



On Localization and the Spectrum of Multi-frequency Quasi-periodic Operators

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Abstract

We study multi-frequency quasi-periodic Schrödinger operators on $\mathbb{Z}$ in the regime of positive Lyapunov exponent and for general analytic potentials. Combining Bourgain's semi-algebraic elimination of multiple resonances (Bourgain: *Geom. Funct. Anal.* **17**, 682–706, 2007) with the method of elimination of double resonances (*Ann. Math. (2)* **173**, 337–475, 2011), we establish exponential finite-volume localization as well as the separation between the eigenvalues. In a follow-up paper (Goldstein et al.: *Invent. Math.* **217**, 603–701, 2019) we develop the method further to show that for potentials given by large generic trigonometric polynomials the spectrum consists of a single interval, as conjectured by Chulaevski and Sinai (*Commun. Math. Phys.* **125**, 91–112, 1989).

Keywords Anderson model · Quasi-periodic potentials · Ergodic shifts on tori · Resonances · Quantum tunneling

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1 Introduction

In their pioneering paper [9], Chulaevsky and Sinai analyzed the spectrum and localized eigenfunctions of Schrödinger operators on $\mathbb{Z}$ with a large quasi-periodic potential given

Dedicated to Carlos Kenig on the occasion of his 70th birthday.

Michael Goldstein Deceased.

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by evaluating a generic smooth function on $\mathbb{T}^2$ along the orbit of an ergodic shift. They conjectured that in contrast to the shift on the one-dimensional torus $\mathbb{T}$, for which the spectrum is typically a Cantor set, for the two-dimensional shift the spectrum can be an interval.

In the 25 years since [9], the theory of quasi-periodic operators has been developed extensively, see [7] and [17]. Techniques from complex and harmonic analysis, the theory of semi-algebraic sets, and the theory of quasi-periodic cocycles played a key role in these developments. For the one-dimensional shift the spectral theory of quasi-periodic Schrödinger cocycles is almost complete. Most of the results have been established *non-perturbatively*, i.e., either in the regime of almost reducibility or in the regime of positive Lyapunov exponent, and Avila's global theory, see [3], gives a qualitative spectral picture, covering both regimes, for generic potentials. In the regime of positive Lyapunov exponent with generic frequency the spectrum is a Cantor set which is Carleson-homogeneous, see [13] and [10]. In this regime finite-volume exponential localization holds outside a set in phase space which is exponentially small in terms of the size of the interval. Moreover, one has a quantitative control on the repulsion between all eigenvalues, see [12]. For the case of the almost Mathieu operator (corresponding to a cosine potential), both localization and the spectrum have been studied in great detail under arithmetic conditions on the frequency, see Jitomirskaya [16] for localization, and Puig [20] and Avila, Jitomirskaya [1, 2] for the Cantor structure of the spectrum.

The spectral theory of Schrödinger cocycles over multidimensional shifts on $\mathbb{T}^d$ turns out to be more intricate to analyze. For $d = 2$ exponential localization was established in [4] almost everywhere in phase space. For $d > 2$ the same result was developed in [7]. However, these results were established directly in infinite volume, whereas localization in finite volume remained unknown. Regarding the structure of the spectrum, we note that Bourgain [6] proved existence of gaps for small potential and atypical frequencies. Numerical investigations by Puig, Simó [21] indicate that the spectral set may range from a Cantor set, to a finite union of intervals, to one interval, depending on the largeness of the potential, which can be tuned through a coupling constant.

In this paper we develop some basic features of the multi-frequency model, which are needed in the resolution of the Chulaevski–Sinai conjecture. Heuristically speaking, gaps in the spectrum of the one-frequency operators are created by resonances between an eigenvalue of one scale, and another shifted eigenvalue of the same scale, see [13, 22]. In contrast to this, the heuristic principle underlying [9] is that the graph of an eigenvalue on finite volume parametrized by the phase is too large to be destroyed along an entire horizontal section by the “forbidden zones” created by resonances. It is clear that some genericity assumption on the potential function is needed for this to be true, since $V(x, y) = v(x)$ has Cantor spectrum. Implementing such an argument, however, appears to be very challenging for a number of reasons. First, long chains of resonances might occur rendering the eigenvalue parametrization hard to handle. Second, the analytical techniques available in finite volume are less favorable (mainly the large deviation theorems and everything that depends on them, see below) as compared to the case of one frequency.

Nevertheless, this paper obtains precisely finite volume localization and level repulsion estimates as in [12, 13] for the case of multiple frequencies. This is made possible by introducing a new device into the analysis, namely Bourgain's semi-algebraic elimination technique from [8]. We emphasize that we do not assume large disorder, so our arguments are non-perturbative and rely only on positive Lyapunov exponents. In a subsequent paper [14] the results obtained here are used in a spectral analysis to show that indeed *level sets of eigenvalues cannot be completely destroyed* for large generic potentials.

To give a more detailed explanation of the difficulties we encounter in the finite interval localization problem for the case of several frequencies, we need to recall some standard definitions and some basic results.

Let V be a real-valued analytic function on the d -dimensional torus $\mathbb{T}^d$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$). We consider the family of discrete Schrödinger operators defined by

$$[H(x, \omega)\psi](n) = -\psi(n+1) - \psi(n-1) + V(x+n\omega)\psi(n)$$

with the frequency vector $\omega \in \mathbb{T}^d$ satisfying the standard Diophantine condition

$$\|k \cdot \omega\| \geq \frac{a}{|k|^b} \quad \text{for all nonzero } k \in \mathbb{Z}^d, \quad (1.1)$$

where $a > 0, b > d$ are some constants (here $\|\cdot\|$ denotes the distance to the nearest integer and $|\cdot|$ stands for the sup-norm on $\mathbb{Z}^d$). It is well known that for any $b > d$, a.e. $\omega \in \mathbb{T}^d$ satisfies (1.1) with some $a = a(\omega)$. We denote by $\mathbb{T}^d(a, b)$ the set of ω which obey (1.1). For integers $k < \ell$ we denote by $H_{[k, \ell]}(x, \omega)$ the operator on the finite interval $[k, \ell]$ with Dirichlet boundary conditions and by

$$f_{[k, \ell]}(x, \omega, E) := \det(H_{[k, \ell]}(x, \omega) - E)$$

the Dirichlet determinants.

The development of finite volume localization starts with *large deviation theorems* (LDT) of the form

$$\text{mes} \left\{ x \in \mathbb{T}^d : |\log |f_{[1, N]}(x, \omega, E)| - NL(\omega, E)| > N^{1-\tau} \right\} < \exp(-N^\sigma), \quad (1.2)$$

where $L(\omega, E)$ stands for the Lyapunov exponent, see [12]. This applies for arbitrary $d \geq 1$. For the one-dimensional torus $\mathbb{T}$, the estimate was sharpened in [12] so that it implies a $(\log N)^A$ -estimate for the local number of zeros of $f_{[1, N]}(z, \omega, E)$ when ω and E are fixed and the phase z runs over a complex neighborhood of the torus. This level of precision allows for a Weierstrass preparation theorem factorization of the determinant $f_{[1, N]}(\cdot, \omega, E)$ with a polynomial factor of degree $(\log N)^A$. This low degree is crucial as it allows, in combination with other tools developed in [13], for an effective control over the *double resonances* of the problem. The latter refers to the phases x for which the (LDT) estimate (1.2) fails twice: at x and at $x + n\omega$ with n in the range $N < |n| < N^A$, $A \gg 1$. In terms of exponentially localized eigenfunctions at scale N , it means that after shifting one of them on $\mathbb{Z}$ by n , this pair will be very close to forming an eigenfunction of a larger scale, say N^C with C large. That is the meaning of *resonance* in an inductive scheme and it is a key feature in [13] and [10].

It is well-known that for $d > 1$ the deviations in the large deviations theorem are much larger than for $d = 1$, because the discrepancy of an orbit of length N of a shift on $\mathbb{T}^d$ cannot be reduced beyond some fixed power of N , whereas in one dimension it can be made essentially logarithmic in N . In particular, for $d > 1$, the (LDT) estimate (1.2) cannot be improved beyond $N^{1/2}$ -deviations. One of the main ideas here is to gain control over the *local deviations* of $\log |f_{[1, N]}(x, \omega, E)|$ rather than insisting on the whole torus $\mathbb{T}^d$. To do so, we rely on Bourgain's [8] analysis of the structure of an arbitrary set $\mathcal{R} \subset \mathbb{Z}$ of shifts of a given semi-algebraic set $\mathcal{A} \subset \mathbb{T}^d \times \mathbb{T}^d \times \mathbb{R}$ with controlled size and complexity, such that

$$\bigcap_{n \in \mathcal{R}} \{x \in \mathbb{T}^d : (x + n\omega, \omega, E) \in \mathcal{A}\} \neq \emptyset. \quad (1.3)$$

The phases x in (1.3) are called multiple resonances. Bourgain's result states that after removing a small set of ω 's *uniformly in* x, E the set $\mathcal{R}$ in (1.3) has a *lacunary structure*.

This structure defines for any given phase x and scale L a suitable size $L \leq N \leq L^A$ for which we show that the local deviations of $\log |f_{[1,N]}(x, \omega, E)|$ behave almost as well as in the one-dimensional case.

The aforementioned statement from [8] is a combination of two techniques: (i) a purely semi-algebraic analysis of the set $\tilde{\mathcal{A}} := \{(x, y) \in \mathbb{T}^{2d} : \mathcal{A}(x) \cap \mathcal{A}(y) \neq \emptyset\}$, stating that this is a small semi-algebraic set (ii) the *method of steep planes* from [4, 7, 8] which implies that it is unlikely that $(x, x + n\omega) \in \tilde{\mathcal{A}}$ for fixed x ; this means that the measure of $\omega \in \mathbb{T}^d$ for which this occurs is small provided n is large enough.

We note that, while Bourgain's technique applies uniformly to all phases and energies, the scales determined by it are sensitive to the choice of these parameters. In and of itself, the measure estimate on resonant ω 's obtained in this fashion is too weak to be useful for finite volume localization and the separation of the eigenvalues. Indeed, it cannot be summed over the range $N \leq n \leq 2N$, since the measure bound is never better than N^{-1} for a single choice of such n . In contrast, the technique from [13], which is based on resultants and the Cartan bound on large negative values of subharmonic functions, gives subexponential bounds which can be summed. However, one needs good control on the degree of the polynomials going into the resultant and a nondegeneracy condition on the resultant, in order to avoid that the resultant is close to 0 everywhere on a disk. For $d = 1$ these issues are addressed by the sharpening of the (LDT), whereas for $d > 1$ we have to take advantage of the flexibility of the methods based on semi-algebraic sets. We gain the needed control on the degrees of the polynomials by working at the scales afforded by the lacunary structure of the set of multiple resonances and we obtain the nondegeneracy condition by the aforementioned semialgebraic-steep-planes elimination method. The combination of all of these techniques in effect allows us to work with *three successive scales* of our inductive procedure.

We now state the main results of the paper. Given an interval $\Lambda \subset \mathbb{Z}$ we use $E_j^\Lambda(x, \omega)$, $\psi_j^\Lambda(x, \omega)$ to denote the eigenpairs of $H_\Lambda(x, \omega)$, with $\psi_j^\Lambda(x, \omega)$ a unit vector. When $\Lambda = [1, N]$ we use the shorter notation $H_N(x, \omega)$, $E_j^{(N)}(x, \omega)$, $\psi_j^{(N)}(x, \omega)$.

Theorem A *Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$. There exists $\sigma = \sigma(a, b)$ and a set Ω_N ,*

$$\text{mes}(\Omega_N) < \exp(-(\log N)^{\varepsilon\sigma}),$$

such that for $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$ there exists a set $\mathcal{B}_{N,\omega}$,

$$\text{mes}(\mathcal{B}_{N,\omega}) < \exp(-\exp((\log N)^{\varepsilon\sigma}))$$

and the following holds. For any $N \geq N_0(V, a, b, \gamma, \varepsilon)$, $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $x \in \mathbb{T}^d \setminus \mathcal{B}_{N,\omega}$, and any eigenvalue $E = E_j^{(N)}(x, \omega)$, such that $L(\omega, E) > \gamma$, there exists an interval

$$I = I(x, \omega, E) \subset [1, N], \quad |I| < \exp((\log N)^{5\varepsilon}),$$

such that

$$\left| \psi_j^{(N)}(x, \omega; m) \right| < \exp\left(-\frac{\gamma}{4} \text{dist}(m, I)\right),$$

provided $\text{dist}(m, I) > \exp((\log N)^{2\varepsilon\sigma})$.

Theorem B *Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$, and Ω_N , $\mathcal{B}_{N,\omega}$ as in Theorem A. For any $N \geq N_0(V, a, b, \gamma, \varepsilon)$, $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $x \in \mathbb{T}^d \setminus \mathcal{B}_{N,\omega}$, and any eigenvalue $E = E_j^{(N)}(x, \omega)$, such that $L(\omega, E) > \gamma$, we have*

$$\left| E_k^{(N)}(x, \omega) - E_j^{(N)}(x, \omega) \right| > \exp(-C(V)|I|)$$

for any $k \neq j$, with $I = I(x, \omega, E)$ as in Theorem A.

Similarly to [13], finite scale localization and separation of eigenvalues allows us to give an effective, quantitative, and detailed description of the spectrum on the whole lattice $\mathbb{Z}$ in terms of the spectrum on finite volume. This is achieved for eigenpairs in Theorem C and for spectral sets in Theorem D. This kind of results are crucial for carrying out a spectral analysis along the lines of [13]. In what follows, when we take the norm of vectors like $\psi_j^{(N)}$, we will always use the ℓ^2 norm.

Theorem C Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$. With $\Omega_N, \mathcal{B}_{N,\omega}$ as in Theorem A, applied on $[-N, N]$ instead of $[1, N]$, let $\hat{\Omega}_{N_0} = \bigcup_{N \geq N_0} \Omega_N$, $\hat{\mathcal{B}}_{N_0,\omega} = \bigcup_{N \geq N_0} \mathcal{B}_{N,\omega}$. The following statements hold for any $N_0 \geq C(V, a, b, \gamma, \varepsilon)$, $\omega \in \mathbb{T}^d(a, b) \setminus \hat{\Omega}_{N_0}$, $x \in \mathbb{T}^d \setminus \hat{\mathcal{B}}_{N_0,\omega}$.

- (a) Let $N_k = N^{2^k}$. If $L(\omega, E_j^{[-N,N]}(x, \omega)) > 2\gamma$ and $I = I(x, \omega, E_j^{[-N,N]}(x, \omega)) \subset [-N/2, N/2]$, then for each $k \geq 1$ there exist j_k such that

$$\begin{aligned} |E_{j_k}^{[-N_k, N_k]}(x, \omega) - E_j^{[-N, N]}(x, \omega)| &< \exp\left(-\frac{\gamma}{10}N\right), \\ \|\psi_{j_k}^{[-N_k, N_k]}(x, \omega) - \psi_j^{[-N, N]}(x, \omega)\| &< \exp\left(-\frac{\gamma}{10}N\right), \end{aligned} \quad (1.4)$$

$$|\psi_{j_k}^{[-N_k, N_k]}(x, \omega; n)| < \exp\left(-\frac{\gamma}{20}\text{dist}(n, I)\right), \quad 3N/4 \leq |n| \leq N_k. \quad (1.5)$$

Furthermore, for any $k' \geq k \geq 0$,

$$\begin{aligned} |E_{j_{k'}}^{[-N_{k'}, N_{k'}]}(x, \omega) - E_{j_k}^{[-N_k, N_k]}(x, \omega)| &< \exp\left(-\frac{\gamma}{10}N_k\right), \\ \|\psi_{j_{k'}}^{[-N_{k'}, N_{k'}]}(x, \omega) - \psi_{j_k}^{[-N_k, N_k]}(x, \omega)\| &< \exp\left(-\frac{\gamma}{10}N_k\right). \end{aligned} \quad (1.6)$$

In particular, the limits

$$E(x, \omega) := \lim_{k \rightarrow \infty} E_{j_k}^{[-N_k, N_k]}(x, \omega), \quad \psi(x, \omega; n) := \lim_{k \rightarrow \infty} \psi_{j_k}^{[-N_k, N_k]}(x, \omega; n), \quad n \in \mathbb{Z}$$

exist, $\|\psi\| = 1$, and

$$|\psi(x, \omega; n)| < \exp\left(-\frac{\gamma}{20}\text{dist}(n, I)\right), \quad 3N/4 \leq |n|. \quad (1.7)$$

- (b) If $L(\omega, E) > 3\gamma$ for all $E \in [E', E'']$, then $P([E', E''])H(x, \omega)P([E', E''])$ has purely pure point spectrum and all the eigenpairs, with unit eigenvector, can be obtained from (a).

By the minimality of the rationally-independent shift on $\mathbb{T}^d$ the spectrum of $H(x, \omega)$ does not depend on x . We denote it by $\mathcal{S}_\omega$. Let

$$\mathcal{S}_{N,\omega} = \bigcup_{x \in \mathbb{T}^d} \text{spec } H_{[-N, N]}(x, \omega).$$

We would like to say that $\mathcal{S}_{N,\omega}$ approximates $\mathcal{S}_\omega$ as $N \rightarrow \infty$. However, it turns out that there can exist large segments in $\mathcal{S}_{N,\omega} \setminus \mathcal{S}_\omega$ that persist as $N \rightarrow \infty$. This segments correspond to eigenvalues that are localized near the edges of $[-N, N]$. We will argue that if we focus only on the eigenvalues localized away from the edges (as in Theorem C), we do get a finite

scale approximation of the spectrum. To this end we replace $S_{N,\omega}$ by a set defined as follows. Let $N \geq 1$, $s > 1$, $k_0 \geq 0$ integers, $\rho = (\rho_0, \dots, \rho_{k_0}) \in \mathbb{R}^{k_0+1}$. We define

$$\mathfrak{S}_{N,\omega}(s, k_0, \rho) = \bigcup_{x \in \mathbb{T}^d} \left(\bigcap_{0 \leq k \leq k_0} (\text{spec } H_{[-N^{(k)}, N^{(k)}]}(x, \omega))^{(\rho_k)} \right), \quad N^{(k)} = N^{s^k}. \quad (1.8)$$

Note that given a set S and $\sigma > 0$, we let

$$S^{(\rho)} = \{x : \text{dist}(x, S) < \sigma\}. \quad (1.9)$$

The motivation behind (1.8) is the fact that any eigenvalue $E_j^{[-N, N]}(x, \omega)$ that is localized away from the edges of $[-N, N]$ is an approximate eigenvalue on any larger interval $[-N', N']$, namely $E_j^{[-N, N]}(x, \omega) \in (\text{spec } H_{[-N', N']}(x, \omega))^{(\sigma)}$, $\sigma = \exp(-cN)$. To fully justify the use of (1.8) we would also need to argue that any energy in $\mathfrak{S}_{N,\omega}(s, k_0, \rho)$ is at least close to an eigenvalue localized away from the edges. This leads to the following type of problem. If $B \subset \mathbb{T}^d$ is a set with small measure, does $E_j^{[-N, N]}(B, \omega)$ also have small measure? This is true for $d = 1$, but it is generally false for $d > 1$. To work around this type of issues we take advantage again of Bourgain's elimination of multiple resonances. As a trade-off we have to be quite particular about the choice of parameters s, k_0, ρ . So, for example we have to take $k_0 = 2^{2d+1} - 1$ instead of the natural choice $k_0 = 1$ (which works only for $d = 1$). However, such technicalities have no effect on the applicability of the result.

Theorem D *Let $\gamma > 0$, $A \geq 1$. For any $N \geq N_0(V, a, b, \gamma, A)$, $s \geq s_0(a, b, A)$, there exists a set $\Omega_N = \Omega_N(s)$, $\text{mes}(\Omega_N) < N^{-A}$, such that the following holds with*

$$\begin{aligned} \mathfrak{S}_{N,\omega} &= \mathfrak{S}_{N,\omega}(s, k_0, \rho_N), \quad k_0 = 2^{2d+1} - 1, \\ \rho_{N,0} &= \exp(-N^{c(a,b)}), \quad \rho_{N,k} = \exp\left(-\frac{\gamma}{10}N\right), \quad k = 1, \dots, k_0. \end{aligned}$$

If $\omega \in \mathbb{T}^d \setminus \Omega_N$ and $L(\omega, E) > \gamma$ for all $E \in [E', E'']$, then

$$\mathcal{S}_\omega \cap [E', E''] \subset \mathfrak{S}_{N,\omega}, \quad (1.10)$$

$$\text{mes}((\mathfrak{S}_{N,\omega} \cap [E', E'']) \setminus \mathcal{S}_\omega) < \exp\left(-\frac{\gamma}{20}N\right). \quad (1.11)$$

Finally, as an application we prove the local homogeneity of the spectrum at a supercritical energy. In particular, this implies that if the Lyapunov exponent is positive for all energies, then the spectrum is homogeneous (in the sense of Carleson; cf. [10]). It would be interesting to establish such a result for all Diophantine frequency vectors.

Theorem E *Let $\gamma > 0$. For any $N \geq N_0(V, a, b, \gamma)$ there exists a set Ω_N , $\text{mes}(\Omega_N) < N^{-1}$, and a constant $\delta_0 = \delta_0(a, b, N)$ such that the following holds. If $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $E \in \mathcal{S}_\omega$, and $L(\omega, E) > \gamma$, then*

$$\text{mes}(\mathcal{S}_\omega \cap (E - \delta, E + \delta)) > \frac{\delta}{2},$$

for any $\delta \in (0, \delta_0]$.

2 Basic Tools

In this section we discuss the basic results we will employ throughout the paper. Many of them can be traced back to [11] and [12], and we will refer to these papers for some of the

proofs. Some of the results were originally derived only for the single frequency case with a strong Diophantine condition, however they easily extend to our more general context. For the convenience of the reader we will sketch the proofs of such results.

We start by recalling some basic information related to the transfer matrix and the Lyapunov exponent. If ψ is a solution of the difference equation

$$-\psi(n+1) - \psi(n-1) + V(x+n\omega)\psi(n) = E\psi(n), \quad n \in \mathbb{Z} \quad (2.1)$$

then for any $a < b$ we have

$$\begin{bmatrix} \psi(b+1) \\ \psi(b) \end{bmatrix} = M_{[a,b]}(x, \omega, E) \begin{bmatrix} \psi(a) \\ \psi(a-1) \end{bmatrix} \quad (2.2)$$

where the transfer matrix is given by

$$M_{[a,b]}(x, \omega, E) = \prod_{n=b}^a \begin{bmatrix} V(x+n\omega) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

The transfer matrix is related to the Dirichlet determinants

$$f_{[a,b]}(x, \omega, E) := \det(H_{[a,b]}(x, \omega) - E)$$

through the following formula

$$M_{[a,b]}(x, \omega, E) = \begin{bmatrix} f_{[a,b]}(x, \omega, E) & -f_{[a+1,b]}(x, \omega, E) \\ f_{[a,b-1]}(x, \omega, E) & -f_{[a+1,b-1]}(x, \omega, E) \end{bmatrix}. \quad (2.3)$$

We let $M_N = M_{[1,N]}$ and

$$L_N(\omega, E) = \frac{1}{N} \int_{\mathbb{T}^d} \log \|M_N(x, \omega, E)\| dx.$$

The sequence L_N is subadditive and the *Lyapunov exponent* is defined as

$$L(\omega, E) = \lim_{N \rightarrow \infty} L_N(\omega, E) = \inf_N L_N(\omega, E).$$

Note that

$$0 \leq \log \|M_N(x, \omega, E)\| \leq C(V, |E|)N \quad (2.4)$$

and therefore

$$0 \leq L_N(\omega, E) \leq C(V, |E|). \quad (2.5)$$

The lower bound in (2.4) is due to the fact that $\det M_N(x, \omega, E) = 1$.

It is known that V extends to be complex analytic on a set

$$\mathcal{A}_\rho := \{x + iy : x \in \mathbb{T}^d, y \in \mathbb{R}^d, |y| < \rho\},$$

with $\rho = \rho(V)$. We use $|\cdot|$ to denote the Euclidean norm. Let

$$\begin{aligned} L_N(y, \omega, E) &= \frac{1}{N} \int_{\mathbb{T}^d} \log \|M_N(x + iy, \omega, E)\| dx, \\ L(y, \omega, E) &= \lim_{N \rightarrow \infty} L_N(y, \omega, E). \end{aligned}$$

For $y = 0$ we reserve the notation $L(\omega, E)$. Most of the results in this section *do not use* the fact that V assumes only real values on the torus $\mathbb{T}^d$ and therefore, they also hold on $\mathbb{T}^d + iy, |y| < \rho$, with $L(y, \omega, E)$ instead of $L(\omega, E)$. In particular, we care that this applies to all the results up to and including the uniform upper estimates in Section 2.4.

Remark 2.1 We briefly comment on the use of constants. Unless stated otherwise, the constants denoted by C might have different values each time they are used. They will be allowed to depend on γ, ω, V, E, d . In most cases we leave the dependence on d implicit. The dependence on V will be through $\rho(V)$ and the sup norm of V on $\mathcal{A}_\rho, \|V\|_\infty$. The dependence on ω will be only on the parameters a, b of the Diophantine condition. Constants depending on E can be chosen uniformly for E in a bounded set. We let $a \lesssim b$ denote $a \leq Cb$ with some positive C and $a \ll b$ denote $a \leq C^{-1}b$ with a sufficiently large positive C . Finally, $a \simeq b$ stands for $a \lesssim b$ and $b \lesssim a$. It will be clear from the context what the implicit constants are allowed to depend on.

2.1 Large Deviations Estimates

The following result, called Large Deviations Theorem (LDT) is the most basic tool in the localization theory. We refer to [4, 11] for two different approaches to its proof.

Theorem 2.2 Assume $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$. There exist $\sigma = \sigma(a, b)$, $\tau = \tau(a, b)$, $\sigma, \tau \in (0, 1)$, such that for $N \geq N_0(V, a, b, |E|)$ one has

$$\text{mes} \left\{ x \in \mathbb{T}^d : |\log \|M_N(x, \omega, E)\| - NL_N(\omega, E)| > N^{1-\tau} \right\} < \exp(-N^\sigma).$$

In [12] it was shown (see [12, Prop. 2.11]) that in the regime of positive Lyapunov exponent, the large deviations estimate extends to the entries of the transfer matrix.

Theorem 2.3 Assume $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. There exist $\sigma = \sigma(a, b)$, $\tau = \tau(a, b)$, $\sigma, \tau \in (0, 1)$, such that for $N \geq N_0(V, a, b, |E|, \gamma)$ one has

$$\text{mes} \left\{ x \in \mathbb{T}^d : |\log |f_N(x, \omega, E)| - NL_N(\omega, E)| > N^{1-\tau} \right\} < \exp(-N^\sigma).$$

The constants σ, τ in the (LDT) for determinants depend on the σ, τ from the (LDT) for the transfer matrix. Since the sharpness of these constants plays no role in our work, we choose them to be the same (by making the constants in Theorem 2.2 smaller). We will refer to either of the deviations estimates as (LDT).

2.2 The Avalanche Principle

The following statement, known as the Avalanche Principle (AP), is another basic tool in the theory of quasi-periodic Schrödinger operators.

Proposition 2.4 [11, Proposition 2.2] Let $A_1, \dots, A_n$ be a sequence of 2×2 -matrices whose determinants satisfy

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1.$$

Suppose that

$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n \quad \text{and}$$

$$\max_{1 \leq j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|] < \frac{1}{2} \log \mu.$$

Then

$$\left| \log \|A_n \dots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C \frac{n}{\mu}$$

with some absolute constant C .

2.3 Estimates for the Lyapunov Exponent

Using (LDT) and (AP) one gets the following estimate on the rate of convergence for the finite scale Lyapunov exponent.

Proposition 2.5 ([11, Lemma 10.1]) *Assume $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. Then for any $N \geq 2$,*

$$0 \leq L_N(\omega, E) - L(\omega, E) < C \frac{(\log N)^{1/\sigma}}{N},$$

where $C = C(V, a, b, |E|, \gamma)$ and σ is as in (LDT).

We will need some estimates on the modulus of continuity of the Lyapunov exponent. A first rough estimate can be obtained from the next lemma.

Lemma 2.6 *Let $N \geq 1$, $(z_i, w_i, E_i) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$, $i = 1, 2$, such that*

$$|\operatorname{Im} z_i|, N|\operatorname{Im} w_i| < \rho(V).$$

Then

$$\begin{aligned} & \|M_N(z_1, w_1, E_1) - M_N(z_2, w_2, E_2)\| \\ & \leq (C(V) + |E_1| + |E_2|)^N (|z_1 - z_2| + |w_1 - w_2| + |E_1 - E_2|). \end{aligned} \quad (2.6)$$

In particular, we have

$$\begin{aligned} & |\log \|M_N(z_1, w_1, E_1)\| - \log \|M_N(z_2, w_2, E_2)\|| \\ & \leq (C(V) + |E_1| + |E_2|)^N (|z_1 - z_2| + |w_1 - w_2| + |E_1 - E_2|). \end{aligned}$$

Proof Let

$$A_{i,n} = \begin{bmatrix} V(z_i + nw_i) - E_i & -1 \\ 1 & 0 \end{bmatrix}, \quad i = 1, 2.$$

Then

$$\begin{aligned} & M_N(z_1, w_1, E_2) - M_N(z_2, w_2, E_2) \\ & = \sum_{n=1}^N A_{2,N} \cdots A_{2,n+1} (A_{1,n} - A_{2,n}) A_{1,n-1} \cdots A_{1,1} \\ & = \sum_{n=1}^N M_{N-n}(z_2 + (n+1)w_2, w_2, E_2) (A_{1,n} - A_{2,n}) M_{n-1}(z_1, w_1, E_1). \end{aligned} \quad (2.7)$$

Now (2.6) follows by using the Mean Value Theorem and the fact that we clearly have

$$\|A_{i,n}\| \leq C(V) + |E_i|.$$

The second inequality follows from (2.6) and the fact that $|\log x - \log y| \leq |x - y|$, provided $x, y \geq 1$. Indeed, we have

$$\begin{aligned} |\log \|M_N(z_1, w_1, E_1)\| - \log \|M_N(z_2, w_2, E_2)\|| & \lesssim \left| \frac{\|M_N(z_1, w_1, E_1)\|}{\|M_N(z_2, w_2, E_2)\|} - 1 \right| \\ & \leq \frac{\|M_N(z_1, w_1, E_1) - M_N(z_2, w_2, E_2)\|}{\|M_N(z_2, w_2, E_2)\|} \end{aligned}$$

The conclusion follows by recalling that

$$\|M_N(z, w, E)\| \geq 1$$

for any z, w, E . □

We refine the result of the previous lemma using (AP) and then we deduce our estimate on the modulus of continuity for $L_N(\omega, E)$.

Lemma 2.7 *Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ be as in (LDT) and let A be a constant such that $\sigma A \geq 1$. Then for all $N \geq N_0(V, a, b, |E_0|, \gamma, A)$,*

$$|\log \|M_N(z, w, E)\| - \log \|M_N(x_0, \omega_0, E_0)\|| < \exp\left(-(\log N)^A\right) \quad (2.8)$$

for any $x_0 \in \mathbb{T}^d \setminus \mathcal{B}_{N, \omega_0, E_0}$, $\text{mes}(\mathcal{B}_{N, \omega_0, E_0}) < \exp(-(\log N)^{\sigma A})$, and $(z, w, E) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$ such that

$$|z - x_0|, |w - \omega_0|, |E - E_0| < \exp\left(-(\log N)^{4A}\right). \quad (2.9)$$

Proof Let $\ell \simeq (\log N)^{A+1}$, $n = [N/\ell]$,

$$A_j = A_j(x, \omega, E) = M_\ell(x + (j-1)\ell\omega, E).$$

To simplify notation we assume that $N = n\ell$. It is easy to see how to adjust the argument for general N .

Using (LDT) and Proposition 2.5 we get that

$$\begin{aligned} \min_{1 \leq j \leq n} \|A_j(x_0, \omega_0, E_0)\| &\geq \exp(\ell L_\ell(\omega_0, E_0) - \ell^{1-\tau}) \geq \exp(\ell\gamma/2) > n, \\ \max_{0 \leq j < n} [\log \|A_{j+1}(x_0, \omega_0, E_0)\| + \log \|A_j(x_0, \omega_0, E_0)\| \\ &\quad - \log \|A_{j+1}(x_0, \omega_0, E_0)A_j(x_0, \omega_0, E_0)\|] \leq \frac{\gamma}{16}\ell, \end{aligned}$$

for all x_0 outside a set $\mathcal{B}_{N, \omega_0, E_0} \subset \mathbb{T}^d$ with

$$\text{mes}(\mathcal{B}_{N, \omega_0, E_0}) \lesssim n \exp(-\ell^\sigma) < \exp(-(\log N)^{\sigma A})$$

(the last inequality holds provided that $\sigma A \geq 1$). Take $x_0 \in \mathbb{T}^d \setminus \mathcal{B}_{N, E_0, \omega_0}$. We apply (AP) with $\mu = \exp(\ell\gamma/2)$ to get

$$\begin{aligned} \left| \log \|M_N(x_0, \omega_0, E_0)\| + \sum_{j=2}^{n-1} \log \|A_j(x_0, \omega_0, E_0)\| \right. \\ \left. - \sum_{j=1}^{n-1} \log \|A_{j+1}(x_0, \omega_0, E_0)A_j(x_0, \omega_0, E_0)\| \right| < Cn \exp(-\gamma\ell/2). \quad (2.10) \end{aligned}$$

Take z, w, E satisfying (2.9). Using Lemma 2.6 it follows that

$$\begin{aligned} \min_{1 \leq j \leq n} \|A_j(z, w, E)\| &\geq \exp(\ell\gamma/4) > n, \\ \max_{0 \leq j < n} [\log \|A_{j+1}(z, w, E)\| + \log \|A_j(z, w, E)\| - \log \|A_{j+1}(z, w, E)A_j(z, w, E)\|] &\leq \frac{\gamma}{8}\ell. \end{aligned}$$

We apply (AP) again, this time with $\mu = \exp(\gamma\ell/4)$, to get

$$\left| \log \|M_N(z, w, E)\| + \sum_{j=2}^{n-1} \log \|A_j(z, w, E)\| - \sum_{j=1}^{n-1} \log \|A_{j+1}(z, w, E)A_j(z, w, E)\| \right| < Cn \exp(-\gamma\ell/4). \quad (2.11)$$

Subtracting (2.10) from (2.11) and applying Lemma 2.6, term-wise, yields (2.8). $\square$

Proposition 2.8 Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ be as in (LDT) and let A be a constant such that $\sigma A \geq 1$. Then for all $N \geq N_0(V, a, b, |E_0|, \gamma, A)$,

$$|L_N(\omega, E) - L_N(\omega_0, E_0)| < \exp\left(-(\log N)^{\sigma A}\right)$$

provided $|\omega - \omega_0|, |E - E_0| \leq \exp\left(-(\log N)^{4A}\right)$.

Proof We have

$$|L_N(\omega, E) - L_N(\omega_0, E_0)| \leq \frac{1}{N} \int_{\mathbb{T}^d} |\log \|M_N(x, \omega, E)\| - \log \|M_N(x, \omega_0, E_0)\|| dx$$

and the conclusion follows from Lemma 2.7. $\square$

As a consequence we obtain the following control on the modulus of continuity of the Lyapunov exponent.

Proposition 2.9 Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. There exists $\varepsilon_0 = \varepsilon_0(V, a, b, |E_0|, \gamma)$ such that if $|E - E_0| < \varepsilon_0$, then $L(\omega_0, E) > \gamma/2$ and

$$|L(\omega_0, E) - L(\omega_0, E_0)| \leq \exp\left(-\frac{1}{2}(-\log |E - E_0|)^{\sigma/4}\right),$$

with σ as in (LDT). Furthermore, if $\omega \in \mathbb{T}^d(a, b) \cap (\omega_0 - \varepsilon_0, \omega_0 + \varepsilon_0)$, then $L(\omega, E_0) > \gamma/2$ and

$$|L(\omega, E_0) - L(\omega_0, E_0)| \leq \exp\left(-\frac{1}{2}(-\log |\omega - \omega_0|)^{\sigma/4}\right).$$

Proof From Propositions 2.5 and 2.8 we have that

$$\begin{aligned} |L(\omega_0, E) - L(\omega_0, E_0)| &< C \frac{(\log N)^{1/\sigma}}{N} \leq \exp\left(-\frac{1}{2} \log(N+1)\right) \\ &\leq \exp\left(-\frac{1}{2}(-\log |E - E_0|)^{\sigma/4}\right) < \frac{\gamma}{2}, \end{aligned}$$

provided

$$\exp(-(\log(N+1))^{4/\sigma}) \leq |E - E_0| \leq \exp(-(\log N)^{4/\sigma})$$

and $N \geq N_0(V, a, b, |E_0|, \gamma)$. The first statement follows by setting

$$\varepsilon_0 = \exp(-(\log N_0)^{4/\sigma}).$$

The second statement follows in the same way (note that we need $\omega \in \mathbb{T}^d(a, b)$ to use Proposition 2.5). $\square$

Remark 2.10 The above result is essentially proved in [11, Prop. 10.2], however the existence of the interval $(E_0 - \varepsilon_0, E_0 + \varepsilon_0)$ is not covered explicitly there.

We are also interested in the modulus of continuity of $L_N(y, \omega, E)$ with respect to y . We will show that L_N is in fact Lipschitz in y . The proof will be based on the following fact.

Lemma 2.11 ([12, Lemma 4.1]) *Let $1 > \rho > 0$ and suppose u is subharmonic on*

$$\mathcal{A}_\rho := \{x + iy : x \in \mathbb{T}, |y| < \rho\},$$

such that $\sup_{\mathcal{A}_\rho} u \leq 1$ and $\int_{\mathbb{T}} u(x) dx \geq 0$. Then for any y, y' so that $-\frac{\rho}{2} < y, y' < \frac{\rho}{2}$ one has

$$\left| \int_{\mathbb{T}} u(x + iy) dx - \int_{\mathbb{T}} u(x + iy') dx \right| \leq C_\rho |y - y'|.$$

Corollary 2.12 *Let $\omega \in \mathbb{T}^d$, $E \in \mathbb{C}$. There exists $C = C(V, |E|)$ such that*

$$|L_N(y, \omega, E) - L_N(\omega, E)| \leq C \sum_{i=1}^d |y_i| \quad \text{for all } |y| < \rho(V),$$

uniformly in N . In particular, the same bound holds with L instead of L_N .

Proof If $d = 1$, then the statement follows directly from Lemma 2.11 applied to $u(z) = \frac{1}{CN} \log \|M_N(z, \omega, E)\|$ (with C as in (2.4)). Let us verify the statement for $d = 2$. Let

$$v_{y_2}(x_1 + iy_1) := \int_{\mathbb{T}} \frac{1}{CN} \log \|M_N(x_1 + iy_1, x_2 + iy_2, \omega, E)\| dx_2.$$

By Lemma 2.11

$$|v_{y_2}(x_1 + iy_1) - v_0(x_1 + iy_1)| \leq C|y_2|.$$

Since v_{y_2} is subharmonic for any fixed y_2 , Lemma 2.11 also implies

$$\left| \int_{\mathbb{T}} v_{y_2}(x_1 + iy_1) dx_1 - \int_{\mathbb{T}} v_{y_2}(x_1) dx_1 \right| \leq C|y_1|.$$

So we have

$$\begin{aligned} |L_N(\omega, E) - L_N(y, \omega, E)| &= \left| \int_{\mathbb{T}} v_0(x_1) dx_1 - \int_{\mathbb{T}} v_{y_2}(x_1 + iy_1) dx_1 \right| \\ &\leq \left| \int_{\mathbb{T}} (v_0(x_1) - v_0(x_1 + iy_1)) dx_1 \right| \\ &\quad + \left| \int_{\mathbb{T}} (v_0(x_1 + iy_1) - v_{y_2}(x_1 + iy_1)) dx_1 \right| \\ &\leq C(|y_1| + |y_2|). \end{aligned}$$

For general d the proof is completely similar with help of induction over d . □

2.4 Uniform Upper Estimates

The upper bound from (2.4) can be improved by invoking the sub-mean value property for pluri-subharmonic functions.

Proposition 2.13 Assume $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. Then for all $N \geq 1$,

$$\sup_{x \in \mathbb{T}^d} \log \|M_N(x, \omega, E)\| \leq NL_N(\omega, E) + CN^{1-\tau}, \quad (2.12)$$

with $C = C(V, a, b, |E|, \gamma)$ and τ as in (LDT).

Proof We only check (2.12) for N large enough, for smaller N we simply choose C large enough. By (LDT),

$$\text{mes} \left\{ x \in \mathbb{T}^d : |\log \|M_N(x + iy, \omega, E)\| - NL_N(y, \omega, E)| > N^{1-\tau} \right\} < \exp(-N^\sigma),$$

and using Corollary 2.12 we have

$$\text{mes} \left\{ x \in \mathbb{T}^d : |\log \|M_N(x + iy, \omega, E)\| - NL_N(\omega, E)| > 2N^{1-\tau} \right\} < \exp(-N^\sigma), \quad (2.13)$$

provided $|y| \leq 1/N$. Applying the sub-mean value property for subharmonic functions iteratively, we have

$$\log \|M_N(x, \omega, E)\| \leq (\pi r^2)^{-d} \int_{\mathcal{P}} \log \|M_N(\xi + iy, \omega, E)\| d\xi dy, \quad (2.14)$$

where $\mathcal{P} = \prod \mathcal{D}(x_j, r) \subset \mathbb{C}^d$, $x = (x_1, \dots, x_d)$, $r = N^{-1}$. Here $\mathcal{D}(x_j, r) \subset \mathbb{C}$ is the Euclidean disk centered at $x_j \in \mathbb{R}$. Denote by $\mathcal{B}_y \subset \mathbb{T}^d$ the set in (2.13). Let

$$\mathcal{B} = \{(\xi, y) \in [0, 1]^d \times (-r, r)^d : \xi \in \mathcal{B}_y\}.$$

Due to (2.13) we have

$$(\pi r^2)^{-d} \int_{\mathcal{P} \setminus \mathcal{B}} \log \|M_N(\xi + iy, \omega, E)\| d\xi dy \leq NL_N(\omega, E) + 2N^{1-\tau}. \quad (2.15)$$

On the other hand, due to (2.4)

$$(\pi r^2)^{-d} \int_{\mathcal{P} \cap \mathcal{B}} \log \|M_N(\xi + iy, \omega, E)\| d\xi dy \leq (\pi r^2)^{-d} CN \text{mes}(\mathcal{B}) < \exp(-N^\sigma/2). \quad (2.16)$$

The conclusion follows by combining (2.14), (2.15), and (2.16). $\square$

For the purpose of Cartan's estimate (see Section 2.5) we need to extend the uniform estimate to complex neighborhoods of (x, ω, E) . It is crucial that the size of the neighborhood in the next result is much larger than what can be obtained by simply applying Lemma 2.6.

Corollary 2.14 Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ, τ as in (LDT). For all $N \geq N_0(V, a, b, |E_0|, \gamma)$ and $(y, w, E) \in \mathbb{R}^d \times \mathbb{C}^d \times \mathbb{C}$ such that

$$|y| < 1/N, \quad |w - \omega_0|, |E - E_0| < \exp(-(\log N)^{8/\sigma}),$$

we have

$$\sup_{x \in \mathbb{T}^d} \log \|M_N(x + iy, w, E)\| \leq NL_N(\omega_0, E_0) + CN^{1-\tau},$$

with $C = C(V, a, b, |E_0|, \gamma)$. In particular, we also have

$$\sup_{x \in \mathbb{T}^d} \log |f_N(x + iy, w, E)| \leq NL_N(\omega_0, E_0) + CN^{1-\tau}.$$

Proof Take y, w, E satisfying the assumptions. Due to Proposition 2.13 and Lemma 2.7 (with $A = 2/\sigma$; both results are applied on $\mathbb{T}^d + iy$), we have that

$$\log \|M_N(x + iy, w, E)\| \leq NL_N(y, \omega_0, E_0) + CN^{1-\tau}$$

for any $x \in \mathbb{T}^d \setminus \mathcal{B}_y$, $\text{mes}(\mathcal{B}_y) < \exp(-(\log N)^2)$. Now we can employ the sub-mean value property for subharmonic functions just as in the proof of Proposition 2.13 and the conclusion follows. $\square$

Another consequence of the uniform upper estimate is an improvement of the stability estimate from Lemma 2.6.

Corollary 2.15 Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ, τ as in (LDT). For all $N \geq N_0(V, a, b, |E_0|, \gamma)$ and $(z_i, w_i, E_i) \in \mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}$ such that

$$|\text{Im } z_i| < 1/N, \quad |w_i - \omega_0|, |E_i - E_0| < \exp(-(\log N)^{8/\sigma}), \quad i = 1, 2$$

we have

$$\begin{aligned} & \|M_N(z_1, w_1, E_1) - M_N(z_2, w_2, E_2)\| \\ & \leq (|z_1 - z_2| + |w_1 - w_2| + |E_1 - E_2|) \exp(NL(\omega_0, E_0) + CN^{1-\tau}), \end{aligned}$$

with $C = C(V, a, b, |E_0|, \gamma)$. In particular,

$$\begin{aligned} & |\log \|M_N(z_1, w_1, E_1)\| - \log \|M_N(z_2, w_2, E_2)\|| \\ & \leq (|z_1 - z_2| + |w_1 - w_2| + |E_1 - E_2|) \exp(NL(\omega_0, E_0) + CN^{1-\tau}), \\ & |\log |f_N(z_1, w_1, E_1)| - \log |f_N(z_2, w_2, E_2)|| \\ & \leq (|z_1 - z_2| + |w_1 - w_2| + |E_1 - E_2|) \exp \frac{(NL(\omega_0, E_0) + CN^{1-\tau})}{\max_i |f_N(z_i, w_i, E_i)|}, \end{aligned} \quad (2.17)$$

provided the right-hand side of (2.17) is less than $1/2$.

Proof The proof is completely analogous to that of Lemma 2.6. The only difference is that now we can use Corollary 2.14 to bound the M_{N-n}, M_{n-1} factors in (2.7). $\square$

In a similar manner one can treat the stability of M_N, f_N under a change of the potential. Let $\tilde{M}_N$ and $\tilde{f}_N$ denote the transfer matrices and the Dirichlet determinants associated to the operator having $\tilde{V}$ as a potential instead of V .

Lemma 2.16 Assume $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. Then for any $x \in \mathbb{T}^d$ and $N \geq 1$ we have

$$\|M_N(x, \omega, E) - \tilde{M}_N(x, \omega, E)\| \leq \|V - \tilde{V}\|_\infty \exp(NL(\omega, E) + CN^{1-\tau}),$$

with $C = C(V, a, b, |E|, \gamma)$. In particular,

$$\begin{aligned} & |\log \|M_N(x, \omega, E)\| - \log \|\tilde{M}_N(x, \omega, E)\|| \leq \|V - \tilde{V}\|_\infty \exp(NL(\omega, E) + CN^{1-\tau}), \\ & |\log |f_N(x, \omega, E)| - \log |\tilde{f}_N(x, \omega, E)|| \leq \|V - \tilde{V}\|_\infty \frac{\exp(NL(\omega, E) + CN^{1-\tau})}{\max(|f_N(x, \omega, E)|, |\tilde{f}_N(x, \omega, E)|)}, \end{aligned}$$

provided the right-hand sides are less than $1/2$.

2.5 Cartan's Estimate

We adopt the definition of Cartan sets from [12].

Definition 2.17 Let $H \gg 1$. For an arbitrary subset $\mathcal{B} \subset \mathcal{D}(z_0, 1) \subset \mathbb{C}$ we say that $\mathcal{B} \in \text{Car}_1(H, K)$ if $\mathcal{B} \subset \bigcup_{j=1}^{j_0} \mathcal{D}(z_j, r_j)$ with $j_0 \leq K$, and

$$\sum_j r_j < e^{-H}.$$

If d is a positive integer greater than 1, and $\mathcal{B} \subset \prod_{i=1}^d \mathcal{D}(z_{i,0}, 1) \subset \mathbb{C}^d$ then we define inductively that $\mathcal{B} \in \text{Car}_d(H, K)$ if for any $1 \leq j \leq d$ there exists $\mathcal{B}_j \subset \mathcal{D}(z_{j,0}, 1) \subset \mathbb{C}$, $\mathcal{B}_j \in \text{Car}_1(H, K)$ so that $\mathcal{B}_z^{(j)} \in \text{Car}_{d-1}(H, K)$ for any $z \in \mathcal{D}(z_{j,0}, 1) \setminus \mathcal{B}_j$, with $\mathcal{B}_z^{(j)} = \{(z_1, \dots, z_d) \in \mathcal{B} : z_j = z\}$.

The above definition appears naturally from the proof of the following generalization of the usual Cartan estimate (see [19, Lecture 11]) to several variables.

Lemma 2.18 ([12, Lemma 2.15]) *Let $\varphi(z_1, \dots, z_d)$ be an analytic function defined in a polydisk $\mathcal{P} = \prod_{j=1}^d \mathcal{D}(z_{j,0}, 1)$, $z_{j,0} \in \mathbb{C}$. Let $M \geq \sup_{\underline{z} \in \mathcal{P}} \log |\varphi(\underline{z})|$, $m \leq \log |\varphi(\underline{z}_0)|$, $\underline{z}_0 = (z_{1,0}, \dots, z_{d,0})$. Given $H \gg 1$ there exists a set $\mathcal{B} \subset \mathcal{P}$, $\mathcal{B} \in \text{Car}_d(H^{1/d}, K)$, $K = C_d(M - m)$, such that*

$$\log |\varphi(z)| > M - C_d H(M - m)$$

for any $z \in \prod_{j=1}^d \mathcal{D}(z_{j,0}, 1/6) \setminus \mathcal{B}$. Furthermore, when $d = 1$ we can take $K = C(M - m)$ and keep only the disks of $\mathcal{B}$ containing a zero of φ in them.

We note that the definition of the Cartan sets gives implicit information about their measure.

Lemma 2.19 *If $\mathcal{B} \in \text{Car}_d(H, K)$ then*

$$\text{mes}_{\mathbb{C}^d}(\mathcal{B}) \leq C(d)e^{-H} \quad \text{and} \quad \text{mes}_{\mathbb{R}^d}(\mathcal{B} \cap \mathbb{R}^d) \leq C(d)e^{-H}.$$

Proof The case $d = 1$ follows immediately from the definition of Car_1 . The case $d > 1$ follows by induction, using Fubini and the definition of Car_d . $\square$

Cartan's estimate allows us to argue that if the (LDT) estimate fails for $f_N(x, \omega, E)$, then x and E are close to zeros of f_N .

Lemma 2.20 *Assume $x \in \mathbb{T}^d$, $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. Let $H \gg 1$ and τ, σ as in (LDT). There exists $C_0 = C_0(V, a, b, |E|, \gamma)$, such that for any $N \geq N_0(V, a, b, |E|, \gamma)$, if*

$$-\infty < \log |f_N(x, \omega, E)| \leq NL_N(\omega, E) - C_0 H N^{1-\tau},$$

then there exists $z \in \mathbb{C}^d$, $|z - x| \lesssim \exp(-(H + N^\sigma/d))$ such that $f_N(z, \omega, E) = 0$. Furthermore,

$$\|(H_N(x, \omega) - E)^{-1}\| \geq c(V) \exp(H + N^\sigma/d).$$

Proof By (LDT) there exists x_0 , $|x - x_0| \lesssim \exp(-N^\sigma/d)$ such that

$$\log |f_N(x_0, \omega, E)| > NL_N(\omega, E) - N^{1-\tau}.$$

Let

$$\phi(\zeta) = f_N \left(x_0 + \frac{C \exp(-N^\sigma/d)\zeta}{|x - x_0|} (x - x_0), \omega, E \right).$$

Let ζ_x be such that $\phi(\zeta_x) = f_N(x, \omega, E)$. Our choice of scaling is such that $\zeta_x \in \mathcal{D}(0, 1/7)$. Using the uniform upper estimate from Corollary 2.14, we can apply Cartan's estimate to get

$$\log |\phi(\zeta)| > NL_N(\omega, E) - CHN^{1-\tau}$$

for all $\zeta \in \mathcal{D}(0, 1/6) \setminus \mathcal{B}$, with $\mathcal{B} \in \text{Car}_1(H, CHN^{1-\tau})$. It follows that $\zeta_x \in \mathcal{D}(\zeta_j, r_j) \subset \mathcal{B}$, with $r_j < \exp(-H)$. By the second part of Lemma 2.18, there exists $\zeta' \in \mathcal{D}(\zeta_j, r_j)$ such that $\phi(\zeta') = 0$. The first statement follows with

$$z = x_0 + \frac{C \exp(-N^\sigma/d)\zeta'}{|x - x_0|} (x - x_0).$$

The second statement follows from the facts that

$$\|H_N(z, \omega) - H_N(x, \omega)\| \leq C(V)|z - x|$$

and that if

$$\|(H_N(x, \omega) - E)^{-1}\| \|H_N(z, \omega) - H_N(x, \omega)\| < 1,$$

then $H_N(z, \omega) - E$ would be invertible. $\square$

We will need the following immediate consequence of the previous lemma. We refer to this result as the *spectral form* of (LDT).

Corollary 2.21 Assume $x \in \mathbb{T}^d$, $\omega \in \mathbb{T}^d(a, b)$, $E \in \mathbb{C}$, and $L(\omega, E) > \gamma > 0$. Let τ, σ as in (LDT). If $N \geq N_0(V, a, b, |E|, \gamma)$ and

$$\|(H_N(x, \omega) - E)^{-1}\| \leq \exp(N^{\sigma/2}),$$

then

$$\log |f_N(x, \omega, E)| > NL_N(\omega, E) - N^{1-\tau/2}.$$

2.6 Poisson's Formula

Recall that for any solution ψ of the difference equation (2.1), Poisson's formula reads

$$\psi(m) = \mathcal{G}_{[a,b]}(x, \omega, E; m, a)\psi(a-1) + \mathcal{G}_{[a,b]}(x, \omega, E; m, b)\psi(b+1), \quad m \in [a, b], \quad (2.18)$$

where $\mathcal{G}_{[a,b]}(x, \omega, E) = (H_{[a,b]}(x, \omega) - E)^{-1}$ is the Green's function. In particular, if ψ is a solution of equation (2.1), which satisfies a zero boundary condition at the left or the right edge, i.e.,

$$\psi(a-1) = 0 \quad \text{or} \quad \psi(b+1) = 0,$$

then

$$\psi(m) = \mathcal{G}_{[a,b]}(x, \omega, E; m, b)\psi(b+1) \quad \text{or} \quad \psi(m) = \mathcal{G}_{[a,b]}(x, \omega, E; m, a)\psi(a-1).$$

Poisson's formula gives us a way to show how the decay of Green's function on an interval can be deduced, via a covering argument, from its decay on smaller subintervals. We let

$$\delta_{m,n} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

Our *covering lemma* is as follows.

Lemma 2.22 Let $x, \omega \in \mathbb{T}^d$, $E \in \mathbb{R}$, and $[a, b] \subset \mathbb{Z}$. If for any $m \in [a, b]$, there exists an interval $I_m = [a_m, b_m] \subset [a, b]$ containing m such that

$$(1 - \delta_{a, a_m}) |\mathcal{G}_{I_m}(x, \omega, E; m, a_m)| + (1 - \delta_{b, b_m}) |\mathcal{G}_{I_m}(x, \omega, E; m, b_m)| < 1, \quad (2.19)$$

then $E \notin \text{spec } H_{[a, b]}(x, \omega)$.

Proof Assume to the contrary that $E \in \text{spec } H_{[a, b]}(x, \omega)$ and let ψ be a corresponding eigenvector. Let $m \in I := [a, b]$ be such that $|\psi(m)| = \max_{n \in I} |\psi(n)|$. The hypothesis together with the Poisson formula (2.18) gives us that

$$|\psi(m)| < \max(|\psi(a_m - 1)|, |\psi(b_m + 1)|)$$

if $a_m \neq a$ and $b_m \neq b$, $|\psi(m)| < |\psi(b_m + 1)|$ if $a_m = a$, and $|\psi(m)| < |\psi(a_m - 1)|$ if $b_m = b$. We arrive at a contradiction, so we must have $E \notin \text{spec } H_{[a, b]}(x, \omega)$. $\square$

Remark 2.23 We comment on the use of Lemma 2.22. By stability considerations, the condition (2.19) will hold for energies E in some interval $(E_0 - \delta, E_0 + \delta)$. The conclusion will then be that $(E_0 - \delta, E_0 + \delta) \cap \text{spec } H_{[a, b]}(x, \omega) = \emptyset$ and therefore $\|\mathcal{G}_{[a, b]}(x, \omega, E_0)\| < \frac{1}{\delta}$. One would then apply the spectral form of (LDT) and the next lemma to obtain the decay of Green's function on $[a, b]$.

The condition (2.19) and its stability in E will be obtained from the following lemma in conjunction with a LDT for the Dirichlet determinants.

Lemma 2.24 Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let τ be as in (LDT), and $K \in \mathbb{R}$. There exists $C_0 = C_0(V, a, b, |E_0|, \gamma)$ such that if $N \geq N_0(V, a, b, |E_0|, \gamma)$ and

$$\log |f_N(x_0, \omega_0, E_0)| > NL_N(\omega_0, E_0) - K,$$

then for any $(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d \times \mathbb{C}$ with $|x - x_0|, |\omega - \omega_0|, |E - E_0| < \exp(-(K + C_0 N^{1-\tau}))$ we have

$$|\mathcal{G}_{[1, N]}(x, \omega, E; j, k)| \leq \exp\left(-\frac{\gamma}{2}|k - j| + K + 2C_0 N^{1-\tau}\right), \quad (2.20)$$

$$\|\mathcal{G}_{[1, N]}(x, \omega, E)\| \leq \exp(K + 3C_0 N^{1-\tau}). \quad (2.21)$$

Proof Take $|x - x_0|, |\omega - \omega_0|, |E - E_0| < \exp(-(K + CN^{1-\tau}))$ with C large enough. Using Corollary 2.15 we have

$$\begin{aligned} \log |f_N(x, \omega, E)| &\geq \log |f_N(x_0, \omega_0, E_0)| - 1 \geq NL_N(\omega_0, E_0) - K - 1 \\ &\geq NL_N(\omega, E) - K - 2 \geq NL(\omega, E) - K - 2. \end{aligned}$$

By Cramer's rule and the uniform upper bound of Proposition 2.13,

$$\begin{aligned} |\mathcal{G}_{[1, N]}(x, \omega, E; j, k)| &= \frac{|f_{j-1}(x, \omega, E)| \cdot |f_{N-k}(x + k\omega, \omega, E)|}{|f_N(x, \omega, E)|} \\ &\leq \exp(-(k - j)L(\omega, E) + CN^{1-\tau} + K + 2) \\ &\leq \exp\left(-\frac{\gamma}{2}(k - j) + K + 2CN^{1-\tau}\right) \end{aligned}$$

(we assumed $j \leq k$ and we also used Proposition 2.9). The estimate (2.21) follows from (2.20). $\square$

Finally, we give a very important application of the covering lemma in combination with the spectral form of (LDT) and with Lemma 2.24. We call it the *covering form* of (LDT).

Lemma 2.25 Assume $N \geq 1$, $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let τ, σ be as in (LDT). Suppose that for each point $m \in [1, N]$ there exists an interval $I_m \subset [1, N]$ such that:

- i) $\text{dist}(m, [1, N] \setminus I_m) \geq |I_m|/100$,
- ii) $\ell_0(V, a, b, |E_0|, \gamma) \leq |I_m|$, with $\ell_0 = N_0$ from the previous lemma,
- iii) $\log |f_{I_m}(x_0, \omega_0, E_0)| > |I_m|L_{|I_m|}(\omega_0, E_0) - |I_m|^{1-\tau/4}$.

Then for any $(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d \times \mathbb{C}$ such that

$$|x - x_0|, N|\omega - \omega_0|, |E - E_0| < \exp\left(-2 \max_m |I_m|^{1-\tau/4}\right),$$

we have

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \exp\left(-2 \max_m |I_m|^{1-\tau/4}\right).$$

Furthermore, if we also have $\omega \in \mathbb{T}^d(a, b)$ and $\max_m |I_m| \leq N^{\sigma/2}$, then

$$\log |f_N(x, \omega, E)| > NL_N(\omega, E) - N^{1-\tau/2}.$$

Proof Due to (iii) and Lemma 2.24 we have that

$$|\mathcal{G}_{I_m}(x, \omega, E; m, k)| \leq \exp\left(-\frac{\gamma}{2}|m - k| + \frac{3}{2}|I_m|^{1-\tau/4}\right),$$

provided

$$|x - x_0|, N|\omega - \omega_0|, |E - E_0| < \exp\left(-\frac{3}{2}|I_m|^{1-\tau/4}\right).$$

This and assumptions (i) and (ii) guarantee that the assumptions of Lemma 2.22 are satisfied, and therefore $E \notin \text{spec } H_N(x, \omega)$, whenever

$$|E - E_0| < \exp\left(-\frac{3}{2} \max_m |I_m|^{1-\tau/4}\right).$$

Therefore, if $|E - E_0| < \exp(-\frac{3}{2} \max_m |I_m|^{1-\tau/4})/2$, then

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \exp\left(-\frac{3}{2} \max_m |I_m|^{1-\tau/4}\right)/2.$$

This proves the first statement.

If we have $\max_m |I_m| \leq N^{\sigma/2}$, then

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \exp(-N^{\sigma/2})$$

and the second statement follows from Corollary 2.21. $\square$

2.7 Wegner's Estimate

We will need an estimate for the probability that there exists an eigenvalue of $H_N(x, \omega)$ in some given interval $(E - \varepsilon, E + \varepsilon)$. From Lemma 2.24 and (LDT) it follows immediately that if $\varepsilon = \exp(-CN^{1-\tau})$, then this probability is less than $\exp(-N^\sigma)$. However, we will need to have control over intervals which are much larger. This is readily achieved by using the covering form of (LDT). The original reference by Wegner is [23].

Proposition 2.26 Assume $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{C}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ be as in (LDT). Let ℓ, N be integers such that $(2 \log N)^{1/\sigma} \leq \ell \leq N$. Then for any $N \geq N_0(V, a, b, |E_0|, \gamma)$ there exists a set $\mathcal{B}_{N, \omega_0, E_0}$, $\text{mes}(\mathcal{B}_{N, \omega_0, E_0}) < \exp(-\ell^\sigma/2)$ such that for any $x \in \mathbb{T}^d \setminus \mathcal{B}_{N, \omega_0, E_0}$ and any $(\omega, E) \in \mathbb{T}^d \times \mathbb{C}$, $N|\omega - \omega_0|, |E - E_0| < \exp(-\ell)$, we have

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \exp(-\ell).$$

Proof Take ω, E satisfying the assumptions. Let $\mathcal{B}_{N, \omega_0, E_0}$ be the set of x such that

$$\log |f_\ell(x + (m-1)\omega_0, \omega_0, E_0)| < \ell L_\ell(\omega_0, E_0) - \ell^{1-\tau}$$

for some $m \in [1, N]$. By (LDT), we have

$$\text{mes}(\mathcal{B}_{N, \omega_0, E_0}) < N \exp(-\ell^\sigma) \leq \exp(-\ell^\sigma/2).$$

By the covering form of (LDT), for any $x \notin \mathcal{B}_{N, \omega_0, E_0}$ we have

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \exp(-\ell),$$

thus concluding the proof. $\square$

An important consequence of Wegner's estimate is that the graphs of the eigenvalues cannot be too flat. Throughout, $E_j^{(N)}(\cdot, \omega)$ denotes the parameterization of the j -th eigenvalue of the Dirichlet Hamiltonian on $[1, N]$.

Lemma 2.27 Let $\omega \in \mathbb{T}^d(a, b)$, $\gamma > 0$, σ be as in (LDT), and ℓ, N be integers such that $(2 \log N)^{1/\sigma} \leq \ell \leq N$. For any $N \geq N_0(V, a, b, \gamma)$ we have that if $S \subseteq \mathbb{T}^d$ is connected and

$$\text{mes}(S) \geq \exp(-\ell^\sigma/3),$$

then with $E_j^{(N)}(S, \omega) := \{E_j^{(N)}(x, \omega) : x \in S\}$,

$$\text{mes}(E_j^{(N)}(S, \omega)) \geq \exp(-\ell)$$

for any $j \in \{1, \dots, N\}$ such that $L(\omega, E) > \gamma$ on $E_j^{(N)}(S, \omega)$.

Proof Since the eigenvalues are continuous in phase, the sets $E_j^{(N)}(S, \omega)$ are intervals. Assume that $L(\omega, E) > \gamma$ on $E_j^{(N)}(S, \omega)$ and let E be the middle point of the interval $E_j^{(N)}(S, \omega)$. Then

$$S \subset \left\{x \in \mathbb{T}^d : \text{dist}(E, \text{spec } H_N(x, \omega)) < \text{mes}\left(E_j^{(N)}(S, \omega)\right)\right\}.$$

By Proposition 2.26, we need to have $\text{mes}(E_j^{(N)}(S, \omega)) \geq \exp(-\ell)$, otherwise the above would imply $\text{mes}(S) < \exp(-\ell^\sigma/2)$. $\square$

2.8 Weierstrass' Preparation Theorem

We also need to discuss shortly a version of Weierstrass' preparation theorem for an analytic function $f(z, w_1, \dots, w_d)$ defined in a polydisk

$$\mathcal{P}_* = \mathcal{D}(z_0, R_0) \times \prod_{j=1}^d \mathcal{D}(w_{j,0}, R_0), \quad z_0, w_{j,0} \in \mathbb{C}, \quad R_0 > 0.$$

Lemma 2.28 Assume that $f(\cdot, w_1, \dots, w_d)$ has no zeros on some circle

$$\{z : |z - z_0| = r\}, \quad 0 < r < R_0/2,$$

for any $\underline{w} = (w_1, \dots, w_d) \in \mathcal{P} = \prod_{j=1}^d \mathcal{D}(w_{j,0}, r_{j,0})$ where $0 < r_{j,0} < R_0$. Then there exist a polynomial $P(z, \underline{w}) = z^k + a_{k-1}(\underline{w})z^{k-1} + \dots + a_0(\underline{w})$ with $a_j(\underline{w})$ analytic in $\mathcal{P}$ and an analytic function $g(z, \underline{w})$, $(z, \underline{w}) \in \mathcal{D}(z_0, r) \times \mathcal{P}$ so that the following properties hold:

- (a) $f(z, \underline{w}) = P(z, \underline{w})g(z, \underline{w})$ for any $(z, \underline{w}) \in \mathcal{D}(z_0, r) \times \mathcal{P}$,
- (b) $g(z, \underline{w}) \neq 0$ for any $(z, \underline{w}) \in \mathcal{D}(z_0, r) \times \mathcal{P}$,
- (c) for any $\underline{w} \in \mathcal{P}$, $P(\cdot, \underline{w})$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(z_0, r)$.

Proof By the usual Weierstrass argument, one notes that

$$b_p(\underline{w}) := \sum_{j=1}^k \zeta_j^p(\underline{w}) = \frac{1}{2\pi i} \oint_{|z-z_0|=r} z^p \frac{\partial_z f(z, \underline{w})}{f(z, \underline{w})} dz$$

are analytic in $\underline{w} \in \mathcal{P}$. Here $\zeta_j(\underline{w})$ are the zeros of $f(\cdot, \underline{w})$ in $\mathcal{D}(z_0, r)$. Since the coefficients $a_j(\underline{w})$ are linear combinations of the b_p , they are analytic in $\underline{w}$. Analyticity of g follows by standard arguments. $\square$

2.9 Resultants

Let us recall the definition of the resultant of two polynomials and deduce a key property that will be needed in conjunction with Weierstrass' Preparation Theorem. Let $f(z) = z^k + a_{k-1}z^{k-1} + \dots + a_0$, $g(z) = z^m + b_{m-1}z^{m-1} + \dots + b_0$ be polynomials with complex coefficients. Let ζ_i , $1 \leq i \leq k$ and η_j , $1 \leq j \leq m$ be the zeros of $f(z)$ and $g(z)$ respectively. The resultant of f and g is defined as follows:

$$\text{Res}(f, g) = \prod_{i,j} (\zeta_i - \eta_j) = \prod_i g(\zeta_i) = (-1)^k \prod_j f(\eta_j). \quad (2.22)$$

The resultant can also be expressed explicitly in terms of the coefficients (see [18, p. 200]):

$$\text{Res}(f, g) = \begin{vmatrix} \overbrace{1 & 0 & \dots}^m & \overbrace{1 & 0 & \dots & 0}^k \\ a_{k-1} & 1 & \dots & b_{m-1} & 1 & \dots & \dots \\ a_{k-2} & a_{k-1} & \dots & b_{m-2} & b_{m-1} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & & & & & \\ 0 & a_0 & \dots & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Lemma 2.29 Let f, g be polynomials as above and set

$$s = \max(k, m), \quad r = \max(\max |\zeta_i|, \max |\eta_j|).$$

Let $\delta \in (0, 1)$. If $|\text{Res}(f, g)| > \delta$ and $r \leq 1/2$, then

$$\max(|f(z)|, |g(z)|) > \left(\frac{\delta}{2}\right)^s \quad \text{for all } z.$$

Proof We argue by contradiction. Suppose there exists z such that

$$\max(|f(z)|, |g(z)|) \leq \left(\frac{\delta}{2}\right)^s.$$

Then there exist ζ_{i_0}, η_{j_0} such that $|z - \zeta_{i_0}|, |z - \eta_{j_0}| \leq \delta/2$. We have that $|\zeta_{i_0} - \eta_{j_0}| \leq \delta$ and by the assumption that $r \leq 1/2$, $|\zeta_i - \eta_j| \leq 1$. Therefore, $|\text{Res}(f, g)| \leq \delta$ and we arrived at a contradiction. $\square$

2.10 Semialgebraic Sets

Recall that a set $S \subset \mathbb{R}^n$ is called semialgebraic if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, a semialgebraic set $S \subset \mathbb{R}^n$ is given by an expression

$$S = \bigcup_j \bigcap_{\ell \in L_j} \{P_{\ell} s_{j\ell} 0\},$$

where $\{P_1, \dots, P_s\}$ is a collection of polynomials of n variables,

$$L_j \subset \{1, \dots, s\} \quad \text{and} \quad s_{j\ell} \in \{>, <, =\}.$$

If the degrees of the polynomials are bounded by d , then we say that the degree of S is bounded by sd . We refer to [7, Chapter 9] for more information on semialgebraic sets.

Semialgebraic sets will be introduced by approximating the potential V with a polynomial $\tilde{V}$. More precisely, given $N \geq 1$, by truncating V 's Fourier series and the Taylor series of the trigonometric functions, one can obtain a polynomial $\tilde{V}$ of degree $\lesssim N^4$ such that

$$\|V - \tilde{V}\|_{\infty} \lesssim \exp(-N^2) \quad (2.23)$$

(see [7, Chapter 10] for some details; the sup norm is taken over $\mathbb{T}^d$). The precise bound on the degree will not be important, as long as the bound is polynomial in N . If we let $\tilde{H}_N(x, \omega)$ be the operator with this truncated potential $\tilde{V}$ and $\tilde{E}_j^{(N)}(x, \omega)$ its eigenvalues, then

$$\left| E_j^{(N)}(x, \omega) - \tilde{E}_j^{(N)}(x, \omega) \right| \leq \|H_N(x, \omega) - \tilde{H}_N(x, \omega)\| \leq \|V - \tilde{V}\|_{\infty}. \quad (2.24)$$

This and Lemma 2.16 will ensure that our estimates at scale N are stable under the change of potential.

For the purpose of semialgebraic approximation we also need to consider the set

$$\mathbb{T}_N^d(a, b) = \left\{ \omega \in \mathbb{T}^d : \|k \cdot \omega\| \geq \frac{a}{|k|^b} \text{ for all } k \in \mathbb{Z}^d, 0 < |k| \leq N \right\}. \quad (2.25)$$

Remark 2.30 All the scale N results presented so far that require the frequency ω to be Diophantine, also work for $\omega \in \mathbb{T}_N^d(a, b)$. Furthermore, whenever the positivity of the Lyapunov exponent is required, the positivity of L_N suffices. This is simply because the proofs of the results at scale N never depend on what happens at larger scales. In particular, this observation applies to (LDT), Wegner's estimate, and the uniform upper bound from Proposition 2.13.

The only remaining issue is having a semialgebraic approximation for $L_N(\omega, E)$. We deal with this in Lemma 2.33, in the same way as in [4], though with a different proof. The proof of Lemma 2.33 will be based on the following result.

Lemma 2.31 ([7, Corollary 9.7]) *Let $\mathcal{S} \subset [0, 1]^d$ be semialgebraic of degree B and $\text{mes}(\mathcal{S}) < \eta$. Let J be an integer such that*

$$\log B \ll \log J < \log \frac{1}{\eta}.$$

Then, for any $x_0 \in \mathbb{T}^d$ and $\omega \in \mathbb{T}_J^d(a, b)$,

$$\#\{j = 1, \dots, J : x_0 + j\omega \in \mathcal{S} \pmod{\mathbb{Z}^d}\} < J^{1-\delta}$$

for some $\delta = \delta(a, b)$.

To apply Lemma 2.31 we need the following lemma.

Lemma 2.32 *Let $N \geq 1$, $\omega \in \mathbb{T}_N^d(a, b)$, $E \in \mathbb{R}$, σ, τ as in (LDT), and*

$$\mathcal{B}_N := \{x \in \mathbb{T}^d : |\log \|M_N(x, \omega, E)\| - NL_N(\omega, E)| \geq 4N^{1-\tau}\}.$$

There exists a semialgebraic set $\mathcal{S}_N$ such that $\mathcal{B}_N \subset \mathcal{S}_N$, $\deg(\mathcal{S}_N) \leq N^C$, and $\text{mes}(\mathcal{S}_N) < \exp(-N^\sigma)$ provided $N \geq N_0(V, a, b, |E|)$.

Proof Take $\tilde{V}$ as in (2.23) and let

$$\mathcal{S}_N := \left\{x \in \mathbb{T}^d : |\log \|\tilde{M}_N(x, \omega, E)\|_{\text{HS}} - NL_N(\omega, E)| \geq 2N^{1-\tau}\right\},$$

where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert–Schmidt norm. Clearly $\deg(\mathcal{S}_N) \leq N^C$, and $\mathcal{B}_N \subset \mathcal{S}_N$ follows from Lemma 2.16. The measure estimate follows from (LDT) and the fact that, by Lemma 2.16,

$$\mathcal{S}_N \subset \{x \in \mathbb{T}^d : |\log \|M_N(x, \omega, E)\| - NL_N(\omega, E)| > N^{1-\tau}\},$$

as desired. $\square$

We can now prove the result that we use to handle $L_N(\omega, E)$.

Lemma 2.33 *Let σ, τ as in (LDT). Then for any $N \geq N_0(V, a, b, |E|)$,*

$$C(a, b) \log N \leq \log J < N^\sigma,$$

$x \in \mathbb{T}^d$, $\omega \in \mathbb{T}_J^d(a, b)$, $E \in \mathbb{R}$, we have

$$\left| \frac{1}{J} \sum_{j=1}^J \log \|M_N(x + j\omega, \omega, E)\| - NL_N(\omega, E) \right| \leq 5N^{1-\tau}.$$

Proof Let $\mathcal{B}_N$ and $\mathcal{S}_N$ be as in Lemma 2.32. For any $x \in \mathbb{T}^d \setminus \mathcal{S}_N$ we have

$$NL_N(\omega, E) - 4N^{1-\tau} \leq \log \|M_N(x, \omega, E)\| \leq NL_N(\omega, E) + 4N^{1-\tau},$$

whereas for $x \in \mathcal{S}_N$ we only have

$$0 \leq \log \|M_N(x, \omega, E)\| \leq C(V, |E|)N.$$

Take J satisfying the assumptions. From the above and Lemma 2.31 we get

$$\begin{aligned} 5N^{1-\tau} &\geq \frac{J - J^{1-\delta}}{J} (4N^{1-\tau}) + \frac{J^{1-\delta}}{J} (CN) \\ &\geq \frac{1}{J} \sum_{j=1}^J \log \|M_N(x + j\omega, \omega, E)\| - NL_N(\omega, E) \\ &\geq \frac{J - J^{1-\delta}}{J} (-4N^{1-\tau}) + \frac{J^{1-\delta}}{J} (-NL_N(\omega, E)) \geq -5N^{1-\tau}. \end{aligned}$$

This concludes the proof. $\square$

Finally, we discuss a particular result on semialgebraic approximation needed for our applications.

Lemma 2.34 *Let $N \geq 1$, $\gamma > 0$, and σ, τ as in (LDT). Let $\mathcal{B}_N$ be the set of $(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d(a, b) \times \mathbb{R}$ such that $L(\omega, E) > \gamma$ and*

$$\log |f_N(x, \omega, E)| \leq NL_N(\omega, E) - N^{1-\tau/2}.$$

Then there exists a semialgebraic set $\mathcal{S}_N$ such that $\mathcal{B}_N \subset \mathcal{S}_N$, $\deg(\mathcal{S}_N) \leq N^{C(a,b)}$, and $\text{mes}(\mathcal{S}_N) < \exp(-N^\sigma/2)$, provided $N \geq N_0(V, a, b, \gamma)$. Furthermore,

$$\text{mes}(\mathcal{S}_N(\omega, E)) < \exp(-N^\sigma), \quad \mathcal{S}_N(\omega, E) = \{x : (x, \omega, E) \in \mathcal{S}_N\}.$$

Proof Note that by the spectral form of (LDT) (see Corollary 2.21), the definition of $\mathcal{B}_N$ implicitly restricts E to be in a bounded interval $[-C(V), C(V)]$ containing $\cup_{x, \omega} \text{spec } H_N(x, \omega)$. Take $\tilde{V}$ as in (2.23) and let $\mathcal{S}_N$ be the set of

$$(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}_J^d(a, b) \times [-C(V), C(V)], \quad J = N^{C(a,b)}$$

such that

$$\begin{aligned} \frac{1}{NJ} \sum_{j=1}^J \log \|\tilde{M}_N(x + j\omega, \omega, E)\|_{\text{HS}} &\geq \frac{\gamma}{2}, \\ \log |\tilde{f}_N(x, \omega, E)| &\leq \frac{1}{J} \sum_{j=1}^J \log \|\tilde{M}_N(x + j\omega, \omega, E)\|_{\text{HS}} - N^{1-\tau/2}/2. \end{aligned}$$

Clearly $\deg(\mathcal{S}_N) \leq N^C$, and $\mathcal{B}_N \subset \mathcal{S}_N$ follows from Lemmas 2.16, 2.33, and Proposition 2.5. For the measure estimate note that for $(x, \omega, E) \in \mathcal{S}_N$ we have $L_N(\omega, E) \geq \gamma/4$ and

$$\log |f_N(x, \omega, E)| \leq NL_N(\omega, E) - N^{1-\tau/2}/4.$$

By (LDT) (recall Remark 2.30), $\text{mes}(\mathcal{S}_N(\omega, E)) < \exp(-N^\sigma)$ and therefore

$$\text{mes}(\mathcal{S}_N) < 2C(V) \exp(-N^\sigma) < \exp(-N^\sigma/2).$$

$\square$

2.11 Perturbation Theory

In this section we state some standard facts on interlacing eigenvalues and basic perturbation theory. They will be needed in Section 5.

Lemma 2.35 ([15, Theorem 4.3.15]) *Let H be a $n \times n$ Hermitian matrix, let r be an integer with $1 \leq r \leq n$, and let H_r be any $r \times r$ principal submatrix of H (obtained by deleting $n - r$ rows and the corresponding columns from H). For each integer k such that $1 \leq k \leq r$ we have*

$$E_k(H) \leq E_k(H_r) \leq E_{k+n-r}(H)$$

($E_k(H)$ denotes H 's k -th eigenvalue; the eigenvalues are arranged in increasing order).

The following statements are standard facts from basic perturbation theory. We give the proofs for completeness.

Lemma 2.36 *If P, Q are two arbitrary projectors on $\mathbb{C}^n$ and $\|P - Q\| < 1$, then $\text{rank } P = \text{rank } Q$.*

Proof Without loss of generality, suppose to the contrary, that $\text{rank } P > \text{rank } Q$. Let $\mathfrak{M} = P(\mathbb{C}^n)$, $\mathfrak{N} = (I - Q)(\mathbb{C}^n)$. Then

$$\dim(\mathfrak{M}) + \dim(\mathfrak{N}) = \text{rank } P + n - \text{rank } Q > n$$

and therefore, there exists $h \in \mathfrak{M} \cap \mathfrak{N}$ with $\|h\| = 1$. Then we reach a contradiction because

$$\|Ph - Qh\| = \|h - 0\| = 1.$$

□

Lemma 2.37 *Let H, H_0 be $n \times n$ matrices, H_0 is Hermitian, $E_0 \in \mathbb{R}$, $r_0 > 0$. Assume the number of eigenvalues of H_0 in $(E_0 - r_0, E_0 + r_0)$ is at most K and*

$$\|H - H_0\| \leq \frac{r_0}{32(K+1)^2}.$$

Then there exist $r_0/2 < r < r_0$, which depends only on H_0 , such that H and H_0 have the same number of eigenvalues in the disk $\mathcal{D}(E_0, r)$. Moreover, neither H nor H_0 have eigenvalues in the region

$$r - \frac{r_0}{8(K+1)} \leq |\zeta - E_0| \leq r + \frac{r_0}{8(K+1)}. \quad (2.26)$$

Proof Clearly, there exists $r_0/2 < r < r_0$ such that H_0 has no eigenvalues in the domain

$$r - \frac{r_0}{4(K+1)} \leq |\zeta - E_0| \leq r + \frac{r_0}{4(K+1)}.$$

Since H_0 is Hermitian,

$$\|(H_0 - \zeta)^{-1}\| \leq \frac{4(K+1)}{r_0} \quad \text{for any } |\zeta - E_0| = r.$$

Recall the basic resolvent expansion estimate:

$$\|(A + B)^{-1} - A^{-1}\| \leq \|A^{-1}\| \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|} \leq 2\|A^{-1}\|^2\|B\|,$$

provided $\|A^{-1}\| \|B\| < 1/2$. This implies

$$\|(H - \zeta)^{-1} - (H_0 - \zeta)^{-1}\| \leq 2\|(H_0 - \zeta)^{-1}\|^2 \|H - H_0\| \leq \frac{1}{r_0} \quad \text{for any } |\zeta - E_0| = r.$$

Consider the Riesz projectors

$$P = \frac{1}{2\pi i} \oint_{|\zeta - E_0| = r} (H - \zeta)^{-1} d\zeta, \quad P_0 = \frac{1}{2\pi i} \oint_{|\zeta - E_0| = r} (H_0 - \zeta)^{-1} d\zeta.$$

Then

$$\|P - P_0\| \leq \frac{r_0^{-1}}{2\pi} \oint_{|\zeta - E_0| = r} d|\zeta| = \frac{r}{r_0} < 1.$$

The previous lemma implies $\text{rank } P = \text{rank } P_0$. The first statement follows by recalling that P, P_0 project onto the sum of the generalized eigenspaces of the eigenvalues in $\mathcal{D}(E_0, r)$.

If ζ is in the region (2.26), then

$$\|(H_0 - \zeta)^{-1}\| \|H - H_0\| \leq \frac{8(K+1)}{r_0} \frac{r_0}{32(K+1)^2} < 1$$

and therefore $H - \zeta$ is invertible and the second statement follows. $\square$

2.12 Stabilization of Eigenvalues and Eigenvectors

In this section we present some basic results on the relation between the eigenvalues and eigenvectors at different scales.

Lemma 2.38 *Let $x, \omega \in \mathbb{T}^d$. For any intervals $\Lambda_0 = [a_0, b_0] \subset \Lambda \subset \mathbb{Z}$ and any j_0 , we have*

$$\text{dist}\left(E_{j_0}^{\Lambda_0}(x, \omega), \text{spec } H_{\Lambda}(x, \omega)\right) \leq \left|\psi_{j_0}^{\Lambda_0}(x, \omega; a_0)\right| + \left|\psi_{j_0}^{\Lambda_0}(x, \omega; b_0)\right|.$$

Proof Let ψ_0 be the extension, with zero entries, of $\psi_{j_0}^{\Lambda_0}(x, \omega)$ to Λ . Since $\|\psi_0\| = 1$, the conclusion follows from the fact that we have

$$\begin{aligned} \left\| \left(H_{\Lambda}(x, \omega) - E_{j_0}^{\Lambda_0}(x, \omega) \right)^{-1} \right\|^{-1} &\leq \left\| \left(H_{\Lambda}(x, \omega) - E_{j_0}^{\Lambda_0}(x, \omega) \right) \psi_0 \right\| \\ &\leq \left| \psi_{j_0}^{\Lambda_0}(x, \omega; a_0) \right| + \left| \psi_{j_0}^{\Lambda_0}(x, \omega; b_0) \right|. \end{aligned}$$

$\square$

Recall the following simple general statement on the finite interval approximation of the spectrum

Lemma 2.39 *If for some $x, \omega \in \mathbb{T}^d$, $E \in \mathbb{R}$, $\rho > 0$, there exist sequences $N'_k \rightarrow -\infty$, $N''_k \rightarrow +\infty$ such that*

$$\text{dist}(E, \text{spec } H_{[N'_k, N''_k]}(x, \omega)) \geq \rho,$$

then

$$\text{dist}(E, \mathcal{S}_{\omega}) \geq \rho.$$

Proof Take arbitrary $\phi \in \ell^2(\mathbb{Z})$ with finite support. For any k large enough so that $\text{supp } \phi \subset (N'_k, N''_k)$ we have

$$(H(x, \omega) - E)\phi(n) = \left(H_{[N'_k, N''_k]}(x, \omega) - E \right) \phi(n), \quad n \in [N'_k, N''_k].$$

Due to the hypothesis,

$$\begin{aligned}\|(H(x, \omega) - E)\phi\| &= \left\| \left(H_{[N'_k, N''_k]}(x, \omega) - E \right) \phi \right\| \\ &\geq \left\| \left(H_{[N'_k, N''_k]}(x, \omega) - E \right)^{-1} \right\|^{-1} \|\phi\| \geq \rho \|\phi\|.\end{aligned}$$

Since this holds for any finite support ϕ it also holds for any $\phi \in \ell^2(\mathbb{Z})$, and the conclusion follows. $\square$

The following standard result is the basis for the stabilization of eigenvectors.

Lemma 2.40 *Let A be a $N \times N$ Hermitian matrix. Let $E, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and suppose there exists $\phi \in \mathbb{R}^N$, $\|\phi\| = 1$, such that*

$$\|(A - E)\phi\| < \varepsilon.$$

Then the following statements hold.

(a) *There exists a normalized eigenvector ψ of A with an eigenvalue E_0 such that*

$$\begin{aligned}E_0 &\in (E - \varepsilon\sqrt{2}, E + \varepsilon\sqrt{2}), \\ |\langle \phi, \psi \rangle| &\geq (2N)^{-1/2}.\end{aligned}$$

(b) *If in addition there exists $\eta > \varepsilon$ such that the subspace of the eigenvectors of A with eigenvalues falling into the interval $(E - \eta, E + \eta)$ is at most of dimension one, then there exists a normalized eigenvector ψ of A with an eigenvalue $E_0 \in (E - \varepsilon, E + \varepsilon)$, such that*

$$\|\phi - \psi\| < \sqrt{2}\eta^{-1}\varepsilon.$$

Proof (a) Let $\psi_j, j = 1, \dots, N$, be an orthonormal basis of eigenvectors of A , $A\psi_j = E_j\psi_j$. Then

$$\sum_{|\langle \phi, \psi_j \rangle| \geq (2N)^{-1/2}} |\langle \phi, \psi_j \rangle|^2 = \|\phi\|^2 - \sum_{|\langle \phi, \psi_j \rangle| < (2N)^{-1/2}} |\langle \phi, \psi_j \rangle|^2 > \frac{1}{2}$$

and

$$\begin{aligned}\varepsilon^2 &> \|(A - E)\phi\|^2 = \sum_j |\langle \phi, \psi_j \rangle|^2 (E_j - E)^2 \geq \sum_{|\langle \phi, \psi_j \rangle| \geq (2N)^{-1/2}} |\langle \phi, \psi_j \rangle|^2 (E_j - E)^2 \\ &\geq \min_{|\langle \phi, \psi_j \rangle| \geq (2N)^{-1/2}} (E_j - E)^2 \sum_{|\langle \phi, \psi_j \rangle| \geq (2N)^{-1/2}} |\langle \phi, \psi_j \rangle|^2 \\ &> \frac{1}{2} \min_{|\langle \phi, \psi_j \rangle| \geq (2N)^{-1/2}} (E_j - E)^2.\end{aligned}$$

This finishes the proof of (a). To prove (b) note that

$$\varepsilon^2 > \sum_j |\langle \phi, \psi_j \rangle|^2 (E_j - E)^2 \geq \min_j (E_j - E)^2.$$

This implies that there is $E_k \in (E - \varepsilon, E + \varepsilon)$. Due to our assumptions, for any $j \neq k$ one has $E_j \notin (E - \eta, E + \eta)$. So

$$\varepsilon^2 > \sum_{j \neq k} |\langle \phi, \psi_j \rangle|^2 (E_j - E)^2 \geq \eta^2 \sum_{j \neq k} |\langle \phi, \psi_j \rangle|^2.$$

Thus,

$$1 - |\langle \phi, \psi_k \rangle|^2 = \|\phi - \langle \phi, \psi_k \rangle \psi_k\|^2 = \sum_{j \neq k} |\langle \phi, \psi_j \rangle|^2 \leq \eta^{-2} \varepsilon^2.$$

The conclusion now follows from the fact that $\|\phi - \psi_k\|^2 = 2(1 - \operatorname{Re} \langle \phi, \psi_k \rangle)$. Note that we can replace ψ_k by $e^{i\theta} \psi_k$ to ensure $\operatorname{Re} \langle \phi, \psi_k \rangle = |\langle \phi, \psi_k \rangle|$. $\square$

We will need the following corollary of the first part in Lemma 2.40.

Corollary 2.41 *Let $x, \omega \in \mathbb{T}^d$ and $[a, b] \subset [c, d]$. Let φ be a normalized eigenvector of $H_{[a,b]}(x, \omega)$, with $H_{[a,b]}(x, \omega)\varphi = E\varphi$. Set*

$$\varepsilon^2 = |\varphi(a)|^2 + |\varphi(b)|^2, \quad N = d - c + 1, \quad \underline{N} = b - a + 1.$$

There exists a normalized eigenvector ψ of $H_{[c,d]}(x, \omega)$ with an eigenvalue E_0 such that

$$\begin{aligned} E_0 &\in (E - \varepsilon\sqrt{2}, E + \varepsilon\sqrt{2}), \\ \max_{n \in [a,b]} |\psi(n)| &\geq (2N\underline{N})^{-1/2}. \end{aligned} \quad (2.27)$$

Proof Set

$$\phi(n) = \begin{cases} \varphi(n), & n \in [a, b], \\ 0, & n \in [c, d] \setminus [a, b]. \end{cases}$$

Note that

$$\|(H_{[c,d]}(x, \omega) - E)\phi\|^2 = |\varphi(a)|^2 + |\varphi(b)|^2 = \varepsilon^2.$$

Due to the first part in Lemma 2.38 there exists an eigenvector ψ_0 of $H_{[c,d]}(x, \omega)$ with an eigenvalue E_0 such that

$$\begin{aligned} E_0 &\in (E - \varepsilon\sqrt{2}, E + \varepsilon\sqrt{2}), \\ |\langle \phi, \psi \rangle| &\geq (2N)^{-1/2}. \end{aligned} \quad (2.28)$$

Since $|\langle \phi, \psi \rangle|^2 \leq \|\psi\|_{[a,b]}^2$, (2.28) implies (2.27). $\square$

3 Localization and Separation of Eigenvalues on a Finite Interval

In this section we discuss finite interval localization and separation of eigenvalues independently of the details of elimination of resonances. Even in this setting we have to deal with issues, absent in [12], stemming from the largeness of the deviation in (LDT). We start by deriving the localization of Dirichlet eigenfunctions at a *given scale* N assuming “no long range double resonances” at a much *smaller scale* ℓ . The precise meaning of “no long range double resonances” is given by the condition (3.1) from the next proposition.

Proposition 3.1 *Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let τ, σ be as in (LDT) and ℓ, N be integers such that $\ell_0(V, a, b, |E_0|, \gamma) \leq \ell \leq N^{\sigma/2}$. Assume that there exists an interval $I = [N', N''] \subset [1, N]$ such that*

$$\log |f_\ell(x_0 + (m-1)\omega_0, \omega_0, E_0)| > \ell L_\ell(\omega_0, E_0) - \ell^{1-\tau/4} \text{ for any } m \in [1, N-\ell+1] \setminus I. \quad (3.1)$$

Then for any $(x, \omega) \in \mathbb{T}^d \times \mathbb{T}^d(a, b)$, $|x - x_0|, |\omega - \omega_0| < \exp(-\ell)$ and any eigenvalue $|E_j^{(N)}(x, \omega) - E_0| < \exp(-\ell)$, the corresponding eigenfunction obeys

$$|\psi_j^{(N)}(x, \omega; n)| < \exp\left(-\frac{\gamma}{4} \operatorname{dist}(n, I)\right),$$

provided $\text{dist}(n, I) \geq \ell^{2/\sigma}$.

Proof Take $x, \omega, E = E_j^{(N)}(x, \omega)$ satisfying the assumptions, and $n \in [1, N] \setminus I$ such that $d = \text{dist}(n, I) \geq \ell^{2/\sigma}$. Assume $n \in [1, N']$. Let $J = [n - d, n + d] \cap [1, N] = [a, N']$. Note that $|J| \geq \ell^{2/\sigma}$. By the covering form of (LDT) (see Lemma 2.25) we have

$$\log |f_J(x, \omega, E)| \geq |J|L_{|J|}(\omega, E) - |J|^{1-\tau/2}. \quad (3.2)$$

Let $\psi = \psi_j^{(N)}(x, \omega)$. By Poisson's formula,

$$\psi(n) = \begin{cases} \mathcal{G}_J(x, \omega, E; n, a)\psi(a-1) + \mathcal{G}_J(x, \omega, E; n, N')\psi(N'+1), & a > 1, \\ \mathcal{G}_J(x, \omega, E; n, N')\psi(N'+1), & a = 1 \end{cases}$$

(recall the Dirichlet boundary condition $\psi(0) = 0$). Now, (3.2) and Lemma 2.24 imply

$$|\psi(n)| \leq 2 \exp\left(-\frac{\gamma}{2}d + C|J|^{1-\tau/2}\right) < \exp\left(-\frac{\gamma}{4}d\right)$$

(recall that ψ is normalized and therefore $|\psi(k)| \leq 1$ for any $k \in [1, N]$). The case $n \in [N'', N]$ is completely analogous. $\square$

The second issue we study in this section is a quantitative estimate for the *separation of the eigenvalues*. More precisely, we consider the separation of a localized eigenvalue $E_j^{(N)}(x, \omega)$ (as in Proposition 3.1) from all the other eigenvalues of $H_N(x, \omega)$. We recall the following basic observation regarding the relation between the Dirichlet determinants and the solutions to the difference equation (2.1). If ψ is a solution of (2.1) that satisfies $\psi(0) = 0, \psi(1) = 1$, then by (2.2) and (2.3) we have $\psi(n) = f_{[1, n-1]}(x, \omega, E), n \geq 1$ (we convene that $f_{[1, 0]} = 1$). In particular, if $E \in \text{spec } H_N(x, \omega)$, then $(f_{[1, n-1]}(x, \omega, E))_{n \in [1, N]}$ is a corresponding eigenfunction. The idea of our method is as follows. The eigenfunctions corresponding to different eigenvalues are orthogonal. As we just saw, each eigenfunction can be expressed in terms of Dirichlet determinants, evaluated at the corresponding eigenvalue. We show that the determinants evaluated at close energies are close themselves. That puts a limitation on how close two different eigenvalues can be. We *cannot* use the estimate from Corollary 2.15 because it is too imprecise and would only give separation of eigenvalues by $\exp(-CN^{1-\tau})$. Instead, we use Harnack's inequality for harmonic functions.

Lemma 3.2 Assume $x_0 \in \mathbb{T}^d, \omega_0 \in \mathbb{T}^d(a, b), E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let τ, σ be as in (LDT) and ℓ, N be integers such that $\ell_0(V, a, b, |E_0|, \gamma) \leq \ell \leq N^{\sigma/2}$. Assume that there exists an interval $I = [N', N''] \subset [1, N]$ such that (3.1) holds. Then for any $(x, \omega) \in \mathbb{T}^d \times \mathbb{T}^d(a, b), |x - x_0|, |\omega - \omega_0| < \exp(-\ell)$ and any $E_i \in \mathbb{R}, |E_i - E_0| < \exp(-\ell), i = 1, 2$, we have

$$|f_n(x, \omega, E_1) - f_n(x, \omega, E_2)| \leq n \exp(\ell) |E_1 - E_2| \max(|f_n(x, \omega, E_1)|, |f_n(x, \omega, E_2)|)$$

for any $\ell^{2/\sigma} \leq n \leq N'$.

Proof Let $n \in [\ell^{2/\sigma}, N']$. Fix x, ω satisfying the assumptions. From (3.1) and the covering form of (LDT) (see Lemma 2.25) it follows that

$$\log |f_n(x, \omega, E)| \geq nL_n(\omega, E) - n^{1-\tau/2} \quad \text{for any } E \in \mathcal{D}(E_0, \exp(-C\ell^{1-\tau})). \quad (3.3)$$

It follows that

$$u(E) = C(V, |E_0|)n - \log |f_n(x, \omega, E)|$$

is harmonic and positive on $\mathcal{D}(E_0, \exp(-C\ell^{1-\tau}))$ (recall (2.4)). We take $r = |E_1 - E_2|$, $R = \exp(-C\ell^{1-\tau})$ (note that $r/R \ll 1/2$) and using Harnack's inequality we get

$$\left(1 - 2\frac{r}{R}\right)u(E_2) \leq \frac{R-r}{R+r}u(E_2) \leq u(E_1) \leq \frac{R+r}{R-r}u(E_2) \leq \left(1 + 4\frac{r}{R}\right)u(E_2).$$

It follows that

$$|\log |f_n(x, \omega, E_1)| - \log |f_n(x, \omega, E_2)|| \leq \frac{Cn}{R}|E_1 - E_2|.$$

By the Mean Value Theorem,

$$||f_n(x, \omega, E_1)| - |f_n(x, \omega, E_2)|| \leq \frac{Cn}{R}|E_1 - E_2| \max(|f_n(x, \omega, E_1)|, |f_n(x, \omega, E_2)|)$$

and the conclusion follows from the fact that $f_n(x, \omega, E)$ does not vanish (due to (3.3)) and hence it has constant sign for $E \in \mathbb{R}$. $\square$

Proposition 3.3 Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let τ, σ be as in (LDT) and ℓ, N be integers such that $\ell_0(V, a, b, |E_0|, \gamma) \leq \ell \leq N^{\sigma/2}$. Assume that there exists an interval $I = [N', N''] \subset [1, N]$ such that (3.1) holds and $|I| \geq \ell^{2/\sigma} + \log N$. Then for any $(x, \omega) \in \mathbb{T}^d \times \mathbb{T}^d(a, b)$, $|x - x_0|, |\omega - \omega_0| < \exp(-\ell)$ and any eigenvalue such that $|E_j^{(N)}(x, \omega) - E_0| < \exp(-\ell)/2$, we have

$$\left|E_j^{(N)}(x, \omega) - E_k^{(N)}(x, \omega)\right| \geq \exp(-C|I|) \quad \text{for any } k \neq j, \quad (3.4)$$

with $C = C(V, |E_0|)$.

Proof We argue by contradiction. Let $E_1 = E_j^{(N)}(x, \omega)$, $E_2 = E_k^{(N)}(x, \omega)$ and assume

$$|E_1 - E_2| < \exp(-C|I|).$$

Note that we therefore have $|E_2 - E_0| < \exp(-\ell)$, so Proposition 3.1 can be applied to both eigenvalues. Let $\psi_i(n) = f_{n-1}(x, \omega, E_i)$. Since ψ_1, ψ_2 are eigenfunctions corresponding to different eigenvalues we have that they are orthogonal and therefore

$$\|\psi_1 - \psi_2\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2.$$

Let $\tilde{I} = \{n \in [1, N] : \text{dist}(n, I) < \ell^{2/\sigma}\}$. By Proposition 3.1,

$$\sum_{n \notin \tilde{I}} |\psi_i(n)|^2 \leq \|\psi_i\|^2 \sum_{n \notin \tilde{I}} \exp\left(-\frac{\gamma}{2} \text{dist}(n, I)\right) \leq \|\psi_i\|^2 \exp\left(-\frac{\gamma}{4} \ell^{2/\sigma}\right)$$

(we used the fact that ℓ is taken to be large enough). Therefore

$$\sum_{n \notin \tilde{I}} |\psi_1(n) - \psi_2(n)|^2 \leq 2 \exp\left(-\frac{\gamma}{4} \ell^{2/\sigma}\right) (\|\psi_1\|^2 + \|\psi_2\|^2). \quad (3.5)$$

Let

$$m = \begin{cases} N' - \ell^{2/\sigma}, & N' \geq 2\ell^{2/\sigma} + 1, \\ 1, & N' < 2\ell^{2/\sigma}. \end{cases}$$

By Lemma 3.2 we have

$$|\psi_1(k) - \psi_2(k)| \leq N \exp(\ell) |E_1 - E_2| \max(|\psi_1(k)|, |\psi_2(k)|) \quad \text{for } k \in \{m-1, m\} \quad (3.6)$$

(note that the estimates hold trivially when $m = 1$). For any $n \in \tilde{I}$ we have

$$\begin{aligned} |\psi_1(n) - \psi_2(n)| &\leq \left\| \begin{bmatrix} \psi_1(n+1) \\ \psi_1(n) \end{bmatrix} - \begin{bmatrix} \psi_2(n+1) \\ \psi_2(n) \end{bmatrix} \right\| \\ &= \left\| M_{[m,n]}(x, \omega, E_1) \begin{bmatrix} \psi_1(m) \\ \psi_1(m-1) \end{bmatrix} - M_{[m,n]}(x, \omega, E_2) \begin{bmatrix} \psi_2(m) \\ \psi_2(m-1) \end{bmatrix} \right\| \\ &\leq \left\| M_{[m,n]}(x, \omega, E_1) \begin{bmatrix} \psi_1(m) - \psi_2(m) \\ \psi_1(m-1) - \psi_2(m-1) \end{bmatrix} \right\| \\ &\quad + \left\| (M_{[m,n]}(x, \omega, E_1) - M_{[m,n]}(x, \omega, E_2)) \begin{bmatrix} \psi_2(m) \\ \psi_2(m-1) \end{bmatrix} \right\|. \end{aligned}$$

Therefore, using (3.6) and Lemma 2.6 we get

$$|\psi_1(n) - \psi_2(n)| \leq N \exp(C(V, |E_0|)(|I| + \ell^{2/\sigma})) |E_1 - E_2| \max(\|\psi_1\|, \|\psi_2\|) \quad \text{for } n \in \tilde{I}$$

and

$$\sum_{n \in \tilde{I}} |\psi_1(n) - \psi_2(n)|^2 \leq \exp(-C|I|)(\|\psi_1\|^2 + \|\psi_2\|^2), \quad (3.7)$$

provided the constant from (3.4) is chosen large enough (recall that $|I| \geq \ell^{2/\sigma} + \log N$). By (3.5) and (3.7),

$$\|\psi_1 - \psi_2\|^2 \ll \|\psi_1\|^2 + \|\psi_2\|^2$$

contradicting the fact that ψ_1, ψ_2 are orthogonal. $\square$

4 (NDR) Condition

In this section we introduce the main new ingredient for the proof of the finite scale localization: the existence of intervals satisfying the following “no double resonances” (NDR) condition.

Definition 4.1 Let σ, τ be as in (LDT). We say that an interval $\Lambda \subset \mathbb{Z}$ is (K, ℓ, C) -(NDR) with respect to x_0, ω_0, E_0 , if there exists another interval $\underline{\Lambda} \subset \Lambda$, $|\underline{\Lambda}| \leq K$, such that

$$|\log |f_\ell(x_0 + (n-1)\omega_0, \omega_0, E_0)| > \ell L_\ell(\omega_0, E_0) - C\ell^{1-\tau/3} \quad \text{for all } n \in \Lambda \setminus \underline{\Lambda}.$$

Furthermore, we require that the connected components of $\Lambda \setminus \underline{\Lambda}$ have length greater than $\ell^{2/\sigma}$. If $C = 1$ we say that Λ is (K, ℓ) -(NDR).

Remark 4.2 We shift by $(n-1)\omega_0$, instead of $n\omega_0$, to make sure that the intervals on which the estimate holds cover $\Lambda \setminus \underline{\Lambda}$. Note that

$$f_\ell(x_0 + (n-1)\omega_0, \omega_0, E_0) = f_{[n, n+\ell-1]}(x_0, \omega_0, E_0).$$

The assumption on the connected components of $\Lambda \setminus \underline{\Lambda}$ is just a matter of convenience (it facilitates the application of the covering form of (LDT)). When this condition is not satisfied, one can simply choose a larger $\underline{\Lambda}$. It goes without saying that for applications we will want K to be small relative to $|\Lambda|$ (to be more precise, we will need that $K \leq |\Lambda|^\varepsilon$ with some sufficiently small $\varepsilon > 0$).

Remark 4.3 The only reason we consider (NDR) intervals with $C \neq 1$ is the following stability property, that will be used in Proposition 7.2. If $C < 1$, Λ is (K, ℓ, C) -(NDR) with

respect to x_0, ω_0, E_0 and $L(\omega_0, E_0) > \gamma > 0$, then, by Corollary 2.15, Λ is (K, ℓ) -(NDR) with respect to x, ω, E , provided

$$|x - x_0|, |\omega - \omega_0|, |E - E_0| < \exp(-C'\ell^{1-\tau/3}).$$

Of course, ℓ also needs to be large enough.

The existence of (NDR) intervals (with small enough K) follows from the work of Bourgain about localization on $\mathbb{Z}^d$ [8]. More specifically, we make use of his result on elimination of multiple resonances. We recall the relevant abstract lemmas from [8] and then we apply them to our concrete situation.

Lemma 4.4 ([8, Lemma 1.18]) *Let $\mathcal{A} \subset [0, 1]^{q+r}$ be semialgebraic of degree B and such that for each $t \in [0, 1]^r$, $\text{mes}_q(\mathcal{A}(t)) < \eta$. Then*

$$\{(x_1, \dots, x_{2r}) : \mathcal{A}(x_1) \cap \dots \cap \mathcal{A}(x_{2r}) \neq \emptyset\} \subset [0, 1]^{q^{2r}}$$

is semialgebraic of degree at most B^C and measure at most

$$\eta_r = B^C \eta^{q^{-r} 2^{-r(r-1)/2}} \quad \text{with } C = C(r).$$

Remark 4.5 Note that in the previous lemma $\mathcal{A}(t) = \{x \in [0, 1]^q : (x, t) \in \mathcal{A}\}$ and $\mathcal{A}(x) = \{t \in [0, 1]^r : (x, t) \in \mathcal{A}\}$.

Lemma 4.6 ([8, Lemma 1.20]) *Let $\mathcal{A} \subset [0, 1]^{r^q}$ be a semialgebraic set of degree B and $\text{mes}_{rq}(\mathcal{A}) < \eta$. Let $\mathcal{N}_1, \dots, \mathcal{N}_{q-1} \subset \mathbb{Z}$ be finite sets with the property that*

$$|n_i| > (B|n_{i-1}|)^C \quad \text{if } n_i \in \mathcal{N}_i \text{ and } n_{i-1} \in \mathcal{N}_{i-1}, \quad 2 \leq i \leq q-1,$$

where $C = C(q, r)$. Assume also

$$\max_{n \in \mathcal{N}_{q-1}} |n|^C < \frac{1}{\eta}.$$

Then

$$\text{mes}\{\omega \in [0, 1]^r : (\omega, n_1\omega, \dots, n_{q-1}\omega) \in \mathcal{A} \text{ for some } n_i \in \mathcal{N}_i\} < B^C \left(\min_{n \in \mathcal{N}_1} |n| \right)^{-1}.$$

Combining Lemmas 4.4 and 4.6 we obtain the following result about the structure of the scale ℓ resonant shifts on the orbit of sub-exponential length for an arbitrary $x \in \mathbb{T}^d$. The result is similar to the Claim from [8, p. 694].

Proposition 4.7 *Let $\ell \geq 1, \gamma > 0$, and σ, τ as in (LDT). Let $\mathcal{N}_1, \dots, \mathcal{N}_{q-1} \subset \mathbb{Z}, q = 2^{2d+1}$, be finite sets with the property that*

$$|n_i| > \left(\ell^{C(d)} |n_{i-1}| \right)^{C'(d)} \quad \text{if } n_i \in \mathcal{N}_i \text{ and } n_{i-1} \in \mathcal{N}_{i-1}, \quad 2 \leq i \leq q-1$$

and

$$\max_{n \in \mathcal{N}_{q-1}} |n| < \exp(c\ell^\sigma), \quad c = c(d).$$

For any $\ell \geq \ell_0(V, a, b, \gamma)$ there exists a set Ω_ℓ , depending on the choice of the finite sets $\mathcal{N}_i$, such that

$$\text{mes}(\Omega_\ell) \leq \ell^{C(a,b)} \left(\min_{n \in \mathcal{N}_1} |n| \right)^{-1}$$

and the following statement holds. For any $x \in \mathbb{T}^d$, $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_\ell$, $E \in \mathbb{R}$, if $L(\omega, E) > \gamma$ and

$$\log |f_\ell(x, \omega, E)| \leq \ell L_\ell(\omega, E) - \ell^{1-\tau/2},$$

then there exists $i \in \{1, \dots, q-1\}$, depending on x, ω, E , such that

$$\log |f_\ell(x + (n-1)\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2} \quad \text{for all } n \in \mathcal{N}_i.$$

Proof Let $\mathcal{B}_\ell, \mathcal{S}_\ell$ be the sets from Lemma 2.34. We have that $\mathcal{S}_\ell$ is semialgebraic, $\mathcal{B}_\ell \subset \mathcal{S}_\ell$, $\deg(\mathcal{S}_\ell) \leq \ell^{C(a,b)}$, $\text{mes}(\mathcal{S}_\ell(\omega, E)) < \exp(-\ell^\sigma)$. Let $\mathcal{T}_\ell$ be the set of, cf. definition (2.25),

$$(y, x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d \times \mathbb{T}_{\ell^C}^d(a, b) \times \mathbb{R}$$

such that $(x + y - \omega, \omega, E) \in \mathcal{S}_\ell$. Clearly, $\mathcal{T}_\ell$ is a semialgebraic set and

$$\deg(\mathcal{T}_\ell) \leq \ell^C, \quad \text{mes}(\mathcal{T}_\ell(x, \omega, E)) < \exp(-\ell^\sigma).$$

By Lemma 4.4 (applied with $\mathcal{A} = \mathcal{T}_\ell$, $t = (x, \omega, E)$, $x = y$, $q = d$, $r = 2d+1$, $B = \ell^C$, $\eta = \exp(-\ell^\sigma)$; note that $\mathcal{A}$ and q are different in our proof), the set

$$\mathcal{A} := \{(y_1, \dots, y_q) : \mathcal{T}_\ell(y_1) \cap \dots \cap \mathcal{T}_\ell(y_q) \neq \emptyset\}$$

is semialgebraic and satisfies $\deg(\mathcal{A}) \leq \ell^C$, $\text{mes}(\mathcal{A}) < \exp(-c\ell^\sigma)$, $c = c(d)$. The conclusion follows from Lemma 4.6 (applied with $\mathcal{A} = \mathcal{A}$, $q = q$, $r = d$, $B = \ell^C$, $\eta = \exp(-c\ell^\sigma)$, $\mathcal{N}_i = \mathcal{N}_i$) by letting

$$\Omega_\ell = \{\omega : (\omega, n_1\omega, \dots, n_{q-1}\omega) \in \mathcal{A} \text{ for some } n_i \in \mathcal{N}_i\}.$$

We leave the remaining details to the reader. $\square$

A typical example of how the previous proposition leads to (NDR) intervals is obtained by considering the sets

$$\mathcal{N}_i = \left\{ n \in \mathbb{Z} : \ell^{\underline{C}_i} \leq |n| \leq \ell^{\overline{C}_i} \right\}, \quad i = 1, \dots, q-1, \quad q = 2^{2d+1}.$$

We can choose the constants

$$1 \ll \underline{C}_1 \ll \overline{C}_1 \ll \dots \ll \underline{C}_{q-1} \ll \overline{C}_{q-1}$$

such that Proposition 4.7 applies. Then (provided $\omega \notin \Omega_\ell$ and $L(\omega, E) > 0$) we have that for each $x \in \mathbb{T}^d$, either

$$\log |f_\ell(x, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2},$$

or there exists $i \in \{1, \dots, q-1\}$, depending on x , such that the interval $[-\ell^{\overline{C}_i}, \ell^{\overline{C}_i}]$ is $(\ell^{\underline{C}_i}, \ell)$ -(NDR) with respect to x, ω, E . The fact that i depends on x poses the following problem. In Proposition 7.2 we will eliminate resonances between (NDR) intervals of the same size, but at this point we cannot guarantee that we can get (NDR) intervals of the same size for any pair of distinct phases, say $x + n\omega$ and $x + m\omega$. We solve this issue in the following corollary.

Corollary 4.8 *Let $\ell \geq 1$, $\gamma > 0$, and σ, τ as in (LDT). Let $\mathcal{N}_1, \dots, \mathcal{N}_{(q-1)^2} \subset \mathbb{Z}$, $q = 2^{2d+1}$, be finite sets with the property that*

$$|n_i| > (\ell^{C(d)} |n_{i-1}|)^{C'(d)} \quad \text{if } n_i \in \mathcal{N}_i \text{ and } n_{i-1} \in \mathcal{N}_{i-1}, \quad 2 \leq i \leq (q-1)^2$$

and

$$\max_{n \in \mathcal{N}_{(q-1)^2}} |n| < \exp(c\ell^\sigma), \quad c = c(d).$$

For any $\ell \geq \ell_0(V, a, b, \gamma)$ there exists a set Ω_ℓ , depending on the choice of the finite sets $\mathcal{N}_i$, such that

$$\text{mes}(\Omega_\ell) \leq \ell^{C(a,b)} \left(\min_{n \in \mathcal{N}_1} |n| \right)^{-1}$$

and the following statement holds. For any $x_1, x_2 \in \mathbb{T}^d$, $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_\ell$, $E \in \mathbb{R}$, if $L(\omega, E) > \gamma$ and

$$\log |f_\ell(x_j, \omega, E)| \leq \ell L_\ell(\omega, E) - \ell^{1-\tau/2}, \quad j = 1, 2$$

then there exists $i \in \{1, \dots, (q-1)^2\}$, depending on x_1, x_2, ω, E , such that

$$\log |f_\ell(x_j + (n-1)\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2}, \quad j = 1, 2, \text{ for all } n \in \mathcal{N}_i.$$

Proof Let Ω_ℓ be the union of the sets of exceptional phases obtained by applying Proposition 4.7 with the following choices of finite sets:

$$\mathcal{N}_{k(q-1)+1}, \dots, \mathcal{N}_{(k+1)(q-1)}, \quad k = 0, \dots, q-2, \quad (4.1)$$

$$\bigcup_{j=1}^{q-1} \mathcal{N}_j, \bigcup_{j=1}^{q-1} \mathcal{N}_{(q-1)+j}, \dots, \bigcup_{j=1}^{q-1} \mathcal{N}_{(q-2)(q-1)+j}. \quad (4.2)$$

The set Ω_ℓ clearly satisfies the stated measure bound. Let $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_\ell$. By Proposition 4.7, with (4.2), there exists $k \in \{0, \dots, q-2\}$ such that

$$\log |f_\ell(x_1 + (n-1)\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2}$$

for all $n \in \bigcup_{j=1}^{q-1} \mathcal{N}_{k(q-1)+j}$. By Proposition 4.7, with (4.1), there exists $j \in \{1, \dots, q-1\}$ such that

$$\log |f_\ell(x_2 + n\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2}$$

for all $n \in \mathcal{N}_{k(q-1)+j}$. The conclusion holds with $i = k(q-1) + j$. $\square$

5 Factorization under (NDR) Condition

In this section we obtain a local factorization for the Dirichlet determinants with respect to the spectral variable, via Weierstrass' Preparation Theorem. It is important that the size of the polydisk on which the factorization holds is not too small and that the degree of the polynomial is not too large. With the aid of the basic tools from Section 2 the factorization at scale N can only be obtained on a polydisk of radius $\exp(-CN^{1-\tau})$ and with a polynomial of degree less than $CN^{1-\tau}$. This is too weak for our purposes. The main reason for considering (NDR) intervals is that for them we can get a much better factorization, as is shown in Proposition 5.2.

First, we show that if Λ is an (NDR) interval, then we have good control on the number of zeroes of $f_\Lambda(z, w, \cdot)$ in a small disk around E_0 .

Lemma 5.1 Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let $K \geq 1$, $\ell \geq \ell_0(V, a, b, |E_0|, \gamma)$, $r_0 = \exp(-\ell)$. Assume Λ is (K, ℓ) -(NDR) with respect to x_0, ω_0, E_0 . There exists $r_0/2 < r < r_0$ such that for any $(z, w) \in \mathbb{C}^d \times \mathbb{C}^d$,

$$|z - x_0| < \frac{c(V)r_0}{(K+1)^2}, \quad |w - \omega_0| < \frac{c(V)r_0}{|\Lambda|(K+1)^2}, \quad (5.1)$$

we have

$$\#\{E \in \mathcal{D}(E_0, r) : f_\Lambda(z, w, E) = 0\} \leq K, \\ \text{dist}(\{E \in \mathbb{C} : f_\Lambda(z, w, E) = 0\}, \{|E - E_0| = r\}) \geq \frac{r_0}{8(K+1)}.$$

Proof Let $\underline{\Lambda}$ be as in Definition 4.1 and $\Lambda' := \Lambda \setminus \underline{\Lambda}$. Let I be any of the connected components of Λ' . By the definition of (NDR) intervals and the covering form of (LDT) (see Lemma 2.25) we have

$$\text{dist}(E, \text{spec } H_I(x_0, \omega_0)) > \exp(-\ell) > 0 \text{ provided } |E - E_0| < \exp(-\ell).$$

Therefore, $H_{\Lambda'}(x_0, \omega_0)$ has no eigenvalues in $(E_0 - r_0, E_0 + r_0)$. Using Lemma 2.35, it follows that $H_\Lambda(x_0, \omega_0)$ has at most $|\underline{\Lambda}|$ eigenvalues in $(E_0 - r_0, E_0 + r_0)$. The condition (5.1) is such that

$$\|H_\Lambda(z, w) - H_\Lambda(x_0, \omega_0)\| \leq C(V)(|z - x_0| + |\Lambda||w - \omega_0|) \leq \frac{r_0}{32(K+1)^2}.$$

So, the conclusion follows by Lemma 2.37. $\square$

Now we can obtain the main result of this section. Note that when we define polydisks, $|\cdot|$ will stand for the maximum norm.

Proposition 5.2 Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let $K \geq 1$, $\ell \geq \ell_0(V, a, b, |E_0|, \gamma)$, $r_0 = \exp(-\ell)$. Assume Λ is (K, ℓ) -(NDR) with respect to x_0, ω_0, E_0 . There exist

$$P(z, w, E) = E^k + a_{k-1}(z, w)E^{k-1} + \cdots + a_0(z, w)$$

with a_j analytic in the polydisk

$$\mathcal{P} := \{|z - x_0| < c(V)r_0(K+1)^{-2}, |w - \omega_0| < c(V)r_0|\Lambda|^{-1}(K+1)^{-2}\},$$

and an analytic function $g(z, w, E)$, on $\mathcal{P} \times \mathcal{D}(E_0, r)$, $r_0/2 < r < r_0$, such that:

- (a) $f_\Lambda(z, w, E) = P(z, w, E)g(z, w, E)$ for any $(z, w, E) \in \mathcal{P} \times \mathcal{D}(E_0, r)$,
- (b) $g(z, w, E) \neq 0$ for any $(z, w, E) \in \mathcal{P} \times \mathcal{D}(E_0, r)$,
- (c) for any $(z, w) \in \mathcal{P}$, $P(z, w, \cdot)$ has no zeros in $\mathbb{C} \setminus \mathcal{D}(E_0, r)$,
- (d) $k \leq K$,
- (e) if $(x, \omega) \in \mathcal{P} \cap (\mathbb{T}^d \times \mathbb{T}^d(a, b))$, $E \in \mathcal{D}(E_0, r)$, and $\frac{r_0}{16(K+1)} \geq \exp(-|\Lambda|^{\sigma/2})$, then

$$\log |g(x, \omega, E)| > |\Lambda|L_{|\Lambda|}(\omega, E) - |\Lambda|^{1-\tau/2}. \quad (5.2)$$

Proof Due to Lemma 5.1 all conditions needed for Lemma 2.28 hold. This implies the statements (a)–(d). We just need to verify (e). Note that $P(x, \omega, E)$ is a product of factors $E - E_j^\Lambda(x, \omega)$, with $E_j^\Lambda(x, \omega) \in (E_0 - r, E_0 + r)$. So, for $E \in \mathcal{D}(E_0, r)$ we have $|P(x, \omega, E)| \leq (2r)^k < 1$. It follows that

$$\log |g(x, \omega, E)| > \log |f_\Lambda(x, \omega, E)|$$

for any $E \in \mathcal{D}(E_0, r)$. Let $r - \frac{r_0}{16(K+1)} < r' < r$. By Lemma 5.1, for any $|E - E_0| = r'$ we have

$$\text{dist}(E, \text{spec } H_N(x, \omega)) \geq \frac{r_0}{16(K+1)} \geq \exp(-|\Lambda|^{\sigma/2})$$

and hence, using Corollary 2.21,

$$\log |f_\Lambda(x, \omega, E)| > |\Lambda| L_{|\Lambda|}(\omega, E) - |\Lambda|^{1-\tau/2}$$

and (5.2) holds. Since the left-hand side of (5.2) is harmonic (in E) and the right-hand side is subharmonic, it follows that the estimate holds for all $E \in \mathcal{D}(E_0, r')$. Since this is true for r' arbitrarily close to r , the conclusion follows. $\square$

6 Elimination of Double Resonances Using Semialgebraic Sets

In this section we obtain a result on elimination of double resonances using semialgebraic sets. This result cannot yield the finite scale localization by itself. Instead, it will only serve as a catalyst for the sharper result on elimination of double resonances between (NDR) intervals from Section 7. The basis for elimination via semialgebraic sets is the following result, see [5, Proposition 5.1], [7, Lemma 9.9], [8, (1.5)].

Lemma 6.1 *Let $S \subset [0, 1]^d$, $d = d_1 + d_2$, be a semialgebraic set of degree B and $\text{mes}_d S < \eta$, $\log B \ll \log \frac{1}{\eta}$. We denote $(x, \omega) \in [0, 1]^{d_1} \times [0, 1]^{d_2}$ the product variable. Fix $\varepsilon > \eta^{\frac{1}{d}}$. Then there is a decomposition*

$$S = S_1 \cup S_2$$

S_1 satisfying

$$\text{mes}_{d_2}(\text{Proj}_\omega S_1) < B^C \varepsilon$$

and S_2 satisfying the transversality property

$$\text{mes}_{d_1}(S_2 \cap L) < B^C \varepsilon^{-1} \eta^{\frac{1}{d}}$$

for any d_1 -dimensional plane L s.t. $\max_{1 \leq j \leq d_2} |\text{Proj}_L(e_j)| < \frac{1}{100} \varepsilon$ (we denote by $e_1, \dots, e_{d_2}$ the ω -coordinate vectors). $\square$

We can now prove our result on semialgebraic elimination. In the sections to follow we will use $\bar{\ell}$ to denote the size of a (K, ℓ) -(NDR) interval. We also use the $\bar{\ell}$ notation in this section because we will only apply its results to (NDR) intervals.

Proposition 6.2 *Let $\alpha \in (0, 1)$, $\gamma > 0$, $\bar{\ell} \geq 1$, and σ as in (LDT). Let Λ_0, Λ_1 be intervals in $\mathbb{Z}$ such that $\bar{\ell}/10 \leq |\Lambda_0|, |\Lambda_1| \leq 10\bar{\ell}$ and $0 \in \Lambda_0 \cap \Lambda_1$. For any $x_0 \in \mathbb{T}^d$, $\bar{\ell} \geq \bar{\ell}_0(V, a, b, \alpha, \gamma)$, $1 \leq t_0 \leq \exp(c_0 \bar{\ell}^{\alpha\sigma})$, there exists a set $\Omega_{\bar{\ell}, t_0, x_0}$, $\text{mes}(\Omega_{\bar{\ell}, t_0, x_0}) < \bar{\ell}^{C(a,b)}/t_0$ such that the following holds. For all $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_{\bar{\ell}, t_0, x_0}$, $E \in \mathbb{R}$, such that $L(\omega, E) > \gamma$, we have*

$$\min(\|(H_{\Lambda_0}(x_0, \omega) - E)^{-1}\|, \|(H_{\Lambda_1}(x_0 + t\omega, \omega) - E)^{-1}\|) \leq \exp(\bar{\ell}^\alpha)$$

for all $t_0 \leq |t| \leq \exp(c_0 \bar{\ell}^{\alpha\sigma})$.

Proof Fix $x_0 \in \mathbb{T}^d$. Let $\mathcal{B}$ be the set of $(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}^d(a, b) \times \mathbb{R}$, such that $L(\omega, E) > \gamma$ and

$$\|(H_{\Lambda_0}(x_0, \omega) - E)^{-1}\| > \exp(\bar{\ell}^\alpha) \quad \text{and} \quad \|(H_{\Lambda_1}(x, \omega) - E)^{-1}\| > \exp(\bar{\ell}^\alpha).$$

Let $\tilde{V}$ be as in (2.23). Then $\mathcal{B}$ is contained in the semialgebraic set $\tilde{\mathcal{S}}$ of

$$(x, \omega, E) \in \mathbb{T}^d \times \mathbb{T}_J^d(a, b) \times [-C(V), C(V)], \quad J = \bar{\ell}^{C(a,b)}$$

defined by

$$\frac{1}{|\Lambda_1|J} \sum_{j=1}^J \log \|\tilde{M}_{|\Lambda_1|}(x + j\omega, \omega, E)\|_{\text{HS}} \geq \frac{\gamma}{2},$$

$$\|(\tilde{H}_{\Lambda_0}(x_0, \omega) - E)^{-1}\|_{\text{HS}} \geq \frac{1}{2} \exp(\bar{\ell}^\alpha), \quad \text{and} \quad \|(\tilde{H}_{\Lambda_1}(x, \omega) - E)^{-1}\|_{\text{HS}} \geq \frac{1}{2} \exp(\bar{\ell}^\alpha).$$

We used (2.24) (cf. proof of Lemma 2.34). Clearly the degree of $\tilde{\mathcal{S}}$ is less than $\bar{\ell}^C$. Furthermore, for $(x, \omega, E) \in \tilde{\mathcal{S}}$ we have $L_{|\Lambda_1|}(\omega, E) \geq \gamma/4$, and

$$\|(H_{\Lambda_0}(x_0, \omega) - E)^{-1}\| > \frac{1}{4} \exp(\bar{\ell}^\alpha), \quad \|(H_{\Lambda_1}(x, \omega) - E)^{-1}\| > \frac{1}{4} \exp(\bar{\ell}^\alpha). \quad (6.1)$$

Applying the Wegner estimate from Proposition 2.26 (recall Remark 2.30) to $H_{\Lambda_1}(x, \omega)$ with $E_0 \in \text{spec } H_{\Lambda_0}(x_0, \omega)$ we obtain that

$$\text{mes}(\text{Proj}_{(x, \omega)} \tilde{\mathcal{S}}) < \exp(-c\bar{\ell}^{\alpha\sigma}).$$

Indeed, if (x, ω, E) satisfies (6.1), then for some $E_0 \in \text{spec } H_{\Lambda_0}(x_0, \omega)$ one has $\text{dist}(\text{spec } H_{\Lambda_1}(x, \omega), E_0) < 8 \exp(-\bar{\ell}^\alpha)$. Applying Proposition 2.26 and taking a union over E_0 proves the claim.

Set $\mathcal{S} := \text{Proj}_{(x, \omega)} \tilde{\mathcal{S}}$. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ be the decomposition of $\mathcal{S}$ afforded by Lemma 6.1 with $\varepsilon = 200/t_0$. Note that to derive the conclusion we just need that $(\{x_0 + t\omega\}, \omega) \notin \mathcal{S}$. Define, with T being integers in the range $t_0 \leq |t| \leq \exp(c_0 \bar{\ell}^{\alpha\sigma})$,

$$\Omega_{\bar{\ell}, t_0, x_0}(\Lambda_0, \Lambda_1) = \text{Proj}_\omega \mathcal{S}_1 \cup \left(\bigcup_{t \in T} \Omega_{\bar{\ell}, t, x_0}(\Lambda_0, \Lambda_1) \right),$$

$$\Omega_{\bar{\ell}, t, x_0}(\Lambda_0, \Lambda_1) := \{\omega : (\{x_0 + t\omega\}, \omega) \in \mathcal{S}_2\},$$

$$\Omega_{\bar{\ell}, t_0, x_0} = \bigcup_{\Lambda_0, \Lambda_1} \Omega_{\bar{\ell}, t_0, x_0}(\Lambda_0, \Lambda_1).$$

Note that $\text{Proj}_\omega \mathcal{S}_1 \supset \{\omega : (\{x_0 + t\omega\}, \omega) \in \mathcal{S}_1\}$ and due to our assumptions there are less than $C\bar{\ell}^4$ possible choices for Λ_0, Λ_1 . We just need to check the estimate on the measure of $\Omega_{\bar{\ell}, t_0, x_0}$. To this end, note that the set of $(\{x_0 + t\omega\}, \omega)$, with $\omega \in [0, 1]^d$, is contained in a union of planes L_i , $i \leq |t|^d$. The planes L_i are parallel to $(t\omega, \omega)$, $\omega \in \mathbb{R}^d$, and therefore

$$|\text{Proj}_{L_i} e_j| \leq \frac{1}{|t|} \leq \frac{1}{t_0} < \frac{\varepsilon}{100} \quad \text{for all } i, j$$

(e_j are as in Lemma 6.1). Then by Lemma 6.1,

$$\text{mes}(\text{Proj}_\omega \mathcal{S}_1) < \bar{\ell}^C/t_0,$$

$$\text{mes}(\Omega_{\bar{\ell}, t, x_0}(\Lambda_0, \Lambda_1)) = \sum_i \text{mes}(\mathcal{S}_2 \cap L_i) \lesssim |t|^d \bar{\ell}^C t_0 \exp(-c\bar{\ell}^{\alpha\sigma}) \leq \exp(-c'\bar{\ell}^{\alpha\sigma}),$$

and the conclusion follows. $\square$

In Proposition 6.2 for every $x_0 \in \mathbb{T}^d$ we eliminate a small set of resonant frequencies, in dependence on x_0 . For the purposes of Section 7 we will now eliminate $x_0 \in \mathbb{T}^d$ in dependence on ω which lie in the complement of a small set in $\mathbb{T}^d$.

Corollary 6.3 *We use the same assumptions and notation as in Proposition 6.2. For any $\bar{\ell} \geq \bar{\ell}_0(V, a, b, \alpha, \gamma)$ and $1 \leq t_0 \leq \exp(c_0 \bar{\ell}^{\alpha\sigma})$, there exists a set $\Omega_{\bar{\ell}, t_0}$, $\text{mes}(\Omega_{\bar{\ell}, t_0}) < \bar{\ell}^{C(a,b)} t_0^{-\frac{1}{2}}$, such that for any $\omega \notin \Omega_{\bar{\ell}, t_0}$ there exists a set $\mathcal{B}_{\bar{\ell}, t_0, \omega}$, $\text{mes}(\mathcal{B}_{\bar{\ell}, t_0, \omega}) < \bar{\ell}^{C(a,b)} t_0^{-\frac{1}{2}}$, and the following holds. For any $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_{\bar{\ell}, t_0}$, $x \in \mathbb{T}^d \setminus \mathcal{B}_{\bar{\ell}, t_0, \omega}$, $E \in \mathbb{R}$, such that $L(\omega, E) > \gamma$, we have*

$$\min(\|(H_{\Lambda_0}(x, \omega) - E)^{-1}\|, \|(H_{\Lambda_1}(x + t\omega, \omega) - E)^{-1}\|) \leq \exp(\bar{\ell}^\alpha)$$

for all $t_0 \leq |t| \leq \exp(c_0 \bar{\ell}^{\alpha\sigma})$.

Proof Let $\mathcal{B}_{\bar{\ell}, t_0}$ be the set of $x \in \mathbb{T}^d$, $\omega \in \mathbb{T}^d(a, b)$ such that $\omega \in \Omega_{\bar{\ell}, t_0, x}$, with $\Omega_{\bar{\ell}, t_0, x}$ as in Proposition 6.2. Using Chebyshev's inequality we get that there exists a set $\Omega_{\bar{\ell}, t_0}$, $\text{mes}(\Omega_{\bar{\ell}, t_0}) \leq \text{mes}(\mathcal{B}_{\bar{\ell}, t_0})^{\frac{1}{2}}$ such that for $\omega \notin \Omega_{\bar{\ell}, t_0}$ we have $\text{mes}(\mathcal{B}_{\bar{\ell}, t_0, \omega}) \leq \text{mes}(\mathcal{B}_{\bar{\ell}, t_0})^{\frac{1}{2}}$, where

$$\mathcal{B}_{\bar{\ell}, t_0, \omega} = \left\{ x : (x, \omega) \in \mathcal{B}_{\bar{\ell}, t_0} \right\}.$$

Now the conclusion follows by Proposition 6.2. $\square$

7 Elimination of Double Resonances under (NDR) Condition

To eliminate double resonances between (NDR) intervals we combine the elimination from Corollary 6.3 with another tool, resultants of polynomials (see Section 2.9). The biggest problem with the estimate in Corollary 6.3 is that, due to the weakness of the measure estimate for the set of resonant phases, we cannot apply it simultaneously to $x_0 + n\omega$, $n \in [1, N]$, as is needed for finite scale localization. In the next lemma we will obtain a sharper (local) measure estimate via Cartan's estimate applied to the resultant of the polynomials from the Weierstrass preparation theorem in the E -variable, under the (NDR) condition, see Proposition 5.2. For Cartan's estimate to be effective, we need a point at which we have a good lower bound on the modulus of the resultant. This point will come from Corollary 6.3.

Lemma 7.1 *Assume $x_0 \in \mathbb{T}^d$, $\omega_0 \in \mathbb{T}^d(a, b)$, $E_0 \in \mathbb{R}$, and $L(\omega_0, E_0) > \gamma > 0$. Let σ, τ as in (LDT) and*

$$1 \leq \ell \leq \bar{\ell} \leq \exp(\ell), \quad 1 \leq K \leq \bar{\ell}^{\frac{1-\tau}{10}}, \quad \exp(\ell^2) \leq |t| \leq \exp(\bar{\ell}^{\frac{\sigma(1-\tau)}{4}}).$$

Let Λ_j , $j = 0, 1$, be intervals in $\mathbb{Z}$ such that $\bar{\ell}/10 \leq |\Lambda_j| \leq 10\bar{\ell}$, $0 \in \Lambda_j$, and Λ_j is (K, ℓ) -(NDR) with respect to x_j , ω_0 , E_0 , $x_j := x_0 + j t \omega_0$, $j = 0, 1$.

For any $\ell \geq \ell_0(V, a, b, |E_0|, \gamma)$ there exists a set $\Omega_{\bar{\ell}}$, $\text{mes}(\Omega_{\bar{\ell}}) < \exp(-\ell^2/4)$, such that if $\omega_0 \notin \Omega_{\bar{\ell}}$ the following holds. There exists a set $\mathcal{B}_{\bar{\ell}, t} = \mathcal{B}_{\bar{\ell}, t}(x_0, \omega_0, E_0)$, such that

$$\text{mes}(\mathcal{B}_{\bar{\ell}, t}) < \exp(-\bar{\ell}^{\frac{1-\tau}{6d}})/|t|^d,$$

and for any $(x, \omega) \in (\mathbb{T}^d \times \mathbb{T}^d(a, b)) \setminus \mathcal{B}_{\bar{\ell}, t}$, $E \in \mathbb{R}$,

$$|x - x_0| < \exp(-4\ell), \quad |\omega - \omega_0| < \exp(-4\ell)/|t|, \quad |E - E_0| < \exp(-\ell)/2, \quad (7.1)$$

we have

$$\max_{j=0,1} (\log |f_{\Lambda_j}(x + jt\omega, \omega, E)| - |\Lambda_j|L_{|\Lambda_j|}(\omega, E) + 2|\Lambda_j|^{1-\tau/2}) > 0.$$

Proof We start by extracting the information we need from the semialgebraic elimination. By Proposition 2.9 we can guarantee that $L(\omega_0, E) > \frac{\gamma}{2}$ for any $|E - E_0| < \exp(-\ell)$ (provided ℓ is large enough). Let $\Omega_{\bar{\ell}, t_0}, \mathcal{B}_{\bar{\ell}, t_0, \omega_0}$ be the sets from Corollary 6.3, with $\alpha = (1 - \tau)/3$, $t_0 = \exp(\ell^2)$, and $\gamma/2$ instead of γ . Since we are assuming $\bar{\ell} \leq \exp(\ell)$ it follows that

$$\text{mes}(\Omega_{\bar{\ell}, t_0}), \text{mes}(\mathcal{B}_{\bar{\ell}, t_0, \omega_0}) < \exp(-\ell^2/4).$$

We set $\Omega_{\bar{\ell}} := \Omega_{\bar{\ell}, t_0}$ and we assume $\omega_0 \notin \Omega_{\bar{\ell}}$. Let $x'_0 \in \mathbb{T}^d$, $|x'_0 - x_0| < \exp(-\ell^2/(5d))$ be such that $x'_0 \notin \mathcal{B}_{\bar{\ell}, t_0, \omega_0}$. Then we have

$$\min (\|(H_{\Lambda_0}(x'_0, \omega_0) - E)^{-1}\|, \|(H_{\Lambda_1}(x'_0 + t\omega_0, \omega_0) - E)^{-1}\|) \leq \exp(\bar{\ell}^{\frac{1-\tau}{3}}) \quad (7.2)$$

for all $|E - E_0| < \exp(-\ell)$ and $\exp(\ell^2) \leq |t| \leq \exp(c\bar{\ell}^{\sigma(1-\tau)/3})$. In particular this holds for t satisfying our assumptions.

Next we collect the relevant facts about the factorization of f_{Λ_j} . Let $f_{\Lambda_j} = P_j g_j$, $j = 0, 1$, be the factorizations from Proposition 5.2. We have that

$$P_j(z, \omega, E) = E^{k_j} + a_{j, k_j-1}(z, \omega)E^{k_j-1} + \cdots + a_{j, 0}(z, \omega), \quad k_j \leq K$$

with $a_{j, i}$ analytic on a polydisk containing

$$\mathcal{P}_j := \{(z, w) \in \mathbb{C}^{2d} : |z - x_j| < \exp(-2\ell), |w - \omega_0| < \exp(-3\ell)\},$$

and all the zeros of $P_j(z, w, \cdot)$, $(z, w) \in \mathcal{P}_j$, are contained in $\mathcal{D}(E_0, \exp(-\ell))$ (we used the assumptions that $K \leq \bar{\ell}^{\frac{1-\tau}{10}}$, $\bar{\ell} \leq \exp(\ell)$, $|\Lambda_j| \leq 10\bar{\ell}$). We also have

$$\log |g_j(x, \omega, E)| > |\Lambda_j|L_{|\Lambda_j|}(\omega, E) - |\Lambda_j|^{1-\tau/2} \quad (7.3)$$

for any $(x, \omega) \in \mathcal{P}_j \cap (\mathbb{T}^d \times \mathbb{T}^d(a, b))$ and $E \in \mathcal{D}(E_0, \exp(-\ell)/2)$. Note that the condition $\frac{r_0}{16(K+1)} \geq \exp(-|\Lambda_j|^{\sigma/2})$ needed for part (e) of Proposition 5.2 is satisfied because our restrictions on t imply $\bar{\ell} \geq \ell^{\frac{8}{\sigma(1-\tau)}}$.

Now we can proceed with the elimination via resultants and Cartan's estimate (recall Sections 2.5 and 2.9). Set

$$R(z, w) = \text{Res}(P_0(z, w, \cdot), P_1(z + tw, w, \cdot)).$$

Then R is well-defined on the polydisk

$$\mathcal{P} := \{(z, w) \in \mathbb{C}^{2d} : |z - x'_0| < \exp(-3\ell), |w - \omega_0| < \exp(-3\ell)/|t|\}.$$

By the definition of the resultant (see (2.22)) and the properties of P_j , we have

$$R(z, w) = \prod (E_i^{\Lambda_0}(z, w) - E_j^{\Lambda_1}(z + tw, w)),$$

where the product is only in terms of eigenvalues contained in $\mathcal{D}(E_0, \exp(-\ell))$. This implies

$$\sup |R(z, w)| \leq (2 \exp(-\ell))^{k_1 k_2} < 1.$$

By (7.2) applied to

$$E \in (\text{spec } H_{\Lambda_0}(x'_0, \omega_0) \cup \text{spec } H_{\Lambda_1}(x'_0 + t\omega_0, \omega_0)) \cap \mathcal{D}(E_0, \exp(-\ell))$$

we get that

$$\left| E_i^{\Lambda_0}(x'_0, \omega_0) - E_j^{\Lambda_1}(x'_0 + t\omega_0, \omega_0) \right| \geq \exp(-\bar{\ell}^{\frac{1-\tau}{3}})$$

for the pairs of eigenvalues contained in $\mathcal{D}(E_0, \exp(-\ell))$. It follows that

$$|R(x'_0, \omega_0)| \geq \exp(-K^2 \bar{\ell}^{\frac{1-\tau}{3}}).$$

Using Cartan's estimate (recall Lemma 2.18) with

$$m = -K^2 \bar{\ell}^{\frac{1-\tau}{3}}, \quad M = 0, \quad H = \bar{\ell}^{\frac{1-\tau}{3}}$$

we have that

$$|R(x, \omega)| > \exp\left(-CK^2 \bar{\ell}^{\frac{2(1-\tau)}{3}}\right)$$

for all $(x, \omega) \in (6^{-1}\mathcal{P} \cap \mathbb{R}^{2d}) \setminus \mathcal{B}$ with, $\mathcal{B} = \mathcal{B}(\Lambda_0, \Lambda_1)$,

$$\text{mes}(\mathcal{B}) \leq \frac{C \exp(-6d\ell)}{|t|^d} \exp(-H^{\frac{1}{2d}}).$$

Note that $6^{-1}\mathcal{P} \cap \mathbb{R}^{2d}$ contains the points (x, ω) satisfying (7.1). Here $\lambda\mathcal{P}$ for $\lambda > 0$ denotes the rescaling of the polydisk $\mathcal{P}$ by λ homothetically about the center of $\mathcal{P}$. We let $\mathcal{B}_{\bar{\ell}, t} = \cup \mathcal{B}(\Lambda_0, \Lambda_1)$. Finally, Lemma 2.29 implies

$$\max(|P_0(x, \omega, E)|, |P_1(x + t\omega, \omega, E)|) \geq \exp\left(-CK^3 \bar{\ell}^{\frac{2(1-\tau)}{3}}\right) > \exp\left(-\bar{\ell}^{1-\tau}\right),$$

for all E and $(x, \omega) \notin \mathcal{B}_{\bar{\ell}, t}$, and the conclusion follows using the factorization of the determinants and (7.3). $\square$

We can now use a covering argument to prove a global version of the previous result.

Proposition 7.2 *Let $\gamma > 0$, σ, τ as in (LDT), and*

$$1 \leq \ell \leq \bar{\ell} \leq \exp(\ell), \quad 1 \leq K \leq \bar{\ell}^{\frac{1-\tau}{10}}, \quad \exp(\ell^2) \leq N.$$

Let Λ_j , $j = 0, 1$, be intervals in $\mathbb{Z}$ such that $\bar{\ell}/10 \leq |\Lambda_j| \leq 10\bar{\ell}$, $0 \in \Lambda_j$, $j = 0, 1$.

For $\ell \geq \ell_0(V, a, b, \gamma)$ there exists $\alpha = \alpha(a, b)$ such that if $N \leq \exp(\bar{\ell}^\alpha)$, there exists a set $\Omega_{N, \bar{\ell}}$, $\text{mes}(\Omega_{N, \bar{\ell}}) < 2 \exp(-\ell^2/4)$, and for every $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_{N, \bar{\ell}}$ there exists a set $\mathcal{B}_{N, \bar{\ell}, \omega}$, $\text{mes}(\mathcal{B}_{N, \bar{\ell}, \omega}) < \exp(-\bar{\ell}^\alpha)$ such that the following statement holds. For any $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_{N, \bar{\ell}}$, $x \in \mathbb{T}^d \setminus \mathcal{B}_{N, \bar{\ell}, \omega}$, $E \in \mathbb{R}$, and $m_0, m_1 \in [1, N]$ such that $|m_0 - m_1| \geq \exp(\ell^2)$, we have that if $L(\omega, E) > \gamma$ and Λ_j , $j = 0, 1$, are $(K, \ell, 1/2)$ -(NDR) intervals with respect to $x + m_j\omega$, ω , E , then

$$\max_{j=0,1} (\log |f_{\Lambda_j}(x + m_j\omega, \omega, E)| - |\Lambda_j| L_{|\Lambda_j|}(\omega, E) + 2|\Lambda_j|^{1-\tau/2}) > 0.$$

Proof By Lemma 2.20, if the conclusion does not hold, then E is in a neighborhood of

$$\text{spec } H_{\Lambda_0}(x + m_0\omega, \omega) \cup \text{spec } H_{\Lambda_1}(x + m_1\omega, \omega).$$

Therefore, it is enough to prove the result with $E \in (-C(V), C(V))$.

Let $\Omega_{\bar{\ell}}$ be the set from Lemma 7.1 (with $\gamma/2$ instead of γ). Let

$$\begin{aligned} \{z : |z - x_k| < \exp(-\ell^{3/2})\}, \quad k \lesssim \exp(d\ell^{3/2}) \\ \{\omega : |\omega - \omega_{k'}| < \exp(-\ell^{3/2})/N\}, \quad k' \lesssim N^d \exp(d\ell^{3/2}) \\ \{E : |E - E_{k''}| < \exp(-\ell^{3/2})\}, \quad k'' \lesssim C(V) \exp(\ell^{3/2}) \end{aligned}$$

be covers of $\mathbb{T}^d$, $\mathbb{T}^d(a, b) \setminus \Omega_{\bar{\ell}}$, and $(-C(V), C(V))$, respectively. Note that if $|\omega - \omega_{k'}| < \exp(-\ell^{3/2})/N$, $|E - E_{k''}| < \exp(-\ell^{3/2})$, and $L(\omega, E) > \gamma > 0$, then, by Proposition 2.9, $L(\omega_{k'}, E_{k''}) > \gamma/2$, provided ℓ is large enough. So, if $L(\omega_{k'}, E_{k''}) > \gamma/2$ we let $\mathcal{B}_{\bar{\ell}, t}(x_k, \omega_{k'}, E_{k''})$ be the set from Lemma 7.1 (with $\gamma/2$ instead of γ). Otherwise we let $\mathcal{B}_{\bar{\ell}, t}(x_k, \omega_{k'}, E_{k''})$ be the empty set. Let

$$\mathcal{B}_{N, \bar{\ell}} = \bigcup_{k, k', k'', t, m} S_m \left(\mathcal{B}_{\bar{\ell}, t}(x_k, \omega_{k'}, E_{k''}) \right),$$

where $S_m(x, \omega) = (x - m\omega, \omega)$, $m \in [1, N]$, $|t| \in [\exp(\ell^2), N]$. Then we have

$$\text{mes} \left(\mathcal{B}_{N, \bar{\ell}} \right) \leq CN^{d+2} \exp((2d+1)\ell^{3/2}) \exp \left(-\bar{\ell}^{\frac{1-\tau}{6d}} \right).$$

Note that, due to the restriction required on $|t|$ by Lemma 7.1 we need to have $N \leq \exp(\bar{\ell}^{\frac{\sigma(1-\tau)}{4}})$. We choose $\alpha = \alpha(\sigma, \tau) = \alpha(a, b)$ such that if $N \leq \exp(\bar{\ell}^\alpha)$, the previous restriction is satisfied and

$$\text{mes}(\mathcal{B}_{N, \bar{\ell}}) < \exp(-2\bar{\ell}^\alpha).$$

Using Chebyshev's inequality we get that there exists a set $\tilde{\Omega}_{N, \bar{\ell}}$, $\text{mes}(\tilde{\Omega}_{N, \bar{\ell}}) < \exp(-\bar{\ell}^\alpha)$ such that for $\omega \notin \tilde{\Omega}_{N, \bar{\ell}}$ we have $\text{mes}(\mathcal{B}_{N, \bar{\ell}, \omega}) < \exp(-\bar{\ell}^\alpha)$, where

$$\mathcal{B}_{N, \bar{\ell}, \omega} = \left\{ x : (x, \omega) \in \mathcal{B}_{N, \bar{\ell}} \right\}.$$

We set $\Omega_{N, \bar{\ell}} = \Omega_{\bar{\ell}} \cup \tilde{\Omega}_{N, \bar{\ell}}$ and we claim that the conclusion follows with our choice of sets. Indeed, let $x, \omega, E, m_j, \Lambda_j$ as in the assumptions. There exist $x_k, \omega_{k'}, E_{k''}$ such that

$$|x + m_0\omega - x_k| < \exp(-\ell^{3/2}), \quad |\omega - \omega_{k'}| < \exp(-\ell^{3/2})/N, \quad |E - E_{k''}| < \exp(-\ell^{3/2})$$

(recall that there's nothing to check if $E \notin (-C(V), C(V))$). Since $L(\omega, E) > \gamma$, we have $L(\omega_{k'}, E_{k''}) > \gamma/2$, and since $\Lambda_j, j = 0, 1$, are $(K, \ell, 1/2)$ -(NDR) with respect to $x + m_j\omega$, ω, E , it follows (recall Remark 4.3) that $\Lambda_j, j = 0, 1$, are (K, ℓ) -(NDR) with respect to

$$x_k + jt\omega_{k'}, \quad \omega_{k'}, \quad E_{k''}, \quad t = m_1 - m_0.$$

The choice of our exceptional sets guarantees $(x + m_0\omega, \omega) \notin \mathcal{B}_{\bar{\ell}, t}(x_k, \omega_{k'}, E_{k''})$ and the conclusion follows from Lemma 7.1. $\square$

8 Finite and Full Scale Localization: Proofs of Theorems A, B, C

Combining elimination of resonances under the (NDR) condition with the covering form of (LDT) yields the following result on elimination of resonances at a given scale, as needed for obtaining Theorem A via Proposition 3.1.

Lemma 8.1 *Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$, and σ, τ as in (LDT). For $N \geq N_0(V, a, b, \gamma, \varepsilon)$, there exists a set Ω_N ,*

$$\text{mes}(\Omega_N) < \exp(-(\log N)^{\varepsilon\sigma}),$$

such that for $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$ there exists a set $\mathcal{B}_{N,\omega}$,

$$\text{mes}(\mathcal{B}_{N,\omega}) < \exp(-\exp((\log N)^{\varepsilon\sigma}))$$

and the following holds for any $\omega \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $x \in \mathbb{T}^d \setminus \mathcal{B}_{N,\omega}$, and $E \in \mathbb{R}$ such that $L(\omega, E) > \gamma$. If $m_0 \in [1, N]$ is such that

$$\log |f_\Lambda(x, \omega, E)| \leq |\Lambda| L_{|\Lambda|}(\omega, E) - |\Lambda|^{1-\tau/4}$$

for all intervals $\Lambda \subset [1, N]$ satisfying $\text{dist}(m_0, [1, N] \setminus \Lambda) > |\Lambda|/100$, $|\Lambda| \geq (\log N)^\varepsilon$, then for any $\bar{\ell} \geq \exp((\log N)^{3\varepsilon\sigma/2})$ we have

$$\log |f_{\bar{\ell}}(x + (m-1)\omega, \omega, E)| > \bar{\ell} L_{\bar{\ell}}(\omega, E) - \bar{\ell}^{1-\tau/2} \quad (8.1)$$

for any $m \in [1, N - \bar{\ell} + 1]$ such that $|m - m_0| \geq \bar{\ell} + 2 \exp(\ell^2)$, $\ell = \lceil (\log N)^{2\varepsilon} \rceil$.

Proof With $q = 2^{2d+1}$, let

$$1 \ll \underline{C}_1 \ll \bar{C}_1 \ll \cdots \ll \underline{C}_{(q-1)^2} \ll \bar{C}_{(q-1)^2}$$

be constants such that the sets

$$\mathcal{N}_k = \{n \in \mathbb{Z} : \exp(\underline{C}_k \ell^{\sigma/2}) \leq |n| \leq \exp(\bar{C}_k \ell^{\sigma/2})\}$$

satisfy the assumptions of Corollary 4.8. Let

$$\bar{\ell}_k = \lfloor \exp(\bar{C}_k \ell^{\sigma/2}) \rfloor, \quad K_k = \lceil \exp(\underline{C}_k \ell^{\sigma/2}) \rceil.$$

We choose the constants $\underline{C}_k, \bar{C}_k$ such that we also have $K_k \leq \frac{1-\tau}{\bar{\ell}_k^{10}}$ as needed for Proposition 7.2. The conclusion will follow by choosing

$$\Omega_N := \Omega_\ell \cup \left(\bigcup_k \Omega_{N, \bar{\ell}_k} \right), \quad \mathcal{B}_{N,\omega} := \bigcup_k \mathcal{B}_{N, \bar{\ell}_k, \omega},$$

where Ω_ℓ is the set from Corollary 4.8 and $\Omega_{N, \bar{\ell}_k}, \mathcal{B}_{N, \bar{\ell}_k, \omega}$ are the sets from Proposition 7.2. Note that

$$\begin{aligned} \text{mes}(\Omega_N) &< \ell^C \exp(-\underline{C}_1 \ell^{1/2}) + 2(q-1)^2 \exp(-\ell^2/4) \\ &< \exp(-\ell^{1/2}) < \exp(-(\log N)^{\varepsilon\sigma}), \\ \text{mes}(\mathcal{B}_{N,\omega}) &< (q-1)^2 \exp(-\bar{\ell}_1^\alpha) < \exp(-\exp(\ell^{1/2})) < \exp(-\exp((\log N)^{\varepsilon\sigma})), \end{aligned}$$

provided we choose $\underline{C}_1, \bar{C}_1$ large enough.

Let $\Lambda_0 = [m'_0, m''_0] \subset [1, N]$, $|\Lambda_0| = \ell$, be an interval such that $\text{dist}(m_0, [1, N] \setminus \Lambda_0) > |\Lambda_0|/100$. Then by our assumptions

$$\log |f_{\Lambda_0}(x, \omega, E)| = \log |f_\ell(x + (m'_0 - 1)\omega, \omega, E)| \leq \ell L_\ell(\omega, E) - \ell^{1-\tau/2}.$$

By Corollary 4.8, for each $m' \in [1, N]$ we either have

$$\log |f_\ell(x + (m' - 1)\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2}$$

or there exists $k = k(m'_0, m')$ such that $[-\bar{\ell}_k, \bar{\ell}_k]$ is $(K_k, \ell, 1/2)$ -(NDR) with respect to both $x + (m'_0 - 1)\omega, \omega, E$ and $x + (m' - 1)\omega, \omega, E$. By our assumptions we must also have

$$\log |f_\Lambda(x + (m'_0 - 1)\omega, \omega, E)| \leq |\Lambda|L_{|\Lambda|}(\omega, E) - 2|\Lambda|^{1-\tau/2},$$

where Λ is any of the intervals $[-\bar{\ell}_k, \bar{\ell}_k] \cap ([1, N] - m'_0 + 1)$. Note that by the definition of (NDR) intervals, if $[-\bar{\ell}_k, \bar{\ell}_k]$ is (NDR) with respect to $x + (m'_0 - 1)\omega, \omega, E$, then $[-\bar{\ell}_k, \bar{\ell}_k] \cap ([1, N] - m'_0 + 1)$ is also (NDR) with respect to $x + (m'_0 - 1)\omega, \omega, E$. It follows from Proposition 7.2 that for any $m' \in [1, N]$ such that $|m' - m'_0| \geq \exp(\ell^2)$, we have

$$\log |f_{\Lambda(m')}(x + (m' - 1)\omega, \omega, E)| > |\Lambda(m')|L_{|\Lambda(m')|}(\omega, E) - 2|\Lambda(m')|^{1-\tau/2}, \quad (8.2)$$

where $\Lambda(m')$ is either $[1, \ell]$ or $[-\bar{\ell}_k, \bar{\ell}_k]$, with $k = k(m'_0, m')$ as above. In addition, Proposition 7.2 guarantees that if $\Lambda(m') = [-\bar{\ell}_k, \bar{\ell}_k]$, then (8.2) also holds for any subintervals of $[-\bar{\ell}_k, \bar{\ell}_k]$ with length greater than $\bar{\ell}_k/10$.

Take $m \in [1, N - \bar{\ell} + 1]$ such that $|m - m_0| \geq \bar{\ell} + 2\exp(\ell^2)$. This choice of m guarantees that

$$[m, m + \bar{\ell} - 1] \subset [1, N] \setminus \{m' : |m' - m'_0| \geq \exp(\ell^2)\}.$$

So (8.1) follows from (8.2) and the covering form of (LDT) (see Lemma 2.25). Indeed, each point of $[m, m + \bar{\ell} - 1]$ is covered by an interval of the form $\Lambda(m') \cap [m, m + \bar{\ell} - 1]$ for some $m' \in [m - 1, m + \bar{\ell} - 1 - \ell]$, on which the large deviations estimate holds. Note that we wanted to make sure that $m' + [1, \ell] \subset [m, m + \bar{\ell} - 1]$, because if a large deviations estimate holds on $m' + [1, \ell]$, it does not necessarily hold on $(m' + [1, \ell]) \cap [m, m + \bar{\ell} - 1]$. On the other hand, we already noted that when a large deviations estimate holds on $m' + [-\bar{\ell}_k, \bar{\ell}_k]$ it also holds on $(m' + [-\bar{\ell}_k, \bar{\ell}_k]) \cap [m, m + \bar{\ell} - 1]$. Finally note that the lower bound on $\bar{\ell}$ is such that

$$|\Lambda(m')| \leq \bar{\ell}_{(q-1)^2} \leq \bar{\ell}^{\sigma/2},$$

as needed for the covering form of (LDT). $\square$

The following propositions are more detailed versions of Theorems A and B.

Proposition 8.2 *Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$, σ as in (LDT), and $\Omega_N, \mathcal{B}_{N,\omega}$ as in Lemma 8.1. For any $N \geq N_0(V, a, b, \gamma, \varepsilon)$, $\omega_0 \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $x_0 \in \mathbb{T}^d \setminus \mathcal{B}_{N,\omega}$, and any eigenvalue $E_0 = E_j^{(N)}(x_0, \omega_0)$, such that $L(\omega_0, E_0) > \gamma$, there exists an interval $I = I(x_0, \omega_0, E_0) \subset [1, N]$,*

$$|I| < \exp((\log N)^{5\varepsilon}),$$

such that for any $(x, \omega) \in \mathbb{T}^d \times \mathbb{T}^d(a, b)$, with

$$|x - x_0|, |\omega - \omega_0| < \exp(-\exp((\log N)^{2\varepsilon\sigma})),$$

we have

$$\left| \psi_j^{(N)}(x, \omega; m) \right| < \exp\left(-\frac{\gamma}{4} \text{dist}(m, I)\right),$$

provided $\text{dist}(m, I) > \exp((\log N)^{2\varepsilon\sigma})$.

Proof Let ψ a choice of normalized eigenvector for E_0 and let m_0 be such that

$$|\psi(m_0)| = \max_{m \in [1, N]} |\psi(m)|.$$

Then m_0 satisfies the assumptions of Lemma 8.1, because otherwise we can use Poisson's formula (2.18) and Lemma 2.24 to contradict the maximality of $|\psi(m_0)|$. So, if we let

$$\bar{\ell} = \lceil \exp((\log N)^{3\varepsilon/2}) \rceil,$$

by Lemma 8.1, (8.1) holds for $m \in [1, N] \setminus I$, $I = [1, N] \cap [m_0 - \bar{\ell} - 2\exp(\ell^2), m_0 + \bar{\ell} + 2\exp(\ell^2)]$, $\ell = \lceil (\log N)^{2\varepsilon} \rceil$. The conclusion follows immediately from Proposition 3.1 (with $\bar{\ell}$ instead of ℓ). $\square$

Proposition 8.3 Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$, and $\Omega_N, \mathcal{B}_{N,\omega}$ as in Proposition 8.2. For any $N \geq N_0(V, a, b, \gamma, \varepsilon)$, $\omega_0 \in \mathbb{T}^d(a, b) \setminus \Omega_N$, $x_0 \in \mathbb{T}^d \setminus \mathcal{B}_{N,\omega}$, and any eigenvalue $E_0 = E_j^{(N)}(x_0, \omega_0)$, such that $L(\omega_0, E_0) > \gamma$, if we take $I = I(x_0, \omega_0, E_0)$ as in Proposition 8.2, then

$$\left| E_k^{(N)}(x, \omega) - E_j^{(N)}(x, \omega) \right| > \exp(-C(V)|I|),$$

for any $k \neq j$ and any $(x, \omega) \in \mathbb{T}^d \times \mathbb{T}^d(a, b)$, with

$$|x - x_0|, |\omega - \omega_0| < \exp(-\exp((\log N)^{2\varepsilon})).$$

Proof From the proof of Proposition 8.2 we know that (8.1) holds for $m \in [1, N] \setminus I$, $I = [1, N] \cap [m_0 - \bar{\ell} - 2\exp(\ell^2), m_0 + \bar{\ell} + 2\exp(\ell^2)]$, $\ell = \lceil (\log N)^{2\varepsilon} \rceil$. The conclusion follows from Proposition 3.3 (with $\bar{\ell}$ instead of ℓ). Since E_0 is restricted to the spectrum, we can choose the constant C from Proposition 3.3 independent of E_0 . $\square$

Next we establish two auxiliary results needed for the proof of Theorem C.

Lemma 8.4 Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$ and $\hat{\Omega}_N, \hat{\mathcal{B}}_{N,\omega}$ as in Theorem C. For $N \geq N_0(V, a, b, \gamma, \varepsilon)$, $\omega \in \mathbb{T}^d(a, b) \setminus \hat{\Omega}_N$, $x \in \mathbb{T}^d \setminus \hat{\mathcal{B}}_{N,\omega}$, if

$$L\left(\omega, E_j^{[-N, N]}(x, \omega)\right) > \frac{3\gamma}{2} \quad \text{and} \quad I = I\left(x, \omega, E_j^{[-N, N]}(x, \omega)\right) \subset [-N/2, N/2]$$

then for any $N \leq N' \leq \exp((\log N)^{\frac{1}{10\varepsilon}})$ there exists $j_{N'}$, such that

$$\left| E_{j_{N'}}^{[-N', N']}(x, \omega) - E_j^{[-N, N]}(x, \omega) \right| < \exp\left(-\frac{\gamma}{9}N\right), \quad (8.3)$$

$$\left\| \psi_{j_{N'}}^{[-N', N']}(x, \omega) - \psi_j^{[-N, N]}(x, \omega) \right\| < \exp\left(-\frac{\gamma}{9}N\right). \quad (8.4)$$

Furthermore, $E_{j_{N'}}^{[-N', N']}(x, \omega)$ is localized, $I(x, \omega, E_{j_{N'}}^{[-N', N']}(x, \omega)) \subset [-3N/4, 3N/4]$, and

$$\left| \psi_{j_{N'}}^{[-N', N']}(x, \omega; n) \right| < \exp\left(-\frac{\gamma}{17}\text{dist}(n, I)\right), \quad N \leq |n| \leq N'.$$

Proof By Theorem A,

$$\left| \psi_j^{[-N, N]}(x, \omega; \pm N) \right| < \exp\left(-\frac{\gamma}{4}\left(\frac{N}{2} - 1\right)\right).$$

Then

$$\left\| (H_{[-N', N']}(x, \omega) - E_j^{[-N, N]}(x, \omega)) \psi_j^{[-N, N]}(x, \omega) \right\| < 2 \exp\left(-\frac{\gamma}{4}\left(\frac{N}{2} - 1\right)\right)$$

and applying Lemma 2.40 (with the aid of Theorem B) we get that there exists $j_{N'}$ such that (8.3) and (8.4) hold. Invoking Proposition 2.9 we have

$$L\left(\omega, E_{j_{N'}}^{[-N', N']}(x, \omega)\right) > \gamma$$

and therefore $E_{j_{N'}}^{[-N', N']}(x, \omega)$ is localized. Using (8.4) and $I \subset [-N/2, N/2]$ we get

$$\sum_{|n| \leq 5N/9} \left| \psi_{j_{N'}}^{[-N', N']}(x, \omega; n) \right|^2 > 1/2.$$

It follows from Theorem A (arguing by contradiction) that

$$I \left(x, \omega, E_{j_{N'}}^{[-N', N']}(x, \omega) \right) \subset [-3N/4, 3N/4]$$

and

$$\begin{aligned} \left| \psi_{j_{N'}}^{[-N', N']}(x, \omega; n) \right| &< \exp \left(-\frac{\gamma}{4} \text{dist}(n, [-3N/4, 3N/4]) \right) \\ &< \exp \left(-\frac{\gamma}{17} \text{dist}(n, I) \right), \end{aligned}$$

for any $N \leq |n| \leq N'$. $\square$

Lemma 8.5 *Let $\varepsilon \in (0, 1/5)$, $\gamma > 0$ and $\hat{\Omega}_N, \hat{\mathcal{B}}_{N, \omega}$ as in Theorem C. The following statement holds for any $N_0 \geq C(V, a, b, \gamma, \varepsilon)$, $\omega \in \mathbb{T}^d(a, b) \setminus \hat{\Omega}_{N_0}$, $x \in \mathbb{T}^d \setminus \hat{\mathcal{B}}_{N_0, \omega}$. If E, ψ is a generalized eigenpair for $H(x, \omega)$, that is*

$$H(x, \omega)\psi = E\psi, \quad \psi \not\equiv 0, \quad |\psi(m)| < C(\psi)(|m| + 1), \quad m \in \mathbb{Z},$$

and $L(\omega, E) > \gamma$, then

$$|\psi(m)| < \exp \left(-\frac{\gamma}{4} |m| \right), \quad |m| \geq C(\psi, V, a, b, \gamma).$$

Proof There exists $C_0(\psi, V, a, b, \gamma)$ such that for any interval $\Lambda \subset \mathbb{Z}$ satisfying

$$|\Lambda| \geq C_0 \quad \text{and} \quad \text{dist}(0, \mathbb{Z} \setminus \Lambda) > |\Lambda|/100$$

we have

$$\log |f_\Lambda(x, \omega, E)| \leq |\Lambda| L_{|\Lambda|}(\omega, E) - |\Lambda|^{1-\tau/4}.$$

Otherwise, we could use Poisson's formula and Lemma 2.24 to show that $\psi \equiv 0$. So, for N large enough we can apply Lemma 8.1 with $m_0 = 0$ to get that (8.1) holds for any

$$\begin{aligned} m \in [-N, N - \bar{\ell} + 1] \setminus I, \quad I = [-\bar{\ell} - 2 \exp(\ell^2), \bar{\ell} + 2 \exp(\ell^2)], \\ \bar{\ell} = \lfloor \exp((\log N)^{2\varepsilon\sigma}) \rfloor, \quad \ell = \lceil (\log(2N + 1))^{2\varepsilon} \rceil. \end{aligned}$$

Let

$$J = \begin{cases} \lfloor m/4 \rfloor, 2m], & m > 0, \\ [2m, \lceil m/4 \rceil], & m < 0. \end{cases}$$

Take m such that $\exp((\log N)^{5\varepsilon}) \leq |m| \leq N/2$. Then $J \subset [-N, N] \setminus I$, and we can use (8.1) and the covering form of (LDT) to get

$$\log |f_J(x, \omega, E)| > |J| L_{|J|}(\omega, E) - |J|^{1-\tau/2}.$$

Poisson's formula and Lemma 2.24 imply

$$|\psi(m)| < 2C(\psi)(2|m| + 1) \exp \left(-\frac{\gamma}{2} \frac{3|m|}{4} + C|m|^{1-\tau} \right) < \exp \left(-\frac{\gamma}{4} |m| \right),$$

provided N is large enough. This concludes the proof. $\square$

Proof of Theorem C (a) Let $N_k = N^{2^k}$. Iterating Theorem 8.4 (and using Proposition 2.9), we get that there exist $j_k, k \geq 0, j_0 = j$, such that

$$\begin{aligned} & \left| E_{j_{k+1}}^{[-N_{k+1}, N_{k+1}]}(x, \omega) - E_{j_k}^{[-N_k, N_k]}(x, \omega) \right| < \exp\left(-\frac{\gamma}{9} N_k\right), \\ & \left\| \psi_{j_{k+1}}^{[-N_{k+1}, N_{k+1}]}(x, \omega) - \psi_{j_k}^{[-N_k, N_k]}(x, \omega) \right\| < \exp\left(-\frac{\gamma}{9} N_k\right), \\ & L\left(\omega, E_{j_{k+1}}^{[-N_{k+1}, N_{k+1}]}(x, \omega)\right) > \frac{3\gamma}{2}, \\ & I_k = I\left(x, \omega, E_{j_{k+1}}^{[-N_{k+1}, N_{k+1}]}(x, \omega)\right) \subset [-3N_k/4, 3N_k/4], \\ & \left| \psi_{j_{k+1}}^{[-N_{k+1}, N_{k+1}]}(x, \omega; n) \right| < \exp\left(-\frac{\gamma}{17} \text{dist}(n, I_k)\right) < \exp\left(-\frac{\gamma}{18} \text{dist}(n, I)\right), \end{aligned}$$

for all $N_k \leq |n| \leq N_{k+1}$. It follows that for any $k' \geq k \geq 0$,

$$\begin{aligned} & \left| E_{j_{k'}}^{[-N_{k'}, N_{k'}]}(x, \omega) - E_{j_k}^{[-N_k, N_k]}(x, \omega) \right| < \sum_{i=k}^{k'-1} \exp\left(-\frac{\gamma}{9} N_i\right) < \exp\left(-\frac{\gamma}{10} N_k\right), \\ & \left\| \psi_{j_{k'}}^{[-N_{k'}, N_{k'}]}(x, \omega) - \psi_{j_k}^{[-N_k, N_k]}(x, \omega) \right\| < \sum_{i=k}^{k'-1} \exp\left(-\frac{\gamma}{9} N_i\right) < \exp\left(-\frac{\gamma}{10} N_k\right). \end{aligned}$$

This proves (1.4), (1.6), and the existence of the limits $E(x, \omega)$, $\psi(x, \omega)$. The fact that $\|\psi\| = 1$ follows immediately.

We already established that (1.5) holds for $N_{k-1} \leq |n| \leq N_k$. For $1 \leq k' \leq k-2$ we have

$$\begin{aligned} & \left| \psi_{j_k}^{[-N_k, N_k]}(x, \omega; n) \right| < \left| \psi_{j_{k'}}^{[-N_{k'}, N_{k'}]}(x, \omega; n) \right| + \sum_{i=k'}^{k-1} \exp\left(-\frac{\gamma}{9} N_i\right) \\ & < \exp\left(-\frac{\gamma}{20} \text{dist}(n, I)\right), \end{aligned} \quad (8.5)$$

for $N_{k'-1} \leq |n| \leq N_{k'}$. This shows (1.5) holds for $N \leq |n| \leq N_k$. By Theorem A,

$$\left| \psi_j^{[-N, N]}(x, \omega; n) \right| < \exp\left(-\frac{\gamma}{17} \text{dist}(n, I)\right), \quad 3N/4 \leq |n| \leq N.$$

Using the reasoning of (8.5) again we conclude that (1.5) holds as stated.

Taking $k' \rightarrow \infty$ in (1.6) we have

$$\left\| \psi(x, \omega) - \psi_{j_k}^{[-N_k, N_k]}(x, \omega) \right\| < \exp\left(-\frac{\gamma}{10} N_k\right)$$

and (1.7) follows in the same way as (1.5).

(b) It is well-known that to get purely pure point spectrum it is enough to show that generalized eigenvectors decay exponentially. We have this by Lemma 8.5, so we just need to argue that all eigenpairs can be obtained as limits from (a). Let E, ψ be an eigenpair for $H(x, \omega)$, $E \in [E', E'']$, $L(\omega, E) > 3\gamma$. By Lemma 8.5,

$$|\psi(m)| < \exp\left(-\frac{\gamma}{4}|m|\right), \quad |m| \geq C(\psi, V, a, b, \gamma). \quad (8.6)$$

Since

$$\|(H_{[-N, N]}(x, \omega) - E)\psi\|^2 = |\psi(-N-1)|^2 + |\psi(N+1)|^2,$$

it follows from Lemma 2.40 that for N large enough there exists j such that

$$\left| E_j^{[-N, N]}(x, \omega) - E \right| < \exp\left(-\frac{\gamma}{5}N\right), \quad \left\| \psi_j^{[-N, N]}(x, \omega) - \psi \right\| < \exp\left(-\frac{\gamma}{5}N\right). \quad (8.7)$$

By Proposition 2.9, $L(\omega, E_j^{[-N, N]}(x, \omega)) > 2\gamma$ and hence $E_j^{[-N, N]}(x, \omega)$ is localized. For N large enough, (8.6) and (8.7) imply that $I = I(x, \omega, E_j^{[-N, N]}(x, \omega)) \subset [-N/2, N/2]$. Let j_k be the sequence from (a). By the same reasoning as above, there exist j'_k such that

$$\left| E_{j'_k}^{[-N_k, N_k]}(x, \omega) - E \right| < \exp\left(-\frac{\gamma}{5}N_k\right), \quad \left\| \psi_{j'_k}^{[-N_k, N_k]}(x, \omega) - \psi \right\| < \exp\left(-\frac{\gamma}{5}N_k\right). \quad (8.8)$$

We just need to argue that $j'_k = j_k$ and that the choice of normalized eigenvector is the same as in (a). This follows by induction. We just check the initial step, the general step being analogous. We have that

$$\left| E_{j'_1}^{[-N_1, N_1]}(x, \omega) - E_{j_1}^{[-N_1, N_1]}(x, \omega) \right| \lesssim \exp\left(-\frac{\gamma}{10}N\right),$$

so Theorem B implies that $j_1 = j'_1$. If the choice of normalized eigenvector in (8.8) is not the same as in (a) we would have

$$\begin{aligned} \left\| \psi_{j_1}^{[-N_1, N_1]}(x, \omega) - \psi_j^{[-N, N]}(x, \omega) \right\| &< \exp\left(-\frac{\gamma}{10}N\right), \\ \left\| -\psi_{j_1}^{[-N_1, N_1]}(x, \omega) - \psi \right\| &< \exp\left(-\frac{\gamma}{5}N_1\right) \end{aligned}$$

which would imply

$$\left\| \psi_{j_1}^{[-N_1, N_1]}(x, \omega) \right\| \lesssim \exp\left(-\frac{\gamma}{10}N\right),$$

contradicting the fact that $\psi_{j_1}^{[-N_1, N_1]}(x, \omega)$ is normalized. $\square$

Remark 8.6 It is possible to prove part (b) of Theorem C in a direct manner by showing that the eigenvectors from part (a) form an orthonormal basis for the spectral subspace corresponding to $[E', E'']$. This argument is straightforward in the case when the Lyapunov exponent is positive on S_ω , see [13, Proposition 13.1], but in general it is more complicated than the above indirect argument.

9 Stabilization of the Spectrum: Proofs of Theorems D, E

We prove the relations (1.10) and (1.11) from Theorem D separately. The relation (1.10) is an immediate consequence of the following lemma.

Lemma 9.1 Assume $\omega \in \mathbb{T}^d(a, b)$ and $L(\omega, E) > \gamma > 0$ for all $E \in [E', E'']$. Let $k_0 \geq 0$, $s > 1$ integers, and σ as in (LDT). Then for all $N \geq N_0(V, a, b, \gamma)$ we have

$$S_\omega \cap [E', E''] \subset \mathfrak{S}_{N, \omega}(s, k_0, \rho),$$

for any $\rho \in \mathbb{R}^{k_0+1}$ such that $\rho_k \geq \exp(-(N^{(k)})^{\sigma/2})$, $k = 0, \dots, k_0$.

Proof Take $E \in [E', E''] \setminus \mathfrak{S}_{N, \omega}(s, k_0, \rho)$. We just need to show $E \notin S_\omega$. For any $x \in \mathbb{T}^d$ there exists

$$I = I(x) \in \{[-N^{(k)}, N^{(k)}] : k = 0, \dots, k_0\}$$

such that

$$\text{dist}(E, \text{spec } H_I(x, \omega)) \geq \exp(-|I|^{\sigma/2}),$$

and by the spectral form of (LDT) (see Corollary 2.21)

$$\log |f_I(x, \omega, E)| > |I|L_I(\omega, E) - |I|^{1-\tau/2}.$$

Given $\overline{N}$ set

$$\Lambda_{\overline{N}} = \bigcup_{m \in [-\overline{N}, \overline{N}]} m + I(m\omega).$$

From the covering form of (LDT) (see Lemma 2.25) we get

$$\text{dist}(E, \text{spec } H_{\Lambda_{\overline{N}}}(0, \omega)) \geq \exp(-N).$$

Since $\overline{N}$ can be chosen arbitrarily large, Lemma 2.39 implies that $\text{dist}(E, \mathcal{S}_\omega) \geq \exp(-N)$. In particular, $E \notin \mathcal{S}_\omega$ and the conclusion follows. $\square$

We will prove relation (1.11) from Theorem D in Proposition 9.5, but first we establish some auxiliary results. We start with an application of Bourgain's elimination of multiple resonances. The next lemma is the reason for the choice of parameters s, k_0, ρ in Theorem D.

Lemma 9.2 *Let $x \in \mathbb{T}^d$, $E \in \mathbb{R}$, $\gamma > 0$, $A \geq 1$, $s > 1$, $N \geq 1$, $N^{(k)} = N^{s^k}$, σ as in (LDT). For any $s \geq s_0(a, b, A)$, $N \geq N_0(V, a, b, \gamma, A, s)$ there exists a set $\Omega_N = \Omega_N(s)$, $\text{mes}(\Omega_N) < N^{-A}$, such that the following holds. If $\omega \in \mathbb{T}^d \setminus \Omega_N$, $L(\omega, E) > \gamma$, and*

$$\text{dist}(E, \text{spec } H_{[-N, N]}(x, \omega)) < \exp(-N^{\sigma/2}),$$

$$\text{dist}(E, \text{spec } H_{[-N^{(k)}, N^{(k)}]}(x, \omega)) < \exp\left(-\frac{\gamma}{10}N\right), \quad k = 1, \dots, k_0, \quad k_0 = 2^{2d+1} - 1,$$

then there exist $k = k(x) \in \{1, \dots, k_0\}$, $j = j(x)$, such that

$$\left| E - E_j^{[-N^{(k)}, N^{(k)}]}(x, \omega) \right| < \exp\left(-\frac{\gamma}{10}N\right), \quad (9.1)$$

$$\left| \psi_j^{[-N^{(k)}, N^{(k)}]}(x', \omega, \pm N^{(k)}) \right| < \exp\left(-\frac{\gamma}{8}N^{(k)}\right), \quad (9.2)$$

$$\text{for all } |x' - x| < \exp(-(\log N)^{3/\sigma}).$$

Proof Let s be a sufficiently large integer, so that the sets

$$\mathcal{N}_k = \left\{ n \in \mathbb{Z} : (N^{(k)})^{1/2} \leq |n| \leq 2N^{(k)} \right\}, \quad k = 1, \dots, k_0$$

satisfy the assumptions of Proposition 4.7, with $\ell = \lfloor (\log N)^{2/\sigma} \rfloor$, for N large enough. We let Ω_N be the set of resonant frequencies from Proposition 4.7. We have

$$\text{mes}(\Omega_N) \leq \ell^C (N^{(1)})^{-1/2} < N^{-A},$$

provided s is large enough. Since

$$\text{dist}(E, \text{spec } H_{[-N, N]}(x, \omega)) < \exp(-N^{\sigma/2}) < \exp(-\ell),$$

it follows from the covering form of (LDT) (see Lemma 2.25) that there exist $m_0 \in [-N, N]$ such that

$$\log |f_\ell(x + m_0\omega, \omega, E)| \leq \ell L_\ell(\omega, E) - \ell^{1-\tau/2}.$$

Applying Proposition 4.7, we get that there exists k such that

$$\log |f_\ell(x + (m-1)\omega, \omega, E)| > \ell L_\ell(\omega, E) - \ell^{1-\tau/2} \quad \text{for any } 2(N^{(k)})^{1/2} \leq |m| \leq N^{(k)}$$

(note that we chose N_k so that we get the above estimate after we take into account the shift by $m_0\omega$). By the hypothesis, there exists $E_j^{[-N^{(k)}, N^{(k)}]}(x, \omega)$ so that

$$\left| E - E_j^{[-N^{(k)}, N^{(k)}]}(x, \omega) \right| < \exp\left(-\frac{\gamma}{10}N\right).$$

The localization estimate (9.2) follows immediately from Proposition 3.1 by noting that

$$\left| E - E_j^{[-N^{(k)}, N^{(k)}]}(x', \omega) \right| < \exp\left(-\frac{\gamma}{10}N\right) + C|x' - x| < \exp(-\ell),$$

provided $|x' - x| < \exp(-(\log N)^{3/\sigma}) < \exp(-\ell^{3/2})$. $\square$

The next result is a technical tool for Lemma 9.4. Its peculiar statement stems from the following issue. The function $\text{mes}(\mathfrak{S}_N(s, k_0, \rho))$ can have jump-discontinuities in ρ , and we have no control over the size of the jumps. This forces us to work with $\text{mes}((\mathfrak{S}_N(s, k_0, \rho))^{(\rho')})$ (recall (1.9)) which is Lipschitz continuous in ρ' .

Lemma 9.3 *Let $N \geq 1$, $\omega \in \mathbb{T}^d$, $s > 1$, $k_0 \geq 1$, $\rho \in \mathbb{R}^{k_0+1}$, $\rho' > 0$, $\varepsilon \in \mathbb{R}^{k_0+1}$, $\varepsilon_k = \exp(-N^{3/2})$. For any $N \geq N_0(V, a, b)$ we have*

$$\text{mes}\left((\mathfrak{S}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon) - \varepsilon))^{(\rho')} \setminus \mathfrak{S}_{N,\omega}(s, k_0, \rho)\right) < (N^{(k_0)})^{C(a,b)} \rho'.$$

Proof Take $\tilde{V}$ as in (2.23) and consider the set of (x, E) such that

$$\left\| \left(\tilde{H}_{[-N^{(k)}, N^{(k)}]}(x, \omega) - E \right)^{-1} \right\|_{\text{HS}} > \rho_k^{-1}, \quad k = 0, \dots, k_0.$$

This set is semialgebraic of degree less than $(N^{(k_0)})^C$. We let $\tilde{\mathfrak{S}}_{N,\omega}(s, k_0, \rho)$ be its projection onto the E -axis. By the Tarski–Seidenberg principle, $\tilde{\mathfrak{S}}_{N,\omega}(s, k_0, \rho)$ is also semialgebraic of degree less than $(N^{(k_0)})^{C'}$ and hence it has less than $(N^{(k_0)})^{C''}$ connected components (see [7, Chapter 9]). Note that

$$\left| E_j^{[-N^{(k)}, N^{(k)}]}(x, \omega) - \tilde{E}_j^{[-N^{(k)}, N^{(k)}]}(x, \omega) \right| < \exp(-N^{3/2}), \quad k = 0, \dots, k_0$$

(recall (2.24)) and therefore

$$\mathfrak{S}_{N,\omega}(s, k_0, \rho) \subset \tilde{\mathfrak{S}}_{N,\omega}(s, k_0, \rho + \varepsilon), \quad \tilde{\mathfrak{S}}_{N,\omega}(s, k_0, \rho) \subset \mathfrak{S}_{N,\omega}(s, k_0, N^{1/2}\rho + \varepsilon)$$

(recall that $\|\cdot\| \leq \|\cdot\|_{\text{HS}} \leq \sqrt{N}\|\cdot\|$ on $\mathbb{C}^{N \times N}$). It follows that

$$\begin{aligned} & (\mathfrak{S}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon) - \varepsilon))^{(\rho')} \setminus \mathfrak{S}_{N,\omega}(s, k_0, \rho) \\ & \subset (\tilde{\mathfrak{S}}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon)))^{(\rho')} \setminus \tilde{\mathfrak{S}}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon)). \end{aligned}$$

Since $\tilde{\mathfrak{S}}_{N,\omega}$ has less than $(N^{(k_0)})^{C''}$ components,

$$\text{mes}\left((\tilde{\mathfrak{S}}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon)))^{(\rho')} \setminus \tilde{\mathfrak{S}}_{N,\omega}(s, k_0, N^{-1/2}(\rho - \varepsilon))\right) \lesssim (N^{(k_0)})^{C''} \rho'$$

and we are done. $\square$

The following is a finite volume version of the second part of Theorem D and it will imply the full scale statement.

Lemma 9.4 Let $\gamma > 0$, $A \geq 1$, σ as in (LDT). For any $s \geq s_0(a, b, A)$, $N \geq N_0(V, a, b, \gamma, A, s)$, there exists a set $\Omega_N = \Omega_N(s)$, $\text{mes}(\Omega_N) < N^{-A}$, such that the following holds with

$$\begin{aligned}\mathfrak{S}_{N,\omega} &= \mathfrak{S}_{N,\omega}(s, k_0, \rho_N), \quad k_0 = 2^{2d+1} - 1, \\ \rho_{N,0} &= \exp(-N^{\sigma/2}), \quad \rho_{N,k} = \exp\left(-\frac{\gamma}{10}N\right), \quad k = 1, \dots, k_0.\end{aligned}$$

If $\omega \in \mathbb{T}^d \setminus \Omega_N$ and $L(\omega, E) > \gamma$ for all $E \in [E', E'']$ and $\overline{N} = \lfloor \exp(N^{1/2}) \rfloor$, then

$$\text{mes}\left((\mathfrak{S}_{N,\omega} \cap [E', E'']) \setminus \mathfrak{S}_{\overline{N},\omega}\right) < \exp\left(-\frac{\gamma}{15}N\right).$$

Proof Let Ω'_N be the set from Lemma 9.2 with $A + 1$ instead of A . Let $\Omega_{\overline{N}}, \mathcal{B}_{\overline{N},\omega}$, be the sets from Theorem A with $\varepsilon = 1/10$, $[-\overline{N}, \overline{N}]$ instead of $[1, \overline{N}]$, and $\gamma/2$ instead of γ . The conclusion will hold with $\Omega_N = \Omega'_N \cup \Omega_{\overline{N}}$. Clearly the measure estimate for Ω_N is satisfied.

Take $E \in \mathfrak{S}_{N,\omega} \cap [E', E'']$. Then by Lemma 9.2, there exist $x \in \mathbb{T}^d$, $k = k(x)$, $j = j(x)$, such that (9.1) and (9.2) hold. Since

$$\text{mes}(\mathcal{B}_{\overline{N},\omega}) < \exp(-\exp((\log \overline{N})^{\sigma/10})),$$

there exists $x' \notin \mathcal{B}_{\overline{N},\omega}$,

$$|x' - x| < \exp(-\exp((\log \overline{N})^{\sigma/20})) < \exp(-(\log N)^{3/\sigma}).$$

Using Corollary 2.41, we have that there exists $E_i^{[-\overline{N}, \overline{N}]}(x', \omega)$ such that

$$\begin{aligned}\left|E_i^{[-\overline{N}, \overline{N}]}(x', \omega) - E_j^{[-N^{(k)}, N^{(k)}]}(x', \omega)\right| &\lesssim \exp\left(-\frac{\gamma}{8}N^{(k)}\right), \\ \max_{n \in [-N^{(k)}, N^{(k)}]} \left|\psi_i^{[-\overline{N}, \overline{N}]}(x', \omega; n)\right| &> (2N^{(k)}\overline{N})^{-1/2}.\end{aligned}$$

Note that

$$\begin{aligned}\left|E - E_i^{[-\overline{N}, \overline{N}]}(x', \omega)\right| &\lesssim \exp\left(-\frac{\gamma}{10}N\right) + C \exp(-\exp((\log \overline{N})^{\sigma/20})) + \exp\left(-\frac{\gamma}{8}N^{(k)}\right) \\ &< \exp\left(-\frac{\gamma}{11}N\right),\end{aligned}$$

so by Proposition 2.9, $L(E_i^{[-\overline{N}, \overline{N}]}(x', \omega), \omega) > \gamma/2$, provided N is large enough. Since $x' \notin \mathcal{B}_{\overline{N},\omega}$, by Theorem A, there exists $I = I(E_i^{[-\overline{N}, \overline{N}]}(x', \omega))$ such that $|I| < \exp((\log \overline{N})^{1/2})$ and

$$\left|\psi_i^{[-\overline{N}, \overline{N}]}(x', \omega; n)\right| < \exp\left(-\frac{\gamma}{4}\text{dist}(n, I)\right),$$

provided $\text{dist}(n, I) > \exp((\log \overline{N})^{\sigma/2})$. It follows that

$$\text{dist}([-N^{(k)}, N^{(k)}], I) \leq \exp((\log \overline{N})^{\sigma/2})$$

and in particular

$$\left|\psi_i^{[-\overline{N}, \overline{N}]}(x', \omega; \pm \overline{N})\right| < \exp\left(-\frac{\gamma}{8}\overline{N}\right).$$

Using Lemma 2.38 we get that

$$E_i^{[-\overline{N}, \overline{N}]}(x', \omega) \in \mathfrak{S}_{\overline{N},\omega}(s, k_0, \rho'), \quad \rho'_k = 4 \exp\left(-\frac{\gamma}{8}\overline{N}\right)$$

and therefore

$$E \in \left(\mathfrak{S}_{\overline{N}, \omega}(s, k_0, \rho') \right)^{(\rho'')}, \quad \rho'' = \exp \left(-\frac{\gamma}{11} N \right).$$

Thus we showed

$$\mathfrak{S}_{N, \omega} \cap [E', E''] \subset \left(\mathfrak{S}_{\overline{N}, \omega}(s, k_0, \rho') \right)^{(\rho'')}.$$

Then

$$\begin{aligned} (\mathfrak{S}_{N, \omega} \cap [E', E'']) \setminus \mathfrak{S}_{\overline{N}, \omega} &\subset \left(\mathfrak{S}_{\overline{N}, \omega}(s, k_0, \rho') \right)^{(\rho'')} \setminus \mathfrak{S}_{\overline{N}, \omega} \\ &\subset \left(\mathfrak{S}_{\overline{N}, \omega}(s, k_0, \overline{N}^{-1/2}(\rho_{\overline{N}} - \varepsilon) - \varepsilon) \right)^{(\rho'')} \setminus \mathfrak{S}_{\overline{N}, \omega}, \quad \varepsilon_k = \exp(-\overline{N}^{3/2}) \end{aligned}$$

and the conclusion follows from Lemma 9.3. $\square$

We can now prove the second part of Theorem D.

Proposition 9.5 *Let $\gamma > 0$, $A \geq 1$. For any $s \geq s_0(a, b, A)$, $N \geq N_0(V, a, b, \gamma, A, s)$, there exists a set $\Omega_N = \Omega_N(s)$, $\text{mes}(\Omega_N) < N^{-A}$, such that the following holds with $\mathfrak{S}_{N, \omega}$ as in Lemma 9.4. If $\omega \in \mathbb{T}^d \setminus \Omega_N$ and $L(\omega, E) > \gamma$ for all $E \in [E', E'']$ then*

$$\text{mes}((\mathfrak{S}_{N, \omega} \cap [E', E'']) \setminus \mathcal{S}_\omega) < \exp \left(-\frac{\gamma}{20} N \right).$$

Proof Let $N_j = \lfloor \exp(N_{j-1}^{1/2}) \rfloor$, $N_0 = N$. Let Ω'_{N_j} be the set from Lemma 9.4 with $A+1$ instead of A . The conclusion will hold with $\Omega_N = \bigcup_j \Omega_{N_j}$. Clearly the measure estimate for Ω_N is satisfied.

Using Lemma 2.38 it follows from Lemma 9.2 that if $E \in \mathfrak{S}_{N, \omega} \cap [E', E'']$, then $\text{dist}(E, \mathcal{S}_\omega) < \exp(-\gamma N/11)$ for N large enough. Then we have

$$\bigcap_{j \geq 1} \mathfrak{S}_{N_j, \omega} \cap [E', E''] \subset \mathcal{S}_\omega$$

and

$$\begin{aligned} (\mathfrak{S}_{N, \omega} \cap [E', E'']) \setminus \mathcal{S}_\omega &\subset (\mathfrak{S}_{N, \omega} \cap [E', E'']) \setminus \left(\bigcap_{j \geq 1} \mathfrak{S}_{N_j, \omega} \cap [E', E''] \right) \\ &= (\mathfrak{S}_{N, \omega} \cap [E', E'']) \setminus \bigcap_{j \geq 1} \mathfrak{S}_{N_j, \omega} \subset \bigcup_{j \geq 0} ((\mathfrak{S}_{N_j, \omega} \cap [E', E'']) \setminus \mathfrak{S}_{N_{j+1}, \omega}). \end{aligned}$$

So the conclusion follows from Lemma 9.4. $\square$

Theorem E is an immediate consequence of the following proposition.

Proposition 9.6 *Let $\gamma > 0$, $A \geq 1$, σ as in (LDT). For any $N \geq N_0(V, a, b, \gamma, A)$ there exists a set Ω_N , $\text{mes}(\Omega_N) < N^{-A}$, such that the following holds. If $\omega \in \mathbb{T}^d \setminus \Omega_N$, $E_0 \in \mathcal{S}_\omega$ and $L(\omega, E_0) > \gamma$, then*

$$\text{mes}((E_0 - \delta_N, E_0 + \delta_N) \cap \mathcal{S}_\omega) > \frac{2}{3} \delta_N, \quad \delta_N = \exp(-(\log N)^{4/\sigma^2}).$$

Proof Let Ω'_N be the exceptional set from Lemma 9.4, with $A+1$ instead of A , $s = s_0$, and $\gamma/2$ instead of γ . The conclusion will hold with $\Omega_N = \bigcup_{k=0}^{k_0} \Omega'_{N(k)}$, $k_0 = 2^{2d+1} - 1$.

Let $\mathfrak{S}_{N,\omega}$ as in Lemma 9.4. By Lemma 9.1, $E_0 \in \mathfrak{S}_{N,\omega}$, and by Lemma 9.2, there exist $x, k = k(x), j = j(x)$, such that

$$\begin{aligned} |E_0 - E_j^{[-N^{(k)}, N^{(k)}]}(x, \omega)| &< \exp\left(-\frac{\gamma}{10}N\right), \\ |\psi_j^{[-N^{(k)}, N^{(k)}]}(x', \omega, \pm N^{(k)})| &< \exp\left(-\frac{\gamma}{8}N^{(k)}\right) \quad \text{for any } |x' - x| < \exp(-(\log N)^{3/\sigma}). \end{aligned}$$

Let $S = \{x' : |x' - x| < \exp(-(\log N)^{3/\sigma})\}$. Using Lemma 2.38 we have

$$E_j^{[-N^{(k)}, N^{(k)}]}(S, \omega) \subset \mathfrak{S}_{N^{(k)}, \omega}.$$

Let $[E', E''] = [E_0 - N^{-1}, E_0 + N^{-1}]$. By Proposition 2.9, $L(\omega, E) > \gamma/2$ for all $E \in [E', E'']$, provided N is large enough. We have $E_j^{[-N^{(k)}, N^{(k)}]}(S, \omega) \subset [E', E'']$, so Lemma 2.27 yields

$$\text{mes}\left(E_j^{[-N^{(k)}, N^{(k)}]}(S, \omega)\right) > \exp(-(\log N)^{4/\sigma^2}).$$

By Proposition 9.5,

$$\text{mes}\left(E_j^{[-N^{(k)}, N^{(k)}]}(S, \omega) \setminus \mathcal{S}_\omega\right) < \exp\left(-\frac{\gamma}{20}N\right).$$

Let

$$I = (E_0 - \delta_N, E_0 + \delta_N) \cap E_j^{[-N^{(k)}, N^{(k)}]}(S, \omega).$$

Then

$$\begin{aligned} \text{mes}((E_0 - \delta_N, E_0 + \delta_N) \cap \mathcal{S}_\omega) &\geq \text{mes}(I \cap \mathcal{S}_\omega) > \text{mes}(I) - \exp\left(-\frac{\gamma}{20}N\right) \\ &> \exp(-(\log N)^{4/\sigma^2}) - 2 \exp\left(-\frac{\gamma}{20}N\right) > \frac{2}{3}\delta_N. \end{aligned}$$

□

Proof of Theorem E Let Ω'_N be the set from Proposition 9.6 with $A = 3$. The conclusion follows immediately from Proposition 9.6, by letting $\Omega_N = \bigcup_{N' \geq N} \Omega'_{N'}$ and $\delta_0 = \exp(-(\log N)^{4/\sigma^2})$ (provided N is large enough). □

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