Bergman Logarithmically Flat and Obstruction Flat Hypersurfaces and Their CR Structures

Peter Ebenfelt and Ming Xiao

This paper is dedicated to Jorge Hounie on the occasion of his 75th birthday anniversary.

1 Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$. Two important geometric objects associated to Ω are the (unique) complete Kähler–Einstein metric g_{KE} with Ricci curvature -1 and the Bergman metric g_B, which is also complete and Kähler. The boundary behaviors of the two metrics are intimately related to the CR geometry of the boundary $\partial\Omega$. The two notions of flatness that we shall consider in this chapter concern the vanishing of an obstruction to C^∞-smoothness up to the boundary of a log potential of the respective metrics. In what follows, the abbreviated terminology "flat CR hypersurface" refers to a strictly pseudoconvex CR hypersurface that is flat in either of these two senses.

In Sect. 2, we first recall the two notions of Bergman logarithmic flatness and obstruction flatness and collect some intriguing open questions. After that, we recall some known results about the two types of such flat CR hypersurfaces. We also establish some new results along the way in our discussion. The 3-dimensional flat CR hypersurfaces are much better understood than higher-dimensional ones. In Sect. 2.4, we discuss some of the known results in the 3-dimensional case. For example, every 3-dimensional Bergman logarithmically flat hypersurface is spherical, which means that their CR structures are locally equivalent. This is

The first author was supported in part by the NSF grant DMS-2154368. The second author was supported in part by the NSF grant DMS-2045104.

P. Ebenfelt (✉) · M. Xiao
Department of Mathematics, University of California at San Diego, La Jolla, CA, USA
e-mail: pebenfelt@ucsd.edu; m3xiao@ucsd.edu

S. Berhanu et al. (eds.), *Geometric Analysis of PDEs and Several Complex Variables*, Latin American Mathematics Series – UFSCar subseries,
https://doi.org/10.1007/978-3-031-69702-9_6

not true for obstruction flat ones, but *compact* 3-dimensional obstruction flat hypersurfaces are expected to be spherical. The latter is sometimes referred to as the Strong Ramadanov Conjecture and is still open.

In Sect. 3, we survey and discuss existence results for flat CR hypersurfaces, noncompact and compact, that are nonspherical in higher dimensions. It remains, however, an interesting question to determine "how many" Bergman logarithmically and obstruction flat (and nonspherical) CR hypersurfaces there are with distinct local CR structures. The question makes sense as the two notions of flatness only depend on the local CR geometry of the hypersurface. A *countably* infinite family of compact such hypersurfaces was constructed in [37] for each real dimension $2n+1$ with $n \geq 2$. The most significant new results of this chapter are probably the ones in Sect. 4. Among other results, we construct an *uncountably* infinite family of flat (in both senses) CR hypersurfaces $\{M_\alpha\}$ in $\mathbb{C}^{n+1}$, for each $n \geq 2$, such that each M_α is homogeneous and has transverse symmetry (but is noncompact) and any pair M_α, M_β with $\alpha \neq \beta$ has distinct local CR structures.

2 Background

2.1 The Kähler–Einstein Metric

Let $\Omega \subset \mathbb{C}^n$ be as in Sect. 1. The (unique) complete Kähler–Einstein metric g_{KE} on Ω with Ricci curvature -1 is given by

$$(g_{KE})_{\alpha\bar{\beta}} = (\log u^{-(n+1)})_{z_\alpha \overline{z_\beta}}, \tag{1}$$

where u is the unique solution in Ω of the Dirichlet problem

$$\begin{cases} J(u) := (-1)^n \det \begin{pmatrix} u & u_{\overline{z_\beta}} \\ u_{z_\alpha} & u_{z_\alpha \overline{z_\beta}} \end{pmatrix} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The uniqueness of a solution u was established by Fefferman [22], who also showed that there are approximate solutions $\rho \in C^\infty(\overline{\Omega})$ to this Dirichlet problem. More precisely, he showed that there are smooth functions ρ in $\overline{\Omega}$ such that $J(\rho) = 1 + O(\rho^{n+1})$, $\rho > 0$ in Ω and $\rho = 0$ on $\partial\Omega$, and any such approximate solution ρ is unique modulo $O(\rho^{n+2})$. Such approximate solutions are called *Fefferman defining functions*.

The existence of an actual solution $u \in C^\infty(\Omega) \cap C^{n+3/2-\epsilon}(\overline{\Omega})$ to (2) was proved by Cheng–Yau [8]. The boundary regularity of u was improved to $C^{n+2-\epsilon}(\overline{\Omega})$ by Lee–Melrose [29], who demonstrated that u has the asymptotic expansion

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k (\rho^{n+1} \log \rho)^k, \quad \eta_k \in C^\infty(\overline{\Omega}), \tag{3}$$

where ρ is a Fefferman defining function for Ω. Graham [23, 24] showed that the coefficients η_k, for $k \geq 1$, are *local* CR invariants of the boundary $\partial\Omega$ modulo $O(\rho^{n+1})$. In particular, the function $\eta_1|_{\partial\Omega}$ is a local CR invariant of $\partial\Omega$, and Graham (*op. cit.*) also proved that, in fact, $\eta_1|_{\partial\Omega} = 0$ if and only if the Cheng–Yau solution u is $C^\infty(\overline{\Omega})$ (which implies that $\eta_k = 0$ in $\overline{\Omega}$ for all $k \geq 1$). The local CR invariant $\eta_1|_{\partial\Omega}$ is called the *obstruction function* and will be denoted by $\mathcal{O}$. We shall say that $\partial\Omega$ is *obstruction flat* if $\mathcal{O} = 0$. The prototype of an obstruction flat hypersurface is the unit sphere or, more generally, any spherical hypersurface, i.e., a hypersurface that is locally CR diffeomorphic to a piece of the unit sphere.

We observe that since the obstruction function $\mathcal{O}$ is determined by the local CR geometry of the strictly pseudoconvex hypersurface $\partial\Omega$, we may define this invariant on a strictly pseudoconvex hypersurface X in a complex manifold in an obvious way. The notion of obstruction flatness then also carries over to strictly pseudoconvex hypersurfaces X in a complex manifold.

2.2 The Bergman Metric

The Bergman metric in Ω is given by

$$(g_B)_{\alpha\bar{\beta}} = (\log K)_{z_\alpha \overline{z_\beta}}, \tag{4}$$

where $K = K(z, \bar{z})$ denotes the Bergman kernel of Ω (the reproducing kernel of the subspace of holomorphic functions in $L^2(\Omega)$) on the diagonal. Fefferman [21] showed that K has an asymptotic expansion

$$K = \frac{\phi}{\rho^{n+1}} + \psi \log \rho, \quad \phi, \psi \in C^\infty(\overline{\Omega}), \tag{5}$$

where we take ρ to be a Fefferman defining function of Ω. The coefficients ϕ mod $O(\rho^{n+1})$ and ψ mod $O(\rho^\infty)$ are determined by the *local* CR geometry of $\partial\Omega$; see [21, 2, 26]. It is further known that $\phi = n!/\pi^n + O(\rho^2)$. We say that $\partial\Omega$ is *Bergman logarithmically flat* if the log coefficient ψ vanishes to infinite order (i.e., $\psi = 0$ mod $O(\rho^\infty)$). By the localization property of the Bergman kernel (cf. [21, 5, 19, 27]), Bergman logarithmic flatness only depends on the local CR geometry of the boundary, and we can define Bergman logarithmic flatness for any germ of strongly pseudoconvex CR hypersurface M in $\mathbb{C}^m$. We say M is Bergman logarithmically flat if, for some (and therefore every, by the localization property of the Bergman kernel) smoothly bounded strongly pseudoconvex domain G with $M \subseteq \partial G$, the coefficient function ψ in the Fefferman expansion (5) of K_G vanishes to the infinite order along M. Furthermore, a CR hypersurface Σ (in a complex manifold) is called Bergman logarithmically flat if it is so as a germ of CR hypersurface at every $p \in \Sigma$. The prototype of a Bergman logarithmically flat hypersurface is the unit sphere or any spherical hypersurface.

In order to observe an analogy with the construction of the Kähler–Einstein metric above, we introduce the function

$$v = K^{-\frac{1}{n+1}}, \tag{6}$$

which is $C^\infty(\Omega) \cap C^{n+2-\epsilon}(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$. In fact, we have an asymptotic expansion of the same type as that for the Cheng–Yau solution u,

$$v \sim \rho \sum_{k=0}^{\infty} \xi_k (\rho^{n+1} \log \rho)^k, \quad \xi_k \in C^\infty(\overline{\Omega}), \tag{7}$$

where we may take, near the boundary $\partial\Omega$,

$$\xi_k = c_k \left(\frac{\psi}{\phi}\right)^k \phi^{-\frac{1}{n+1}} \tag{8}$$

for some combinatorial constants $c_k \neq 0$ with $c_0 = 1$. We note that in this case C^∞-smoothness up to $\partial\Omega$ is obstructed by the coefficient ψ of the log term in Fefferman's asymptotic expansion of K, as ξ_k, for $k \geq 1$, vanishes to infinite order at the boundary if and only if ψ does (cf. [23]). Thus, the function v is $C^\infty(\overline{\Omega})$ if and only if ψ vanishes to infinite order at $\partial\Omega$. Moreover, in analogy with the Kähler–Einstein metric above, we can express the Bergman metric as

$$(g_B)_{\alpha\bar{\beta}} = (\log v^{-(n+1)})_{z_\alpha \overline{z_\beta}}. \tag{9}$$

While the Cheng–Yau solution u and the function v associated with the Bergman kernel have similar asymptotic expansions at $\partial\Omega$, a major difference between the two functions is that u solves the partial differential equation $J(u) = 1$, whereas v does not (in any apparent way) come from a differential equation. In fact, we have the following result, which is a slight sharpening of a key step in Huang–Xiao's resolution [28] of the Cheng Conjecture.

Proposition 2.1 *Let $\Omega \subset \mathbb{C}^n$, with $n \geq 3$, be a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$, and let v be the function given by (6). Then,*

$$J(v) = \frac{\pi^n}{n!} + O(\rho^2) \tag{10}$$

and the following are equivalent:

(i) It holds that

$$J\,(v) = \frac{\pi^n}{n!} + o(\rho^2). \tag{11}$$

(ii) It holds that

$$J(v) = \frac{\pi^n}{n!} + O(\rho^{n+2}). \tag{12}$$

(iii) $\partial\Omega$ *is spherical.*

Remark 2.2 The implication (i) $\Longrightarrow$ (iii) is contained in [34], but in terms of the so-called Bergman function instead of $J(v)$; see the remark at the end of this subsection.

Proof of Proposition 2.1 First, we recall the following result that follows from Fefferman's calculation on pp. 399–400 in [22]:

Lemma 2.3 *Let* χ *and* η *be* C^2*-smooth functions in a neighborhood of* $\partial\Omega$ *such that* $\chi = 0$ *on* $\partial\Omega$*. Then, for any* $s \geq 2$*,* $v = \chi + \eta\chi^s$ *satisfies*

$$J(v) = \left(1 + (n+2-s)s\eta\chi^{s-1}\right) J(\chi) + O(\chi^s). \tag{13}$$

By repeating this calculation, *mutatis mutandi*, with $v = \chi + \eta\chi^s \log\chi$ instead of $v = \chi + \eta\chi^s$, we also obtain the following:

Lemma 2.4 *Let* χ *and* η *be* C^2*-smooth functions in a neighborhood of* $\partial\Omega$ *such that* $\chi = 0$ *on* $\partial\Omega$*. Then, for any* $s \geq 3$*,* $v = \chi + \eta\chi^s \log\chi$ *satisfies*

$$J(v) = \left(1 + (n+2-s)\eta\chi^{s-1}s\log\chi + (n+2-2s)\eta\chi^{s-1})\right) J(\chi) + O(\chi^s \log\chi). \tag{14}$$

We shall now apply Lemma 2.4 to the function v given by (6). We observe that, by the asymptotic expansion (7), there is a function $\tilde{\eta} \in C^{n+1-\epsilon}$ near $\partial\Omega$ such that $v = \rho\xi_0 + \tilde{\eta}\rho^{n+2}\log\rho$. We note at any point $p \in \partial\Omega$, and it holds that $\rho(p) = 0$, $\xi_0(p) = c := (\frac{n!}{\pi^n})^{-\frac{1}{n+1}}$ and $\tilde{\eta}(p) = a$ for some $a \in \mathbb{R}$. We want to rewrite this in the form $v = \chi + \eta\chi^{n+2}\log\chi$, where $\chi = \rho\xi$ for some $\xi \in C^{n+1-\epsilon}$ near $\partial\Omega$. By comparing the two expressions, we note that this can be done if we can solve the two equations

$$\begin{aligned} &\xi + \eta\xi^{n+2}\rho^{n+1}\log\xi = \xi_0 \\ &\eta\xi^{n+2} = \tilde{\eta}, \end{aligned} \tag{15}$$

for $\xi = \xi(\rho, \xi_0, \tilde{\eta})$ and $\eta = \eta(\rho, \xi_0, \tilde{\eta})$ near the point $(\rho, \xi_0, \tilde{\eta}; \xi, \eta) = (0, c, a; c, \frac{a}{c^{n+2}})$. One can readily verify the system (15) is nondegenerate in (ξ, η) at this point. Therefore a straightforward application of the Implicit Function Theorem guarantees that this is the case. Moreover, the functions $\xi = \xi(\rho, \xi_0, \tilde{\eta})$ and $\eta = \eta(\rho, \xi_0, \tilde{\eta})$ are smooth (in fact, real analytic) in the arguments $(\rho, \xi_0, \tilde{\eta})$; thus ξ, η becomes $C^{n+1-\epsilon}$ functions near $\partial\Omega$ after the composition. Moreover, by the first equation in (15), we see $\xi = \xi_0 + O(\rho^{n+1})$. We now apply Lemma 2.4 to the function $v = \chi + \eta\chi^{n+2}\log\chi$ with $\chi = \rho\xi$ as above. Lemma 2.4 implies that

$$J(v) = J(\rho\xi) + O(\rho^{n+1}). \tag{16}$$

Since $\rho\xi = \rho\xi_0 + O(\rho^{n+2})$, Lemma 2.3 with $s = n+2$ then implies $J(\rho\xi) = J(\rho\xi_0) + O(\rho^{n+2})$. Consequently,

$$J(v) = J(\rho\xi_0) + O(\rho^{n+1}). \tag{17}$$

If we introduce the function

$$P_2 = \frac{\phi - n!/\pi^n}{\rho^2},$$

then we can write $\phi = n!/\pi^n + P_2\rho^2$, which implies that $\rho\xi_0 = \rho\phi^{-\frac{1}{n+1}} = \chi + \eta\chi^3$, where

$$\chi = \left(\frac{\pi^n}{n!}\right)^{1/(n+1)} \rho$$

and where

$$\eta|_{\partial\Omega} = -\frac{1}{n+1}\left(\frac{\pi^n}{n!}\right)^{(n-1)/(n+1)} P_2|_{\partial\Omega}. \tag{18}$$

If we apply Lemma 2.3 with $v = \rho\xi_0 = \chi + \eta\chi^3$, then we conclude that

$$J(\rho\xi_0) = (1 + 3(n-1)\eta\chi^2)J(\chi) + O(\rho^3) = (1 + 3(n-1)\eta\chi^2)\frac{\pi^n}{n!} + O(\rho^3), \tag{19}$$

since $J(\chi) = \pi^n/n! J(\rho) = \pi^n/n! + O(\rho^{n+1})$ and $n \geq 3$. Here we have used the fact that ρ is a Fefferman defining function. The conclusion (10) follows from (17) and (19).

It remains to establish the equivalence of the statements (i)–(iii). We first prove (i) $\Longrightarrow$ (iii). Equations (17) and (19) together imply that (11) holds if and only if $\eta|_{\partial\Omega} = 0$, which by (18) is equivalent to $P_2|_{\partial\Omega} = 0$. On the other hand, $P_2|_{\partial\Omega}$ is known to be a nonzero constant multiple of $\|S\|^2$, where $S = S_{\alpha\bar{\beta}\nu\bar{\mu}}$ is the Cartan–Chern–Moser CR curvature tensor of $\partial\Omega$ and $\|S\|^2 = S_{\alpha\bar{\beta}\nu\bar{\mu}}S^{\alpha\bar{\beta}\nu\bar{\mu}}$ (cf. [10, 23, 28]). Since being spherical can be characterized by the vanishing of $\|S\|^2$, the implication (i) $\Longrightarrow$ (iii) of the proposition is proved.

To prove (iii) $\Longrightarrow$ (ii), we recall that ϕ mod $O(\rho^{n+1})$ and ψ (mod $O(\rho^\infty)$) are determined by the local CR geometry of $\partial\Omega$. Thus, if $\partial\Omega$ is spherical, then

$\phi = n!/\pi^n + O(\rho^{n+1})$ and $\psi = 0$ to infinite order at $\partial\Omega$. Moreover, the obstruction function $\mathcal{O} = 0$, which implies that the Cheng–Yau solution u is $C^\infty(\overline{\Omega})$. Thus, the Cheng–Yau solution u is a Fefferman defining function. Taking $\rho = u$ in (5), by (6), we get

$$v = u\phi^{-\frac{1}{n+1}} + O(u^\infty) = u\left(\frac{\pi^n}{n!}\right)^{\frac{1}{n+1}} + O(u^{n+2}).$$

The conclusion (12) now follows from Lemma 2.3 with $s = n + 2$.

To complete the proof of the equivalence of (i)–(iii) and, hence, that of the proposition, we simply note that (ii) obviously $\Longrightarrow$ (i). □

We end this subsection with some remarks that are somewhat tangential to the main theme of this chapter. We recall that, for $n \geq 3$, a boundary point $p \in \partial\Omega$ is said to *CR umbilical* if the CR curvature tensor $S_{\alpha\bar{\beta}\nu\bar{\mu}}$ vanishes at that point. Thus, being spherical is equivalent to being CR umbilical in an open neighborhood. We observe that the proof of Proposition 2.1 readily yields the following pointwise result.

Proposition 2.5 *Let $\Omega \subset \mathbb{C}^n$, with $n \geq 3$, be a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$ and let v be the function given by (6). Then, a point $p \in \partial\Omega$ is CR umbilical if and only if*

$$\lim_{z\to p} \rho(z)^{-2}\left(J\,(v(z)) - \frac{\pi^n}{n!}\right) = 0. \tag{20}$$

For our second remark, we observe that if (11) holds, then the Bergman metric is *asymptotically Kähler–Einstein* (with Ricci equal to -1) in the sense that $Ric(g_B) = -g_B + o(1)$, i.e., every component of the tensor $Ric(g_B) + g_B$ is $o(1)$ as $z \to \partial\Omega$. The converse can also be verified. Indeed, a pointwise result along the lines of Proposition 2.5 can be established using the same ideas as in the proof of Proposition 2.1.

Proposition 2.6 *Let $\Omega \subset \mathbb{C}^n$, with $n \geq 3$, be a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$. Then, a point $p \in \partial\Omega$ is CR umbilical if and only if*

$$\lim_{z\to p} (Ric(g_B)(z) + g_B(z)) = 0. \tag{21}$$

Proof It follows easily from the identity $Ric(g) = -(\log G)_{z_\alpha\overline{z_\beta}}$, where $G = \det g$, and the well-known formula

$$\det[(-\log f)_{z_\alpha\overline{z_\beta}}] = \frac{J(f)}{f^{n+1}} \tag{22}$$

that the components of the tensor $Ric(g_B)(z) + g_B(z)$ can be expressed as

$$(Ric(g_B)(z) + g_B(z))_{\alpha\bar{\beta}} = -(\log J(v))_{z_\alpha \overline{z_\beta}}. \tag{23}$$

It follows from (17) and (19) in the proof of Proposition 2.1 that

$$J(v) = (1 + 3(n-1)\eta\chi^2)\frac{\pi^n}{n!} + O(\rho^3), \tag{24}$$

where χ and η are as in that proof. It follows that there is a C^2-smooth function $\tilde{P}_2$ such that

$$\log J(v) = \rho^2 \tilde{P}_2. \tag{25}$$

Moreover, in view of (18) and the arguments in the proof of Proposition 2.1, it holds that $\tilde{P}_2|_{\partial\Omega}$ is a nonzero constant times the norm squared of the CR curvature, $\|S\|^2$. Thus, (23) implies that

$$\lim_{z\to p} (Ric(g_B)(z) + g_B(z))_{\alpha\bar{\beta}} = c\rho_{z_\alpha}(z)\rho_{\overline{z_\beta}}(p)\|S(p)\|^2, \quad \forall\, 1 \le \alpha, \beta \le n, \tag{26}$$

for some nonzero constant c. The proposition then follows from the above identity. □

As our final remark, we note that by using the identity (22), it is not difficult to show that

$$J(v) = \frac{1}{(n+1)^n}\frac{G}{K},$$

where $G = \det g_B$ as above. The function $B = G/K$ is sometimes called the *Bergman function*. This is the function referred to in Remark 2.2.

2.3 *The Ramadanov Conjecture and an Analogous Question for Obstruction Flat Boundaries*

We first recall the Ramadanov Conjecture [35]:

Ramadanov Conjecture *If $\Omega \subset \mathbb{C}^n$, $n \ge 2$, is a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$ and if $\partial\Omega$ is Bergman logarithmically flat, then $\partial\Omega$ is spherical.*

The Ramadanov Conjecture has been established in $\mathbb{C}^2$ by the work of Burns, Graham, and Boutet de Monvel [24, 4]. Indeed, in this case it suffices that ψ =

$O(\rho^2)$ for $\partial\Omega$ to be spherical. The Ramadanov Conjecture is still open in $\mathbb{C}^n$ with $n \geq 3$, but it is known to fail if we instead consider domains Ω in more general complex manifolds [20] or Stein spaces [17] as long as the complex dimension n of the space is at least 3.

We observe that the Ramadanov Conjecture can be formulated in terms of the flatness of the first log coefficient ξ_1 in the asymptotic expansion of the function v introduced in (6), since ξ_1 is flat if and only if ψ is flat. As observed above, flatness of ψ is also equivalent to C^∞-smoothness up to the boundary of v.

One may ask the same question in the context of the Kähler–Einstein metric. Asking for C^∞-smoothness up to the boundary of the Cheng–Yau solution is, as mentioned above, equivalent to obstruction flatness. Thus, the analogue of the Ramadanov Conjecture for the Cheng–Yau solution would be:

Obstruction Flatness Conjecture *If $\Omega \subset \mathbb{C}^n$, $n \geq 2$, is a bounded, strictly pseudoconvex domain with C^∞-smooth boundary $\partial\Omega$ and if $\partial\Omega$ is obstruction flat, then $\partial\Omega$ is spherical.*

The Obstruction Flatness Conjecture is open in all dimensions $n \geq 2$, although there is ample evidence for its correctness when $n = 2$. For example, the first author and Curry have shown that in $\mathbb{C}^2$ the conjecture holds if one assumes in addition that $\partial\Omega$ has an infinitesimal CR symmetry [15, 13] or, more generally, an approximate infinitesimal automorphism [11]. We recall that $\partial\Omega$ has an infinitesimal CR symmetry if and only if there exists a holomorphic vector field near $\partial\Omega$ whose real part is tangent along $\partial\Omega$. Possessing an *approximate* infinitesimal automorphism is equivalent to the existence of a holomorphic vector field such that its real part is, roughly speaking, more tangent than normal along $\partial\Omega$; the reader is referred to [11] for the precise definition.

2.4 The Two-Dimensional Case

For domains $\Omega \subseteq \mathbb{C}^2$ (or more generally in a complex surface), there is a precise and direct relation between the two notions of flatness. Graham [24] has shown that in this case, the obstruction function $\mathcal{O}$ equals a nonzero constant times the restriction to $\partial\Omega$ of the log term coefficient ψ of the Bergman kernel K. Thus, Bergman logarithmic flatness implies obstruction flatness. We recall that in $\mathbb{C}^2$ (or in a complex surface), the Ramadanov Conjecture holds and, in fact, a stronger statement holds: If $\psi = O(\rho^2)$ along the boundary $\partial\Omega$, then $\partial\Omega$ is spherical. The Obstruction Flatness Conjecture can then be reformulated as asserting the stronger statement that $\psi = O(\rho)$ implies that $\partial\Omega$ is spherical. Hence, in $\mathbb{C}^2$ the Obstruction Flatness Conjecture has sometimes been called the *Strong Ramadanov Conjecture*.

For domains $\Omega \subset \mathbb{C}^2$, a *local* version of the Ramadanov Conjecture holds. Indeed, if $\psi = O(\rho^2)$ along an open piece U of the boundary $\partial\Omega$, then U is spherical. This follows from the proof of this conjecture in [24, 4]. The corresponding local version of the Obstruction Flatness Conjecture (a.k.a. the Strong Ramadanov Conjecture) is false, however, as was first demonstrated by Graham [23]. In fact, the following, which is a consequence of Theorem 3.2 in [16] and Chern–Moser's normal forms theory [9], holds true:

Theorem 2.7 *There are uncountably infinite families of obstruction flat strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ that are pairwise locally CR inequivalent.*

Proof Recall from [9, pp. 246–247] that a real analytic hypersurface X in $\mathbb{C}^2$ with a non-umbilical point $p_0 \in X$ can be put in a special Chern–Moser normal form at $p_0 = (0, 0)$ as follows:

$$\operatorname{Im} w = |z|^2 + 2 \operatorname{Re} z^4 \bar{z}^2 \left(1 + A^0_{5\bar{2}} z + i A^1_{4\bar{2}} \operatorname{Re} w\right) + O(8), \tag{27}$$

where $A^0_{5\bar{2}} \in \mathbb{C}$, $A^1_{4\bar{2}} \in \mathbb{R}$ are such that $(A^0_{5\bar{2}})^2$ and $A^1_{4\bar{2}}$ are invariants, and $O(8)$ denotes terms that are of (unweighted) order at least 8 in $(z, \bar{z}, \operatorname{Re} w)$. By the proof of Theorem 3.2 in [16], we may pose a Cauchy problem for the totally characteristic partial differential operator K from [23], associated with obstruction flat hypersurfaces, with the data

$$\rho = \operatorname{Im} w - |z|^2 - 2 \operatorname{Re} z^4 \bar{z}^2 \left(1 + A^0_{5\bar{2}} z + i A^1_{4\bar{2}} \operatorname{Re} w\right) \tag{28}$$

on the real hyperplane $\operatorname{Re} z = 0$, and produce, via the Cauchy–Kowalevski Theorem, a unique obstruction flat hypersurface whose Chern–Moser normal form at 0 is of the special form (27). Clearly, using this construction, the existence of families of obstruction flat hypersurfaces as in the statement of Theorem 2.7 follows. □

It was also shown in [16] that there are nonspherical obstruction flat hypersurfaces in $\mathbb{C}^2$ with transverse infinitesimal CR symmetries, which implies that the local versions of the positive results on the Obstruction Flatness Conjecture in [15, 13, 11] do not hold.

2.5 Flat Rigidity of the Unit Sphere

While both the Ramadanov Conjecture in $\mathbb{C}^n$, for $n \geq 3$, and the Obstruction Flatness Conjecture, for all $n \geq 2$, are open, they are known to hold for small deformations of the unit sphere. To describe this more precisely, we let $Z_1, \ldots, Z_n$ denote a local frame, near some point $p \in S^{2n-1}$, for the $(1, 0)$-vector fields that define the standard CR structure on the unit sphere $S^{2n-1} = \partial\mathbb{B}^n$ in $\mathbb{C}^n$. Any abstract smooth CR structure on the sphere S^{2n-1} whose underlying contact structure is isotopic to the standard one is CR diffeomorphic to a deformation of the standard structure given, near p, by the vector fields

$$Z_\alpha^{(\mu)} = Z_\alpha + {\mu_\alpha}^{\bar\beta} Z_{\bar\beta}, \tag{29}$$

where $\mu = {\mu_\alpha}^{\bar\beta}$ is a smooth deformation tensor with L^∞-norm < 1; and we are using the summation convention. The deformation tensor $\mu = {\mu_\alpha}^{\bar\beta}$, defined locally in this way near p, defines a global deformation tensor, i.e., a global section of $(T^{1,0}S^{2n-1})^* \otimes T^{1,0}S^{2n-1}$. Since the underlying contact structure for these deformations is fixed (to be the standard one on S^{2n-1}), the deformation tensor μ completely determines the CR structure, and we shall denote the deformed structure by (S^{2n-1}, μ).

We shall be concerned with CR structures that are close to the standard structure $(S^{2n-1}, 0)$. More precisely, we shall require the C^k-norm of μ, $\|\mu\|_k$ for a sufficiently large k (e.g., $k \geq 6$ suffices), to be small; in particular, the CR structure (S^{2n-1}, μ) will remain strictly pseudoconvex. We observe a fundamental difference between the lowest dimensional case, $n = 2$, and $n \geq 3$. In the former case, most deformations (S^3, μ) are not CR diffeomorphic to a CR structure on S^3 obtained by deforming the unit sphere inside $\mathbb{C}^2$ (indeed, they are not even embeddable locally) and the ones that form a proper Fréchet submanifold inside the space of all deformations; the reader is referred to [7, 30, 12] for a discussion. We point out that even when the CR structure (S^3, μ) is not locally embeddable, the CR invariant $\mathcal{O}$ can still be defined, and, hence, the notion of obstruction flatness carries over to this situation. In this case, however, the connection with an ambient complete Kähler–Einstein metric is no longer valid, although a connection to a different ambient metric problem does exist; the reader is referred, e.g., to [14] for a discussion. The notion of Bergman logarithmically flat makes less sense for CR structures that are not locally embeddable.

In the case $n \geq 3$, it follows from Boutet de Monvel's CR embedding theorem [3] and Lempert's stability theorem [30] that *any* deformed structure (S^{2n-1}, μ) is CR diffeomorphic to one obtained by deforming the unit sphere inside $\mathbb{C}^n$.

By combining the results of Curry and the first author [13, 11, 11] (for the obstruction flat statement) and the resolution of the Ramadanov Conjecture (for the Bergman logarithmically flat statement) in the case $n = 2$ and that of Hirachi [25] in the case $n \geq 3$, one obtains the following:

Theorem 2.8 *Let (S^{2n-1}, μ), for $n \geq 2$, be a sufficiently small deformation of the standard CR structure $(S^{2n-1}, 0)$ of the unit sphere. If (S^{2n-1}, μ) is obstruction flat or Bergman logarithmically flat (in the locally embeddable case), then (S^{2n-1}, μ) is CR diffeomorphic to $(S^{2n-1}, 0)$.*

While Theorem 2.8 offers evidence of the validity of the Ramadanov Conjecture and the Obstruction Flatness Conjecture for domains in $\mathbb{C}^n$, we note that the conjectures can also be posed for smoothly bounded (relatively compact) strictly pseudoconvex domains Ω in more general complex manifolds and, in this context, both conjectures would fail when $n \geq 3$. The remainder of this chapter will focus on investigating this phenomenon more closely in the special context of disk bundles over Kähler manifolds.

3 Flat Hypersurfaces in Higher Dimension

In this section, we shall investigate the local CR geometry of compact, obstruction flat, strictly pseudoconvex hypersurfaces in higher dimensions. While the Obstruction Flatness Conjecture is still open in two dimensions, i.e., for 3-dimensional CR manifolds (embeddable in complex manifolds or not), there is ample evidence for its validity. If true, then the local CR geometry of 3-dimensional, compact, obstruction flat, strictly pseudoconvex hypersurfaces is trivial; they are all spherical. As mentioned above, however, both the Ramadanov and Obstruction Flatness Conjecture fail for boundaries of domains in complex manifolds of higher dimension. Hence, the next challenge is to understand the (local) CR equivalence classes of compact, obstruction flat, strictly pseudoconvex hypersurfaces. As a first step in this direction, we shall consider a special class of such, namely, unit circle bundles over Kähler manifolds.

3.1 The Setup

Let M be a Kähler manifold of dimension m, endowed with a Kähler metric g. The associated Kähler form will be denoted by ω. Let $L \to M$ be a negative holomorphic line bundle equipped with a Hermitian metric h "quantizing" (M, g); i.e., the metric g is induced by the curvature of $(L, h) \to M$ via

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h.$$

We shall denote the dimension of the complex manifold L by $n = m + 1$. The Ricci curvature $Ric(g)$ induces an endomorphism $Ric(g)^{\sharp} \colon TM \to TM$, given in a local coordinate chart $z = (z_1, \ldots, z_m)$ by

$$X^{\alpha} \mapsto R_{\gamma}{}^{\alpha} X^{\gamma},$$

where

$$X = X^{\gamma}\partial/\partial z_{\gamma}, \qquad R_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z_{\alpha}\overline{z_{\beta}}}\log G, \qquad G = \det g,$$

and where we raise and lower indices by the metric; e.g., $R_\alpha{}^\beta = g^{\beta\bar{\mu}} R_{\alpha\bar{\mu}}$ with $g^{-1} = g^{\alpha\bar{\beta}}$ denoting the inverse of the matrix $g = g_{\alpha\bar{\beta}}$.

We shall denote by $D(M, h)$ and $\Sigma(M, h)$ the unit disk bundle and the unit circle bundle, respectively, i.e.,

$$\begin{aligned} D(M, h) &= \{v \in L \colon |v|_h < 1\}, \\ \Sigma(M, h) &= \{v \in L \colon |v|_h = 1\}. \end{aligned} \tag{30}$$

When M and $(L, h) \to M$ are fixed, we shall abbreviate $D(M, h)$ and $\Sigma(M, h)$ by D and Σ, respectively. It is a well-known result of Grauert that Σ is a smooth strictly pseudoconvex hypersurface in L. If M is compact, then Σ is compact and comprises the boundary ∂D. If M is noncompact, then in general Σ is merely a proper subset of ∂D, and ∂D may be neither smooth nor strictly pseudoconvex.

3.2 Sufficient Conditions for Obstruction Flatness

The endomorphism $Ric(g)^\sharp$ has, pointwise, m real eigenvalues $\lambda_1 \leq \cdots \leq \lambda_m$ that form real-valued functions on M. We have the following sufficient condition for obstruction flatness from the authors' joint work with H. Xu [18]:

Theorem 3.1 *Let $(L, h) \to M$ be a negative Hermitian holomorphic line bundle over a Kähler manifold (M, g) such that the curvature of h induces the metric g as in Sect. 3.1. If the eigenvalues of the Ricci endomorphism $Ric(g)^\sharp$ are constant, then $\Sigma(M, h)$ is obstruction flat.*

Remark 3.2 The prototype of a situation (but not the only one) where the Ricci eigenvalues are constant is where (M, g) is Kähler–Einstein or, more generally, a product of Kähler–Einstein manifolds. If, for example, $M = M_1 \times \ldots \times M_k$ is a product of Kähler–Einstein manifolds (M_i, g^i) such that $Ric(g^i) = \mu_i g^i$, then the Ricci eigenvalues are $\mu_1, \ldots, \mu_k$ with each eigenvalue having multiplicity $=$ dim M_i. Another prototype is when (M, g) is a homogeneous Kähler manifold.

3.3 Examples of Bergman Logarithmic Flatness

Recall the only Bergman logarithmically flat hypersurfaces in $\mathbb{C}^2$ are the spherical ones, by the work of Burns, Graham, and Boutet de Monvel [23, 4]. In higher dimensions, there exist noncompact Bergman logarithmically flat hypersurfaces in $\mathbb{C}^n$ that are not spherical. There are also compact examples realized as circle bundles $\Sigma(M, h)$ that arise from a negative line bundle (L, h) quantizing a compact Kähler manifold (M, g) as in (30). The first such examples were constructed by

Engliš and Zhang [20]. They showed that the corresponding circle bundle must be Bergman logarithmically flat if (M, g) is an irreducible Hermitian symmetric space of compact type. Furthermore, this circle bundle is spherical precisely when M is the projective space (note this fact also follows from the combined work of Webster [36] and Bryant [6], cf. Theorem 4.1). Later Loi–Mossa–Zuddas [32] proved that $\Sigma(M, h)$ is Bergman logarithmically flat if (M, g) is a compact simply connected homogeneous Kähler–Einstein manifold of classical type.

It is then natural to ask about the case when (M, g) is a Hermitian symmetric space of noncompact type, i.e., a bounded symmetric domain. We pause to recall some preliminaries on bounded symmetric domains. A Hermitian symmetric space of noncompact type can be realized as a convex and circular bounded domain containing 0 in some complex Euclidean space, via the Harish–Chandra realization, equipped with the (possibly normalized) Bergman metric. Such realizations will be called bounded symmetric domains. A bounded symmetric domain is called irreducible if it cannot be written as the product of two of such domains. We next recall the notion of generic norms. Let Ω_0 be an irreducible bounded symmetric domain and denote by $K(z, \overline{z})$ its Bergman kernel. Write V_{Ω_0} for the volume of Ω_0 in the Euclidean measure, and γ for the genus of Ω_0 (which is a positive integer associated to Ω_0, cf. [33]). Then $(V_{\Omega_0} K(z, \overline{z}))^{-\frac{1}{\gamma}}$ gives a Hermitian polynomial, denoted by $N(z, \overline{z})$. The polynomial N is called the generic norm of Ω_0.

Now let $\Omega = \Omega_1 \times \cdots \times \Omega_l$ be a bounded symmetric domain in $\mathbb{C}^n_z$, where each Ω_i, $1 \leq i \leq l$, is an irreducible bounded symmetric domain in $\mathbb{C}^{n_i}_{z_i}$ with $n = \sum_{i=1}^{l} n_i$. Denote by $N_i(z_i, \overline{z_i})$, $1 \leq i \leq l$, the generic norm of Ω_i. Next let $L = \Omega \times \mathbb{C}$ be the trivial line bundle over Ω. Equip L with the Hermitian metric h, where

$$h(z, \overline{z}) = \prod_{i=1}^{l} \left(N_i(z_i, \overline{z_i})\right)^{-k_i}. \tag{31}$$

Here every k_i is a positive real number. Note the negative of the Chern class $-c_1(L, h) = \partial\overline{\partial} \log h$ induces a complete homogeneous Kähler metric ω_Ω on Ω. Note ω_Ω is also a product of Kähler–Einstein metrics on Ω_i's. Moreover, the disk and circle bundles of (L, h) are given by the following, respectively:

$$D(\Omega, h) := \left\{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^{l} \left(N_i(z_i, \overline{z_i})\right)^{k_i}\right\} \subseteq \mathbb{C}^{n+1}; \tag{32}$$

$$\Sigma(\Omega, h) := \left\{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^{l} \left(N_i(z_i, \overline{z_i})\right)^{k_i}\right\} \subseteq \mathbb{C}^{n+1}. \tag{33}$$

The Bergman kernel $K_D(z, \xi, \overline{z}, \overline{\xi})$ of $D(\Omega, h)$ was computed by Ahn–Park [1]. Write

$$r(z, \xi, \overline{z}, \overline{\xi}) = 1 - \frac{|\xi|^2}{\prod_{i=1}^{l} \left(N_i(z_i, \overline{z_i})\right)^{k_i}}.$$

Note r is a local defining function of $\Sigma(\Omega, h)$. By Theorem 2.5 in [1], we see $r^{n+1} K_D$ extends smoothly across $\Sigma(\Omega, h)$. Let $D' \subset D(\Omega, h)$ be a small smoothly bounded strongly pseudoconvex domain that shares an open piece of boundary $M \subset \Sigma(\Omega, h)$ with $D(\Omega, h)$. Then writing $K_{D'}$ for the Bergman kernel of D', by the localization property of the Bergman kernel (see, e.g., Theorem 4.2 in [19]), $r^{n+1} K_{D'}$ extends smoothly across M. By inspecting the Fefferman expansion of $K_{D'}$, we conclude that M is Bergman logarithmically flat; since M can be arbitrarily chosen, so is $\Sigma(\Omega, h)$. Thus we have observed the following fact.

Proposition 3.3 *Let $\Omega = \Omega_1 \times \cdots \times \Omega_l$ be a bounded symmetric domain in $\mathbb{C}^n_z$, where each $\Omega_i, 1 \le i \le l$, is an irreducible bounded symmetric domain in $\mathbb{C}^{n_i}_{z_i}$ with $n = \sum_{i=1}^{l} n_i$. Denote by $N_i(z_i, \overline{z_i}), 1 \le i \le l$, the generic norm of Ω_i. If all $k_i, 1 \le i \le l$, are positive real numbers, then $\Sigma(\Omega, h)$ in (33) is Bergman logarithmically flat.*

4 CR Equivalence Classes of Flat Hypersurfaces

Although Theorem 3.1 allows for a wide range of compact (or noncompact, of course) locally isometrically inequivalent Kähler manifolds (M, g) as base manifolds, it is not at all clear that the corresponding unit sphere bundles $\Sigma(M, h)$ are locally CR inequivalent. To elucidate the issue, and also characterize those (M, g) in the context of Theorem 3.1 that give rise to spherical unit circle bundles $\Sigma(M, h)$, we state the following result, which is a consequence of the work of Webster [36] and Bryant [6] combined with Theorem 3.1:

Theorem 4.1 *Let $(L, h) \to M$ be a negative Hermitian holomorphic line bundle over a Kähler manifold (M, g) such that the curvature of h induces the metric g as in Sect. 3.1. Assume that the eigenvalues of the Ricci endomorphism $Ric(g)^\sharp$ are constant. Then, $\Sigma(M, h)$ is obstruction flat. Furthermore, $\Sigma(M, h)$ is spherical if and only if (M, g) is locally holomorphically isometric to one of the following:*

(i) $(\mathbb{B}^n, \mu g_{-1})$ *for some* $\mu \in \mathbb{R}^+$
(ii) $(\mathbb{CP}^n, \mu g_1)$ *for some* $\mu \in \mathbb{R}^+$
(iii) $(\mathbb{C}^n, g_0)$
(iv) $(\mathbb{B}^l \times \mathbb{CP}^{n-l}, \mu g_{-1} \times \mu g_1)$ *for some* $1 \le l \le n-1$ *and some* $\mu \in \mathbb{R}^+$

where g_c denotes the canonical complete metric of constant holomorphic curvature c on the corresponding space form.

We note that the base manifolds within each category (i)–(iv) are conformally isometric, whereas two manifolds from different categories are locally and conformally isometrically inequivalent. Nevertheless, their unit sphere bundles are all spherical and, hence, locally CR equivalent. We shall see in the following two subsections that this spherical case is in a sense special. In particular, we shall consider more general products of base manifolds with constant holomorphic sectional curvature metrics and provide conditions that guarantee that the unit circle bundles are locally CR equivalent if and only if the base manifolds are conformally isometrically equivalent.

4.1 Rigidity of Unit Circle Bundles Over Bounded Symmetric Domains

As a special case of Theorem 3.1, if a negative line bundle (L, h) quantizes a homogeneous Kähler manifold (M, g), then the circle bundle Σ is obstruction flat. In general, homogeneous manifolds are more tractable than the broader class of Kähler manifolds with constant Ricci eigenvalues. Among homogeneous Kähler manifolds, one can further distinguish the Hermitian symmetric spaces. Recently, the second author studied the CR geometry of the corresponding circle bundle when the base manifold (M, g) is a Hermitian symmetric space of noncompact type, i.e., a bounded symmetric domain. A rigidity phenomenon was discovered for local CR mappings between such circle bundles. Using this rigidity result, one can construct countably infinite families of compact, obstruction flat CR manifolds that are pairwise locally CR inequivalent (see Theorem 4.3).

Following the notions in Sect. 3.3, let $\Omega = \Omega_1 \times \cdots \times \Omega_l$ be a bounded symmetric domain in $\mathbb{C}^n_z$, where each Ω_i, $1 \leq i \leq l$, is an irreducible bounded symmetric domain in $\mathbb{C}^{n_i}_{z_i}$ with $n = \sum_{i=1}^{l} n_i$. Denote by $N_i(z_i, \overline{z_i})$, $1 \leq i \leq l$, the generic norm of Ω_i. Let $L = \Omega \times \mathbb{C}$ be the trivial line bundle over Ω. Equip L with the Hermitian metric h as defined in (31), where every k_i, there is a positive real number. Note the negative of the Chern class $-c_1(L, h) = \partial\overline{\partial} \log h$ of (L, h) induces a complete homogeneous Kähler metric ω_Ω on Ω, which is a product of Kähler–Einstein metrics on Ω_i's. Let $D(\Omega, h)$ and $\Sigma(\Omega, h)$ be as defined in (32) and (33), respectively. From Proposition 3.3, we see $\Sigma(\Omega, h)$ is Bergman logarithmically flat; from Theorem 3.1 (see also Remark 3.2), we know $\Sigma(\Omega, h)$ is obstruction flat as well.

We also consider a similar set of disk and circle bundles as above. Let $\widetilde{\Omega} = \widetilde{\Omega}_1 \times \cdots \times \widetilde{\Omega}_m$ be a bounded symmetric domain in $\mathbb{C}^n$, where each $\widetilde{\Omega}_j$, $1 \leq j \leq m$, is an irreducible bounded symmetric domain in $\mathbb{C}^{\widetilde{n}_j}_{w_j}$ with $n = \sum_{j=1}^{m} \widetilde{n}_j$. Denote by $\widetilde{N}_j(w_j, \overline{w_j})$, $1 \leq j \leq m$, the generic norm of $\widetilde{\Omega}_j$. Next let $\widetilde{L} = \widetilde{\Omega} \times \mathbb{C}$ be the trivial line bundle over $\widetilde{\Omega}$, and equip $\widetilde{L}$ with the Hermitian metric $\widetilde{h}$, where

$$\widetilde{h}(w, \bar{w}) = \prod_{i=1}^{m} \left(\widetilde{N}_i(w_i, \overline{w_i})\right)^{-\widetilde{k}_i}, \quad \widetilde{k}_i \in \mathbb{R}, \widetilde{k}_i > 0.$$

As before, the negative of the Chern class $-c_1(\widetilde{L}, \widetilde{h}) = \partial\overline{\partial} \log \widetilde{h}$ of $(\widetilde{L}, \widetilde{h})$ induces a complete homogeneous Kähler metric $\omega_{\widetilde{\Omega}}$ on $\widetilde{\Omega}$. The disk bundle of $(\widetilde{L}, \widetilde{h})$ is given by

$$D(\widetilde{\Omega}, \widetilde{h}) := \{(w, \eta) \in \Omega \times \mathbb{C} : |\eta|^2 < \prod_{i=1}^{t} \left(\widetilde{N}_i(w_i, \overline{w_i})\right)^{\widetilde{k}_i}\} \subseteq \mathbb{C}^{n+1}.$$

The corresponding circle bundle $\Sigma(\widetilde{\Omega}, \widetilde{h})$ is defined similarly by replacing the inequality with an equation. In [37], the second author proved the following rigidity result.

Theorem 4.2 *Let all k_i's and $\widetilde{k}_i$'s in the above be positive integers, and assume at least one k_i and at least one $\widetilde{k}_i$ equal to* 1. *Let F be a nonconstant continuous CR map from an open connected piece of $\Sigma(\Omega, h)$ to $\Sigma(\widetilde{\Omega}, \widetilde{h})$. Then the following statements hold:*

(1) The map F extends to a rational biholomorphism from $D(\Omega, h)$ to $D(\widetilde{\Omega}, \widetilde{h})$.
(2) The two bounded symmetric domains (Ω, ω_Ω) and $(\widetilde{\Omega}, \omega_{\widetilde{\Omega}})$ are holomorphically isometric.

Note $\Sigma(\Omega, h)$ is not compact in $\mathbb{C}^{n+1}$, and its (compact) closure in $\mathbb{C}^{n+1}$ is not a smooth hypersurface in general. On the other hand, we may consider the circle bundles over compact quotients of bounded symmetric domains. Let $X = X_1 \times \cdots \times X_l$ be a compact quotient of $\Omega = \Omega_1 \times \cdots \times \Omega_l$, where X_i, $1 \leq i \leq l$, is a compact manifold covered by Ω_i. Note X_i naturally inherits the Kähler–Einstein metric from Ω_i. Write L_i, $1 \leq i \leq l$, for the anti-canonical line bundle of X_i, which locally is the same as that of Ω_i. Let $q = (q_1, \cdots, q_l)$ be a tuple of positive integers and consider the line bundle

$$L_q := L_1^{q_1} \otimes \ldots \otimes L_l^{q_l}.$$

It is routine to verify that the circle bundle of L_q is locally CR equivalent to the hypersurface defined by

$$|\xi|^2 = \prod_{i=1}^{l} \left(N_i(z_i, \overline{z_i})\right)^{\gamma_i q_i}.$$

Here γ_i is the genus of Ω_i. By picking $q_i = \frac{k_i \prod_{j=1}^{l} \gamma_j}{\gamma_i}$, $1 \leq i \leq l$ for some positive integers k_i's, and taking an appropriate root of ξ locally, we further see the above hypersurface is locally CR equivalent to the one defined in (33). Using this observation and Theorem 4.2, we can prove the following result (Theorem 1.14 in [37]). See [37] for the detailed proof.

Theorem 4.3 *Let $n \geq 2$. There exists a countably infinite family $\mathcal{F}$ of compact real analytic CR hypersurfaces (realized as circle bundles over complex manifolds) of real dimension $2n+1$ such that the following hold:*

(1) Every CR hypersurface $M \in \mathcal{F}$ is obstruction flat and Bergman logarithmically flat. Moreover, every $M \in \mathcal{F}$ is locally homogeneous and has transverse symmetry.
(2) For every pair of (distinct) CR hypersurfaces $M_1, M_2 \in \mathcal{F}$, any open pieces $U \subseteq M_1$ and $V \subseteq M_2$ are not CR diffeomorphic.

4.2 CR Curvature Eigenvalues of Unit Circle Bundles over Products of Space Forms

Consider two Kähler manifolds (M, g) and (M', g') quantized by two negative line bundles (L, h) and (L', h'), respectively, as in Sect. 3.1. We observe that if M and M' are locally (holomorphically) conformally isometric, then their corresponding circle bundles $\Sigma(M, h)$ and $\Sigma(M', h')$ are locally CR equivalent. To see that, assume f is a local holomorphic map at $p_0 \in M$ from M to M' and preserves the metric: $f^*(g') = \mu g$ up to a conformal constant $\mu > 0$. Note in some local charts at $p_0 \in M$ and $q_0 := f(p_0) \in M'$, the circle bundles $\Sigma(M, h)$ and $\Sigma(M', h')$ can be written as follows, respectively:

$$|\xi|^2 h(z, \overline{z}) = 1; \; |\eta|^2 h'(w, \overline{w}) = 1.$$

Here z and w denote the local coordinates of M and M', while ξ, η denote the local fiber coordinates of L, L', respectively. By the conformal isometry assumption, we have

$$\partial\overline{\partial} \log h'(f, \overline{f}) = \mu \partial\overline{\partial} \log h(z, \overline{z}).$$

This yields that $h'(f, \overline{f}) = h^{\mu} |e^{\phi}|^2$ for some local holomorphic function ϕ at p_0. It follows that we have a local CR diffeomorphism from $\Sigma(M, h)$ and $\Sigma(M', h')$ over p_0 and q_0 given by

$$(z, \xi) \to (f(z), e^{-\phi} \xi^{\mu}). \tag{34}$$

It is natural to ask whether the converse of the above observation holds. By Theorem 4.1, the converse fails in general, even for space forms and their products. In the following, we shall show the converse still holds for products of three or more space forms under some mild conditions: If $\Sigma(M, h)$ and $\Sigma(M', h')$ are locally CR diffeomorphic, then (M, g) and (M', g') are locally conformally isometric.

For that, we shall consider unit circle bundles over a base manifold of the form

$$M = M_{c_1} \times \ldots \times M_{c_p},$$

where each M_{c_i} is a complex space form (always taken to be simply connected) of dimension m_i equipped with a complete metric g^i of constant holomorphic sectional curvature c_i and where M is equipped with the product metric $g = g^1 \times \ldots \times g^p$. Let (M', g') be another such product of space forms. We note that for the two products of space forms (M, g) and (M', g'), being locally conformally isometric is the same as being (globally) conformally isometric. The latter means there is a biholomorphism from M and M' that preserves the metric: $f^*(g') = \mu g$ up to a conformal constant $\mu > 0$. We also observe the following easy fact, whose proof will be omitted.

Lemma 4.4 *Let (M, g) and (M', g') be two products of space forms as above, where $M = M_{c_1} \times \ldots \times M_{c_p}$ and $M' = M_{c'_1} \times \ldots \times M_{c'_q}$. Write the two tuples $c = (c_1, \cdots, c_p)$ and $c' = (c'_1, \cdots, c'_q)$. Assume $p = q$ and, after a permutation of $M'_{c_i}s$ and a scaling of c by a factor $a > 0 : c \to ac$, we have $\dim M_{c_i} = \dim M_{c'_i}$ for all i and $c = c'$. Then (M, g) and (M', g') are conformally isometric.*

We also assume that there is a Hermitian holomorphic line bundle $(L, h) \to M$ inducing the metric g as in Sect. 3.1; we can, e.g., take $L = L^1 \otimes \ldots \otimes L^p$ with metric $h = h^1 \ldots h^p$, where each $(L^i, h^i) \to M_{c_i}$ is as in Sect. 3.1. (Note the existence of such a line bundle poses an extra condition on those $c_i > 0$.) The unit circle bundle $\Sigma(M, h)$ gives rise to a strictly pseudoconvex hypersurface in L which is obstruction flat by Theorem 3.1. The CR dimension of $\Sigma(M, h)$ is $m = n - 1$. Moreover, since M is a simply connected homogeneous Kähler manifold, $\Sigma(M, h)$ is a homogeneous CR hypersurface.

We shall now assume that $n = \dim L = m + 1 = m_1 + \ldots + m_p + 1 \geq 3$; in other words, the dimension m of M is at least 2. A fundamental CR invariant of $\Sigma = \Sigma(M, h)$ is then the Chern–Moser CR curvature tensor $S_{\alpha}{}^{\beta}{}_{\nu\bar{\mu}}$, which can be defined using either Moser's normal form [9], Cartan–Chern's parabolic geometry construction, or Webster's pseudohermitian geometry [36]; the reader is referred to these papers for details of the construction. We lower the index β using the Levi form to obtain a trace-free (2, 2)-tensor $S_{\alpha\bar{\beta}\nu\bar{\mu}}$ with Hermitian curvature symmetries and with the property that the CR manifold is spherical if and only if the CR curvature vanishes in a neighborhood. When defined relative to a pseudohermitian structure, i.e., with a given choice of contact form θ, the CR curvature tensor $S_{\alpha}{}^{\beta}{}_{\nu\bar{\mu}}$ remains invariant under changes of contact form $\tilde{\theta} = e^u\theta$, whereas the Levi form $g_{\alpha\bar{\beta}}$ changes by $\tilde{g}_{\alpha\bar{\beta}} = e^u g_{\alpha\bar{\beta}}$. Using the Levi form to raise and lower indices, we obtain an endomorphism

$$S^\sharp \colon T^{1,0}\Sigma \odot T^{1,0}\Sigma \to T^{1,0}\Sigma \odot T^{1,0}\Sigma,$$

where $\odot$ denotes the symmetric tensor product, given in a local frame $Z_1, \ldots, Z_m$ for $T^{1,0}\Sigma$ and with a choice of θ by

$$X \odot Y \mapsto S_{\gamma}{}^{\alpha}{}_{\nu}{}^{\beta} X^{\gamma} Y^{\nu},$$

where $X = X^{\alpha} Z_{\alpha}$, $Y = Y^{\beta} Z_{\beta} \in T^{1,0}\Sigma$. This endomorphism is a weighted CR invariant in the sense that it is a pseudohermitian invariant, which transforms under changes of contact form $\tilde{\theta} = e^u \theta$ by $\tilde{S}^{\sharp} = e^{-u} S^{\sharp}$, as is easily verified. In particular, if we denote by $\lambda_1, \ldots, \lambda_m$ the eigenvalues (repeated according to multiplicity) of $S^{\sharp}$ at a point $p \in \Sigma$ relative to some contact form θ, then the m-tuple $(\lambda_1, \ldots, \lambda_m)$ is a CR invariant modulo permutations and scaling $(\lambda_1, \ldots, \lambda_m) \mapsto a(\lambda_1, \ldots, \lambda_m)$ for $a > 0$. We shall make use of this observation to study the CR equivalence classes of $\Sigma = \Sigma(M, h)$ for $M = M_{c_1} \times \ldots \times M_{c_p}$ as above.

Now, let $M = M_{c_1} \times \ldots \times M_{c_p}$, take $\rho = |\xi|^2 h(z, \bar{z}) - 1$ as a local defining function for $\Sigma = \Sigma(M, h)$ in some local chart and trivialization, and choose the contact form $\theta = i\bar{\partial} \log |\xi|^2 h(z, \bar{z})$. Then (see [36]), the Levi form of Σ is S^1-invariant and coincides with the Kähler metric g; the pseudohermitian curvature of Σ is S^1-invariant and coincides with the curvature tensor $R_{\alpha}{}^{\beta}{}_{\nu\bar{\mu}}$ of M; the CR curvature tensor $S_{\alpha}{}^{\beta}{}_{\nu\bar{\mu}}$ is S^1-invariant as well, coincides with the Bochner tensor of M, and is obtained by the following formula:

$$S_{\alpha\bar{\beta}\nu\bar{\mu}} = R_{\alpha\bar{\beta}\nu\bar{\mu}} - \frac{1}{m+2}(R_{\alpha\bar{\beta}} g_{\nu\bar{\mu}} + R_{\alpha\bar{\mu}} g_{\nu\bar{\beta}} + R_{\nu\bar{\beta}} g_{\alpha\bar{\mu}} + R_{\nu\bar{\mu}} g_{\alpha\bar{\beta}}) + \frac{R}{(m+1)(m+2)}(g_{\alpha\bar{\beta}} g_{\nu\bar{\mu}} + g_{\nu\bar{\beta}} g_{\alpha\bar{\mu}}), \qquad (35)$$

where $R_{\alpha\bar{\beta}} = R_{\gamma}{}^{\gamma}{}_{\alpha\bar{\beta}}$ is the Ricci tensor and $R = R_{\gamma}{}^{\gamma}$ is the scalar curvature.

Next, recall that if (M, g) is a Kähler manifold of constant holomorphic sectional curvature c (and not the product manifold for a moment), then the curvature tensor is given by

$$R_{\alpha\bar{\beta}\nu\bar{\mu}} = c(g_{\alpha\bar{\beta}} g_{\nu\bar{\mu}} + g_{\nu\bar{\beta}} g_{\alpha\bar{\mu}}).$$

Thus, for our product manifold $M = M_{c_1} \times \ldots \times M_{c_p}$, the curvature tensor can be written

$$R_{\alpha\bar{\beta}\nu\bar{\mu}} = \sum_{i=1}^{p} c_i \left(\hat{g}^i_{\alpha\bar{\beta}} \hat{g}^i_{\nu\bar{\mu}} + \hat{g}^i_{\nu\bar{\beta}} \hat{g}^i_{\alpha\bar{\mu}} \right), \qquad (36)$$

where the tensors $\hat{g}^i_{\alpha\bar{\beta}}$ are defined as follows. For each $i = 1, \ldots, p$, let $Z^i_{\alpha^i}$, $\alpha^i = 1, \ldots, m_i$, be a local frame for M_{c_i} and Z_{α}, $\alpha = 1, \ldots, m$, the corresponding local frame for M such that $(\pi_i)_* Z_{\alpha} = Z^i_{\alpha-(m_1+\ldots+m_{i-1})}$ when $\alpha \in \{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i\}$ and $= 0$ otherwise; here, π_i denotes the projection $M \to M_{c_i}$.

We define

$$\hat{g}^i_{\alpha\bar{\beta}} = g^i((\pi_i)_* Z_\alpha, (\pi_i)_* Z_\beta).$$

One readily checks that

$$R_{\alpha\bar{\beta}} = \sum_{i=1}^{p} c_i(m_i+1)\hat{g}^i_{\alpha\bar{\beta}}, \qquad R = \sum_{i=1}^{p} c_i m_i(m_i+1). \tag{37}$$

By an abuse (but great simplification) of notation, we shall denote by Z_α a collection of $(1,0)$-vector fields on Σ such that the projection $\pi\colon \Sigma \to M$ identifies the various curvature tensors on Σ with those on M as explained above.

To investigate the eigenvalues of the CR curvature endomorphism $S^\sharp$, we note that the collection of local sections

$$\zeta^{ij}_{\alpha^i\beta^j} = Z_\alpha \odot Z_\beta, \qquad i \le j = 1,\dots,p, \quad \alpha^i = 1,\dots,m_i, \quad \beta^j = 1,\dots,m_j, \tag{38}$$

where $\alpha = m_1 + \ldots + m_i + \alpha^i$ and $\beta = m_1 + \ldots + m_j + \beta^j$, forms a frame for $T^{1,0}\Sigma \odot T^{1,0}\Sigma$. It is straightforward to check, using (35) and (37), that the $\zeta^{ij}_{\alpha^i\beta^j}$ are also eigenvectors for $S^\sharp$. Moreover, the eigenvalues are readily computed and seen to be constant as functions of $p \in \Sigma$. The details of these calculations are left to the diligent reader. We summarize these observations in the following:

Proposition 4.5 *Let $M = M_{c_1} \times \ldots \times M_{c_p}$, $(L,h) \to M$, and $\Sigma = \Sigma(M,h)$, be as described above. Then, $\zeta^{ij}_{\alpha^i\beta^j}$ given by (38) form a complete system of eigenvectors for the CR curvature operator $S^\sharp$. The eigenvalues λ_{ij} and their multiplicities d_{ij}, for which $\zeta^{ij}_{\alpha^i\beta^j}$, $\alpha^i = 1,\dots,m_i$, $\beta^j = 1,\dots,m_j$, are the eigenvectors, are constant as functions of $p \in \Sigma$ and given by*

$$\lambda_{ii} = \frac{2}{(m+1)(m+2)} \sum_{l=1}^{p} m_l(m_l+1)c_l + \frac{2}{m+2}(m-2m_i)c_i, \tag{39}$$

$$d_{ii} = \frac{m_i(m_i+1)}{2},$$

and for $i < j$,

$$\lambda_{ij} = \frac{2}{(m+1)(m+2)} \sum_{l=1}^{p} m_l(m_l+1)c_l - \frac{2}{m+2}\left((m_i+1)c_i + (m_j+1)c_j\right), \tag{40}$$

$$d_{ij} = m_i m_j.$$

Remark 4.6 We observe that it may happen that there is an eigenvalue λ such that $\lambda = \lambda_{ij}$ for multiple different pairs (i, j). In this case, the multiplicity of λ is then (of course) the sum of the corresponding individual multiplicities d_{ij}.

Remark 4.7 If there is only one factor (i.e., M itself has constant holomorphic sectional curvature), then there is only one eigenvalue and that is $\lambda = 0$; in other words, the CR curvature is identically zero, implying that Σ is spherical. If there are two factors, one checks readily that all eigenvalues are multiples of $c_1 + c_2$. In particular, if $c_1 + c_2 = 0$, then the CR curvature is identically zero and Σ is again spherical. This is consistent with Theorem 4.1 above.

We temporarily fix p and $m_1, \cdots, m_p$, and as above let $m = \sum_{i=1}^p m_i$. Denote by $\Lambda\colon \mathbb{R}^p \to \mathbb{R}^p$ the linear map whose ith component Λ_i, $1 \le i \le p$, is given by

$$\Lambda_i(c_1, \ldots, c_p) = \frac{2}{(m+1)(m+2)} \sum_{l=1}^{p} m_l(m_l+1)c_l + \frac{2}{m+2}(m-2m_i)c_i. \tag{41}$$

As noted in Remark 4.7, when $p = 1$, the rank of Λ is zero; when $p = 2$, the rank is either zero or one. In neither case is Λ invertible. However, for $p \ge 3$, we have the following:

Lemma 4.8 *Fix $p \ge 3$ and $m_1, \cdots, m_p \ge 1$, and let $m = \sum_{i=1}^p m_i$. Assume that $m - 2m_i \ge 0$ for $i = 1, \ldots, p$. Then, the linear map $\Lambda\colon \mathbb{R}^p \to \mathbb{R}^p$ given by (41) is invertible. Consequently, if $\Lambda(c) = a\Lambda(c')$ for some $c, c' \in \mathbb{R}^p$ and $a \in \mathbb{R}$, then $c = ac'$.*

Proof Let us denote by

$$\alpha_i = \frac{2m_i(m_i+1)}{(m+1)(m+2)}, \qquad \beta_i = \frac{2(m-2m_i)}{m+2}.$$

We observe that the algebraic structure of Λ noted above, we may assume (after a permutation of the components if necessary) that $\beta_1 \ge \beta_2 \ge \ldots \ge \beta_p \ge 0$. Moreover, only the last (smallest) number β_p can equal 0 (since $p \ge 3$). A simple row reduction scheme now shows that

$$\det \Lambda = \beta_1 \ldots \beta_{p-1} \left(\alpha_p + \beta_p \left(1 + \sum_{l=1}^{p-1} \frac{\alpha_l}{\beta_l} \right) \right). \tag{42}$$

The conclusion easily follows, since $\alpha_i > 0$ for all $i = 1, \ldots, p$ and $\beta_i > 0$ for $i \le p - 1$. □

Let (M, g) be a product of space forms and $\Sigma = \Sigma(M, h)$ is the unit circle bundle in $(L, h) \to (M, g)$ as above. We will show under some conditions, we can recover (M, g), up to conformal isometry, from the CR curvature of Σ. The strategy can be roughly summarized as follows: First under some mild conditions, the value of p can be recovered, and the subcollection of the p eigenvalues λ_{ii}, for $i = 1, \ldots, p$, can be identified among the full collection of eigenvalues, relative to some contact form θ; and the λ'_{ii}s are mutually distinct. From this we can then recover the dimensions of factor M'_{c_i}s from the multiplicities d_{ii} of λ_{ii}. Finally with the help of Lemma 4.8, the set of the holomorphic sectional curvatures $\{c_1, \ldots, c_p\}$ can be uniquely determined modulo scaling $a \to \{ac_1, \ldots, ac_p\}$ for $a > 0$.

The first technical problem in this context, which is alluded to in Remark 4.6, occurs when the CR curvature endomorphism $S^\sharp$ has an eigenvalue λ that corresponds to multiple λ'_{ij}s. In this case, one would need to properly split the eigenvalue λ and its multiplicity into individual λ_{ij}s and d_{ij}s, which a priori could potentially be done in multiple ways. To address this issue, we introduce the full eigenvalue map $\hat{\Lambda} \colon \mathbb{R}^p \to \mathbb{R}^{p(p+1)/2}$, given in components $\hat{\Lambda}_{ij}$ for $i \leq j$ by the right-hand sides of (39) and (40); we are using here pairs (i, j) with $i \leq j$ to index the components of a vector $\xi \in \mathbb{R}^{p(p+1)/2}$; we note that $\hat{\Lambda}_{ii} = \Lambda_i$. The following is easy to check, and we omit its proof.

Lemma 4.9 *Write $\mathcal{S}$ for the set of $c \in \mathbb{R}^p$, $p \geq 3$, for which either $\hat{\Lambda}_{ii}(c) = \hat{\Lambda}_{i'j'}(c)$ for some i and $i' \neq j'$, or $\hat{\Lambda}_{ii}(c) = \hat{\Lambda}_{i'i'}(c)$ for some $i \neq i'$. Then $\mathcal{S}$ is a finite union of $(p-1)$-dimensional subspaces of $\mathbb{R}^p$.*

The scheme described above to recover the holomorphic sectional curvatures c_i and the dimensions m_i of the M_i in the product $M = M_{c_1} \times \ldots \times M_{c_p}$, provided that we can identify the subcollection of eigenvalues λ_{ii}, now works for $c = (c_1, \ldots, c_p) \in \mathbb{R}^p \setminus \mathcal{S}$.

We proceed by identifying two situations where this scheme to recover the conformal isometry class of the base manifold can be executed. The first result is the following:

Proposition 4.10 *Fix $m \geq 3$. Let $\mathcal{F}_1$ be the family of unit circle bundles $\Sigma(M, h)$ of CR dimension m, which arises from a negative line bundle (L, h) quantizing a Kähler manifold (M, g) satisfying the following conditions:*

- (i) *(M, g) is a product of space forms: $M = M_{c_1} \times \ldots \times M_{c_p}$, for some $p = p(M) \geq 3$ and M'_{c_i}s, depending on M. Besides, their dimensions satisfy $m - 2m_i \geq 0$, for $i = 1, \ldots p$.*
- (ii) *$c_i \geq 0$ (or $c_i \leq 0$), for all $i = 1, \ldots p$.*
- (iii) *$c = (c_1, \ldots, c_p) \notin \mathcal{S}$, where $\mathcal{S}$ is given by Lemma 4.9.*

Then $\Sigma(M, h), \Sigma(M', h') \in \mathcal{F}_1$ are locally CR equivalent if and only if (M, g) and (M', g') are conformally isometric.

Proof By the discussion at the beginning of this subsection, it suffices to prove the "only if" part. For that, we first give the algorithm of recovering M, up to conformal isometry, from the eigenvalues of the CR curvature operator $S^\sharp$ of $\Sigma(M, h)$. Note that conditions (ii) and (iii) guarantee that the p largest (or smallest) eigenvalues of $S^\sharp$ correspond to the λ_{ii} for $i = 1, \ldots, p$, and the λ'_{ii}s are mutually distinct. Without loss of generality, we assume they are the largest ones. Furthermore, by rearranging the factors M'_{c_i}s in M, we can assume $\lambda_{11} > \cdots > \lambda_{pp}$.

Next pick the largest eigenvalue of $S^\sharp$. Call it λ_1 and write k_1 for its multiplicity. Then $\lambda_1 = \lambda_{11}$, and by Proposition 4.5, k_1 can be uniquely written as $k_1 = \frac{m_1(m_1+1)}{2}$. Note m_1 is the dimension of M_{c_1}. Then pick the second largest eigenvalue λ_2 of $S^\sharp$, which must equal to λ_{22}, and write k_2 for its multiplicity. Again by Proposition 4.5, k_2 can be uniquely written as $k_2 = \frac{m_2(m_2+1)}{2}$, and m_2 is the dimension of M_{c_2}. Keep doing this until the m_i's add up to m. Note the number of m_i's is then precisely p. In this way, we recover both p and m_i's. Once they are determined, we can define the linear map $\Lambda\colon \mathbb{R}^p \to \mathbb{R}^p$ as in (41). Finally, since we know the values of $\lambda_{11} > \cdots > \lambda_{pp}$ (relative to some contact form θ), by Lemma 4.8, we can recover the values of c_i's modulo positive factor scaling.

If $\Sigma(M,h), \Sigma(M',h') \in \mathcal{F}_1$ are locally CR equivalent, then their CR curvature operators have the same set of eigenvalues, up to a positive factor scaling. Moreover, if we sort both sets of eigenvalues in descending order, the corresponding multiplicities are equal. Write $M' = M_{c_1'} \times \ldots \times M_{c_q'}$. Repeating the above algorithm to (M', g'), we see M and M' satisfy the conditions in Lemma 4.4. Hence, they must be conformally isometric. □

Remark 4.11 The proof of Proposition 4.10 does not provide much information on the structure of the local CR diffeomorphism between $\Sigma(M,h), \Sigma(M',h')$ (if exists). We stress that, unlike Theorem 4.2, one cannot expect the rationality and rigidity result to hold for the local CR mappings in Proposition 4.10, as the local CR diffeomorphism may take the form as in (34). In general, this map is not rational and does not extend to a biholomorphism between the corresponding disk bundles.

Our second result along these lines allows mixed signs of the holomorphic sectional curvatures, but at the expense of fixing the dimensions of the factors:

Proposition 4.12 *Fix $p \geq 3$ and $m = pm_0$ for some $m_0 \geq 2$. Consider the family $\mathcal{F}_1$ of unit circle bundles $\Sigma(M,h)$, where $M = M_{c_1} \times \ldots \times M_{c_p}$, $(L,h) \to M$, and $\Sigma(M,h)$ are as in Proposition 4.5 and where $m_i = m_0$ for $i = 1, \ldots p$ and $c = (c_1, \ldots, c_p) \notin \mathcal{S}$. Then $\Sigma(M,h), \Sigma(M',h') \in \mathcal{F}_1$ are locally CR equivalent if and only if (M,g) and (M',g') are conformally isometric.*

Proof In this case, we identify the eigenvalues λ_{ii} by considering the multiplicities of the eigenvalues of $S^\sharp$. Let λ be an eigenvalue of $S^\sharp$. If λ corresponds to some λ_{ii}, then it corresponds to precisely one λ_{ij} (as $c \notin \mathcal{S}$), and thus its multiplicity equals to $\frac{m_0(m_0+1)}{2}$. If λ corresponds to some λ_{ij} with $i < j$, then it cannot correspond to any λ_{ii}, but it may correspond to multiple λ_{ij}'s with $i < j$: $\lambda_{i_1 j_1}, \cdots, \lambda_{i_k j_k}$ for some $k \geq 1$. Then its multiplicity equals to km_0^2. Since $m_0 \geq 2$, we have $km_0^2 > m_0(m_0+1)/2$. Thus the multiplicity $\frac{m_0(m_0+1)}{2}$ identifies the λ_{ii}. The proof is concluded as above and the details are left to the reader. □

Using the above results, we can construct some families of obstruction flat CR manifolds that are pairwise locally CR inequivalent. To construct examples of compact types, in the settings of Propositions 4.10 and 4.12, we may use as the building blocks (M_{c_i}, g^i) of $(\mathbb{P}^{m_i}, \mu_i g_1)$, and compact quotients of $(\mathbb{B}^{m_i}, \mu_i g_{-1})$

and $(\mathbb{C}^{m_i}, g_0)$. Observe that since we insist that the metrics are induced by holomorphic line bundles $(L_i, h_i) \to M_{c_i}$, we are restricted to, e.g., taking L_i to be integral powers of the (anti-)canonical line bundle, which means that we obtain holomorphic sectional curvatures of the form $c_i = \pm 1/k_i$ (or 0) for integers k_i. These families are necessarily only countably infinite. Since this compact case was already covered by Theorem 4.3, we will not elaborate it here. It remains an interesting question to ask whether there exists an uncountably infinite family of compact hypersurfaces with the properties in Theorem 4.3. In the remainder of the section, we shall concentrate on the construction of noncompact obstruction flat CR manifolds.

Theorem 4.13 *Let $n \geq 2$. There exists a family $\mathcal{F}$, parameterized by*

$$\mathcal{J} = \{(\lambda_2, \cdots, \lambda_n) : 1 > \lambda_2 > \cdots > \lambda_n > 0\},$$

of (noncompact) real analytic CR hypersurfaces in $\mathbb{C}^{n+1}$ such that the following hold:

(1) *Every CR hypersurface $M \in \mathcal{F}$ is obstruction flat and Bergman logarithmically flat. Moreover, every $M \in \mathcal{F}$ is homogeneous and has transverse symmetry.*
(2) *For every pair of (distinct) CR hypersurfaces $M_1, M_2 \in \mathcal{F}$, any open pieces $U \subseteq M_1$ and $V \subseteq M_2$ are not CR diffeomorphic.*

Remark 4.14 We recall that in $\mathbb{C}^2$, every Bergman logarithmically flat CR hypersurface must be spherical; and there exist (noncompact) real analytic nonspherical CR hypersurfaces that are all obstruction flat and transversally symmetric (see [23, 24, 16]).

Proof of Theorem 4.13 We will utilize a simple and useful case of Proposition 4.10 where all $m_i = 1$ and $c_i < 0$ (so that $m = p$). In this case, each M_i is a unit disk equipped with the (normalized) Poincaré metric. We first fix various notations and make some observations. Fix $n \geq 2$ and λ be a $(n-1)$ tuple of real numbers $\lambda = (\lambda_2, \cdots, \lambda_n) \in \mathcal{J}$, where $\mathcal{J} = \{(\lambda_2, \cdots, \lambda_n) : 1 > \lambda_2 > \cdots > \lambda_n > 0\}$. □

Consider the homogeneous CR hypersurface

$$\Sigma_\lambda = \Big\{(z, \xi) = (z_1, \cdots, z_n, \xi) \in \Delta^n \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2) \prod_{j=2}^{n} (1 - |z_j|^2)^{\lambda_j} \Big\} \subset \mathbb{C}^{n+1}.$$

We first note:

Lemma 4.15 *Every $\Sigma_\lambda \subset \mathbb{C}^{n+1}, \lambda \in \mathcal{J}$, is a homogeneous, obstruction flat and Bergman logarithmically flat, strongly pseudoconvex real hypersurface with transverse symmetry.*

Proof Write $h(z,\bar{z}) = (1-|z_1|^2)^{-1}\prod_{j=2}^{n}(1-|z_j|^2)^{-\lambda_j}$. Note with M being the n-dimensional polydisk Δ^n, we have Σ_λ is indeed the circle bundle $\Sigma(M,h)$ of the Hermitian line bundle $(M\times\mathbb{C},h)$. Consequently, Σ_λ has transverse symmetry. By earlier discussions of this subsection, Σ_λ is homogeneous and obstruction flat. Furthermore, note the polydisk is a special case of bounded symmetric domains and the generic norm of a disk is $1-|z_j|^2$. Then Σ_λ is a special case of $\Sigma(\Omega,h)$, which is defined in (33). By Proposition 3.3, Σ_λ is Bergman logarithmically flat. □

We now prove Theorem 4.13. For that, we first consider the case $n\geq 3$.

Lemma 4.16 *Let $n\geq 3$. Then $\mathcal{F} := \{\Sigma_\lambda\subset\mathbb{C}^{n+1}|\lambda\in\mathcal{J}\}$ gives a family of CR hypersurfaces in $\mathbb{C}^{n+1}$ with mutually distinct local CR structure. More precisely, Σ_λ and $\Sigma_{\lambda'}$ in $\mathcal{F}$ are locally CR equivalent if and only if $\lambda=\lambda'$.*

Proof It suffices to prove for the "only if" part. For that, we shall apply Proposition 4.10 for the special case where all $m_i=1$ and $c_i<0$ (so that $m=p$), and $p=n\geq 3$. For $c<0$, denote by Δ_c the unit disk equipped with the normalized Poincaré metric of constant Gaussian curvature c. Write M for the product of the normalized Poincaré disks $M=\Delta_{-2}\times\Delta_{-\frac{2}{\lambda_2}}\times\cdots\Delta_{-\frac{2}{\lambda_n}}, \lambda\in\mathcal{J}$. As already observed, the metric on M is induced by the Kähler form $\partial\bar{\partial}\log h$, where as above, $h(z,\bar{z}) = (1-|z_1|^2)^{-1}\prod_{j=2}^{n}(1-|z_j|^2)^{-\lambda_j}$. Moreover, Σ_λ is the circle bundle $\Sigma(M,h)$. We also note that conditions (i) and (ii) of Proposition 4.10 are satisfied in this setting. Next, let the set $\mathcal{S}$ be as in condition (iii) there, which is a finite union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$, and write

$$\mathcal{T} := \left\{(\lambda_2,\cdots,\lambda_n)\in\mathcal{J}\;\middle|\; c := \left(-2,-\frac{2}{\lambda_2},\cdots,-\frac{2}{\lambda_n}\right)\in\mathcal{S}\right\}.$$

Assume Σ_λ and $\Sigma_{\lambda'}$, where $\lambda,\lambda'\in\mathcal{J}\setminus\mathcal{T}$, are locally CR equivalent. Write the two tuples

$$c := (c_1,\cdots,c_n) = \left(-2,-\frac{2}{\lambda_2},\cdots,-\frac{2}{\lambda_n}\right),\quad \text{and}$$
$$c' := (c_1',\cdots,c_n') = \left(-2,-\frac{2}{\lambda_2'},\cdots,-\frac{2}{\lambda_n'}\right).$$

Then it follows from (the proof of) Proposition 4.10 that, after a permutation of the factors if necessary, c and c' are proportional by a constant $a>0$. But c and c' are already in the decreasing order, and the largest component for both tuples is -2. We thus must have $a=1$, and $\lambda_i=\lambda_i'$ for all i, i.e., $\lambda=\lambda'$.

Finally to finish the proof of the lemma, it suffices to prove that $\mathcal{T}=\emptyset$. We recall $\mathcal{S}$ is defined in Lemma 4.9 in terms of $\hat{\Lambda}_{ij}$ for $i\leq j$, where the latter is given by the right-hand sides of (39) and (40). By these expressions, as all $m_i=1$ and $c_i<0$, we have $\hat{\Lambda}_{ii}(c)<\hat{\Lambda}_{i'j'}(c)$ for any i and $i'\neq j'$. Moreover, for $\lambda\in\mathcal{J}$, the corresponding c satisfies $c_i\neq c_j$ for $i\neq j$. This yields that $\hat{\Lambda}_{ii}(c)\neq\hat{\Lambda}_{jj}(c)$ for $i\neq j$. Consequently, $\mathcal{T}=\emptyset$. □

It is clear that Theorem 4.13 follows from Lemmas 4.15 and 4.16 when $n \geq 3$. It remains to prove it for the case $n = 2$. In this case, $\Sigma_\lambda \subset \mathbb{C}^3$ is defined by

$$|\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)^\lambda, \ (z_1, z_2) \in \Delta^2, \lambda \in (0, 1).$$

By taking a local logarithm of the above equation and applying a simple change of coordinates, we see Σ_λ is locally CR diffeomorphic to the hypersurface

$$H_\lambda := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : \mathrm{Im}\xi = \log(1 - |z_1|^2) + \lambda \log(1 - |z_2|^2)\}, \lambda \in (0, 1).$$

The latter hypersurface was studied by Loboda in [31, Theorem 1], where it was proved that for $\lambda, \lambda' \in (0, 1)$, H_λ and $H_{\lambda'}$ are locally CR equivalent if and only if $\lambda = \lambda'$. Consequently, we have the following lemma which corresponds to $n = 2$.

Lemma 4.17 *Write $\mathcal{G} := \{\Sigma_\lambda \subset \mathbb{C}^3 : \lambda \in (0, 1)\}$. Then $\mathcal{G}$ is a family of CR hypersurfaces with mutually distinct local CR structure. More precisely, Σ_λ and $\Sigma_{\lambda'}$ in $\mathcal{G}$ are locally CR equivalent if and only if $\lambda = \lambda'$.*

We finally combine Lemmas 4.15 and 4.17 to establish Theorem 4.13 for $n = 2$. □

References

1. Heungju Ahn and Jong-Do Park. The explicit forms and zeros of the Bergman kernel function for Hartogs type domains. *Journal of Functional Analysis*, 262(8):3518–3547, 2012.
2. Toby N. Bailey, Michael G. Eastwood, and C. Robin Graham. Invariant theory for conformal and CR geometry. *Ann. of Math. (2)*, 139(3):491–552, 1994.
3. L. Boutet de Monvel. Intégration des équations de Cauchy-Riemann induites formelles. In *Séminaire Goulaouic-Lions-Schwartz 1974–1975; Équations aux derivées partielles linéaires et non linéaires*, pages Exp. No. 9, 14. 1975.
4. L. Boutet de Monvel. Le noyau de Bergman en dimension 2. In *Séminaire sur les Équations aux Dérivées Partielles 1987–1988*, pages Exp. No. XXII, 13. École Polytech., Palaiseau, 1988.
5. L. Boutet de Monvel and J. Sjöstrand. Sur la singularité des noyaux de Bergman et de Szegő. In *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, pages 123–164. Astérisque, No. 34–35. Soc. Math. France, Paris, 1976.
6. Robert L. Bryant. Bochner-Kähler metrics. *J. Amer. Math. Soc.*, 14(3):623–715, 2001.
7. Daniel M. Burns and Charles L. Epstein. Embeddability for three-dimensional CR-manifolds. *J. Amer. Math. Soc.*, 3(4):809–841, 1990.
8. Shiu Yuen Cheng and Shing Tung Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.
9. Shiing-Shen Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta Math.*, 133:219–271, 1974.
10. H. J. Christoffers. The Bergman kernel and Monge–Ampere approximations. *ProQuest LLC, Ann Arbor, Ph.D. Thesis, The University of Chicago, Chicago*, 1981.
11. Sean N. Curry and Peter Ebenfelt. Bounded strictly pseudoconvex domains in $\mathbb{C}^2$ with obstruction flat boundary II. *Adv. Math.*, 352:611–631, 2019.
12. Sean N. Curry and Peter Ebenfelt. Deformations and embeddings of three-dimensional strictly pseudoconvex CR manifolds. *Math. Ann.*, to appear, https://arxiv.org/abs/2007.00547, 2020.

13. Sean N. Curry and Peter Ebenfelt. Bounded strictly pseudoconvex domains in $\mathbb{C}^2$ with obstruction flat boundary. *Amer. J. Math.*, 143(1):265–306, 2021.
14. Sean N. Curry and Peter Ebenfelt. Obstruction flat rigidity of the CR 3-sphere. *J. Reine Angew. Math.*, 781:105–126, 2021.
15. Peter Ebenfelt. The log term in the Bergman and Szego kernels in strictly pseudoconvex domains in $\mathbb{C}^2$. *Doc. Math.*, 23:1659–1676, 2018.
16. Peter Ebenfelt, Ming Xiao, and Hang Xu. On the analytic and geometric aspects of obstruction flatness. *preprint, arXiv:2212.04034*, 2022.
17. Peter Ebenfelt, Ming Xiao, and Hang Xu. On the classification of normal stein spaces and finite ball quotients with Bergman-Einstein metrics. *Int. Math. Res. Not. IMRN*, (19):15240–15270, 2022.
18. Peter Ebenfelt, Ming Xiao, and Hang Xu. Kähler-Einstein metrics and obstruction flatness of circle bundles. *Journal de Mathématiques Pures et Appliquées*, 177:368–414, 2023.
19. Miroslav Engliš. Pseudolocal estimates for $\overline{\partial}$ on general pseudoconvex domains. *Indiana Univ. Math. J.*, 50(4):1593–1607, 2001.
20. Miroslav Engliš and Genkai Zhang. Ramadanov conjecture and line bundles over compact Hermitian symmetric spaces. *Math. Z.*, 264(4):901–912, 2010.
21. Charles L. Fefferman. The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.*, 26:1–65, 1974.
22. Charles L. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.
23. C. Robin Graham. Higher asymptotics of the complex Monge-Ampère equation. *Compositio Math.*, 64(2):133–155, 1987.
24. C. Robin Graham. Scalar boundary invariants and the Bergman kernel. In *Complex analysis, II (College Park, Md., 1985–86)*, volume 1276 of *Lecture Notes in Math.*, pages 108–135. Springer, Berlin, 1987.
25. Kengo Hirachi. Personal communication. *Announced in talk at 'Symmetry and Geometric Structures' at IMPAN, Warsaw, November 12–18, 2017*.
26. Kengo Hirachi. Construction of boundary invariants and the logarithmic singularity of the Bergman kernel. *Ann. of Math. (2)*, 151(1):151–191, 2000.
27. Xiaojun Huang and Xiaoshan Li. Bergman-Einstein metric on a Stein space with a strongly pseudoconvex boundary. *arXiv:2008.03645v1*, 2020.
28. Xiaojun Huang and Ming Xiao. Bergman-Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2021(770):183–203, 2021.
29. John M. Lee and Richard Melrose. Boundary behaviour of the complex Monge-Ampère equation. *Acta Math.*, 148:159–192, 1982.
30. László Lempert. Embeddings of three-dimensional Cauchy-Riemann manifolds. *Math. Ann.*, 300(1):1–15, 1994.
31. A V Loboda. Homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$ with two-dimensional isotropy groups. *Sbornik: Mathematics*, 192(12):1741, Dec 2001.
32. Andrea Loi, Roberto Mossa, and Fabio Zuddas. The log-term of the Bergman kernel of the disc bundle over a homogeneous Hodge manifold. *Ann. Global Anal. Geom.*, 51(1):35–51, 2017.
33. Ottmar Loos. Bounded symmetric domains and jordan pairs. 1977. Math. Lectures, University of California, Irvine.
34. A. H. Martin. Boundary asymptotics for convex and strongly pseudoconvex domains. *Ph.D. Thesis, The University of California Riverside*, 2021.
35. I. P. Ramadanov. A characterization of the balls in $\mathbf{C}^n$ by means of the Bergman kernel. *C. R. Acad. Bulgare Sci.*, 34(7):927–929, 1981.
36. S. M. Webster. Pseudo-Hermitian structures on a real hypersurface. *J. Differential Geom.*, 13(1):25–41, 1978.
37. Ming Xiao. A new Poincaré type rigidity phenomenon with applications. *preprint, arXiv:2309.13768*, 2023.