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Performance-Guaranteed Solutions for Multi-Agent Optimal Coverage Problems using Submodularity, Curvature, and Greedy Algorithms

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***Abstract*— We consider a class of multi-agent optimal coverage problems in which the goal is to determine the optimal placement of a group of agents in a given mission space so that they maximize a coverage objective that represents a blend of individual and collaborative event detection capabilities. This class of problems is extremely challenging due to the non-convex nature of the mission space and of the coverage objective. With this motivation, *greedy algorithms* are often used as means of getting feasible coverage solutions efficiently. Even though such greedy solutions are suboptimal, the *submodularity* (diminishing returns) property of the coverage objective can be exploited to provide performance bound guarantees. Moreover, we show that improved performance bound guarantees (beyond the standard (1-1/e) performance bound) can be established using various *curvature measures* of the coverage problem. In particular, we provide a brief review of all existing popular applicable curvature measures, including a recent curvature measure that we proposed, and discuss their effectiveness and computational complexity, in the context of optimal coverage problems. We also propose novel computationally efficient techniques to estimate some curvature measures. Finally, we provide several numerical results to support our findings and propose several potential future research directions.**

I. INTRODUCTION

Our research focuses on multi-agent optimal coverage problems, which often arise in critical applications such as (but not limited to) surveillance, security, agriculture, and search and rescue [2], [7]. In these problems, the overall goal is to find an effective placement (decision variable) for the agent team so that they can optimally "cover" (i.e., individually and/or collaboratively detect events of interest randomly occurring in) the mission space.

Due to their wide applicability, several variants of multi-agent optimal coverage problems have been extensively studied in the literature [5], [11], [14]. Typically, these are formulated as continuous optimization problems inspired by real-world conditions. However, their corresponding solutions are computationally expensive unless significant simplifying assumptions are made regarding the particular coverage problem setup. This is mainly due to the overall challenging nature of coverage problems resulting from the often non-linear, non-convex, and non-smooth coverage objectives and non-convex mission spaces involved.

In this paper, we adopt an alternative approach that has been adopted in the literature [8], [9], and formulate the multi-agent optimal coverage problem as a combinatorial optimization problem by discretizing the associated mission/decision space. The coverage objective function, in this setting, is proven to be a *submodular* set function. In other words, the coverage objective function shows *diminishing returns* when the deployed set of agents is expanded. While this combinatorial formulation simplifies the coverage problem to a certain level, it now takes the form of a submodular maximization problem that is known to be NP-hard [6].

Greedy algorithms are commonly used to solve submodular maximization problems due to their simplicity and computational efficiency. Most importantly, the resulting greedy solutions, even though suboptimal, entertain performance bounds that characterize their proximity to the global optimal solution. The seminal work in [6] has established a $1-(1-\frac{1}{N})^N$ performance bound, which becomes $(1-\frac{1}{e}) \simeq 0.6321$ as the solution set size (i.e., in the coverage problem, the number of agents) $N \to \infty$. Hence the greedy solution is not worse than 63.21% of the global optimal solution. Recent literature has focused on developing improved performance bounds beyond this fundamental performance bound.

To this end, various *curvature measures* have been proposed to further characterize any given submodular maximization problem [3], [4], [10], [13]. These curvature measures provide corresponding performance bounds, which may or may not significantly improve upon the fundamental performance bound - depending on the nature of the considered problem/application. However, often these curvature measures are computationally expensive to evaluate. Moreover, given the variety of curvature measures available and the variations in their effectiveness with respect to problem parameters, selecting a curvature measure that is likely to provide significantly improved performance bounds for a particular application is challenging.

Our previous work in [9] considered a widely studied multi-agent optimal coverage problem [11], [14] and showed that the *total curvature* [3] and *elemental curvature* [10] can provide improved performance bounds. The subsequent work in [8] considered a slightly different coverage problem with a team selection element and showcased the effectiveness of the *greedy curvature* [3] and *partial curvature* [4] in providing improved performance bounds. Our most recent work in [13] considered the general submodular maximization problem and proposed a new curvature measure called the *extended greedy curvature*. Then, using the coverage problem in [9], its effectiveness compared to said other curvature measures was illustrated. In this paper, we consider a more general coverage problem than before and investigate the effectiveness and complexity of all these curvature measures

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through theoretical analysis and numerical experiments.

In particular, our contributions are as follows: **(1)** We consider a more general coverage problem (compared to those in [9], [13]); **(2)** Submodularity and several other key properties of the considered coverage problem are established (in Theorems 1 and 2, respectively); **(3)** We provide a comparative review of five curvature measures that are applicable to the considered coverage problem (to the best of our knowledge, this review is exhaustive); **(4)** We exploit the established properties to propose novel techniques for numerical evaluation of some complex curvature measures for the considered coverage problems (in Props. 1 and 2); **(5)** We detail the effectiveness and complexity of different curvature metrics with respect to various coverage problem parameters; and **(6)** We implement the proposed coverage problem setup in a simulation environment and evaluate different curvature measures and their performance bounds under different problem conditions.

Organization: We introduce the considered coverage problem in Section II. Some notations, preliminary concepts, and the proposed greedy solution are presented in Section III. Different curvature measures found in the literature, along with discussions on their effectiveness and complexity in the context of optimal coverage problems, are provided in Section IV. A summary and some future research directions are given in Section V. Several numerical results are reported in Section VI before concluding the paper in Section VII. All proofs are omitted here but can be found in [12].

II. Multi-Agent Optimal Coverage Problem

The goal of the considered coverage problem is to determine an optimal placement for a given team of agents (e.g., sensors, cameras, guards, etc.) in a given mission space that maximizes the probability of detecting events that occur randomly over the mission space.

We model the *mission space* Ω as a convex polytope in $\mathbb{R}^n$ that may also contain h polytopic (and possibly non-convex) *obstacles* $\{\Psi_i : \Psi_i \subset \Omega, i \in \mathbb{N}_h\}$ (note that: $\mathbb{N}_n \triangleq \{1,2,\ldots,n\}$). The obstacles (1) limit the agent placement to the feasible space $\Phi \triangleq \Omega \backslash \cup_{i\in\mathbb{N}_h} \Psi_i$, (2) constrain the sensing capabilities of agents via obstructing their line of sight, and (3) occupy areas where no events of interest occur.

To model the likelihood of random *events* occurring over the mission space, an *event density function* $R : \Omega \to \mathbb{R}_{\geq 0}$ is used, where $R(x) = 0, \forall x \notin \Phi$ and $\int_\Omega R(x)dx < \infty$. If no prior information on $R(x)$ is known, one can use $R(x) = 1, \forall x \in \Phi$.

To detect these random events, N *agents* are to be placed inside the feasible space Φ, where their placement (i.e., the decision variable) is denoted by a matrix $s \triangleq [s_1, s_2, \ldots, s_N] \in \mathbb{R}^{m\times N}$ or a set $S \triangleq \{s_i : i \in \mathbb{N}_N, s_i \in \mathbb{R}^m\}$, where each $s_i, i \in \mathbb{N}_N$ represents an agent placement such that $s_i \in \Phi$.

The ability of an agent to *detect* events is limited by its sensing capabilities and visibility obstruction from obstacles. In particular, for an agent at $s_i \in \Phi$, its *visibility region* is defined as $V(s_i) \triangleq \{x : (qx + (1-q)s_i) \in \Phi, \forall q \in [0,1]\}$. Moreover, agents are assumed to be homogeneous in their *sensing capabilities*. In particular, each agent has a finite *sensing radius* $\delta \in \mathbb{R}_{\geq 0}$ and *sensing decay rate* $\lambda \in \mathbb{R}_{\geq 0}$ that defines the probability of an agent at $s_i \in \Phi$ detecting an event at $x \in \Phi$ via a *sensing function* of the form

$$p(x, s_i) \triangleq e^{-\lambda\|x-s_i\|} \cdot \mathbf{1}_{\{x\in V(s_i)\}}. \tag{1}$$

Given agent team placement s (or, equivalently, S), their ability to detect an event at $x \in \Phi$ is described by a *detection function* $P(x,s)$. For this, the *joint detection function*:

$$P_J(x,s) \triangleq 1 - \prod_{i\in\mathbb{N}_N} (1 - p(x,s_i)), \tag{2}$$

is a popular choice that represents the probability of detection by at least one agent (assuming independently detecting agents). Moreover, the *max detection function* given by

$$P_M(x,s) \triangleq \max_{i\in\mathbb{N}_N} p(x,s_i), \tag{3}$$

is also a widely used choice that represents the maximum probability of detection by any agent. The following remark summarizes the pros and cons of using (2) or (3) as $P(x,s)$.

Remark 1: The joint detection function (2) offers a complete but computationally intensive view of coverage by combining all agent efforts. Thus it is suited for scenarios where collaborative detection is needed. Conversely, the max detection function (3) focuses on the top-performing agent at each point, providing a simpler yet non-smooth and potentially under-utilizing approach. Thus it is suited for scenarios where individual yet maximum detection is needed.

Motivated by the contrasting nature of the joint and max detection functions, we propose the detection function

$$P(x,s) \triangleq \theta P_J(x,s) + (1-\theta)P_M(x,s), \tag{4}$$

where $\theta \in [0,1]$ is a predefined weight (e.g., see Fig. 1).

Using the defined event density and detection functions, the considered optimal coverage problem can be stated as

$$s^* = \underset{s:s_i\in\Phi, i\in\mathbb{N}_N}{\arg\max}\ H(s) \triangleq \int_\Omega R(x)P(x,s)dx, \tag{5}$$

where $H(s)$ (or equivalently, $H(S)$) is the *coverage function*.

Continuous Optimization Approach: The optimal coverage problem (5) involves a non-convex feasible space and a non-convex, non-linear, and non-smooth objective function. Therefore, it is extremely difficult to solve without using: (1) standard global optimization solvers that are computationally expensive, or (2) systematic gradient-based solvers that require extensive domain knowledge.

Combinatorial Optimization Approach: Motivated by the said challenges, here we take a combinatorial optimization approach to solve (5). This requires reformulating (5) as a set function maximization problem (in set variable S).

First, we discretize the feasible space Φ formulating a *ground set* $X = \{x_l : x_l \in \Phi, l \in \mathbb{N}_M\}$. Second, we replace the matrix variable s with the *set variable* $S \triangleq \{s_i : i \in \mathbb{N}\}$ in detection and coverage functions defined in (2)-(5), to obtain their respective set detection and set coverage functions. To limit the cardinality of S (denoted by $|S|$) to N, we introduce the constraint $S \in \mathscr{I}^N \triangleq \{Y : Y \subseteq X, |Y| \leq N\}$. Finally, we restate the optimal coverage problem (5) as a set function maximization problem:

$$S^* = \underset{S \in \mathscr{I}^N}{\arg\max}\ H(S) \triangleq \int_\Omega R(x)P(x,S)dx. \quad (6)$$

Clearly, the size of the search space of (6) is combinatorial as $|\mathscr{I}^N| = \sum_{r=0}^{N}\binom{M}{r}$. Therefore, obtaining an optimal solution S^* for it is impossible without significant simplifying assumptions. Hence our goal here is to obtain a candidate solution for (6) (say S^G) in an efficient manner with some guarantees on its coverage performance $H(S^G)$ with respect to the optimal coverage performance $H(S^*)$.

To efficiently obtain such a candidate solution, we use a vanilla *greedy algorithm* as given in Alg. 1. Note that it uses the notion of *marginal coverage function* defined as

$$\Delta H(y|S^{i-1}) \triangleq H(S^{i-1} \cup \{y\}) - H(S^{i-1}), \quad (7)$$

to iteratively determine optimal individual agent placements until N such agent placements have been chosen.

Motivated by the linear relationship between the set coverage function $H(S)$ (6) and set detection function $P(x,S)$ (4), we define the notion of *marginal detection function* as

$$\Delta P(x,y|S^{i-1}) \triangleq P(x,S^{i-1} \cup \{y\}) - P(x,S^{i-1}). \quad (8)$$

Through (6), it is easy to see that a similar linear relationship also exists between the marginal functions $\Delta H(y|S)$ (7) and $\Delta P(x,y|S)$ (8). In the sequel, we exploit these linear relationships to infer certain set function properties of $H(S)$ using those of $P(x,S)$ and $\Delta P(x,y|S)$.

Finally, we point out that, the notations in (7) and (8) will be used more liberally by replacing y and S^{i-1} respectively with sets A and B, where $A,B \subseteq X$ (e.g., see Th. 2). The notation $S^i \triangleq \{s^1,s^2,\ldots,s^i\}$ used to represent the greedy solution after i greedy iterations in Alg. 1 will also be used more liberally for any $i \in \mathbb{N}_M^0 \triangleq \mathbb{N}_M \cup \{0\}$ (e.g., see (21)).

III. The Greedy Solution with Performance Bound Guarantees

In this section, we show that the greedy solution S^G given by Alg. 1 for the optimal coverage problem (6) not only efficient but also entertains performance guarantees with respect to the global optimal performance $H(S^*)$. For this, we first need to introduce some standard set function properties.

Definition 1: [13] Let $F : 2^Y \to \mathbb{R}$ be an arbitrary set function defined over a finite ground set Y, and $\Delta F(y|A) \triangleq F(A \cup \{y\}) - F(A)$ be the corresponding marginal gain function. This set function F is: **(1)** *normalized* if $F(\emptyset) = 0$; **(2)** *monotone* if $\Delta F(y|A) \geq 0$ for all y,A where $A \subset Y$ and $y \in Y \backslash A$, or equivalently, if $F(B) \leq F(A)$ for all B,A where $B \subseteq A \subseteq Y$; **(3)** *submodular* if $\Delta f(y|A) \leq \Delta f(y|B)$ for all y,A,B where $B \subseteq A \subset Y$ and $y \in Y \backslash A$, or equivalently, if $F(A \cup B) + F(A \cap B) \leq F(A) + F(B)$ for all $A,B \subseteq Y$; **(4)** a *polymatroid* set function if all the above properties hold [1].

Algorithm 1 The greedy algorithm to solve (6)

1: $i = 0$; $S^i = \emptyset$; ▷ Greedy iteration index and solution
2: **for** $i = 1,2,3,\ldots,N$ **do**
3: $\quad s^i = \arg\max_{\{y: S^{i-1} \cup \{y\} \in \mathscr{I}^N\}} \Delta H(y|S^{i-1})$; ▷ New item
4: $\quad S^i = S^{i-1} \cup \{s^i\}$; ▷ Append the new item
5: **end for**
6: $S^G := S^N$; **Return** S^G;

The following lemma and theorem establish that the coverage function $H(S)$ in (6) is a polymatroid set function.

Lemma 1: With respect to a common ground set, any positive linear combination of arbitrary polymatroid set functions is also a polymatroid set function.

Theorem 1: The set coverage function $H(S)$ in (6) is a polymatroid set function.

This polymatroid nature of set coverage function $H(S)$ (6) enables establishing *performance bounds* (denoted by β) for the greedy solution S^G (given by Alg. 1). Formally, a performance bound is defined as a theoretically established lower bound for the ratio $\frac{H(S^G)}{H(S^*)}$, i.e., $\beta \leq \frac{H(S^G)}{H(S^*)}$. Having a performance bound $\beta \simeq 1$ implies that the performance of the greedy solution S^G is close to that of the global optimal solution S^*. Thus, β is an indicator of the effectiveness of the greedy approach to solving the coverage problem (6).

The seminal work [6] has established a performance bound (henceforth called the *fundamental performance bound*, and denoted by β_f) for polymatroid set function maximization problems. This, in light of Th. 1, is applicable to the optimal coverage problem (6) as:

$$\beta_f \triangleq 1 - \left(1 - \frac{1}{N}\right)^N \leq \frac{H(S^G)}{H(S^*)}. \quad (9)$$

While β_f decreases with the number of agents N, it is lower-bounded by $1 - \frac{1}{e} \simeq 0.6321$, because $\lim_{N\to\infty} \beta_f = (1 - \frac{1}{e})$. This implies that the coverage performance of the greedy solution will always be not worse than 63.21% of the maximum achievable coverage performance.

As we will see in the next section, further improved performance bounds beyond β_f can be achieved by exploiting certain characteristics called *curvature measures* of the interested set function maximization problem.

Before moving on, we provide a theorem that establishes the polymatroid nature of the marginal coverage function $\Delta H(B|A)$ with respect to both of its set arguments A and B.

Theorem 2: For a fixed set $A \subset X$, the marginal coverage function $G_A(B) \triangleq \Delta H(B|A)$ is a polymatroid set function over $B \subseteq X \backslash A$. Also, for a fixed set $B \subset X$, the affine negated marginal coverage function $\bar{G}_B(A) \triangleq -\Delta H(B|A) + H(B)$ is a polymatroid set function over $A \subseteq X \backslash B$.

The above result further emphasizes the deep polymatroid features of optimal coverage problems. Moreover, as we will see in the sequel, it enables achieving computationally efficient estimates for some (otherwise computationally intractable) curvature measures discussed in the next section.

IV. Improved Performance Bound Guarantees using Curvature Measures

In this section, we discuss several improved performance bounds (i.e., closer to 1 compared to β_f in (9)) applicable for the greedy solution S^G given by Alg. 1 for the optimal coverage problem in (6). This is important as such an improved performance bound will accurately characterize the proximity of S^G to S^* and thus enable making informed decisions regarding spending extra resources (e.g., computational power, agents, and sensing capabilities) to seek further improved coverage solutions beyond S^G.

In particular, curvature measures are used to obtain such improved performance bounds, and they are dependent purely on the underlying objective function, the ground set, and the feasible space, which, in the considered optimal coverage problem, are $H(S)$, X, and $\mathscr{I}^N$, respectively. Here, we will review five established curvature measures and their respective performance bounds, outlining their characteristics, strengths, and weaknesses in the context of optimal coverage problems (6).

a) Total Curvature [3]: By definition, the *total curvature* of (6) is given by

$$\alpha_t \triangleq \max_{y\in X}\left[1-\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)}\right]. \quad (10)$$

The corresponding performance bound β_t is given by

$$\beta_t \triangleq \frac{1}{\alpha_t}\left[1-\left(1-\frac{\alpha_t}{N}\right)^N\right] \leq \frac{H(S^G)}{H(S^*)}. \quad (11)$$

From (11) and (9), it is easy to see that: (1) when $\alpha_t \to 1$, $\beta_t \to \beta_f$ (i.e., no improvement); (2) when $\alpha_t \to 0$, $\beta_t \to 1$ (i.e., a significant improvement); and (3) β_t is monotonically decreasing in α_t. Using these three facts and (10), it is easy to see that the improvement in the performance bound is correlated with the magnitude of: $\gamma_t \triangleq \min_{y\in X}\left[\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)}\right]$. Note that, the submodularity of H implies $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \leq 1, \forall y \in X$. Thus, γ_t is large only when $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X$.

In other words, a significantly improved performance bound from the total curvature measure can only be obtained when H is just "weakly" submodular (i.e., when H is closer to being modular rather than submodular). This is also clear from simplifying the condition $\frac{\Delta H(y|X\backslash\{y\})}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X$, which leads to the condition:

$$H(X) \simeq H(y)+H(X\backslash\{y\}), \quad \text{for all } y \in X, \quad (12)$$

which holds whenever H is modular.

As H is the set coverage function (6), (12) holds when an agent deployed at any $y \in X$ and all other agents deployed at $X\backslash\{y\}$ contribute to the global coverage objective independently in a modular fashion. This happens when the ground set X is very sparse and/or when the agents have significantly weak non-overlapping sensing capabilities (i.e., small range δ and high decay λ in (1)).

However, (12) is easily violated if $H(X) \ll H(y) + H(X\backslash\{y\})$ for some $y \in X$. To interpret this case using (6), let us consider the analogous detection function (4) requirement: $P(x,X) \ll P(x,\{y\}) + P(x,X\backslash\{y\})$ for some $y \in X$, for a majority of $x \in \Phi$. This requirement can be simplified to: $0 \ll \theta(p(x,y)(1-\prod_{s_i\in X\backslash\{y\}}(1-p(x,s_i))) + (1-\theta)(\min\{\max_{s_i\in X\backslash\{y\}} p(x,s_i), p(x,y)\})$. Since $\theta \in [0,1]$, we need to consider both terms in the above requirement separately. However, they both lead to a common requirement: $0 \ll p(x,y)$ and $0 \ll p(x,s_i)$, for some $s_i \in X\backslash\{y\}$. In all, the total curvature measure leads to poor performance bounds when there exists some $y \in X$ and $s_i \in X\backslash\{y\}$ so that $0 \ll p(x,y) \simeq p(x,s_i) \simeq 1$, for many feasible space locations $x \in \Phi$. Notably, this condition holds when the ground set X is dense and the agents have significantly strong sensing capabilities (i.e., large range δ and small decay λ in (1)).

One final remark on the total curvature α_t (10) is that it requires an evaluation of $H(X)$ and $M(\triangleq|X|)$ evaluations of $H(X\backslash\{y\})$ terms. In certain coverage applications, this might be ill-defined [8] or computationally expensive as often $H(S)$ is of the complexity $O(|S|\bar{M})$ (where $\bar{M}$ is the size of the discretization used to evaluate the coverage integral in (6)).

b) Greedy Curvature [3]: The *greedy curvature* of (6) is given by

$$\alpha_g \triangleq \max_{0\leq i\leq N-1}\left[\max_{y\in X^i}\left(1-\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)}\right)\right], \quad (13)$$

where $X^i \triangleq \{y : y \in X\backslash S^i, (S^i\cup\{y\}) \in \mathscr{I}^N\}$ (i.e., the set of feasible options at the $(i+1)^{\text{th}}$ greedy iteration). The corresponding performance bound β_g is given by

$$\beta_g \triangleq 1-\alpha_g\left(1-\frac{1}{N}\right) \leq \frac{H(S^G)}{H(S^*)}. \quad (14)$$

Note that β_g monotonically decreases with α_g, and $\alpha_g \in [0,1]$ (as H is submodular). Therefore, as $\alpha_g \to 0$, $\beta_g \to 1$, and as $\alpha_g \to 1$, $\beta_g \to \frac{1}{N} < \beta_f$. Using these facts and (13), it is easy to see that the improvement in the performance bound is correlated with the magnitude of $\gamma_g \triangleq \min_{0\leq i\leq N-1}\left[\min_{y\in X^i}\left(\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)}\right)\right]$. Similar to before, the submodularity of H implies that γ_g is large only when $\frac{\Delta H(y|S^i)}{\Delta H(y|\emptyset)} \simeq 1, \forall y \in X^i, i \in \mathbb{N}_{N-1}^0$. In other words, similar to the total curvature, the greedy curvature provides a significantly improved performance bound when H is weakly submodular.

In fact, as observed in [8], when H is significantly weakly submodular, it can provide better performance bounds even compared to those provided by the total curvature, i.e., $\beta_f \ll \beta_t \leq \beta_g \simeq 1$. This observation can be theoretically justified using γ_t and γ_g as follows. Due to submodularity, $\Delta H(y|X\backslash\{y\}) \leq \Delta H(y|S^i)$ for any y and S^i, and thus, $\gamma_t \leq \gamma_g$. This, with weak submodularity of H leads to $\alpha_t \geq \alpha_g \simeq 0$. Now, noticing that the growth of β_g is faster as $\alpha_g \to 0$ compared to that of β_t as $\alpha_t \to 0$, we get $\beta_f \ll \beta_t \leq \beta_g \simeq 1$.

We can follow the same steps and arguments as before to show that such improved performance bounds can only be achieved when the ground set is sparse and/or when the agents have weak sensing capabilities. On the other hand, when the ground set is dense and when the agents have strong sensing capabilities, greedy curvature provides poor performance bounds (often, it may even be worse than β_f).

Nevertheless, compared to the total curvature, greedy curvature has two key redeeming qualities: it is always fully defined, and it can be computed efficiently using only the evaluations of H executed in the greedy algorithm.

c) Elemental Curvature [10]: The *elemental curvature* of (6) is given by

$$\alpha_e \triangleq \max_{\substack{(S,y_i,y_j):S\subset X,\\ y_i,y_j\in X\backslash S,\ y_i\neq y_j.}} \left[\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)}\right]. \quad (15)$$

The corresponding performance bound β_e is given by

$$\beta_e \triangleq 1-\left(\frac{\alpha_e+\alpha_e^2+\cdots+\alpha_e^{N-1}}{1+\alpha_e+\alpha_e^2+\cdots+\alpha_e^{N-1}}\right)^N \leq \frac{H(S^G)}{H(S^*)}. \quad (16)$$

It can be shown that β_e monotonically decreases with α_e, and due to the submodularity of H, $0\leq\alpha_e\leq 1$. Therefore, when $\alpha_e\to 0$, $\beta_e\to 1$ and when $\alpha_e\to 1$, $\beta_e\to\beta_f$.

According to [6, Prop. 2.1], submodularity of H also implies that $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)}\leq 1$, for all feasible (S,y_i,y_j) considered in (15). Therefore, when there exists some feasible (S,y_i,y_j) such that $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)}\simeq 1$, i.e. when H is weakly submodular (closer to being modular) in that region, based on (15), $\alpha_e\simeq 1$ - which implies $\beta_e\simeq\beta_f$ (i.e., no improvement).

As explained earlier, H is weakly submodular (which now implies $\beta_f\simeq\beta_e\ll 1$) when the ground set X is sparse and/or when agents have weak sensing capabilities. Therefore, elemental curvature contrasts from total and greedy curvature - where weakly submodular scenarios of H led to significantly improved performance bounds $\beta_f\ll\beta_t\leq\beta_g\simeq 1$.

On the other hand, the elemental curvature provides an improved performance bound when $\frac{\Delta H(y_i|S\cup\{y_j\})}{\Delta H(y_i|S)}\ll 1$ over all feasible (S,y_i,y_j) considered in (15). To further interpret this condition, let us consider the corresponding marginal detection function requirement:

$$\Delta P(x,y_i|S\cup\{y_j\})\ll\Delta P(x,y_i|S), \quad \forall(S,y_i,y_j) \quad (17)$$

which should hold for a majority of $x\in\Phi$.

Since each $\Delta P=\theta\Delta P_J+(1-\theta)\Delta P_M$ where $\theta\in[0,1]$, we first consider (17) with $\theta=1$. It can be shown that (omitting a few steps that can be found in [12]): $\Delta P_J(x,y_i|S\cup\{y_j\})\ll\Delta P_J(x,y_i|S)\iff 0\ll p(x,y_i)p(x,y_j)\prod_{s_i\in S}(1-p(x,s_i))$. This condition holds if for all feasible (S,y_i,y_j),

$$0\ll p(x,y_i)\simeq p(x,y_j)\simeq 1 \text{ with } 0\simeq p(x,s_i)\ll 1, \quad (18)$$

for some $s_i\in S$ over many feasible space locations $x\in\Phi$.

Now, let us consider (17) with $\theta=0$. Using (3), it can be shown that (again omitting a few steps that can be found in [12]): $\Delta P_M(x,y_i|S\cup\{y_j\})\ll\Delta P_M(x,y_i|S)\iff\max_{s_i\in S}p(x,s_i)\ll p(x,y_i)\simeq p(x,y_j)$. This condition also holds under the same condition obtained in (18).

In all, the elemental curvature measure leads to significantly improved performance bounds, if for all feasible (S,y_i,y_j), $0\simeq p(x,s_i)\ll p(x,y_i)\simeq p(x,y_j)\simeq 1$ holds for some $s_i\in S$, over a majority of $x\in\Phi$. Clearly, this requirement holds when the ground set X is dense and when the agents have significantly strong overlapping sensing capabilities (i.e., large range δ and small decay λ in (1)).

Finally, note that the evaluation of the elemental curvature α_e (15) is computationally expensive as it involves solving a set function maximization problem. The following proposition provides a computationally efficient upper bound for α_e, which can be used in (16) to obtain a lower bound for β_e which then can serve as a performance bound for (6).

Proposition 1: An upper-bound for the elemental curvature α_e in (15) is given by $\alpha_e\leq\bar{\alpha}_e\triangleq 1-\min_{y_i\in X,y_j\in X\backslash\{y_i\},x\in\Phi,p(x,y_i)\neq 0}p(x,y_j)\mathbf{1}_{\{\theta=1\}}$.

Remark 2: The proposed elemental curvature upper-bound $\bar{\alpha}_e$ in Prop. 1 becomes trivial (i.e., $\bar{\alpha}_e=1$) under two scenarios. The first is when two agents can be placed in the ground set (i.e., find $y_i,y_j\in X$) such that there is no complete overlapping in their sensing regions (i.e., when $\exists x\in\Omega$ such that $p(x,y_i)\neq 0$ but $p(x,y_j)=0$). The second scenario is when the max detection function is used in the coverage objective (i.e., when $\theta\neq 1$ in (4)). Note, however, that $\bar{\alpha}_e=1$ does not imply $\alpha_e=1$. Therefore, ongoing research is directed towards addressing these two challenging scenarios.

d) Partial Curvature [4]: The *partial curvature* of (6) is given by

$$\alpha_p=\max_{(S,y):y\in S\in\mathscr{I}^N}\left[1-\frac{\Delta H(y|S\backslash\{y\})}{\Delta H(y|\emptyset)}\right]. \quad (19)$$

The corresponding performance bound β_p is given by

$$\beta_p\triangleq\frac{1}{\alpha_p}\left[1-\left(1-\frac{\alpha_p}{N}\right)^N\right]\leq\frac{H(S^G)}{H(S^*)}. \quad (20)$$

This partial curvature α_p (19) provides an alternative to the total curvature α_t (10), particularly when the $H(X)$ term involved in α_t is ill-defined. Basically, α_p can be used when the domain of H is constrained to be some $\mathscr{I}\subseteq\mathscr{I}^N\subset 2^X$.

Due to the similarities in the forms of α_p and α_t (and β_p and β_t), we can directly conclude that β_p will provide significantly improved performance bounds (i.e., $\beta_p\simeq 1$) when H is weakly submodular, i.e., when the ground set is sparse and/or agent sensing capabilities are weak. On the other hand, β_p will provide poor performance bounds (i.e., $\beta_p\simeq\beta_f$) when H is strongly submodular, i.e., when the ground set is dense, and agent sensing capabilities are strong.

It should be noted that the above β_p (20) is only valid under a few additional technical conditions on H, X and $\mathscr{I}^N$ (which are omitted here, but can be found in [4]). The work in [4] has also established that $\beta_p\geq\beta_t$, i.e., β_p, always provide a better performance bound than β_t.

Similar to α_e (15), evaluating α_p (19) is extremely computationally expensive as it involves solving a set function maximization problem. The following proposition provides a computationally efficient upper bound for α_p exploiting special polymatroid properties of the considered optimal coverage problem established in Th. 2. This upper-bound for α_p, when used in (20), provides a lower bound for β_p, which then can serve as a performance bound for (6).

Proposition 2: An upper-bound for the partial curvature α_p in (19) is given by $\alpha_p\leq\bar{\alpha}_p=\frac{1}{\beta_f}\max_{y\in X}(1-\frac{\Delta H(y|A_y^G)}{H(\{y\})})$, where A_y^G is the greedy solution for the polymatroid maximization problem: $A_y^*=\arg\max_A(-\Delta H(y|A)+H(\{y\}))$, subject to constraints: $A\subseteq X\backslash\{y\}$ and $|A|=N-1$.

*e) **Extended Greedy Curvature [13]:*** The *extended greedy curvature*, as the name suggests, requires executing some extra greedy iterations in the greedy algorithm (i.e., Alg. 1). This is not an issue as Alg. 1 can be executed beyond N iterations until $M \triangleq |X|$ iterations - analogous to a scenario where more than N agents are to be deployed to the mission space in a greedy fashion.

Recall that we used S^i and s^i to denote the greedy set and greedy element at the i^{th} greedy iteration, where $i \in \mathbb{N}_M^0$. Let $m \triangleq \text{floor}(\frac{M}{N})$, and for any $n \in \mathbb{N}_{m-1}^0$, let $S_n^G \triangleq S^{(n+1)N} \backslash S^{nN} = \{s^{nN+1}, s^{nN+2}, \ldots, s^{nN+N}\}$, $X_n \triangleq X \backslash S^{nN}$, and $\mathscr{I}_n^N \triangleq \{S : S \subseteq X_n, |S| \leq N\}$. The extended greedy curvature of (6) is

$$\alpha_u \triangleq \min_{i \in Q} \alpha_u^i, \tag{21}$$

where $Q \subseteq \bar{Q} \triangleq \bar{Q}_1 \cup \bar{Q}_2 \cup \bar{Q}_3$, $\bar{Q}_1 \triangleq \{i : i = nN+1, n \in \mathbb{N}_{m-1}^0\}$, $\bar{Q}_2 \triangleq \{i : i = nN, n \in \mathbb{N}_m\}$, and $\bar{Q}_3 \triangleq \{M\}$, with

$$\alpha_u^i \triangleq \begin{cases} H(S^{i-1}) + \max\limits_{S \in \mathscr{I}_{(i-1)/N}^N} \left[\sum_{y \in S} \Delta H(y|S^{i-1})\right], & i \in \bar{Q}_1 \\ H(S^{i-N}) + \frac{1}{\beta_f}\left[H(S^i) - H(S^{i-N})\right], & i \in \bar{Q}_2 \\ H(S^i), & i \in \bar{Q}_3 \end{cases}$$

The corresponding performance bound β_u is given by

$$\beta_u \triangleq \frac{H(S^G)}{\alpha_u} \leq \frac{H(S^G)}{H(S^*)}. \tag{22}$$

Note that $\bar{Q}$ is a fixed set of greedy iteration indexes, where for each $i \in \bar{Q}$, a corresponding α_u^i value can be computed using the byproducts of greedy iterations. Q is an arbitrary subset of $\bar{Q}$ selected based on the user preference.

To characterize the effectiveness of the performance bound β_u (22) in the context of optimal coverage problem (6), let us first consider $\alpha_u^1 = H(S^0) + \max_{S \in \mathscr{I}_0^N} \left[\sum_{y \in S} \Delta H(y|S^0)\right] = \max_{S \in \mathscr{I}^N} \left[\sum_{y \in S} H(y)\right]$. Note that, when H is weakly submodular (closer to being modular), $\alpha_u^1 \simeq H(S^*)$. Through (21) and (22), this implies that $\frac{H(S^G)}{\beta_u} = \alpha_u \leq \alpha_u^1 \simeq H(S^*) \implies \beta_u \simeq 1$. Therefore, when H is weakly submodular, i.e., when the ground set is sparse and/or agent sensing capabilities are weak, β_u provides significantly improved performance bounds (similar to β_t, β_g, and β_p).

Now consider $\alpha_u^{2N} = H(S^G) + \frac{1}{\beta_f}\left[H(S^{2N}) - H(S^N)\right]$. Note that when H is strongly submodular (diminish in the returns is severe), $H(S^i)$ should quickly saturate with the greedy iterations i, and thus, $\alpha_u^{2N} \simeq H(S^G)$ as $\frac{1}{\beta_f}(H(S^{2N}) - H(S^N)) \simeq 0$. Through (21) and (22), this implies that $\frac{H(S^G)}{\beta_u} = \alpha_u \leq \alpha_u^1 \simeq H(S^G) \implies \beta_u \simeq 1$. Therefore, when H is strongly submodular, i.e., when the ground set is dense, and agent sensing capabilities are strong, β_u provides significantly improved performance bounds (similar to β_e).

In all, the extended greedy curvature-based performance bound β_u is computationally efficient and provides significantly improved performance bounds under both weak and strong agent sensing capabilities. This versatile behavior of β_u contrasts with that of $\beta_t, \beta_g, \beta_e$ and β_p discussed before.

V. Discussion

In this section, we summarize our findings on the effectiveness and computational complexity of different curvature-based performance bounds in optimal coverage problems (extra details on complexity analysis can be found in [12]). Our key findings have been summarized in Tab. I.

In terms of effectiveness, based on our analysis, total, greedy, and partial curvature measures provide improved performance bounds when agents have low sensing capabilities (i.e., high decay λ and/or low range δ). Conversely, the elemental curvature measure provides improved performance bounds when agents have strong sensing capabilities (i.e., low decay λ and/or high range δ). Most importantly, the extended greedy curvature distinguishes itself by being able to provide improved performance bounds regardless of the weak or strong nature of agent sensing capabilities.

In terms of computational complexity, the greedy curvature measure is the most efficient as it can be computed directly using the byproducts of the greedy algorithm (thus, it has a complexity ($O(N)$)). The total curvature exhibits a complexity of $O(M^2\bar{M})$ mainly due to the involved $H(X)$ computation ($\bar{M}$ denotes the number of discrete points in Ω used for the evaluation of the coverage integral (6)). The conservative upper-bound proposed for the elemental curvature has the same complexity $O(M^2\bar{M})$. In contrast, the original elemental and partial curvatures measures have the highest computational complexities, $O(M^3 2^M \bar{M})$ and $O(M^N \bar{M})$, respectively. The proposed upper-bound estimate of the partial curvature has a higher complexity $O(N^2M^2\bar{M})$ than that of the elemental curvature. The complexity of the extended greedy curvature is lower compared to that of elemental and partial curvature. However, it is of comparable complexity with respect to that of total curvature and conservative upper bound estimates of elemental and partial curvature measures.

To summarize, this review has highlighted three main challenges in using curvature-based performance bounds for optimal coverage problems: (1) the inherent dependence of their effectiveness on the strong or weak nature of the submodularity property of the considered optimal coverage problem, (2) the computational complexity associated with computing the curvature measures, and (3) the technical conditions required for the successful application of curvature-based performance bounds (e.g., see Remark 2). Towards addressing these challenges, the recently proposed extended greedy curvature concept [13] has shown promising advances. This curvature measure takes a data-driven approach and utilizes only the information observed during a selected number of extra greedy iterations - offering a computationally efficient performance bound without inherent or technical limitations.

In light of these findings, we believe that future research should be directed toward finding more *data-driven curvature measures* (like α_u) to directly address computational challenges faced by standard *theoretical curvature measures* (like α_e, α_p). However, in such a pursuit, a crucial challenge would be in establishing theoretical guar-

TABLE I: Characteristics of different curvature-based performance metrics in the context of optimal coverage problems

β	$\beta_f \ll \beta \simeq 1$ when: Agent Sensing (1)		Denseness of X		Complexity	Remarks $H(S) \sim O(\lvert S\rvert\bar{M})$ Alg. 1 $\sim O(N^2M\bar{M})$
	Low $\delta\downarrow, \lambda\uparrow$	High $\delta\uparrow, \lambda\downarrow$	Low $(M\downarrow)$	High $(M\uparrow)$		
β_t	✓	✗	✓	✗	$O(M^2\bar{M})$	
β_g	✓	✗	✓	✗	$O(N)$	
β_e	✗	✓	✗	✓	$O(M^3 2^M \bar{M})$	Prop 1: $O(M^2\bar{M})$
β_p	✓	✗	✓	✗	$O(M^N\bar{M})$	Prop 2: $O(N^2M^2\bar{M})$
β_e	✓	✓	✓	✓	$O(n^2N^2M\bar{M})$	(i.e., for nN extra iter.) Worst Case: $O(M^3\bar{M})$

antees/characterizations on their effectiveness/performance. This challenge motivates exploring *hybrid curvature measures* that have elements rooted in both data-driven and theoretical curvature measures (for more details, see [12]).

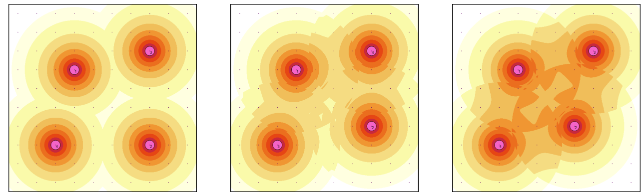
VI. Case Studies

In our numerical experiments, we considered size 600×600 square mission spaces with two obstacle arrangements named "Blank" and "Maze," as can be seen in Figs. 1-2 and 3, respectively. In such figures, obstacles are shown as dark green-colored blocks, candidate agent locations (ground set X with $|X| = 100$) are shown as small black dots, and agent locations are shown as numbered pink-colored circles. Light-colored areas indicate low coverage levels, while dark-colored areas indicate the opposite. The event density function was assumed to be uniform: $R(x) = 1, \forall x \in \Phi$ in (5).

The main attributes and functionalities of the considered optimal coverage problem, the greedy algorithm (Alg. 1), and the reviewed performance bounds β_f (9), β_t (11), β_g (14), β_e (16), β_p (20) and β_u (22) were all implemented in an interactive JavaScript simulator which is available at `https://github.com/shiran27/P2-Submod_Coverage`.

a) Impact of the weight parameter θ used in* (4)*: First, we show the impact of θ using the Blank mission space with $N = 4$ agents. Each agent was assumed to have a sensing range $\lambda = 200$ and a decay $\delta = 0.012$ (see (1)). The observed greedy solutions, coverage level patterns, and performance bounds, when $\theta \in \{0, 0.5, 1\}$ are reported in Fig. 1.

As stated in Rm. 1, choosing $\theta = 1$ motivates cooperation while choosing $\theta = 0$ motivates individualism in sensing. This behavior is confirmed by the observations reported in Fig. 1. In particular, notice that when $\theta \simeq 0$ (Fig. 1(a)), agents are spread out in the mission space - leaving a blind region in the middle but covering a broad area in the mission space. In contrast, when $\theta = 1$ (Fig. 1(c)), agents are flocked together without leaving a blind region in the middle but failing to cover some outer regions of the mission space. Note also that θ affects the performance bounds (in this case, β_u - which were the tightest). This implies that, with respect to the greedy approach, the optimal coverage problem defined with a max detection function (3) is harder to solve than that with a joint detection function (2). This conclusion is intuitive as max detection functions significantly increase the non-smooth nature of the optimal coverage problems. Note that, in the sequel, we have used $\theta = 0.5$ unless stated otherwise.

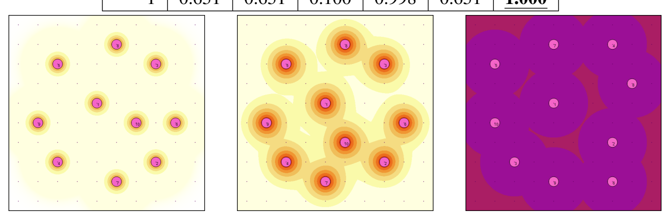


(a) $\theta = 0, \beta_u = 0.84$ (b) $\theta = 0.5, \beta_u = 0.87$ (c) $\theta = 1, \beta_u = 0.92$

Fig. 1: Greedy solutions, coverage level patterns, and the tightest performance bounds observed under different weight parameters $\theta \in [0, 1]$ in the Blank mission space with $N = 4$ agents with sensing range $\delta = 200$ and decay $\lambda = 0.012$.

TABLE II: Performance bounds observed under different sensing decay λ values in the Blank mission space with $N = 10$ agents with sensing range $\delta = 800$.

Perf. bounds with respect to θ at $\lambda = 0.05$						
θ	β_f	β_t	β_g	β_e	β_p	β_u
0	0.651	0.745	0.595	0.651	0.676	**0.943**
0.5	0.651	0.790	0.753	0.651	0.753	**0.965**
1	0.651	0.840	0.872	0.651	0.829	**0.992**
Perf. bounds with respect to λ at $\theta = 0.5$						
λ	β_f	β_t	β_g	β_e	β_p	β_u
0.05	0.651	0.790	0.753	0.651	0.753	**0.965**
0.045	0.651	0.765	0.714	0.651	0.720	**0.951**
0.035	0.651	0.713	0.617	0.651	0.651	**0.901**
0.025	0.651	0.670	0.493	0.651	0.651	**0.795**
0.015	0.651	0.655	0.324	0.651	0.651	**0.656**
0.005	0.651	0.652	0.118	0.651	0.651	**0.912**
0.001	0.651	0.651	0.100	0.651	0.651	**0.986**
Perf. bounds with respect to θ at $\lambda = 0.001$						
θ	β_f	β_t	β_g	β_e	β_p	β_u
0	0.651	0.651	0.101	0.651	0.651	**0.967**
0.5	0.651	0.651	0.100	0.651	0.651	**0.986**
1	0.651	0.651	0.100	0.998	0.651	**1.000**



(d) $\lambda = 0.05$ (e) $\lambda = 0.025$ (f) $\lambda = 0.001$

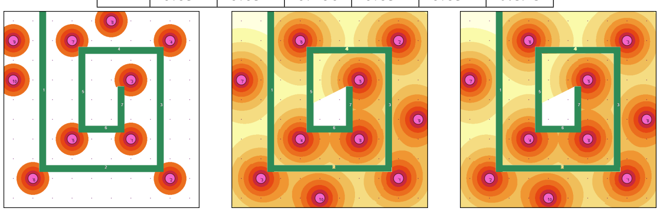
Fig. 2: Greedy solutions and coverage level patterns observed under different sensing decay λ values considered in Tab. II.

b) Impact of the agent sensing capabilities λ, δ: We next show the impact of agent sensing capabilities (characterized by their sensing decay λ and range δ) on different curvature-based performance bounds. For this purpose, in one experiment, we used the Blank mission space with $N = 10$ agents, where we kept $\delta = 800$ (fixed) and varied λ from $\lambda = 0.05$ to $\lambda = 0.001$. The observed performance bounds and a few selected greedy solutions are reported in Tab. II and the accompanying Fig. 2, respectively. In the second experiment, we used the Maze mission space with $N = 10$ agents, where we kept $\lambda = 0.012$ (fixed) and varied δ from $\delta = 50$ to $\delta = 800$. The observed performance bounds and a few selected greedy solutions are reported in Tab. III and the accompanying Fig. 3, respectively.

In Tabs. II and III, we have further explored the impact

TABLE III: Performance bounds observed under different sensing range δ values in the Maze mission space with $N = 10$ agents with sensing decay $\lambda = 0.012$.

Perf. bounds with respect to θ at $\delta = 50$						
θ	β_f	β_t	β_g	β_e	β_p	β_u
0	0.651	0.694	0.682	0.651	0.651	**0.992**
0.5	0.651	0.718	0.729	0.651	0.651	**0.992**
1	0.651	0.742	0.775	0.651	0.672	**0.992**
Perf. bounds with respect to δ at $\theta = 0.5$						
δ	β_f	β_t	β_g	β_e	β_p	β_u
50	0.651	0.718	0.729	0.651	0.651	**0.992**
100	0.651	0.660	0.401	0.651	0.651	**0.976**
200	0.651	0.657	0.342	0.651	0.651	**0.892**
350	0.651	0.656	0.329	0.651	0.651	**0.841**
400	0.651	0.656	0.329	0.651	0.651	**0.837**
600	0.651	0.656	0.328	0.651	0.651	**0.828**
800	0.651	0.656	0.329	0.651	0.651	**0.834**
Perf. bounds with respect to θ at $\delta = 800$						
θ	β_f	β_t	β_g	β_e	β_p	β_u
0	0.651	0.660	0.191	0.651	0.651	**0.789**
0.5	0.651	0.656	0.329	0.651	0.651	**0.834**
1	0.651	0.652	0.460	0.651	0.651	**0.898**



(a) $\delta = 50$ (b) $\delta = 350$ (c) $\delta = 800$

Fig. 3: Greedy solutions and coverage level patterns observed under different sensing range δ values considered in Tab. III.

of θ on performance bounds at extreme cases of agent sensing capabilities, i.e., when $\lambda \in \{0.05, 0.001\}$ and $\delta \in \{50, 800\}$, respectively (otherwise, by default, $\theta = 0.5$). The observations in each case are given in smaller sub-tables located above and below the main table. In each table (and sub-table), the highest performance bound values observed in each row and column have been highlighted. Also, the tables have been arranged so that when going from top to bottom, the sensing capabilities of the agents increase.

Recall that, based on our analysis, the performance bounds β_t, β_g, β_p, and β_u should provide significant improvements beyond β_f when agent sensing capabilities are low (i.e., when λ is high and δ is low). The results in Tabs. II and III validate this conclusion (e.g., see the respective results for $\lambda = 0.05$ and $\delta = 50$). On the other hand, the performance bounds β_e and β_u should provide significant improvements beyond β_f when agent sensing capabilities are high (i.e., when λ is low and δ is high). In this case, with regard to β_e, as pointed out in Rm. 2, we also need the mission space to be obstacle-free and $\theta = 1$. Again, the results in Tabs. II and III validate this conclusion (e.g., see the respective results for $\lambda = 0.001$ and $\delta = 800$, particularly with $\theta = 1$). Moreover, as expected from our analysis, the observations in Tabs. II and III also confirm that the performance bound β_u provides significant improvements beyond β_f regardless of the agent sensing capabilities.

VII. Conclusion

In this paper, we considered a generalized class of multi-agent optimal coverage problems and established its several polymatroid features. These properties enabled efficient solving of the considered optimal coverage problem via greedy algorithms with performance-bound guarantees. To obtain further improved performance bounds, we reviewed five curvature measures found in the literature. In particular, we identified their effectiveness and computational complexity features and proposed novel techniques to efficiently estimate candidates for some of such curvature measures. We also implemented the proposed coverage problem setup, its solution, and performance bounds in an interactive simulator. The obtained numerical results validated our findings. Ongoing research activities explore meaningful ways to combine the strengths of data-driven and theoretical curvature measures.

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