorem 4.1.1][12, pg. 126, Proposition][27, Theorem 3.6] [19, Section 8, the Remarque on pg. 65] *Let $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ be a weak non-constant solution to (1.5) of finite energy. Then ω is smooth and extends as a smooth harmonic map from the sphere to itself of nonzero degree, which minimizes the energy (1.1) in its degree class with $E(\omega) = 4\pi|\deg(\omega)|$.*

Remark 2.2. The regularity statement above is due to Hélein and holds in the more general setting of weak harmonic maps $\omega \in H^1(\mathcal{M}, \mathcal{N})$, where $\mathcal{M}$ is a closed, two-dimensional, orientable Riemannian manifold and $\mathcal{N}$ is a smooth compact Riemannian manifold. The extension of a smooth, finite energy harmonic map $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ to a smooth, finite energy harmonic map $\widetilde{\omega} : \mathbb{S}^2 \to \mathbb{S}^2$ is a consequence of the conformal equivalence between $\mathbb{R}^2$ and $\mathbb{S}^2 \setminus \{p_0\}$ via the stereographic projection map and the Removable Singularity Theorem of Sacks-Uhlenbeck [27]. Here we also use the fact, due to Eells and Sampson [12], that in the case of orientable two-dimensional Riemannian manifolds $\mathcal{M}, \mathcal{N}$, if $\omega : \mathcal{M} \to \mathcal{N}$ is smooth and $\phi : \mathcal{M} \to \mathcal{M}$ is a conformal diffeomorphism, then $\omega \circ \phi$ is harmonic if and only if ω is, and moreover, $E(\omega) = E(\omega \circ \phi)$. The relationship between the topological degree and the energy (energy quantization) generalizes to harmonic maps between closed, two-dimensional, orientable, Riemannian manifolds $\omega : \mathcal{M} \to \mathcal{N}$, where we have $E(\omega) = \text{Area}(\mathcal{N})|\deg(\omega)|$; see, for example, Lemaire [19, Section 8, the Remarque on pg. 65].

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2.1.1. The scale and center of a harmonic map

Given a non-constant harmonic map $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, recall the notion of scale $\lambda(\omega;\gamma_0)$ and center $a(\omega;\gamma_0)$ from Definition 1.3 and Defintion 1.4.

Lemma 2.3 (Center and scale). *Let $\gamma_0 \in (0, 2\pi)$ and let $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ be a non-constant harmonic map. Then $\lambda(\omega) = \lambda(\omega,\gamma_0)$ as in Definition 1.3 is uniquely defined and strictly positive. Moreover, there exists $a(\omega) = a(\omega,\gamma_0)$ as in Definition 1.4. For all $b \in \mathbb{R}^2$ and $\mu \in (0,\infty)$,*

$$\lambda\big(\omega\big(\frac{\cdot - b}{\mu}\big)\big) = \lambda(\omega)\mu, \ \ \textit{and}\ \ \Big|a\big(\omega\big(\frac{\cdot - b}{\mu}\big)\big) - b - a(\omega)\mu\Big| \le 2\lambda(\omega)\mu. \tag{2.1}$$

Proof. Since $E(\omega; D(0,R)) \to E(\omega)$ as $R \to \infty$, it follows that the scale $\lambda(\omega)$ is well-defined. If $\lambda(\omega) = 0$, then there exists $a_n \in \mathbb{R}^2$ so that

$$E(\omega; D(a_n, 1/n)) \ge E(\omega) - \gamma_0 \qquad \forall\, n \ge 1. \tag{2.2}$$

If $n \neq m$, the $D(a_n, 1/n) \cap D(a_m, 1/m) \neq \emptyset$. Indeed, otherwise,

$$E(\omega) \ge E(\omega; D(a_n, 1/n)) + E(\omega; D(a_m, 1/m)) \ge 2E(\omega) - 2\gamma_0,$$

whence $E(\omega) \le 2\gamma_0 < 4\pi$, which contradicts that ω is not constant. Therefore, $\{a_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^2$, and $a_n \to a_\infty$. Passing to the limit in (2.2) gives a contradiction.

To see that a center $a(\omega)$ can be chosen, take $\lambda_n \to \lambda(\omega)$ and $a_n \in \mathbb{R}^2$ such that

$$E(\omega; D(a_n, \lambda_n)) \ge E(\omega) - \gamma_0.$$

As before, we conclude that no two disks $\{D(a_n, \lambda_n)\}_{n=1}^\infty$ can be disjoint. Thus, $a_n \in \mathbb{R}^2$ lie in a compact set, and we may assume that $a_n \to a_\infty$ as $n \to \infty$, which is a desired center. We note that $\lambda(\omega)$ is uniquely defined, but $a(\omega)$ is defined only up to a distance of $2\lambda(\omega)$. The properties (2.1) are immediate from the definitions. □

Lemma 2.4 (Decay of harmonic maps). *There exists $\gamma_0 \in (0, 2\pi)$ with the following property. For any $0 < \gamma \le \gamma_0$ and any harmonic map $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, the exterior energy decays at the following rate:*

$$E(\omega; \mathbb{R}^2 \setminus D(a(\omega;\gamma); R\lambda(\omega,\gamma))) \le \pi R^{-2} \tag{2.3}$$

for all $R \ge 2$.

We use the following ϵ-compactness result of Ding and Tian [9] in the proof of Lemma 2.4.

Lemma 2.5 (ϵ-compactness). *[9, Lemma 2.1] Let $y \in \mathbb{R}^2$ and let $u : D(y,1) \to \mathbb{S}^2 \subset \mathbb{R}^3$ belong to the class $W^{2,2}(D(y,1); \mathbb{S}^2)$. Then there exists $\epsilon_0 > 0, C > 0$ such that if $E(u; D(y,1)) < \epsilon_0$, then*

$$\|u - u_{\textrm{avg}}\|_{W^{2,2}(D(y,1/2))} \le C\Big(\sqrt{E(u; D(y,1))} + \|\mathcal{T}(u)\|_{L^2(D(y,1))}\Big), \tag{2.4}$$

where $u_{\textrm{avg}}$ denotes the mean of u over the disc $D(y,1)$. In particular,

$$\|u - u_{\textrm{avg}}\|_{L^\infty(D(y,1/2))} \le C\Big(\sqrt{E(u; D(y,1))} + \|\mathcal{T}(u)\|_{L^2(D(y,1))}\Big). \tag{2.5}$$

Proof of Lemma 2.4. With loss of generality, $a(\omega;\gamma) = 0$ and $\lambda(\omega;\gamma) = 1$. Consider the harmonic map $\tilde{\omega}(z) = \omega(1/z)$ for which 0 is a removable singularity (see [27]) and $E(\tilde{\omega}; D(0;1)) \le \gamma$. Applying Lemma 2.5, we conclude that

$$\|D^2\, \tilde{\omega}\|_{L^2(D(0,1/2))} \lesssim \sqrt{\gamma},$$

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whence $\tilde{\omega} \in W^{1,p}(D(0,1/2))$ for any $2 < p < \infty$. From the equation $\Delta\tilde{\omega} + \tilde{\omega}|\nabla\tilde{\omega}|^2 = 0$, it follows that $D^2\tilde{\omega} \in L^p(D(0,1/2))$ for any $2 < p < \infty$. In particular, there exists an absolute constant $\gamma_0 > 0$ such that $0 < \gamma \le \gamma_0$ ensures that

$$|\nabla\tilde{\omega}(z)| \le 1 \quad \forall\ |z| \le 1/2,$$

whence $E(\omega, \mathbb{R}^2 \setminus D(0,1/r)) = E(\tilde{\omega}, D(0,r)) \le \pi r^2$ for all $r \le \frac{1}{2}$. □

Lemma 2.6 (Energy of multi-bubbles). *Let $y_n \in \mathbb{R}^2$, $\rho_n > 0$ be sequences, and $M \in \mathbb{N}$. Let $\omega_\infty \in \mathbb{S}^2$ be a constant, let $\omega_1, \ldots, \omega_M$ be nontrivial harmonic maps, and let $b_{n,j} \in D(y_n, \rho_n)$ and $\mu_{n,j} \in (0,\infty)$ for $j \in \{1,\ldots,M\}$ be sequences so that*

$$\lim_{n\to\infty} \left[\sum_{j\neq k} \left(\frac{\mu_{n,j}}{\mu_{n,k}} + \frac{\mu_{n,k}}{\mu_{n,j}} + \frac{|b_{n,j} - b_{n,k}|}{\mu_{n,j}} \right)^{-1} + \sum_{j=1}^{M} \frac{\mu_{n,j}}{\operatorname{dist}(b_{n,j}, \partial D(y_n,\rho_n))} \right] = 0. \tag{2.6}$$

Then

$$\lim_{n\to\infty} E\Big(\mathcal{Q}\big(\omega_\infty, \omega_1\big(\frac{\cdot - b_{n,1}}{\mu_{n,1}}\big), \ldots, \omega_M\big(\frac{\cdot - b_{n,M}}{\mu_{n,M}}\big)\big); D(y_n,\rho_n)\Big) = \sum_{j=1}^{M} E(\omega_j). \tag{2.7}$$

Proof of Lemma 2.6. To simplify notation within the proof, we use the shorthand $\omega_{n,j} = \omega_j\big(\frac{\cdot - b_{n,j}}{\mu_{n,j}}\big)$. Expanding the energy, we obtain

$$E(\mathcal{Q}(\omega_{n,1},\ldots,\omega_{n,M}); D(y_n,\rho_n)) = \sum_{j=1}^{M} E(\omega_{n,j}; D(y_n,\rho_n)) + \frac{1}{2}\sum_{j\neq k} \int_{D(y_n,\rho_n)} \nabla\omega_{n,j}\nabla\omega_{n,k}.$$

By the separation condition (2.6) with respect to $\partial D(y_n,\rho_n)$ and Lemma 2.4,

$$E(\omega_{n,j}; D(y_n,\rho_n)) = E(\omega_{n,j}) + o_n(1)$$

as $n \to \infty$. However, if $j \neq k$, then

$$\Big| \int_{D(y_n,\rho_n)} \nabla\omega_{n,j}\nabla\omega_{n,k} \Big| \le \int |\nabla\omega_{n,j}||\nabla\omega_{n,k}| = o_n(1) \tag{2.8}$$

by the first term of (2.6). □

2.2. *Properties of the harmonic map heat flow*

2.2.1. Well-posedness

The starting point for our analysis of the HMHF is the classical result of Struwe [28], which says that the initial value problem is well-posed for data in the energy space and solutions are regular up to their maximal time.

Following Struwe, we introduce the space,

$$\begin{aligned} \mathcal{V}_\tau^T := \Big\{ & u : [\tau, T] \times \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3 \ \big|\ u \text{ is measurable, and} \\ & \sup_{t\in[\tau,T]} E(u(t)) + \int_\tau^T \|\partial_t u(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 \,\mathrm{d}t < \infty \Big\}. \end{aligned} \tag{2.9}$$

We use the shorthand $\mathcal{V}^T = \mathcal{V}_0^T$.

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Theorem 2.7 (Local well-posedness). *[28, Theorem 4.1] Let $u_0 \in \mathcal{E}$. Then there exists a maximal time of existence $T_+ = T_+(u_0)$ and a unique solution $u \in \bigcap_{T<T_+} \mathcal{V}^T$ to (1.2) with $u(0) = u_0$. The solution $u(t)$ is regular (e.g., C^2) on the open interval $(0, T_+)$.*

A finite maximal time $T_+ < \infty$ is characterized by the existence of an integer $L \geq 1$, a number $\epsilon_0 > 0$ and points $\{x_\ell\}_{\ell=1}^L \subset \mathbb{R}^2$ such that

$$\limsup_{t \to T_+} E(u(t); D(x_\ell, R)) \geq \epsilon_0, \quad \forall\, R > 0, \quad \forall 1 \leq \ell \leq L. \tag{2.10}$$

The $\{x_\ell\}_{\ell=1}^L$ are called bubbling points, and there are at most finitely many. There exists a finite energy mapping $u^* : \mathbb{R}^2 \to \mathbb{S}^2$, called the body map, such that $u(t) \rightharpoonup u^*$ as $t \to T_+$ weakly in $H^1(\mathbb{R}^2; \mathbb{S}^2)$ and strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{x_\ell\}_{\ell=1}^L; \mathbb{S}^2)$.

The energy $E(u(t))$ is continuous and nonincreasing as a function of $t \in [0, T_+)$, and for any $t_1 \leq t_2 \in [0, T_+)$, there holds

$$E(u(t_2)) + \int_{t_1}^{t_2} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t = E(u(t_1)). \tag{2.11}$$

In particular, there exists $E_+ := \lim_{t \to T_+} E(u(t))$, and

$$\int_0^{T_+} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t < \infty. \tag{2.12}$$

Remark 2.8. Lemma 2.7 is proved by Struwe for the HMHF in the case of maps from a closed Riemannian surface $\mathcal{M}$ to a compact Riemannian manifold $\mathcal{N}$; see [28, Theorem 4.1]. The same arguments hold for $\mathcal{M} = \mathbb{R}^2$.

Lemma 2.9 (Localized energy inequality). *There exists a constant $C > 0$ with the following property. Let $u(t)$ be a solution to (1.2) with initial data u_0 as in Lemma 2.7, on its maximal interval $I_{\max} = [0, T_+)$. Let $0 < t_1 < t_2 < T_+$. Let $R > 0$, $\phi \in C_0^\infty(\mathbb{R}^2)$ satisfy $0 \leq \phi(x) \leq 1$ and $|\nabla \phi| \leq R^{-1}$. Then*

$$\int_{\mathbb{R}^2} |\nabla u(t_2, x)|^2 \phi(x)^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^2} |\nabla u(t_1, x)|^2 \phi(x)^2 \, \mathrm{d}x + C E(u_0) \frac{t_2 - t_1}{R^2} \tag{2.13}$$

and

$$\int_{\mathbb{R}^2} |\nabla u(t_2, x)|^2 \phi(x)^2 \, \mathrm{d}x \geq \int_{\mathbb{R}^2} |\nabla u(t_1, x)|^2 \phi(x)^2 \, \mathrm{d}x - C\Big(E(u_0) \frac{(t_2 - t_1)}{R^2} + |E(u(t_1)) - E(u(t_2))|\Big). \tag{2.14}$$

Proof of Lemma 2.9. Take the dot product of the equation (1.2) with $\partial_t u \phi^2$ and integrate by parts to obtain the identity

$$\|\partial_t u(t) \phi\|_{L^2}^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla u(t) \phi\|_{L^2}^2 = -\sum_{j=1}^2 \int_{\mathbb{R}^2} \partial_j u(t, x) \cdot \partial_t u(t, x) \partial_j \phi(x) \phi(x) \, \mathrm{d}x. \tag{2.15}$$

Integrating the above from t_1 to t_2 we obtain the identity

$$\int_{t_1}^{t_2} \|\partial_t u(t) \phi\|_{L^2}^2 \, \mathrm{d}t + \frac{1}{2} \|\nabla u(t_2) \phi\|_{L^2}^2 = \frac{1}{2} \|\nabla u(t_1) \phi\|_{L^2}^2 - \int_{t_1}^{t_2} \sum_{j=1}^2 \int_{\mathbb{R}^2} \partial_j u(t, x) \cdot \partial_t u(t, x) \partial_j \phi(x) \phi(x) \, \mathrm{d}x, \mathrm{d}t. \tag{2.16}$$

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The right-hand side above is bounded by

$$\int_{t_1}^{t_2} \Big| \sum_{j=1}^{2} \int_{\mathbb{R}^2} \partial_j u(t,x) \cdot \partial_t u(t,x) \partial_j \phi(x) \phi(x) \, \mathrm{d}x \, \mathrm{d}t \Big| \lesssim \int_{t_1}^{t_2} \frac{\sqrt{E(u_0)}}{R} \| \partial_t u(t) \phi \|_{L^2} \, \mathrm{d}t. \tag{2.17}$$

The lemma readily follows after an application of Cauchy Schwarz, where we note that in obtaining (2.14), we also make use of the energy identity (2.11). □

Lemma 2.10. *Let $u_n(t)$ be a sequence of HMHFs with initial data $u_{n,0} \in \mathcal{E}$ defined on time intervals $I_n := [0, \tau_n]$ for a sequence $\tau_n > 0$ with $\lim_{n\to\infty} \tau_n = 0$, and satisfying $\limsup_{n\to\infty} E(u_{n,0}) < \infty$. Let ω be a harmonic map and let $R_n > 0$ be a sequence such that $\lim_{n\to\infty} \tau_n R_n^{-2} = 0$. Suppose that*

$$\lim_{n\to\infty} E(u_{n,0} - \omega; D(0, 2R_n)) = 0. \tag{2.18}$$

Then

$$\lim_{n\to\infty} E(u_n(\tau_n) - \omega; D(0, R_n)) = 0. \tag{2.19}$$

Next, let $\epsilon_n > 0$ be a sequence with $\epsilon_n < R_n$ for all n and such that $\lim_{n\to\infty} \tau_n \epsilon_n^{-2} = 0$. Let $L \geq 1$ be an integer and let $\{x_\ell\}_{\ell=1}^{L} \subset \mathbb{R}^2$ be such that the discs $D(x_\ell, \epsilon_n)$ are disjoint and satisfy $D(x_\ell, \epsilon_n) \subset D(0, R_n)$ for each n and each $\ell \in \{1, \dots, L\}$. Moreover, $|x_\ell - x_m| \geq 100\epsilon_n$ if $\ell \neq m$. Suppose that

$$\lim_{n\to\infty} E\Big(u_{n,0} - \omega; D(0, 2R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, 2^{-1}\epsilon_n)\Big) = 0. \tag{2.20}$$

Then

$$\lim_{n\to\infty} E\Big(u_n(\tau_n) - \omega; D(0, R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, \epsilon_n)\Big) = 0. \tag{2.21}$$

Proof of Lemma 2.10. The proof is very similar to the Proof of Lemma 2.9. We prove the estimate (2.21), as the proof of (2.19) is analogous. Set $v_n(t) := u_n(t) - \omega$. Then

$$\partial_t v_n - \Delta v_n = u_n |\nabla u_n|^2 - \omega |\nabla \omega|^2. \tag{2.22}$$

Let $\phi_n \in C_0^\infty(\mathbb{R}^2)$ and take the dot product of the above with $\partial_t v_n \phi_n^2$. Recalling that $\partial_t v_n = \partial_t u_n \perp u_n$, integrating by parts, and integrating in time from 0 to τ_n, we obtain the inequality

$$\begin{aligned} \frac{1}{2} \| \nabla v_n(\tau_n) \phi_n \|_{L^2}^2 + \int_0^{\tau_n} \| \partial_t v_n(t) \phi_n \|_{L^2}^2 \, \mathrm{d}t &\leq \frac{1}{2} \| \nabla u_{n,0} \phi_n \|_{L^2}^2 \\ &+ \int_0^{\tau_n} \int_{\mathbb{R}^2} |\nabla v_n(t)| |\partial_t v_n(t)| |\nabla \phi_n| \phi_n \, \mathrm{d}x \, \mathrm{d}t + \int_0^{\tau_n} \int_{\mathbb{R}^2} |\nabla \omega|^2 |\partial_t v_n| \phi_n^2 \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{2.23}$$

Now, let ϕ_n be cutoffs supported in the region $D(0, 2R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, 2^{-1}\epsilon_n))$ and $= 1$ in the region $D(0, R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, \epsilon_n))$, satisfying the bound $|\nabla \phi_n| \lesssim \epsilon_n^{-1}$. The first term of the last line above satisfies

$$\int_0^{\tau_n} \int_{\mathbb{R}^2} |\nabla v_n(t)| |\partial_t v_n(t)| |\nabla \phi_n| \phi_n \, \mathrm{d}x \, \mathrm{d}t \lesssim (E(u_{n,0}) + E(\omega)) \frac{\tau_n}{\epsilon_n^2} + \frac{1}{2} \int_0^{\tau_n} \| \partial_t v_n(t) \phi_n \|_{L^2}^2 \, \mathrm{d}t, \tag{2.24}$$

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and the second term on the right above can be absorbed into the left-hand side of (2.23). Similarly,

$$\int_0^{\tau_n} \int_{\mathbb{R}^2} |\nabla\omega|^2 |\partial_t v_n| \phi_n^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim \tau_n \|\nabla\omega\|_{L^4}^4 + \frac{1}{2}\int_0^{\tau_n} \|\partial_t v_n(t)\phi_n\|_{L^2}^2 \,\mathrm{d}t, \tag{2.25}$$

and the second term on the right above can be absorbed into the left-hand side of (2.23). The limit (2.21) readily follows. □

2.2.2. Local L^∞ estimates for the heat flow

We use the notation $e^{t\Delta}$ to denote the heat propagator in $\mathbb{R}^2$ – that is,

$$e^{t\Delta} v(x) := \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}} v(y)\,\mathrm{d}y, \tag{2.26}$$

where here $v : \mathbb{R}^2 \to \mathbb{R}^3$. We also recall Duhamel's formula,

$$v(t) = e^{t\Delta} v(0) + \int_0^t e^{(t-s)\Delta}(\partial_s v(s) - \Delta v(s))\,\mathrm{d}s. \tag{2.27}$$

Lemma 2.11 (Parabolic Strichartz estimates). [29, Lemma 2.5] *There exists a constant $C_0 > 0$ with the following property. Let $v_0 \in L^2(\mathbb{R}^2;\mathbb{R}^3)$. Let $T > 0$, $I := [0,T]$ and let $F \in L^1([0,T]; L^2(\mathbb{R}^2;\mathbb{R}^3))$. Let $v(t)$ denote the unique solution to the linear heat equation*

$$\begin{aligned} \partial_t v - \Delta v &= F \\ v(0) &= v_0. \end{aligned} \tag{2.28}$$

Then

$$\|v\|_{L^2(I;L^\infty(\mathbb{R}^2;\mathbb{R}^3))} \le C_0\big(\|v_0\|_{L^2} + \|F\|_{L^1(I;L^2(\mathbb{R}^2;\mathbb{R}^3))}\big). \tag{2.29}$$

Proof. Setting $(Tf)(t) := e^{t\Delta} f$ for $t \ge 0$, one has $T^*F = \int_0^\infty e^{s\Delta} F(s)\,ds$. Starting from the two-dimension estimate $\|(Tf)(t)\|_\infty \lesssim t^{-1}\|f\|_1$, we have

$$(TT^*F)(t) = \int_0^\infty e^{(t+s)\Delta} F(s)\,ds,$$

whence

$$\|(TT^*F)(t)\|_\infty \lesssim \int_0^\infty (t+s)^{-1}\|F(s)\|_1\,ds = \int_0^\infty (1+u)^{-1}\|F(ut)\|_1\,du.$$

The right-hand side is $L^2_t((0,\infty))$ bounded and we conclude that

$$\|TT^*F\|_{L^2((0,\infty),L^\infty(\mathbb{R}^2))} \lesssim \|F\|_{L^2((0,\infty),L^1(\mathbb{R}^2))}.$$

Thus, $\langle TT^*F, F\rangle = \|T^*F\|_2^2 \lesssim \|F\|^2_{L^2((0,\infty),L^1(\mathbb{R}^2))}$, and by duality, we obtain the $F = 0$ case of (2.29), viz. $\|Tv_0\|_{L^2((0,\infty),L^\infty(\mathbb{R}^2))} \lesssim \|v_0\|_2$. However, if $v_0 = 0$, then

$$v(t) = \int_0^t e^{(t-s)\Delta} F(s)\,ds = \int_0^\infty \chi_{[s\le t]}\, e^{(t-s)\Delta} F(s)\,ds,$$

whence

$$\|v(t)\|_{L^2((0,\infty),L^\infty(\mathbb{R}^2))} \le \int_0^\infty \|e^{(t-s)\Delta}F(s)\|_{L^2((s,\infty),L^\infty(\mathbb{R}^2))}\,ds \lesssim \int_0^\infty \|F(s)\|_{L^2(\mathbb{R}^2)}\,ds, \tag{2.30}$$

as claimed. □

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Lemma 2.12. *Let $u_n(t)$ be a sequence of solutions to (1.2) with initial data $u_{n,0} \in \mathcal{E} \cap C^0(\mathbb{R}^2;\mathbb{R}^3)$ and $\limsup_{n\to\infty} E(u_{n,0}) < \infty$, defined on time intervals $I_n := [0,\tau_n]$ for a sequence $\tau_n > 0$ with $\lim_{n\to\infty}\tau_n = 0$. Let ω be a harmonic map (possibly constant) and let $R_n > 0$ be a sequence so that $\lim_{n\to\infty}\tau_n R_n^{-2} = 0$. Suppose that*

$$\lim_{n\to\infty}\Big(\|u_{n,0}-\omega\|_{L^\infty(D(0,4R_n))} + E(u_{n,0}-\omega; D(0,4R_n))\Big) = 0. \tag{2.31}$$

Then

$$\lim_{n\to\infty}\|u_n(\tau_n)-\omega\|_{L^\infty(D(0,R_n))} = 0. \tag{2.32}$$

Next, let $\epsilon_n > 0$ be a sequence with $\epsilon_n < R_n$ for all n and such that $\lim_{n\to\infty}\tau_n\epsilon_n^{-2} = 0$. Let $L \geq 1$ be an integer and let $\{x_\ell\}_{\ell=1}^L \subset \mathbb{R}^2$ be such that the discs $D(x_\ell,\epsilon_n)$ are disjoint and satisfy $D(x_\ell,\epsilon_n) \subset D(0,R_n)$ for each n and each $\ell \in \{1,\dots,L\}$. Moreover, $|x_\ell - x_m| \geq 100\epsilon_n$ if $\ell \neq m$. Suppose that

$$\lim_{n\to\infty}\Big(\|u_{n,0}-\omega\|_{L^\infty(D(0,4R_n)\setminus\bigcup_{\ell=1}^L D(x_\ell,4^{-1}\epsilon_n))} + E\Big(u_{n,0}-\omega; D(0,4R_n)\setminus\bigcup_{\ell=1}^L D(x_\ell,4^{-1}\epsilon_n)\Big)\Big) = 0. \tag{2.33}$$

Then

$$\lim_{n\to\infty}\|u_n(\tau_n)-\omega\|_{L^\infty(D(0,R_n)\setminus\bigcup_{\ell=1}^L D(x_\ell,\epsilon_n))} = 0. \tag{2.34}$$

Proof. We begin with a solution $u_n(t)$ of the heat flow satisfying

$$\|u_n(0)-\omega\|_{L^\infty(D(0,4R_n))} + E(u_n(0)-\omega, D(0,4R_n)) = o_n(1).$$

We pick ϕ_1 to be the ground state of the Dirichlet Laplacian on the disk $D(0,3R_n/2)$ and set $v(t) = (u(t)-\omega)\phi_1$, dropping the index n for simplicity. We normalize $\phi_1(0) = 1$, which means that $\|\phi_1\|_\infty = 1$. Then $-\Delta\phi_1 = \lambda_1^2\phi_1$, $\lambda_1^2 \simeq R_n^{-2}$ and

$$\partial_t v - \Delta v = (|\nabla u|^2 u - |\nabla\omega|^2\omega)\phi_1 - 2\nabla(u-\omega)\nabla\phi_1 + \lambda_1^2 v.$$

Since ϕ_1 is not globally smooth, we cannot solve this heat equation on the plane but rather need to use the heat flow on the region $\Omega = \overline{D(0,3R_n/2)}$ with Dirichlet boundary conditions. By the Beurling-Deny theorem (see Davies [5, Theorem 1.3.5]), the heat flow is a contraction on $L^\infty(\Omega)$, and we conclude that

$$\begin{aligned}\max_{0\le t\le T}\|v(t)\|_\infty \le \|v(0)\|_\infty &+ \int_I \|(|\nabla u(s)|^2 u(s) - |\nabla\omega|^2\omega)\phi_1\|_\infty\,\mathrm{d}s \\ &+ 2\int_I \|\nabla(u(s)-\omega)\nabla\phi_1\|_\infty\,\mathrm{d}s + \lambda_1^2|I|\|v\|_{L^\infty(I\times\mathbb{R}^2)}\end{aligned} \tag{2.35}$$

with $[0,\tau_n] = I$. If $\lambda_1^2|I| \le \frac{1}{2}$, then the final term gets absorbed to the left-hand side. Next,

$$\begin{aligned}\|(|\nabla u(s)|^2 u(s) - |\nabla\omega|^2\omega)\phi_1\|_\infty &\le 2(\|\phi_2\nabla(u(s)-\omega)\|_\infty^2 + \|\phi_2\nabla\omega\|_\infty^2)\|v(s)\|_\infty \\ &+ \|\omega\|_\infty\|\phi_2\nabla(u-\omega)\|_\infty(\|\phi_2\nabla(u-\omega)\|_\infty + 2\|\phi_2\nabla\omega\|_2),\end{aligned} \tag{2.36}$$

where ϕ_2 is a smooth cutoff to $D(0,2R_n)$ with $\phi_1\phi_2 = \phi_1$. We further bound

$$\|\nabla(u(s)-\omega)\nabla\phi_1\|_\infty \lesssim R_n^{-1}\|\phi_2\nabla(u(s)-\omega)\|_\infty$$

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using that $\|\nabla\phi_1\|_\infty \lesssim R_n^{-1}$, which follows by scaling. Define $w(s) := \phi_2\nabla(u(s)-\omega)$ and let $X := L^\infty(I;L^\infty(\mathbb{R}^2))$, $Y := L^2(I;L^\infty(\mathbb{R}^2;\mathbb{R}^3))$. Then (2.35) implies that

$$\|v\|_X \lesssim o(1) + \tau_n + \|v\|_X(\|w\|_Y^2 + \tau_n) + \|w\|_Y^2 + \sqrt{\tau_n R_n^{-2}}\|w\|_Y,$$

which in turn simplifies to

$$\|v\|_X \lesssim o(1) + (\|v\|_X + 1)\|w\|_Y^2. \tag{2.37}$$

To bound w, we use the PDE

$$\begin{aligned}\partial_t w - \Delta w = \phi_2\nabla(u|\nabla u|^2) - 2\sum_{j=1}^{2}\nabla\partial_j(u-\omega)\partial_j\phi_2 \\ - \nabla(u-\omega)\Delta\phi_2 - \phi_2\nabla(\omega|\nabla\omega|^2) =: G.\end{aligned} \tag{2.38}$$

By (2.29), and with $Z := L^1(I;L^2(\mathbb{R}^2;\mathbb{R}^3))$,

$$\|w\|_Y \lesssim \|\nabla(u(0)-\omega)\phi_2\|_2 + \|G\|_Z \lesssim o(1) + \|G\|_Z. \tag{2.39}$$

To bound G, we estimate with a smooth cutoff ϕ_3 to $D(0,3R_n)$ so that $\phi_2\phi_3 = \phi_2$,

$$\begin{aligned}\|\phi_2\nabla(u|\nabla u|^2)\|_2 &\lesssim \|\phi_2\nabla u\|_\infty\|\phi_3\nabla u\|_4^2 + \|\phi_2\nabla u\|_\infty\|\phi_3 D^2 u\|_2 \\ \|\nabla\partial_j(u-\omega)\partial_j\phi_2\|_2 &\lesssim R_n^{-1}\|\phi_3 D^2(u-\omega)\|_2 \\ \|\Delta\phi_2\nabla(u-\omega)\|_2 &\lesssim R_n^{-2}\|\phi_3\nabla(u-\omega)\|_2\end{aligned} \tag{2.40}$$

as well as $\|\phi_2\nabla(\omega|\nabla\omega|^2)\|_2 \lesssim 1$. Furthermore,

$$\begin{aligned}\|\phi_2\nabla u\|_\infty &\lesssim \|w\|_\infty + 1 \\ \|\phi_3\nabla u\|_4 &\lesssim \|\phi_3\nabla(u-\omega)\|_4 + 1 \\ \|\phi_3 D^2 u\|_2 &\lesssim \|\phi_3 D^2(u-\omega)\|_2 + 1\end{aligned} \tag{2.41}$$

uniformly in R_n. By (2.39) therefore,

$$\|w\|_Y \lesssim o(1) + \|w\|_Y^2 + \int_I \|\phi_3\nabla(u(s)-\omega)\|_4^4\,\mathrm{d}s + \int_I \|\phi_3 D^2(u(s)-\omega)\|_2^2\,\mathrm{d}s. \tag{2.42}$$

To perform energy estimates on $u-\omega$, we apply the methods of Struwe [28] to the PDE

$$\partial_t(u-\omega) - \Delta(u-\omega) = (u-\omega)|\nabla\omega|^2 + u(|\nabla u|^2 - |\nabla\omega|^2). \tag{2.43}$$

Integrating by parts against $\phi_3^2\partial_t(u-\omega)$ implies that (with $T=\tau_n$)

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^2}|\partial_t(u-\omega)|^2\phi_3^2\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\nabla(u(t)-\omega)\cdot\nabla[\partial_t(u-\omega)\phi_3^2]\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T\int_{\mathbb{R}^2}(u-\omega)|\nabla\omega|^2\cdot\partial_t(u-\omega)\phi_3^2\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}u\cdot\partial_t(u-\omega)\phi_3^2(|\nabla u(t)|^2 - |\nabla\omega|^2)\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.44}$$

which implies

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} |\partial_t (u-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} |\nabla(u(T)-\omega)|^2 \phi_3^2 \,\mathrm{d}x \\
&\quad \lesssim o(1) + \int_0^T \int_{\mathbb{R}^2} \big||\nabla u(t)|^2 - |\nabla\omega|^2\big|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^2 |\nabla\phi_3|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad \lesssim o(1) + \int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^4 \phi_3^2 \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.45}
$$

The final term on the second line of (2.45) is dominated by $TR_n^{-2}(E(u(0)) + E(\omega))$ and so can be absorbed in the $O(\tau_n R_n^{-2})$. Multiplying (2.43) by $-\phi_3^2 \Delta(u-\omega)$ and integrating by parts yields

$$
\begin{aligned}
&\sum_{j=1}^{2} \int_0^T \int_{\mathbb{R}^2} \partial_t |\partial_j (u-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2} |\Delta(u(t)-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad = \sum_{j=1}^{2} \int_0^T \int_{\mathbb{R}^2} \partial_j [(u-\omega)|\nabla\omega|^2 \phi_3^2] \cdot \partial_j (u-\omega)\, dxdt - 2\sum_{j=1}^{2} \int_0^T \int_{\mathbb{R}^2} \partial_t (u-\omega) \phi_3 \partial_j \phi_3 \cdot \partial_j (u-\omega) \,\mathrm{d}x\mathrm{d}t \\
&\qquad - \int_0^T \int_{\mathbb{R}^2} \Delta(u-\omega) \cdot [\phi_3^2 (|\nabla u(t)|^2 - |\nabla\omega|^2) u] \,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{2.46}
$$

which implies

$$
\begin{aligned}
&-\frac{1}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_t (u-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} |\nabla(u(T)-\omega)|^2 \phi_3^2 \, dx + \frac{1}{2} \int_0^T \int_{\mathbb{R}^2} |\Delta(u(t)-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad \lesssim o(1) + \int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^4 \phi_3^2 \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.47}
$$

Adding this to (2.45), we obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} |\partial_t (u-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} |\nabla(u(T)-\omega)|^2 \phi_3^2 \, dx + \int_0^T \int_{\mathbb{R}^2} |D^2(u(t)-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad \lesssim o(1) + \int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^4 \phi_3^2 \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{2.48}
$$

For the third term on the left-hand side, we used another integration by parts to bound

$$
\int_0^T \int_{\mathbb{R}^2} |D^2(u(t)-\omega)|^2 \phi_3^2 \, dxdt \lesssim \int_0^T \int_{\mathbb{R}^2} |\Delta(u(t)-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \tau_n R_n^{-2}.
$$

By [28, Lemma 3.2],

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^4 \phi_3^2 \, dxdt &\lesssim \sup_{0\le t\le T} \int_{D(0,3R_n)} |\nabla(u(t,x)-\omega(x))|^2 \,\mathrm{d}x \\
&\qquad \Big(\int_0^T \int_{\mathbb{R}^2} |D^2(u(t)-\omega)|^2 \phi_3^2 \, dxdt + \tau_n R_n^{-2} \Big).
\end{aligned} \tag{2.49}
$$

By Lemma 2.10, the local energy on $D(0,3)$ is small. Hence, we conclude from (2.48) and the bound (2.49) that

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$$\int_0^T \int_{\mathbb{R}^2} |\partial_t(u-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2} |\nabla(u(T)-\omega)|^2 \phi_3^2 \,dx + \int_0^T \int_{\mathbb{R}^2} |D^2(u(t)-\omega)|^2 \phi_3^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}^2} |\nabla(u(t)-\omega)|^4 \,\phi_3^2 \,\mathrm{d}x\mathrm{d}t \lesssim o(1). \tag{2.50}$$

Inserting this bound into (2.42) yields $\|w\|_Y = o(1)$, whence from (2.37), finally $\|v\|_X = o(1)$. This finishes the proof for disks.

For the punctured disks, we would like to proceed in the same fashion. As a first step, it appears that we would need to obtain bounds on the suitably normalized ground state eigenfunction ϕ_1 of the Dirichlet Laplacian on the punctured disk

$$D^*(0,4R_n) = D(0,4R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, \epsilon_n/4),$$

where x_ℓ are as stated in the lemma. This turns out to be misguided as we will now see, in addition to being delicate in terms of obtaining the needed bounds on ϕ_1 uniformly in the choice of holes. In fact, it suffices to select ϕ_1 smooth on $\Omega = \overline{D^*(0,4R_n)}$, vanishing on $\partial\Omega$ so that $-\Delta\phi_1 = V\phi_1$ with $\|V\|_{L^\infty(\Omega)} \lesssim \epsilon_n^{-2}$ uniformly in all parameters. By rescaling, it will also suffice to set $R_n = 1$.

We define, with $r = |x|$ and $r_\ell = |x - x_\ell|$,

$$\phi_1(x) := \chi_0(r) \prod_{\ell=1}^{L} \chi_\ell(r_\ell) \tag{2.51}$$

with smooth functions $\chi_j > 0$ on the interior of Ω, $0 \le j \le L$ that we now specify. First, $\chi_0(r) = 1$ for $r \le 2$, and on the annulus $3 \le r \le 4$, $\chi_0(r)$ agrees with the L^∞-normalized Dirichlet ground state of the disk $D(0,4)$. To define χ_ℓ, consider *all radial* Dirchlet eigenfunctions, $\{\psi_n\}_{n=1}^\infty$ on the annulus $D(0,1) \setminus D(0,\gamma)$. Then $\psi_n(r) = a_n J_0(\mu_n r) + b_n Y_0(\mu_n r)$, where μ_n^2 is the eigenvalue for $n \ge 1$ and $a_n, b_n \in \mathbb{R}$. Since $a_n^2 + b_n^2 > 0$, and $\psi_n(\gamma) = \psi_n(1) = 0$, the spectrum is characterized by the conditions

$$J_0(\mu_n\gamma)Y_0(\mu_n) - Y_0(\mu_n\gamma)J_0(\mu_n) = 0.$$

Note that the ratio $R(x) := Y_0(x)/J_0(x)$ is strictly increasing on the interval $(0,\rho_1)$ where $\rho_1 > 0$ is the smallest positive zero of J_0, as well as on any subsequent interval (ρ_j, ρ_{j+1}), $j \ge 1$. This follows from the fact that the Wronskian $Y_0'(x)J_0(x) - Y_0(x)J_0'(x) = 2/(\pi x) > 0$. The first crossing of the graphs, which determines the smallest energy $\mu_1 > 0$, is determined by $R(x) = R(\gamma x)$. The expansion $R(x) = \frac{2}{\pi}\log x + O(1)$ for $x \to 0+$ shows that $R(x) > R(x\gamma)$ for all $0 < x < \rho_1$, and the first crossing occurs at $x \in (\rho_1, \rho_2)$ and so the ground state energy $\mu_1 \in (\rho_1, \rho_2)$. Similarly, we find the other energies $\mu_j \in (\rho_j, \rho_{j+1})$, for $j \ge 1$. We select an eigenfunction ψ_k with $\mu_k \simeq \gamma^{-1}$. This is possible due to the zeros of $J_0(x)$ forming, to leading order, an arithmetic progression.

We can now define χ_ℓ in (2.51) by centering this ψ_k for $\gamma = \epsilon_\ell$ at x_ℓ and gluing it smoothly with the constant 1 at a distance $\simeq \epsilon_\ell$ away from the x_ℓ hole. The resulting function $\chi_\ell > 0$ will then satisfy $\chi_\ell(x) \ge \frac{1}{2}$ provided $|x - x_\ell| \gtrsim \epsilon_\ell$.

The associated cutoff function ϕ_1 satisfies

$$\phi_1(x) \ge c_0\|\phi_1\|_\infty \quad \text{for all} \quad x \in D(0,3R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, \epsilon_n/2) \tag{2.52}$$

with some absolute constant $c_0 > 0$, independently of R_n, ϵ_n and the choice of the centers x_ℓ as above. It is clear from the preceding that the ground state would have energy $\simeq 1$ and does not satisfy (2.52). Furthermore, $\|\nabla\phi_1\|_\infty \lesssim \epsilon_n^{-1}\|\phi_1\|_\infty$ and most importantly,

$$\|\phi_1^{-1}\Delta\phi_1\|_\infty \le \sum_{\ell=0}^{L} \|\chi_\ell^{-1}\Delta\chi_\ell\|_\infty \lesssim \epsilon_n^{-2},$$

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as desired. Because of our assumption $\tau_n\epsilon_n^{-2} \to 0$ as $n \to \infty$, the proof above applies. In the final step, we use (2.21) to control the L^4-norm as before, with one modification: we apply [28, Lemma 3.2] locally on ϵ_n-disks and then cover the punctured disk $D^*(0, 4R_n)$ with ϵ_n-disks followed by a summation over the disks in the cover. Cf. [28, Lemma 3.1, 3.3]. □

2.2.3. Concentration properties of the heat flow

Here we record the fact that the harmonic map heat flow cannot concentrate energy at the self-similar scale. The case of finite time blow-up was treated by Topping in [31], and the global in time case follows quickly from a local energy inequality as in Lemma 2.9.

Lemma 2.13 (No self-similar concentration in the blow-up case). [31, Proof of Theorem 1.6, page 288] *Let $u(t)$ be the solution to (1.2) with maximal time of existence $T_+ < \infty$ and initial data $u_0 \in \mathcal{E}$. Let $x_0 \in \mathbb{R}^2$ denote a bubbling point in the sense of Lemma 2.7 and suppose that $r > 0$ is sufficiently small so that $D(x_0, r)$ does not contain any other bubbling point. Then*

$$\lim_{t\to T_+} E(u(t); D(x_0, r) \setminus D(x_0, \alpha\sqrt{T_+ - t})) = E(u^*; D(x_0, r)) \tag{2.53}$$

for any $\alpha > 0$, where u^* is as in Lemma 2.7. In particular, there exist $T_0 < T_+$ and functions $\nu, \xi : [T_0, T_+) \to (0, \infty)$ such that $\lim_{t\to T_+}(\nu(t) + \xi(t)) = 0$ and

$$\lim_{t\to T_+} \left(\frac{\xi(t)}{\sqrt{T_+ - t}} + \frac{\sqrt{T_+ - t}}{\nu(t)} \right) = 0, \tag{2.54}$$

and so that

$$\lim_{t\to T_+} E(u(t); D(x_0, \nu(t)) \setminus D(x_0, \xi(t))) = 0. \tag{2.55}$$

Lemma 2.14 (No self-similar concentration in the global case). *Let $u(t)$ be the solution to (1.2) with initial data $u_0 \in \mathcal{E}$. Suppose that $T_+ = \infty$. Let $y \in \mathbb{R}^2$. Then*

$$\lim_{t\to\infty} E(u(t); \mathbb{R}^2 \setminus D(y, \alpha\sqrt{t})) = 0 \tag{2.56}$$

for any $\alpha > 0$. In particular, there exist $T_0 < \infty$ and a function $\xi : [T_0, \infty) \to (0, \infty)$ such that

$$\lim_{t\to T_+} \frac{\xi(t)}{\sqrt{t}} = 0, \tag{2.57}$$

and so that

$$\lim_{t\to T_+} E(u(t); \mathbb{R}^2 \setminus D(y, \xi(t))) = 0. \tag{2.58}$$

Proof. Fix $y \in \mathbb{R}^2$ and $\alpha > 0$. Let $\epsilon > 0$ and, using (2.12), choose $T_0 > 0$ so that

$$\frac{4\sqrt{E(u(0))}}{\alpha}\Big(\int_{T_0}^{\infty} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t\Big)^{\frac{1}{2}} \le \frac{\epsilon}{2}. \tag{2.59}$$

Next, let $T_1 \ge T_0$ be sufficiently large so that

$$E\big(u(T_0); \mathbb{R}^2 \setminus D\big(y, \frac{\alpha\sqrt{T}}{4}\big)\big) \le \frac{\epsilon}{2} \tag{2.60}$$

for all $T \ge T_1$. Fixing any such T, we set

$$\phi_T(|x|) = 1 - \chi(4|x|/\alpha\sqrt{T}), \tag{2.61}$$

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where $\chi(r)$ is a smooth function on $(0,\infty)$ such that $\chi(r) = 1$ for $r \le 1$, $\chi(r) = 0$ if $r \ge 4$, and $|\chi'(r)| \le 1$ for all $r \in (0,\infty)$. We now use the identity (2.16) on the time interval $[T_0, T]$ and with the function $\phi = \phi_T$ to obtain the inequality

$$\begin{aligned}\frac{1}{2}\|\nabla u(T)\phi_T\|_{L^2}^2 &\le \frac{1}{2}\|\nabla u(T_0)\phi_T\|_{L^2}^2 + \int_{T_0}^{T} |\nabla u(t)|\,|\partial_t u(t)|\,|\nabla\phi_T|\,\phi_T\,\mathrm{d}t\\ &\le E(u(T_0);\mathbb{R}^2\setminus D(y,\frac{\alpha\sqrt{T}}{4})) + \frac{4\sqrt{E(u(0))}}{\alpha}\Big(\int_{T_0}^{\infty}\|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t\Big)^{\frac{1}{2}} \le \epsilon, \qquad (2.62)\end{aligned}$$

which holds for all $T \ge T_1$, completing the proof. □

2.3. *Localized sequential bubbling*

The following localized sequential bubbling lemma proved in a series of works by Struwe [28], Qing [23], Ding-Tian [9], Wang [37], Qing-Tian [24] and Lin-Wang [20]. We state as a lemma below a summary of these works, which can be found, for example, in Topping's paper [30, Theorem 1.1].

Theorem 2.15 (Compactness Lemma). [23, Theorem 1.2], [30, Theorem 1.1] *Let $u_n : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ be a sequence of C^2 maps such that $\limsup_{n\to\infty} E(u_n) < \infty$. Let $\rho_n \in (0,\infty)$ be a sequence and suppose that*

$$\lim_{n\to\infty} \rho_n \|\mathcal{T}(u_n)\|_{L^2} = 0. \qquad (2.63)$$

Then, for every sequence $y_n \in \mathbb{R}^2$, there exists a sequence $R_n \to \infty$ a fixed integer $M \ge 0$, a constant $C > 0$, a harmonic map ω_0 (possibly constant), non-constant harmonic maps $\omega_1, \dots, \omega_M$, and sequences of vectors $b_{1,n}, \dots, b_{M,n} \in D(y_n, C\rho_n)$ and scales $\mu_{1,n}, \dots, \mu_{M,n} \in (0,\infty)$ so that $\max_j \mu_{j,n}/\rho_n \to 0$ as $n \to \infty$ and

$$\begin{aligned}\lim_{n\to\infty}\Bigg[&E\Big(u_n - \omega_0\big(\frac{\cdot - y_n}{\rho_n}\big) - \sum_{j=1}^{M}\big(\omega_j\big(\frac{\cdot - b_{j,n}}{\mu_{j,n}}\big) - \omega_j(\infty)\big); D(y_n, R_n\rho_n)\Big)\\ &+ \Big\|u_n - \omega_0\big(\frac{\cdot - y_n}{\rho_n}\big) - \sum_{j=1}^{M}\big(\omega_j\big(\frac{\cdot - b_{j,n}}{\mu_{j,n}}\big) - \omega_j(\infty)\big)\Big\|_{L^\infty(D(y_n,R_n\rho_n))} \qquad (2.64)\\ &+ \sum_{j\ne k}\Big(\frac{\mu_{j,n}}{\mu_{k,n}} + \frac{\mu_{k,n}}{\mu_{j,n}} + \frac{|b_{k,n} - b_{j,n}|^2}{\mu_{j,n}\mu_{k,n}}\Big)^{-1} + \sum_{j=1}^{M}\frac{\mu_{j,n}}{\mathrm{dist}(b_{j,n}, \partial D(y_n, C\rho_n))}\Bigg] = 0.\end{aligned}$$

In particular, $\lim_{n\to\infty}\delta(u_n; D(y_n, \widetilde{R}_n\rho_n)) = 0$ for any sequence $1 \ll \widetilde{R}_n \ll R_n$. There exist $L \le M$ points $x_1, \dots, x_L \in D(0, C)$ so that

$$\begin{aligned}&u_n(y_n + \rho_n\cdot) \rightharpoonup \omega_0 \text{ weakly in } H^1(D(0,C);\mathbb{S}^2)\\ &u_n(y_n + \rho_n\cdot) \to \omega_0 \text{ strongly in } W^{2,2}_{\mathrm{loc}}(D(0,C)\setminus\{x_1,\dots,x_L\};\mathbb{S}^2). \qquad (2.65)\end{aligned}$$

For each $j \in \{1, \dots, M\}$, there exist a finite set of points S_j, possibly empty and with $\#S_j \le M - 1$, such that

$$u_n(b_{j,n} + \mu_{j,n}\cdot) \to \omega_j \text{ strongly in } W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2\setminus S_j;\mathbb{S}^2). \qquad (2.66)$$

Finally, there exists an integer $K \ge 0$ so that

$$\lim_{n\to\infty} E(u_n; D(y_n, R_n\rho_n)) = 4\pi K. \qquad (2.67)$$

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Remark 2.16. Theorem 2.15 can be combined with Lemmas 2.13, 2.14 and the bound (2.12) to prove a sequential decomposition as in Theorem 1.1 along the *well-chosen* sequence of times described in Section 1.1; see, for example, [31, Section 2]. We note that the second statement (2.64) gives L^∞ convergence on the whole disc $D(y_n, R_n\rho_n)$ rather than just at the scales of the bubbles, which is all that is required for $\boldsymbol{\delta}(u_n; D(y_n, \widetilde{R}_n\rho_n))$ to tend to zero for $1 \ll \widetilde{R}_n \ll R_n$; see Definition 1.6.

Remark 2.17. Parker [22] proved an earlier version of Theorem 2.15 in the case when the sequence u_n consists of harmonic maps (i.e., $\mathcal{T}(u_n) = 0$ for each n). We use this restricted version of Theorem 2.15 (for sequences consisting only of harmonic maps) at several instances in the next section.

3. Proofs of the main results

3.1. *The minimal collision energy*

For the remainder of the paper, we fix a solution $u(t)$ of (1.2), defined on the time interval $I_+ = [0, T_+)$, where $T_+ < \infty$ in the finite time blow-up case and $T_+ = \infty$ in the global case. We fix $\gamma_0 > 0$ such that $\gamma_0 \le \min\{\frac{1}{100}, \frac{1}{100E(u_0)}\}$ and sufficiently small so that Lemma 2.4 holds. From now on, we omit the subscript γ_0 from $\mathbf{d}_{\gamma_0}$, and $\boldsymbol{\delta}_{\gamma_0}$ and for a harmonic map, ω we write $\lambda(\omega) = \lambda(\omega; \gamma_0)$ and $a(\omega) = a(\omega; \gamma_0)$ for the scale and center.

Our strategy is to study collisions of bubbles, which we define as follows.

Definition 3.1 (The minimal collision energy). Let K be the smallest natural number with the following properties. There exist sequences $y_n \in \mathbb{R}^2$, $\rho_n, \epsilon_n \in (0, \infty)$, $\sigma_n, \tau_n \in (0, T_+)$ and $\eta > 0$, with $\epsilon_n \to 0$, $0 < \sigma_n < \tau_n < T_+$, $\sigma_n, \tau_n \to T_+$, so that

1. $\boldsymbol{\delta}(u(\sigma_n); D(y_n, \rho_n)) \le \epsilon_n$;
2. $\boldsymbol{\delta}(u(\tau_n); D(y_n, \rho_n)) \ge \eta$;
3. the interval $I_n := [\sigma_n, \tau_n]$ satisfies $|I_n| \le \epsilon_n \rho_n^2$;
4. $E(u(\sigma_n); D(y_n, \rho_n)) \to 4K\pi$ as $n \to \infty$;

We call $[\sigma_n, \tau_n]$ a sequence of collision intervals associated to K and the parameters y_n, ρ_n, ϵ_n and η, and we write $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \epsilon_n, \eta)$.

Remark 3.2. By Definition 1.6 and Property (1) in Definition 3.1, we can associate to each sequence of collision intervals $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \epsilon_n, \eta)$ sequences $\xi_n, \nu_n \in (0, \infty)$ with $\lim_{n\to\infty}(\frac{\xi_n}{\rho_n} + \frac{\rho_n}{\nu_n}) = 0$, and a sequence of constants $\omega_n \in \mathbb{S}^2$ so that

$$\lim_{n\to\infty}\Big(E(u(\sigma_n); D(y_n, 4\nu_n) \setminus D(y_n, 4^{-1}\xi_n)) + \|u(\sigma_n) - \omega_n\|_{L^\infty(D(y_n,4\nu_n)\setminus D(y_n,4^{-1}\xi_n))}\Big) = 0. \quad (3.1)$$

Using Property (3) in Definition 3.1, we can always ensure (by enlarging the excised discs above) that

$$|I_n| = \tau_n - \sigma_n \ll \xi_n^2. \quad (3.2)$$

Then, by Lemma 2.9 and Lemma 2.12, the limits above can be propagated throughout the whole collision interval I_n yielding

$$\lim_{n\to\infty} \sup_{t\in[\sigma_n,\tau_n]} E(u(t); D(y_n, \nu_n) \setminus D(y_n, \xi_n)) + \|u(t) - \omega_n\|_{L^\infty(D(y_n,\nu_n)\setminus D(y_n,\xi_n))} = 0. \quad (3.3)$$

Moreover, the above holds after enlarging ξ_n or shrinking ν_n (i.e, for any $\widetilde{\xi}_n, \widetilde{\nu}_n$ with $\xi_n \ll \widetilde{\xi}_n \ll \rho_n \ll \widetilde{\nu}_n \ll \nu_n$).

Lemma 3.3 (Existence of $K \ge 1$). *If Theorem 1.8 is false, then K is well-defined and $K \ge 1$.*

Proof of Lemma 3.3. Assume Theorem 1.8 is false (in either the case $T_+ < \infty$ or $T_+ = \infty$). Then we can find $\eta > 0$, sequences $\tau_n \to T_+$, $y_n \in \mathbb{R}^2$, $0 < \rho_n < \infty$ with $\rho_n \le \sqrt{T_+ - t_n}$ in the case $T_+ < \infty$ and $\rho_n \le \sqrt{t_n}$ in the case $T_+ = \infty$ so that

$$\boldsymbol{\delta}(u(\tau_n); D(y_n, \rho_n)) \ge \eta, \quad \forall n, \tag{3.4}$$

and sequences $\alpha_n \to 0$ and $\beta_n \to \infty$ so that

$$\lim_{n\to\infty} E(u(\tau_n); D(y_n, \beta_n\rho_n) \setminus D(y_n, \alpha_n\rho_n)) = 0. \tag{3.5}$$

In case $\rho_n \simeq \sqrt{T_+ - \tau_n}$ or $\rho_n \simeq \sqrt{t_n}$, the existence of α_n, β_n as above is guaranteed by Lemma 2.13 or Lemma 2.14.

We claim that there exists a sequence of times $\sigma_n < \tau_n$, $\sigma_n \to T_+$, such that

$$|[\sigma_n, \tau_n]| \ll \rho_n^2, \quad \text{and} \quad \lim_{n\to\infty} \rho_n^2 \|\mathcal{T}(u(\sigma_n))\|_{L^2}^2 = 0. \tag{3.6}$$

If not, we could find numbers $c, c_0 > 0$ and a subsequence of the τ_n so that

$$\rho_n^2 \|\mathcal{T}(u(t))\|_{L^2}^2 \ge c_0, \quad \forall t \in [\tau_n - c\rho_n^2, \tau_n]. \tag{3.7}$$

But then

$$\int_0^{T_+} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t \ge \sum_n \int_{\tau_n - c\rho_n^2}^{\tau_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t \ge c_0 \sum_n \int_{\tau_n - c\rho_n^2}^{\tau_n} \rho_n^{-2} \, \mathrm{d}t = \infty, \tag{3.8}$$

and the above contradicts (2.12).

Using (2.14) from Lemma 2.9 and the fact that $|E(u(\sigma_n)) - E(u(\tau_n))| \to 0$ since $\sigma_n, \tau_n \to T_+$ (see Lemma 2.7), we see that (3.5) can be used to ensure that

$$\lim_{n\to\infty} E(u(\sigma_n); D(y_n, 2^{-1}\beta_n\rho_n) \setminus D(y_n, 2\alpha_n\rho_n)) = 0. \tag{3.9}$$

Given the sequence σ_n in (3.6), we can apply the Compactness Lemma 2.15 to $u(\sigma_n)$ and conclude that after passing to a subsequence (which we still denote by σ_n), we see that a bubble decomposition as in (2.64) holds for some sequence $R_n \to \infty$. Because of (3.9), we see that the harmonic map ω_0 in (2.64) must be constant (i.e., $\omega_0(x) = \omega \in \mathbb{S}^2$), and we can conclude that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(\sigma_n); D(y_n, \rho_n)) = 0. \tag{3.10}$$

By Lemma 2.6, we can find an integer $K \ge 0$ so that

$$E(u(\sigma_n); D(y_n, \rho_n)) \to 4\pi K. \tag{3.11}$$

We have shown that Properties (1)–(4) hold for the intervals $[\sigma_n, \tau_n]$. This proves that K is well-defined and ≥ 0.

We claim that $K \ge 1$. Suppose $K = 0$ and $y_n, \rho_n, \epsilon_n, \sigma_n, \tau_n$ are as in Definition (3.1). But then, we can find ξ_n, ν_n and $\omega \in \mathbb{S}^2$ as in (3.3) in Remark 3.2. By Lemma 2.9, we have

$$E(u(\tau_n); D(y_n, \rho_n)) = o_n(1), \tag{3.12}$$

and by (3.3) in Remark 3.2, we have

$$\|u(\tau_n) - \omega\|_{L^\infty(D(y_n,\nu_n)\setminus D(y_n,\xi_n))} + \frac{\xi_n}{\rho_n} + \frac{\rho_n}{\nu_n} = o_n(1), \tag{3.13}$$

which makes it impossible for (2) in Definition 3.1 to be satisfied. This proves that $K \ge 1$. □

3.2. Lengths of collision intervals

We assume that Theorem 1.8 is false. Let $K \geq 1$ be as in Lemma 3.3 and let $y_n \in \mathbb{R}^2$, $\rho_n \in (0,\infty)$, $\epsilon_n \to 0$, $0 < \sigma_n < \tau_n < T_+$ with $\sigma_n, \tau_n \to T_+$, and $\eta > 0$ be a choice of parameters given by Definition 3.1 (i.e., $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \epsilon_n, \eta)$).

Lemma 3.4 (Length of a collision interval). *There exists $\eta_0 > 0$ sufficiently small so that for each $\eta \in (0, \eta_0]$, there exists $\epsilon > 0$ and $c_0 > 0$ with the following properties. Let $[\sigma, \tau] \subset [\sigma_n, \tau_n]$ be any subinterval such that*

$$\boldsymbol{\delta}(u(\sigma); D(y_n, \rho_n)) \leq \epsilon, \quad \text{and} \quad \boldsymbol{\delta}(u(\tau); D(y_n, \rho_n)) \geq \eta, \tag{3.14}$$

and let $\omega \in \mathbb{S}^2$ and $\omega_1, \ldots, \omega_M$ be any collection of non-constant harmonic maps, and $\vec{\nu} = (\nu, \nu_1, \ldots, \nu_M), \vec{\xi} = (\xi, \xi_1, \ldots, \xi_M) \in (0,\infty)^{M+1}$ any admissible vectors in the sense of Definition 1.6 such that

$$\epsilon \leq \mathbf{d}(u(\sigma), \mathcal{Q}(\boldsymbol{\omega}); D(y_n, \rho_n); \vec{\nu}, \vec{\xi}) \leq 2\epsilon. \tag{3.15}$$

Then

$$\tau - \sigma \geq c_0 \max_{j \in \{1,\ldots,M\}} \lambda(\omega_j)^2. \tag{3.16}$$

Corollary 3.5. *Let $\eta_0 > 0$ be as in Lemma 3.4, $\eta \in (0, \eta_0]$, and $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \epsilon_n, \eta)$. Then there exist $\epsilon \in (0, \eta)$, $c_0 > 0$ $n_0 \in \mathbb{N}$ and $s_n \in (\sigma_n, \tau_n)$ such that for all $n \geq n_0$, the following conclusions hold. First,*

$$\boldsymbol{\delta}(u(s_n); D(y_n, \rho_n)) = \epsilon. \tag{3.17}$$

Moreover, for each $n \geq n_0$, let $M_n \in \mathbb{N}$, and $\mathcal{Q}(\boldsymbol{\omega}_n)$ be any sequence of M_n-bubble configurations, and let $\vec{\nu}_n = (\nu_n, \nu_{1,n}, \ldots, \nu_{M,n}), \vec{\xi}_n = (\xi_n, \xi_{1,n}, \ldots, \xi_{M,n}) \in (0,\infty)^{M+1}$ be any admissible sequences in the sense of Definition 1.6 such that

$$\epsilon \leq \mathbf{d}(u(s_n), \mathcal{Q}(\boldsymbol{\omega}_n); D(y_n, \rho_n), \vec{\nu}_n, \vec{\xi}_n) \leq 2\epsilon \tag{3.18}$$

for each n, and define

$$\lambda_{\max,n} = \lambda_{\max}(s_n) := \max_{j=1,\ldots,M_n} \lambda(\omega_{j,n}). \tag{3.19}$$

Then $s_n + c_0 \lambda_{\max}(s_n)^2 \leq \tau_n$, and

$$\boldsymbol{\delta}(u(t); D(y_n, \rho_n)) \geq \epsilon, \quad \forall\, t \in [s_n, s_n + c_0 \lambda_{\max}(s_n)^2]. \tag{3.20}$$

We make the following definitions.

Definition 3.6. We say that two triples $(\omega_j, a_{j,n}, \lambda_{j,n})$ and $(\omega_{j'}, a_{j',n}, \lambda_{j',n})$ where $\omega_j, \omega_{j'}$ are nontrivial harmonic maps, $a_{j,n}, a_{j',n} \in \mathbb{R}^2$ are sequences of vectors in $\mathbb{R}^2$, and $\lambda_{j,n}, \lambda_{j',n} \in (0,\infty)$ are sequences of scales, are *asymptotically orthogonal* if

$$\lim_{n\to\infty} \Big(\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|a_{j,n} - a_{j',n}|^2}{\lambda_{j,n}\lambda_{j',n}} \Big) = \infty. \tag{3.21}$$

Definition 3.7. We say that a sequence of nontrivial harmonic maps $\mathfrak{h} = \{\omega_n\}_{n=1}^\infty$ is a *descendant* sequence of an *ancestor* sequence of harmonic maps $\mathfrak{H} = \{\Omega_n\}_{n=1}^\infty$ if $\frac{\lambda(\Omega_n)}{\lambda(\omega_n)} \to \infty$, and there exists a constant $C > 0$ so that the discs $D(a(\omega_n), \lambda(\omega_n)) \subset D(a(\Omega_n), C\lambda(\Omega_n))$ for all sufficiently large n. We denote this relation by $\{\omega_n\} \prec \{\Omega_n\}$, and $\{\omega_n\} \preceq \{\Omega_n\}$ allows for equality. Given a natural number M, a collection of sequences of harmonic maps $(\mathfrak{h}_1, \ldots, \mathfrak{h}_M) = (\{\omega_{1,n}\}_{n=1}^\infty, \ldots, \{\omega_{1,n}\}_{n=1}^\infty)$

with asymptotically orthogonal centers and scales are partially ordered by $\preceq$. The *roots* are defined to be the maximal elements relative to this partial order. In other words, a sequence of harmonic maps $\mathfrak{h}_j$ is a root if it is not a descendant sequence of any ancestor sequence $\mathfrak{h}_{j'}$ for any $j' \in \{1, \dots, M\}$. We denote by

$$\mathcal{R} := \{j \in \{1, \dots, M\} \mid \mathfrak{h}_j \text{ is a root}\}. \tag{3.22}$$

Finally, to each root $\mathfrak{h}_j$, we can associate a bubble tree $\mathcal{T}(j) := \{\mathfrak{h}_{j'} \mid \mathfrak{h}_{j'} \preceq \mathfrak{h}_j\}$.

Proof of Lemma 3.4. If the Lemma were false, we could find intervals $[s_n, t_n] \subset [\sigma_n, \tau_n]$ so that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(s_n); D(y_n, \rho_n)) = 0, \quad \lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(y_n, \rho_n)) > 0, \tag{3.23}$$

integers $M_n \geq 0$, sequences of M_n-bubble configurations $\mathcal{Q}(\omega_n)$, and sequences of vectors $\vec{\nu}_n = (\nu_n, \nu_{1,n}, \dots, \nu_{M_n,n}) \in (0,\infty)^{M_n+1}, \vec{\xi}_n = (\xi_n, \xi_{1,n}, \dots, \xi_{M_n,n}) \in (0,\infty)^{M_n+1}$ such that

$$\lim_{n\to\infty} \mathbf{d}(u(s_n), \mathcal{Q}(\omega_n); D(y_n, \rho_n); \vec{\nu}_n, \vec{\xi}_n) = 0, \tag{3.24}$$

and so that for $\lambda_{\max,n} := \max_{j=1,\dots,M_n} \lambda(\omega_{j,n})$, we have

$$\lim_{n\to\infty} \frac{(t_n - s_n)^{\frac{1}{2}}}{\lambda_{\max,n}} = 0. \tag{3.25}$$

Passing to a subsequence, we may assume that $M_n = M$ is a fixed integer and $\omega_n = \omega \in \mathbb{S}^2$ is a fixed constant.

Consider the sequences of harmonic maps, $\mathfrak{h}_j = \{\omega_{j,n}\}_{n=1}^\infty$, for $j = 1, \dots, M$, together with sequences of centers $a(\omega_{j,n})$ and scales $\lambda(\omega_{j,n})$, and the partial order $\prec$ on $(\mathfrak{h}_1, \dots, \mathfrak{h}_M)$ as in Definition 3.7. Using the language of Definition 3.7, we observe that, after passing to a subsequence in n, there exists a sequence $\widetilde{R}_n \to \infty$ so that for any root sequences $\mathfrak{h}_j = \{\omega_{j,n}\}_{n=1}^\infty$, $\mathfrak{h}_{j'} = \{\omega_{j,n}\}_{n=1}^\infty$ with $j, j' \in \mathcal{R}$, the discs $D(a(\omega_{j,n}), 4R_n\lambda(\omega_{j,n}))$ and $D(a(\omega_{j',n}), 4R_n\lambda(\omega_{j',n}))$ are disjoint for each n for any sequence $R_n \leq \widetilde{R}_n$. By Lemma 2.4,

$$\lim_{n\to\infty} E(\omega_{j,n}; \mathbb{R}^2 \setminus D(a(\omega_{j,n}); 4^{-1}R_n\lambda(\omega_{j,n}))) = 0 \tag{3.26}$$

for each $j \in \mathcal{R}$ and for any sequence $R_n \to \infty$, and hence, by (3.24),

$$\lim_{n\to\infty} E(u(s_n); D(y_n, \rho_n) \setminus \bigcup_{j\in\mathcal{R}} D(a(\omega_{j,n}), 4^{-1}R_n\lambda(\omega_{j,n}))) = 0 \tag{3.27}$$

for any sequence $R_n \to \infty$.

Each of the sequences $\{\omega_{j,n}\}_{n=1}^\infty$ for $j \in \{1, \dots, M\}$ satisfies the hypothesis of the Compactness Lemma 2.15 (noting that $\mathcal{T}(\omega_{j,n}) = 0$ since $\omega_{j,n}$ is harmonic), and passing to a (joint) subsequence, we can find non-negative integers M_j, a sequence $\check{R}_n \leq \widetilde{R}_n$ with $1 \ll \check{R}_n \ll \xi_n\lambda_{\max,n}^{-1}$, harmonic maps $\omega_{j,0}$ (possibly constant), nontrivial harmonic maps $\theta_{j,k}$, scales $\mu_{j,k,n} \ll \lambda(\omega_{j,n})$ and centers $b_{j,k,n} \in D(a(\omega_{j,n}), C\lambda(\omega_{j,n}))$ for each j and where $k \in \{1, \dots, M_j\}$, satisfying (2.65), (2.66), and so that

$$\begin{aligned}\lim_{n\to\infty} \Bigg[& E\Big(\omega_{j,n} - \omega_{j,0}\Big(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\Big) - \sum_{k=1}^{M_j} \Big(\theta_{j,k}\Big(\frac{\cdot - b_{j,k,n}}{\mu_{j,k,n}}\Big) - \theta_{j,k}(\infty)\Big); D_{j,n}\Big) \\ & + \Big\| \omega_{j,n} - \omega_{j,0}\Big(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\Big) - \sum_{k=1}^{M_j} \Big(\theta_{j,k}\Big(\frac{\cdot - b_{j,k,n}}{\mu_{j,k,n}}\Big) - \theta_{j,k}(\infty)\Big)\Big\|_{L^\infty(D_{j,n})} \\ & + \sum_{k\neq k'} \Big(\frac{\mu_{j,k,n}}{\mu_{j,k',n}} + \frac{\mu_{j,k',n}}{\mu_{j,k,n}} + \frac{|b_{j,k,n} - b_{j,k',n}|^2}{\mu_{j,k,n}\mu_{j,k',n}}\Big)^{-1} + \sum_{k=1}^{M_j} \frac{\mu_{j,k,n}}{\operatorname{dist}(b_{j,k,n}, \partial D(a(\omega_{j,n}), C\lambda(\omega_{j,n}))} \Bigg] = 0,\end{aligned} \tag{3.28}$$

where $D_{j,n} := D(a(\omega_{j,n}), 4R_n\lambda(\omega_{j,n}))$, $C > 0$ is some finite constant, and R_n is a sequence, to be fixed below, such that $1 \ll R_n \leq \breve{R}_n$. In this decomposition, we distinguish the (possibly constant) harmonic maps $\omega_{j,0}$, which arise as the weak limits $\omega_{j,n}(\lambda(\omega_{j,n})(\cdot + a(\omega_{j,n}))) \rightharpoonup \omega_{j,0}$, and we call these the body maps associated to the sequence $\mathfrak{h}_j = \{\omega_{j,n}\}_{n=1}^{\infty}$.

Define the set of indices

$$\mathcal{J}_{\max} := \left\{ j \in \{1, \dots, M\} \mid C_j^{-1} \leq \frac{\lambda_{\max,n}}{\lambda(\omega_{j,n})} \leq C_j, \text{ for each } n \text{ for some } C_j > 1 \right\} \tag{3.29}$$

and let

$$4\pi K_0 := \sum_{j \in \mathcal{J}_{\max}} E(\omega_{j,0}). \tag{3.30}$$

That is, $4\pi K_0$ is the sum of the energies of the body maps associated to the $\omega_{j,n}$ arising from indices $j \in \mathcal{J}_{\max}$. Note that $\mathcal{J}_{\max}$ is a (possibly strict) subset of the set of indices $\mathcal{R}$ associated to the roots.

Case 1: First suppose that $K_0 = K$, which means that $\mathcal{J}_{\max} = \mathcal{R} = \{1, \dots, M\}$ and all of the energy in $D(y_n, \rho_n)$ is captured by the body maps. In this case, the sequences $\{\omega_{j,n}\}$ have no concentrating bubbles – that is, $M_j = 0$ for each j, and we have

$$\lim_{n\to\infty} E\left(\omega_{j,n} - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\right); D(a(\omega_{j,n}), 4R_n\lambda(\omega_{j,n}))\right) = 0 \tag{3.31}$$

and

$$\lim_{n\to\infty} \left\| \omega_{j,n} - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda_{\max,n}}\right) \right\|_{L^\infty(D(a(\omega_{j,n}), 4R_n\lambda_{\max,n}))} = 0 \tag{3.32}$$

for each $j \in \{1, \dots, M\}$. Using (3.24), the fact that $\lambda(\omega_{j,n}) \simeq \lambda_{\max,n}$ for each $j \in \{1, \dots, M\}$, and the above, we can now fix (for Case 1) a sequence $R_n \leq \breve{R}_n$ so that

$$\lim_{n\to\infty} E\left(u(s_n) - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\right); D(a(\omega_{j,n}), 4R_n\lambda_{\max,n})\right) = 0 \tag{3.33}$$

and

$$\lim_{n\to\infty} \left\| u(s_n) - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda_{\max,n}}\right) \right\|_{L^\infty(D(a(\omega_{j,n}), 4R_n\lambda_{\max,n}))} = 0 \tag{3.34}$$

for each $j \in \{1, \dots, M\}$ (i.e., we need to additionally ensure that $4R_n\lambda_{\max,n} \leq \min\{\nu_{j,n}\}_{j=1}^{M}$). Using Lemma 2.10 and Lemma 2.12 along with the fact that $(t_n - s_n)^{\frac{1}{2}} \ll \lambda_{\max,n}$, we can propagate these estimates to time t_n – that is,

$$\lim_{n\to\infty} E\left(u(t_n) - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\right); D(a(\omega_{j,n}), R_n\lambda_{\max,n})\right) = 0 \tag{3.35}$$

and

$$\lim_{n\to\infty} \left\| u(t_n) - \omega_{j,0}\left(\frac{\cdot - a(\omega_{j,n})}{\lambda_{\max,n}}\right) \right\|_{L^\infty(D(a(\omega_{j,n}), R_n\lambda_{\max,n}))} = 0 \tag{3.36}$$

for each $j \in \{1, \dots, M\}$. Using Lemma 2.9 and that $(t_n - s_n)^{\frac{1}{2}} \ll \lambda_{\max,n}$, we can also propagate (3.27) to time t_n, deducing

$$\lim_{n\to\infty} E\left(u(t_n); D(y_n, \rho_n) \setminus \bigcup_{j=1}^{M} D(a(\omega_{j,n}); R_n\lambda_{\max,n})\right) = 0. \tag{3.37}$$

Combining (3.35), (3.36), (3.37), the disjointness of the discs $D(a(\omega_{j,n}), R_n\lambda(\omega_{j,n}))$, the asymptotic orthogonality of the triples $(\omega_{j,0}, a(\omega_{j,n}), \lambda(\omega_{j,n}))$, and Remark 3.2, we find that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(y_n, \rho_n)) = 0, \tag{3.38}$$

which contradicts (3.23).

Case 2: Next, consider the case $K_0 < K$. We show this case leads to a contradiction with the minimality of K. Again, we will need $R_n \to \infty$ such that $4R_n\lambda_{\max,n} \le \min\{\nu_{j,n}\}_{j\in\mathcal{J}_{\max}}$ and $R_n \le \breve{R}_n$.

We claim there exists an integer $L \ge 1$, sequences $\{x_{\ell,n}\}_{\ell=1}^{L}$ with $x_{\ell,n} \in D(y_n, \xi_n)$ for each n and each $\ell \in \{1, \dots, L\}$, and a sequence r_n such that

$$(t_n - s_n)^{\frac{1}{2}} \ll r_n \ll \lambda_{\max,n}, \tag{3.39}$$

such that the discs $D(x_{\ell,n}, r_n)$ are disjoint for $\ell \in \{1, \dots, L\}$ and satisfy

$$\lim_{n\to\infty} E\Big(u(s_n); \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 4\pi K - 4\pi K_0 \tag{3.40}$$

as well as

$$\lim_{n\to\infty} \frac{|x_{\ell,n} - x_{\ell',n}|}{r_n} = \infty \tag{3.41}$$

for $\ell \neq \ell'$, and finally such that there exist sequences $\alpha_n \to 0, \beta_n \to \infty$ so that

$$\lim_{n\to\infty} \sum_{\ell=1}^{L} E(u(s_n); D(x_{\ell,n}, \beta_n r_n) \setminus D(x_{\ell,n}, \alpha_n r_n)) = 0 \tag{3.42}$$

and a sequence $\breve{\xi}_n$ so that

$$\xi_n \ll \breve{\xi}_n \ll \rho_n \quad \text{and} \quad D(x_{\ell,n}, \beta_n r_n) \subset D(y_n, \breve{\xi}_n). \tag{3.43}$$

We construct a set of sequences $\mathcal{P} := \{\{x_{\ell,n}\} : 1 \le \ell \le L\}$ and the radii $\{r_n\}$ as follows. Any root $\mathfrak{h}_j$ with $j \in \mathcal{J}_{\max}$ we call a dominant root. For any dominant root $\mathfrak{h}_{j_0}$, we define $\mathcal{T}(j_0) = \{\mathfrak{h}_j \preceq \mathfrak{h}_{j_0}\}$ as the bubble tree with root $\mathfrak{h}_{j_0}$, and $\mathcal{D}(j_0)$ as the maximal elements of the pruned tree $\mathcal{T}(j_0) \setminus \{\mathfrak{h}_{j_0}\}$.

We define $\mathcal{P}_0$ as the points $y_{\ell,n}$ for $\ell \in \{1, \dots, L'\}$ as an enumeration of all (i) $a(\omega_{j,n})$ with $\mathfrak{h}_j \in \mathcal{R} \setminus \mathcal{J}_{\max}$ (i.e., the centers of the roots that are not dominant), (ii) $a(\omega_{j,n})$ with $\mathfrak{h}_j \in \mathcal{D}(j_0)$ for some $j_0 \in \mathcal{J}_{\max,n}$ and (iii) sequences $b_{j_0,k,n}$ associated to harmonic maps $\theta_{j_0,k}\big(\frac{\cdot - b_{j_0,k,n}}{\mu_{j_0,k,n}}\big)$ for some $j_0 \in \mathcal{J}_{\max,n}$ that are

- asymptotically orthogonal to every $\mathfrak{h}_j \in \mathcal{D}(j_0)$
- not descendants of any $\mathfrak{h}_j \in \mathcal{D}(j_0)$.

Passing to a joint subsequence, we can assume that the limits

$$\lim_{n\to\infty} \frac{(t_n - s_n)^{\frac{1}{2}}}{\operatorname{dist}(y_{\ell',n}, y_{\ell,n})} \tag{3.44}$$

exist in $[0, \infty]$ for all $\ell \neq \ell' \in \{1, \dots, L'\}$. We define $\mathcal{P}$ by means of $\mathcal{P}_0$ by the following algorithm: we include the sequence $y_{\ell_0,n} \in \mathcal{P}_0$ in the set $\mathcal{P}$ if

$$\lim_{n\to\infty} \frac{(t_n - s_n)^{\frac{1}{2}}}{\operatorname{dist}(y_{\ell_0,n}, y_{\ell,n})} = 0, \quad \forall \ell \in \{1, \dots, L')\} \setminus \ell_0. \tag{3.45}$$

For those $y_{\ell_0,n} \in \mathcal{P}_0$ for which the above does not hold, we define the sets

$$\mathcal{B}(\ell_0) := \Big\{\ell_0 \text{ and any } \ell \text{ for which } \lim_{n\to\infty} \frac{(t_n - s_n)^{\frac{1}{2}}}{\mathrm{dist}(y_{\ell_0,n}, y_{\ell,n})} \neq 0\Big\}. \tag{3.46}$$

An index ℓ can be in at most one set $\mathcal{B}(\ell_0)$ (i.e., the sets $\mathcal{B}(\ell) = \mathcal{B}(\ell')$ if $\ell' \in \mathcal{B}(\ell)$). For each of the sets $\mathcal{B}(\ell_0)$, we let, for each n, $x_{\ell_0,n}$ denote the barycenter of the points $y_{\ell,n}$ associated to indices $\ell \in \mathcal{B}(\ell_0)$. We include the points $x_{\ell_0,n}$ in the set $\mathcal{P}$. This completes the construction of the set $\mathcal{P}$, which consists of finitely many (say $L \in \mathbb{N}$) sequences $\{x_{\ell,n}\} \subset D(y_n, \xi_n)$ for $\ell \in \{1, \dots, L\}$.

We choose r_n to be any sequence such that

$$\begin{aligned} (t_n - s_n)^{\frac{1}{2}} &\ll r_n \ll \lambda_{\max,n}, \\ R_n\lambda(\omega_{j,n}) &\ll r_n \quad \forall j \notin \mathcal{J}_{\max}, \\ \max(\mu_{j,k,n}, \xi_{j,n}) &\ll r_n \quad \forall j \in \mathcal{J}_{\max}, \forall k \in \{1, \dots, M_j\}, \end{aligned} \tag{3.47}$$

and such that the discs $D(x_{\ell,n}, r_n)$ satisfy (3.41). In view of the definition of $j_0 \in \mathcal{J}_{\max}$,

$$\lambda_{\max,n}^{-1} |a(\omega_{j_0,n}) - a(\omega_{j,n})| \to \infty$$

for all $j \in \mathcal{R} \setminus \mathcal{J}_{\max}$. This ensures that for any $x_{\ell,n}$, which is one of the sequences $a(\omega_{j,n})$ for $j \in \mathcal{R} \setminus \mathcal{J}_{\max}$, the disc $D(x_{\ell,n}, r_n)$ is separated from any of the discs $D(a(\omega_{j_0,n}), R_n\lambda_{\max,n})$ for $j_0 \in \mathcal{J}_{\max}$ by an amount $\gg r_n$ (we are free to take $R_n \to \infty$ to be diverging as slowly as needed).

We claim that the sequences of discs $D(x_{\ell,n}, r_n)$ with $x_{\ell,n} \in \mathcal{P}$ satisfy (3.40). To see this, first note that for any $j_0 \in \mathcal{J}_{\max}$,

$$\lim_{n\to\infty} E\Big(u(s_n) - \omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big); D(a(\omega_{j_0,n}), 4R_n\lambda_{\max,n}) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 0, \tag{3.48}$$

which follows from the construction of the set $\{x_{\ell,n}\}_{\ell=1}^{L}$, the limit in (3.28) and the choice of r_n. Note also that $r_n \ll \lambda_{\max,n}$ means that

$$\lim_{n\to\infty} E\Big(\omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big); \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 0. \tag{3.49}$$

We can conclude from the above, (3.48), (3.30) and (3.27) that

$$\lim_{n\to\infty} E\Big(u(s_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 4\pi K_0. \tag{3.50}$$

The condition (3.40) follows then from above and the disjointness of the discs $D(x_{\ell,n}, r_n)$. The condition (3.42) and the existence of the sequence $\breve{\xi}_n$ as in (3.43) follows from the construction of the set $\mathcal{P}$ and the choice of r_n.

We claim that there must exist $\ell_1 \in \{1, \dots, L\}$, $\eta_1 > 0$ so that, up to passing to a subsequence in n, we have

$$\boldsymbol{\delta}(u(t_n); D(x_{\ell_1,n}, r_n)) \geq \eta_1. \tag{3.51}$$

To see this, we argue by contradiction. If (3.51) fails, then we would have

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(x_{\ell,n}, r_n)) = 0, \quad \forall \ell \in \{1, \dots, L\}. \tag{3.52}$$

We will use the above to show that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(y_n, \rho_n)) = 0, \tag{3.53}$$

which contradicts (3.23). To start, $(t_n - s_n)^{\frac{1}{2}} \ll r_n$ means we can use Lemma 2.9 and (3.42) to propagate (3.40), (3.50) and (3.48) to time t_n, giving

$$\lim_{n\to\infty} E\Big(u(t_n); \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 4\pi K - 4\pi K_0, \tag{3.54}$$

$$\lim_{n\to\infty} E\Big(u(t_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 4\pi K_0 \tag{3.55}$$

$$\lim_{n\to\infty} E\Big(u(t_n) - \omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big); D(a(\omega_{j_0,n}), R_n\lambda_{\max,n}) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n)\Big) = 0, \tag{3.56}$$

for all $j_0 \in \mathcal{J}_{\max}$, where in the last line we remark that for each $\ell \in \{1, \dots, L\}$, either the disc $D(x_{\ell,n}; r_n)$ is completely contained in $D(a(\omega_{j_0,n}), R_n\lambda_{\max,n})$ or disjoint from it.

Next, using $\lambda_{\max,n}R_n \leq \min\{\nu_{j,n}\}_{j\in\mathcal{J}_{\max}}$ and $\max(\mu_{j,k,n}, \xi_{j,n}) \ll r_n \quad \forall j \in \mathcal{J}_{\max}, \forall k \in \{1, \dots, M_j\}$, we see that (3.24) can be combined with the middle line of (3.28) to yield

$$\lim_{n\to\infty} \Big\| u(s_n) - \omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big)\Big\|_{L^\infty(D(a(\omega_{j_0,n}), 4R_n\lambda_{\max,n})\setminus\bigcup_{\ell=1}^{L} D(x_{\ell,n}, 4^{-1}r_n))} = 0 \tag{3.57}$$

for all $j_0 \in \mathcal{J}_{\max}$. Since $(t_n - s_n)^{\frac{1}{2}} \ll r_n$, Lemmas 2.12, (3.48) and (3.42) allow us to propagate the above to time t_n, yielding

$$\lim_{n\to\infty} \Big\| u(t_n) - \omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big)\Big\|_{L^\infty(D(a(\omega_{j_0,n}), R_n\lambda_{\max,n})\setminus\bigcup_{\ell=1}^{L} D(x_{\ell,n}, r_n))} = 0. \tag{3.58}$$

Using again Lemma 2.10 and (3.27), the construction of the sequences $\{x_{\ell,n}\}$ and the choice of $\lambda_{\max,n} \gg r_n \gg (t_n - s_n)^{\frac{1}{2}}$ as well as $r_n \gg R_n\lambda(\omega_{j,n})$ for all $j \notin \mathcal{J}_{\max}$, we have

$$\lim_{n\to\infty} E\Big(u(t_n); D(y_n, \rho_n) \setminus \Big[\bigcup_{j\in\mathcal{J}_{\max}} D(a(\omega_{j,n}), R_n\lambda_{\max,n}) \cup \bigcup_{\ell=1}^{L} D(x_{\ell,n}; r_n)\Big]\Big) = 0. \tag{3.59}$$

Now, by (3.52), after passing to a joint subsequence in n, for each $\ell \in \{1, \dots, L\}$, we can find an integer $\widetilde{M}_\ell \geq 0$, a sequence of $\widetilde{M}_\ell$-bubble configurations $\mathcal{Q}(\boldsymbol{\Omega}_{\ell,n})$, and sequences of vectors $\vec{\nu}_{\ell,n} = (\nu_{\ell,n}, \nu_{\ell,1,n}, \dots, \nu_{\ell,\widetilde{M}_\ell,n})$ and $\vec{\xi}_{\ell,n} = (\xi_{\ell,n}, \xi_{\ell,1,n}, \dots, \xi_{\ell,\widetilde{M}_\ell,n})$ so that

$$\lim_{n\to\infty} \mathbf{d}(u(t_n), \mathcal{Q}(\boldsymbol{\Omega}_{\ell,n}); D(x_{\ell,n}, r_n); \vec{\nu}_{\ell,n}, \vec{\xi}_{\ell,n}) = 0. \tag{3.60}$$

Here, $\boldsymbol{\Omega}_{\ell,n} = (\Omega_{\ell,n}, \Omega_{\ell,1,n}, \dots, \Omega_{\ell,\widetilde{M}_\ell,n})$. Dropping the constants $\Omega_{\ell,n} \in \mathbb{S}^2$ in the $\widetilde{M}_\ell$-bubble configurations, consider finally the sequence (in n) of multi-bubbles formed by the constant $\omega \in \mathbb{S}^2$ and the harmonic maps

$$\{\Omega_{\ell,k,n}\}_{\ell=1,k=1}^{\ell=L,k=\widetilde{M}_\ell}, \ \{\omega_{j,0,n}\}_{j\in\mathcal{J}_{\max}} := \Big\{\omega_{j,0}\big(\frac{\cdot - a(\omega_{j,n})}{\lambda(\omega_{j,n})}\big)\Big\}_{j\in\mathcal{J}_{\max}}. \tag{3.61}$$

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For each $j \in \mathcal{J}_{\max}$, we define $\nu_{j,n} := R_n$ and $\xi_{j,n} = r_n$, and then defining

$$\vec{\widetilde{\nu}}_n := (\nu_n, (\nu_{\ell,n})_{\ell=1}^L, (\nu_{j,n})_{j\in\mathcal{J}_{\max}}), \qquad \vec{\widetilde{\xi}}_n := (\breve{\xi}_n, (\xi_{\ell,n})_{\ell=1}^L, (\xi_{j,n})_{j\in\mathcal{J}_{\max}}),$$

we claim that

$$\lim_{n\to\infty} \mathbf{d}\big(u(t_n), \mathcal{Q}(\omega, (\Omega_{\ell,k,n})_{\ell=1,k=1}^{\ell=L,k=\widetilde{M}_\ell}, (\omega_{j,0,n})_{j\in\mathcal{J}_{\max}}); D(y_n,\rho_n); \vec{\widetilde{\nu}}_n, \vec{\widetilde{\xi}}_n\big) = 0, \tag{3.62}$$

which would yield (3.53). Indeed, by (3.60) and since all of the $D(x_{\ell,n}, r_n)$ are disjoint and satisfy (3.41), any distinct triples $(\Omega_{\ell,k,n}, a(\Omega_{\ell,k,n}), \lambda(\Omega_{\ell,k,n}))$ and $(\Omega_{\ell',k',n}, a(\Omega_{\ell',k',n}), \lambda(\Omega_{\ell',k',n}))$ are asymptotically orthogonal for $(\ell,k) \neq (\ell',k')$. Moreover, for any ℓ and $j_0 \in \mathcal{J}_{\max}$ for which $D(x_{\ell,n}, r_n) \subset D(a(\omega_{j_0,n}), R_n\lambda(\omega_{j_0,n}))$, the triples

$$(\Omega_{\ell,k,n}, a(\Omega_{\ell,k,n}), \lambda(\Omega_{\ell,k,n})) \quad \text{and} \quad \Big(\omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big), a(\omega_{j_0,n}), \lambda(\omega_{j_0,n})\Big)$$

are asymptotically orthogonal since $r_n \ll \lambda_{\max,n}$. Indeed,

$$\lim_{n\to\infty} E\Big(\omega_{j_0,0}\big(\frac{\cdot - a(\omega_{j_0,n})}{\lambda(\omega_{j_0,n})}\big); D(x_{\ell,n}, r_n)\Big) = 0, \quad \forall j_0 \in \mathcal{J}_{\max}, \quad \forall \ell \in \{1,\dots,L\} \tag{3.63}$$

and

$$\lim_{n\to\infty} E(\Omega_{\ell,k,n}; D(y_n,\rho_n) \setminus D(x_{\ell,n}, r_n)) = 0, \quad \forall\, \ell \in \{1,\dots,L\},\ k \in \{1,\dots,\widetilde{M}_\ell\}. \tag{3.64}$$

These observations, together with (3.56), (3.58), the estimate (3.59) and Remark 3.2 (using now $\breve{\xi}_n$ instead of ξ_n), yield (3.62). This completes the proof of (3.51).

Having established (3.51), we claim that there exist times $\widetilde{\sigma}_n < t_n$ so that

$$t_n - \widetilde{\sigma}_n \ll r_n^2 \quad \text{and} \quad \lim_{n\to\infty} r_n \|\mathcal{T}(u(\widetilde{\sigma}_n))\|_{L^2} = 0. \tag{3.65}$$

If not, we could find $c, c_1 > 0$ and a subsequence of the t_n for which

$$r_n^2 \|\mathcal{T}(u(t))\|_{L^2}^2 \geq c_1 \quad \forall t \in [t_n - cr_n^2, t_n]. \tag{3.66}$$

But then we would have

$$\sum_n \int_{t_n - cr_n^2}^{t_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \geq c_1 \sum_n \int_{t_n - cr_n^2}^{t_n} r_n^{-2} \,\mathrm{d}t \geq cc_1 \sum_n 1 = \infty, \tag{3.67}$$

which contradicts (2.12). Given the sequence $\widetilde{\sigma}_n$ as in (3.65), we can apply the Compactness Lemma 2.15, so that after passing to a subsequence in n (still denoted by $\widetilde{\sigma}_n, t_n$), we have a bubble decomposition as in (2.64) for some sequence $\hat{R}_n \to \infty$. The estimate (3.42) can be propagated to time $\widetilde{\sigma}_n$ using Lemma 2.9, which gives

$$\lim_{n\to\infty} E\big(u(\widetilde{\sigma}_n); D(x_{\ell,n}; 2^{-1}\beta_n r_n) \setminus D(x_{\ell,n}; 2\alpha_n r_n)\big) = 0. \tag{3.68}$$

The above ensures that the harmonic map in (2.64) at scale r_n must be constant, which we denote by $\widetilde{\omega} \in \mathbb{S}^2$, and so we can conclude that in fact,

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(\widetilde{\sigma}_n); D(x_{\ell_1,n}, r_n)) = 0, \tag{3.69}$$

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By (2.67), we can find an integer $K_1 \geq 0$ so that

$$E(u(\widetilde{\sigma}_n); D(x_{\ell_1,n}, r_n)) \to 4\pi K_1 \quad \text{as} \quad n \to \infty. \tag{3.70}$$

Because of (3.51), we must have $K_1 \geq 1$ (since $t_n - \widetilde{\sigma}_n \ll r_n^2$).

Consider the time intervals $[\widetilde{\sigma}_n, t_n]$ and the discs $D(x_{\ell_1,n}; r_n)$. Property (1) from Definition 3.1 is given by the first line in (3.69). Property (2) is given by (3.51). Property (3) is satisfied because of the first estimate in (3.65), and property (4) because of (3.70).

Lastly, we claim that $K_1 < K$. This is clear if $K_0 > 0$ since in that case, some energy lies at the scale $\simeq \lambda_{\max,n} \gg r_n$. If $K_0 = 0$ and $K_1 = K$, then all of the energy in the larger discs $D(y_n, \rho_n)$ would be captured within the sequence of discs $D(x_{\ell_1,n}, r_n)$. However, recall that there is at least one index j_0 such that $\lambda(\omega_{j_0,n}) = \lambda_{\max,n}$, and we have chosen r_n so that $r_n \ll \lambda_{\max,n} = \lambda(\omega_{j_0,n})$, which (by Definition 1.3) implies at least 3π in energy concentrates outside the discs $D(x_{\ell_1,n}, r_n)$, a contradiction.

We conclude that $K_1 < K$ and that $[\widetilde{\sigma}_n, t_n] \in \mathcal{C}_{K_1}(x_{\ell_1,n}, r_n, \epsilon_{1,n}, \eta_1)$ for some sequence $\epsilon_{1,n} \to 0$, contradicting the minimality of K. This completes the proof. □

Proof of Corollary 3.5. Let η_0 be as in Lemma 3.4 and fix an $\eta \in (0, \eta_0]$. Let $\epsilon > 0$ be given by Lemma 3.4 and define s_n by

$$s_n := \inf\{t \in [\sigma_n, \tau_n] \mid \boldsymbol{\delta}(u(\tau); D(y_n, \rho_n)) \geq \epsilon, \quad \forall \tau \in [t, \tau_n]\}, \tag{3.71}$$

which is well-defined for all sufficiently large n. Then $\boldsymbol{\delta}(u(s_n); D(y_n, \rho_n)) = \epsilon$. Define $\lambda_{\max}(s_n)$ as in the statement of the result. By Lemma 3.4, it follows that $s_n + c_0\lambda_{\max}(s_n)^2 \leq \tau_n$ for all sufficiently large n. The remaining claims hold by the choice of s_n. □

3.3. *Proof of Theorem 1.8*

Proof of Theorem 1.8. Assume the theorem is false. Let $K \geq 1$ and fix collision intervals $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \epsilon_n, \eta)$ as in Definition 3.1 and Lemma 3.3. We assume that $\eta > 0$ is sufficiently small as in Lemma 3.4 and let $\epsilon > 0$ and s_n be given by Corollary 3.5, so we have

$$\boldsymbol{\delta}(u(s_n), D(y_n, \rho_n)) = \epsilon. \tag{3.72}$$

Let M_n be a sequence of non-negative integers, $\mathcal{Q}(\omega_n)$ a sequence of M_n-bubble configurations, and $\vec{\nu}_n \in (0, \infty)^{M_n+1}$, $\vec{\xi}_n \in (0, \infty)^{M_n+1}$ sequences so that

$$\epsilon \leq \mathbf{d}(u(s_n), \mathcal{Q}(\omega_n); D(y_n, \rho_n); \vec{\nu}_n, \vec{\xi}_n) \leq 2\epsilon. \tag{3.73}$$

We fix a choice of ξ_n, ν_n (the first components of the vectors $\vec{\xi}_n, \vec{\nu}_n$) as in Remark 3.2 so that (3.2) and (3.3) hold. Defining $\lambda_{\max,n} = \lambda_{\max}(s_n)$ as in Corollary 3.5, we have that $[s_n, s_n + c_0\lambda_{\max,n}^2] \subset [\sigma_n, \tau_n]$ and moreover that

$$\boldsymbol{\delta}(u(t); D(y_n, \rho_n)) \geq \epsilon, \quad \forall t \in [s_n, s_n + c_0\lambda_{\max,n}^2] \tag{3.74}$$

for all n sufficiently large. Since $\sup_{t<T_+} E(u(t)) < \infty$ we can, after passing to a subsequence, assume $M_n = M$ for some fixed integer M and that the constant $\omega_n \in \mathbb{S}^2$ in the M-bubble configuration $\mathcal{Q}(\omega_n)$ are fixed (i.e., $\omega_n = \omega \in \mathbb{S}^2$).

We claim there exists $c_1 > 0$ such that for all $n \geq n_0$,

$$\lambda_{\max,n}^2 \|\mathcal{T}(u(t))\|_{L^2}^2 \geq c_1, \quad \forall t \in [s_n, s_n + c_0\lambda_{\max,n}^2]. \tag{3.75}$$

If not, we could find a sequence $t_n \in [s_n, s_n + c_0\lambda_{\max,n}^2] \subset [\sigma_n, \tau_n]$ such that

$$\lim_{n\to\infty} \lambda_{\max,n} \|\mathcal{T}(u(t_n))\|_{L^2} = 0. \tag{3.76}$$

By the Compactness Lemma 2.15, for all $x_n \in \mathbb{R}^2$, there exists a subsequence of the $u(t_n)$ and a sequence $R_n(x_n) \to \infty$, such that, for any sequence $1 \ll \breve{R}_n \ll R_n(x_n)$,

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(x_n, \breve{R}_n \lambda_{\max,n})) = 0. \tag{3.77}$$

By Lemma 2.9, we also have that

$$\lim_{n\to\infty} E(u(t_n); D(y_n, \rho_n)) = 4K\pi, \tag{3.78}$$

where here we have used that $|[\sigma_n, \tau_n]| \ll \rho_n^2$ to propagate Property (4) from Definition 3.1 from time σ_n to time t_n. Note also that $\rho_n^2 \gg \xi_n^2 \gg \tau_n - \sigma_n$ and Corollary 3.5 ensure that $\xi_n \gg \lambda_{\max,n}$.

We claim that after passing to a subsequence, there exists an integer $L > 0$, sequences $x_{\ell,n}$ for each $\ell \in \{1, \ldots, L\}$, a number $R \geq 2$, and a sequence $1 \ll \widetilde{R}_n \ll \lambda_{\max,n}^{-1}\xi_n$ so that

$$E\Big(u(s_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, R\lambda_{\max,n}\Big) \leq \frac{\pi}{2}, \tag{3.79}$$

and

$$D(x_{\ell,n}, \widetilde{R}_n \lambda_{\max,n}) \cap D(x_{\ell',n}, \widetilde{R}_n \lambda_{\max,n}) = \emptyset \tag{3.80}$$

for any $\ell \neq \ell'$. We find the points $x_{\ell,n}$ as follows. Passing to a subsequence, we can assume the existence of the limits

$$\lim_{n\to\infty} \frac{|a(\omega_{j,n}) - a(\omega_{k,n})|}{\lambda_{\max,n}} \in [0, \infty] \tag{3.81}$$

for each $j \neq k$. We define the index sets

$$\mathcal{L}(j) := \Big\{ j \text{ and any index } k \in \{1, \ldots, M\} \text{ such that } \lim_{n\to\infty} \frac{|a(\omega_{j,n}) - a(\omega_{k,n})|}{\lambda_{\max,n}} < \infty \Big\} \tag{3.82}$$

and note that for any distinct indices j, j' either $\mathcal{L}(j) = \mathcal{L}(j')$ or they are disjoint. For each n and for each of the sets $\mathcal{L}(j)$, we let $x_{\mathcal{L}(j),n}$ denote the barycenter of the points $a(\omega_{j_1,n}), \ldots, a(\omega_{j_{\#\mathcal{L}(j)},n})$ where each $j_k \in \mathcal{L}(j)$. There are $L \leq M$ many distinct index sets $\mathcal{L}(j)$, and we let $\{x_{\ell,n}\}_{\ell=1}^{L}$ be an enumeration of the distinct $x_{\mathcal{L}(j),n}$.

Next, from (3.73), Lemma 2.4 and the definitions of $\mathbf{d}$ and $\lambda_{\max,n}$ we can find $R_1 \geq 2$ so that

$$E\Big(u(s_n); D(y_n, \rho_n) \setminus \bigcup_{j=1}^{M} D(a(\omega_{j,n}), R_1\lambda_{\max,n})\Big) \leq \frac{\pi}{2} \tag{3.83}$$

for all sufficiently large n. From the above and the definition of the $x_{\ell,n}$, we can find $R \geq R_1$ so that (3.79) holds. The existence of a sequence $1 \ll \widetilde{R}_n \ll \lambda_{\max,n}^{-1}\xi_n$ so that (3.80) holds follows from definition of the $x_{\ell,n}$.

Consider each of the sequences $x_{\ell,n}$ as the x_n in (3.77) and find corresponding sequences $R_{\ell,n}$ so that for any sequence $\breve{R}_n \leq R_{\ell,n}$,

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(x_{\ell,n}, \breve{R}_n \lambda_{\max,n})) = 0, \quad \ell = 1, \ldots, L. \tag{3.84}$$

Enlarge the sequence ξ_n to a sequence $\widetilde{\xi}_n$ as in Remark (3.2) (i.e., so that $\xi_n \ll \widetilde{\xi}_n \ll \rho_n$). Then, since all of the $x_{\ell,n} \in D(y_n, \xi_n)$ and $\lambda_{\max,n} \ll \xi_n$, we have

$$\lim_{n\to\infty} \frac{\lambda_{\max,n}}{\operatorname{dist}(x_{\ell,n}, \partial D(y_n, \widetilde{\xi}_n))} = 0 \tag{3.85}$$

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for each ℓ. We can thus find a sequence $R_n \le \min\{\widetilde{R}_n, R_{\ell,n}\}_{\ell=1,\dots,L}$ such that $D(x_{\ell,n}, R_n\lambda_{\max,n}) \subset D(y_n, \widetilde{\xi}_n)$ for each ℓ.

Enlarging the excised discs (replacing R by R_n) in (3.79), we obtain

$$E\Big(u(s_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, R_n\lambda_{\max,n}\Big) \le \frac{\pi}{2}. \tag{3.86}$$

We use Lemma 2.9 to propagate this bound forward to time t_n, giving

$$E\Big(u(t_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, R_n\lambda_{\max,n}\Big) \le \pi. \tag{3.87}$$

However, by (3.84) (replacing $R_{1,n}$ by R_n), we can find integers K_ℓ so that

$$E(u(t_n); D(x_{\ell,n}, R_n\lambda_{\max,n})) \longrightarrow 4K_\ell\pi \ \text{ as } \ n \to \infty \tag{3.88}$$

for each $\ell \in \{1, \dots, L\}$. Combining the above with (3.78), we see that

$$\lim_{n\to\infty} E\Big(u(t_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, R_n\lambda_{\max,n})\Big) = 4K\pi - \sum_{\ell=1}^{L} 4K_\ell\pi. \tag{3.89}$$

Comparing the above with (3.87), it follows that $\sum_\ell 4K_\ell\pi = 4K\pi$, and thus,

$$\lim_{n\to\infty} E\Big(u(t_n); D(y_n, \rho_n) \setminus \bigcup_{\ell=1}^{L} D(x_{\ell,n}, R_n\lambda_{\max,n})\Big) = 0. \tag{3.90}$$

From (3.84) and the definition of R_n, we have

$$\lim_{n\to\infty} \sum_{\ell=1}^{L} \boldsymbol{\delta}(u(t_n); D(x_{\ell,n}, R_n\lambda_{\max,n})) = 0 \tag{3.91}$$

and moreover that the discs $D(x_{\ell,n}, R_n\lambda_{\max,n})$ are disjoint by (3.80) and the choice of $R_n \le \widetilde{R}_n$. Combining (3.91), (3.85), the disjointness of the discs $D(x_{\ell,n}, R_n\lambda_{\max,n})$, (3.90), and Remark 3.2, we conclude that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(y_n, \rho_n)) = 0, \tag{3.92}$$

which contradicts (3.74), proving (3.75).

By (3.75), we have

$$\sum_n \int_{s_n}^{s_n + c_0\lambda_{\max}(s_n)^2} \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \ge c_1 \sum_n \int_{s_n}^{s_n + c_0\lambda_{\max}(s_n)^2} \lambda_{\max}(s_n)^{-2} \,\mathrm{d}t \ge c_0 c_1 \sum_n 1 = \infty.$$

However, since the intervals $[\sigma_n, \tau_n]$ are disjoint, the above contradicts the bound (2.12) – that is,

$$\sum_n \int_{s_n}^{s_n + c_0\lambda_{\max}(s_n)^2} \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \le \int_0^{T_+} \|\mathcal{T}(u(t)\|_{L^2}^2 \,\mathrm{d}t < \infty, \tag{3.93}$$

which completes the proof. □

3.4. *Proof of Theorem 1.1*

In this section, we prove Theorem 1.1 using Theorem 1.8 as a main ingredient in the proof.

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Proof of Theorem 1.1. We consider the case of finite time blow-up (i.e., $T_+ < \infty$), noting that the analysis for the global case is similar.

Let $L \geq 1$ and $\{x_\ell\}_{\ell=1}^L$ be the bubbling points given by the local theory of Struwe in Theorem 2.7. Let $\rho_0 > 0$ be sufficiently small so that $D(x_\ell; 2\rho_0) \cap D(x_m; 2\rho_0) = \emptyset$ for each $\ell \neq m$. By Theorem 2.7, we have that

$$\lim_{t \to T_+} E\Big(u(t) - u^*; \mathbb{R}^2 \setminus \bigcup_{\ell=1}^{L} D(x_\ell; \rho_0)\Big) = 0. \tag{3.94}$$

By Lemma 2.13, we know that for each ℓ,

$$\lim_{t \to T_+} E\big(u(t) - u^*; D(x_\ell; \rho_0) \setminus D(x_\ell; \sqrt{T_+ - t})\big) = 0, \tag{3.95}$$

and since $u^* \in \mathcal{E}$,

$$\lim_{t \to T_+} E\big(u^*; D(x_\ell; \sqrt{T_+ - t})\big) = 0 \tag{3.96}$$

for each $\ell \in \{1, \dots, L\}$. Hence, it suffices to examine the solution $u(t)$ in the discs $D(x_\ell; \sqrt{T_+ - t})$ for each $\ell \in \{1, \dots, L\}$. Fix an ℓ and, to ease notation, we write $y = x_\ell$ below. By Theorem 1.8, we know that for $\rho(t) := \sqrt{T_+ - t}$, we have,

$$\lim_{t \to T_+} \boldsymbol{\delta}(u(t); D(y, \rho(t))) = 0. \tag{3.97}$$

Now, let $t_n \to T_+$ be any sequence of times. By the above, we can find a sequence $1 \leq M_n \leq (4\pi)^{-1} E(u_0)$, a sequence of M_n-bubble configurations $\mathcal{Q}(\boldsymbol{\omega}_n)$, and sequences $\vec{\nu}_n = (\nu_n, \nu_{1,n}, \dots, \nu_{M_n,n})$, $\vec{\xi}_n = (\xi_n, \xi_{1,n}, \dots, \xi_{M_n,n})$ such that

$$\lim_{n \to \infty} \mathbf{d}(u(t_n), \mathcal{Q}(\boldsymbol{\omega}_n); D(y, \rho(t_n)); \vec{\nu}_n, \vec{\xi}_n) = 0. \tag{3.98}$$

Passing to a subsequence of the t_n, we may assume that $M_n = M$ is a fixed integer and that the constants $\omega_n \in \mathbb{S}^2$ in the M-bubble configurations $\mathcal{Q}(\boldsymbol{\omega}_n)$ are fixed (i.e., $\omega_n = \omega \in \mathbb{S}^2$). This proves the estimate (1.10).

Since each of the $\omega_{j,n}$ is a harmonic map (and thus $\mathcal{T}(\omega_{j,n}) = 0$), these sequences satisfy the hypothesis of the Compactness Lemma 2.15. Therefore, after passing to a joint subsequence, for each $j \in \{1, \dots, M\}$, we can find integers $M_j \geq 0$, harmonic maps $\theta_{j,0}, \theta_{j,1}, \dots \theta_{j,M_j}$ (where only $\theta_{j,0}$ is possibly constant), along with sequences of vectors $b_{j,k,n} \in D(a(\omega_{j,n}), C_j\lambda(\omega_{j,n}))$ and scales $\mu_{j,k,n} \ll \lambda(\omega_{j,n})$ satisfying (2.65), (2.66) and so that

$$\begin{aligned}\lim_{n \to \infty} \Bigg[& E\Big(\omega_{j,n} - \theta_{j,0}\big(\frac{\cdot - a(\omega_{j,})}{\lambda(\omega_{j,n})}\big) - \sum_{k=1}^{M_j} \Big(\theta_{j,k}\big(\frac{\cdot - b_{j,k,n}}{\mu_{j,k,n}}\big) - \theta_{j,k}(\infty)\Big); D_{j,n}\Big) \\ & + \Big\| \omega_{j,n} - \theta_{j,0}\big(\frac{\cdot - a(\omega_{j,})}{\lambda(\omega_{j,n})}\big) - \sum_{k=1}^{M_j} \Big(\theta_{j,k}\big(\frac{\cdot - b_{j,k,n}}{\mu_{j,k,n}}\big) - \theta_{j,k}(\infty)\Big) \Big\|_{L^\infty(D_{j,n})} \Bigg] = 0,\end{aligned} \tag{3.99}$$

where $D_{j,n} := D(a(\omega_{j,n}), R_n\lambda(\omega_{j,n}))$ for some sequence $R_n \to \infty$, and where for each fixed j,

$$\lim_{n \to \infty} \sum_{k \neq k'} \Big(\frac{\mu_{j,k,n}}{\mu_{j,k',n}} + \frac{\mu_{j,k',n}}{\mu_{j,k,n}} + \frac{|b_{j,k,n} - b_{j,k',n}|^2}{\mu_{j,k,n}\mu_{j,k',n}} \Big)^{-1} = 0. \tag{3.100}$$

To make the notation for the scales and centers of the harmonic maps above more uniform, we also introduce the notation

$$\mu_{j,0,n} := \lambda(\omega_{j,n}), \quad b_{j,0,n} := a(\omega_{j,n}). \tag{3.101}$$

Our goal is to find a collection of asymptotically orthogonal triples $(\omega_j, a_{j,n}, \lambda_{j,n})$ as in the statement of Theorem 1.1. The sequences $\{(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})\}_{j=1,k=0}^{j=M,k=M_j}$ are not guaranteed to be such a collection. While (3.100) holds for each fixed j, the triples $(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})$ and $(\theta_{j',k'}, b_{j',k',n}, \mu_{j',k',n})$ with $j \neq j'$ might not be asymptotically orthogonal. As in Definition 3.7 and the proof of Lemma 3.4, we define the set of indices $\mathcal{R}$ to be those associated to the roots (i.e., the maximal elements of the sequences $\mathfrak{h}_j = \{\omega_{j,n}\}$ of harmonic maps under the partial order $\preceq$). For each root $\mathfrak{h}_{j_0}$, we define the bubble tree

$$\mathcal{T}(j_0) := \{\mathfrak{h}_j \preceq \mathfrak{h}_{j_0}\}.$$

Let $C_0 > 0$ be large enough so that $\mathfrak{h}_j \prec \mathfrak{h}_{j_0}$ implies $D(a(\omega_{j,n}), \lambda(\omega_{j,n})) \subset D(a(\omega_{j_0,n}), C_0\lambda(\omega_{j_0,n}))$ for all n. The collection of all harmonic maps, together with scales and centers, concentrating inside the discs $D(a(\omega_{j_0,n}), C_0\lambda(\omega_{j_0,n}))$ equals

$$\bigcup_{\mathfrak{h}_j \in \mathcal{T}(j_0)} \{(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})\}_{k=0}^{M_j}. \tag{3.102}$$

We let

$$\begin{aligned}\mathcal{K}(j,k) := \big\{(j,k) \text{ and any } (j',k') \text{ associated to a triple } (\omega_{j',k'}, b_{j',k',n}, \mu_{j',k',n})\\ \text{not asymptotically orthogonal to } (\omega_{j,k}, b_{j,k,n}, \mu_{j,k,n})\big\}\end{aligned} \tag{3.103}$$

If $\#\mathcal{K}(j,k) = 1$, we keep the triple $(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})$ in our final collection – note that (1.9) and (1.14) will be consequences of (3.98) (3.99), and (2.65) (2.66). Now consider a set of indices (j_1, k_1) with $\mathfrak{h}_{j_1} \in \mathcal{T}(j_0)$ and such that $\#\mathcal{K}(j_1, k_1) \geq 2$. After performing a fixed (in n) rescaling and translation of each harmonic map $\theta_{j,k}$ associated to an index $(j,k) \in \mathcal{K}(j_1, k_1)$, we may assume that

$$b_{j,k,n} = b_{j_1,k_1,n} \text{ and } \mu_{j,k,n} = \mu_{j_1,k_1,n}, \quad \forall (j,k) \in \mathcal{K}(j_1, k_1), \tag{3.104}$$

and to simplify notation below, we simply write $b_n = b_{j_1,k_1,n}$ and $\mu_n = \mu_{j_1,k_1,n}$. By (3.98) and (3.99), we can also find $r_n \to \infty$ a number $C_1 > 0$, an integer $L_1 \geq 0$, and a finite number of sequences of discs $D(c_{\ell,n}, \rho_{n,\ell}) \subset D(b_n, C_1\mu_n)$ for $\ell \in \{1, \dots, L_1\}$ and with

$$\frac{\rho_{\ell,n}}{\operatorname{dist}(c_{\ell,n}, \partial D(b_n, C_1\mu_n))} \to 0 \text{ as } n \to \infty \tag{3.105}$$

so that

$$\lim_{n\to\infty} E(u(t_n); D(c_{\ell,n}, 2\rho_{\ell,n}) \setminus D(c_{\ell,n}, \frac{1}{2}\rho_{\ell,n})) = 0 \tag{3.106}$$

and

$$\lim_{n\to\infty} E(u_n; D(b_n, 2r_n\mu_n) \setminus D(b_n, \frac{1}{2}r_n\mu_n)) = 0 \tag{3.107}$$

and

$$\lim_{n\to\infty} E\Big(u(t_n) - \sum_{(j,k)\in\mathcal{K}(j_1,k_1)} \Big(\theta_{j,k}(\frac{\cdot - b_n}{\mu_n}) - \theta_{j,k}(\infty)\Big); D(b_n, r_n\mu_n) \setminus \bigcup_{\ell=1}^{L_1} D(c_{n,\ell}, \rho_{n,\ell})\Big) = 0. \tag{3.108}$$

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By Theorem 1.8 and (3.107), we know that,

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); D(b_n, r_n\mu_n)) = 0. \tag{3.109}$$

This means that, after passing to a subsequence, we can find an integer M_{j_1,k_1}, a sequence of M_{j_1,k_1}-bubble configurations $\mathcal{Q}(\boldsymbol{\Omega}_n)$, with the nontrivial harmonic maps denoted by $\Omega_{m,n}$ for $m \in \{1, \ldots, M_{j_1,k_1}\}$, sequences $\vec{\widetilde{\nu}}_n = (\widetilde{\nu}_n, \widetilde{\nu}_{1,n}, \ldots, \widetilde{\nu}_{M_{j_1,k_1},n})$ and $\vec{\widetilde{\xi}}_n = (\widetilde{\xi}_n, \widetilde{\xi}_{1,n}, \ldots, \widetilde{\xi}_{M_{j_1,k_1},n})$, so that

$$\mathbf{d}(u(t_n), \mathcal{Q}(\Omega_{\infty,n}, \boldsymbol{\Omega}_n); D(b_n, r_n\mu_n); \vec{\widetilde{\nu}}_n, \vec{\widetilde{\xi}}_n) \to 0 \text{ as } n \to \infty. \tag{3.110}$$

Consider the centers and scales $a(\Omega_{m,n}), \lambda(\Omega_{m,n})$ associated to the harmonic maps $\Omega_{m,n}$. Using (3.105), (3.106) (3.107) and (3.108), we see that there are only two possible cases.

Case 1: All of the harmonic maps $\Omega_{n,m}$ concentrate within the discs $\bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})$ (i.e., for each $m \in \{1, \ldots, M_{j_1,k_1}\}$, we have $D(a(\Omega_{m,n}), \lambda(\Omega_{m,n})) \subset D(c_{\ell,n}, \rho_{\ell,n})$ for some $\ell \in \{1, \ldots, L_1\}$). This means that

$$\lim_{n\to\infty} E\Big(u(t_n); D(b_n, r_n\mu_n) \setminus \bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})\Big) = 0. \tag{3.111}$$

In fact, comparing the above with (3.108), one can deduce that the harmonic maps

$$\sum_{(j,k)\in\mathcal{K}(j_1,k_1)} (\theta_{j,k}(x) - \theta_{j,k}(\infty)) = \text{constant} \in \mathbb{R}^3.$$

In this case, we discard all of the harmonic maps with indices $(j,k) \in \mathcal{K}(j_1,k_1)$ from the final collection.

Case 2: Exactly one of the harmonic maps $\Omega_{m_1,n}$ has scale $\lambda(\Omega_{m_1,n}) \simeq \mu_n$ and center $|a(\Omega_{m_1,n}) - b_n| \lesssim \mu_n$, and the rest concentrate within the discs $\bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})$. We then have

$$\lim_{n\to\infty} E\Big(u(t_n) - \Omega_{m_1,n}; D(b_n, r_n\mu_n) \setminus \bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})\Big) = 0 \tag{3.112}$$

and

$$\lim_{n\to\infty} \Big\|u(t_n) - \Omega_{m_1,n}\Big\|_{L^\infty(D(b_n,r_n\mu_n)\setminus\bigcup_{\ell=1}^{L_1} D(c_{\ell,n},\rho_{\ell,n}))} = 0. \tag{3.113}$$

By another application of the Compactness Lemma 2.15, we can find a nontrivial harmonic map, which we label Θ_{j_1,k_1}, a non-negative integer P, scales $\nu_{p,n} \ll \mu_n$, centers $d_{p,n}$, and nontrivial harmonic maps Θ_p, satisfying (2.65) and (2.66), and so that

$$\begin{aligned}\lim_{n\to\infty}\Bigg[&E\Big(\Omega_{m_1,n} - \Theta_{j_1,k_1}\big(\frac{\cdot - b_n}{\mu_n}\big) - \sum_{p=1}^{P}\big(\Theta_p\big(\frac{\cdot - d_{p,n}}{\nu_{p,n}}\big) - \Theta_p(\infty)\big); D(b_n, r_n\mu_n)\Big)\\ &+ \Big\|\Omega_{m_1,n} - \Theta_{j_1,k_1}\big(\frac{\cdot - b_n}{\mu_n}\big) - \sum_{p=1}^{P}\big(\Theta_p\big(\frac{\cdot - d_{p,n}}{\nu_{p,n}}\big) - \Theta_p(\infty)\big)\Big\|_{L^\infty(D(b_n,r_n\mu_n))}\Bigg] = 0\end{aligned} \tag{3.114}$$

We know that the harmonic map Θ_{j_1,k_1} must be nontrivial because of (3.112) together with (3.108), where the latter ensures that energy cannot concentrate within the region

$$D(b_n, r_n\mu_n) \setminus \bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})$$

at scales smaller than μ_n. Indeed, by (3.108), the scales and centers of the nontrivial harmonic maps Θ_p must all concentrate within the discs $\bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})$, and we can conclude that

$$\lim_{n\to\infty} E\Big(u(t_n) - \Theta_{j_1,k_1}(\frac{\cdot - b_n}{\mu_n}); D(b_n, r_n\mu_n) \setminus \bigcup_{\ell=1}^{L_1} D(c_{\ell,n}, \rho_{\ell,n})\Big) = 0 \tag{3.115}$$

and

$$\lim_{n\to\infty} \Big\| u(t_n) - \Theta_{j_1,k_1}(\frac{\cdot - b_n}{\mu_n}) \Big\|_{L^\infty(D(b_n, r_n\mu_n)\setminus \bigcup_{\ell=1}^{L_1} D(c_{\ell,n},\rho_{\ell,n}))} = 0 \tag{3.116}$$

In this case, we discard all the triples $(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})$ with indices $(j,k) \in \mathcal{K}(j_1,k_1)$ from the final collection and replace them with the triple $(\Theta_{j_1,k_1}, b_{j_1,k_1,n}, \mu_{j_1,k_1,n})$.

To summarize, we keep for the final decomposition any triples $(\theta_{j,k}, b_{j,k,n}, \mu_{j,k,n})$ with $\mathfrak{h}_j \in \mathcal{T}(j_0)$ if $\#\mathcal{K}(j,k) = 1$. If $\#\mathcal{K}(j,k) > 1$, we discard all of the triples $(\theta_{j',k'}, b_{j',k',n}, \mu_{j',k',n})$ with indices $j' \in \mathcal{K}(j,k)$ and, in the event of Case 2 above, we replace them with $\Theta_{j,k}, b_{j,k,n}, \mu_{j,k,n}$. We perform this analysis for each index $j_0 \in \mathcal{R}$, resulting in a final collection of triples that are mutually asymptotically orthogonal and satisfy the conclusions of the theorem. □

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References

[1] A. Bernand-Mantel, C. B. Muratov and T M. Simon, 'A quantitative description of skyrmions in ultrathin ferromagnetic films and rigidity of degree ±1 harmonic maps from $\mathbb{R}^2$ to S^2', *Arch. Ration. Mech. Anal.* **239**(1) (2021), 219–299.

[2] H. Brezis and J. M. Coron, 'Convergence of solutions of H-systems or how to blow bubbles', *Arch. Ration. Mech. Anal.* **89** (1985), 21–56.

[3] K.-C. Chang, W. Y. Yue Ding and R. Ye, 'Finite-time blow-up of the heat flow of harmonic maps from surfaces', *J. Differential Geom.* **36**(2) (1992), 507–515.

[4] J.-M. Coron and J.-M. Ghidaglia, 'Explosion en temps fini pour le flot des applications harmoniques', *C. R. Acad. Sci. Paris Sér. I Math.* **308**(12) (1989), 339–344.

[5] E. B. Davies, *Heat Kernels and Spectral Theory* (Cambridge Tracts in Mathematics) vol. 92 (Cambridge University Press, Cambridge, 1990).

[6] J. Dávila, M. del Pino and J. Wei, 'Singularity formation for the two-dimensional harmonic map flow into S^2', *Invent. Math.* **219**(2) (2020), 345–466.

[7] M. del Pino, M. Musso and J. Wei, 'Existence and stability of infinite time bubble towers in the energy critical heat equation', *Anal. PDE* **14**(5) (2021), 1557–1598.

[8] B. Deng, L. Sun and J. Wei, 'Quantitative stability of harmonic maps from $\mathbb{R}^2$ to S^2 with higher degree', to appear in *Calc. Var. PDE.*

[9] W. Y. Ding and G. Tian, 'Energy identity for a class of approximate harmonic maps from surfaces', *Comm. Anal. Geom.* **3**(3–4) (1995), 543–554.

[10] J. Eells and L. Lemaire, 'A report on harmonic maps', *Bull. London Math. Soc.* **10**(1) (1978), 1–68.

[11] J. Eells and J. C. Wood, 'Restrictions on harmonic maps of surfaces', *Topology* **15**(3) (1976), 263–266.

[12] J. Eells, Jr. and J. H. Sampson, 'Harmonic mappings of Riemannian manifolds', *Amer. J. Math.* **86** (1964), 109–160.

[13] M. Guan, S. Gustafson and T.-P. Tsai, 'Global existence and blow-up for harmonic map heat flow', *J. Differential Equations* **246**(1) (2009), 1–20.

[14] S. Gustafson, K. Nakanishi and T.-P. Tsai, 'Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$', *Comm. Math. Phys.* **300**(1) (2010), 205–242.

[15] F. Hélein, *Harmonic Maps, Conservation Laws and Moving Frames* (Cambridge Tracts in Mathematics) vol. 150, second edn. (Cambridge University Press, Cambridge, 2002). Translated from the 1996 French original, with a foreword by James Eells.

[16] J. Jendrej and A. Lawrie, 'Bubble decomposition for the harmonic map heat flow in the equivariant case', *Calc. Var. PDE* **62**(9) (2023), 36pp.

[17] J. Jendrej and A. Lawrie, 'Soliton resolution for the energy-critical nonlinear wave equation in the radial case', *Ann. PDE* **9**(2) (2023), 117pp.

[18] J. Jendrej and A. Lawrie, 'Soliton resolution for energy–critical wave wave maps in the equivariant case', *J. Amer. Math. Soc.*, to appear.

[19] L. Lemaire, 'Applications harmoniques de surfaces Riemanniennes', *J. Differential Geometry* **13**(1) (1978), 51–78.

[20] F. Lin and C. Wang, 'Energy identity of harmonic map flows from surfaces at finite singular time', *Calc. Var. PDE* **6**(4) (1998), 369–380.

[21] F. Lin and C. Wang, *The Analysis of Harmonic Maps and Their Heat Flows* (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008).

[22] T. H. Parker, 'Bubble tree convergence for harmonic maps', *J. Differential Geom.* **44**(3) (1996), 595–633.

[23] J. Qing, 'On singularities of the heat flow for harmonic maps from surfaces into spheres', *Comm. Anal. Geom.* **3** (1995), 297–316.

[24] J. Qing and G. Tian, 'Bubbling of the heat flows for harmonic maps from surfaces', *Comm. Pure Appl. Math.* **50**(4) (1997), 295–310.

[25] P. Raphaël and R. Schweyer, 'Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow', *Comm. Pure Appl. Math.* **66**(3) (2013), 414–480.

[26] P. Raphaël and R. Schweyer, 'Quantized slow blow-up dynamics for the corotational energy-critical harmonic heat flow', *Anal. PDE* **7**(8) (2014), 1713–1805.

[27] J. Sacks and K. Uhlenbeck, 'The existence of minimal immersions of 2-spheres', *Ann. of Math. (2)* **113**(1) (1981), 1–24.

[28] M. Struwe, 'On the evolution of harmonic mappings of riemannian surfaces', *Comment. Math. Helv.* **60**(4) (1985), 558–581.

[29] T. Tao, 'Global regularity of wave maps IV. Absence of stationary or self-similar solutions in the energy class', Preprint, 2008, arXiv:0806.3592.

[30] P. Topping, 'Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow', *Ann. of Math.* **159**(2) (2004), 465–534.

[31] P. Topping, 'Winding behaviour of finite-time singularities of the harmonic map heat flow', *Math. Z.* **247**(2) (2004), 279–302.

[32] P. M. Topping, 'The harmonic map heat flow from surfaces', PhD thesis, University of Warwick, 1996.

[33] P. M. Topping, 'Rigidity in the harmonic map heat flow', *J. Diff. Geom.* **45**(3) (1997), 593–610.

[34] P. Topping, 'An example of a nontrivial bubble tree in the harmonic map heat flow', in *Harmonic Morphisms, Harmonic Maps, and Related Topics (Brest, 1997)* (Chapman & Hall/CRC Res. Notes Math.) vol. 413 (Chapman & Hall/CRC, Boca Raton, FL, 2000), 185–191.

[35] P. Topping, 'Reverse bubbling and nonuniqueness in the harmonic map flow', *Int. Math. Res. Not.* **10** (2002), 505–520.

[36] P. M. Topping, 'A rigidity estimate for maps from s^2 to s^2 via the harmonic map flow', *Bull. Lond. Math. Soc.* **55**(Issue 1) (2023), 338–343.

[37] C. Wang, 'Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets', *Houston J. Math.* **22**(3) (1996), 559–590.

[38] J. Wei, Q. Zhang and Y. Zhou, 'Trichotomy dynamics of the 1-equivariant harmonic map flow', Preprint, 2023, arXiv:2301.09221.

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