EXOTIC EIGENVALUES OF SHRINKING METRIC GRAPHS

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Abstract. Eigenvalue spectrum of the Laplacian on a metric graph with arbitrary but fixed vertex conditions is investigated in the limit as the lengths of all edges decrease to zero at the same rate. It is proved that there are exactly four possible types of eigenvalue asymptotics. The number of eigenvalues of each type is expressed via the index and nullity of a form defined in terms of the vertex conditions.

1. Introduction

Analysis of differential operators on a metric graph or network is a burgeoning area of research due to its numerous applications in physics and engineering. The first step in solving, say, a control problem on a metric network [CZ98] is to understand the spectral properties of the time-independent operator, which in many cases is the Laplacian or Sturm–Liouville operator on every edge augmented by suitable matching conditions on every vertex [BK13, M14].

Many interesting mathematical problems arise here due to richness of the topology of the network and of the physically realizable [CET10] matching conditions. To give an example, the Laplacian on a metric graph can have eigenmodes localizing exactly on a sub-structure, such as a cycle or a path connecting two vertices of degree one [SK03, CdV15]. Such eigenmodes play havoc with inverse and/or control problems: a localized eigenmode is not detectable or controllable via the edges where it is identically zero [K13]. Fortunately, for the Laplacian with the so-called standard (or Neumann–Kirchhoff) matching conditions, such exactly localized eigenstate disappear under small perturbations of the edge metric [BL17] — with some important exceptions.

Until recently, mathematical analyses of the Laplacian on a metric graph usually assumed a uniform lower bound on the edge length. However, spectral properties of graph models with short lengths are important in applications, such as in creation of periodic materials with special wave transmission properties [K05, BET21, CEK$^+$22, LTC22]. Limits of operators on metric graphs with short edges and general matching conditions started[1] receiving significant analytical attention with concurrent publication of the works by Cacciapuoti [C19] and by Berkolaiko, Latushkin and Sukhtaiev [BLS19]. These publications treated slightly different models, which were subsequently generalized by Borisov [B22]. In these works, convergence of operators (in the norm-resolvent sense) is established under certain non-resonance conditions. Counter-examples to convergence use a clever choice of matching conditions to produce what we call "exotic states": eigenfunction that localize on the shrinking edges of the graph. The limiting operator has those edges shrunk to a point which cannot support eigenfunctions. The non-resonance conditions (such as [BLS19, Cond. 3.2] reviewed in equation (2.9) below) are

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[1]Spectral analysis of the *discrete Laplacian* on a graph with shrinking edges has a more substantial history, including works such as [CdV90, CdV18].

sufficient to exclude exotic eigenfunctions, but to what extent such conditions are necessary is an open problem.

In this note, we consider perhaps the simplest model of a graph with short edge lengths — the Laplacian on a graph where *all* lengths are shrinking uniformly. The aim of this simplification is to gain complete understanding of the number of the exotic states, and of the eigenvalue asymptotics in general. In particular, we find that, in this context, the non-resonance condition [BLS19, Cond. 3.2] is both necessary and sufficient in that it provides the exact count of the exotic states.

To be more precise, we consider the Laplacian on a metric graph Γ_ϵ with arbitrary but fixed self-adjoint matching conditions and edge lengths that are uniformly shrinking: the length of an edge e is $\epsilon\ell_e$, where $\ell_e > 0$ is fixed. Intuitively, a shorter string produces a higher pitch, so the eigenvalues should increase as $\epsilon \to 0$. From dimensional considerations of the eigenvalue equation $-\frac{d^2}{dx^2}f = \lambda f$, one can expect the asymptotic behavior $\lambda \sim \epsilon^{-2}$. This intuition very quickly runs into trouble: a Neumann Laplacian on an interval has a constant eigenvalue 0, Robin Laplacian may have eigenvalues scaling as ϵ^{-1} (positive or negative), or even converging to a non-zero constant, see examples is Appendix A. In this note we explain why these are the only options in our setting and give a quantitative description of different behaviors.

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2. Notation and the main result

Denoting by $\mathcal{E}$ the finite edge set of the graph in question, we consider the operator H_ϵ acting as $-d^2/dx^2$ in the space $\mathcal{H}_\epsilon := \bigoplus_{e\in\mathcal{E}} L_2(0, \epsilon\ell_e)$, where $\ell_e > 0$ are assumed to be fixed. The domain of the operator H_ϵ is an appropriate subspace of $\mathcal{H}_\epsilon$ which we now describe.

Let $E \simeq \mathbb{C}^{2|\mathcal{E}|}$ be the space of boundary values of functions on graph edges and $E' \simeq \mathbb{C}^{2|\mathcal{E}|}$ be the space of derivative values. More precisely, denoting by e^- and e^+ the two endpoints of the edge $e \in \mathcal{E}$, we set

$$f_{e^-} = f(0), \qquad f_{e^+} = f(\epsilon\ell_e), \qquad \partial f_{e^-} = f'(0), \qquad \partial f_{e^+} = -f'(\epsilon\ell_e), \tag{2.1}$$

which are well-defined when $f \in H^2(0, \epsilon\ell_e)$. Corresponding to a function f, we then have two vectors

$$F := \left(f_{e^\pm}\right)_{e\in\mathcal{E}} \in E \qquad \text{and} \qquad F := \left(\partial f_{e^\pm}\right)_{e\in\mathcal{E}} \in E'. \tag{2.2}$$

The edge endpoints are ordered arbitrarily but in the same way for F and F'. The mapping from $f \in \bigoplus_{e\in\mathcal{E}} H^2(0, \epsilon\ell_e)$ to $F \in E$ will be denoted by γ_D.

Self-adjoint realizations of $-d^2/dx^2$ may be described using the theory of boundary triplets [S12, BHdS20] which use Green's formula to define a symplectic structure on $E \oplus E'$, with E' being viewed as a dual space to E — with a natural identification between them. With respect to this (complex!) symplectic structure, the self-adjoint realizations of $-d^2/dx^2$ are in one-to-one correspondence with the Lagrangian subspaces of $E \oplus E'$ [vj74, KS99, H00]. However, in our setting, more directly usable parametrization of self-adjoint matching conditions can be written as

$$P_D F = 0, \tag{2.3}$$

$$P_N F' = 0, \tag{2.4}$$

$$P_R F' = \Lambda P_R F. \tag{2.5}$$

Here P_D, P_N and P_R are orthogonal projectors such as $P_D + P_N + P_R = I$ (in particular, the projectors are mutually orthogonal); Λ is an invertible self-adjoint operator acting on the space $P_R E$; equation (2.5) assumes the natural identification of E' and E. Some examples of the matching conditions can be found in Appendix A. We remark that the connectivity among edges (i.e. "vertices") is encoded in the matching conditions (2.3)-(2.5) and will not play a direct role in the subsequent analysis. For this reason we did not introduce the notion of a vertex and use the term "matching conditions" instead of perhaps more widespread "vertex conditions".

Let

$$D_0 := \{F \in E \colon F_{e^-} = F_{e^+} \ \text{ for all } e \in \mathcal{E}\}, \tag{2.6}$$

$$N_0 := \{F' \in E' \colon F'_{e^-} = -F'_{e^+} \ \text{ for all } e \in \mathcal{E}\}. \tag{2.7}$$

Our main results is the following.

Theorem 2.1. *Let*

$$\mathcal{F}_0 := D_0 \cap \operatorname{Ker} P_D, \qquad m_0 := \dim \mathcal{F}_0. \tag{2.8}$$

Denote by n_-, n_0 and n_+ the number of negative, zero and positive eigenvalues of the sesquilinear form $q[G, F] := \langle G, P_R \Lambda P_R F\rangle$ restricted to $\mathcal{F}_0$. Then the spectrum $\{\lambda_n(\epsilon)\}_{n\in\mathbb{N}}$ of H_ϵ consists of

1. *Infinitely many "normal" or "fast" eigenvalues which tend to $+\infty$ at the rate $1/\epsilon^2$,*
$$\lambda_n(\epsilon) = \mu_n \epsilon^{-2} + O(\epsilon^{-1}), \quad n > m_0, \qquad \epsilon \to 0,$$
2. *and m_0 lowest "slow" eigenvalues, which are further boken into three types,*
   - (a) *n_- "slow negative" eigenvalues tending to $-\infty$ at the rate $1/\epsilon$,*
   - (b) *n_0 "exotic" eigenvalues tending to a finite* non-positive *limit,*
   - (c) *n_+ "slow negative" eigenvalues tending to $+\infty$ at the rate $1/\epsilon$.*

It is interesting to compare this result with the convergence results established in [BLS19, Thm. 3.6]. That theorem predicts spectral convergence on any fixed interval, to the spectrum of the Laplacian on a metric graph which, in the present context, is empty. The sufficient condition for this convergence is the "Non-Resonance Condition" [BLS19, Cond. 3.2], which, adjusted to the present context, reads

$$\dim \operatorname{Proj}_E \big(\mathcal{L} \cap (D_0 \oplus N_0)\big) = 0. \tag{2.9}$$

Here $\mathcal{L}$ is the subspace of $E \oplus E'$ consisting of F and F' satisfying matching conditions (2.3)-(2.5) (assumed to be independent of ϵ). Comparing with part (2)(2b) of Theorem 2.1, we conclude that the Non-Resonance Condition (2.9) should be the sufficient condition for the absense of exotic eigenvalues. It turns out to be a necessary condition as well, as shown by the following.

Proposition 2.2. *The space* $\mathrm{Proj}_E\big(\mathcal{L}\cap(D_0\oplus N_0)\big)$ *is the null space of the sesquilinear form* $q[G,F]:=\langle G, P_R\Lambda P_R F\rangle$ *restricted to* $\mathcal{F}_0$. *In particular, a uniformly shrinking graph* H_ϵ *has no exotic eigenvalues if and only if the Non-Resonance Condition* (2.9) *holds.*

3. Counting eigenvalues

Rescaling the edges to constant lengths we obtain an operator R_ϵ on a fixed graph Γ_1 with lengths $\big(\ell_e\big)_{e\in\mathcal{E}}$. It acts as $-d^2/dx^2$ but with matching conditions that now depend on ϵ,

$$P_D F = 0, \tag{3.1}$$

$$P_N F' = 0, \tag{3.2}$$

$$P_R F' = \epsilon\Lambda P_R F. \tag{3.3}$$

The eigenvalues of H_ϵ and R_ϵ are related by

$$\lambda\big(H_\epsilon\big) = \frac{\lambda(R_\epsilon)}{\epsilon^2}. \tag{3.4}$$

We can now view the rescaled operator R_ϵ as a regular perturbation problem for the operator R_0, obtained by simply setting $\epsilon = 0$ in (3.1)-(3.3). The fast eigenvalues of Theorem 2.1 part (1) correspond to non-zero eigenvalues of R_0. The various slow eigenvalues branch out of the zero eigenspace of R_0.

We will show that $m_0 = \dim\operatorname{Ker} R_0$ and use Rellich–Kato Selection Theorem [K95, Thms. II.5.4 and VII.3.9] (known as Hellmann-Feynman formula for degenerate eigenvalues in the physics literature) to classify how the eigenvalues branch out from zero. This information is contained in the spectrum of the restriction of q to $\operatorname{Ker} R_0$. More precisely, negative and positive eigenvalues of the restricted q correspond to "slow negative" and "slow positive" eigenvalues of Theorem 2.1 part (1). Zero eigenvalues of the restricted q are responsible for the "exotic" eigenvalues in Theorem 2.1 part (2)

3.1. **Spectrum of the rescaled unperturbed problem.** We first aim to understand the unperturbed problem R_0. Setting $\epsilon = 0$ in the matching conditions (3.1)-(3.3), we obtain the matching conditions for the unperturbed problem:

$$P_D F = 0, \tag{3.5}$$

$$(I - P_D)F' = 0, \tag{3.6}$$

where we used that $P_D + P_N + P_R = I$.

Lemma 3.1. *The rescaled unperturbed operator* R_0 *is a non-negative self-adjoint operator with discrete spectrum. The kernel of* R_0 *is the span of all edgewise constant functions satisfying* $P_D F = 0$. *Its dimension is equal to* $m_0 = \dim\left(D_0\cap\operatorname{Ker} P_D\right)$.

Proof. For the purpose of establishing Lemma 3.1, it is much simpler to work with the quadratic form corresponding to the operator R_0. The form Q_0 is

$$Q_0[f] = \int_{\Gamma_1} |f(x)'|^2 dx, \qquad \mathrm{dom}[Q_0] = \left\{ f\in\bigoplus_{e\in\mathcal{E}} H^1(0,\ell_e)\colon P_D F = 0\right\}. \tag{3.7}$$

It is immediate that $Q_0 \geq 0$. That this form corresponds to a self-adjoint operator on the graph Γ_1 is a well-known fact (see, e.g., [BK13, Sec. 1.4.3]). The spectrum is discrete due to compactness of Γ_1 [BK13, Sec. 3.1.1].

To be in the kernel of Q_0, the function f must have $f' \equiv 0$, as can be seen directly from (3.7). We conclude that f is edge-wise constant, which immediately gives $F \in D_0$; f also must belong to the domain of Q_0, i.e. satisfy the condition $P_D F = 0$.

Conversely, any $G \in D_0 \cap \operatorname{Ker} P_D$ corresponds to an edge-wise constant function g which belongs to the domain of R_0 and obviously satisfies $R_0 g = 0$. □

3.2. **Proof of Theorem 2.1.** From [BK12] (see also [BK13, Sec. 3.1.2]) we know that the eigenvalues of R_ϵ depend analytically on the parameter ϵ. In particular, for all $n > m_0$, the n-th eigenvalue can be represented as

$$\lambda_n\big(R_\epsilon\big) = \lambda_n^0 + O(\epsilon), \qquad \lambda_n^0 := \lambda_n\big(R_0\big) > 0. \tag{3.8}$$

Note that here we have perturbation with one parameter only. In case of eigenvalue multiplicity of R_0 we are effectively using the Rellich–Kato Selection Theorem, but only from one side, $\epsilon > 0$. Part (1) of Theorem 2.1 now follows from (3.4).

The lowest m_0 eigenvalues bifurcate from the eigenvalue 0 of R_0. Here we apply the Rellich–Kato Selection Theorem in earnest, to determine the slopes of eigenvalue bifurcation. The appropriate version of the theorem, for perturbations in the domain of the self-adjoint operator, was established only recently [LS23, Thm. 3.23] (see also [LS22, Thm. 2.10]). In our case, it boils down to finding the eigenvalues of the derivative of the quadratic form

$$Q_\epsilon[f] = \int_{\Gamma_1} |f(x)'|^2 dx + \epsilon\big\langle P_R(\gamma_D f), \Lambda P_R(\gamma_D f)\big\rangle \tag{3.9}$$

of the operator R_ϵ on the kernel of the operator R_0. By Lemma 3.1, the eigenvalues of the quadratic form $\big\langle P_R(\gamma_D f), \Lambda P_R(\gamma_D f)\big\rangle$ on $\operatorname{Ker} R_0$ are the same as the eigenvalues of $\big\langle P_R F, \Lambda P_R F\big\rangle$ on $D_0 \cap \operatorname{Ker} P_D$. These eigenvalues, which we denote by $\{\mu_n\}_{n=1}^{m_0}$, yield the next term in the expansion of $\lambda_n(R_\epsilon)$ for $n \le m_0$,

$$\lambda_n\big(R_\epsilon\big) = 0 + \mu_n \epsilon + O(\epsilon^2), \qquad n \le m_0. \tag{3.10}$$

Combined with (3.4), we get part (2) of Theorem 2.1.

Finally, for n such that $\mu_n = 0$, we want to understand the sign of the ϵ^2 term in the expansion (3.10). We know a priori [BK12, KZ19] that the eigenvalues and eigenfunctions are analytic in ϵ (or can be chosen to be so in the cases of multiplicity). Let $n \in \{n_- + 1, \ldots, n_- + n_0\}$ and consider an eigenvalue with small ϵ expansion

$$\lambda_n\big(R_\epsilon\big) = 0 + \nu_n \epsilon^2 + O(\epsilon^3), \tag{3.11}$$

and let

$$f(x;\epsilon) = f_0(x) + \epsilon f_1(x) + O(\epsilon^2) \tag{3.12}$$

be the corresponding eigenfunction. We will collect some information about f_0 and f_1 and then obtain $\lambda_n\big(R_\epsilon\big)$ as $Q_\epsilon[f(x;\epsilon)]$.

Since $f_0 \in \operatorname{Ker} R_0$, we know it is edgewise constant. Since $-\frac{d^2}{dx^2} f(x;\epsilon) = \lambda_n f(x;\epsilon)$, we get that $-\frac{d^2}{dx^2} f_1 = 0$ and therefore

$$f_1\big|_e = b_e x + c_e. \tag{3.13}$$

Let $F_1' \in E'$, $F_1 \in E$ and $F_0 \in E$ be the vectors of the Neumann values of f_1 and the Dirichlet values of f_1 and f_0, correspondingly (as remarked above, Neumann values of f_0 are zero).

Since the eigenfunction $f(x;\epsilon)$ satisfies the ϵ-dependent matching conditions (3.1)-(3.3), we have

$$P_D F_1 = 0, \quad P_N F_1' = 0, \quad P_R F_1' = \Lambda P_R F_0, \tag{3.14}$$

which will become useful shortly.

Substituting expansions (3.11) and (3.12) into $\lambda_n(R_\epsilon) = Q_\epsilon[f(x;\epsilon)]$ and extracting the term of order ϵ^2, we get

$$\nu_n = \sum_{e\in\mathcal{E}} \int_{\ell_e} |b_e|^2 dx + \langle P_R F_0, \Lambda P_R F_1\rangle + \langle P_R F_1, \Lambda P_R F_0\rangle \tag{3.15}$$

$$= \sum_{e\in\mathcal{E}} \ell_e |b_e|^2 + 2\,\mathrm{Re}\,\langle P_R F_1, P_R F_1'\rangle\,, \tag{3.16}$$

where we used (3.14) to get rid of $\Lambda P_R F_0$ (note that Λ is Hermitian). We can simplify the inner product in (3.16) further by noting that

$$P_R F_1 = (I - P_D - P_N)F_1 = (I - P_N)F_1,$$
$$P_R F_1' = (I - P_D - P_N)F_1' = (I - P_D)F_1',$$

and therefore

$$\langle P_R F_1, P_R F_1'\rangle = \langle (I-P_N)F_1, (I-P_D)F_1'\rangle = \langle F_1, F_1'\rangle\,, \tag{3.17}$$

both using (3.14) as well as the mutual orthogonality of the projectors P_N and P_D.

We now compute F_1 and F_1' more explicitly,

$$F_1'\big|_{e^-} = b_e, \qquad\qquad F_1'\big|_{e^+} = -b_e \tag{3.18}$$

$$F_1\big|_{e^-} = -\frac{1}{2}\ell_e b_e + c_e, \qquad\qquad F_1\big|_{e^+} = \frac{1}{2}\ell_e b_e + c_e, \tag{3.19}$$

where we assumed without loss of generality that the local coordinate x on the edge e runs from $-\ell_e/2$ at the endpoint e^- to $\ell_e/2$ at the endpoint e^+. In particular, we have $F_1' \in N_0$ and

$$F_1 = -\frac{1}{2} L F_1' + C, \tag{3.20}$$

where $C \in D_0$ is a vector of c_e and L is the diagonal $2|\mathcal{E}| \times 2|\mathcal{E}|$ matrix with ℓ_e on the diagonal. We substitute (3.20) into (3.17) to obtain

$$2\,\mathrm{Re}\,\langle F_1, F_1'\rangle = 2\,\mathrm{Re}\left\langle -\frac{1}{2}LF_1' + C, F_1'\right\rangle = -\,\mathrm{Re}\,\langle LF_1', F_1'\rangle \tag{3.21}$$

$$= -\sum_{e\in\mathcal{E}} 2|b_e|^2 \ell_e, \tag{3.22}$$

where we used that $C \perp F_1'$ (because $D_0 \perp N_0$) and the 2 appeared because each edge contributes two endpoints to the inner product.

To summarize, with (3.16) we get

$$\nu_n = -\sum_{e\in\mathcal{E}} 2|b_e|^2 \ell_e \geq 0, \tag{3.23}$$

which is establishes the last part of Theorem 2.1, namely that the exotic eigenvalues are non-positive.

3.3. **Counting exotic eigenvalues: proof of Proposition 2.2.** Assuming

$$(F, F') \in \mathcal{L} \cap (D_0 \oplus N_0), \tag{3.24}$$

we want to show then F belongs to the kernel of $q[G, F]$ restricted to $\mathcal{F}_0 = D_0 \cap \operatorname{Ker} P_D$. Namely, we need to check that $F \in \mathcal{F}_0$ and that

$$\langle P_R\Phi, \Lambda P_R F\rangle = 0 \qquad \text{for all } \Phi \in \mathcal{F}_0. \tag{3.25}$$

That $F \in \mathcal{F}_0$ follows directly from (3.24). Furthermore, since $F' \in N_0 = D_0^\perp$ (assuming the natural identification of E' and E), we get

$$0 = \langle \Phi, F'\rangle = \langle P_N\Phi + P_R\Phi, P_D F' + P_R F'\rangle = \langle P_R\Phi, P_R F'\rangle = \langle P_R\Phi, \Lambda P_R F\rangle.$$

Here, in the second equality we used that $P_D + P_N + P_R = I$, $P_D\Phi = 0$ and $P_N F' = 0$; in the second we used mutual orthogonality of P_D, P_N and P_R; in the third we used equation (2.5). Note that this calculation is analogous with the one used to derive (3.17).

For the converse, we assume that an $F \in \mathcal{F}_0$ satisfies and (3.25), and we seek an $F' \in N_0 = D_0^\perp$ such that (3.24) holds. We will look for F' of the form $F' = P_D G + \Lambda P_R F$, which immediately yields $P_N F' = 0$ and $P_R F' = \Lambda P_R F$, fulfilling (3.24).

The condition $P_D G + \Lambda P_R F \in D_0^\perp$ is equivalent to finding $G \in E$ such that

$$\langle P_D\Psi, G\rangle = -\langle \Psi, \Lambda P_R F\rangle \tag{3.26}$$

for all $\Psi \in D_0$. But equation (3.26) is solvable for G since the right hand-side is a well-defined functional of $P_D\Psi$. Indeed, let $\Psi_1, \Psi_2 \in D_0$ be such that $\Psi_1 - \Psi_2 \in \operatorname{Ker} P_D$. Then $\Psi_1 - \Psi_2 \in \mathcal{F}_0$ and (3.25) gives

$$-\langle \Psi_1 - \Psi_2, \Lambda P_R F\rangle = 0, \tag{3.27}$$

finishing the proof.

Appendix A. Some basic examples

In this section we list some basic examples illustrating different asymptotic behaviors of the eigenvalues on a shrinking graph. In fact, in all examples the graph is just an interval.

Example A.1. Consider the interval $(0, \epsilon)$ with Dirichlet matching conditions,

$$f(0) = 0, \qquad f(\epsilon) = 0. \tag{A.1}$$

The eigenvalues are $\lambda_n = (\pi n)^2\epsilon^{-2}$, $n \in \mathbb{N}$, i.e. all eigenvalues are of "fast" type, Theorem 2.1 (1). To put conditions (A.1) in the form (2.3)-(2.5), we would set $P_D = I_2$ (the 2×2 identity matrix), $P_N = P_R = 0$.

Example A.2. The eigenvalues the interval $(0, \epsilon)$ with Neumann matching conditions,

$$f'(0) = 0, \qquad -f'(\epsilon) = 0, \tag{A.2}$$

are $\lambda_1 = 0$ and $\lambda_{n+1} = (\pi n)^2\epsilon^{-2}$, $n \in \mathbb{N}$, i.e. all but one are "fast" and one is "exotic". Conditions (A.2) are equivalent to $P_N = I_2$, $P_D = P_R = 0$.

Example A.3. If we change one of the vertices to Robin condition, namely

$$f'(0) = \gamma f(0), \qquad -f'(\epsilon) = 0, \tag{A.3}$$

where γ is real, then all but one eigenvalues are "fast" while $\lambda_1 \approx \gamma/\epsilon$, i.e. a "slow" eigenvalue with a sign which depends on γ.

Notably, when the other vertex is Dirichlet, namely

$$f'(0) = \gamma f(0), \qquad f(\epsilon) = 0, \tag{A.4}$$

then for every fixed γ all eigenvalues are eventually positive and "fast". The influence of the condition at ϵ comes through $\operatorname{Ker} P_D$ which, in the case of conditions (A.4) is disjoint from D_0 resulting in $m_0 = \dim \mathcal{F}_0 = 0$.

Conditions (A.3) are equivalent to

$$P_D = 0, \quad P_N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \gamma & \cdot \\ \cdot & \cdot \end{pmatrix}, \tag{A.5}$$

while conditions (A.4) have the above P_D and P_N swapped.

Example A.4. Finally, Robin conditions at both ends may produce more variety, including a non-zero exotic eigenvalue $-\gamma^2$ when

$$f'(0) = \gamma f(0), \qquad -f'(\epsilon) = -\gamma f(\epsilon). \tag{A.6}$$

The latter can be seen to correspond to the eigenfunction $f(x) = \exp(\gamma x)$.

The projectors in this example are

$$P_D = 0, \quad P_N = 0, \quad P_R = I_2, \quad \Lambda = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}. \tag{A.7}$$

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