Distributed State Estimation of Jointly Observable Linear Systems under Directed Switching Networks

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***Abstract*— This paper investigates the distributed state estimation problem by a group of networked agents for jointly observable linear systems in the context of directed dynamic interaction topologies. In this setting, no individual agent can fully observe the system state using only its own measurements. We extend existing decentralized observer design methodologies to more complex scenarios, where the communication topology is directed, aperiodic and switching, and the system measurements may be available either continuously or at sporadic time instants. Leveraging the joint observability condition, and assuming a mild mutual reachability condition over specific time intervals, we demonstrate that the unobservable subspaces of each agent can be reconstructed using the observable subspaces of other agents in the network. The effectiveness of the proposed approach is illustrated through a numerical example.**

I. Introduction

The objective of the distributed state estimation problem is to design distributed observers to reconstruct the state of the plant to be observed by using the local measurements of each observer agents and the information received from its neighbors [1]. The primary challenge in the distributed state estimation problem arises from the limitation that none of the observer agents can independently reconstruct the plant state from its own local measurements.

The distributed state estimation problem under joint observability condition for static networks is studied in [1], [2], and for switching networks in [3]–[8], where graph topologies were considered to be undirected [3], [6], [8], periodic [9], or strongly connected [7]. Our proposed solution in this work addresses the challenges posed by the following constraints:

1) directed, aperiodic switching communication networks;
2) limited observability of the system state by each agent;
3) continuous or sporadic availability of system measurements.

Compensating for the unobservable subspaces of individual agents under these communication constraints presents a significant challenge. Building on the decentralized observer design in [1], utilizing joint observability property, and assuming mutual reachability of the agents over some frequent time intervals, the proposed distributed state estimation protocol enables observer agents to collaboratively reconstruct the plant state despite switching communication and partial observability.

Our approach utilizes the joint observability condition to project each agent's estimated state vector onto unique observable subspaces. This enables the decomposition of the distributed state estimation problem into multiple decoupled estimation subproblems, each associated with a distinct observable subspace. Despite the dynamic nature of the interaction topologies, we demonstrate that if the union of directed, switched graphs over some frequent time intervals guarantees mutual reachability among all agents, then the distributed state estimation can be achieved asymptotically for both continuous and sporadic system-agent measurements.

Notation. The set of nonnegative real numbers is denoted as $\mathbb{R}_{\geq 0}$, and nonnegative integers as $\mathbb{Z}_+$. For a subspace $\mathcal{U} \subset \mathbb{R}^n$, the orthogonal complement of $\mathcal{U}$ is defined as $\mathcal{U}^\perp := \{x \in \mathbb{R}^n \mid x^\top u = 0 \ \forall u \in \mathcal{U}\}$. The Kronecker product of matrices is given by the symbol $\otimes$. A vector $\mathbf{1}_N$ is an N dimensional column vector of all ones. The symbol I_n denotes an identity matrix with order $n \times n$, while $\mathbf{0}$ denotes a zero matrix with appropriate dimensions. For a matrix $A \in \mathbb{R}^{m \times n}$, $\text{Ker}(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$ and $\text{Im}(A) = \{z \in \mathbb{R}^m \mid z = Ax, x \in \mathbb{R}^n\}$ denote the null space and image space of A, respectively. For square matrices A_i, $i \in \{1, 2, \cdots, N\}$, of compatible dimensions, $A = \text{diag}(A_1, A_2, \cdots, A_N)$ represent a block-diagonal matrix with diagonal elements A_i. For matrices $A_i \in \mathbb{R}^{m_i \times n}$, $i \in \{1, 2, \cdots, N\}$, $(A_1, A_2, \cdots, A_N) = \begin{bmatrix} A_1^\top, A_2^\top, \cdots, A_N^\top \end{bmatrix}^\top$. Given two vectors $u, v \in \mathbb{R}^n$, the notation (u, v) is equivalent to $\begin{bmatrix} u^\top & v^\top \end{bmatrix}^\top$. Given two symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, the symbol $A \prec B$ indicates that the matrix $B - A$ is positive definite. For a vector x and a matrix M, the symbols $|x|$ and $|M|$ represent the Euclidean norm and the induced matrix 2-norm, respectively.

II. Problem Statement

A. Problem Description

Consider the continuous-time linear time-invariant system

$$\dot{x} = Ax \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector. A network of N agents receives partial measurements of x. For each $i \in \mathcal{V} := \{1, 2, \cdots, N\}$,

$$y_i = C_i x \in \mathbb{R}^{p_i} \tag{2}$$

is the partial output of system (1) continuously available to agent i. We collect these measurements in $y := (y_1, y_2, \cdots, y_N) \in \mathbb{R}^p$, where $p = \sum_{i=1}^N p_i$, and define

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$\bar{C} := (C_1, C_2, \cdots, C_N)$. We assume that none of the agents can independently estimate the state x from y_i, i.e., for each $i \in \mathcal{V}$, the pair (A, C_i) is not observable. Therefore, the goal of these N agents system is to collectively yield an unbiased estimate of the state x through local inter-agent communication. To solve this problem, we consider the following joint observability assumption, given below.

Assumption 1: The pair $(A, \bar{C})$ is observable.

Remark 1: For each $i \in \mathcal{V}$, let $\nu_i \in \mathbb{Z}_+$ be the dimension of the observable subspace $\mathcal{S}_i$ of the pair (A, C_i), i.e., $\text{rank}\,(\mathcal{O}_i) = \nu_i$, where $\mathcal{O}_i := (C_i, C_iA, \cdots, C_iA^{n-1})$ is the observability matrix associated with (A, C_i). Then, the unobservable subspace $\mathcal{U}_i := \text{Ker}(\mathcal{O}_i)$ is of dimension $n-\nu_i$. Since $\mathcal{S}_i$ is in the column space of $\mathcal{O}_i$, then $\mathcal{U}_i = \mathcal{S}_i^\perp$. Furthermore, the joint observability condition on the pair $(A, \bar{C})$ in Assumption 1 implies that $\bigcap_{i=1}^N \mathcal{U}_i = \{\mathbf{0}\}$.

Assumption 2: The matrix A in (1) is neutrally stable[1].

Remark 2: Under Assumption 2, there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix A is similar to a block diagonal matrix $\bar{A} := P^{-1}AP$ with each block being a real Jordan block of the form $J_k = \begin{bmatrix} 0 & -b_k \\ b_k & 0 \end{bmatrix}$. Similar to [3], [6], with no loss of generality, we thus consider $\bar{A}^\top = -\bar{A}$.

B. Graph Theory Preliminaries

Let $\sigma : [0, \infty) \mapsto \mathcal{P} := \{1, 2, \ldots, \rho_{\max}\}$ be a piecewise constant, right-continuous function that switches at the points in the sequence $\{t_s : s \in \{0, 1, 2, \ldots\}\}$ with a minimum dwell-time bound $T_1 > 0$, i.e., $t_{s+1} - t_s \geq T_1$. For each $\sigma \in \mathcal{P}$, let the communication network topology of the agents be described by a directed communication graph (digraph) $\mathcal{G}_\sigma := (\mathcal{V}, \mathcal{E}_\sigma)$, where $\mathcal{V}$ represents the set of agent nodes, and $\mathcal{E}_\sigma \subseteq \mathcal{V}\times\mathcal{V}$ denotes the set of edges representing communication links between agents. For $l \in \{0, 1, 2, \cdots\}$, if σ switches $k \in \mathbb{Z}_+$ times in the interval $[t_l, t_{l+k})$, then a $k-$union digraph $\bigcup_{r=l}^{l+k} \mathcal{G}_{\sigma(t_r)}$ is defined as a simple digraph[2] with node set $\mathcal{V}$ and edge set given by the union of the edge sets of all the graphs in the collection $\{\mathcal{G}_{\sigma(t_l)}, \mathcal{G}_{\sigma(t_{l+1})}, \cdots, \mathcal{G}_{\sigma(t_{l+k})}\}$.

For each $\sigma \in \mathcal{P}$, the weighted adjacency matrix $\mathcal{A}_\sigma := [a_{ij,\sigma}] \in \mathbb{R}^{N\times N}$ of the digraph $\mathcal{G}_\sigma$ is a nonnegative matrix with $a_{ii,\sigma} = 0$ and $a_{ij,\sigma} = 1$ for all $(j, i) \in \mathcal{E}_\sigma$, i.e., an agent i receives information from agent j during the switched configuration of agents $\mathcal{G}_\sigma$. The set of neighbors of agent i is denoted by $\mathcal{N}_{i,\sigma} := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_\sigma\}$. A directed path from a node i_l to i_{l+k} in $\mathcal{G}_\sigma$ is a sequence of directed edges $\{(i_l, i_{l+1}), (i_{l+1}, i_{l+2}), \cdots, (i_{l+k-1}, i_{l+k})\}$ from i_l to i_{l+k} and the node i_{l+k} is reachable from node i_l. A digraph is strongly connected if there exists a directed path from one node to another. A digraph is said to have a directed spanning tree if every node has exactly one parent except for one node, called the root, which has no parent and from which there is a directed path to every other node. We denote the Laplacian of a directed dynamic graph $\mathcal{G}_\sigma$ as a zero row sum matrix $\mathcal{L}_\sigma \in \mathbb{R}^{N\times N}$ with $l_{ij,\sigma} = -a_{ij,\sigma}$ for $i \neq j$ and $l_{ii,\sigma} = \sum_{j=1}^N a_{ij,\sigma}$.

C. Leader-follower Communication Digraph Preliminaries

Consider a digraph $\bar{\mathcal{G}}_\sigma = (\bar{\mathcal{V}}, \bar{\mathcal{E}}_\sigma)$ of $N + 1$ nodes with one leader node[3], designated as node 0 for simplicity, and N nodes of the set $\mathcal{V}$. If the outgoing communication link from node 0 to a node $i \in \mathcal{V}$ is represented as an edge $(0, i)$, then $\bar{\mathcal{E}}_\sigma = \{(0, i)\} \cup \mathcal{E}_\sigma$ for all $i \in \mathcal{N}_{0,\sigma}$. Let $\Delta_\sigma \in \mathbb{R}^{N\times N}$ be a nonnegative diagonal matrix whose i^{th} diagonal element is $a_{i0,\sigma}$ for each $i \in \mathcal{V}$. Then, the Laplacian matrix $\bar{\mathcal{L}}_\sigma$ of $\bar{\mathcal{G}}_\sigma := (\bar{\mathcal{V}}_\sigma, \bar{\mathcal{E}}_\sigma)$ is partitioned as follows:

$$\bar{\mathcal{L}}_\sigma := \left[\begin{array}{c|c} 0 & \mathbf{0} \\ \hline -\Delta_\sigma \mathbf{1}_N & H_\sigma \end{array}\right], \tag{3}$$

where the matrix $H_\sigma \in \mathbb{R}^{N\times N}$ has non-positive off-diagonal entries, and therefore $-H_\sigma$ is a Metzler matrix[4]. The zero first row is due to the fact that $a_{0j,\sigma} = 0$ for all σ. Furthermore, $\bar{\mathcal{L}}_\sigma \mathbf{1}_{N+1} = \mathbf{0}$ yields $\Delta_\sigma \mathbf{1}_N = H_\sigma 1_N$.

D. Graph Theoretic Results

We now include a graph theoretic result that plays a key role in establishing our main results. The following lemma is closely related to [11, Theorem 3.12], and the detailed proof is thus omitted here for the sake of brevity.

Lemma 1: Consider a linear switched system

$$\dot{z} = -\mu(H_{\sigma(t)} \otimes I_n)z \tag{4}$$

where $t \mapsto \sigma(t)$ is the switching signal, $z = (z_1, z_2, \cdots, z_N)$, $z_i \in \mathbb{R}^n$ for all $i \in \mathcal{V}$, $\mu > 0$ is an arbitrary constant, and $H_\sigma \in \mathbb{R}^{N\times N}$ is given in (3). If there exists a sequence $\{s_k\}_{k=0}^\infty \in \mathbb{Z}_+$ such that, for some $v > 0$, $t_{s_{k+1}} - t_{s_k} < v$, where t_s is switching time of $t \mapsto \sigma(t)$, and the union digraph $\bigcup_{r=s_k}^{s_{k+1}-1} \bar{\mathcal{G}}_{\sigma(t_r)}$ has a directed spanning tree rooted at node 0, then the origin of the system (4) is globally exponentially stable.

Remark 3: Since the origin is exponentially stable for (4), then, it is easy to verify that for any σ satisfying the conditions in Lemma 1, the origin of the switched system

$$\dot{\bar{z}} = [(I_N \otimes A) - \mu(H_\sigma \otimes I_n)]\,\bar{z}, \tag{5}$$

with $\bar{z} = (I_N \otimes e^{At})z$ is also exponentially stable, where the matrix A satisfies Assumption 2. Since, by Assumption 2, $A^\top = -A$ and consequently $|e^{At}| = 1$, therefore, each solution $t \mapsto \bar{z}(t)$ of (5) yields

$$|\bar{z}(t)| \leq |z(t)| \leq k_0 e^{-k_1 t}|z(0)| = k_0 e^{-k_1 t}|\bar{z}(0)|, \tag{6}$$

where $k_0,\ k_1 > 0$. Thus, the origin of the system (5) is exponentially stable as well.

[1]All eigenvalues of A are on the imaginary axis and semi-simple (algebraic multiplicity = geometric multiplicity).

[2]A digraph is simple if it does not contain either a self-loop or multiple directed edges between the same pair of vertices.

[3]A node that does not have a parent node.

[4]A matrix whose off-diagonal elements are positive or zero is referred to as a Metzler matrix [10]. So, a negated Metzler matrix has nonpositive off-diagonal entries.

E. Distributed State Estimation Protocol

In this section, we present the distributed observers for system (1) with locally available outputs (2) for directed dynamic network topologies. The distributed observer consists of N local observers with the dynamics of the i^{th} observer being given as

$$\dot{\hat{x}}_i = A\hat{x}_i + G_i(y_i - C_i\hat{x}_i) + \mu M_i \sum_{j\in\mathcal{N}_{i,\sigma(t)}} a_{ij,\sigma(t)}(\hat{x}_j - \hat{x}_i) \qquad i\in\mathcal{V}, \quad (7)$$

where $\hat{x}_i \in \mathbb{R}^n$ denotes the estimate of x generated by the i^{th} agent, and $A \in \mathbb{R}^{n\times n}$ is the system matrix in (1). Additionally, $G_i \in \mathbb{R}^{n\times p_i}$ and $M_i \in \mathbb{R}^{n\times n}$ are the observer gain and weighting matrices, respectively, which are to be designed. The scaling factor $\mu > 0$ is any arbitrary positive real number.

The structure of the distributed state estimation protocol (7) was initially proposed in [1] for static graphs, where σ is constant, and later extended to undirected switching graphs [3], [4], [6] with symmetric weights, namely, $a_{ij,\sigma} = a_{ji,\sigma}$ for all $\sigma \in \mathcal{P}$. In [5], [9], the switching signal characterizes periodicity of networks, i.e., $\sigma(t + T) = \sigma(t)$ for all $t \geq 0$ and a fixed time-period $T > 0$. In this paper, we extend this estimation framework to accommodate a more general $t \mapsto \sigma(t)$ allowing for switches between directed and aperiodic network topologies. Our proposed solution relaxes the requirement in [5], [7] of strong connectivity of $\mathcal{G}_{\sigma(t)}$ for all $t \geq 0$.

Problem 1: Given the system dynamics in (1) and measured outputs in (2), design gain matrices G_i, M_i for each $i \in \mathcal{V}$, and scaling factor μ in the distributed state estimation protocol (7), such that, for any arbitrary switching signal $t \mapsto \sigma(t)$ dictating a transition between directed dynamic network topologies $\mathcal{G}_{\sigma(t)}$ at switching instants t_s with $t_{s+1} - t_s \geq T_1 > 0$ for all $s \in \{0, 1, 2, \cdots\}$, the set

$$\mathcal{A} \coloneqq \{\tilde{x} \in \mathbb{R}^{Nn} \ : \ \tilde{x}_i = \mathbf{0} \ \forall i \in \mathcal{V}\}. \quad (8)$$

is globally exponentially stable for the dynamics of the concatenated error $\tilde{x} \coloneqq (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_N)$ with $\tilde{x}_i \coloneqq \hat{x}_i - x$.

III. A Distributed State Estimator with Continuous Measurements

A. Observability Decomposition

For each $i \in \mathcal{V}$, let $U_i \in \mathbb{R}^{n\times(n-\nu_i)}$ be a matrix whose columns form an orthonormal basis of $\mathcal{U}_i \in \mathbb{R}^{n\times(n-\nu_i)}$ such that $\text{Im}(U_i) = \mathcal{U}_i = \ker(\mathcal{O}_i)$, and $S_i \in \mathbb{R}^{n\times\nu_i}$ be a matrix whose columns form an orthonormal basis for $\mathcal{S}_i = \mathcal{U}_i^{\perp}$. Then, by construction, it follows that $\text{Im}(S_i) = \mathcal{S}_i = \text{Im}(\mathcal{O}_i^{\top})$. For each $i \in \mathcal{V}$, let us now define an orthogonal transformation matrix Σ_i as

$$\Sigma_i \coloneqq \begin{bmatrix} S_i & U_i \end{bmatrix} \in \mathbb{R}^{n\times n}, \ \Sigma_i^{\top}\Sigma_i = I_n, \quad (9)$$

which satisfies a variant of the Kalman observability decomposition lemma, given below, the proof of which is given in [6, Lemma 3.2].

Lemma 2: Under Assumption 2, for each $i \in \mathcal{V}$,

$$\Sigma_i^{\top} A \Sigma_i = \begin{bmatrix} \bar{A}_i & \mathbf{0} \\ \mathbf{0} & \hat{A}_i \end{bmatrix}, \ C_i\Sigma_i = \begin{bmatrix} \bar{C}_i & \mathbf{0} \end{bmatrix}, \quad (10)$$

where $\bar{A}_i \in \mathbb{R}^{\nu_i\times\nu_i}$, $\hat{A}_i \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$, $\bar{C}_i \in \mathbb{R}^{p_i\times\nu_i}$ are defined such that:

(I) $(\bar{A}_i, \bar{C}_i)$ is the observable pair,

(II) $\bar{A}_i$, $\hat{A}_i$ are skew-symmetric matrices,

(III) $S_i^{\top} A S_i = \bar{A}_i$, $U_i^{\top} A U_i = \hat{A}_i$.

To solve Problem 1, let us introduce the following assumption on the switched communication digraphs $\mathcal{G}_{\sigma(t)}$.

Assumption 3: There exists $\{s_k\}_{k=0}^{\infty} \in \mathbb{Z}_+$ and $v > 0$, such that $t_{s_{k+1}} - t_{s_k} < v$ for each $k \in \mathbb{Z}_+$, where t_s is switching time of $t \mapsto \sigma(t)$, and the union digraph $\bigcup_{r=s_k}^{s_{k+1}-1} \mathcal{G}_{\sigma(t_r)}$ is strongly connected.

B. Distributed Observer Design

For each $i \in \mathcal{V}$ and each $\sigma \in \mathcal{P}$, the evolution of the estimation error $\tilde{x}_i$, based on (1) and (7), is given by

$$\dot{\tilde{x}}_i = (A - G_iC_i)\tilde{x}_i + \mu M_i \sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}(\tilde{x}_j - \tilde{x}_i), \quad (11)$$

where G_i and M_i are constructed by following the decentralized design approach in [1]. Specifically,

$$G_i = \Sigma_i \begin{bmatrix} \bar{G}_i \\ \mathbf{0} \end{bmatrix}, \ M_i = \Sigma_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{n-\nu_i} \end{bmatrix} \Sigma_i^{\top}, \quad (12)$$

where $n - \nu_i$ is the dimension of $\mathcal{U}_i$. The specific structure of G_i and M_i in (12) arise from the fact that the neighborhood state estimates solely impact the unobservable mode, whereas each agent can independently estimate its own observable component [1]. By Item I of Lemma 2, the matrix $\bar{G}_i \in \mathbb{R}^{n\times\nu_i}$ in G_i is selected such that $\bar{A}_i - \bar{G}_i\bar{C}_i$ is Hurwitz. Next, by using the transformation (9), and substituting G_i and M_i from (12) into (11), we obtain, for each $\sigma \in \mathcal{P}$,

$$S_i^{\top}\dot{\tilde{x}}_i = \left(\bar{A}_i - \bar{G}_i\bar{C}_i\right) S_i^{\top}\tilde{x}_i, \quad (13)$$

$$U_i^{\top}\dot{\tilde{x}}_i = \hat{A}_iU_i^{\top}\tilde{x}_i + \mu \sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}U_i^{\top}(\tilde{x}_j - \tilde{x}_i). \quad (14)$$

Let each agent $i \in \mathcal{V}$ have unique observable subspace $\mathcal{S}_i$ that collectively span $\mathbb{R}^n$, i.e., $\mathcal{S}_i \bigcap \mathcal{S}_j = \{0\}$ (or $S_i^{\top}S_j = \mathbf{0}$) and $\bigcup_{i\in\mathcal{V}} \mathcal{S}_i = \mathbb{R}^n$ (or equivalently $\sum_{i=1}^{N} \nu_i = n$). Then, from Assumption 1, for each $i \in \mathcal{V}$,

$$\mathcal{U}_i = \bigcup_{j\in\mathcal{V}\setminus\{i\}} \mathcal{S}_j. \quad (15)$$

By appropriately exploiting the joint observability condition and mutual reachability of agents in Assumption 3, the proposed estimation protocol (7) can also be extended to the case where multiple agents may have identical observable subspaces.

Theorem 1: Let Assumptions 1 – 3 hold. Consider the system (1) with measurements y_i in (2) for each $i \in \mathcal{V}$. The distributed state estimation protocol (7), where the weighting matrix M_i and the gain matrix G_i are designed according to (12), with $\bar{G}_i$ such that the matrix $\bar{A}_i - \bar{G}_i\bar{C}_i$ is Hurwitz, solves Problem 1, i.e., the estimation error dynamics $\tilde{x}$ in (11) is globally exponentially stable, uniformly in σ.

Sketch proof: There are three major steps in this proof, which are given as follows.

(I) *Joint observability property*: The unobservable subspace of each agent is the same as the observable subspace of the remaining $N-1$ agents by Assumption 1. For each $i \in \mathcal{V}$, let the matrix $S_{-i} \in \mathbb{R}^{n\times(n-\nu_i)}$ be constructed by concatenating matrices S_k where $k \in \mathcal{V}\setminus\{i\}$. Therefore, the column spaces of S_{-i} form an orthonormal basis for the subspace $\mathcal{S}_{-i} := \bigcup_{j\in\mathcal{V}\setminus\{i\}} \mathcal{S}_j$ in (15). However, these basis vectors may not be identical to those spanning the subspace $\mathcal{U}_i$. Then, the change of orthonormal basis vectors between two identical subspaces $\mathcal{U}_i$ and $\mathcal{S}_{-i}$ yields that there exists an orthogonal transformation $\hat{\Gamma}_i \in \mathbb{R}^{(n-\nu_i)\times(n-\nu_i)}$ such that

$$U_i^\top = \hat{\Gamma}_i S_{-i}^\top. \tag{16}$$

By substituting (16) in (14), and using Lemma 2, we obtain, for each $\sigma \in \mathcal{P}$,

$$S_{-i}^\top\dot{\tilde{x}}_i = (S_{-i}^\top A S_{-i})S_{-i}^\top\tilde{x}_i + \mu\sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}S_{-i}^\top(\tilde{x}_j - \tilde{x}_i). \tag{17}$$

Since S_{-i} is a concatenation of the matrices $S_k,\ k\in\mathcal{V}\setminus\{i\}$, then by the Item (III) of Lemma 2, for each $i \in \mathcal{V}$ and each $k \in \mathcal{V}\setminus\{i\}$, (17) yields

$$\begin{aligned} S_k^\top\dot{\tilde{x}}_i &= \bar{A}_k S_k^\top\tilde{x}_i + \mu\sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}S_k^\top(\tilde{x}_j - \tilde{x}_i),\\ &= \left(\bar{A}_k - \mu\sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}\right)S_k^\top\tilde{x}_i + \mu\sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma}S_k^\top\tilde{x}_j. \end{aligned} \tag{18}$$

(II) *Projections along unique observable subspaces*: We reorganize (13) and (18) in accordance with the projections onto the same subspace $\mathcal{S}_i$ for each $i\in\mathcal{V}$. To this end, for each $i \in \mathcal{V}$, the dynamics associated with the projections onto $\mathcal{S}_i$ are

$$S_i^\top\dot{\tilde{x}}_i = \left(\bar{A}_i - \bar{G}_i\bar{C}_i\right)S_i^\top\tilde{x}_i, \tag{19}$$

$$\begin{aligned} S_i^\top\dot{\tilde{x}}_l &= \left(\bar{A}_i - \mu\sum_{k\in\mathcal{N}_{l,\sigma}} a_{lk,\sigma}\right)S_i^\top\tilde{x}_l + \mu a_{li,\sigma}S_i^\top\tilde{x}_i,\\ &\quad + \mu\sum_{k\in\mathcal{N}_{l,\sigma}\setminus\{i\}} a_{lk,\sigma}S_i^\top\tilde{x}_k \qquad \forall l \in \mathcal{V}\setminus\{i\}. \end{aligned} \tag{20}$$

Based on (15), the estimation problem with state $\hat{x}_i \in \mathbb{R}^n$ in (7), or equivalently the error state $\tilde{x}_i \in \mathbb{R}^n$ in (11), reduces to N decoupled estimation problems with projected error vector components (19) and (20).

(III) *Mutual reachability of agents*: In each of these N decoupled estimation problems, there is one leader node with the dynamics (19) which does not have an incoming edge, thanks to the structure of the gain and weighting matrices (12). Let $\mathcal{L}_\sigma^i \in \mathbb{R}^{N\times N}$ be the Laplacian matrix representing the connectivity among N nodes at the i^{th} decoupled problem. For a leader-follower architecture, the matrix $\mathcal{L}_\sigma^i$ has a structure similar to (3), where Δ_σ and H_σ are replaced with Δ_σ^i and H_σ^i, respectively, with the leader node i. Since, the leader node for each decoupled problem is unique, the matrix $\mathcal{L}_\sigma^i$ is distinct for all N decoupled estimation problems.

Let $\tilde{\boldsymbol{x}}_{-i} := (I_{N-1}\otimes S_i^\top)\tilde{x}_{-i}$ with $\tilde{x}_{-i} := (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_N)$ be an augmented state vector with $N-1$ components $\tilde{x}_k$ for all $k\in\mathcal{V}\setminus\{i\}$. Then, from (20), we obtain

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}}_{-i} &= \left[(I_{N-1}\otimes\bar{A}_i) - \mu(H_\sigma^i\otimes I_{\nu_i})\right]\tilde{\boldsymbol{x}}_{-i}\\ &\quad + \mu\left(\Delta_\sigma^i\mathbf{1}_{N-1}\otimes S_i^\top\right)\tilde{x}_i. \end{aligned} \tag{21}$$

Since the matrix $\bar{A}_i - \bar{G}_i\bar{C}_i$ is Hurwitz by an appropriate selection of $\bar{G}_i$, then, for all $i\in\mathcal{V}$, each solution $t\mapsto\tilde{x}_i(t)$ to (19) is such that $\lim_{t\to\infty}\left(\Delta_\sigma^i\mathbf{1}_{N-1}\otimes S_i^\top\right)\tilde{x}_i(t) = \mathbf{0}$ exponentially. Then, by [12, Lemma 1], each solution $t\mapsto\tilde{\boldsymbol{x}}_{-i}(t)$ of (21) can be shown to converge to zero exponentially for any initial condition $\tilde{\boldsymbol{x}}_{-i}(0)\in\mathbb{R}^{(N-1)\nu_i}$, if the origin of

$$\dot{\tilde{\boldsymbol{x}}}_{-i} = \left[(I_{N-1}\otimes\bar{A}_i) - \mu(H_\sigma^i\otimes I_{\nu_i})\right]\tilde{\boldsymbol{x}}_{-i} \tag{22}$$

is exponentially stable.

The union digraph corresponding to $\sum_{r=s_k}^{s_{k+1}-1}\mathcal{L}_{\sigma(t_r)}^i$ features a leader node i that has a directed path to the rest $N-1$ nodes, thanks to Assumption 3, which implies that the corresponding union digraph contains a directed spanning tree rooted at node i in (19), resembling a leader-follower architecture. Consequently, by Lemma 2 and Remark 4, the origin of (22) is globally exponentially stable. Since this holds for all $i\in\mathcal{V}$, the estimation error vector $\tilde{x}_i$ in (22), or equivalently $\tilde{x}$, converges globally and exponentially to the origin.

IV. A Distributed State Estimator with Sporadic System-Agent Measurements

In contrast to the setting in Section III, where measurements $y_i,\ i\in\mathcal{V}$, in (2) are continuously available to the agents, now we consider the scenario where these measurements are only available at the discrete switching instants $t_s,\ s\in\mathbb{Z}_+$, with the inter-transmission interval $t_{s+1}-t_s$ being constrained within the bounds $[T_1, T_2],\ T_1 > 0$ as follows:

$$T_1 \le t_{s+1} - t_s \le T_2 \qquad \forall s\in\mathbb{Z}_+. \tag{23}$$

While system-agent communication takes place sporadically at discrete instants t_s, inter-agent communication still occurs in a piecewise continuous manner, as in Section III. Just to keep the exposition simple, for the sake of brevity, we consider that y_i in (2) for all $i\in\mathcal{V}$ are measured synchronously at every t_s for all $s\in\mathbb{Z}_+$.

A. System Description

Given the intermittent nature of the measurements y_i, we design the distributed state estimation protocol of (1) with

N observer agents exhibiting the flow (if $t \neq t_s$) and jump dynamics (if $t = t_s$) as

$$\dot{\hat{x}}_i = A\hat{x}_i + \mu M_i \sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma(t)}(\hat{x}_j - \hat{x}_i), \tag{24}$$

$$\hat{x}_i^+ = \hat{x}_i + G_i(y_i - C_i\hat{x}_i), \tag{25}$$

where $\mu > 0$, G_i, M_i are designed as in (12). Using the orthogonal transformation (10), similar to (13) and (14), the estimation error dynamics for agent i and for each $\sigma \in \mathcal{P}$, with $\tilde{x}_i = \hat{x}_i - x$, evolves during flows as

$$S_i^\top \dot{\tilde{x}}_i = \bar{A}_i S_i^\top \tilde{x}_i, \tag{26}$$

$$U_i^\top \dot{\tilde{x}}_i = \hat{A}_i U_i^\top \tilde{x}_i + \mu \sum_{j\in\mathcal{N}_{i,\sigma}} a_{ij,\sigma} U_i^\top(\tilde{x}_j - \tilde{x}_i), \tag{27}$$

and at jumps as

$$S_i^\top \tilde{x}_i^+ = \left(I - \bar{G}_i\bar{C}_i\right) S_i^\top \tilde{x}_i, \tag{28}$$

$$U_i^\top \tilde{x}_i^+ = U_i^\top \tilde{x}_i, \tag{29}$$

where $\bar{C}_i$, $\bar{G}_i$ are derived from C_i in (10), G_i in (12), respectively, by the orthogonal transformation Σ_i in (9). The dynamics associated with the unobservable component $U_i^\top \tilde{x}_i$ do not exhibit a change in jumps, while those governing the observable counterpart $S_i^\top \tilde{x}_i$ are truly hybrid.

By exploiting the joint observability property (15), as in the proof for Theorem 1, the distributed estimation dynamics (26) – (29) can be transformed into N decoupled estimation problems. At the i^{th} decoupled problem, the dynamics of N nodes associated with the projections onto $\mathcal{S}_i$ yield, during flows

$$S_i^\top \dot{\tilde{x}}_i = \bar{A}_i S_i^\top \tilde{x}_i, \tag{30}$$

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}_{-i} &= \left[(I_{N-1}\otimes \bar{A}_i) - \mu(H_\sigma^i \otimes I_{\nu_i})\right]\tilde{\boldsymbol{x}}_{-i} \\ &\quad + \mu\left(\Delta_\sigma^i \mathbf{1}_{N-1}\otimes S_i^\top\right)\tilde{x}_i,\end{aligned} \tag{31}$$

and at jumps as

$$S_i^\top \tilde{x}_i^+ = \left(I - \bar{G}_i\bar{C}_i\right) S_i^\top \tilde{x}_i, \tag{32}$$

$$\tilde{\boldsymbol{x}}_{-i}^+ = \tilde{\boldsymbol{x}}_{-i}, \tag{33}$$

where $\tilde{\boldsymbol{x}}_{-i} = (I_{N-1}\otimes S_i^\top)\tilde{x}_{-i}$ in (21).

B. Stability Results

Since (30) and (32) describe the dynamics of an autonomous hybrid system, we can analyse its stability independently from (31) and (33), using the framework proposed in [13]. Given that each agent i receives a measurement $y_i(t)$ from the system (1) at time instants $t = t_s,\ s \in \mathbb{Z}_+$, we introduce a centralized timer variable τ to keep track of the sequence of event times $\{t_s\}_{s=1}^{\infty}$. Specifically, we make τ decrease as ordinary time t increases until it reaches zero, at which point τ is reset to any value within the interval $[T_1, T_2]$, where T_1 and T_2 represent the minimum and maximum dwell-time bound, respectively, as introduced in (23). For each $i \in \mathcal{V}$, let $\tilde{x}_{iS} := S_i^\top \tilde{x}_i$. Then, the system with $\xi_i := (\tilde{x}_{iS}, \tau) \in \mathbb{R}^{\nu_i}\times\mathbb{R}_{\geq 0}$ can be represented as

$$\mathcal{H}_i \begin{cases} \dot{\xi}_i = f_i(\xi_i) & \xi_i \in C_{\mathcal{H}_i} \\ \xi_i^+ = g_i(\xi_i) & \xi_i \in D_{\mathcal{H}_i}, \end{cases} \tag{34}$$

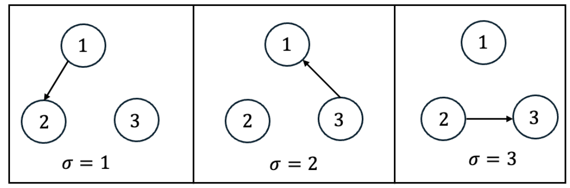

Fig. 1: The switching digraphs $\mathcal{G}_i$, $i \in \{1, 3\}$.

where the flow and jump sets are given by

$$C_{\mathcal{H}_i} := \{\xi_i \in \mathbb{R}^{\nu_i}\times\mathbb{R}_{\geq 0} : \tau \in [0, T_2]\}, \tag{35}$$

$$D_{\mathcal{H}_i} := \{\xi_i \in \mathbb{R}^{\nu_i}\times\mathbb{R}_{\geq 0} : \tau = 0\}, \tag{36}$$

and the flow and jump maps by

$$f_i(\xi_i) := \begin{bmatrix} \bar{A}_i\tilde{x}_{iS} \\ -1 \end{bmatrix} \qquad \forall \xi_i \in C_{\mathcal{H}_i} \tag{37}$$

$$g_i(\xi_i) := \begin{bmatrix} \left(I - \bar{G}_i\bar{C}_i\right)\tilde{x}_{iS} \\ [T_1, T_2] \end{bmatrix} \qquad \forall \xi_i \in D_{\mathcal{H}_i}. \tag{38}$$

The set $\mathcal{A}_i := \{\xi_i \in \mathbb{R}_i^{\nu}\times\mathbb{R}_{\geq 0} : \tilde{x}_{iS} = \mathbf{0}, \tau \in [0, T_2]\}$ is to be rendered globally exponentially stable [13] for $\mathcal{H}_i$ in (34) for all $i \in \mathcal{V}$. To this end, we consider the Lyapunov function candidate

$$V_i(\xi_i) := \tilde{x}_{iS}^\top e^{\bar{A}_i\tau} P_i e^{\bar{A}_i\tau}\tilde{x}_{iS} \qquad \forall \xi_i \in \mathbb{R}^{\nu_i}\times\mathbb{R}_{\geq 0} \tag{39}$$

where $P_i \in \mathbb{R}^{\nu_i} \succ \mathbf{0}$. Then, by [14, Theorem 1], the set $\mathcal{A}_i$ is globally exponentially stable for $\mathcal{H}_i$ if

$$(I - \bar{G}_i\bar{C}_i)^\top e^{\bar{A}_i^\top v_i} P_i e^{\bar{A}_i v_i}(I - \bar{G}_i\bar{C}_i) - P_i \prec \mathbf{0} \tag{40}$$

for all $v_i \in [T_1, T_2]$. We have the following main result.

Theorem 2: Let Assumptions 1 and 2 hold. Given the system (1) with measurements y_i in (2) that are sporadically measured at instants $t_s,\ s \in \mathbb{Z}_+$, with inter-transmission interval bounds T_1, $T_2 > 0$ satisfying (23), the distributed state estimation protocol (24), (25), with G_i, M_i given in (12) and the gain matrix $\bar{G}_i$ satisfying (40) for all $v_i \in [T_1, T_2]$, solves the distributed state estimation problem under any switching signal $t \mapsto \sigma(t)$ that satisfies Assumption 3, i.e., the estimation error dynamics $\tilde{x}$ in (26)–(29) is globally exponentially stable.

V. Illustrative Example

Let us now consider a numerical example of a triple frequency harmonic oscillator system in (1) with

$$A = \operatorname{diag}(5.85, 4.47, 2.4)\otimes\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{41}$$

which is monitored by three sensor agents. Let the measurements of the sensor agents be given in the form (2) with:

$$C_1 = \begin{bmatrix} 1 & \mathbf{0}_{1\times 5}\end{bmatrix}, C_2 = \begin{bmatrix}\mathbf{0}_{1\times 3} & 1 & \mathbf{0}_{1\times 2}\end{bmatrix}, C_3 = \begin{bmatrix}\mathbf{0}_{1\times 4} & 1 & 0\end{bmatrix}.$$

It is easy to verify that none of the pairs $(A, C_j),\ j \in \{1, 3\}$ is observable. However, the unobservable subspace of each agent is the collective observable subspace of the other two agents. Therefore, the matrix pair $(A, \bar{C})$ is observable, satisfying Assumption 1.

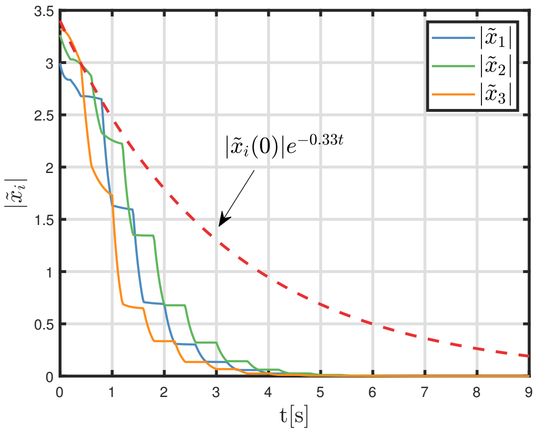

Fig. 2: Normed estimation error vectors for three agents.

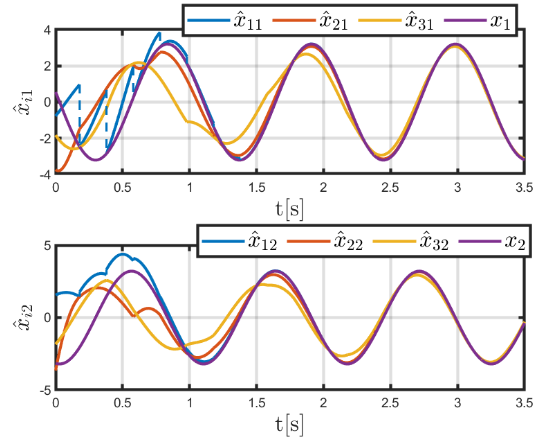

Fig. 3: The trajectory of the first two components of the state of each agent.

Let us assume that three agents communicate over a switching digraph $\mathcal{G}_\sigma$, shown in Figure 1, where

$$\sigma(t) = \text{mod}\left(\frac{s}{3}\right) + 1 \;\forall t \in \left[\frac{sT}{3}, \frac{(s+1)T}{3}\right), s \in \mathbb{Z}_+, \quad (42)$$

that repeats periodically in every $T = 0.6$ seconds, i.e., dwell-time bounds are $T_1 = T_2 = 0.6$. It is easy to verify that the union digraph in every $[sT, (s+1)T)$ interval is strongly connected, satisfying Assumption 3.

Next, we construct the distributed observers in the form (7), with scaling factor $\mu = 5$, and the observer gains and weighting matrices designed as follows

$$\begin{aligned} G_1 &= \begin{bmatrix}5.29 & -4.83 & \mathbf{0}\end{bmatrix}^\top, \; M_1 = \text{diag}(\mathbf{0}, I_4), \\ G_2 &= \begin{bmatrix}\mathbf{0} & 2.78 & 4.41 & \mathbf{0}\end{bmatrix}^\top, \; M_2 = \text{diag}(I_2, \mathbf{0}, I_2), \\ G_3 &= \begin{bmatrix}\mathbf{0} & 5 & 1.09\end{bmatrix}^\top, \; M_3 = \text{diag}(I_4, \mathbf{0}). \end{aligned} \quad (43)$$

These choices allow the observer at each agent to provide an exponentially converging estimate of the state vector x despite communication constraints, as seen from the normed estimation error in Figure 2. Next, for the second scenario, where the system measurements are available sporadically at $t_s = sT/3$, using the distributed estimation protocol (24), (25) with M_i designed as in (43), and

$$\begin{aligned} G_1 &= \begin{bmatrix}1 & -0.1 & \mathbf{0}\end{bmatrix}^\top, G_3 = \begin{bmatrix}\mathbf{0} & 1 & 0.51\end{bmatrix}^\top, \\ G_2 &= \begin{bmatrix}\mathbf{0} & -0.16 & 1 & \mathbf{0}\end{bmatrix}^\top, \end{aligned} \quad (44)$$

satisfying (40), the observer agent successfully estimates x, as observed from Figure 3.

VI. Conclusion

In this paper, we investigate the distributed state estimation problem for a linear system that is jointly observable by a network of observer agents. These agents communicate over directed dynamic network topologies, with system measurements available either continuously or sporadically. Motivated by the decentralized observer design in [1], leveraging the joint observability condition, and assuming mutual reachability of agents over some frequent time-intervals, the unobservable subspace of an agent can be recovered from the observable subspaces of the rest of the agents in the network. In our future work, we aim to extend the piecewise continuous inter-agent communication studied here to a more general setting, where agents communicate with their neighbors only at sporadic time instants.

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