Statistics and Probability Letters 213 (2024) 110172

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Isotropic random tangential vector fields on spheres

Tianshi Lu

Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033, USA

ARTICLE INFO

MSC:
60G60

Keywords:
Isotropic random vector field
Karhunen–Loève expansion
Helmholtz–Hodge decomposition

ABSTRACT

In this paper we characterized isotropic random tangential vector fields on d-spheres for $d \geq 1$ by the cross-covariance, and derived their Karhunen–Loève expansion. The tangential vector field can be decomposed into a curl-free part and a divergence-free part by the Helmholtz–Hodge decomposition. We proved that the two parts can be correlated on a 2-sphere, while they must be uncorrelated on a d-sphere for $d \geq 3$. On a 3-sphere, the divergence-free part can be further decomposed into two isotropic flows.

1. Introduction

Random tangential vector fields on a sphere have applications in terrestrial physics such as oceanic and wind currents (Fan et al., 2018; Hutchinson et al., 2021). Isotropic random currents in $\mathbb{R}^d$ have been characterized in Ref. Ito (1951, 1956) and Wong and Zakai (1989). In this paper, we will study random tangential vector fields, a.k.a random 1-currents or random flows. Fan et al. (2018) constructed a family of isotropic random tangential vector fields on $\mathbb{S}^2$. We will characterize isotropic random flows on $\mathbb{S}^d$ for $d \geq 1$.

In this paper, a random vector field V on $\mathbb{S}^d = \{\mathbf{x} \in \mathbb{R}^{d+1}, \|\mathbf{x}\| = 1\}$ is isotropic if

$$\mathrm{E}V(Q\mathbf{x}) = \mathrm{E}V(\mathbf{x}), \quad \mathrm{E}V(Q\mathbf{x})V^T(Q\mathbf{y}) = \mathrm{E}V(\mathbf{x})V^T(\mathbf{y}), \tag{1.1}$$

for any $\mathbf{x} \in \mathbb{S}^d$, $\mathbf{y} \in \mathbb{S}^d$, and $Q \in \mathrm{SO}(d+1)$. It is analogous to the definition of two-weakly isotropic random fields on a sphere (Marinucci and Peccati, 2011). Ref. Lang and Schwab (2015) studied isotropic Gaussian random fields on the sphere, and Ref. Ma and Malyarenko (2020) studied isotropic random vector fields on compact two-point homogeneous spaces. A tangential vector field, a.k.a flow, on $\mathbb{S}^d$ is a vector field $\mathbf{J}(\mathbf{x}) : \mathbb{S}^d \to \mathbb{R}^{d+1}$ such that $\mathbf{x} \cdot \mathbf{J}(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbb{S}^d$. A random flow $\mathbf{J}$ on $\mathbb{S}^d$ is isotropic if

$$\mathrm{E}\mathbf{J}(Q\mathbf{x}) = \mathrm{E}Q\mathbf{J}(\mathbf{x}), \quad \mathrm{E}\mathbf{J}(Q\mathbf{x})\mathbf{J}^T(Q\mathbf{y}) = \mathrm{E}Q\mathbf{J}(\mathbf{x})\mathbf{J}^T(\mathbf{y})Q^T, \tag{1.2}$$

for any $\mathbf{x} \in \mathbb{S}^d$, $\mathbf{y} \in \mathbb{S}^d$, and $Q \in \mathrm{SO}(d+1)$. If Eq. (1.2) holds for any $Q \in \mathrm{O}(d+1)$, $\mathbf{J}$ is called reflexive. By Eq. (1.2), $\mathbf{J}$ is mean-zero except on $\mathbb{S}^1$. A matrix kernel $\mathbf{C}$ on $\mathbb{S}^d$ is called isotropic if $\mathbf{C}(Q\mathbf{x}, Q\mathbf{y}) = Q\mathbf{C}(\mathbf{x}, \mathbf{y})Q^T$ for any $Q \in \mathrm{SO}(d+1)$. An isotropic random flow on $\mathbb{S}^1$ has the same covariance and the Karhunen–Loève (KL) expansion as an isotropic field. The KL expansion can be decomposed into the gradient flow ($l > 0$ modes) and the curl flow ($l = 0$ mode).

In Section 2, we investigate reflexive isotropic random flows on $\mathbb{S}^d$ for $d \geq 2$. In Section 3, we derive the general form of isotropic random flows on $\mathbb{S}^2$. In Section 4, we characterize isotropic random flows on $\mathbb{S}^3$. In Section 5, we provide the proofs. In Section 6, we draw conclusions.

E-mail address: tianshi.lu@wichita.edu.

https://doi.org/10.1016/j.spl.2024.110172
Received 7 March 2024; Received in revised form 4 April 2024; Accepted 29 May 2024
Available online 3 June 2024

2. Isotropic random flows on $\mathbb{S}^d$ for $d \geq 2$

We use wedge products to construct cross-covariances of random flows on $\mathbb{S}^d$. The inner product of two wedge products satisfies $(u_1 \wedge \cdots \wedge u_k) \cdot (v_1 \wedge \cdots \wedge v_k) = \det(u_i^T v_j)$. Let

$$Z_l(\mathbf{x}) = \sum_{m=1}^{\dim H_l} N_{lm} S_{lm}(\mathbf{x}) \sqrt{\frac{\omega_d}{\dim H_l}}, \tag{2.1}$$

where S_{lm} are real spherical harmonics on $\mathbb{S}^d$, N_{lm} are iid standard normal random variables,

$$\omega_d = \mu(\mathbb{S}^d) = 2\pi^{\frac{d+1}{2}} / \Gamma(\frac{d+1}{2}), \tag{2.2}$$

and H_l is the eigenspace of the Laplacian on $\mathbb{S}^d$ corresponding to $\lambda_l = l(l+d-1)$,

$$\dim H_l = \frac{(2l+d-1)\Gamma(l+d-1)}{\Gamma(d)\Gamma(l+1)}. \tag{2.3}$$

Then $\mathrm{E} Z_l(\mathbf{x}) Z_l(\mathbf{y}) = p_{\alpha,l}(\mathbf{x}^T\mathbf{y})$, where $p_{\alpha,l}(z)$ is the normalized Jacobi polynomial, $P_l^{(\alpha,\alpha)}(z)/P_l^{(\alpha,\alpha)}(1)$, with $\alpha = d/2 - 1$. For $l \geq 1$, let $\mathbf{A}_l$ be the matrix kernel on $\mathbb{S}^d$ defined by

$$\mathbf{A}_l(\mathbf{x}, \mathbf{y}) = \mathrm{E}(\nabla Z_l(\mathbf{x}))(\nabla Z_l(\mathbf{y}))^T, \tag{2.4}$$

where ∇ is the gradient along $\mathbb{S}^d$, and $\mathbf{B}_l$ is the matrix kernel on $\mathbb{S}^d$ such that for any $\mathbf{x}_1$ and $\mathbf{y}_1$ in $\mathbb{R}^{d+1}$,

$$\mathbf{x}_1^T \mathbf{B}_l(\mathbf{x}, \mathbf{y}) \mathbf{y}_1 = \mathrm{E}(\mathbf{x} \wedge \mathbf{x}_1 \wedge \nabla Z_l(\mathbf{x})) \cdot (\mathbf{y} \wedge \mathbf{y}_1 \wedge \nabla Z_l(\mathbf{y})). \tag{2.5}$$

On $\mathbb{S}^2$, it is equivalent to

$$\mathbf{B}_l(\mathbf{x}, \mathbf{y}) = \mathrm{E}(\mathbf{x} \times \nabla Z_l(\mathbf{x}))(\mathbf{y} \times \nabla Z_l(\mathbf{y}))^T. \tag{2.6}$$

Since Z_l is isotropic, $\mathbf{A}_l$ and $\mathbf{B}_l$ are isotropic. To express $\mathbf{A}_l(\mathbf{x}, \mathbf{y})$ and $\mathbf{B}_l(\mathbf{x}, \mathbf{y})$ explicitly, let

$$\mathbf{n}_{\mathbf{x}} = \frac{\mathbf{y} - \mathbf{x}(\mathbf{x}^T\mathbf{y})}{\sqrt{1 - (\mathbf{x}^T\mathbf{y})^2}}, \quad \mathbf{n}_{\mathbf{y}} = \frac{\mathbf{y}(\mathbf{x}^T\mathbf{y}) - \mathbf{x}}{\sqrt{1 - (\mathbf{x}^T\mathbf{y})^2}}, \quad \mathbf{x} \neq \pm\mathbf{y}. \tag{2.7}$$

The vectors $\mathbf{n}_{\mathbf{x}}$ and $\mathbf{n}_{\mathbf{y}}$ are the unit vectors at $\mathbf{x}$ and $\mathbf{y}$, respectively, along the shorter geodesic from $\mathbf{x}$ to $\mathbf{y}$ on $\mathbb{S}^d$. For $\mathbf{x} = \mathbf{y}$ ($\mathbf{x} = -\mathbf{y}$), $\mathbf{n}_{\mathbf{x}}$ is an arbitrary unit vector in $\mathbb{R}^{d+1}$ that is orthogonal to $\mathbf{x}$, and $\mathbf{n}_{\mathbf{y}} = \mathbf{n}_{\mathbf{x}}$ ($\mathbf{n}_{\mathbf{y}} = -\mathbf{n}_{\mathbf{x}}$). Notice that $\mathbf{x} \wedge \mathbf{n}_{\mathbf{x}} = \mathbf{y} \wedge \mathbf{n}_{\mathbf{y}}$. We also introduce the vectors $\{\mathbf{m}_i\}_{i=1}^{d-1}$ such that $(\mathbf{x}, \mathbf{n}_{\mathbf{x}}, \mathbf{m}_1, \ldots, \mathbf{m}_{d-1})$ is a positively oriented orthonormal basis of $\mathbb{R}^{d+1}$. The next lemma gives the matrix form of $\mathbf{A}_l$ and $\mathbf{B}_l$.

Lemma 2.1. *For the matrix kernels $\mathbf{A}_l$ and $\mathbf{B}_l$ on $\mathbb{S}^d$ defined by Eqs. (2.4) and (2.5), with $z = \mathbf{x}^T\mathbf{y}$,*

$$\mathbf{A}_l(\mathbf{x}, \mathbf{y}) = \left[\lambda_l p_{\alpha,l}(z) - (d-1) z p'_{\alpha,l}(z)\right] \mathbf{n}_{\mathbf{x}} \mathbf{n}_{\mathbf{y}}^T + p'_{\alpha,l}(z) \sum_{i=1}^{d-1} \mathbf{m}_i \mathbf{m}_i^T, \tag{2.8}$$

$$\mathbf{B}_l(\mathbf{x}, \mathbf{y}) = (d-1) p'_{\alpha,l}(z) \mathbf{n}_{\mathbf{x}} \mathbf{n}_{\mathbf{y}}^T + \left[\lambda_l p_{\alpha,l}(z) - z p'_{\alpha,l}(z)\right] \sum_{i=1}^{d-1} \mathbf{m}_i \mathbf{m}_i^T. \tag{2.9}$$

For two isotropic kernels $\mathbf{C}_1$ and $\mathbf{C}_2$ on $\mathbb{S}^d$, let their inner product be

$$(\mathbf{C}_1, \mathbf{C}_2) = \frac{1}{\omega_d} \mathrm{tr} \int_{\mathbb{S}^d} \mathbf{C}_1(\mathbf{x}, \mathbf{y}) \mathbf{C}_2(\mathbf{y}, \mathbf{x}) d\mathbf{y}. \tag{2.10}$$

The inner product is independent of $\mathbf{x}$ due to the isotropy. The following lemma shows that $\{\mathbf{A}_l, \mathbf{B}_l\}_{l=1}^{\infty}$ is an orthogonal set of kernels on $\mathbb{S}^d$ and represents the Helmholtz–Hodge decomposition.

Lemma 2.2. *For the matrix kernels $\mathbf{A}_l$ and $\mathbf{B}_l$ defined by Eqs. (2.4) and (2.5), the image space of $\mathbf{A}_l$ and $\mathbf{B}_l$ are curl-free and divergence-free, respectively, and*

$$(\mathbf{A}_l, \mathbf{A}_{l'}) = \frac{\lambda_l^2}{\dim H_l} \delta_{ll'}, \quad (\mathbf{B}_l, \mathbf{B}_{l'}) = \frac{(d-1)(l+1)(l+d-2)\lambda_l}{\dim H_l} \delta_{ll'}, \quad (\mathbf{A}_l, \mathbf{B}_{l'}) = 0. \tag{2.11}$$

Since $\mathbf{A}_l$ and $\mathbf{B}_l$ are positive semidefinite, their image spaces are orthogonal. The following theorems characterize reflexive isotropic flows on $\mathbb{S}^d$ and their cross-covariances.

Theorem 2.3. *A mean-square continuous reflexive isotropic random flow on $\mathbb{S}^d$ for $d \geq 2$ has the cross-covariance,*

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \sum_{l=1}^{\infty} (a_l \mathbf{A}_l(\mathbf{x}, \mathbf{y}) + b_l \mathbf{B}_l(\mathbf{x}, \mathbf{y})), \tag{2.12}$$

where $\mathbf{A}_l$ and $\mathbf{B}_l$ are given by Eqs. (2.4) and (2.5), $a_l \geq 0$, $b_l \geq 0$, $\sum_{l=1}^{\infty} a_l l^2$ and $\sum_{l=1}^{\infty} b_l l^2$ converge. The series in Eq. (2.12) converge uniformly.

Theorem 2.4. *The Funk–Hecke formula for a reflexive isotropic matrix kernel* $\mathbf{C}$ *on* $\mathbb{S}^d$ *for* $d \geq 2$ *is*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d} \mathbf{C}(\mathbf{x},\mathbf{y})\nabla S_{lm}(\mathbf{y})d\mathbf{y} = \frac{a_l\lambda_l}{\dim H_l}\nabla S_{lm}(\mathbf{x}), \quad \frac{1}{\omega_d}\int_{\mathbb{S}^d} \mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{F}_{lm}(\mathbf{y})d\mathbf{y} = \frac{(d-1)b_l\lambda_l}{\dim K_l}\mathbf{F}_{lm}(\mathbf{x}), \tag{2.13}$$

where a_l *and* b_l *are given in Eq. (2.12),* K_l *is the image space of* $\mathbf{B}_l$*,*

$$\dim K_l = \frac{(2l+d-1)(l+d-1)\Gamma(l+d-2)}{(l+1)\Gamma(d-1)\Gamma(l)}, \tag{2.14}$$

and $\{\mathbf{F}_{lm}\}$ *is an orthogonal basis of* K_l *normalized by* $\int_{\mathbb{S}^d}\|\mathbf{F}_{lm}(\mathbf{x})\|^2 d\mathbf{x} = \lambda_l$*. A mean-square continuous reflexive isotropic random flow* $\mathbf{J}(\mathbf{x})$ *on* $\mathbb{S}^d$ *with* $\mathbf{C}(\mathbf{x},\mathbf{y})$ *given by Eq. (2.12) has the KL expansion,*

$$\mathbf{J}(\mathbf{x}) = \sum_{l=1}^{\infty}\sum_{m=1}^{\dim H_l} a_{lm}\nabla S_{lm}(\mathbf{x}) + \sum_{l=1}^{\infty}\sum_{m=1}^{\dim K_l} b_{lm}\mathbf{F}_{lm}(\mathbf{x}), \tag{2.15}$$

where

$$a_{lm} = \frac{1}{\lambda_l}\int_{\mathbb{S}^d}\mathbf{J}^T(\mathbf{x})\nabla S_{lm}(\mathbf{x})d\mathbf{x}, \quad b_{lm} = \frac{1}{\lambda_l}\int_{\mathbb{S}^d}\mathbf{J}^T(\mathbf{x})\mathbf{F}_{lm}(\mathbf{x})d\mathbf{x}. \tag{2.16}$$

The covariances of the coefficients are $\mathrm{cov}(a_{lm}, b_{l'm'}) = 0$*, and*

$$\mathrm{cov}(a_{lm}, a_{l'm'}) = \delta_{ll'}\delta_{mm'}\frac{a_l\omega_d}{\dim H_l}, \quad \mathrm{cov}(b_{lm}, b_{l'm'}) = \delta_{ll'}\delta_{mm'}\frac{(d-1)b_l\omega_d}{\dim K_l}. \tag{2.17}$$

Theorem 2.4 holds for isotropic random flows on $\mathbb{S}^d$ for $d \geq 4$ because they are always reflexive. The isotropic random flows on $\mathbb{S}^2$ and $\mathbb{S}^3$ have richer structures as shown in the following sections.

3. Isotropic random flows on $\mathbb{S}^2$

Theorem 3.1. *A mean-square continuous isotropic random flow on* $\mathbb{S}^2$ *has the cross-covariance,*

$$\mathbf{C}(\mathbf{x},\mathbf{y}) = \sum_{l=1}^{\infty}(a_l\mathbf{A}_l(\mathbf{x},\mathbf{y}) + b_l\mathbf{B}_l(\mathbf{x},\mathbf{y}) + c_l\mathbf{C}_l(\mathbf{x},\mathbf{y})), \tag{3.1}$$

where $a_l \geq 0$*,* $b_l \geq 0$*,* $|c_l| \leq \sqrt{a_l b_l}$*,* $\sum_{l=1}^{\infty} a_l l^2$ *and* $\sum_{l=1}^{\infty} b_l l^2$ *converge, and*

$$\begin{aligned} &\mathbf{A}_l(\mathbf{x},\mathbf{y}) = s_l(z)\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T + t_l(z)\mathbf{m}\mathbf{m}^T, \qquad \mathbf{B}_l(\mathbf{x},\mathbf{y}) = t_l(z)\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T + s_l(z)\mathbf{m}\mathbf{m}^T, \\ &\mathbf{C}_l(\mathbf{x},\mathbf{y}) = (s_l(z) - t_l(z))(\mathbf{n}_\mathbf{x}\mathbf{m}^T + \mathbf{m}\mathbf{n}_\mathbf{y}^T), \end{aligned} \tag{3.2}$$

in which $z = \mathbf{x}^T\mathbf{y}$*,* $\mathbf{m} = \mathbf{x}\times\mathbf{n}_\mathbf{x}$*,* $s_l(z) = l(l+1)P_l(z) - zP_l'(z)$*,* $t_l(z) = P_l'(z)$*, where* $P_l(z)$ *is the Legendre polynomial of degree* l*. The series in Eq. (3.1) converge uniformly.*

Theorem 3.2. *A mean-square continuous isotropic random flow on* $\mathbb{S}^2$ *has the spectral expansion,*

$$\mathbf{J}(\mathbf{x}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}(a_{lm}\nabla S_{lm}(\mathbf{x}) + b_{lm}\mathbf{x}\times\nabla S_{lm}(\mathbf{x})), \tag{3.3}$$

with

$$a_{lm} = \frac{1}{l(l+1)}\int_{\mathbb{S}^2}\mathbf{J}(\mathbf{x})\cdot\nabla S_{lm}(\mathbf{x})d\mathbf{x}, \quad b_{lm} = \frac{1}{l(l+1)}\int_{\mathbb{S}^2}\mathbf{J}(\mathbf{x})\cdot(\mathbf{x}\times\nabla S_{lm}(\mathbf{x}))d\mathbf{x}. \tag{3.4}$$

If the cross-covariance of $\mathbf{J}$ *is given by Eq. (3.1), the covariance of the coefficients are*

$$\begin{aligned} &\mathrm{cov}(a_{lm}, a_{l'm'}) = \frac{4\pi}{2l+1}a_l\delta_{ll'}\delta_{mm'}, \quad \mathrm{cov}(b_{lm}, b_{l'm'}) = \frac{4\pi}{2l+1}b_l\delta_{ll'}\delta_{mm'}, \\ &\mathrm{cov}(a_{lm}, b_{l'm'}) = \frac{4\pi}{2l+1}c_l\delta_{ll'}\delta_{mm'}. \end{aligned} \tag{3.5}$$

We can rewrite Eq. (3.3) as $\mathbf{J}(\mathbf{x}) = \nabla X(\mathbf{x}) + \mathbf{x}\times\nabla Y(\mathbf{x})$, where

$$X(\mathbf{x}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}S_{lm}(\mathbf{x}), \quad Y(\mathbf{x}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} b_{lm}S_{lm}(\mathbf{x}). \tag{3.6}$$

The cross-covariance of the random vector field $V = (X, Y)^T$ is

$$\mathrm{cov}(V(\mathbf{x}), V(\mathbf{y})) = \sum_{l=1}^{\infty}\begin{pmatrix} a_l & c_l \\ c_l & b_l \end{pmatrix} P_l(z). \tag{3.7}$$

Since $\sum_{l=1}^{\infty} a_l l^2$ and $\sum_{l=1}^{\infty} b_l l^2$ converge, V is mean-square differentiable, hence the following corollary.

Corollary 3.2.1. *A mean-square continuous isotropic random flow on $\mathbb{S}^2$ can be written as*

$$\mathbf{J}(\mathbf{x}) = \nabla X(\mathbf{x}) + \mathbf{x} \times \nabla Y(\mathbf{x}), \tag{3.8}$$

in which $V = (X, Y)^T$ is a mean-square differentiable isotropic random vector field on $\mathbb{S}^2$.

An example called Tangential Matérn Model was given by Fan et al. (2018).

4. Isotropic random flows on $\mathbb{S}^3$

We will use the following lemma in the construction of isotropic cross-covariances on $\mathbb{S}^3$.

Lemma 4.1. *For $\mathbf{x}$ and $\mathbf{y}$ on $\mathbb{S}^d$, $\mathbf{x}_1$ and $\mathbf{y}_1$ in $\mathbb{R}^{d+1}$,*

$$\frac{\dim H_l}{\omega_d} \int_{\mathbb{S}^d} p'_{\alpha,l}(\mathbf{x}^T\mathbf{z})p'_{\alpha,l}(\mathbf{z}^T\mathbf{y})(\mathbf{x} \wedge \mathbf{x}_1 \wedge \mathbf{z}) \cdot (\mathbf{y} \wedge \mathbf{y}_1 \wedge \mathbf{z})d\mathbf{z} = \mathbf{x}_1^T \mathbf{B}_l(\mathbf{x}, \mathbf{y})\mathbf{y}_1. \tag{4.1}$$

Theorem 4.2. *A mean-square continuous isotropic random flow on $\mathbb{S}^3$ has the cross-covariance,*

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \sum_{l=1}^{\infty} (a_l \mathbf{A}_l(\mathbf{x}, \mathbf{y}) + b_l \mathbf{B}_l(\mathbf{x}, \mathbf{y}) + c_l \mathbf{C}_l(\mathbf{x}, \mathbf{y})), \tag{4.2}$$

with $a_l \geq 0$, $b_l \geq 0$, $|c_l| \leq b_l$, $\sum_{l=1}^{\infty} a_l l^2$ and $\sum_{l=1}^{\infty} b_l l^2$ converge, $\mathbf{A}_l$ and $\mathbf{B}_l$ given in Eqs. (2.4) and (2.5),

$$\mathbf{C}_l(\mathbf{x}, \mathbf{y}) = (l + 1)p'_{\alpha,l}(z)\sqrt{1 - z^2}(\mathbf{m}_2\mathbf{m}_1^T - \mathbf{m}_1\mathbf{m}_2^T), \tag{4.3}$$

where $z = \mathbf{x}^T\mathbf{y}$. The series in Eq. (4.2) converges uniformly.

On $\mathbb{S}^3$, $\dim H_l = (l + 1)^2$, $\dim K_l = 2l(l + 2)$. By Theorem 4.2, $\mathbf{B}_l^{\pm} = (\mathbf{B}_l \pm \mathbf{C}_l)/2$ is positive semidefinite, and $\{\mathbf{A}_l, \mathbf{B}_l^+, \mathbf{B}_l^-\}_{l=1}^{\infty}$ is an orthogonal basis of the isotropic matrix kernels on $\mathbb{S}^3$. We obtain the KL expansion for isotropic random flows on $\mathbb{S}^3$ by rewriting Eq. (4.2) as

$$\mathbf{C}(\mathbf{x}, \mathbf{y}) = \sum_{l=1}^{\infty} (a_l \mathbf{A}_l(\mathbf{x}, \mathbf{y}) + (b_l + c_l)\mathbf{B}_l^+(\mathbf{x}, \mathbf{y}) + (b_l - c_l)\mathbf{B}_l^-(\mathbf{x}, \mathbf{y})). \tag{4.4}$$

Theorem 4.3. *A mean-square continuous isotropic random flow on $\mathbb{S}^3$ has the KL expansion,*

$$\mathbf{J}(\mathbf{x}) = \sum_{l=1}^{\infty} \left(\sum_{m=1}^{(l+1)^2} a_{lm} \nabla S_{lm}(\mathbf{x}) + \sum_{m=1}^{l(l+2)} [b_{lm}^+ \mathbf{F}_{lm}^+(\mathbf{x}) + b_{lm}^- \mathbf{F}_{lm}^-(\mathbf{x})] \right), \tag{4.5}$$

with $\mathbf{F}_{lm}^{\pm}$ being orthogonal kernels in the image space of $\mathbf{B}_l^{\pm}$ with $\int_{\mathbb{S}^3} \|\mathbf{F}_{lm}^{\pm}(\mathbf{x})\|^2 d\mathbf{x} = l(l + 2)$, and

$$a_{lm} = \frac{1}{l(l + 2)} \int_{\mathbb{S}^3} \mathbf{J}^T(\mathbf{x})\nabla S_{lm}(\mathbf{x})d\mathbf{x}, \quad b_{lm}^{\pm} = \frac{1}{l(l + 2)} \int_{\mathbb{S}^3} \mathbf{J}^T(\mathbf{x})\mathbf{F}_{lm}^{\pm}(\mathbf{x})d\mathbf{x}. \tag{4.6}$$

The covariance of the coefficients is $\operatorname{cov}(a_{lm}, b_{l'm'}^{\pm}) = 0$, $\operatorname{cov}(b_{lm}^{\mp}, b_{l'm'}^{\pm}) = 0$, and

$$\operatorname{cov}(a_{lm}, a_{l'm'}) = \delta_{ll'}\delta_{mm'}\frac{2\pi^2 a_l}{(l + 1)^2}, \quad \operatorname{cov}(b_{lm}^{\pm}, b_{l'm'}^{\pm}) = \delta_{ll'}\delta_{mm'}\frac{2\pi^2(b_l \pm c_l)}{l(l + 2)}, \tag{4.7}$$

for the cross-covariance given in Eq. (4.2).

5. Proofs

Proof of Lemma 2.1. For $\mathbf{x}^T\mathbf{x}_1 = 0$ and $\mathbf{y}^T\mathbf{y}_1 = 0$,

$$\mathbf{x}_1^T \mathbf{A}_l(\mathbf{x}, \mathbf{y})\mathbf{y}_1 = \mathrm{E}\mathbf{x}_1^T \nabla Z_l(\mathbf{x})(\nabla Z_l(\mathbf{y}))^T \mathbf{y}_1 = p'_{\alpha,l}(\mathbf{x}^T\mathbf{y})(\mathbf{x}_1^T\mathbf{y}_1) + p''_{\alpha,l}(\mathbf{x}^T\mathbf{y})(\mathbf{x}_1^T\mathbf{y})(\mathbf{x}^T\mathbf{y}_1). \tag{5.1}$$

As a result, for $1 \leq i \leq d - 1$ and $1 \leq j \leq d - 1$,

$$\mathbf{n}_\mathbf{x}^T \mathbf{A}_l(\mathbf{x}, \mathbf{y})\mathbf{n}_\mathbf{y} = zp'_{\alpha,l}(z) - (1 - z^2)p''_{\alpha,l}(z), \quad \mathbf{m}_i^T \mathbf{A}_l(\mathbf{x}, \mathbf{y})\mathbf{m}_j = p'_{\alpha,l}(z)\delta_{ij}, \quad \mathbf{m}_i^T \mathbf{A}_l(\mathbf{x}, \mathbf{y})\mathbf{n}_\mathbf{y} = 0. \tag{5.2}$$

Eq. (2.8) follows from the differential equation with $\alpha = d/2 - 1$,

$$(1 - z^2)p''_{\alpha,l}(z) - 2(\alpha + 1)zp'_{\alpha,l}(z) + \lambda_l p_{\alpha,l}(z) = 0. \tag{5.3}$$

$$\mathbf{n}_\mathbf{x}^T \mathbf{B}_l(\mathbf{x}, \mathbf{y})\mathbf{n}_\mathbf{y} = \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \mathbf{m}_i^T \nabla Z(\mathbf{x})\mathbf{m}_j^T \nabla Z(\mathbf{y})(\mathbf{x} \wedge \mathbf{n}_\mathbf{x} \wedge \mathbf{m}_i) \cdot (\mathbf{y} \wedge \mathbf{n}_\mathbf{y} \wedge \mathbf{m}_j) = (d - 1)p'_{\alpha,l}(z). \tag{5.4}$$

$$\begin{aligned}\mathbf{m}_i^T\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{m}_i =& \sum_{j=1}^{d-1}\mathbf{m}_j^T\nabla Z(\mathbf{x})\mathbf{m}_j^T\nabla Z(\mathbf{y})(\mathbf{x}\wedge\mathbf{m}_i\wedge\mathbf{m}_j)\cdot(\mathbf{y}\wedge\mathbf{m}_i\wedge\mathbf{m}_j)\\ &+\mathbf{n}_\mathbf{x}^T\nabla Z(\mathbf{x})\mathbf{n}_\mathbf{y}^T\nabla Z(\mathbf{y})(\mathbf{x}\wedge\mathbf{m}_i\wedge\mathbf{n}_\mathbf{x})\cdot(\mathbf{y}\wedge\mathbf{m}_i\wedge\mathbf{n}_\mathbf{y})\\ =&(d-1)zp'_{\alpha,l}(z)-(1-z^2)p''_{\alpha,l}(z)=\lambda_l p_{\alpha,l}(z)-zp'_{\alpha,l}(z).\end{aligned} \tag{5.5}$$

Also, $\mathbf{m}_i^T\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{m}_j=0$ for $i\neq j$, and $\mathbf{m}_i^T\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{n}_\mathbf{y}=\mathbf{n}_\mathbf{x}^T\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{m}_i=0$. □

Proof of Lemma 2.2. The image space of $\mathbf{A}_l$ is spanned by the curl-free flows $\nabla S_{lm}(\mathbf{x})$ with $1\le m\le \dim H_l$, because

$$\mathbf{A}_l(\mathbf{x},\mathbf{y})=\frac{\omega_d}{\dim H_l}\sum_{m=1}^{\dim H_l}\nabla S_{lm}(\mathbf{x})(\nabla S_{lm}(\mathbf{y}))^T. \tag{5.6}$$

The image space of $\mathbf{B}_l$ is divergence-free, because for any $\mathbf{y}_1\in\mathbb{R}^{d+1}$,

$$\nabla_\mathbf{x}\cdot\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{y}_1=\sum_{i=1}^{d}\sum_{j=1}^{d}\mathrm{E}(\mathbf{x}_i^T\nabla)(\mathbf{x}_j^T\nabla)Z_l(\mathbf{x})(\mathbf{x}\wedge\mathbf{x}_i\wedge\mathbf{x}_j)\cdot(\mathbf{y}\wedge\mathbf{y}_1\wedge\nabla Z_l(\mathbf{y}))=0, \tag{5.7}$$

in which $(\mathbf{x},\mathbf{x}_1,\dots,\mathbf{x}_d)$ is a positively oriented orthonormal basis of $\mathbb{R}^{d+1}$. The image space of $\mathbf{B}_l$ is orthogonal to the space of curl-free flows, because for any field $\phi(\cdot)$ on $\mathbb{S}^d$,

$$\int_{\mathbb{S}^d}\nabla\phi(\mathbf{x})\cdot\mathbf{B}(\mathbf{x},\mathbf{y})d\mathbf{x}=\int_{\mathbb{S}^d}[\nabla_\mathbf{x}\cdot(\phi(\mathbf{x})\mathbf{B}(\mathbf{x},\mathbf{y}))-\phi(\mathbf{x})\nabla_\mathbf{x}\cdot\mathbf{B}_l(\mathbf{x},\mathbf{y})]d\mathbf{x}=0. \tag{5.8}$$

So $(\mathbf{A}_l,\mathbf{B}_{l'})=0$ for any l and l'. By

$$\int_{\mathbb{S}^d}(\nabla S_{lm}(\mathbf{y}))^T\nabla S_{l'm'}(\mathbf{y})d\mathbf{y}=-\int_{\mathbb{S}^d}S_{lm}(\mathbf{y})\nabla^2 S_{l'm'}(\mathbf{y})d\mathbf{y}=\lambda_l\delta_{ll'}\delta_{mm'}, \tag{5.9}$$

we get

$$(\mathbf{A}_l,\mathbf{A}_{l'})=\lambda_l\delta_{ll'}\frac{\omega_d}{(\dim H_l)^2}\sum_{m=1}^{\dim H_l}(\nabla S_{lm}(\mathbf{x}))^T\nabla S_{lm}(\mathbf{x})=\frac{\lambda_l^2\delta_{ll'}}{\dim H_l}. \tag{5.10}$$

By Eq. (2.9), we can write

$$\mathbf{B}_l(\mathbf{x},\mathbf{y})=f_l(z)\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T+g_l(z)\sum_{i=1}^{d-1}\mathbf{m}_i\mathbf{m}_i^T, \tag{5.11}$$

where $f_l(z)=(d-1)p'_{\alpha,l}(z)$, $g_l(z)=\lambda_l p_{\alpha,l}(z)-zp'_{\alpha,l}(z)$, $z=\mathbf{x}^T\mathbf{y}$. It gives

$$(\mathbf{B}_l,\mathbf{B}_{l'})=\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}(f_l(z)f_{l'}(z)+(d-1)g_l(z)g_{l'}(z))(1-z^2)^\alpha dz. \tag{5.12}$$

Since $p'_{\alpha,l}(1)=\lambda_l/d$, $f_l(1)=g_l(1)$. By the parity of $p_{\alpha,l}$, $f_l(-1)=-g_l(-1)$. Therefore, for $l\ge 1$,

$$f_l(z)=a_l+b_lz+(1-z^2)\hat{f}_l(z),\quad g_l(z)=b_l+a_lz+(1-z^2)\hat{g}_l(z), \tag{5.13}$$

for numbers a_l and b_l, and polynomials $\hat{f}_l(z)$ and $\hat{g}_l(z)$ of order up to $l-2$. By the identity (Abramowitz and Stegun, 1965), $dp'_{\alpha,l}(z)=\lambda_l p_{\alpha+1,l-1}(z)$, we have that for $1\le l'<l$,

$$\int_{-1}^{1}f_l(z)\hat{f}_{l'}(z)(1-z^2)^{\alpha+1}dz=0,\quad \int_{-1}^{1}g_l(z)\hat{g}_{l'}(z)(1-z^2)^{\alpha+1}dz=0, \tag{5.14}$$

$$\int_{-1}^{1}(f_l(z)+(d-1)g_l(z)z)(1-z^2)^\alpha dz=(d-1)\int_{-1}^{1}(\lambda_l p_{\alpha,l}(z)z(1-z^2)^\alpha+p'_{\alpha,l}(z)(1-z^2)^{\alpha+1})dz=0, \tag{5.15}$$

$$\int_{-1}^{1}(f_l(z)z+(d-1)g_l(z))(1-z^2)^\alpha dz=(d-1)\int_{-1}^{1}\lambda_l p_{\alpha,l}(z)(1-z^2)^\alpha dz=0. \tag{5.16}$$

Hence $(\mathbf{B}_l,\mathbf{B}_{l'})=0$ for $1\le l'<l$. Eq. (2.11) follows from the identities (Abramowitz and Stegun, 1965),

$$\begin{aligned}&\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}(p_{\alpha,l}(z))^2(1-z^2)^\alpha dz=\frac{1}{\dim H_l},\quad \frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}(p'_{\alpha,l}(z))^2(1-z^2)^\alpha dz=\frac{2l+d-1}{d}\frac{\lambda_l}{\dim H_l},\\ &\frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}zp'_{\alpha,l}(z)p_{\alpha,l}(z)(1-z^2)^\alpha dz=\frac{l}{\dim H_l},\quad \frac{\omega_{d-1}}{\omega_d}\int_{-1}^{1}(zp'_{\alpha,l}(z))^2(1-z^2)^\alpha dz=\frac{2l-1}{d}\frac{\lambda_l}{\dim H_l}.\end{aligned} \tag{5.17}$$

□

Proof of Theorem 2.3. For given $\mathbf{x}$ and $\mathbf{y}$, let $Q_i=I-2\mathbf{m}_i\mathbf{m}_i^T$ for $1\le i\le d-1$. Since $Q_i\in\mathrm{O(d+1)}$, and $\mathbf{C}$ is isotropic and reflexive,

$$\mathbf{n}_\mathbf{x}^T\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{m}_i=(Q_i\mathbf{n}_\mathbf{x})^T\mathbf{C}(Q_i\mathbf{x},Q_i\mathbf{y})Q_i\mathbf{m}_i=-\mathbf{n}_\mathbf{x}^T\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{m}_i=0. \tag{5.18}$$

Similarly, $\mathbf{m}_i^T\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{n}_\mathbf{y} = 0$ and $\mathbf{m}_i^T\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{m}_j = 0$ for $i \neq j$. Therefore, $\mathbf{C}(\mathbf{x},\mathbf{y})$ can be written as

$$\mathbf{C}(\mathbf{x},\mathbf{y}) = f(\mathbf{x}^T\mathbf{y})\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T + g(\mathbf{x}^T\mathbf{y})\sum_{i=1}^{d-1}\mathbf{m}_i\mathbf{m}_i^T. \tag{5.19}$$

So $\{\mathbf{A}_l, \mathbf{B}_l\}_{l=1}^\infty$ is an orthogonal basis of the set of reflexive isotropic matrix kernels on $\mathbb{S}^d$, and the series in Eq. (2.12) converges in L_2 norm to $\mathbf{C}(\mathbf{x},\mathbf{y})$. Since $\mathbf{A}_l$ and $\mathbf{B}_l$ are positive semidefinite, and

$$\mathrm{tr}\mathbf{A}_l(\mathbf{x},\mathbf{x}) = \lambda_l, \quad \mathrm{tr}\mathbf{B}_l(\mathbf{x},\mathbf{x}) = (d-1)\lambda_l, \tag{5.20}$$

$a_l \geq 0$, $b_l \geq 0$, and $\sum_{l=1}^\infty a_l l^2$, $\sum_{l=1}^\infty b_l l^2$ converge. The series in Eq. (2.12) is absolutely summable, thus uniformly converges to $\mathbf{C}(\mathbf{x},\mathbf{y})$. □

Proof of Theorem 2.4. Eq. (2.13) follows from Eqs. (5.6) and (5.9). We have

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d}\mathbf{B}_l(\mathbf{x},\mathbf{z})\mathbf{B}_l(\mathbf{z},\mathbf{y})d\mathbf{z} = C\mathbf{B}_l(\mathbf{x},\mathbf{y}), \tag{5.21}$$

for a constant C, because the left-hand-side is a reflexive isotropic matrix kernel whose image space is in K_l, which must be a multiple of $\mathbf{B}_l$ by Theorem 2.3. Therefore, K_l is the eigenspace of $\mathbf{B}_l$, and

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d}\mathbf{B}_l(\mathbf{x},\mathbf{y})\mathbf{F}_{lm}(\mathbf{y})d\mathbf{y} = \frac{\mathrm{tr}\mathbf{B}_l(\mathbf{x},\mathbf{x})}{\dim K_l}\mathbf{F}_{lm}(\mathbf{x}) = \frac{(d-1)\lambda_l}{\dim K_l}\mathbf{F}_{lm}(\mathbf{x}). \tag{5.22}$$

By Eqs. (2.11) and (2.3),

$$\dim K_l = \frac{(\mathrm{tr}\mathbf{B}_l(\mathbf{x},\mathbf{x}))^2}{(\mathbf{B}_l,\mathbf{B}_l)} = \frac{(2l+d-1)(l+d-1)\Gamma(l+d-2)}{(l+1)\Gamma(d-1)\Gamma(l)}. \tag{5.23}$$

The KL expansion is a consequence of the Funk–Hecke formula. □

Proof of Theorem 3.1. The cross-covariance $\mathbf{C}(\mathbf{x},\mathbf{y})$ on $\mathbb{S}^2$ can be written as

$$\mathbf{C}(\mathbf{x},\mathbf{y}) = C_{11}(\mathbf{x},\mathbf{y})\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T + C_{12}(\mathbf{x},\mathbf{y})\mathbf{n}_\mathbf{x}\mathbf{m}^T + C_{21}(\mathbf{x},\mathbf{y})\mathbf{m}\mathbf{n}_\mathbf{y}^T + C_{22}(\mathbf{x},\mathbf{y})\mathbf{m}\mathbf{m}^T. \tag{5.24}$$

By the isotropy, $\mathbf{n}_\mathbf{x}^T\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{m} = \mathbf{n}_\mathbf{y}^T\mathbf{C}(\mathbf{y},\mathbf{x})\mathbf{m}$, so $C_{12}(\mathbf{x},\mathbf{y}) = C_{21}(\mathbf{x},\mathbf{y})$. Therefore,

$$\mathbf{C}(\mathbf{x},\mathbf{y}) = f(z)\mathbf{n}_\mathbf{x}\mathbf{n}_\mathbf{y}^T + g(z)\mathbf{m}\mathbf{m}^T + h(z)(\mathbf{n}_\mathbf{x}\mathbf{m}^T + \mathbf{m}\mathbf{n}_\mathbf{y}^T), \tag{5.25}$$

where $z = \mathbf{x}^T\mathbf{y}$. By the identities (Abramowitz and Stegun, 1965),

$$(s_i, s_j) + (t_i, t_j) = \delta_{ij}2i^2(i+1)^2/(2i+1), \quad (s_i, t_j) + (t_i, s_j) = 0, \quad i \geq 1,\ j \geq 1, \tag{5.26}$$

where $(f,g) = \int_{-1}^1 f(z)g(z)dz$, we have

$$(\mathbf{C}_i, \mathbf{C}_j) = (s_i - t_i, s_j - t_j) = (s_i, s_j) + (t_i, t_j) = 2(\mathbf{A}_i, \mathbf{A}_i)\delta_{ij}. \tag{5.27}$$

By Lemma 2.2 and the orthogonality between $\{\mathbf{A}_l, \mathbf{B}_l\}_{l=1}^\infty$ and $\{\mathbf{C}_l\}_{l=1}^\infty$, $\{\mathbf{A}_l, \mathbf{B}_l, \mathbf{C}_l\}_{l=1}^\infty$ is an orthogonal basis of the set of isotropic matrix kernels on $\mathbb{S}^2$. By Theorem 2.3, $a_l \geq 0$, $b_l \geq 0$, $\sum_{l=1}^\infty a_l l^2$ and $\sum_{l=1}^\infty b_l l^2$ converge. For $a \geq 0$ and $b \geq 0$, let

$$\mathbf{U}_{a,b}^\pm(\mathbf{x}) = \sqrt{a}\nabla Z_l(\mathbf{x}) \pm \sqrt{b}\mathbf{x} \times \nabla Z_l(\mathbf{x}). \tag{5.28}$$

The cross-covariance of $\mathbf{U}_{a,b}^\pm$ is

$$\mathbf{D}_{a,b}^\pm(\mathbf{x},\mathbf{y}) = a\mathbf{A}_l(\mathbf{x},\mathbf{y}) + b\mathbf{B}_l(\mathbf{x},\mathbf{y}) \pm \sqrt{ab}\mathbf{C}_l(\mathbf{x},\mathbf{y}), \tag{5.29}$$

which is positive semidefinite. If $\mathbf{C}$ is positive semidefinite,

$$(\mathbf{C}, \mathbf{D}_{b_l,a_l}^\pm) = 2(a_l b_l \pm c_l\sqrt{a_l b_l})(\mathbf{A}_l, \mathbf{A}_l) \geq 0, \tag{5.30}$$

which implies that $|c_l| \leq \sqrt{a_l b_l}$. On the other hand, if $|c_l| \leq \sqrt{a_l b_l}$, the series in Eq. (3.1) is positive semidefinite and absolutely summable, thus converges uniformly to $\mathbf{C}(\mathbf{x},\mathbf{y})$. □

Proof of Theorem 3.2. The covariances $\mathrm{cov}(a_{lm}, a_{l'm'})$ and $\mathrm{cov}(b_{lm}, b_{l'm'})$ follow Theorem 2.4.

$$\begin{aligned}\mathbf{C}_l(\mathbf{x},\mathbf{y}) &= \mathrm{E}(\nabla Z_l(\mathbf{x})(\mathbf{y} \times \nabla Z_l(\mathbf{y}))^T + \mathbf{x} \times \nabla Z_l(\mathbf{x})(\nabla Z_l(y))^T) \\ &= \frac{4\pi}{2l+1}\sum_{m=-l}^{l}[\nabla S_{lm}(\mathbf{x})(\mathbf{y} \times \nabla S_{lm}(\mathbf{y}))^T + (\mathbf{x} \times \nabla S_{lm}(\mathbf{x}))(\nabla S_{lm}(\mathbf{y}))^T].\end{aligned} \tag{5.31}$$

Therefore,

$$\int_{\mathbb{S}^d}\mathbf{C}_l(\mathbf{x},\mathbf{y})\nabla S_{lm}(\mathbf{y})d\mathbf{y} = \frac{4\pi\lambda_l}{2l+1}\mathbf{x} \times \nabla S_{lm}(\mathbf{x}), \tag{5.32}$$

which gives the covariance $\mathrm{cov}(a_{lm}, a_{l'm'})$ in Eq. (3.5). □

Proof of Lemma 4.1. Applying $(\mathbf{x} \wedge \mathbf{x}_1 \wedge \nabla_{\mathbf{x}}) \cdot (\mathbf{y} \wedge \mathbf{y}_1 \wedge \nabla_{\mathbf{y}})$ to the identity (Abramowitz and Stegun, 1965),

$$\frac{\dim H_l}{\omega_d} \int_{\mathbb{S}^d} p_{\alpha,l}(\mathbf{x}^T\mathbf{z}) p_{\alpha,l}(\mathbf{z}^T\mathbf{y}) d\mathbf{z} = p_{\alpha,l}(\mathbf{x}^T\mathbf{y}), \tag{5.33}$$

we get

$$\begin{aligned} &\frac{\dim H_l}{\omega_d} \int_{\mathbb{S}^d} p'_{\alpha,l}(\mathbf{x}^T\mathbf{z}) p'_{\alpha,l}(\mathbf{z}^T\mathbf{y}) (\mathbf{x} \wedge \mathbf{x}_1 \wedge \mathbf{z}) \cdot (\mathbf{y} \wedge \mathbf{y}_1 \wedge \mathbf{z}) d\mathbf{z} \\ &= (\mathbf{x} \wedge \mathbf{x}_1 \wedge \nabla_{\mathbf{x}}) \cdot (\mathbf{y} \wedge \mathbf{y}_1 \wedge \nabla_{\mathbf{y}}) p_{\alpha,l}(\mathbf{x}^T\mathbf{y}) = \mathbf{x}_1^T \mathbf{B}_l(\mathbf{x},\mathbf{y}) \mathbf{y}_1. \quad \square \end{aligned} \tag{5.34}$$

Proof of Theorem 4.2. For an isotropic random flow on $\mathbb{S}^3$, $\mathbf{n}_{\mathbf{x}}^T \mathbf{C}(\mathbf{x},\mathbf{y}) \mathbf{m}_i = \mathbf{m}_i^T \mathbf{C}(\mathbf{x},\mathbf{y}) \mathbf{n}_{\mathbf{y}} = 0$ for $i = 1, 2$. Let $Q = \mathbf{x}\mathbf{x}^T + \mathbf{n}_{\mathbf{x}}\mathbf{n}_{\mathbf{x}}^T + \mathbf{m}_1\mathbf{m}_2^T - \mathbf{m}_2\mathbf{m}_1^T$. Since $Q \in \mathrm{SO}(4)$,

$$\mathbf{m}_1^T \mathbf{C}(\mathbf{x},\mathbf{y}) \mathbf{m}_2 = (Q\mathbf{m}_1)^T \mathbf{C}(Q\mathbf{x}, Q\mathbf{y})(Q\mathbf{m}_2) = -\mathbf{m}_2^T \mathbf{C}(\mathbf{x},\mathbf{y}) \mathbf{m}_1. \tag{5.35}$$

The cross-covariance can be written as

$$\mathbf{C}(\mathbf{x},\mathbf{y}) = f(z)\mathbf{n}_{\mathbf{x}}\mathbf{n}_{\mathbf{y}}^T + g(z)(\mathbf{m}_1\mathbf{m}_1^T + \mathbf{m}_2\mathbf{m}_2^T) + h(z)(\mathbf{m}_2\mathbf{m}_1^T - \mathbf{m}_1\mathbf{m}_2^T), \tag{5.36}$$

where $z = \mathbf{x}^T\mathbf{y}$. Since $p'_{\alpha,l}(z) = (\lambda_l/d) p_{\alpha+1,l-1}(z)$, by Eq. (5.17) we have $(\mathbf{C}_l, \mathbf{C}_{l'}) = 2l(l+2)\delta_{ll'}$. Since $\{\mathbf{C}_l\}_{l=1}^{\infty}$ is orthogonal to $\{\mathbf{A}_l, \mathbf{B}_l\}_{l=1}^{\infty}$, combined with Lemma 2.2, we see that $\{\mathbf{A}_l, \mathbf{B}_l, \mathbf{C}_l\}_{l=1}^{\infty}$ is an orthogonal basis of the set of isotropic matrix kernels on $\mathbb{S}^3$. Notice that for any vectors $\mathbf{x}_1$ and $\mathbf{y}_1$ in $\mathbb{R}^4$,

$$\mathbf{x}_1^T \mathbf{C}_l(\mathbf{x},\mathbf{y}) \mathbf{y}_1 = (l+1) p'_{\alpha,l}(\mathbf{x}^T\mathbf{y}) (\mathbf{x} \wedge \mathbf{x}_1 \wedge \mathbf{y} \wedge \mathbf{y}_1)^*, \tag{5.37}$$

where the Hodge star is a linear mapping such that $(\mathbf{x} \wedge \mathbf{n}_{\mathbf{x}} \wedge \mathbf{m}_1 \wedge \mathbf{m}_2)^* = 1$. By Lemma 4.1,

$$\begin{aligned} &\int_{\mathbb{S}^3} \mathbf{x}_1^T \mathbf{C}_l(\mathbf{x},\mathbf{z}) \mathbf{C}_l(\mathbf{z},\mathbf{y}) \mathbf{y}_1 d\mathbf{z} \\ &= (l+1)^2 \int_{\mathbb{S}^3} p'_{\alpha,l}(\mathbf{x}^T\mathbf{z}) p'_{\alpha,l}(\mathbf{z}^T\mathbf{y}) (\mathbf{x} \wedge \mathbf{x}_1 \wedge \mathbf{z}) \cdot (\mathbf{z} \wedge \mathbf{y} \wedge \mathbf{y}_1) d\mathbf{z} = \omega_d \mathbf{x}_1^T \mathbf{B}_l(\mathbf{x},\mathbf{y}) \mathbf{y}_1. \end{aligned} \tag{5.38}$$

Since $\mathbf{C}_l$ is symmetric, its image space is the same as that of $\mathbf{B}_l$. By Eq. (2.13),

$$\int_{\mathbb{S}^3} \mathbf{B}_l(\mathbf{x},\mathbf{z}) \mathbf{B}_l(\mathbf{z},\mathbf{y}) d\mathbf{z} = \omega_d \mathbf{B}_l(\mathbf{x},\mathbf{y}), \quad \int_{\mathbb{S}^3} \mathbf{B}_l(\mathbf{x},\mathbf{z}) \mathbf{C}_l(\mathbf{z},\mathbf{y}) d\mathbf{z} = \omega_d \mathbf{C}_l(\mathbf{x},\mathbf{y}). \tag{5.39}$$

Let $\mathbf{B}_l^{\pm} = (\mathbf{B}_l \pm \mathbf{C}_l)/2$, then

$$\int_{\mathbb{S}^3} \mathbf{B}_l^{\pm}(\mathbf{x},\mathbf{z}) \mathbf{B}_l^{\pm}(\mathbf{z},\mathbf{y}) d\mathbf{z} = \omega_d \mathbf{B}_l^{\pm}(\mathbf{x},\mathbf{y}), \quad \int_{\mathbb{S}^3} \mathbf{B}_l^{\pm}(\mathbf{x},\mathbf{z}) \mathbf{B}_l^{\mp}(\mathbf{z},\mathbf{y}) d\mathbf{z} = 0. \tag{5.40}$$

Therefore, $\{\mathbf{A}_l, \mathbf{B}_l^+, \mathbf{B}_l^-\}_{l=1}^{\infty}$ is a positive semidefinite orthogonal basis of the set of isotropic matrix kernels on $\mathbb{S}^3$. The series in Eq. (4.2) is positive semidefinite if and only if $|c_l| \le b_l$. Given that $|c_l| \le b_l$, by the positive semidefiniteness of $\mathbf{B}_l^{\pm}$, the series is absolutely summable, thus converges uniformly. $\square$

6. Conclusion

In this paper, we derived the cross-covariance of isotropic random flows on the sphere $\mathbb{S}^d$ for $d \ge 1$. We also derived the KL expansion of the isotropic random flows. On $\mathbb{S}^d$ with $d \ge 4$, the curl-free part of the flow is uncorrelated with the divergence-free part of the Helmholtz–Hodge decomposition. On $\mathbb{S}^2$, the two parts can be correlated. On $\mathbb{S}^3$, the divergence-free part can be further decomposed into two isotropic flows. In subsequent works, we will study random flows on other symmetric spaces.

Data availability

No data was used for the research described in the article.

Acknowledgment

The author was supported by U.S. National Science Foundation Grant DMS-2008154.

References

Abramowitz, M., Stegun, I. (Eds.), 1965. Handbook of Mathematical Functions. Dover, New York.
Fan, M., Paul, D., Lee, T.C.M., Matsuo, T., 2018. Modeling tangential vector fields on a sphere. J. Amer. Statist. Assoc. 113 (524), 1625–1636.
Hutchinson, M., Terenin, A., Borovitskiy, V., et al., 2021. Vector-valued Gaussian processes on Riemannian manifolds via gauge independent projected kernels. Adv. Neural Inf. Process. Syst. 34, 17160–17169.
Ito, S., 1951. On the canonical form of turbulence. Nagoya Math. J. 2, 83–92.
Ito, K., 1956. Isotropic random current. In: Berkeley Symposium on Math. Stat. and Probab.. vol. 3.2, pp. 125–132.

Lang, A., Schwab, C., 2015. Isotropic Gaussian random fields on the sphere: Regularity, fast simulation and stochastic partial differential equations. Ann. Appl. Probab. 25 (6), 3047–3094.
Ma, C., Malyarenko, A., 2020. Time-varying isotropic vector random fields on compact two-point homogeneous spaces. J. Theor. Probab. 33, 319–339.
Marinucci, D., Peccati, G., 2011. Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications 389. Cambridge University Press.
Wong, E., Zakai, M., 1989. Spectral representation of isotropic random currents. Sémin. Probab. Strasbourg 23, 503–526.