2024 Eighteenth International Congress on Artificial Materials for Novel Wave Phenomena (Metamaterials) | 979-8-3503-7349-3/24/$31.00 ©2024 IEEE | DOI: 10.1109/Metamaterials62190.2024.10703230

A Second Order Homogenized Dispersive Wave Equation in a Quasiperiodic Medium

S. Guenneau[1,2], E. Cherkaev[3] and N. Wellander[4,5]

[1]Imperial College London, The Blackett Laboratory, Physics Department, London SW7 2AZ, United Kingdom
[2]UMI 2004 Abraham de Moivre-CNRS, Imperial College London, London SW7 2AZ, United Kingdom
[3]University of Utah, Department of Mathematics, 155 South 1400 East, JWB 233, Salt Lake City, UT 84112, USA
[4] Luleå University of Technology, SE-971 87 Luleå, Sweden
[5]Swedish Defence Research Agency (FOI), SE-581 11 Linköping, Sweden
niklas.wellander@foi.se

Abstract – **Wave propagation in a quasiperiodic medium is modeled by a second order homogenized scalar wave equation. The multiscale asymptotic expansion method for high-order homogenization of periodic structures in [1] is adapted to the quasiperiodic (cut-and-projection) setting. A periodic medium in a higher spatial dimension is used to model a quasiperiodic material by applying the cut-and-project procedure [2]. The partial differential operators (gradient and divergence) in the higher dimensional space are projected onto operators acting on quasiperiodic functions in a lower dimensional physical space. The second-order homogenized wave equation is dispersive, which is reflected by a fourth-order Burnett tensor for quasiperiodic structures.**

I. Introduction

Wave propagation in a heterogeneous medium can be modeled by the initial value wave equation

$$(\mathcal{P})\begin{cases} -\nabla\cdot\left(a_\varepsilon(\boldsymbol{x})\nabla u_\varepsilon(t,\boldsymbol{x})\right)+\frac{\partial^2}{\partial t^2}u_\varepsilon(t,\boldsymbol{x})=f(t,\boldsymbol{x}) & \text{, in } [0,\infty)\times\mathbb{R}^n \\ u_\varepsilon(0,\boldsymbol{x})=u_0(\boldsymbol{x}), \quad \frac{\partial}{\partial t}u_\varepsilon(0,\boldsymbol{x})=v_0(\boldsymbol{x}) & \end{cases}$$

Here $\varepsilon > 0$ is a (small) parameter that is given by the relation between the spatial macroscopic and microscopic scales. The solution u_ε depends on this parameter. The aim of this paper is to replace the heterogeneous problem in the case when the medium at the microscopic scale is quasiperiodic, modeled by $(\mathcal{P}_\varepsilon)$ below, by an effective anisotropic medium as $\varepsilon \to 0$. The effective medium turns out to be dispersive. We assume $u_0 \in W^{1,2}(\mathbb{R}^n), v_0 \in L^2(\mathbb{R}^n), f \in W^{1,2}([0,\infty), L^2(\mathbb{R}^n))$ with compact support in $\mathbb{R}^n$. We also assume that the solutions satisfy certain radiation conditions. The original problem $(\mathcal{P})$ is replaced with the following problem:

$$(\mathcal{P}_\varepsilon)\begin{cases} -\nabla\cdot\left(a(\frac{\mathbf{R}\boldsymbol{x}}{\varepsilon})\nabla u_\varepsilon(t,\boldsymbol{x})\right)+\frac{\partial^2}{\partial t^2}u_\varepsilon(t,\boldsymbol{x})=f(t,\boldsymbol{x}) & \text{, in } [0,+\infty)\times\mathbb{R}^n \\ u_\varepsilon(0,\boldsymbol{x})=u_{init}(\boldsymbol{x}), \quad \frac{\partial}{\partial t}u_\varepsilon(0,\boldsymbol{x})=v_{init}(\boldsymbol{x}) & \end{cases} \tag{1}$$

where u_{init} and v_{init} are initial wave amplitude and velocity. We assume that $a(\boldsymbol{y})$ is a Y^m-periodic symmetric matrix valued function such that for some $0 < \alpha \leq \beta$, $\alpha \mid \xi \mid^2 \leq a_{ij}(\boldsymbol{y})\xi_i\xi_j \leq \beta \mid \xi \mid^2$ for all $i,j = 1,\cdots n$, $n < m$, a.e. $\boldsymbol{y} \in Y^m$, with $Y^m =]0,1[^m$ a periodic cell in higher dimensional space $\mathbb{R}^m$. The matrix $\mathbf{R}$ that maps $\mathbb{R}^n$ to $\mathbb{R}^m$, satisfies the criterion

$$\mathbf{R}^T\boldsymbol{k} \neq \mathbf{0}\ ,\ \forall \boldsymbol{k} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}\ . \tag{2}$$

The mapping $\mathbf{R}$ is illustrated in Figure 1 as the mapping of the oblique line to the 2-dimensional unit cube. The scaling of $a(\mathbf{R}\boldsymbol{x}/\varepsilon)$, with $\boldsymbol{x} \in \mathbb{R}^n$ yields a quasiperiodic function.

Following [10], [8], it is useful to decompose the higher-dimensional periodic space Y^m into the n-dimensional subspace $Y^m_{\|} = \{\boldsymbol{y} \in \mathbb{R}^m \mid \left(\mathbf{I}_m - \mathbf{R}\mathbf{R}^T\right)\ \boldsymbol{y} = \mathbf{0}\}$ and its orthogonal complement $Y^m_{\perp} = \{\boldsymbol{y} \in \mathbb{R}^m \mid \mathbf{R}\mathbf{R}^T\boldsymbol{y} = \mathbf{0}\}$. Such decomposition is the essence of the cut-and-projection method. This is illustrated in Figure 1 where

$Y_{\|}^m$ corresponds to the blue line and $Y_{\perp}^m$ is thus a line perpendicular to it (not shown). The coefficient a in $(\mathcal{P}_\varepsilon)$ satisfies the Carathéodory assumptions i)-ii):

i) $a(\boldsymbol{x}, \cdot)$ is Y-periodic in $\mathbb{R}^m$, is Lebesgue measurable on $Y_{\|}^m$ and continuous on $Y_{\perp}^m$, for every $\boldsymbol{x} \in \mathbb{R}^n$.

ii) $a(\cdot, \boldsymbol{y})$ is continuous for almost every $\boldsymbol{y} \in \mathbb{R}^m$.

Assumption $i)$ implies that $a(\boldsymbol{x}, \cdot) = \tilde{a}(\boldsymbol{x}, \cdot, \cdot)$, where $\tilde{a}(\boldsymbol{x}, \cdot, \cdot)$ is a function of $Y_{\perp}^m \times Y_{\|}^m$ such that $\tilde{a}(\boldsymbol{x}, y_\perp, \cdot)$ is Lebesgue measurable for every $y_\perp \in Y_\perp$ and $\tilde{a}(\boldsymbol{x}, \cdot, y_\|)$ is continuous for almost every $y_\| \in Y_\|$. Standard estimates yield $\|\partial_t u\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \leq C$ and $\|\nabla u\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} \leq C$ uniformly with respect to ε, [5]. As ε tends to zero, we want to replace the quasiperiodic structure with an effective anisotropic and dispersive medium.

Asymptotic expansion, using the same scaling as for coefficient a, of the solutions $\boldsymbol{u}_\varepsilon(t, \boldsymbol{x})$ of the PDE takes the form

$$\boldsymbol{u}_\varepsilon(t, \boldsymbol{x}) = \boldsymbol{u}_0(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\varepsilon) + \varepsilon \boldsymbol{u}_1(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\varepsilon) + \varepsilon^2 \boldsymbol{u}_2(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\varepsilon) + \ldots , \tag{3}$$

where $\boldsymbol{u}_i(t, \boldsymbol{x}, \boldsymbol{y})$, $i = 0, 1, ...$, are Y-periodic functions in $\boldsymbol{y}$ with enough regularity in t, $\boldsymbol{x}$ and $\boldsymbol{y}$. We note that the rescaled gradient acting on two-scale functions $\boldsymbol{u}_i$ is such that

$$\nabla\ \boldsymbol{u}_i(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta) = \nabla_{\boldsymbol{x}}\ \boldsymbol{u}_i(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta) + \eta^{-1} \nabla_{\mathbf{R}}\ \boldsymbol{u}_i(t, \boldsymbol{x}, \mathbf{R}\boldsymbol{x}/\eta) , \tag{4}$$

where we have defined the so-called cut-and-projection gradient operator $\nabla_{\mathbf{R}}\ \boldsymbol{u}_i(t, \boldsymbol{x}, \boldsymbol{y}) := \mathbf{R}^T \nabla_{\boldsymbol{y}}\ \boldsymbol{u}_i(t, \boldsymbol{x}, \boldsymbol{y})$. Note that the gradient operator $\mathrm{grad}_{\mathbf{R}} := \nabla_{\mathbf{R}} = \mathbf{R}^T \nabla$ is a directional derivative given by the projection on $\mathbb{R}^n$ of the usual gradient in $\mathbb{R}^m$. The divergence is obtained using the same projection in combination with the usual nabla rules. We shall see that one can then carry out an asymptotic analysis of the wave equation in a way similar to what was done in [11] for the periodic case. The leading order homogenized wave equation reads

$$(\mathcal{P}_{hom}) = \begin{cases} -\nabla \cdot (a_{hom} \nabla u) = -\frac{\partial^2}{\partial t^2} u(t, \boldsymbol{x}) + f(t, \boldsymbol{x}) & , \text{ in } [0, \infty) \times \mathbb{R}^n \\ u(0, \boldsymbol{x}) = u_{init}(\boldsymbol{x}), \quad \frac{\partial}{\partial t} u(0, \boldsymbol{x}) = v_{init}(\boldsymbol{x}) & \end{cases}$$

with the homogenized coefficient given by

$$a_{hom} = \langle a(\boldsymbol{y})(\mathbf{I}_d - \nabla_{\mathbf{R}} \boldsymbol{W}^{(1)}(\boldsymbol{y})) \rangle_{Y^m} \tag{5}$$

where $\langle \cdot \rangle_{Y^m}$ is the average over Y^m, i.e., $\langle g \rangle_{Y^m} = \int_{Y^m} g(\boldsymbol{y})\ \mathrm{d}\boldsymbol{y}$ and the projected gradients $\mathbf{R}^T \nabla_{\boldsymbol{y}} W^j\ , j \in \{1, \cdots, n\}$ of the potential $\boldsymbol{W}^{(1)} = (W^1, \cdots, W^n)$, are unique solutions in $L^2_\sharp(Y^m)^n$ of one of the following n problems

$$\mathrm{div}_{\mathbf{R}} \left(a(\boldsymbol{y}) \nabla_{\mathbf{R}} W^i(\boldsymbol{y}) \right) = \mathrm{div}_{\mathbf{R}} \left(a(\boldsymbol{y}) \mathbf{e}_i \right) \tag{6}$$

The homogenized equation can be improved by adding higher-order correction terms. This has been done in the periodic case [11], here extended to the quasiperiodic case. We get

$$-\nabla \cdot (a_{hom} \nabla v_\varepsilon(t, \boldsymbol{x})) + \varepsilon^2 D_{hom} \nabla^4 v_\varepsilon(t, \boldsymbol{x}) = -\frac{\partial^2}{\partial t^2} v_\varepsilon(t, \boldsymbol{x}) + f(t, \boldsymbol{x}) + \varepsilon^2 \nabla \cdot (d_{hom} \nabla f(t, \boldsymbol{x})) + O(\varepsilon^4)$$

where D_{hom} is a fourth-order tensor, known as the Burnett tensor in the periodic case, and d_{hom} is a second-order tensor. Note that the coefficients are constant; hence, it is a model of an effective metamaterial, in contrast to the original equation $(\mathcal{P}_\varepsilon)$. Notice that as in the periodic case [11], it is essential to take into account the corrector $\varepsilon^2 \nabla \cdot (d_{hom} \nabla f(t, \boldsymbol{x}))$ for modeling the long-time behavior of the solution of the second order homogenized wave equation. The notation $D\nabla^4$ means the full contraction

$$D\nabla^4 = \sum_{i,j,k,l=1}^{d} D_{i,j,k,l} \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} \tag{7}$$

This term is not only important from a purely theoretical and numerical standpoint (as it allows for a better approximation of the solution at long times), but it also indicates the effective medium is inherently dispersive [11].

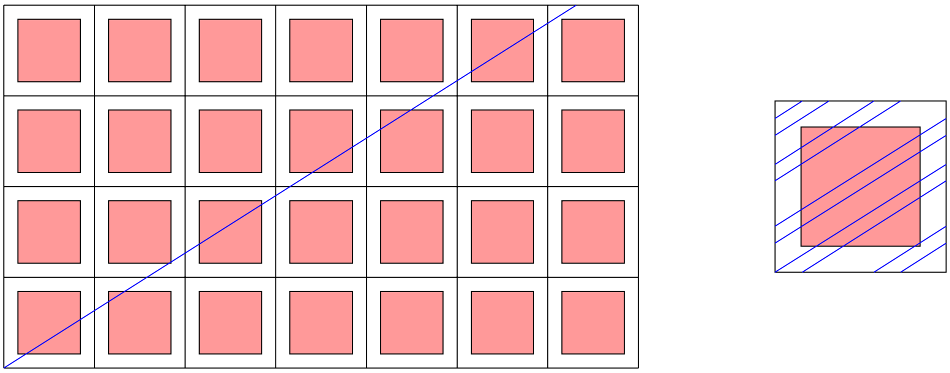

Fig. 1: Principle of cut-and-projection method: the projection of a periodic structure in higher dimensional space R^m ($m = 2$ here) onto a hyperplane (represented by a straight line) or a lower dimensional space $\mathbb{R}^n$ ($n = 1$ here), generates a quasiperiodic structure in $\mathbb{R}^n$, when the slope is irrational. With a rational slope, the line is folded back onto a finite number of segments in the periodic cell Y^m, whereas with an irrational slope, as shown here, the set of segments is countable and dense in Y^m, as a consequence of Kronecker's approximation theorem [Theorem 444 in [14]], as noted in [13]).

II. Conclusion

The paper discusses a higher-order homogenized model of scalar wave propagation in a quasiperiodic metamaterial with dispersion represented by a fourth-order Burnett tensor. This dispersion is not captured by the leading order approximation achieved in our former works [6, 7, 2]. The present work opens up a path for higher-order homogenized models of electromagnetic and mechanical quasiperiodic metamaterials.

Acknowledgement

EC acknowledges support from the U.S. NSF grant DMS-2111117.

References

[1] A. Bensoussan, J.L. Lions and G. Papanicolaou, *Asymptotic analysis for periodic structures,* 1st edition, Amsterdam, The Netherlands: North-Holland, 1978.
[2] E. Cherkaev, S. Guenneau, H. Hutridurga and N. Wellander, "Quasiperiodic Composites: Multiscale Reiterated Homogenization", IEEE proceedings of *Metamaterials'2019,* pp. X086-X088, Rome, Italy, 16-19 September 2019.
[3] G. Bouchitté, S. Guenneau and F. Zolla, "Homogenization of dielectric photonic quasi crystals," *Multiscale Modeling and Simulation* vol. 8, pp. 1862–1881, 2010.
[4] M. Duneau and A. Katz, "Quasiperiodic patterns," *Phys. Rev. Lett.* vol. 54, 2688–2691, 1985.
[5] L.C. Evans, Partial differential equations, American Mathematical Society, 2010
[6] N. Wellander, S. Guenneau and E. Cherkaev, "Two-scale cut-and-projection convergence; homogenization of quasiperiodic structures," *Mathematical Methods in the Applied Sciences* vol. 41, 1101–1106, 2018.
[7] N. Wellander, S. Guenneau and E. Cherkaev, "Homogenization of quasiperiodic structures and two-scale cut-and-projection convergence," *arXiv preprint 2019*arXiv:1911.03560
[8] N. Wellander, S. Guenneau and E. Cherkaev, "Two-scale cut-and-projection convergence for quasiperiodic monotone operators," *European Journal of Mechanics-A/Solids* vol. 100, 104796, 2023.
[9] D. Shechtman, I. Blech, D. Gratias, and J.W. Cahn, "Metallic phase with long-range orientational order and no translational symmetry," *Phys. Rev. Lett.* vol. 53, 1951–1953, 1984.
[10] N.G. de Bruijn, "Algebraic theory of Penrose's non-periodic tilings of the plane," *Indag. Math.* vol. 43, 39-66, 1981
[11] G. Allaire, T. Yamada, "Optimization of dispersive coefficients in the homogenization of the wave equation in periodic structures," *Numerische Mathematik, Springer Verlag* vol. 140, 265–326, 2018.
[12] C. Conca, J. San Martin, L. Smaranda and M. Vaninnathan, "On Burnett coefficients in periodic media in low contrast regime", *J. Math. Phys.* vol. 49, 053514, 2008.
[13] P. Amenoagbadji, S. Fliss and P. Joly, "Wave propagation in one-dimensional quasiperiodic media", *HAL preprint 2022* hal-03786730
[14] Godfrey Harold Hardy et al., An introduction to the theory of numbers. 6th ed. Oxford university press, 2009.