SMOOTH AND ANALYTIC ACTIONS OF SL$(n, \mathbf{R})$ AND SL$(n, \mathbf{Z})$ ON CLOSED n-DIMENSIONAL MANIFOLDS

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Abstract. The main theorem is a classification of smooth actions of $\mathrm{SL}(n, \mathbf{R})$, $n \geq 3$, or connected groups locally isomorphic to it, on closed n-manifolds, extending a theorem of Uchida [33]. We also construct new exotic actions of $\mathrm{SL}(n, \mathbf{Z})$ on the n-torus and connected sums of n-tori, and we formulate a conjectural classification of actions of lattices in $\mathrm{SL}(n, \mathbf{R})$ on closed n-manifolds. We prove some related results about invariant rigid geometric structures for $\mathrm{SL}(n, \mathbf{R})$-actions.

In memory of Fuichi Uchida (1938–2021)

1. Introduction

1.1. **Classification of $\mathrm{SL}(n, \mathbf{R})$-actions.** Any smooth—even continuous,—faithful action of $\mathrm{SL}(n, \mathbf{R})$ on an $(n-1)$-dimensional manifold is the transitive action on $\mathbf{S}^{n-1}$, and the only other nontrivial action is the quotient action of $\mathrm{PSL}(n, \mathbf{R})$ on $\mathbf{RP}^{n-1}$. In 1979 F. Uchida constructed an infinite family of real-analytic actions of $\mathrm{SL}(n, \mathbf{R})$ on $\mathbf{S}^n$ and proved his construction yields all of them [33]. Previously, C.R. Schneider classified all C^ω actions of $\mathrm{SL}(2, \mathbf{R})$ on closed surfaces and $\mathbf{R}^2$ [29], see also [31]. A key role is played in both proofs by the linearizability theorem for real-analytic actions of semisimple Lie groups on $(\mathbf{R}^n, 0)$ due to Guillemin–Sternberg [16] and Kushnirenko [21]. This theorem was partially improved to C^k linearizability of C^k actions of $\mathrm{SL}(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ for $k \geq 1$ and $n \geq 2$ by Cairns–Ghys [6]. Relying partly on this result, we classify smooth actions of G on closed n-manifolds, where G is connected and locally isomorphic to $\mathrm{SL}(n, \mathbf{R})$ and $n \geq 3$.

The actions are of two types—aside from a few exceptional transitive actions in dimensions 3 and 4—, depending on the existence of G-fixed points. Let $Q < \mathrm{SL}(n, \mathbf{R})$ be the stablizer of a line in the standard representation on $\mathbf{R}^n$. The actions without fixed points are induced from Q or Q^0-actions on S^1, yielding circle bundles over $\mathbf{RP}^{n-1}$ or $\mathbf{S}^{n-1}$. These are analogous to Schneider's actions on $\mathbf{T}^2$ or $\mathbf{K}^2$ for $n = 2$. The actions with G-fixed points are actions on $\mathbf{S}^n$ or $\mathbf{RP}^n$, all arising from the smooth version of Uchida's

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construction. See constructions I and II in Section 3.1 and the classification theorems 4.2 and 4.3 below. Although the smooth linearization theorem of Cairns–Ghys may permit a classification of smooth $\mathrm{SL}(2, \mathbf{R})$-actions on surfaces, our classification will not be valid as there is another family of actions in the case $n = 2$.

A consequence of Theorems 4.2 and 4.3 is that non-transitive real-analytic actions can be parametrized with real-analytic vector fields on $\mathbf{S}^1$, which in turn are given by finitely-many real and discrete parameters thanks to [17], along with some finite additional data; see Corollary 5.2 below. This constitutes a *smooth classification*, in the set-theoretic sense (see [28]), of analytic $\mathrm{SL}(n, \mathbf{R})$-actions on closed n-manifolds, up to analytic conjugacy.

1.2. **Motivation from the Zimmer Program.** An important motivation for our classification are Zimmer's conjectures on actions by semisimple Lie groups G, with no $\mathbf{R}$-rank-one local factors, and their lattices on low-dimensional closed manifolds. See for example [40, 11, 12]. The lowest possible dimension for a nontrivial action of such a lattice should be the minimal dimension $\alpha(G)$ of G/Q where Q is a maximal parabolic. For non-isometric, volume-preserving actions, the conjectured minimal dimension is the minimal dimension $\rho(G)$ of a locally faithful linear representation of G. In general, the bound $\alpha(G) \leq \rho(G) - 1$ can have a significant gap; for $G = \mathrm{SL}(n, \mathbf{R})$, they are equal—that is $\alpha(G) = n - 1$ while $\rho(G) = n$. For lattices in $\mathrm{SL}(n, \mathbf{R})$, $n \geq 3$, in joint work with A. Brown and S. Hurtado, the first author proved both conjectured bounds [3, 4, 5]. They moreover proved dimension bounds for actions of lattices in many other higher-rank simple Lie groups; their results are sharp for lattices in $\mathrm{SL}(n, \mathbf{R})$, $n \geq 3$, and $\mathrm{Sp}(2n, \mathbf{R})$, $n \geq 2$. These results resolved a major portion of Zimmer's most famous conjecture.

Zimmer's Program asks more generally to what extent actions of higher-rank semisimple Lie groups and their lattices on closed manifolds arise from algebraic constructions. The basic building blocks of such constructions are actions on G/H where H is a closed, cocompact subgroup, or actions of a lattice Γ on N/Λ, where G acts by automorphisms of a nilpotent group N and Γ normalizes a cocompact lattice Λ. Brown–Rodriguez-Hertz–Wang have announced a proof that any infinite action of a lattice $\Gamma < \mathrm{SL}(n, \mathbf{R})$ on a closed $(n-1)$-manifold, where $n \geq 3$, extends to the standard action of $\mathrm{SL}(n, \mathbf{R})$ on $\mathbf{S}^{n-1}$ or $\mathbf{RP}^{n-1}$.

We propose to consider the above question for actions of $\mathrm{SL}(n, \mathbf{R})$ and its lattices on closed n-manifolds, for $n \geq 3$. That this case is central to further progress is already emphasized in [12]. Theorems 4.2 and 4.3 fully describe the Γ-actions that extend to $\mathrm{SL}(n, \mathbf{R})$. The well-known action of the second type above, which does not extend to $\mathrm{SL}(n, \mathbf{R})$, is that of $\mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n \cong \mathbf{R}^n/\mathbf{Z}^n$. In 1996 Katok–Lewis famously constructed exotic $\mathrm{SL}(n, \mathbf{Z})$-actions on $\mathbf{T}^n$ in which the fixed point corresponding to 0 is blown up. They also show that the weight determining the action on the normal bundle of the blow-up can be freely chosen and that one choice gives a volume-preserving exotic action [18].

We construct new exotic actions of $\mathrm{SL}(n,\mathbf{Z})$, and its finite-index subgroups, by gluing in "exotic disks" to $\mathbf{T}^n$ at 0 or along periodic orbits, and by forming connected sums of n-tori along "exotic tubes." We conjecture the following classification of actions of lattices in $\mathrm{SL}(n,\mathbf{R})$ on closed n-manifolds, for $n \geq 3$. See Section 3.2.2 below for the relevant definitions.

Conjecture 3.6 *Let $\Gamma < \mathrm{SL}(n,\mathbf{R})$ be a lattice and M a compact manifold of dimension n. Then any action $\rho : \Gamma \to \mathrm{Diff}(M)$ either*

(1) *extends to an action of $\mathrm{SL}(n,\mathbf{R})$ or $\widetilde{\mathrm{SL}(n,\mathbf{R})}$,*
(2) *factors through a finite quotient of Γ, or*
(3) *is an action built from tori, G-tubes, G-disks, blow-ups, and two-sided blow-ups, with Γ a finite-index subgroup of $\mathrm{SL}(n,\mathbf{Z})$.*

None of the actions in (1) and few of the actions in (3) are volume-preserving (Proposition 3.7). Conjecture 3.8 below asserts that volume-preserving actions of Γ as above are finite or are actions of finite-index subgroups of $\mathrm{SL}(n,\mathbf{Z})$ built from tori, blow-ups, and two-sided blow-ups with weight n on the normal bundle, as in [18].

1.3. **Invariant rigid geometric structures.** Largely inspired by Zimmer's results and conjectures, Gromov, together with D'Ambra, proposed a program to investigate to what extent actions of "large"—for example, noncompact—Lie groups on closed manifolds preserving a rigid geometric structure arise from algebraic constructions (see [15, 8]). Benveniste–Fisher proved that Katok–Lewis' actions do not preserve any rigid geometric structure of algebraic type in the sense of Gromov [2]. We prove:

Theorem 6.7. *Let G be locally isomorphic to $\mathrm{SL}(n,\mathbf{R})$, acting smoothly on a compact n-manifold M, preserving a projective structure $[\nabla]$. Then $(M,[\nabla])$ is equivalent to*

- *$\mathbf{S}^n$ or $\mathbf{RP}^n$ with the standard projective structure; or*
- *a Hopf manifold, diffeomorphic to a flat circle bundle over $\mathbf{RP}^{n-1}$ with either trivial or $\mathbf{Z}_2$ monodromy.*

On the other hand, we show in Proposition 6.1 that all $\mathrm{SL}(n,\mathbf{R})$-actions on closed n-manifolds are 2-rigid in the sense of Gromov.

Pecastaing proved in [26] that if a uniform lattice in a simple Lie group G of $\mathbf{R}$-rank $\geq n$ admits an infinite action by projective transformations of a closed $(n-1)$-manifold, then G is locally isomorphic to $\mathrm{SL}(n,\mathbf{R})$, and Γ acts by the restriction of the standard action on $\mathbf{S}^{n-1}$ or $\mathbf{RP}^{n-1}$. Two interesting questions that remain are:

Question 1.1. *Are the projective actions identified in Theorem 6.7 the only smooth actions of $\mathrm{SL}(n,\mathbf{R})$ on a closed n-manifold preserving a rigid geometric structure of algebraic type?*

Question 1.2. *Given a non-affine, projective action of* $\mathrm{SL}(n, \mathbf{Z})$ *on a closed* n*-manifold, does it always extend to* $\mathrm{SL}(n, \mathbf{R})$*?*

1.4. **Other simple Lie groups.** This work might be considered a special case of a more general problem:

Problem 1.3. *For* G *a simple Lie group of noncompact type, classify the smooth* G*-actions on compact manifolds of dimension* $\alpha(G)+1$ *up to smooth conjugacy.*

As above, $\alpha(G)$ is the minimal codimension of a maximal parabolic subgroup of G. This article concerns only groups locally isomorphic $\mathrm{SL}(n, \mathbf{R})$, $n \geq 3$, because this family is central in current research on the Zimmer Program, and because the complexity of actions in Problem 1.3 depends on the local isomorphism type of G.

The only complete classification not mentioned so far is due to Uchida for $G \cong \mathrm{SL}(n, \mathbf{C})$ [35]. In that case, there are no faithful G-representations in dimension $2n - 1 = \alpha(G) + 1$. The actions in Uchida's classification correspondingly have no global fixed points and are as in our Construction I below; these are induced from actions of the maximal parabolic Q or Q^0 on S^1.

We do not yet have a general conjectural picture for all G. Uchida has numerous results for other simple groups [34, 36, 37, 38], though none of these papers contains a complete classification. The only case in which we expect the classification to be more or less analogous to the one presented here is for $G = \mathrm{Sp}(2n, \mathbf{R})$, $n \geq 2$. An interesting case is $G = \mathrm{SO}(p, q)$, which has a faithful representation in dimension $p + q = \alpha(G) + 2$. In the projectivization, the stabilizers in open orbits are reductive, in contrast to what occurs in our case (see Theorem 2.1). The other family of actions obtained by Schneider [29] referred to in Subsection 1.1 arises from the isomorphism $\mathrm{PSL}(2, \mathbf{R}) \cong \mathrm{SO}^0(1, 2)$.

2. Linearization and classification of orbit types

A celebrated result of Guillemin–Sternberg [16] and Kushnirenko [21] states that a real-analytic action of a semisimple Lie group G on a real-analytic manifold M is linearizable near any fixed point $p \in M$. Linearization means that there is a diffeomorphism Φ from a neighborhood U of p to a neighborhood V of $0 \in T_pM$ such that for all $g \in G$, the germ of g at p equals the germ of $\Phi^{-1} \circ D_pg \circ \Phi$. Alternatively, for all vector fields X arising from the G-action, Φ_*X is the linear vector field D_pX on $V \subset T_pM$.

Uchida's classification of analytic $\mathrm{SL}(n, \mathbf{R})$-actions on $\mathbf{S}^n$ for $n \geq 3$ relies on this analytic linearization result. Our improvement to smooth actions is enabled by the smooth linearization result of Cairns–Ghys [6] for $\mathrm{SL}(n, \mathbf{R})$-actions on $\mathbf{R}^n$ fixing 0. Also very useful for our arguments is a classification of orbit types up to dimension n. We recall their results in this section, together with selected proofs.

2.1. **Classification of orbit types.** The following smooth orbit classification of [6] will play a key role in the sequel.

Theorem 2.1 (Cairns–Ghys [6] Thm 3.5)**.** *Let G be connected and locally isomorphic to* $\mathrm{SL}(n, \mathbf{R})$ *with $n \geq 3$, and assume G acts continuously on a topological manifold M. For any $x \in M$, any orbit $G.x$ of dimension $\leq n$ is equivariantly homeomorphic to one of the following:*

(1) *a point;*
(2) $\mathbf{S}^{n-1}$ *or* $\mathbf{RP}^{n-1}$ *with the projective action;*
(3) $\mathbf{R}^n \backslash \{0\}$ *with the restricted linear action or* $(\mathbf{R}^n \backslash \{0\})/\Lambda$*, for Λ a discrete subgroup of the group of scalars* $\mathbf{R}^*$*;*
(4) *one of the following closed, exceptional orbits, or a finite cover:*
   - *For $n = 3$, $\mathcal{F}^3_{1,2}$, the variety of complete flags in* $\mathbf{R}^3$
   - *For $n = 4$, $Gr(2, 4) = \mathcal{F}^4_2$, the Grassmannian of 2-planes in* $\mathbf{R}^4$

Remark 2.2. Some details:

- The actions in (2) are faithful for $\widetilde{\mathrm{SL}}(n, \mathbf{R})$ and $\mathrm{PSL}(n, \mathbf{R})$, respectively, while $\mathrm{SL}(n, \mathbf{R})$ does not act faithfully on any $(n-1)$-dimensional manifold.
- Similarly, the actions in (3) are faithful for $\mathrm{SL}(n, \mathbf{R})$ and $\mathrm{PSL}(n, \mathbf{R})$, respectively, while $\widetilde{\mathrm{SL}}(n, \mathbf{R})$ does not have a faithful n-dimensional representation.
- The fundamental group of $\mathcal{F}^3_{1,2}$ is the quaternion group Q_8. The universal cover is $\mathbf{S}^3$, on which $\widetilde{\mathrm{SL}}(3, \mathbf{R})$ acts faithfully.
- The fundamental group of $\mathrm{Gr}(2, 4)$ is $\mathbf{Z}_2$ (see, *e.g.*, [22]). The universal cover is $S^2 \times S^2$, which can be identified with the space of oriented 2-planes in $\mathbf{R}^4$, on which $\mathrm{SL}(4, \mathbf{R})$ acts faithfully.

To correct some oversights and provide additional details, we present the proof here, more or less following the arguments of [6].

Proof. An orbit $\mathcal{O}_x = G.x$ is a homogeneous space of G, identified with G/G_x, for $G_x \leq G$ closed; thus the orbit is smooth with smooth G-action. A maximal compact subgroup K is locally isomorphic to $\mathrm{SO}(n)$, with dimension $n(n-1)/2$, and preserves a Riemannian metric on $\mathcal{O}_x$. By [20, Thm II.3.1], the isometry group of an m-dimensional Riemannian manifold has dimension at most $m(m+1)/2$, with equality if and only if it is $\mathbf{S}^m$ or $\mathbf{RP}^m$ with the standard $\mathrm{SO}(m+1)$- or $\mathrm{PO}(m+1)$-action, respectively. Thus any orbit of dimension less than n is as in parts (1) and (2) of the theorem; in particular, all such orbits are closed.

Now assume that $\mathcal{O}_x$ is n-dimensional and consider $\mathfrak{g}_x \otimes \mathbf{C}$. There is no reductive subalgebra of $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(n, \mathbf{C})$ of complex codimension less than or equal n. Indeed, a suitable Cartan decomposition $\mathfrak{k}_x + \mathfrak{p}_x$ of the reductive subalgebra would align with that of $\mathfrak{g}$, so the compact form $\mathfrak{k}_x + i\mathfrak{p}_x$ would be contained in that of $\mathfrak{g}$. By the same dimension arguments as in the previous paragraph, there is no closed subgroup of $\mathrm{SU}(n)$ of codimension less than or equal n, assuming $n > 2$. Thus there is no compact subalgebra of $\mathfrak{su}(n)$ of codimension less than or equal n for $n > 2$—a contradiction.

Assuming now that $\mathfrak{g}_x \otimes \mathbf{C}$ is not reductive, the isotropy representation on $V = (\mathfrak{g}/\mathfrak{g}_x)_{\mathbf{C}}$ will be reducible. Assume there is an invariant p-dimensional complex subspace, for $0 < p < n$. The stabilizer in $\mathfrak{sl}(n, \mathbf{C})$ of a p-dimensional subspace has codimension $p(n-p)$, so $n \geq p(n-p)$. Then $p = 1$ or $n - 1$ and $\mathfrak{g}_x \otimes \mathbf{C}$ has codimension 1 in the subspace stabilizer, or $n = 4$ and $p = 2$.

In the case $n = 4$ and $p = 2$, our dimension assumptions imply that $\mathfrak{g}_x \otimes \mathbf{C}$ equals the full stabilizer in $\mathfrak{sl}(n, \mathbf{C})$ of a 2-dimensional complex subspace $W \subset V \cong \mathbf{C}^4$. Because $\mathfrak{g}_x \otimes \mathbf{C}$ does not preserve any proper subspace of W, the intersection $W_0 = W \cap \overline{W}$ is real-even-dimensional and $\mathfrak{g}_x$-invariant. Since W is assumed to be a proper subspace, $W_0 \neq V_0$.

If dim $W_0 = 2$, then $\mathfrak{g}_x$ is contained in the stabilizer of a 2-dimensional subspace of $\mathbf{R}^4$. As this stabilizer has real codimension 4, it is equal to $\mathfrak{g}_x$. The orbit $\mathcal{O}_x$ is the real Grassmannian $\mathrm{Gr}(2, 4)$ or a finite covering space.

The remaining possibility is that $W_0 = 0$. This means $V = W \oplus \overline{W}$. Given $v_0 \in V_0$, there is a unique $w \in W$ such that

$$v_0 = \frac{1}{2}(w + \bar{w})$$

Then

$$J(v_0) = \frac{i}{2}(\bar{w} - w)$$

defines a $\mathfrak{g}_x$-equivariant automorphism of V_0 with $J^2 = -\mathrm{Id}$. Now $\mathfrak{g}_x$ is contained in a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$. The codimension of $\mathfrak{sl}(2, \mathbf{C})$ in $\mathfrak{sl}(4, \mathbf{R})$ is 9, so this case does not arise under our assumptions.

Next consider $p = 1$ and $n \geq 3$. For the invariant complex line W, if $W \cap \overline{W} = 0$, then $\mathfrak{g}_x \otimes \mathbf{C}$ preserves a flag $W \subset U = W \oplus \overline{W} \subset \mathbf{C}^n$. The stabilizer of such a flag has codimension $2n - 3$, which is less than or equal n only for $n = 3$. In the case $n = 3$, this flag is a full flag, and the stabilizer has codimension 3, so it equals $\mathfrak{g}_x \otimes \mathbf{C}$. The intersection $U_0 = U \cap \overline{U}$ is real-two-dimensional and $\mathfrak{g}_x$-invariant. As in the previous paragraph, the $\mathfrak{g}_x \otimes \mathbf{C}$-invariant decomposition $U = W \oplus \overline{W}$ defines an $\mathfrak{g}_x$-invariant complex structure on U_0. Now $\mathfrak{g}_x$ is contained in the stabilizer of a 2-plane U_0 in $\mathbf{R}^3$ together with a complex structure on U_0. The codimension of this stabilizer in $\mathfrak{sl}(3, \mathbf{R})$ is 4. Thus this case does not arise under our assumptions.

Now we can assume $W_0 = W \cap \overline{W}$ is a real line, and $\mathfrak{g}_x$ has codimension 1 in its stabilizer. As in the previous paragraph, $\mathfrak{g}_x$ must be irreducible on V_0/W_0 unless $n = 3$, in which case, if $\mathfrak{g}_x$ is reducible on this quotient, it is the Borel subalgebra of $\mathfrak{sl}(3, \mathbf{R})$. This case corresponds to $\mathcal{O}_x \cong \mathcal{F}^3_{1,2}$ or a finite cover.

Now we assume $\mathfrak{g}_x$ is irreducible on V_0/W_0, and it is a codimension-one subalgebra of the stabilizer $\mathfrak{q}$ of a line in $\mathbf{R}^n$. The image of Q on V_0/W_0 is $\mathrm{GL}(n-1, \mathbf{R})$, and $Q \cong \mathrm{GL}(n-1, \mathbf{R}) \ltimes \mathbf{R}^{n-1}$. If the intersection $G_x \cap \mathbf{R}^{n-1}$ were a proper subspace, then G_x would project onto $\mathrm{GL}(n-1, \mathbf{R})$

by dimension considerations. But this would not be consistent with G_x being a subgroup, because $\mathrm{GL}(n-1,\mathbf{R})$ is irreducible on $\mathbf{R}^{n-1}$. Therefore, G_x has full intersection with this kernel and projects onto a closed, codimension-one, irreducible subgroup of $\mathrm{GL}(n-1,\mathbf{R})$. Our earlier arguments show that $\mathrm{SL}(n-1,\mathbf{R})$ has no closed, codimension-one subgroup for $n \geq 4$, while for $n = 3$, the unique such subgroup is reducible. Finally, we conclude that the projection of $\mathfrak{g}_x$ to $\mathfrak{gl}(n-1,\mathbf{R})$ equals $\mathfrak{sl}(n-1,\mathbf{R})$, and $\mathfrak{g}_x \cong \mathfrak{sl}(n-1,\mathbf{R}) \ltimes \mathbf{R}^{n-1} \lhd \mathfrak{q}$. Let E^0 be the connected, normal subgroup of Q isomorphic to $\mathrm{SL}(n-1,\mathbf{R}) \ltimes \mathbf{R}^{n-1}$, and $E_\Lambda \lhd Q$ the inverse image of $\Lambda < Q/E^0 \cong \mathbf{R}^*$. The possibilities in (3) correspond to $G_x = E^0$ or E_Λ, respectively, for Λ a nontrivial discrete subroup of $\mathbf{R}^*$.

In the cases with $p = n-1$ and $n \geq 3$ the outer automorphism $g \mapsto (g^{-1})^t$ gives an equivariant diffeomorphism from $\mathcal{O}_x$ to one of the orbits in (3) or (4). □

2.2. **Fixed Points and linearization.** Let $K < G$ be a maximal compact subgroup. There is a K-invariant Riemannian metric on M—it can be obtained by averaging any Riemannian metric on M over K with respect to the Haar measure. We will denote this metric κ. The K-action near a K-fixed point is linearizable, via the exponential map of κ.

Proposition 2.3. *Let G be connected and locally isomorphic to $SL(n,\mathbf{R})$. For a nontrivial smooth action of G on a connected manifold M of dimension n, the fixed set $Fix(G)$ is discrete. In particular, if M is compact, then $Fix(G)$ is finite.*

Proof. Let $x \in M$ be a G-fixed point. First suppose the isotropy representation of G at x is trivial. Then via the exponential map of κ, we deduce that the K-action is trivial in a neighborhood of x. Then K is trivial on all of M, and so is G.

Now assume the isotropy representation of G at x is nontrivial; then it is irreducible and factors through $\mathrm{SL}(n,\mathbf{R})$. The isotropy of K, which is isogeneous to $\mathrm{SO}(n)$, is also locally faithful and thus irreducible; in particular, there are no nontrivial fixed vectors. Now by linearization of the K-action on a neighborhood, say, U, of x, there are no K-fixed points other than x in U. In particular, there are no G-fixed points other than x in U. □

We recall here the smooth linearization theorem for $\mathrm{SL}(n,\mathbf{R})$ of [6], exactly as stated there.

Theorem 2.4 (Cairns–Ghys [6] Thm 1.1)**.** *For all $n > 1$ and for all $k = 1,\ldots,\infty$, every C^k-action of* $\mathrm{SL}(n,\mathbf{R})$ *on* $(\mathbf{R}^n, 0)$ *is C^k-linearizable.*

There are two nontrivial $\mathrm{SL}(n,\mathbf{R})$-representations on $\mathbf{R}^n$, the standard one, which we will denote ρ, and $\rho^*(g) = \rho((g^{-1})^t)$. Under ρ, there is a Q-invariant line, pointwise fixed by E^0, where Q and E^0 are the subgroups introduced in the proof of Theorem 2.1. Under ρ^*, there is no Q-invariant line when $n \geq 3$; rather, Q acts irreducibly on an invariant $(n-1)$-dimensional subspace.

3. Constructions of smooth actions

In this section we construct all smooth, non-transitive, nontrivial $\mathrm{SL}(n, \mathbf{R})$-actions on n-dimensional compact manifolds for $n \geq 3$. The proof that the list is complete will be given in the next section. We will also give a conjecturally complete description of smooth actions of lattices $\Gamma < \mathrm{SL}(n, \mathbf{R})$ on compact n-dimensional manifolds.

Throughout this section, $G = \mathrm{SL}(n, \mathbf{R})$ and $n \geq 3$. The actions will be faithful or will factor through faithful actions of $\mathrm{PSL}(n, \mathbf{R})$. Recall that a maximal parabolic subgroup $Q < G$ is isomorphic to $\mathrm{GL}(n-1, \mathbf{R}) \ltimes \mathbf{R}^{n-1}$. Let $L \cong GL(n-1, \mathbf{R})$ be a Levi subgroup of Q and $\pi : Q \to L$ the projection. Let $\{a^t\}$ be the one-parameter subgroup generating the identity component of the center of L. Let $C \cong O(n-1)$ be a maximal compact subgroup of L. Let $\sigma \in C$ project to -1 in $\mathbf{R}^* \cong L/[L, L]$. Let Q^0 and L^0 be the identity components of Q and L, respectively; note that $\pi(Q^0) = L^0$. Define homomorphisms

$$\begin{aligned} \nu_0 : Q^0 \to \mathbf{R} \qquad & q \mapsto \ln(\det(\pi(q))) \\ \nu : Q \to \mathbf{R}^* \cong \mathbf{Z}_2 \times \mathbf{R} \qquad & q \mapsto (\mathrm{sgn}(\det(\pi(q))), \ln|\det(\pi(q))|) \end{aligned}$$

The kernel of ν is E_Λ with $\Lambda = \{\pm 1\}$, which will henceforth be denoted E; its identity component E^0 is the kernel of ν_0.

3.1. Constructions of G-actions on closed n-manifolds.

There are two families of non-transitive actions of $\mathrm{SL}(n, \mathbf{R})$ on compact n-manifolds. The first family have no G-fixed points and are circle bundles over $\mathbf{RP}^{n-1}$ or $\mathbf{S}^{n-1}$. Actions in the second family have two or one G-fixed points and are diffeomorphic to $\mathbf{S}^n$ or $\mathbf{RP}^n$, respectively.

3.1.1. *Construction I: without global fixed points.* Let Σ^0 be a smooth circle.

Lemma 3.1. *If τ is a nontrivial smooth involution of Σ^0, then it has 0 or 2 fixed points.*

Proof. This is a fact of topology, but since our action is smooth, we will use the existence of a τ-invariant metric. The fixed set of τ is closed and equals Σ^0 or is finite. Assuming τ is not trivial, the differential at these fixed points is $-\mathrm{Id}_1$. Then if $\mathrm{Fix}(\tau)$ is nonempty, the complement has exactly two connected components, and $\mathrm{Fix}(\tau)$ comprises two points. □

Let $\{\psi_X^t\}$ be a smooth flow on Σ^0, generated by a vector field X. Let τ be a smooth involution on Σ^0 commuting with X, or the involution of $\Sigma^0 \times \{1, -1\}$ exchanging the two components. Let $\Sigma = \Sigma^0$ in the first case, and $\Sigma^0 \times \{-1, 1\}$ in the second. In the second case, extend X to Σ by pushing forward via τ to the other component.

Define an action of $\mathbf{R}^*$ on Σ by

$$t \mapsto \psi_X^{\ln|t|} \circ \tau^{(1-\mathrm{sgn}(t))/2}$$

Via $\nu : Q \to \mathbf{R}^*$, this lifts to an action of Q on Σ, which we will denote $\mu_{X,\tau}$. Then

$$M = G \times_Q \Sigma$$

is a closed manifold with smooth G-action.

Construction I with $\Sigma = \Sigma^0$, $X = 0$, and $\tau = \mathrm{Id}$, gives the standard action on $\mathbf{RP}^{n-1}$ product the trivial action on $\mathbf{S}^1$, while $\Sigma = \Sigma^0 \times \{-1, 1\}$, $X = 0$, and $\tau|_{\Sigma^0} = \mathrm{Id}$ gives the standard action on $\mathbf{S}^{n-1}$ product with the trivial action on $\mathbf{S}^1$. If $\tau|_{\Sigma^0} = \mathrm{Id}$ and X is a constant nonvanishing vector field, then M is homogeneous and equivalent to E_Λ as in Theorem 2.1 (3), for Λ a lattice in $\mathbf{R}^*$. These are called Hopf manifolds and will be significant in Theorem 6.7 below.

3.1.2. *Construction II: with global fixed points.* Next we construct SL(n, $\mathbf{R}$)-actions on $\mathbf{S}^n$. These are the same as those constructed by Uchida in [33, Sec 2], except they are allowed to be only smooth.

Let $\Sigma_+ = [-1, 1]$. Let X be a smooth vector field on Σ_+ vanishing at -1 and 1, such that $D_{-1}X = 1 = D_1X$. Notice that X is nonvanishing on a nonempty open interval with -1 as endpoint, and similarly for 1; moreover, X has at least one zero in $(-1, 1)$. Concretely, there is $z_- > -1$ the minimum of the zero set of X in $(-1, 1)$ and $z_+ < 1$ the maximum of the zero set of X in $(-1, 1)$. It could be that $z_- = z_+$.

Define a Q^0-action on $(-1, 1)$ by letting $\{a^t\}$ act by the flow $\{\psi_X^t\}$ and then pulling back via the epimorphism $\nu_0 : Q^0 \to \mathbf{R}^*_{>0}$. Let $M' = G \times_{Q^0} (-1, 1)$, a bundle over $\mathbf{S}^{n-1}$ with interval fibers.

In $(\mathbf{R}^n, 0)$ with the standard action of SL(n, $\mathbf{R}$), let ℓ_0 be one of the two Q^0-invariant rays from the origin, pointwise fixed by E^0. The restriction of $\{a^t\}$ to ℓ_0 is smoothly equivalent to $\{\psi_X^t\}$ on $(-1, z_-)$. Identifying ℓ_0 with $(-1, z_-)$ by this equivalence and extending G-equivariantly gives a smooth gluing of $(\mathbf{R}^n, 0)$ to M', resulting in a manifold again diffeomorphic to $\mathbf{R}^n$. Similarly gluing another copy of $(\mathbf{R}^n, 0)$ along ℓ_0 to $(z_+, 1)$ in a Q^0-equivariant way yields a closed manifold M, diffeomorphic to $\mathbf{S}^n$, on which SL(n, $\mathbf{R}$) acts with two global fixed points.

Last, we construct actions of SL(n, $\mathbf{R}$) on $\mathbf{RP}^n$. Let $\Sigma_+ = [-1, 0]$ and let X be a smooth vector field vanishing at -1 and 0, such that $D_{-1}X = 1$. As above, define a Q_0-action on Σ_+ by composing the flow ψ_X^t with ν_0. Let $M' = G \times_{Q^0} (-1, 0]$. As above, glue $\mathbf{R}^n$ to M' by gluing ℓ_0 to $(-1, z_-)$, for $z_- \leq 0$ the minimum of the zeros of X on $(-1, 0]$. The result is a manifold with boundary, diffeomorphic to $\mathbf{D}^n$. The antipodal map corresponds to $[(g, x)] \mapsto [(g\sigma, x)]$, which is well-defined because $\sigma \in Q$ and ν_0 is invariant under conjugation by σ. The Q^0-action on the Σ_+-fibers is equivariant with respect to this involution. Now quotient by the antipodal map restricted to the boundary of the disk, mapping the boundary onto $\mathbf{RP}^{n-1}$. The resulting space is diffeomorphic to $\mathbf{RP}^n$, with smooth, faithful $\mathrm{SL}(n, \mathbf{R})$-action. Note that these actions on $\mathbf{RP}^n$ can be obtained as two-fold quotients from actions on $\mathbf{S}^n$ when X is invariant under $-\mathrm{Id}_1$ on Σ_+. The standard $SL(n, \mathbf{R})$-representation ρ on $\mathbf{R}^n$ product with a one-dimensional trivial representation yields, after projectivization, the "standard action" on $\mathbf{RP}^n$, obtained from a standard embedding of $\mathrm{SL}(n, \mathbf{R})$ in $\mathrm{SL}(n+1, \mathbf{R})$. This action is obtained from X vanishing only at -1 and 0, with derivative -1 at 0. The standard action on $\mathbf{S}^n$ is the double cover, which arises from X vanishing at $-1, 0,$ and 1 only, with derivative -1 at 0.

3.2. **New examples of lattice actions in supraminimal dimension.** The goal of this section is to develop new examples of actions of $\mathrm{SL}(n, \mathbf{Z})$ and its finite-index subgroups on manifolds of dimension n, including new actions on the n-dimensional torus $\mathbf{T}^n$. A sample result is

Theorem 3.2. *Let $\Gamma \leq \mathrm{SL}(n, \mathbf{Z})$ be a finite-index subgroup. Then for $r > 2$ and $r = \omega$ there exist uncountably many C^r actions of Γ on $\mathbf{T}^n$, none of which is C^1-conjugate to another.*

3.2.1. *Preliminaries: G-actions on blow-ups, disks and tubes.* From the constructions in the previous section, we will obtain exotic G-actions on $\mathbf{D}^n$ and on $\mathbf{S}^{n-1} \times I$, for I a closed interval, which we will call *G-disks* and *G-tubes*, respectively (Defs 3.3, 3.4). These will be patched into the standard action on the torus to build more general actions than previously constructed.

The blow-up of $\mathbf{R}^n$ at the origin is constructed as the following algebraic subvariety of $\mathbf{R}^n \times \mathbf{RP}^{n-1}$:

$$B = \{(x, [v]) \in \mathbf{R}^n \times \mathbf{RP}^{n-1} \ : \ x = cv \text{ for some } c \in \mathbf{R}\}$$

The G-action on $\mathbf{R}^n \times \mathbf{RP}^{n-1}$ preserves B. The projection onto the second coordinate exhibits B as the tautological line bundle over $\mathbf{RP}^{n-1}$. In particular, B is a manifold with an analytic G-action. The points of B projecting to 0 in the first factor form a subvariety $E \cong \mathbf{RP}^{n-1}$, called the exceptional divisor.

Let BS be the universal cover of B. The 2-to-1 covering $BS \to B$ is G-equivariant. Note that although BS is diffeomorphic to $\mathbf{R}^n \backslash \{0\}$, the G-action is not the restriction of the linear action from $\mathbf{R}^n$. A construction of BS analogous with that of B is as follows: view $\mathbf{S}^{n-1}$ as $\mathbf{R}^n \backslash \{0\} / \mathbf{R}^*_{>0}$ and define

$$BS = \{(x, [v]) \in \mathbf{R}^n \times \mathbf{S}^{n-1} \ : \ x = cv \text{ for some } c \in \mathbf{R}\}$$

As for B, the projection on the second factor exhibits BS as a line bundle over $\mathbf{S}^{n-1}$. It is the tautological line bundle and is trivial. The covering $BS \to B$ corresponds to taking the quotient by the diagonal action of -1. The subset of BS projecting to 0 in the first factor will also be denoted E. We define

$$BS^+ = \{(x, [v]) \in BS \ : \ x = cv \text{ for some } c \in \mathbf{R}^*_{>0}\}$$

and similarly for BS^-.

Denote by $\ell_B \subset B$ the fiber over the unique Q-fixed point in $\mathbf{RP}^{n-1}$. The Q-action on ℓ_B factors through the homomorphism $\nu : Q \to \mathbf{R}^* \cong \mathbf{Z}_2 \times \mathbf{R}$ and includes a flow, generated by a vector field X_B. For $\{a^t\} < Q$ the connected component of the center of L, the flow along X_B is the $\{a^t\}$-action on ℓ_B. For the linear G-action via ρ on $\mathbf{R}^n$, let ℓ be the unique Q-invariant line, on which Q acts via ν. The action of $\nu_0(Q^0)$ on ℓ is generated by a vector field X vanishing at 0, which can be normalized so that $D_0X = 1$. Under the projection $B \to \mathbf{R}^n$, the line ℓ_B is mapped to ℓ. Up to normalization, we can assume that $D_0X_B = 1$.

The G-action on B is equivalent to the induced action on $G \times_Q \ell_B$. Similarly, on BS there is a unique Q^0-invariant line ℓ_{BS}, and the G-action is the induced action on $G \times_{Q^0} \ell_{BS}$. Moreover, ℓ_{BS} is Q-invariant. The deck transformations of the covering $BS \to B$ can be realized as a Q/Q^0-action commuting with the G-action (as in the construction in the previous section of the action on $\mathbf{RP}^n$ as a quotient of an action on $\mathbf{D}^n$).

Modifying the vector field X yields different G-actions on B and BS. Given any X on ℓ_B invariant under the $\mathbf{Z}_2$-action and vanishing only at 0, the resulting G-space $G \times_Q \ell_B$ will be denoted B_X. The order of vanishing and derivatives of X at 0 can be arbitrary. Similarly, any vector field X on ℓ_{BS} vanishing only at 0 yields a G-space diffeomorphic to BS, which we will denote BS_X. If X is moreover invariant under the $\mathbf{Z}_2$-action on ℓ_{BS}, there is again a G-equivariant covering $BS_X \to B_X$.

The G-space BS_X, or in some cases just BS_X^+, will serve as a patch between the standard torus action and the building blocks for our exotic actions. The blow-ups in Katok–Lewis' construction in [18] are obtained by gluing a space B_X into $\mathbf{T}^n \backslash \{0\}$. They present B_X and the gluing in coordinates. We will provide a coordinate-free construction of their actions below. Here are the building blocks for our exotic actions.

Definition 3.3. Let X be a vector field on $I = [-1,1]$ with $X(-1) = X(1) = 0$. Let Q^0 act on I via ν_0 followed by the flow along X. The induced G-space $G \times_{Q^0} I$ is called a *G-tube*.

Now let X be a vector field on $[-1,1]$ invariant by $-\mathrm{Id}_1$, with $D_0X = 1$. Let $\dot{D} = G \times_{Q^0} (0,1]$. As in subsection 3.1.2, the linear G-action on $\mathbf{R}^n$ can be glued equivariantly onto an open subset of $\dot{D}$, yielding a G-action on $\mathbf{D}^n$.

Definition 3.4. These G-actions on $\mathbf{D}^n$ are called *G-disks*.

3.2.2. *Lattice actions on compact manifolds.* By gluing G-disks in place of fixed points in $\mathbf{T}^n$, we will construct the examples of Theorem 3.2 and prove the claim that they are C^1-distinct. We will provide a common context for the famous examples of Katok–Lewis [18] and our new examples, as well as an additional construction using G-tubes to glue together tori. The section finishes with a conjecture on the classification of Γ-actions on closed n-manifolds, for $\Gamma < \mathrm{SL}(n,\mathbf{R})$ a lattice.

For the standard action of $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ on $\mathbf{T}^n$, local modifications near the fixed point 0 can be achieved by modifying the $\mathrm{SL}(n, \mathbf{Z})$-action on $\mathbf{R}^n$ near 0. Given a fundamental domain $F \subset \mathbf{R}^n$ containing the set $U = (-\frac{1}{2}, \frac{1}{2})^n$, let, for each $\gamma \in \Gamma$, V_γ equal $\gamma^{-1}(U) \cap U \subset \mathbf{R}^n$. On V_γ the covering map $\pi : \mathbf{R}^n \to \mathbf{T}^n$ is a diffeomorphism such that $\pi(\gamma(x)) = \gamma(\pi(x))$. Changing the Γ-action on $\mathbf{R}^n$ on a neighborhood of 0 contained in U yields a well-defined action on $\mathbf{T}^n$. The same procedure is valid at any fixed point p for Γ or a finite-index subgroup. By such local modification a G-disk D as in Definition 3.4 can be glued into the standard action $\mathbf{T}^n \backslash \{0\}$, using a suitable BS_X as the patch between them.

To begin with, assume the vector field X on $[-1, 1]$ determining D vanishes at 1 with derivative 1 there. Identify a collar neighborhood of E in BS^+ radially with a neighborhood of the puncture in $\dot{U} = \pi(U \backslash \{0\})$ in $\mathbf{T}^n$, thus gluing BS to the punctured torus. The resulting space inherits a well-defined, smooth Γ-action. Next, identify a collar neighborhood of ∂D in D radially with a collar neighborhood of E in BS^-, thus gluing D to BS, while retaining a smooth Γ-action. The result of performing both gluings on one copy of BS is a closed manifold M diffeomorphic to $\mathbf{T}^n$ with a well-defined Γ-action. There is an invariant hypersphere corresponding to E, such that the Γ-action in a neighborhood of this hypersphere in M is equivalent to the Γ-action near E in BS. The Γ-action is thus smooth in this neighborhood, and on all of M. In fact, if the G-action on D is real-analytic, then this gluing yields a C^ω action of Γ on $M \cong \mathbf{T}^n$.

A general G-disk can be glued in to a punctured torus by the following procedure. Let h be a diffeomorphism of $\mathbf{T}^n \backslash \{\pi(0)\}$ that is the identity outside $\dot{U}$ and is radial on its support. Conjugate the $\mathrm{SL}(n, \mathbf{Z})$-action on $\mathbf{T}^n \backslash \{\pi(0)\}$ by h. Near the puncture, this action coincides with a modified $\mathrm{SL}(n, \mathbf{Z})$-action μ on $\mathbf{R}^n \backslash \{0\}$. Note that by Theorem 2.4, the local linearization theorem, this action will not in most cases extend to $\mathbf{R}^n$. It does, however, extend over a suitable B_X or BS_X patched into the puncture. Indeed, the restriction of Q^0 to $\ell_0 \backslash \{0\}$ corresponds to a vector field X_0 that extends to a vector field on ℓ_0 vanishing at the origin, which we will also denote by X_0. Identifying ℓ_0 smoothly with ℓ_B pushes X_0 forward to a vector field X_B vanishing at 0. Then inducing over Q_0 defines the G-action on BS_X. The restriction to BS_X^+ is equivalent to μ near 0, because h is radial. Gluing $BS_X \backslash BS_X^-$ into $\mathbf{R}^n \backslash \{0\}$ along BS_X^+ gives a smooth G-action on $\mathbf{R}^n$ with an open ball removed, which is equivalent to μ on the complement of the boundary. Making the local identification with the torus, this gives a modified $\mathrm{SL}(n, \mathbf{Z})$-action on the torus minus an open ball, equivalent to the originally modified action on the complement of the boundary. Now gluing a collar neighborhood of ∂D into BS_X^- exactly as before yields an action on $\mathbf{T}^n$, such that the Γ-action on $M \backslash D$ is equivalent to the original action on $\mathbf{T}^n \backslash \{0\}$.

Proposition 3.5. *Two actions of* $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ *on* $\mathbf{T}^n$ *obtained by gluing a* G*-disk in place of* $\pi(0)$ *as above are conjugate if and only if the vector fields on* $[-1, 1]$ *determining the* G*-disks are conjugate.*

Proof. Consider a G-disk D determined by $\nu_0 : Q^0 \to \mathbf{R}$. For a conjugate $\hat{Q}^0 < G$, there is a unique $\hat{Q}^0$-invariant interval $\hat{I} \subset D$ on which the $\hat{Q}^0$-action is given by $\hat{\nu}_0 : \hat{Q}^0 \to \mathbf{R}$. The given G-action on D is the same as that induced from $\hat{Q}_0$ by $\hat{\nu}_0$. We will choose $\hat{Q}_0$ so that $\Gamma \cap \hat{Q}^0$ has dense image in $\mathbf{R}$ under $\hat{\nu}_0$. This implies that the Γ-action determines the vector field X on $\hat{I}$ for some, and hence any, choice of conjugate $\hat{Q}^0$, which suffices to prove the proposition.

By [27, Thm 1] of Prasad–Rapinchuk, there is a $\mathbf{Q}$-irreducible Cartan subgroup—often referred to as a $\mathbf{Q}$-irreducible torus—in Γ. Denote it by T. Irreducibility here means that T contains no nontrivial, proper, algebraic subtorus. There is a conjugate $\hat{Q}^0$ of Q^0 in G containing T. It suffices to show that $\hat{\nu}_0$ is faithful on $T_\Gamma = T \cap \Gamma$, since then the image will necessarily be dense in $\mathbf{R}$. If $\ker(\hat{\nu_0}) \cap T_\Gamma$ is nontrivial, let T^0 be its Zariski closure. It is the kernel of the rational map $\hat{\nu}_0$ restricted to T. Since Γ, and therefore $\ker(\hat{\nu_0}) \cap T_\Gamma$, consists of $\mathbf{Z}$-points in G, T^0 is defined over $\mathbf{Q}$. It is thus an algebraic subgroup of G, contained in T, contradicting $\mathbf{Q}$-irreducibility. □

Here are additional constructions of Γ-actions, for $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ or a finite subgroup, on closed n-manifolds.

Blow-up and two-sided blow-up. For a vector field X on ℓ_{BS} invariant under the deck group of the cover $BS_X \to B_X$, patching BS_X into $\mathbf{T}^n \backslash \{\pi(0)\}$ and dividing by the deck group yields an $\mathrm{SL}(n, \mathbf{Z})$-action on $\mathbf{T}^n$ with $\pi(0)$ blown up. Katok–Lewis' blow-up examples correspond to X vanishing to first order at $0 \in \ell_{BS}$. They computed that choosing X with derivative n at 0 yields a volume-preserving action [18].

A related construction is what we will call a *two-sided blow-up.* (It is discussed briefly in [18] and in more detail in [2, 13].) Start with two tori with $\pi(0)$ removed, $\dot{\mathbf{T}}^n_+$ and $\dot{\mathbf{T}}^n_-$. Patch them together with BS_X by gluing BS_X^+ into the punctured neighborhood $\dot{U}_+$ and BS_X^- into $\dot{U}_-$. The resulting space is the connected sum of two tori with a smooth $\mathrm{SL}(n, \mathbf{Z})$-action. A second variant involves a single torus punctured at two points which are fixed by a finite-index subgroup $\Gamma < \mathrm{SL}(n, \mathbf{Z})$. Patching neighborhoods of the two punctures together with BS_X yields another smooth Γ-space. Some, but not all, of these examples are equivariant covers of blow-up actions as constructed above, in which case the full $\mathrm{SL}(n, \mathbf{Z})$ acts on the cover. As for the blow-ups, these actions are volume-preserving if X has derviative n at 0.

Connected sum along G-tube. Let T be a G-tube with defining vector field X. Let BS_X^a and BS_X^b be two copies of BS_X with exceptional divisors E_a and E_b, respectively. For each of the two boundary components of T, identify a collar neighborhood radially with a collar neighborhood of E_i

in $(BS^i_X)^-$, for $i = a, b$, to glue BS^a_X and BS^b_X to T, one on each end. Let $\mathbf{T}^n_a$ and $\mathbf{T}^n_b$ be two tori with $\pi(0)$ removed. Then identify collar neighborhoods of E_i in $(BS^i_X)^+$ radially with neighborhoods of the punctures in $\mathbf{T}^n_i$, for $i = a, b$, respectively. The result is two punctured tori connected along the G-tube T, with smooth (or even real-analytic) Γ-action.

Multiple G-disks along a finite orbit. Given a finite $\mathrm{SL}(n, \mathbf{Z})$-orbit O, a finite-index subgroup Γ fixes each point of O. Gluing G-disks into some or all of these Γ-fixed points by the procedure explicated above yields additional Γ-actions on $\mathbf{T}^n$. One can also perform conjugate gluings along the periodic orbit to obtain an $\mathrm{SL}(n, \mathbf{Z})$-action with multiple G-disks which are permuted by the action.

Connecting points of a finite orbit by a G-tube. Given a finite orbit O as above, pointwise fixed by a finite-index subgroup $\Gamma < \mathrm{SL}(n, \mathbf{Z})$, gluing G-tubes between some pairs of distinct points of O by the procedure above yields further Γ-actions.

Combinations. Given a finite collection of tori $T_1, \ldots, T_k$, each with finite orbits O_i, for $i = 1, \ldots, k$, let $\Gamma < \mathrm{SL}(n, \mathbf{Z})$ be a finite-index subgroup pointwise fixing $O = \cup_i O_i$. Combinations of G-tubes and two-sided blow-ups between distinct points of O and G-disks or blow-ups at points of O yield closed, connected n-manifolds with smooth Γ-action.

Equivariant gluings along periodic orbits. Up to restriction to a subgroup of finite index, we believe the constructions described so far are a complete list. But our constructions admit equivariant extensions to $\mathrm{SL}(n, \mathbf{Z})$ along periodic orbits. The simplest version of this was already noted by Katok and Lewis: instead of blowing up a fixed point, one can blow up all the points in a periodic orbit and still obtain an action of the full $\mathrm{SL}(n, \mathbf{Z})$. A more complicated construction is illustrated by Farb and Shalen in [9, Sec 3] where multiple tori are glued over two-sided blow ups along any finite $\mathrm{SL}(n, \mathbf{Z})$-invariant set in an initial torus. (Their construction is for $n = 3$ but generalizes to arbitrary $n \geq 3$.) One can also insert G-disks and G-tubes equivariantly along finite orbits, or finite invariant sets. These constructions that modify the manifold equivariantly along a finite invariant set restrict on a finite-index subgroup to examples we have already discussed above. Farb and Shalen also continuously glue certain G-disks into tori, attributing that construction to S. Weinberger, [9, Section 3.1]. They thus obtain an exotic action by homeomorphisms. They ask whether their construction can yield analytic and volume-preserving actions, a question which is positively answered by our constructions here.

We refer to any action of a finite-index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ constructed by finite iteration of the operations described above as an *action built from tori, G-disks, G-tubes, blow-ups and two-sided blow-ups.* Finite iterations of a subset of these operations yields a subset of these actions; for example, the actions in Theorem 3.2 are actions built from tori and G-disks.

Conjecture 3.6. *Let $\Gamma < \mathrm{SL}(n,\mathbf{R})$ be a lattice and M a compact manifold of dimension n. Then any action $\rho : \Gamma \to \mathrm{Diff}(M)$ either*

(1) *extends to an action of $\mathrm{SL}(n,\mathbf{R})$ or $\widetilde{\mathrm{SL}(n,\mathbf{R})}$;*
(2) *factors through a finite quotient of Γ; or*
(3) *is an action built from tori, G-tubes, G-disks, blow-ups and two-sided blow-ups, with Γ a finite-index subgroup of $\mathrm{SL}(n,\mathbf{Z})$*

Actions as in item (1) are classified by Theorems 4.2 and 4.3 below, so this conjecture amounts to a full description of Γ-actions in dimension n.

We can formulate a much more restrictive conjecture for Γ-actions preserving a finite volume, thanks to the following proposition.

Proposition 3.7. *Let $\Gamma \leq \mathrm{SL}(n,\mathbf{Z})$ be a finite-index subgroup acting on a closed manifold M^n preserving a finite volume. Then the Γ-action does not extend to $\mathrm{SL}(n,\mathbf{R})$ or $\widetilde{\mathrm{SL}(n,\mathbf{R})}$. If it is built from tori, G-disks, G-tubes, blow-ups, and two-sided blow-ups, then it is in fact built only from volume-preserving blow-ups and two-sided blow-ups.*

Recall that the volume-preserving blow-ups and two-sided blow-ups are those with definiing vector field X having derivative n at 0.

Proof. Let $G \cong \mathrm{SL}(n,\mathbf{R})$. For any lattice $\Gamma < G$, the Γ-action on $\mathbf{R}^n\backslash\{0\}$ is ergodic by the Howe-Moore theorem (see [39, Thm 2.2.6]). Any volume form on $\mathbf{R}^n\backslash\{0\}$ is $f\lambda$, for λ the G-invariant Lebesgue measure and f a smooth function. Thus the only Γ-invariant volume forms on $\mathbf{R}^n\backslash\{0\}$ are constant multiples of λ, all having infinite total volume.

Now let Γ act on M as in the statement of the proposition. If the action is volume-preserving, there are no open sets on which the Γ-action is conjugate to the standard action on $\mathbf{R}^n\backslash\{0\}$. Any action extending to a non-transitive action of $\mathrm{SL}(n,\mathbf{R})$ or $\widetilde{\mathrm{SL}(n,\mathbf{R})}$, or any action containing G-tubes or G-disks, always contain such open sets. The exceptional homogeneous spaces of Theorem 2.1 (4) do not have any Γ-invariant finite volume, also by the Howe-Moore Theorem. □

Here is the resulting conjecture for volume-preserving Γ-actions on closed n-manifolds:

Conjecture 3.8. *Let $\Gamma < \mathrm{SL}(n,\mathbf{R})$ be a lattice and M a compact manifold of dimension n. Then any action $\rho : \Gamma \to \mathrm{Diff}(M)$ either*

(1) *factors through a finite quotient of Γ; or*
(2) *$\Gamma \leq \mathrm{SL}(n,\mathbf{Z})$ is a finite-index subgroup, and the action is built from tori and volume-preserving blow-ups and two-sided blow ups—that is, for which all vector fields in the construction have derivative n at 0.*

While this version of the conjecture does not appear anywhere in the literature, it seems to be widely believed by experts. The more general Conjecture

3.6 is less established, mainly because the examples involving G-tubes and G-disks were previously unkown.

4. Classification of smooth G-actions

Let G be connected and locally isomorphic to $\mathrm{SL}(n, \mathbf{R})$. In this section we prove that, aside from the exceptional homogeneous spaces listed in Theorem 2.1, all nontrivial G-actions on closed n-manifolds are obtained from constructions I or II from Section 3.1.

It follows from Theorem 2.1 that a nontrivial non-transitive action of G is a faithful action of $\mathrm{SL}(n, \mathbf{R})$ or $\mathrm{PSL}(n, \mathbf{R})$. We set $G = \mathrm{SL}(n, \mathbf{R})$ for the remainder of this section and assume it acts nontrivially and non-transitively on a compact manifold M.

4.1. Compact subgroups and fixed circle.

Let the subgroups Q, L, and C be as in previous sections, and let $C^0 \cong \mathrm{SO}(n-1)$. As above, let K be a maximal compact subgroup of G, containing C, and κ a K-invariant metric on M.

Proposition 4.1. *Assume the G-action on M is not transitive. Let $\Sigma \subset M$ comprise the C^0-fixed points. It is a nonempty, finite union of circles.*

Proof. The G-orbits in Theorem 2.1 except those in (4) contain C^0-fixed points, and those in (4) are ruled out by our hypotheses. Thus $\Sigma \neq \emptyset$.

By classical results, Σ is a closed, totally geodesic submanifold for κ. It remains to verify that each connected component has dimension 1. Let $x \in \Sigma$ and refer to Theorem 2.1. If x is a G-fixed point, then the K-action is linearizable near x. The isotropy representation of K extends to G; it is the standard representation of K on $\mathbf{R}^n$, in which the C^0-fixed set has dimension 1. If $\mathcal{O}_x$ has dimension $n-1$, then C^0 is irreducible on $T_x\mathcal{O}_x$ and trivial on the κ-orthogonal. Then Σ coincides with the κ-geodesic orthogonal to $\mathcal{O}_x$ in a neighborhood of x. In the case $\mathcal{O}_x$ has dimension n, then by (3) of Theorem 2.1, the fixed set of C^0 in $\mathcal{O}_x$ is, as in the linear action on $\mathbf{R}^n \backslash \{\mathbf{0}\}$, of dimension one. □

4.2. Classification in the absence of G-fixed points.

Theorem 4.2. *Let $G \cong \mathrm{SL}(n, \mathbf{R})$, acting non-trivially on a closed n-manifold M. Assume that the G-action is not transitive and has no global fixed points. Then the G-action on M is as in Construction I—that is,*

$$M = G \times_Q \Sigma$$

where Q acts via $\mu_{(X,\tau)}$, yielding one of the following:

1. *M is diffeomorphic to $\mathbf{S}^{n-1} \times \mathbf{S}^1$, with faithful, fiber-preserving G-action.*
2. *M is diffeomorphic to $\mathbf{RP}^{n-1} \times \mathbf{S}^1$, with fiber-preserving action factoring through $\mathrm{PSL}(n, \mathbf{R})$.*

(3) *M is a flat circle bundle with $\mathbf{Z}_2$ monodromy over $\mathbf{RP}^{n-1}$, with faithful G-action.*
(4) *M is diffeomorphic to the blow-up of $\mathbf{RP}^n$ at a point. The G-action is faithful, leaves invariant the exceptional divisor and another hypersurface diffeomorphic to $\mathbf{RP}^{n-1}$, and preserves an $\mathbf{S}^{n-1}$-fibration on the complement of these two.*

Proof. Our assumptions, together with Theorem 2.1, imply that all point stabilizers are conjugate in G into Q.

Let Σ be the fixed set of C^0, as in Proposition 4.1. For $x \in \Sigma$, the stabilizer of x contains C^0; denote this stabilizer by G_x. Let $h \in G$ be such that $hG_xh^{-1} \leq Q$; in particular, $hC^0h^{-1} \leq Q$. The following homogeneous spaces are K-equivariantly diffeomorphic:

$$G/Q \cong \mathbf{RP}^n \cong K/C$$

Now

$$C^0 \leq \mathrm{Stab}_G(hQ) \cap K = \mathrm{Stab}_K(h'C)$$

for some $h' \in K$, where here the stabilizers are for the action by translation on left-coset spaces. Then $h' \in N_K(C^0) = C$. It follows that $h'C = C$ and thus $hQ = Q$. We conclude that $G_x \leq Q$ for all $x \in \Sigma$.

As the subgroups E^0, E, and Q^0 are each normal in Q, the stabilizer $G_x \in \{E^0, E, Q^0, Q\}$ for all $x \in \Sigma$, using Theorem 2.1. These stabilizers all contain C^0. It follows that $Q.\Sigma = \Sigma$. The normal subgroup E^0 is trivial in restriction to Σ. Thus, the Q-action on Σ factors through the epimorphism $\nu : Q \to \mathbf{R}^*$.

Let Σ^0 be a connected component of Σ. Now $\mathbf{R}^*_{>0}$ preserves Σ^0; this action is a smooth flow $\{\psi^t_X\}$, the restriction of $\{a^t\}$. Let σ be an involution in C mapping under ν to -1; it leaves Σ invariant. Let $\tau = \sigma|_{\Sigma}$. Depending on τ, let $\Sigma = \Sigma^0 \times \{-1, 1\}$ or $\Sigma = \Sigma^0$ Let $\mu_{(X,\tau)}$ be the corresponding Q-action on Σ.

For this Q-action on Σ, define the G-equivariant map

$$\Phi : G \times_Q \Sigma \to M \qquad [(g, x)] \mapsto g.x$$

The image of Φ is closed because the fiber product is compact. Let $g.x$ be in the image of Φ, with $g \in G$ and $x \in \Sigma$. If the orbit $G.x$ is n-dimensional, it is open; by dimension comparison, the differential of Φ at $[(g, x)]$ is an isomorphism. If $G.x$ is $(n-1)$-dimensional, corresponding to $G_x = Q^0$ or Q, then $G.x$ is equivariantly diffeomorphic to $\mathbf{RP}^{n-1}$ or $\mathbf{S}^{n-1}$, and the C^0-fixed set in $G.x$ is 0-dimensional. Thus $G.x$ is transverse to Σ at x, and the differential of Φ at $[(e, x)]$ is onto T_xM. By equivariance of Φ, the differential at $[(g, x)]$ is also onto $T_{g.x}M$. We conclude that Φ is open, so it is a surjective local diffeomorphism—in this case, a covering map.

The Iwasawa Decomposition is a diffeomorphism

$$K \times A \times N \to \mathrm{SL}(n, \mathbf{R})$$

where $K \cong \mathrm{SO}(n)$, as above, A is the identity component of the diagonal subgroup, and N is the group of unipotent upper-triangular matrices (see [19, Thm VI.6.46]). As $N < E^0$ and $A/(A \cap E^0) \cong \{a^t\}$, the Iwasawa Decomposition gives a normal form for elements of $G \times_Q \Sigma$: for any $g = ka'n \in G$ and $x \in \Sigma$,

$$(g, x) \sim (k, a^t.x) \sim (k\sigma, \tau a^t.x)$$

where $a' = a^t a''$ with $a'' \in A \cap E^0$. Every $[(g, x)] \in G \times_Q \Sigma$ is represented by (k, x) with $k \in K$ and $x \in \Sigma^0$.

If $\Phi([(k, x)] = \Phi([(k', x')])$ with $k, k' \in K$ and $x, x' \in \Sigma^0$, then $k' = kq$ and $x' = q^{-1}.x$ for $q = k^{-1}k'$ in the normalizer of C^0, which intersects K in C. Thus $[(k, x)] = [(k', x')]$. We conclude that Φ is injective, hence a diffeomorphism.

Now suppose $\Sigma = \Sigma^0 \times \{1, -1\}$. Represent a point $p \in M$ by $[(k, x)]$ with $k \in K$ and $x \in \Sigma^0$. The assignment $p \mapsto (kC^0, x) \in K/C^0 \times \Sigma^0$ is well-defined, because the stabilizer of Σ^0 intersect K equals C^0 in this case. It is easy to verify that this map is a diffeomorphism $M \to \mathbf{S}^{n-1} \times \mathbf{S}^1$, corresponding to case (1).

If τ is trivial, then M has a well-defined diffeomorphism to $K/C \times \Sigma^0 \cong \mathbf{RP}^{n-1} \times \mathbf{S}^1$, corresponding to case (2).

Next assume $\Sigma = \Sigma^0$ and τ acts freely. In this case, the stabilizer in K of Σ^0 is C; note also that $C = \langle \sigma, C^0 \rangle$ and σ normalizes C^0. Given $p \in M$ corresponding to $[(k, x)]$ with $k \in K$ and $x \in \Sigma^0$, there is a well-defined map to the orbit $\{(kC^0, x), (k\sigma C^0, \tau.x)\} \in (K/C^0 \times \Sigma^0)/\langle \sigma \rangle$. This is case (3).

In the last case, when τ has two fixed points, say x_0 and x_1, on Σ^0, then τ permutes the two components of $\Sigma^0 \backslash \{x_0, x_1\}$. Let I_0 be one component. There is a well-defined map on $M \backslash (G.x_0 \cup G.x_1)$ sending $[(k, x)]$ to (kC^0, x) with $x \in I_0$. The image is diffeomorphic to $\mathbf{S}^{n-1} \times I_0$. The orbits $G.x_i$ are $\mathbf{RP}^{n-1}$. The manifold M can be obtained from $\mathbf{S}^{n-1} \times (\{x_0\} \cup I_o \cup \{x_1\})$ by gluing $\mathbf{S}^{n-1} \times \{x_i\}$ to $\mathbf{RP}^{n-1} \times \{x_i\}$ by the standard covering, for $i = 0, 1$. This is case (4). □

4.3. **Classification of actions with G-fixed points.**

Theorem 4.3. *Let $G \cong \mathrm{SL}(n, \mathbf{R})$, acting non-trivially on a closed n-manifold M. Assume that the G-action is not transitive and has at least one global fixed point. Then the G-action on M is as in Construction II and has one or two fixed points.*

1. *In the case of two fixed points, it is obtained from an induced action on $G \times_{Q^0} (-1, 1)$ by attaching two copies of $\mathbf{R}^n$ and is diffeomorphic to $\mathbf{S}^n$.*
2. *In the case of one fixed point, it is a two-fold quotient of an action as in (1), diffeomorphic to $\mathbf{RP}^n$.*

In either case, the G-action is faithful.

Proof. Suppose that $x_0 \in M$ is G-fixed, and let Σ^0 be the connected component of Σ containing x_0. This is a Q-invariant curve through x_0, so there is a Q-invariant line ℓ_0 tangent to Σ^0 in the isotropy representation of G at x_0. The isotropy is thus the standard representation ρ. Note also that $G \cong \mathrm{SL}(n, \mathbf{R})$. Let $\{a^t\}$, as above, be the one-parameter subgroup in the center of $L \cong \mathrm{GL}(n-1, \mathbf{R}) < Q$. In a suitable parametrization $\rho(a^t)$ has eigenvalue e^t on ℓ_0. Thus there is a neighborhood of x_0 in Σ^0 in which x_0 is the only Q-fixed point. Let $\sigma \in C$ be as above, so that $\rho(\sigma)$ acts as $-\mathrm{Id}_1$ on ℓ_0. Both $\{a^t\}$ and σ have no fixed points on $\Sigma^0 \backslash \{x_0\}$ in a neighborhood of x_0. Thus in this neighborhood, points of $\Sigma^0 \backslash \{x_0\}$ have stabilizer contained in E^0, which means, thanks to Theorem 2.1, that these stabilizers are E^0 and the corresponding orbits are $\mathbf{R}^n \backslash \{0\}$. Then an n-dimensional G-orbit fills a punctured neighborhood of x_0. Finally, a neighborhood U_0 of x_0 is G-equivariantly homeomorphic to $(\mathbf{R}^n, 0)$.

Now [6, Thm 1.1], stated here as Theorem 2.4, applies to give that the G-action on U_0 is smoothly equivalent to the representation ρ. Let $I_0 = \Sigma^0 \cap U_0$, the open interval corresponding in these coordinates to the line ℓ_0 through the origin pointwise fixed by E^0 and invariant by Q.

Let $\tau = \sigma|_{\Sigma^0}$. The involution τ has exactly one other fixed point, call it x_1, in Σ^0, by Lemma 3.1.

Proposition 4.4. *The standard* $\mathrm{SL}(n, \mathbf{R})$*-representation* ρ *on* $\mathbf{R}^n$ *does not extend to a smooth action on any smooth one-point compactification.*

Proof. Assume $n \geq 3$. Let $\{a^t\}$ be the one-parameter subgroup as above, oriented such that $\|\mathrm{Ad}\ a^t\| > 1$ on $\mathfrak{u}^+$, the unipotent radical of $\mathfrak{q}$, for $t > 0$. This implies that a^t is expanding on the Q-invariant line ℓ_0 for $t > 0$.

Suppose that for a smooth structure on the one-point compactification $\mathbf{R}^n \cup \{x_1\}$ the $\mathrm{SL}(n, \mathbf{R})$-action extends smoothly. In the linearization at x_1 given by Theorem 2.4, the image of $I_0 \cup \{x_1\}$ contains a Q-invariant line ℓ_1. Then the representation in this linearization is ρ. That means a^t is expanding on ℓ_1 for $t > 0$. Then the union of the curves corresponding to ℓ_0 and ℓ_1 is a circle containing exactly two $\{a^t\}$-fixed points, both of which are expanding, a contradiction.

Though we do not need it here, we note the proof requires modification when $n = 2$. In that case, direct computation shows that in ρ^*, the action of a^t on the Q-invariant line also moves points away from the origin. So when $n = 2$, the contradiction is similar. □

As $\{a^t\}$ normalizes C and commutes with C^0, it leaves $\mathrm{Fix}(\tau) = \{x_0, x_1\}$ invariant. Thus x_1 is also $\{a^t\}$-fixed.

Corollary 4.5. *At the* τ*-fixed point* x_1*, the* $\{a^t\}$*-action is expanding on* Σ^0*. The point* x_1 *does not lie on the boundary of* I_0*.*

As in the proof above, the existence of a Q-invariant 1-manifold through x_1 forces the linearization at x_1 to be ρ, so $\{a^t\}$ is expanding. Since $\{a^t\}$ is also expanding on Σ^0 at x_0, this precludes $x_1 \in \partial I_0$.

We now proceed with the identification of the action on M. Let $\Sigma^0_\pm$ be the two connected components of $\Sigma^0 \backslash \{x_0, x_1\}$. The stabilizers of all points of Σ^0_+ are conjugate in G to Q, Q^0, E, or E^0. By the same argument as in Section 4.2, the stabilizers are in fact equal to one of these subgroups. One consequence is that Σ^0_+ is Q^0-invariant. As $\tau.\Sigma^0_+ = \Sigma^0_-$, the union $\Sigma^0_+ \cup \Sigma^0_-$ is Q-invariant, and stabilizers of points in Σ^0_+ are in fact one of Q^0 or E^0.

Now we can define

$$\Phi : G \times_{Q^0} \Sigma^0_+ \to M \qquad \Phi : [(g,x)] \mapsto g.x$$

As in the proof of Theorem 4.2, Φ is a local diffeomorphism; as such, it has open image in M.

Recall that $G/Q^0 \cong K/C^0$. There is in fact a natural K-equivariant diffeomorphism

$$K \times_{C^0} \Sigma^0_+ \to G \times_{Q^0} \Sigma^0_+$$

mapping the C^0-orbit of $(k,x) \in K \times \Sigma^0_+$ to the corresponding Q^0-orbit in $G \times \Sigma^0_+$. This map is well-defined and injective because $C^0 = K \cap Q^0$. It is easy to see the map is open. Surjectivity follows from the Iwasawa Decomposition: write any $g \in G$ as a product $ka'n$, with $k \in K$, $a' = a^t a'' \in A$, $a'' \in A \cap E^0$ and $n \in N < E^0$; then, given any $x \in \Sigma^0_+$, we have $[(g,x)] = [(k, a^t.x)]$. The composition of this diffeomorphism with Φ is $[(k,x)] \mapsto k.x$, which is injective because the stabilizer in K of any $x \in \Sigma^0_+$ equals C^0. We conclude that Φ is a diffeomorphism onto its image, which is in turn diffeomorphic to $\mathbf{S}^{n-1} \times \Sigma^0_+$.

Let $(I_0)_\pm = U_0 \cap \Sigma^0_\pm$, so $(I_0)_+ \cup (I_0)_- = I_0 \backslash \{x_0\}$. The restriction of Φ to $G \times_{Q^0} (I_0)_+$ is a G-equivariant diffeomorphism to $U_0 \backslash \{x_0\}$. Under the K-equivariant identification with $\mathbf{S}^{n-1} \times (I_0)_+$, the fibers $\{p\} \times (I_0)_+$ are K-equivariantly identified with the rays from the origin in $U_0 \cong \mathbf{R}^n$. Thus $U_0 \cup \mathrm{Im}\ \Phi$ is K-equivariantly diffeomorphic to $\mathbf{R}^n$.

Suppose x_1 is a G-fixed point, and let U_1 be an open neighborhood of x_1 in M on which the G-action is equivalent to the linear action by ρ. Let $(I_1)_+ = U_1 \cap \Sigma^0_+$. As in the previous paragraph, Φ restricted to $G \times_{Q^0} (I_1)_+ \cong \mathbf{S}^{n-1} \times (I_1)_+$ identifies fibers $\{p\} \times (I_1)_+$ with rays from the origin in $U_1 \backslash \{x_1\}$ in a K-equivariant manner. The fibers $\{p\} \times (I_1)_+$ are in turn identified K-equivariantly with infinite segments of rays from the origin in $\mathbf{R}^n$ under its identification with $U_0 \cup \mathrm{Im}\ \Phi$. Thus $U_1 \cup \mathrm{Im}\Phi \cup U_0$ is K-equivariantly diffeomorphic to $\mathbf{S}^n$. It is moreover open and closed in M, so it equals M. We conclude that M is G-equivariantly diffeomorphic to the G-action on $\mathbf{S}^n$ in Construction II with $\{\psi^t_X\}$ equal $\{a^t\}$ restricted to Σ^0_+.

Now suppose x_1 is not G-fixed. The stabilizer of x_1 contains C^0, σ, and $\{a^t\}$. Thus it equals Q, and the orbit of x_1 is $G/Q \cong K/C \cong \mathbf{RP}^{n-1}$. The K-invariant metric κ determines a normal bundle to $G.x_1$, and an identification of a neighborhood of the zero section with a normal neighborhood $U_1 \cong K/C \times (-\epsilon, \epsilon)$ of the orbit. The fiber over x_1, call it I_1, comprises C^0-fixed points and is contained in Σ^0. Thus $I_1 \backslash \{x_1\}$ intersects $U_0 \cup \mathrm{Im}\ \Phi$

in two components, $(I_1)_+$ and $(I_1)_-$, contained in Σ^0_+ and Σ^0_-, respectively. The saturation $K.(I_1)_+$ equals $U_1 \backslash G.x_1$, and each distinct translate $k.(I_1)_+$ is identified with an infinite segment of a unique ray from the origin in $U_0 \cup \text{Im } \Phi \cong \mathbf{R}^n$. The resulting K-equivariant gluing of the normal bundle of $G.x_1$ to $U_0 \cup \text{Im } \Phi$ is equivalent to the gluing of the normal bundle of $\mathbf{RP}^{n-1}$ to $\mathbf{R}^n$ yielding $\mathbf{RP}^n$. We obtain that $M = U_0 \cup \text{Im } \Phi \cup U_1$ is diffeomorphic to $\mathbf{RP}^n$, with the G-action on $\mathbf{RP}^n$ in Construction II corresponding to $\{\psi^t_X\}$ equal $\{a^t\}$ restricted to $\Sigma^0_+ \cup \{x_1\}$. □

5. Analytic classification

In construction I of section 3.1, the actions are determined by the vector field X on $\Sigma^0 \cong \mathbf{S}^1$ and the involution τ of Σ commuting with X. There are four possibilities for τ, corresponding to the four possible diffeomorphism types in Theorem 4.2. In construction II, the action is determined by the vector field X on the interval $\Sigma_+ = [-1, 1]$; the actions on $\mathbf{RP}^n$ correspond to X being invariant by $x \mapsto -x$. By doubling Σ_+ and gluing at the endpoints -1 and 1, the vector field in this case determines a vector field on $\mathbf{S}^1$ invariant by a reflection (invariant by two reflections in the case corresponding to an action on $\mathbf{RP}^n$).

Thus, aside from the aforementioned finite data, G-actions on closed n-manifolds are determined by a smooth vector field on a circle, with some additional symmetries according to the type and subtype.

Proposition 5.1. *The vector field X and the involution τ are real-analytic, rather than just smooth, if and only if the resulting n-manifold and* SL$(n, \mathbf{R})$*-action are real analytic.*

Proof. Let $G = \text{SL}(n, \mathbf{R})$. First assume $M = G \times_Q \Sigma$ as in Construction I. If the vector field X and involution τ are C^ω, then the resulting $\mathbf{R}^*$-action on Σ^0 is C^ω. As $\nu : Q \to \mathbf{R}^*$ is a C^ω homomorphism, the lifted Q-action on Σ^0 is C^ω. Next, $Q < G$ is an analytic—in fact, algebraic—subgroup, so the diagonal Q-action on $G \times \Sigma^0$ is C^ω. We conclude that M, the quotient by this action, is C^ω.

If M is built from a C^ω vector field X on $\Sigma_+ = [-1, 1]$, then the resulting Q^0-action is C^ω on $(-1, 1)$, so $M' = G \times_{Q^0} (-1, 1)$ is C^ω. The diffeomorphisms from ℓ_0 to $(-1, z_-)$ and $(z_+, 1)$ conjugating $\{a^t\}$ to the respective restrictions of $\{\psi^t_X\}$ are then C^ω, as are their G-equivariant extensions. The gluings are then C^ω quotient maps, so the resulting action on $\mathbf{S}^n$, or the two-fold quotient, $\mathbf{RP}^n$, is C^ω. (The analyticity in this paragraph was previously proved by Uchida [33, Sec 2].)

Now suppose M^n is C^ω with real-analytic G-action. We are assuming the G-action is not transitive. The compact subgroup $C^0 < G$ is analytic, so the fixed set Σ is, too. As shown in the proofs of Theorems 4.2 and 4.3, Σ is Q-invariant. The restriction of the one-parameter subgroup $\{a^t\}$ to any component of Σ is C^ω. Similarly, the restriction of the involution $\sigma \in C$ to any component is C^ω. These yield the vector field X and the involution τ, respectively, so this data is real-analytic. □

N. Hitchin gave a complete set of invariants for C^ω vector fields on S^1 in [17, Thm 3.1]. They are as follows, for $X \in \mathcal{X}^\omega(S^1)$:

- A nonnegative integer *number* $k \in \mathbf{N}$ *of zeros* of X.

  Given a choice of cyclic ordering of the zeros,
- An *orientation* $\sigma \in \{\pm 1\}$.
- A list $m_1, \ldots, m_k$ of positive integers, the *orders of vanishing* of X at each zero
- A list $r_1, \ldots, r_k$ of real numbers, the *residues* of X at each zero. When X vanishes to order 1 at x_i, the residue r_i is the reciprocal of the derivative of X at x_i. The residues are defined analytically in general, see [17, Sec 1].
- A *global invariant* $\mu \in \mathbf{R}$. For $X = f\partial\theta$ nonvanishing, this is the integral around S^1 of $d\theta/f$. For general f it is analytically defined, see [17, Sec 2].

Different choices of orientation and cyclic ordering of the zeros correspond to the dihedral group D_k. More precisely,

$$\left(\{\pm 1\} \times \mathbf{R} \times \coprod_{k=0}^{\infty} (\mathbf{N}^k \times \mathbf{R}^k) \right) / \coprod_{k=0}^{\infty} D_k$$

is a Borel subset of a Polish space, providing a *smooth classification* of analytic vector fields on S^1 up to analytic conjugacy (see [28, 14] for background on this set-theoretic notion).

Corollary 5.2. *Real-analytic actions of* $\mathrm{SL}(n, \mathbf{R})$ *on closed, analytic* n*-manifolds are classified up to equivariant, real-analytic diffeomorphism by the following set of invariants:*

(1) *A type, I or II, or one of the finitely-many transitive actions in Theorem 2.1.*
(2) *For type I, one of four possible conjugacy classes for the analytic involution* τ*, and Hitchin's set of invariants for the analytic vector field* X*, commuting with* τ*.*
(3) *For type II, one of two homotopy types of* M *and Hitchin's set of invariants for the analytic vector field* X *on* S^1*, having at least two zeros of order one with identical positive residues, invariant by reflection in this pair of zeros, and additionally invariant by the antipodal map in the case* M *is not simply connected.*

The assignment of types and conjugacy classes of suitable vector fields on S^1 to $\mathrm{SL}(n, \mathbf{R})$-actions on closed n-manifolds, up to equivariant diffeomorphism (in the smooth or C^ω category), is a *Borel reduction* (again, see [28, 14]); recall, this assignment is by the restriction of a certain one-parameter subgroup $\{a^t\} < \mathrm{SL}(n, \mathbf{R})$ to a component Σ^0 of the fixed set of

$C^0 \cong \mathrm{SO}(n-1)$. The reverse assignment, starting from a type and a conjugacy class of compatible vector fields and constructing a closed n-manifold with $\mathrm{SL}(n, \mathbf{R})$-action, up to equivariant diffeomorphism, is also a Borel reduction. Our construction and classification result give a *Borel bireduction* between these two equivalence relations, whether in the smooth or analytic category. Thanks to Hitchin's result we obtain in the analytic case a smooth classification (in the set-theoretic sense above) of analytic $SL(n, \mathbf{R})$-actions on closed manifolds of dimension n up to analytic conjugacy. The subject of smooth classification in dynamical systems has recently received considerable attention.

Smooth vector fields, in contrast to analytic ones, do not admit a smooth classification: E_0, a particular Borel equivalence relation on a standard Borel space known not to admit a smooth classification, can be Borel reduced to it. It follows that $\mathrm{SL}(n, \mathbf{R})$-actions on closed n-manifolds in the smooth category do not admit a smooth classification (again, see [28, 14] for these notions and the definition of E_0). We thank Christian Rosendal for very helpful conversations on this topic.

6. Invariant Geometric Structures

The linear action of $\mathrm{SL}(n, \mathbf{R})$ on $\mathbf{R}^n$ preserves the standard, flat affine structure, while the transitive action on $\mathbf{S}^{n-1}$ preserves the standard, flat projective structure.

A *projective structure* is an equivalence class of torsion-free connections, where $\nabla \sim \nabla'$ means there is a 1-form ω on M such that

$$\nabla'_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X$$

for all $X, Y \in \mathcal{X}(M)$. Equivalent connections determine the same geodesic curves, up to reparametrization. See [30, Ch 8] or [20, Ch IV].

All actions of $\mathrm{SL}(n, \mathbf{R})$ on closed n-manifolds are built from projective actions on $\mathbf{R}^n$, $\mathbf{R}^n \backslash \{0\}$, and $\mathbf{S}^{n-1}$, but only a few well-known examples preserve a projective structure. We will prove this in Section 6.2 below. These actions all do, however, preserve a *rigid geometric structure of order two*, a much more flexible notion due to Gromov.

6.1. **Invariant 2-rigid geometric structure.**

For $k \geq 0$, denote by $\mathcal{F}^{(k)}M$ the bundle of k-frames on M of order k, with $\mathcal{F}^{(0)}M = M$. A k-frame at $x \in M$ is the k-jet at 0 of a coordinate chart $(\mathbf{R}^n, 0) \to (M, x)$. These form a principal $\mathrm{GL}^{(k)}(n, \mathbf{R})$-bundle, where this is the group of k-jets at 0 of local diffeomorphisms of $\mathbf{R}^n$ fixing 0.

Proposition 6.1. *Given any nontrivial, smooth action of* $\mathrm{SL}(n, \mathbf{R})$ *on a compact, n-dimensional manifold M, the action of* $\mathrm{SL}(n, \mathbf{R})$ *on* $\mathcal{F}^{(2)}M$ *is free and proper. In particular, the action preserves a* 2*-rigid geometric structure in the sense of Gromov.*

For the definition of *rigid geometric structure of order k* we refer to [15, 0.3], [1, Def 3.7], or [10, Sec 4]. For the equivalence for a smooth Lie group action between preserving a k-rigid geometric structure and acting freely and properly on $\mathcal{F}^{(k)}M$, see [15, 0.4] or [1, Thm 3.22]

Remark 6.2. Gromov asserts in [15, 0.4.C3] that any real-analytic action of a semisimple Lie group with finite center is 3-rigid. Benveniste-Fisher assert the 2-rigidity of a specific $\mathrm{SL}(n, \mathbf{R})$-action on a manifold of type (4) in Theorem 4.2, obtained by blowing up the origin in the standard $\mathrm{SL}(n, \mathbf{R})$-action on $\mathbf{RP}^n$ [2, Sec 3].

Lemma 6.3. *Let $\pi : M' \to N$ be a smooth submersion and $k \geq 0$. Suppose π is equivariant with respect to smooth actions of a group G on M' and N. If G acts freely and properly on $\mathcal{F}^{(k)}N$, then it acts freely and properly on $\mathcal{F}^{(k)}M'$.*

Proof. Let $m = \dim M'$ and $n = \dim N$. Let $\mathcal{S} \subset \mathcal{F}^{(k)}M'$ comprise the k-jets of coordinate charts $\varphi : (\mathbf{R}^m, 0) \to (M', x)$ for which $(\pi \circ \varphi)(0 \times \mathbf{R}^{m-n})$ is constant to order k at 0, for all $x \in M'$; in other words, $\varphi(0 \times \mathbf{R}^{m-n})$ is tangent to the π-fiber of x to order k. The set $\mathcal{S}$ is G-invariant and closed—in fact, it is a subbundle of $\mathcal{F}^{(k)}M'$.

Each k-frame in $\mathcal{S}_x$ gives a k-frame to N at $\pi(x)$, for all $x \in M'$, by restricting a representative coordinate chart to $\mathbf{R}^n \times 0$ and composing with π. This association is in fact a G-equivariant map $\mathcal{S} \to \mathcal{F}^{(k)}N$. By the elementary fact that freeness and properness of an action pulls back by equivariant maps, we see that G acts freely and properly on $\mathcal{S}$.

Recall that $\mathcal{S}$ is a closed subbundle of $\mathcal{F}^{(k)}M'$. The group $\mathrm{GL}^{(k)}(m, \mathbf{R})$ acts transitively on fibers of $\mathcal{F}^{(k)}M'$, commuting with the G-action. The stabilizer in G of $\xi \in \mathcal{F}^{(k)}M'$ is thus equal the stabilizer of $\xi.h$ for any $h \in \mathrm{GL}^{(k)}(m, \mathbf{R})$. Since G acts freely on $\mathcal{S}$, it acts freely on all of $\mathcal{F}^{(k)}M'$.

Let $K \subset \mathcal{F}^{(k)}M'$ be a compact subset and consider G_K, comprising all $g \in G$ with $g.K \cap K \neq \emptyset$. Let $\bar{K}$ be the projection of K to M' and cover $\bar{K}$ with finitely-many compact sets $\bar{U}_i$ over which the bundle $\mathcal{F}^{(k)}M'$ is trivializable. Let $U_i \subset \mathcal{S}$ be sections over $\bar{U}_i$. There are compact subsets $H_i \subset \mathrm{GL}^{(k)}(m, \mathbf{R})$ such that $K \subseteq \cup_i U_i.H_i$. Now

$$G_K \subseteq \bigcup_{i,j} G_{U_i.H_i, U_j.H_j}$$

where $G_{A,B}$ comprises the elements g with $g.A \cap B \neq \emptyset$. These subsets in turn can be expressed

$$G_{U_i.H_i, U_j.H_j} = G_{U_i, U_j.(H_j H_i^{-1})} = G_{U_i, V}$$

where $V = U_j.(H_j H_i^{-1}) \cap \mathcal{S}$, because $U_i \subset \mathcal{S}$ and $\mathcal{S}$ is G-invariant. Now because $\mathcal{S}$ is closed, and by properness of the G-action on $\mathcal{S}$, the set $G_{U_i,V}$ is compact. We conclude that G_K is compact, so G acts properly on all of $\mathcal{F}^{(k)}M'$. ∎

Lemma 6.4. *Let $U \subset M$ equal the closure of its interior $\mathring{U}$, and assume that $\partial U = D$ is a smooth hypersurface, not necessarily connected. Let G act smoothly on M leaving U invariant. For any $k \geq 0$, if G acts freely and properly on $\mathcal{F}^{(k)}\mathring{U}$ and on $\mathcal{F}^{(k)}D$, then G acts freely and properly on $\mathcal{F}^{(k)}M\big|_U$.*

Proof. Let $n = \dim M$. Let $\mathcal{S}$ comprise the k-frames in $\mathcal{F}^{(k)}M\big|_D$ at $x \in D$ arising from coordinate charts φ such that $\varphi(\mathbf{R}^{n-1} \times 0)$ is tangent at x to D up to order k—in other words, if F is a defining function for D in a neighborhood of x in M, then $F \circ \varphi$ restricted to $\mathbf{R}^{n-1} \times 0$ vanishes to order k at 0. Now $\mathcal{S}$ is a closed, G-invariant subbundle of $\mathcal{F}^{(k)}M\big|_D$. Each k-frame in $\mathcal{S}$ determines a k-frame of D, and this correspondence is a G-equivariant map from $\mathcal{S}$ to $\mathcal{F}^{(k)}D$. As in the previous proof, we conclude that G acts freely and properly on $\mathcal{S}$ and then, using the $\mathrm{GL}^{(k)}(n, \mathbf{R})$-action, that G acts freely and properly on the entire $\mathcal{F}^{(k)}M\big|_D$.

Note that $\mathcal{F}^{(k)}\mathring{U} = \mathcal{F}^{(k)}M\big|_{\mathring{U}}$. Thus G acts freely on $\mathcal{F}^{(k)}M\big|_U = \mathcal{F}^{(k)}M\big|_{\mathring{U}} \cup \mathcal{F}^{(k)}M\big|_D$.

Given a compact subset K of $\mathcal{F}^{(k)}M\big|_U$, suppose first that the projection of K to U lies in $\mathring{U}$. Then G_K is compact by our assumption on $\mathcal{F}^{(k)}\mathring{U}$. Otherwise, the projection of K has nontrivial intersection with D. Let $K' = K \cap \big(\mathcal{F}^{(k)}M\big|_D\big)$. Since the latter set is closed, K' is also compact. Then $G_K \subseteq G_{K'}$, which is compact by properness of the G-action on $\mathcal{F}^{(k)}M\big|_D$. This completes the proof. □

Lemma 6.5. *Let $M = U \cup V$ be a smooth manifold, and U and V closed subsets. Let G act smoothly on M, leaving U and V invariant. If G acts freely and properly on $\mathcal{F}^{(k)}M\big|_U$ and $\mathcal{F}^{(k)}M\big|_V$, then G acts freely and properly on $\mathcal{F}^{(k)}M$, for any $k \geq 0$.*

Proof. This proof proceeds easily from the decomposition of $\mathcal{F}^{(k)}M$ into closed sets $\mathcal{F}^{(k)}M\big|_U$ and $\mathcal{F}^{(k)}M\big|_V$. □

Here is the proof of Proposition 6.1.

Proof. Projective structures are 2-rigid geometric structures in the sense of Gromov (see [20, Ch 4]). Thus G acts freely and properly on $\mathcal{F}^{(2)}N$ for $N = \mathbf{S}^{n-1}$ or $\mathbf{RP}^{n-1}$. The actions in construction I have G-equivariant maps to $G/Q = \mathbf{RP}^{n-1}$. They satisfy the conclusion of the proposition by Lemma 6.3.

Now assume M arises from construction II. The subset $\mathring{V} = G \times_{Q^0} (z_-, z_+)$ is open and G-invariant, where z_- and z_+ are the first and last zeroes of the vector field X inside $(-1, 1)$; if $z_- = z_+$, then take $\mathring{V} = \emptyset$. First assume $\mathring{V} \neq \emptyset$. There is a G-equivariant submersion $\mathring{V} \to G/Q^0 = \mathbf{S}^{n-1}$, so G acts freely and properly on $\mathcal{F}^{(2)}\mathring{V}$ by Lemma 6.3. Let V be the closure, with boundary a union of two hyperspheres. As G preserves a projective structure on ∂V, it is free and proper on $\mathcal{F}^{(2)}\partial V$. Lemma 6.4 gives that G is free and proper on $\mathcal{F}^{(2)}M\big|_V$.

Let U_- be the closure in M of the copy of $\mathbf{R}^n$ glued along ℓ_0 to $(-1, z_-)$. This is a closed disk, with the standard linear G-action on the interior $\mathring{U}_-$, the standard action on the boundary $\mathbf{S}^{n-1}$-fiber over z_-, and the normal action to this boundary determined by the germ of X at z_-. Regardless of this germ, G is free and proper on $\mathcal{F}^{(2)}M\big|_{U_-}$. Indeed, the G-action on $\mathring{U}_-$ is affine, so it is free and proper on $\mathcal{F}^{(1)}\mathring{U}_-$ and thus also on $\mathcal{F}^{(2)}\mathring{U}_-$. Then Lemma 6.4 applies to give the desired conclusion. Similarly for U_+, the closure in M of the copy of $\mathbf{R}^n$ glued along ℓ_0 to $(z_+, 1)$, the G-action on $\mathcal{F}^{(2)}M\big|_{U_+}$ is free and proper. Still assuming $\mathring{V} \neq \emptyset$, two applications of Lemma 6.5 on $M = U_- \cup V \cup U_+$, lead to the conclusion that G acts freely and properly on $\mathcal{F}^{(2)}M$.

If $\mathring{V} = \emptyset$, then Lemma 6.5 applies to $M = U_- \cup U_+$ to yield the same conclusion. We have proved 2-rigidity of the sphere actions in construction II. The $\mathbf{RP}^n$-actions are double-covered by sphere actions, so the conclusion applies to them, as well. □

Recall from Section 1.3 of the introduction that Benveniste–Fisher proved in [2] nonexistence of an invariant rigid geometric structure of algebraic type for certain exotic $\mathrm{SL}(n, \mathbf{Z})$-actions on $\mathbf{T}^n$ constructed by Katok–Lewis in [18]. That proof relied on the affine local action of $\mathbf{R}^n$ on $\mathbf{T}^n \backslash \{0\}$, which is not available for the $\mathrm{SL}(n, \mathbf{R})$-actions of Theorems 4.2 and 4.3. We formulate here a variant of Question 1.1:

Question 6.6. *Which smooth* $\mathrm{SL}(n, \mathbf{R})$*-actions on closed* n*-manifolds preserve a rigid geometric structure of algebraic type?*

We identify some in the next section, and we expect that these are the only ones.

6.2. **No invariant projective structure for nonstandard actions.** A projective structure on a manifold M^n determines a canonical Cartan geometry modeled on $\mathbf{RP}^n$, comprising a principal Q_{n+1}-bundle over M equipped with an $\mathfrak{sl}(n+1, \mathbf{R})$-valued 1-form satisfying three axioms (see [30, Thm 3.8] or [20, Thm 4.2]). Here $Q_{n+1} < \mathrm{SL}(n+1, \mathbf{R})$ is the maximal parabolic subgroup stabilizing a line of $\mathbf{R}^{n+1}$ in the standard representation, as usual. A consequence is that the isotropy in the group of projective transformations at any point of M admits an injective homomorphism to Q_{n+1}.

If the projective Weyl curvature of a projective structure on M^n vanishes, then M is *projectively flat* and has a $(\mathrm{PSL}(n+1, \mathbf{R}), \mathbf{RP}^n)$-structure. Such a structure corresponds to a projective map $\delta : \widetilde{M} \to \mathbf{S}^n$ called the *developing map*, a local diffeomorphism, equivariant with respect to a *holonomy homomorphism* $\rho : \pi_1(M) \to \mathrm{SL}(n+1, \mathbf{R})$. See [32, 25] for more about these structures.

An n-dimensional *Hopf manifold* is a compact quotient $(\mathbf{R}^n\backslash\{0\})/\Lambda$ for Λ a lattice in the group of scalars $\mathbf{R}^*$, such as $\Lambda = \{2^k \cdot \mathrm{Id}_n \ : \ k \in \mathbf{Z}\}$. The transitive $\mathrm{SL}(n, \mathbf{R})$-action preserves the flat connection on these spaces inherited from $\mathbf{R}^n$. Note that the connection on the quotient is not the Levi-Civita connection of any metric, because it is not unimodular. These actions are projective. Hopf manifolds arise from Construction I with X a nonvanishing vector field on $\Sigma^0 = S^1$.

The standard action of $\mathrm{SL}(n, \mathbf{R})$ on $\mathbf{S}^n$ preserves the standard projective structure, which can be viewed as a projective compactification of two copies of $\mathbf{R}^n$ by $\mathbf{S}^{n-1}$. It arises from Construction II from a vector field X with a single zero in $(-1, 1)$ of order one and derivative -1.

Theorem 6.7. *Let G be locally isomorphic to $\mathrm{SL}(n, \mathbf{R})$, acting smoothly on a compact n-manifold M, preserving a projective structure $[\nabla]$. Then $(M, [\nabla])$ is equivalent to*

- *$\mathbf{S}^n$ or $\mathbf{RP}^n$ with the standard projective structure*
- *a Hopf manifold, diffeomorphic to a flat circle bundle over $\mathbf{RP}^{n-1}$ or $\mathbf{S}^{n-1}$ with trivial or $\mathbf{Z}_2$ monodromy.*

Proof. If the G-action is transitive and $n > 4$, then it is type I and M is a quotient of $\mathbf{R}^n\backslash\{0\}$ by a cocompact, discrete group of scalar matrices, by Theorem 2.1—a Hopf manifold. For $n = 4$, if M is not a Hopf manifold then it equals the Grassmannian $\mathcal{F}^4_2$, up to finite covers. Similarly, if $n = 3$ and M is not a Hopf manifold, then it equals the flag variety $\mathcal{F}^3_{1,2}$, up to finite covers. We will show that these homogeneous spaces do not have an invariant projective structure.

The stabilizer $P^4_2 < \mathrm{SL}(4, \mathbf{R})$ of a point of $\mathcal{F}^4_2$ is a semidirect product $S(\mathrm{GL}(2, \mathbf{R}) \times \mathrm{GL}(2, \mathbf{R})) \ltimes U$, where U is isomorphic to the abelian group of linear endomorphisms of $\mathbf{R}^2$. The fact that $\mathrm{Ad}\, U$ is trivial on $\mathfrak{sl}(4, \mathbf{R})/\mathfrak{p}^4_2$ corresponds to the differentials of all elements of U being trivial at the P^4_2-fixed point in $\mathcal{F}^4_2$. The stabilizer $P^3_{1,2}$ of a point of $\mathcal{F}^3_{1,2}$ is a semidirect product $(\mathbf{R}^*)^2 \ltimes N$ with N isomorphic to the 3-dimensional Heisenberg group. The center $Z(N)$ acts trivially via Ad on $\mathfrak{sl}(3, \mathbf{R})/\mathfrak{p}^3_{1,2}$, which corresponds to its differential being trivial on $\mathcal{F}^3_{1,2}$ at the $P^3_{1,2}$-fixed point. These projective transformations with trivial differential are called *strongly essential* (see [7], [23]).

Nagano–Ochiai proved that if there is a strongly essential 1-parameter subgroup of the stabilizer of a point $p \in M$ in the projective group, then a neighborhood of p in M is projecively flat [24, Lem 5.6]. This gives a projective local diffeomorphism from a neighborhood of any point of $\mathcal{F}^4_2$ or $\mathcal{F}^3_{1,2}$ to an open subset of $\mathbf{S}^4$ or $\mathbf{S}^3$, respectively. All local projective transformations of $\mathbf{S}^n$ are restrictions of elements of $\mathrm{SL}(n+1, \mathbf{R})$ (see [30, Thm 5.5.2]). By transitivity of the projective G-actions, the developing map would be a G-equivariant projective embedding of $\mathcal{F}^4_2$ or $\mathcal{F}^3_{1,2}$—or a finite cover—into $\mathbf{S}^4$ or $\mathbf{S}^3$, respectively, for some monomorphism $G \to \mathrm{SL}(n +$

$1, \mathbf{R})$. There is no closed, n-dimensional orbit of $\mathrm{SL}(n, \mathbf{R})$ in the projective action on $\mathbf{S}^n$, so this is a contradiction.

Next assume the G-action on M is not transitive. By Theorems 4.2 and 4.3, the action arises from Construction I or II. In either case there is a closed $(n-1)$-dimensional orbit O, equivalent to $\mathbf{S}^{n-1}$ or $\mathbf{RP}^{n-1}$ by Theorem 2.1.

There is a 1-parameter group of strongly essential projective transformations in this case, too. Let p_0 be a Q^0-fixed point in O. The unipotent radical U of Q^0 is in the kernel of the differential along O at p_0. The Q^0-invariant curve of C^0-fixed points runs through p_0 transversal to O; denote it Σ. The Q^0-action on Σ factors through ν^0, so it is pointwise fixed by U. Thus U is in the kernel of the full differential at p_0.

Now [24, Lem 5.6] again says that the projective structure on M is flat in a neighborhood of p_0. By G-invariance of the projective structure, it is projectively flat in a neighborhood V of O. The developing map $\delta : \widetilde{V} \to \mathbf{S}^n$ is a local diffeomorphism. Here $\widetilde{V}$ can be assumed diffeomorphic to $\mathbf{S}^{n-1} \times (-\epsilon, \epsilon)$. The vector fields generating the G-action on $\widetilde{V}$ are conjugated by δ to projective vector fields on $\mathbf{S}^n$, forming a subalgebra of $\mathfrak{sl}(n+1, \mathbf{R})$ isomorphic to $\mathfrak{sl}(n, \mathbf{R})$. Let G' be the corresponding subgroup of $\mathrm{SL}(n+1, \mathbf{R})$. The developing image of $\widetilde{O}$ is an $(n-1)$-dimensional orbit O' of $G' \cong \mathrm{SL}(n, \mathbf{R})$. Up to a projective transformation of $\mathbf{S}^n$, it must be the hypersphere $\mathbf{S}^{n-1} \subset \mathbf{S}^n$. Now δ restricts to an equivariant diffeomorphism $\widetilde{O} \to O'$.

Next, $V' = \delta(\widetilde{V})$ is diffeomorphic to $\mathbf{S}^{n-1} \times (-\epsilon, \epsilon)$; moreover δ is a diffeomorphism $\widetilde{V} \to V'$. The saturation $G.\widetilde{V}$ is projectively flat, and its developing image is the saturation $G'.V'$. The latter set is the complement of the two $\mathrm{SL}(n, \mathbf{R})$-fixed points in $\mathbf{S}^n$.

Replace V with $G.V$ and V' with $G'.V'$, and consider a point p on the boundary of V. The orbit $N = G.p$ is necessarily closed. If it is $\mathbf{S}^{n-1}$ or $\mathbf{RP}^{n-1}$, then the argument above implies that $\widetilde{N}$ develops onto a hypersphere in $\mathbf{S}^n$. Because δ is a local diffeomorphism and maps $\widetilde{V}$ to V' diffeomorphically, the image $\delta(\widetilde{N})$ must be on the boundary of V', which comprises only points. We conclude that the boundary of V comprises G-fixed points. In Construction II there are at most two G-fixed points. If M has one, then it is equivalent to $\mathbf{RP}^n$ with the standard action, and if M has two, then it is equivalent to $\mathbf{S}^n$ with the standard action. □

References

1. W. Ballmann, *Geometric structures*, `http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/geostr00.pdf`.
2. E. J. Benveniste and D. Fisher, *Nonexistence of invariant rigid structures and invariant almost rigid structures*, Comm. Anal. Geom. **13** (2005), no. 1, 89–111.
3. A. Brown, D. Fisher, and S. Hurtado, *Zimmer's conjecture: Subexponential growth, measure rigidity, and strong property (T)*, arXiv.org:1608.04995.
4. ———, *Zimmer's conjecture for SL(m,Z)*, Inv. Math. **221** (2020), no. 3, 1001–1060.

5. ——, *Zimmer's conjecture for non-uniform lattices and escape of mass*, arxiv:2105.14541, 2021.
6. G. Cairns and É. Ghys, *The local linearization problem for smooth $SL(n)$-actions*, Enseign. Math. **43** (1997), 133–171.
7. A. Čap and K. Melnick, *Essential killing fields of parabolic geometries: projective and conformal structures*, Cent. Eur. J. Math. **11** (2013), no. 12, 2053–2061.
8. G. D'Ambra and M. Gromov, *Lectures on transformation groups: geometry and dynamics*, Surveys in Differential Geometry (Lehigh University, Bethlehem, PA, 1990) (Cambrige, MA), 1991, pp. 19–111.
9. Benson Farb and Peter Shalen, *Lattice actions, 3-manifolds and homology*, Topology **39** (2000), no. 3, 573–587. MR 1746910
10. R. Feres, *Rigid geometric structures and actions of semisimple Lie groups*, Rigidité, Groupe Fondamentale, et Dynamique, vol. 13, Société Mathématique de France, 2002, pp. 121–166.
11. D. Fisher, *Groups acting on manifolds: Around the Zimmer program*, Geometry, Rigidity, and Group Actions (London) (D. Fisher B. Farb, ed.), University of Chicago Press, 2011, pp. 74–93.
12. ——, Group actions in ergodic theory, geometry and toplogy, selected papers of Robert J. Zimmer (London) (D. Fisher, ed.), University of Chicago Press, 2020.
13. D. Fisher and K. Whyte, *Continuous quotients for lattice actions on compact spaces*, Geom. Dedicata **87** (2001), no. 1-3, 181–189.
14. M. Foreman, *The complexity of the structure and classification of dynamical systems*, arXiv:2203.10655, 2022.
15. M. Gromov, *Rigid transformations groups*, Géométrie Différentielle (Paris, 1986) (D. Bernard and Y. Choquet-Bruhat, eds.), Hermann, Paris, 1988, pp. 65–139.
16. V. Guillemin and S. Sternberg, *Remarks on a paper of Hermann*, Trans. Amer. Math. Soc. **130** (1968), 110–116.
17. N. Hitchin, *Mechanics, analysis and geometry: 200 years after lagrange*, ch. Vector fields on the circle, pp. 359–378, Elsevier Sci. Pub., 1991, Ed. M. Francaviglia.
18. A. Katok and J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume-preserving actions*, Israel J. Math. **93** (1996), 253–280.
19. A.W. Knapp, *Lie groups: Beyond an introduction*, Birkhäuser, Boston, MA, 1996.
20. S. Kobayashi, *Transformation groups in differential geometry*, Springer, Berlin, 1995.
21. A.G. Kushnirenko, *Linear-equivalent action of a semisimple Lie group in the neighborhood of a stationary point*, Functional Anal. Appl. 1 (1967), 89–90.
22. K. Lin, *The real Grassmannian $Gr(2,4)$*, `https://math.harvard.edu/~klin/Gr24.pdf`.
23. K. Melnick and K. Neusser, *Strongly essential flows for irreducible parabolic geometries*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 8079–8110.
24. T. Nagano and T. Ochiai, *On compact Riemannian manifolds admitting essential projective transformations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), no. 2, 233–246.
25. V.Y. Ovsienko and S.L Tabachnikov, *Projective geometry old and new*, Cambridge Tracts in Mathematics, no. 165, Cambridge Univ. Press, Cambridge, 2005.
26. V. Pecastaing, *Projective and conformal closed manifolds with a higher-rank lattice action*, arxiv:1910.06199, 2019.
27. G. Prasad and A.S. Rapinchuk, *Irreducible tori in semisimple groups*, Internat. Math. Res. Notices (2001), no. 23, 1229–1242.
28. C. Rosendal, *Descriptive classification theory and separable Banach spaces*, Notices Amer. Math. Soc. **58** (2011), no. 9, 1251–1262.
29. C.R. Schneider, *$SL(2,\mathbf{R})$-actions on surfaces*, Amer. J. Math. **96** (1974), no. 4, 511–528.

30. R.W. Sharpe, *Differential geometry : Cartan's generalization of Klein's Erlangen program*, Springer, New York, 1996.
31. D.C. Stowe, *Real analytic actions of* $SL(2, \mathbf{R})$ *on a surface*, Erg. Th. Dynam. Sys. **3** (1983), 447–499.
32. W.P. Thurston, *Three-dimensional geometry and topology*, vol. 1, Princeton Univ. Press, 1997, ed. S. Levy.
33. F. Uchida, *Classification of real analytic* $SL(n, \mathbf{R})$ *actions on* n*-sphere*, Osaka J. Math. **16** (1979), 561–579.
34. Fuichi Uchida, *On smooth* $\mathrm{SO}_0(p, q)$*-actions on* S^{p+q-1}, Osaka J. Math. **26** (1989), no. 4, 775–787. MR 1040424
35. ———, *Smooth* $\mathrm{SL}(n, \mathbf{C})$ *actions on* $(2n-1)$*-manifolds*, Hokkaido Math. J. **21** (1992), no. 1, 79–86. MR 1153753
36. ———, *On smooth* $\mathrm{SO}_0(p, q)$*-actions on* S^{p+q-1}*. II*, Tohoku Math. J. (2) **49** (1997), no. 2, 185–202. MR 1447181
37. ———, *On smooth* $\mathrm{Sp}(p, q)$*-actions on* $S^{4p+4q-1}$, Osaka J. Math. **39** (2002), no. 2, 293–314. MR 1914295
38. Fuichi Uchida and Kazuo Muk**o**yama, *Smooth actions of non-compact semi-simple Lie groups*, Current trends in transformation groups, K-Monogr. Math., vol. 7, Kluwer Acad. Publ., Dordrecht, 2002, pp. 201–215. MR 1941644
39. R.J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.
40. ———, *Actions of semisimple groups and discrete subgroups*, Proceedings of the ICM (Berkeley, California, 1986) (Providence, RI), AMS, 1987, pp. 1247–1258.

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