Some converse Lyapunov-like results for strong forward invariance

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***Abstract*— In the setting of a differential inclusion, strong forward invariance of a closed or a compact set is studied. Main results are novel necessary Lyapunov-like conditions for this property. They involve time-varying and autonomous Lyapunov/barrier functions that are smooth everywhere or at least outside the invariant set and are decreasing or at least not increasing faster than exponentially.**

I. Introduction

Invariance is an important concept in dynamics and control. When uniqueness of solutions to the dynamics is not guaranteed, one may need to distinguish between weak invariance and strong invariance. Strong forward invariance of a set is the property that no (forward) solution to the dynamics leaves the set. This paper studies strong forward invariance of a closed or compact set in the setting of differential inclusions. The main results, proved under Basic Assumptions on the differential inclusion, with and without its Lipschitz continuity, show that for strongly forward invariant closed or compact sets there exist Lyapunov-like functions that are positive definite with respect to the invariant set, are reasonably smooth, and certify the invariance.

Necessary and sufficient conditions for strong forward invariance of a set, in terms of tangent cones to the set, are reasonably well-known [1]. Sufficient conditions involving Lyapunov-like functions appeared in [2], [3], [4], [5]. Necessary conditions involving Lyapunov-like functions, especially smooth ones, appear to not have been studied. Proposing such conditions is the main contribution of this paper.

When the invariant set is compact, Theorem 3.1 provides a smooth time-varying Lyapunov-like function. Its proof relies on augmenting the differential inclusion and applying, to the resulting hybrid system, a converse Lyapunov result of [6]. For the more general case of a closed set, Theorem 3.2 provides an autonomous Lyapunov-like function that is smooth outside the invariant set K and does not increase faster than exponentially. Its proof relies on a smooth Lyapunov-like barrier function [7] that certifies the invariance of the open complement of the invariant set K for the reverse dynamics. Theorem 3.4, for Lipschitz dynamics, provides an autonomous Lyapunov-like function that is continuously differentiable everywhere and does not increase faster than exponentially. Theorem 3.7 characterizes, through the existence of an exponentially decreasing smooth Lyapunov function, strong forward invariance that persists under nonvanishing perturbations. This is facilitated by an idea of [8], which links this property to asymptotic stability under smaller, but still nonvanishing, perturbations.

Related results include the existence of smooth Lyapunov-like or barrier functions that verify no finite-time blow-up, or forward completeness, of the solutions on open sets; see [9] for differential equations and, essentially, Lipschitz continuous differential inclusions, and [7] for hybrid inclusions.[1] A different, closely related, body of work is on barrier functions and safety, including [12], [13], [14], [15], [8], [16], [17], [18] and [19]. Section IV-A below explores the connections to safety in some detail. For a survey of converse Lyapunov results related to asymptotic stability, see [20].

II. Setting and Preliminaries

The setting for most of this paper is that of a differential inclusion

$$\dot{x} \in F(x), \tag{1}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping: for every $x \in \mathbb{R}^n$, $F(x) \subset \mathbb{R}^n$ is a set. Solutions to (1) are understood in the Caratheodory sense: $\phi : I \to \mathbb{R}^n$ is a *solution* to (1) if $I \subset \mathbb{R}$ is an interval, ϕ is locally absolutely continuous on I, and, for almost every $t \in I$, $\dot{\phi}(t) \in F(\phi(t))$.

Let $S \subset \mathbb{R}^n$ be a set. The main property of interest is defined below:

> The set S is *strongly forward invariant* for (1) if every solution $\phi : [0, T] \to \mathbb{R}^n$ of (1) with $\phi(0) \in S$ satisfies $\phi(t) \in S$ for every $t \in [0, T]$; equivalently, if there exists no solution $\phi : [0, T] \to \mathbb{R}^n$ of (1) such that $\phi(0) \in S$ and $\phi(T) \notin S$.

Note that strong forward invariance does not insist on the existence of solutions. As an extreme example, one can consider (1) where $F(x) = \emptyset$ for all $x \in \mathbb{R}^n$, and then every set S is strongly forward invariant for (1) as there is no solution to check. A common understanding of weak forward invariance of a set S (see, e.g., [21]) insists on the existence of at least one forward complete solution, i.e., a solution with domain unbounded to the right, that remains in S, from every initial condition in S. One should be aware then that, in general, strong forward invariance does not ensure weak forward invariance in this sense.

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[1] Early necessary and sufficient conditions for no finite-time blow-up, for time-varying differential equations, with or without uniqueness, and involving Lyapunov-like functions that are not smooth, are in [10] and [11].

A. Sufficient conditions

Below, a sufficient Lyapunov-like condition for strong forward invariance of a closed set is given. Another one, involving a time-varying Lyapunov-like function, is included later in Theorem 3.1. The main purpose of this paper is to provide converse results to these sufficient conditions.

Proposition 2.1: *(general sufficient condition) Let $K \subset \mathbb{R}^n$ be a nonempty closed set. Suppose that there exist an open neighborhood $U \subset \mathbb{R}^n$ of the boundary ∂K of K, a continuous function $V : U \to \mathbb{R}$ that is continuously differentiable on $U \setminus K$ and such that $V(x) = 0$ if $x \in \partial K$ and $V(x) > 0$ if $x \in U \setminus K$, and $\lambda \in \mathbb{R}$ so that*

$$\nabla V(x) \cdot f \leq \lambda V(x) \quad \forall x \in U \setminus K,\ \forall f \in F(x).$$

Then, K is strongly forward invariant for (1).

The right-hand side of the Lyapunov inequality in Proposition 2.1 involves the function $v \mapsto \lambda v$. More general functions that guarantee the same outcome can be used; see for example the so-called "uniqueness functions" in [22, Definition 4] and the earlier references therein. The sufficient conditions in Proposition 2.1 can be generalized by considering functions V that need not be smooth, for example just locally Lipschitz, and using generalized differentiation techniques, for example the Clarke generalized gradient.

B. Background results

From now on, the following assumption is in place:

Assumption 2.2: *(Standing Assumption) The set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfies the Basic Assumptions: for every $x \in \mathbb{R}^n$, $F(x)$ is nonempty, closed, and convex, and F is outer semicontinuous and locally bounded on $\mathbb{R}^n$.*

Equivalently, F has nonempty, compact, and convex values at every $x \in \mathbb{R}^n$ and is upper semicontinuous. All set-valued analysis terminology used here is from [23], see also [24, Chapter 5] or [25]. The outer semicontinuity and local boundedness of F amounts to the graph $\{(x, y) \in \mathbb{R}^{2n} \mid y \in F(x)\}$ of F being closed and the set $\bigcup_{x\in S} F(x)$ being bounded for every bounded $S \subset \mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be a set. Two notions related to strong forward invariance are defined next:

> The differential inclusion (1) has *no finite-time blow-up relative to S* if there exists no solution $\phi : [0, T) \to \mathbb{R}^n$ of (1) such that $\phi(t) \in S$ for all $t \in [0, T)$ and either $\lim_{t\to T} \|\phi(t)\| = \infty$ or there exists a sequence $t_i \nearrow T$ such that $\lim_{i\to\infty} \phi(t_i) \notin S$.

The case of $S = \mathbb{R}^n$ deserves a slightly different name:

> The differential inclusion (1) has *no finite-time blow-up to ∞* if there exists no solution $\phi : [0, T) \to \mathbb{R}^n$ of (1) such that $\lim_{t\to T} \|\phi(t)\| = \infty$.

Sufficient conditions for no finite-time blow-up to ∞ include:

(i) linear growth of F: the existence of $a, b > 0$ such that

$$F(x) \subset (a + b\|x\|)\mathbb{B} \quad \forall x \in \mathbb{R}^n;$$

(ii) existence of a compact and global asymptotically stable set for (1);

and more. Mappings F that satisfy the Basic Assumptions and have linear growth are often called "Marchaud mappings" [1].

A basic existence result for (1), which can be found in [26], and some relationships between strong forward invariance and finite-time blow-up are summarized next.

Theorem 2.3: *(existence and completeness) Let $O \subset \mathbb{R}^n$ be an open set. Then:*

(a) *For every $x_0 \in O$ there exists $T > 0$ and a solution $\phi : [0, T] \to O$ of* (1) *with $\phi(0) = x_0$.*
(b) *If* (1) *has no finite-time blow-up relative to O then every solution to* (1) *can be extended to be forward complete, and O is strongly forward invariant for* (1).
(c) *If* (1) *has no finite-time blow-up to ∞ and O is strongly forward invariant for* (1) *then* (1) *has no finite-time blow up relative to O.*

The property concluded in (b) above is sometimes called *forward completeness* of (1) on O. The implication opposite to Theorem 2.3(b), from strong forward invariance to no blow-up is not true in general. For example, the differential equation $\dot{x} = x^2$ on $\mathbb{R}$ has the open set $O = \mathbb{R}$ strongly forward (and backward) invariant but no solution from $x_0 > 0$ is forward complete: they all blow-up in finite time.

Let $O \subset \mathbb{R}^n$ be an open set. A function $\omega : O \to (0, \infty)$ is *proper with respect to O* if it is continuous and $\omega(x_i) \to \infty$ for every sequence of points $x_i \in O$ such that $\|x_i\| \to \infty$ or $x_i \to x$ for some $x \notin O$.

Theorem 2.4: *(Lyapunov characterization of forward completeness) Let $O \subset \mathbb{R}^n$ be open. Then, the following are equivalent:*

(a) (1) *has no finite-time blow-up relative to O;*
(b) (1) *is forward complete on O;*
(c) *For every proper with respect to O function $\omega : O \to (0, \infty)$, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 and a smooth function $V : O \to (0, \infty)$ such that:*

$$\alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x)), \tag{2}$$

$$\nabla V(x) \cdot f \leq V(x) \quad \forall f \in F(x),$$

for all $x \in O$.

The equivalence of (b) and (c) follows from the generalization [7, Theorem 8.1] to hybrid dynamics of [9, Theorem 2] given for differential equations with perturbations.

The next background result is a converse Lyapunov result for a hybrid inclusion; see [6, Theorem 3.13], [24, Corollary 7.32]. A symbolic representation of a hybrid inclusion is

$$\begin{aligned} x \in C \quad & \dot{x} \in F(x) \\ x \in D \quad & x^+ \in G(x). \end{aligned} \tag{3}$$

Roughly, solutions to (3) may flow according to (1) while in C, and jump according to the difference inclusion $x^+ \in G(x)$ from D. A formal definition of a solution to (3), of Hybrid Basic Assumptions, which generalize Assumption 2.2

to the setting of (3), and stability concepts for (3) used below are as in [24]. In casual words, global pre-asymptotic stability of a set $\mathcal{A}$ means that solutions that start close to $\mathcal{A}$ remain close to it, all solutions are bounded, the forward complete ones converge to $\mathcal{A}$, and "pre" allows for maximal solutions to (3) to be not forward complete. The function $d_{\mathcal{A}}$ below is the distance from $\mathcal{A}$.

Theorem 2.5: *(hybrid converse Lyapunov) Suppose that the data* (F, C, G, D) *of* (3) *satisfies the Hybrid Basic Assumptions. Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty compact set that is globally pre-asymptotically stable for* (3). *Then, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 and a smooth function $V : \mathbb{R}^n \to [0, \infty)$ such that* (2) *hold for all $x \in \mathbb{R}^n$, with $\omega(x) = d_{\mathcal{A}}(x)$, and*

$$\nabla V(x) \cdot f \le -V(x) \quad \forall x \in C,\ \forall f \in F(x);$$

$$V(g) \le \frac{1}{e} V(x) \quad \forall x \in D,\ \forall g \in G(x).$$

III. Converse results

This section contains the main results of the paper. Recall that Assumption 2.2 is in place.

A. Time-varying Lyapunov-like function

The result below first gives a sufficient condition for strong forward invariance of a closed set K in terms of a smooth time-varying Lyapunov-like function that does not increase faster than exponentially. The condition turns out necessary when K is compact and then the Lyapunov-like function is in fact decreasing. Key steps in the proof are augmenting the differential inclusion (1) to form a hybrid inclusion

$$\begin{array}{ll} (x,\tau) \in \mathbb{R}^n \times [0,1] & \begin{bmatrix} \dot{x} \\ \dot{\tau} \end{bmatrix} \in \begin{bmatrix} F(x) \\ 1 \end{bmatrix}, \\ (x,\tau) \in \mathbb{R}^n \times [0,1] & \begin{bmatrix} x^+ \\ \tau^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \end{bmatrix}. \end{array} \quad (4)$$

The x part of a solution to (4) flows according to (1) until the timer τ reaches 1, and then x jumps into K and the timer resets. This results in the augmented invariant set

$$\mathcal{A} := K \times [0,1]$$

being globally pre-asymptotically stable for (4), and one can invoke the hybrid converse result, Theorem 2.5.

Theorem 3.1: *(time-varying certificate of invariance) Let $K \subset \mathbb{R}^n$ be closed. If*

(a) *there exist an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$, $\lambda \in \mathbb{R}$, and a continuous function $V : \mathcal{N} \to \mathbb{R}$ that is continuously differentiable on $\mathcal{N} \setminus (K \times [0,1])$ such that*
   - (i) *$V(x,\tau) = 0$ for all $(x,\tau) \in \partial K \times [0,1]$,*
   - (ii) *$V(x,\tau) > 0$ for all $(x,\tau) \in \mathcal{N} \setminus (K \times [0,1])$,*
   - (iii) *$\nabla V(x,\tau) \cdot (f,1) \le \lambda V(x,\tau)$ for all $(x,\tau) \in \mathcal{N} \setminus (K \times [0,1])$ and all $f \in F(x)$,*

then

(b) *K is strongly forward invariant for* (1).

If K is nonempty and compact, and (b) holds, then there exist an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$ and a smooth function $V : \mathcal{N} \to [0,\infty)$ such that (i), (ii) above hold and

(iv) *$\nabla V(x,\tau) \cdot (f,1) \le -V(x,\tau)$ for all $(x,\tau) \in \mathcal{N}$ and all $f \in F(x)$.*

Pre-asymptotic stability of a compact set, under hybrid basic assumptions, is robust; see [24, Theorem 7.21]. From the proof of Theorem 3.1, one can then deduce that the strong forward invariance of a compact K for (1) is robust, in the following sense: there exists a continuous $\rho : \mathbb{R}^n \to [0,\infty)$ such that $\rho(x) > 0$ if $x \notin K$ such that K is strongly forward invariant for

$$\dot{x} \in F_\rho(x), \quad (5)$$

where the *inflation* $F_\rho : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of F is the set-valued mapping given at each $x \in \mathbb{R}^n$ by

$$F_\rho(x) := \operatorname{con} F(x + \rho(x)\mathbb{B}) + \rho(x)\mathbb{B}. \quad (6)$$

B. Autonomous Lyapunov-like function

For any set $K \subset \mathbb{R}^n$, K is strongly forward invariant for (1) if and only if $\mathbb{R}^n \setminus K$ is strongly forward invariant for

$$\dot{x} \in -F(x). \quad (7)$$

Below, no blow-up to ∞ is assumed about (7), and not (1). If F has linear growth, so does $-F$, and then both inclusions (1) and (7) have no finite-time blow-up to ∞.

Theorem 3.2: *(autonomous certificate of invariance) Suppose that* (7) *has no finite-time blow-up to ∞. Let $K \subset \mathbb{R}^n$ be a nonempty closed set. The following are equivalent:*

(a) *K is strongly forward invariant for* (1).

(b) *There exists a continuous function $V : \mathbb{R}^n \to [0,\infty)$ that is smooth on $\mathbb{R}^n \setminus K$ such that $V(x) = 0$ if and only if $x \in K$ and*

$$\nabla V(x) \cdot f \le V(x) \quad \forall x \in \mathbb{R}^n \setminus K,\ \forall f \in F(x).$$

The key step in the proof of the implication from (a) to (b), is observing that the open complement of K, $O := \mathbb{R}^n \setminus K$, is invariant for (7) and this property admits a Lyapunov-like characterization by a function V as in Theorem 2.4. Considering $1/V(x)$ on O and extending the resulting function to $\mathbb{R}^n$ produces the function required in (b).

For a compact invariant set K, one can dispose of the extra assumption of no blow-up to ∞, with the price to pay being that the Lyapunov-like function is constructed not on $\mathbb{R}^n$ but on a neighborhood of K.

Corollary 3.3: *(local autonomous certificate of invariance) Let $K \subset \mathbb{R}^n$ be a nonempty compact set. The following are equivalent:*

(a) *K is strongly forward invariant for* (1).

(b) *For any bounded and open neighborhood $U \subset \mathbb{R}^n$ of K there exists a continuous function $V : U \to [0,\infty)$ that is smooth on $U \setminus K$ such that $V(x) = 0$ if and only if $x \in K$ and*

$$\nabla V(x) \cdot f \le V(x) \quad \forall x \in U \setminus K,\ \forall f \in F(x).$$

C. Lipschitz dynamics

Under the Lipschitz assumption, the assumption of convexity of the values of F can be dropped. Indeed, under all other conditions from Basic Assumptions, and with Lipschitz continuity of F, it follows from the Filippov-Ważewski relaxation theorem (see, for example, [27, Theorem 10.4.4]) that strong forward invariance for (1) implies that property for the "relaxed" inclusion $\dot{x} \in \operatorname{con} F(x)$, which satisfies all Basic Assumptions and is Lipschitz continuous.

The previous two converse results, Theorem 3.1 and Theorem 3.2, rely on already available results on the existence of smooth Lyapunov or Lyapunov-like functions certifying certain properties. Below, a natural candidate for a Lyapunov-like function, namely the distance from the invariant set K, requires a smoothing procedure.

Theorem 3.4: *(autonomous certificate of invariance) Suppose that F is Lipschitz continuous on $\mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a nonempty compact set. The following are equivalent:*

- (a) *K is strongly forward invariant for (1).*
- (b) *There exists a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$, smooth on $\mathbb{R}^n \setminus K$, such that $V(x) = 0$ if and only if $x \in K$, and $\lambda > 0$ such that*

$$\nabla V(x) \cdot f \leq \lambda V(x) \quad \forall x \in \mathbb{R}^n,\ \forall f \in F(x).$$

The proof of the implication from (a) to (b), relies on distance function $d_K(x) := \min_{y \in K} \|x - y\|$. Its Dini/contingent derivative satisfies

$$Dd_K(x; f) := \liminf_{v \to f, t \searrow 0} \frac{d_K(x + tv) - d_K(x)}{t} \leq L d_K(x),$$

where L is a Lipschitz constant for F. The function d_K is Lipschitz continuous, hence differentiable almost every $x \in \mathbb{R}^n$, and at such x,

$$\nabla d_K(x) \cdot f \leq L d_K(x).$$

This inequality, and the smoothing of d_K using [28, Lemma 16], inspired by an earlier result by [29], leads to function that, when squared, provides the desired V. This function additonally satisfies, for every $x \in \mathbb{R}^n$,

$$a_1 (d_K(x))^2 \leq V(x) \leq a_2 (d_K(x))^2 \tag{8}$$

for $0 < a_1 < a_2$ and

$$\|\nabla V(x)\| \leq 3 d_K(x). \tag{9}$$

Bounds (8), (9) are expected to be useful in application of the Lyapunov-like functions as above to interconnections.

D. Strongly robust invariance

Strong forward invariance of a compact set is robust, in the sense that it persists for the inflated dynamics (5) given by the inflation (6), where the inflation size ρ vanishes on the compact set. This subsection addresses strong robustness, where the inflation size is positive everywhere.

The result below is a minor variation of the nice observation in [8, Theorem 19], and a special case of a hybrid inclusion version of [8, Theorem 19], provided in [18, Proposition 3.8]. It pertains to the differential inclusions

$$\dot{x} \in L(x) + \rho_1(x)\mathbb{B}, \tag{10}$$

$$\dot{x} \in L(x) + \rho_2(x)\mathbb{B}. \tag{11}$$

Proposition 3.5: *(from invariance to stability) Let $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy the Basic Assumptions and be locally Lipschitz continuous. Let $\rho_1 : \mathbb{R}^n \to (0, \infty)$ be continuous and $\rho_2 : \mathbb{R}^n \to (0, \infty)$ be locally Lipschitz continuous and such that $\rho_2(x) < \rho_1(x)$ for all $x \in \mathbb{R}^n$. If a set $X \subset \mathbb{R}^n$ is strongly forward invariant for (10) then its closure $\overline{X}$ is (locally) asymptotically stable for (11).*

The next result can be deduced from [28, Lemma 8], which extracted the essential conclusion from [29, Proposition 3.5], and [24, Lemma 7.36].

Lemma 3.6: *(Lipschitz inflation) For any continuous function $\rho : \mathbb{R}^n \to (0, \infty)$ there exist*

- (i) *a set-valued mapping $L : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ that satisfies the Basic Assumptions and is locally Lipschitz continuous;*
- (ii) *continuous functions $\rho_0, \rho_1 : \mathbb{R}^n \to (0, \infty)$*

such that, for every $x \in \mathbb{R}^n$,

$$F(x) \subset F_{\rho_0}(x) \subset L(x) \subset L(x) + \rho_1(x)\mathbb{B} \subset F_\rho(x). \tag{12}$$

This lemma lets one overapproximate F with a Lipschitz L, as described in (12). Combined with Proposition 3.5, this leads to the result below.

Theorem 3.7: *(certificate of strongly robust invariance) Let $K \subset \mathbb{R}^n$ be a nonempty and compact set. If*

- (a) *K is strongly robustly strongly forward invariant, i.e., there exists a continuous function $\rho : \mathbb{R}^n \to (0, \infty)$ such that K is strongly forward invariant for (5),*

then

- (b) *there exists a smooth function $V : \mathbb{R}^n \to [0, \infty)$, positive definite with respect to K and proper, and a neighborhood $N \subset \mathbb{R}^n$ of K, such that*

$$\nabla V(x) \cdot f \leq -V(x) \quad \forall x \in N,\ \forall f \in F(x).$$

IV. Connections and Applications

A. Barriers and safety

Consider $X_g, X_b \subset \mathbb{R}^n$ with $X_g \cap X_b = \emptyset$. The differential inclusion (1) is *safe with respect to* (X_g, X_b) if there exists no solution from X_g that reaches X_b. Some immediate connections of the concept of safety to the concept of strong forward invariance are as follows: Obviously, for any set $K \subset \mathbb{R}^N$, (1) is safe with respect to $(K, \mathbb{R}^n \setminus K)$ if and only if K is strongly forward invariant for (1). More broadly, with X_g, X_b as above, if there exists a strongly forward invariant set $K \subset \mathbb{R}^n$ for (1) such that

$$X_g \subset K \subset \mathbb{R}^n \setminus X_b, \tag{13}$$

then (1) is safe with respect to (X_g, X_b). The opposite implication is true too: if (1) is safe with respect to (X_g, X_b), then the strongly forward invariant (essentially by its definition) infinite horizon reachable set from X_g plays the role of a set K such that (13) holds.

A natural sufficient condition for safety, dating back to [30] and given originally for a differential equation, involves a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ such that $B(x) \leq 0$ for $x \in X_g$, $B(x) > 0$ for $x \in X_b$, and

$$\nabla B(x) \cdot f \leq 0 \quad \forall x \in \mathbb{R}^n,\ \forall f \in F(x). \tag{14}$$

Such B is said to be a barrier function. Naturally, (14) renders the set

$$K := \{x \in \mathbb{R}^n \,:\, B(x) \leq 0\} \tag{15}$$

strongly forward invariant. This property, and safety with respect to (X_g, X_b), remains true if (14) is changed to a strict inequality but only at each $x \in \partial K$.

Already in [30], (14) was shown to be necessary, for a differential equation, and subject to compactness of X_g, X_b and of the whole state space, and a further somewhat restrictive assumption. Versions and extensions of this sufficient condition, and converse statements, appear in [31], [14], [32], [15], [19] [8], and [16]. For example, in the current setting of a differential inclusion subject to the Basic Assumptions, [19] provides a necessary and sufficient condition for safety with a time-varying, lower semicontinuous, and nonincreasing barrier function. Characterizations of strongly robust safety are discussed below, after Proposition 4.2.

The converse results producing barrier functions yield functions that are not increasing. Most of these results cannot be immediately applied to a strongly forward invariant closed set $K =: X_g$ and its complement $X_b := \mathbb{R}^n \setminus K$, because such X_b is not closed and because often, for example in [31], [14], the state space itself is compact and not $\mathbb{R}^n$, and further other assumptions are made. In fact, for the simple example of $\dot{x} = x$ for which $K := \{0\}$ is strongly forward invariant there is no positive definite function that is nonincreasing along all solutions. Accordingly, the converse results in Section III allow for a not-too-fast increase.

One path to applying the converse results in Section III in the context of safety is as follows. Following [1], given a set $S \subset \mathbb{R}^n$, the (strong forward) *invariance kernel* of S is the largest closed subset of S that is strongly forward invariant for (1). From [1, Theorem 5.4.2], it follows that:

Theorem 4.1: *(invariance kernel) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be Lipschitz continuous. Then for any closed $S \subset \mathbb{R}^n$ there exists the (possibly empty) invariance kernel. It consists of all initial conditions in S from which all solutions stay in S.*

Thus, if X_b is open and if the invariance kernel of $\mathbb{R}^n \setminus X_b$ contains X_g, then (1) is safe with respect to (X_g, X_b).

Proposition 4.2: *(invariance kernel for safety) Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be Lipschitz continuous. Let $X_g, X_b \subset \mathbb{R}^n$ be nonempty and suppose that* (1) *is safe with respect to $(X_g, \widetilde{X_b})$, where $\widetilde{X_b}$ is an open set containing X_b. Then, $K \subset \mathbb{R}^n$ defined as the invariance kernel of $\mathbb{R}^n \setminus \widetilde{X_b}$ is nonempty and a strongly forward invariant for* (1) *closed set such that* (13) *holds.*

Now, one can apply the results in Section III to K. The additional margin of safety assumed in Proposition 4.2, through the open set $\widetilde{X_b}$, resembles what is sometimes done in the barrier/safety literature, like [14], or can be deduced from robustness assumptions, like in [8].

In the context of strongly robust safety for (1), made precise in the result below, [17] states a converse result that yields a continuously differentiable barrier B that is nonpositive on X_g, positive on X_b, and $\nabla B(x) \cdot f < 0$ for all x in the boundary of K in (15). This generalizes earlier results, [15, Theorem 1] and [8, Theorem 16], from differential equations or Lipschitz differential inclusions, to more general dynamics. The proof of [17, Theorem 2] relies on an unpublished converse result [33] with a technical proof. Under stronger assumptions, a version of [17, Theorem 2] and a generalization of [8, Theorem 16] can be deduced from Theorem 3.7. This is done below, where the construction of V from [8] is applied to a Lipschitz inflation of F.

Theorem 4.3: *(certificate of safety) Let $X_g, X_b \subset \mathbb{R}^n$ and suppose that* (1) *is strongly robustly safe with respect to (X_g, X_b), in the sense that there exists a continuous function $\rho : \mathbb{R}^n \to (0, \infty)$ such that* (5) *is safe with respect to (X_g, X_b). Suppose that $\overline{\mathcal{R}}$ is bounded and such that $\overline{\mathcal{R}} \cap \overline{X_b} = \emptyset$, where $\mathcal{R}$ is the (infinite horizon) reachable set from X_g for* (5). *Then, there exists a smooth function $B : \mathbb{R}^n \to [0, \infty)$ and $c > 0$ such that*

(i) *$B(x) < 0$ for all $x \in X_g$, and $B(x) > 0$ for all $x \in X_b$.*
(ii) *$\nabla B(x) \cdot f \leq -c$ for all $f \in F(x)$ and x such that $B(x) = 0$.*

B. State constraints

In anticipation of treating, in future work, strong forward invariance for hybrid inclusions, some of the results from the previous section are now extended to the setting of a differential inclusion with a constraint:

$$x \in C \quad \dot{x} \in F(x). \tag{16}$$

The following is posed throughout this subsection:

Assumption 4.4: *$C \subset \mathbb{R}^n$ is a nonempty and closed set.*

Solutions to (16) are solutions $\phi : I \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ that also satisfy $\phi(t) \in C$ for all $t \in I$.

Proposition 4.5: *(time-varying certificate of invariance) Let $K \subset \mathbb{R}^n$ be closed. If*

(a) *there exists an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $(K \cap C) \times [0, 1]$, $\lambda \in \mathbb{R}$, and a continuous function $V : \mathcal{N} \to \mathbb{R}$ that is smooth on an open set containing $(C \cap \mathcal{N}) \setminus (K \times [0, 1])$ such that*
  (i) *$V(x, \tau) = 0$ for all $(x, \tau) \in (C \cap \partial K) \times [0, 1]$,*
  (ii) *$V(x, \tau) > 0$ for all $(x, \tau) \in (C \cap \mathcal{N}) \setminus (K \times [0, 1])$,*
  (iii) *$\nabla V(x, \tau) \cdot (f, 1) \leq \lambda V(x, \tau)$ for all $(x, \tau) \in (C \cap \mathcal{N}) \setminus (K \times [0, 1])$ and $f \in F(x)$,*

then

(b) *K is strongly forward invariant for* (16).

If K is nonempty and compact, and (b) holds, then there exist a an open neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of $K \times [0,1]$ and a smooth function $V : \mathcal{N} \to [0, \infty)$ such that (i), (ii) above hold and

(iv) $\nabla V(x, \tau) \cdot (f, 1) \leq -V(x, \tau)$ *for all* $(x, \tau) \in \mathcal{N} \cap (C \times [0,1])$ *and* $f \in F(x)$.

The proof of Proposition 4.5 is essentially the same as that of Theorem 3.1. The difference is that the hybrid system (4) that augmented (1) in the proof of Theorem 3.1 requires here a different flow set. The first occurrence of $(x, \tau) \in \mathbb{R}^n \times [0,1]$ in (4) needs to be replaced here by $(x, \tau) \in C \times [0,1]$.

Proposition 4.6: *(autonomous certificate of invariance) Suppose that $\dot{x} \in -F(x)$, $x \in C$ has no finite-time blow-up to ∞. Let $K \subset \mathbb{R}^n$ be a nonempty closed set. The following are equivalent:*

(a) *K is strongly forward invariant for* (16).

(b) *There exists a continuous function $V : \mathbb{R}^n \to [0, \infty)$ that is smooth on $\mathbb{R}^n \setminus (K \cap C)$ such that $V(x) = 0$ if and only if $x \in K \cap C$ and*

$$\nabla V(x) \cdot f \leq V(x) \quad \forall x \in C \setminus K, \ f \in F(x).$$

The proof of Proposition 4.6 is similar to that of Theorem 3.2. One considers the open set $O := \mathbb{R}^n \setminus (K \cap C)$ and relies on [7, Theorem 8.1], rather than on its consequence stated in this paper, Theorem 2.4. The hybrid inclusion to which one applies [7, Theorem 8.1] is

$$\begin{array}{ll} x \in C \cap O & \dot{x} \in -F(x) \\ x \in O & x^+ = y \end{array}$$

where $y \in \mathbb{R}^n \setminus (K \cap C)$ is any a priori picked point. As for Corollary 3.3, one can dispose of the extra assumption of no blow-up and obtain a local result.

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