LEAVES OF FOLIATED PROJECTIVE STRUCTURES

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Abstract. The $\mathrm{PSL}(4, \mathbb{R})$ Hitchin component of a closed surface group $\pi_1(S)$ consists of holonomies of properly convex foliated projective structures on the unit tangent bundle of S. We prove that the leaves of the codimension-1 foliation of any such projective structure are all projectively equivalent if and only if its holonomy is Fuchsian. This implies constraints on the symmetries and shapes of these leaves.

We also give an application to the topology of the non-T_0 space $\mathfrak{C}(\mathbb{RP}^n)$ of projective classes of properly convex domains in $\mathbb{RP}^n$. Namely, Benzécri asked in 1960 if every closed subset of $\mathfrak{C}(\mathbb{RP}^n)$ that contains no proper nonempty closed subset is a point. Our results imply a negative resolution for $n \geq 2$.

1. Introduction

A $\mathrm{PSL}(4, \mathbb{R})$ Hitchin representation ρ of a closed surface group Γ induces a curious Γ-invariant curve $\mathfrak{s}_\rho$ from the Gromov boundary $\partial\Gamma$ to the space $\mathfrak{C}$ of projective classes of properly convex domains in $\mathbb{RP}^2$. We call $\mathfrak{s}_\rho$ the *leaf map* of ρ, and study it here.

As for other equivariant maps from $\partial\Gamma$ arising from geometry (e.g. [4, 8, 9, 10, 20, 37]), the regularity and irregularities of $\mathfrak{s}_\rho$ are salient and interesting. The relevant aspects of our setting have an idiosyncratic character due to the point-set topological richness of $\mathfrak{C}$. Namely, $\mathfrak{C}$ is non-separated (i.e. not T_0) and contains both large families of closed one-point sets and dense one-point sets (see [6, 18, 27]).

We prove $\mathfrak{s}_\rho$ is constant if and only if ρ is Fuchsian. A proposition of Benoist [5] then implies that, for non-Fuchsian ρ, images of leaf maps are closed in $\mathfrak{C}$, are not points, and are *minimal* in the sense that they contain no proper nonempty closed subset. It follows that non-point minimal closed sets exist in the space $\mathfrak{C}(\mathbb{RP}^n)$ of projective classes of properly convex domains in $\mathbb{RP}^n$ ($n \geq 2$). The existence of non-point minimal closed sets is a basic question for a non-separated space. It has been open for $\mathfrak{C}(\mathbb{RP}^n)$ since Benzécri posed the question in 1960 ([6] §V.3).

Let us be more detailed. By work of Guichard-Wienhard [23], $\mathrm{PSL}(4, \mathbb{R})$ Hitchin representations are exactly the holonomies of properly convex foliated projective structures on the unit tangent bundle T^1S, which are a refinement of $(\mathrm{PSL}(4, \mathbb{R}), \mathbb{RP}^3)$ structures on T^1S. See §3.1.1 for definitions of Hitchin and Fuchsian representations and §3.2.2 for properly convex foliated projective structures. By definition, the developing map of a properly convex foliated projective structure maps leaves of the stable foliation $\overline{\mathcal{F}}$ of $T^1\widetilde{S}$ to properly convex domains in projective planes. The leaf space of $\overline{\mathcal{F}}$ is identified with $\partial\Gamma$, and $\mathfrak{s}_\rho(x)$ is defined for $x \in \partial\Gamma$ as $[\mathrm{dev}_\rho x] \in \mathfrak{C}$. For ρ Fuchsian, $\mathfrak{s}_\rho$ is constant with value the ellipse.

Leaf maps exhibit counter-intuitive phenomena. For instance, $\mathfrak{s}_\rho$ maps any nonempty open set $U \subset \partial\Gamma$ onto all of $\mathfrak{s}_\rho(\partial\Gamma)$. In general, determining when leaf maps are constant is made difficult by the non-separation of $\mathfrak{C}$. We resolve the matter:

Theorem 1.1. *Let $\rho \in \mathrm{Hit}_4(S)$. The following are equivalent:*

(1) ρ is Fuchsian,
(2) The leaf map $\mathfrak{s}_\rho$ is constant,
(3) The leaf map $\mathfrak{s}_\rho$ has countable image,

(4) There exists a leaf $\mathfrak{s}_\rho(x)$ that is an ellipse, is divisible, is a closed point of $\mathfrak{C}$, or has non-discrete projective automorphism group.

Recall that a properly convex domain is *divisible* if it admits a cocompact action by a discrete subgroup of $\mathrm{SL}(3,\mathbb{R})$. Condition (4) considerably limits the symmetries of leaves of non-Fuchsian ρ. It is in contrast to the observation that some leaves have symmetries: automorphism groups of leaves $\mathfrak{s}_\rho(\gamma^\pm)$ of fixed points $\gamma^\pm \in \partial\Gamma$ for $\gamma \in \Gamma - \{e\}$ contain $\mathbb{Z}$.

Theorem 1.1 is a rigidity result for Fuchsian representations among $\mathrm{PSL}(4,\mathbb{R})$ Hitchin representations in terms of the behavior of the leaf map $\mathfrak{s}_\rho$. It is a theme in the better-studied setting of *convex* projective geometry that highly symmetric examples, namely ellipsoids, exhibit numerous rigidity phenomena. See e.g. [4] Proposition 6.1, [13] Theorems 0.5-0.6, [14] Theorem 1.1, [44] Theorem 1.35, and [45] Theorem 1.4. For examples of rigidity phenomena in the study of discrete subgroups of Lie groups more generally, see e.g. [3] Théorème 1.2.b, [32] Theorem 1.1, [37] Theorem D, and [40] Theorem B.

Though non-constancy of $\mathfrak{s}_\rho$ for non-Fuchsian ρ may appear intuitive, it implies leaf maps exhibit a rather dramatic phenomenon, impossible for any map to a T_1 space:

Theorem 1.2. *For non-Fuchsian $\rho \in \mathrm{Hit}_4(S)$, the leaf map $\mathfrak{s}_\rho : \partial\pi_1 S \to \mathfrak{C}$ is continuous, constant on $\pi_1 S$ orbits, and not constant.*

Note in the above theorem that all $\pi_1 S$ orbits in $\partial\pi_1 S$ are dense.

Benoist has proved that $\mathfrak{s}_\rho$ has closed image in $\mathfrak{C}$ (in unpublished work; see §4.4 for details). From the continuity of $\mathfrak{s}_\rho$ and the minimality of the action of Γ on $\partial\Gamma$, it follows that the image $\mathfrak{s}_\rho(\partial\Gamma)$ is a *minimal* closed set in $\mathfrak{C}$, in the sense that it is closed and contains no proper nonempty closed subset. By taking cones over leaves of non-Fuchsian $\mathrm{PSL}(4,\mathbb{R})$ Hitchin representations, non-point minimal closed subsets of $\mathfrak{C}(\mathbb{RP}^n)$ can be constructed for all $n \geq 2$ (§4.6).

All prior examples of minimal closed sets in $\mathfrak{C}(\mathbb{RP}^n)$, such as divisible domains [6], are points. So our results imply:

Theorem 1.3. *For all $n \geq 2$, $\mathfrak{C}(\mathbb{RP}^n)$ contains minimal closed sets that are not points.*

Benzécri concludes his seminal thesis, in which his namesake compactness theorem is proved and the topology of $\mathfrak{C}$ is first seriously studied, with a few questions on $\mathfrak{C}(\mathbb{RP}^n)$ for $n \geq 2$ ([6] §V.3). The first was whether all minimal closed subsets of $\mathfrak{C}(\mathbb{RP}^n)$ are points.

Among the experts aware of $\mathfrak{s}_\rho$ having closed image, Theorems 1.1-1.3 were expected to be true. However, no proof that $\mathfrak{s}_\rho$ is non-constant for non-Fuchsian ρ had been found. This ends up being the main difficulty, and presents technical challenges. Our proof uses a range of methods, for instance relying on the Baire category theorem and the classification of Zariski closures of Hitchin representations.

1.0.1. *Shapes of Leaves.* Our results place further restrictions on the geometry of individual leaves of non-Fuchsian properly convex foliated projective structures, which we explain here. First, they prevent any boundary point of a leaf from being too regular without being very flat.

Corollary 1.4. *Let ρ be non-Fuchsian and $x \in \partial\pi_1 S$. Then the leaf $\mathfrak{s}_\rho(x)$ has no C^2 boundary point of nonvanishing curvature.*

This is analogous to a classical result of Benzécri for divisible domains. It is notable in that it constrains arbitrary boundary points. This is in contrast to the constraints accessible with standard methods to study boundary regularity of similar objects, which control the worst-behaved points (e.g. [20] Theorem 22, [37] Theorem D, and [43] Theorem 1.1).

Pairing Theorem 1.1 with the closedness of the collection of all leaves in $\mathfrak{C}$ results in constraints on how complicated and asymmetric the boundary behavior a leaf may be. For instance, Benzécri showed in [6] (§V.3, p.321) that there are dense one-point sets in $\mathfrak{C}$. The following implies that any such domain cannot occur as a leaf.

Corollary 1.5. *If $\rho \in \mathrm{Hit}_4(S)$ is non-Fuchsian and $x \in \partial\Gamma$, then $\mathrm{Cl}_{\mathfrak{C}}\{\mathfrak{s}_\rho(x)\}$ contains no closed point.*

In the remainder of the introduction we outline our proof and situate our results in the context of broader projects in higher Teichmüller theory.

1.1. **Outline of Proof of Theorem 1.1.** A rough outline of our proof is that after addressing regularity of varying projective equivalences with a Baire category argument, the closed subgroup theorem forces major constraints on the eigenvalues of ρ when $\mathfrak{s}_\rho(x)$ is constant. These constraints, when paired with the classification of Zariski closures of Hitchin representations [41] allow us to deduce Theorem 1.1.

It proves useful to do case analysis on the size of the projective automorphism group of $\mathfrak{s}_\rho(x)$. The most involved case is when $\mathfrak{s}_\rho(x)$ has discrete automorphism group. This case is ill-suited to productive use of Benzécri's compactness theorem, and is a place where we must contend with the non-separation of $\mathfrak{C}$. This appears in the form that there are discontinuous paths $A_t : [0,1] \to \mathrm{SL}(3,\mathbb{R})$ and domains Ω in $\mathbb{RP}^2$ so that $A_t\overline{\Omega}$ is continuous in the Hausdorff topology.[1]

Our argument in this case to obtain constraints on eigenvalues of ρ if $\mathfrak{s}_\rho$ is constant hinges on the Baire category theorem. Using it, we show that the above pathology may be avoided on a nonempty open subset $U \subset \partial\Gamma$ in the sense that we may arrange for representatives of the equivalence classes $\mathfrak{s}_\rho(x)$ to vary by a continuous family of projective equivalences on U. Paired with this basic regularity, the closed subgroup theorem forces a compatibility of the action of $\rho(\Gamma)$ on leaves and our family of projective equivalences.

This compatibility implies certain actions of $\rho(\gamma)$ on fixed subspaces are projectively conjugate, which places major constraints on the eigenvalues of $\rho(\gamma)$ for $\gamma \in \Gamma - \{e\}$. This in turn places constraints on the Zariski-closure of the image of $\rho(\Gamma)$. The argument is concluded by comparing the constraints we obtain and Guichard's classification of Zariski closures of Hitchin representations (see [41]).

1.2. **Context and Related Results.**

1.2.1. *Properly Convex Projective Structures.* Some notable analogues to Theorems 1.1 and 1.2 occur in the study of properly convex projective structures on surfaces. These structures parameterize $\mathrm{SL}(3,\mathbb{R})$ Hitchin components [11, 18].

Briefly, a projective structure $(\mathrm{dev}, \mathrm{hol})$ on S is said to be *properly convex* if dev is a homeomorphism of $\widetilde{S}$ onto a properly convex domain Ω of $\mathbb{RP}^2$. In this case, Γ acts properly discontinuously and without fixed-points on Ω through hol.

A similar statement to Theorem 1.2 that is much easier to prove is the observation that in the above notation, $\partial\Omega$ is topologically a circle and the map $\mathrm{reg} : \partial\Omega \to (1,2]$ associating to $x \in \partial\Omega$ the optimal pointwise C^α regularity of $\partial\Omega$ at x (see e.g. §2) is a Γ-invariant map that is constant on all orbits of Γ, and only constant if hol is in the Fuchsian locus of $\mathrm{Hit}_3(S)$.

[1] A simple example of this phenomenon is as follows. Let Ω be a domain with nontrivial projective automorphism group and let A_t be a discontinuous path of projective automorphisms of Ω. Then $A_t\Omega = \Omega$ is constant but A_t is not continuous. We must also contend with e.g. the possibility that for a divergent sequence $A_t \in \mathrm{SL}(3,\mathbb{R})$ the domains $A_t\overline{\Omega}$ converge to $\overline{\Omega}$.

Of course this is an imperfect analogue to Theorem 1.2 since the target, $(1, 2]$, of reg is much better-separated than $\mathfrak{C}$, and there is no aspect of continuity present. Nevertheless, there is a theme here that the local projective geometry of domains of discontinuity for non-Fuchsian $\mathrm{PSL}(n, \mathbb{R})$ Hitchin representations is quite complicated (c.f. also [38]).

The geometry of properly convex projective structures is well-studied, and much of the structure in this setting (e.g. [4, 20]) is due to the presence of divisibility. It is not clear to what extent the geometry of leaves $\mathfrak{s}_\rho(x)$ is similar. One expects similarities due to the closedness of the image of $\mathfrak{s}_\rho$.

1.2.2. *Geometric Structures and Hitchin Representations.* For all split real forms G of complex simple centerless Lie groups, the G-Hitchin components are parametrized by holonomies of connected components of spaces of geometric structures on manifolds M_G associated to S [24]. Understanding the qualitative geometry of these geometric structures is a program within higher rank Teichmüller theory, into which this work falls. The basic question of the topological type of M_G has seen major recent progress in cases of special interest in [1] and more generally in [2] and [16]. There is no qualitative characterization of these connected components of geometric structures currently known in general.

In fact, the only Lie group G as above of rank at least 3 where M_G is known and the geometric structures corresponding to Hitchin representations are qualitatively characterized is $\mathrm{PSL}(4, \mathbb{R})$. Since the analytic tools that are often used to study these geometric structures in low rank (e.g. [12]) break down in rank 3 [39], the $\mathrm{PSL}(4, \mathbb{R})$ Hitchin component is a natural candidate for study in developing expectations for the general geometry of Hitchin representations.

1.2.3. *The Mapping Class Group Action on Hitchin Components.* A long-standing question in higher Teichmüller theory is to understand the structure of the action of the mapping class group $\mathrm{Mod}(S)$ on Hitchin components. A conjecture that would have settled this question was due to Labourie ([30], Conjecture 1.6). Labourie's conjecture holds for Hitchin components for Lie groups G as above of rank 2 [31], and was disproved in rank at least 3 as the culmination of a series of papers by Marković, Sagman, and Smillie [34, 35, 39].

However, the negative resolution to Labourie's conjecture does not appear to directly yield information about the $\mathrm{Mod}(S)$ action on Hitchin components, and leaves open what we shall call the fibration conjecture ([42], Conjecture 14). To state the fibration conjecture, let $\mathcal{Q}^k(S)$ denote the holomorphic bundle over Teichmüller space of holomorphic k-adic differentials (see e.g. [7]).

Question 1.6 (Fibration Conjecture)**.** *Is the* $\mathrm{PSL}(n, \mathbb{R})$ *Hitchin component naturally* $\mathrm{Mod}(S)$*-equivariantly diffeomorphic to the bundle sum* $\bigoplus_{k=3}^{n} \mathcal{Q}^k(S)$*?*

Work of the author [36] implies that a conjecture of Fock and Thomas on higher degree complex structures [17] is equivalent to the fibration conjecture. The connection of the fibration conjecture to this paper is through its prediction that there should be canonical projections $\mathrm{Hit}_n(S) \to \mathrm{Hit}_k(S)$ for $2 \leq k < n$. The only known such projections have $k = 2$ (e.g. [29, 33, 25]).

In their paper ([23], §1) introducing properly convex foliated projective structures, Guichard and Wienhard suggest that perhaps these geometric objects could be used to approach the fibration conjecture for $\mathrm{PSL}(4, \mathbb{R})$. The question that motivated the investigations leading to this paper was if examining the leaves of properly convex foliated projective structures gave rise to a projection $\mathrm{Hit}_4(S) \to \mathrm{Hit}_3(S)$. This would have been evidence in favor of the Fock-Thomas and fibration conjectures.

More specifically, properly convex subsets of $\mathbb{RP}^2$ are the setting of the geometric structures corresponding to the $\mathrm{SL}(3,\mathbb{R})$ Hitchin component, and also appear as leaves of properly convex foliated projective structures. One might hope, after noticing that $\mathfrak{s}_\rho$ is continuous and constant on Γ-orbits that $\mathfrak{s}_\rho$ was constant, $\mathfrak{s}_\rho(x)$ was divisible, and examining the action of $\rho \in \mathrm{Hit}_4(S)$ on the value of $\mathfrak{s}_\rho(x)$ gave an element of $\mathrm{Hit}_3(S)$. Theorem 1.1 shows that this hope fails.

Organization. Following the introduction are two sections on background: §2 on convex domains in $\mathbb{RP}^2$ and §3 on Hitchin representations and properly convex foliated projective structures. In §4 we prove Theorems 1.1-1.3 and present a proof, following Benoist and printed here with his permission, that $\mathfrak{s}_\rho(\partial\Gamma)$ is closed in $\mathfrak{C}$.

Acknowledgements. This paper would not have been written were it not for the reading group on Anosov representations at Rice University in 2021 and Rice's RTG geometry-topology seminar. Among the participants of these, I would like to specifically thank Chris Leininger, Mike Wolf, Alan Reid, and Sara Edelman-Muñoz.

This paper has benefitted a great deal from conversations with various mathematicians, in particular with Olivier Guichard, Max Riestenberg, Jean-Philippe Burelle, Colin Davalo, and Teddy Weisman. I would like to thank Yves Benoist for sharing his proof that leaf maps have closed image, and his permission to print it here. It is my pleasure to further thank Mike Wolf for his support and guidance. I am grateful to the anonymous referee for their careful reading of this paper and helpful suggestions. In particular, I thank the referee for pointing out a simplification of the main argument of the paper.

This material is based upon work supported by the National Science Foundation under Grant No. 1842494 and Grant No. 2005551.

2. Properly Convex Domains in $\mathbb{RP}^2$

In this section we recall the foundational facts about properly convex subsets of $\mathbb{RP}^2$ that are essential to our later arguments.

We begin by introducing definitions and notation. A *projective line* in $\mathbb{RP}^2$ is the collection of lines contained in a plane in $\mathbb{R}^3$. We refer to intervals contained in projective lines as *line segments* in $\mathbb{RP}^2$. A set $\Omega \subset \mathbb{RP}^2$ is *convex* if for any pair of points $p, q \in \Omega$ there is a line segment contained in Ω between p and q. A *domain* is an open connected subset of $\mathbb{RP}^2$. A convex domain Ω is said to be *properly convex* if $\overline{\Omega}$ is contained in a single affine chart, and is said to be *strictly convex* if for every $p, q \in \overline{\Omega}$, a line segment connecting p and q in $\overline{\Omega}$ can be taken to be contained in Ω except at its endpoints.

2.1. Spaces of Properly Convex Sets.

Let $\mathcal{C}$ denote the collection of properly convex domains in $\mathbb{RP}^2$. Let $\mathcal{C}^*$ denote the collection of pointed properly convex domains in $\mathbb{RP}^2$, that is, pairs (Ω, p) where $\Omega \in \mathcal{C}$ and $p \in \Omega$. We give $\mathcal{C}$ the topology induced by the Hausdorff topology on closures, and $\mathcal{C}^*$ the topology induced from the product $\mathcal{C} \times \mathbb{RP}^2$. Both spaces are Hausdorff. Note that $\mathrm{SL}(3,\mathbb{R})$ takes lines in $\mathbb{RP}^2$ to lines in $\mathbb{RP}^2$, and so acts on $\mathcal{C}$ and $\mathcal{C}^*$. We denote the quotients of $\mathcal{C}$ and $\mathcal{C}^*$ by the action of $\mathrm{SL}(3,\mathbb{R})$ by $\mathfrak{C}$ and $\mathfrak{C}^*$, respectively.

The topology of $\mathfrak{C}$ only separates some points—one-point sets in $\mathfrak{C}$ need not be closed. This phenomenon plays a prominent role in this paper. A first example of non-closed points in $\mathfrak{C}$ is as follows.

Example 2.1. *Let e_1, e_2, e_3 be a basis for $\mathbb{R}^3$. Work in an affine chart containing $[e_1]$, $[e_2]$, and $[e_3]$. Let Ω be a strictly convex domain contained in this affine chart preserved by $A = \mathrm{diag}(e^{\lambda}, e^{\eta}, e^{-\lambda-\eta})$ for some $\lambda > \eta \geq 0$. For instance Ω may be an ellipse if $\eta = 0$.*

Let ℓ denote the line segment from $[e_1]$ to $[e_3]$ in this affine chart and $p \in \ell - \{e_1, e_3\}$. Let ℓ' denote the line determined by $[e_2]$ and p. Then ℓ' bisects Ω. Let Ω' be the component of $\Omega - \ell'$ whose closure contains $[e_3]$. Then Ω' is not projectively equivalent to Ω as its boundary contains a line segment, but $A^n\overline{\Omega'}$ converges to $\overline{\Omega}$ in the Hausdorff topology. So $[\Omega] \in \overline{\{[\Omega']\}}$.

The closures of points in $\mathfrak{C}$ vary a great deal: it is a consequence of Benzécri's compactness theorem below that all divisible domains are closed points, while Benzécri also showed ([6] §V.3, p.321) there there exist dense one-point sets in $\mathfrak{C}$. The topology of $\mathfrak{C}$ is quite complicated, and is rich enough that the continuity of a map with target $\mathfrak{C}$ has nontrivial content.

On the other hand, all of the poor separation in $\mathfrak{C}$ is caused by divergent sequences of elements of $\mathrm{SL}(3,\mathbb{R})$ for the tautological reason that if $K \subset \mathrm{SL}(3,\mathbb{R})$ is compact and $\Omega \in \mathcal{C}$, then the orbit of Ω under K represents a single point in $\mathfrak{C}$. As a consequence, if one is able to gain finer control on a sequence $\Omega_n \in \mathcal{C}$ than convergence in $\mathcal{C}$, it can be tractable to understand the limiting projective geometry of Ω_n in spite of the non-separation of points in $\mathfrak{C}$.

The typical way this is done in practice is by gaining control over a single point of the domains Ω_n in question, working with the space $\mathfrak{C}^*$ instead of $\mathfrak{C}$. It follows from the below fundamental result of Benzécri ([6], see also [19] Theorem 4.5.4) that this is enough to guarantee uniqueness of limits.

Theorem 2.2 (Benzécri Compactness)**.** $\mathrm{SL}(3,\mathbb{R})$ *acts properly and co-compactly on* $\mathcal{C}^*$.

As an immediate corollary, we have:

Corollary 2.3. $\mathfrak{C}^*$ *is a compact Hausdorff space.*

3. Properly Convex Foliated Projective Structures and Hitchin Representations

In this section, we recall the relevant features of Hitchin representations and the theory of properly convex foliated projective structures developed by Guichard and Wienhard in [23] to our later discussion. We also prove a few basic lemmata and set conventions for later use. §3.3 is the only portion of this section not contained in existing literature.

Notation. *Through the rest of the paper, S is a closed, oriented surface of genus $g \geq 2$, $\Gamma = \pi_1(S)$, and $\overline{\Gamma} = \pi_1(T^1S)$ where T^1S is the unit tangent bundle of S.*

3.1. Hitchin Representations.

We recall the definitions of Hitchin components and Labourie-Guichard's characterization of Hitchin representations in terms of the geometry of certain special invariant curves.

3.1.1. *Fuchsian and Hitchin Representations.* Let $\mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ be the $\mathrm{PSL}(n,\mathbb{R})$-*character variety of* Γ, i.e. the collection of conjugacy classes[2] of representations $\Gamma \to \mathrm{PSL}(n,\mathbb{R})$. For $n = 2$, there are two connected components of $\mathfrak{X}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))$ that consist entirely of discrete and faithful representations. Denote their union by $\mathcal{T}(S)$. Each is identified with the Teichmüller space of isotopy classes of hyperbolic structures on S, and the existence of two components corresponds to a choice of orientation.

Classical Lie group representation theory shows that there is a unique conjugacy class of embeddings of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSL}(n,\mathbb{R})$ whose images act irreducibly on $\mathbb{R}^n$ (see Example 3.2 below). Let $\iota_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ be such an embedding.

[2]There is some variance in convention on what equivalence relation to use in defining character varieties. Namely, the coarser relation that $\rho_1 \sim \rho_2$ if the closures of the representations' conjugation orbits intersect is often used. These equivalence relations coincide for the representations we consider.

Definition 3.1. *A representation $\rho \in \mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ is* Fuchsian *if it is contained in $\iota_n(\mathcal{T}(S))$. The* $\mathrm{PSL}(n,\mathbb{R})$-Hitchin component(s) $\mathrm{Hit}_n(S)$ *are the connected components of $\mathfrak{X}(\Gamma, \mathrm{PSL}(n,\mathbb{R}))$ containing Fuchsian representations. Representations in $\mathrm{Hit}_n(S)$ are called* Hitchin representations.

There are one (if n is odd) or two (if n is even) Hitchin components in $\mathrm{PSL}(n,\mathbb{R})$, each homeomorphic to a ball of dimension $(2g-2)(n^2-1)$ where g is the genus of S [26].

3.1.2. *Hitchin Representations and Hyperconvex Frenet Curves.* While efficient, Definition 3.1 does not illuminate the structure of Hitchin representations. We now recall another characterization of Hitchin representations, in terms of the geometry of special equivariant curves, due to Labourie and Guichard. This characterization supplies a great deal of structure to us and is central to our methods.

Let $\mathcal{F}(\mathbb{R}^n)$ denote the space of full flags of nested subspaces of $\mathbb{R}^n$, i.e. $(n-1)$-tuples $(V_1, ..., V_{n-1})$ of nested subspaces of $\mathbb{R}^n$ with $\dim V_i = i$ for $i = 1, ..., n-1$. A continuous curve $\xi = (\xi^1, ..., \xi^{n-1}) : \partial\Gamma \to \mathcal{F}(\mathbb{R}^n)$ is a *hyperconvex Frenet curve* if:

1. (Convexity) For any $k_1, ..., k_j$ with $\sum_{l=1}^{j} k_l \leq n$, and distinct $x_1, ..., x_j \in \partial\Gamma$, the vector space sum $\xi^{k_1}(x_1) + ... + \xi^{k_j}(x_j)$ is direct;
2. (Osculation) For any $x \in \partial\Gamma$ and $k_1, ..., k_j$ with $K = \sum_{l=1}^{j} k_l < n$ we have that $\xi^K(x) = \lim_{m\to\infty} \left[\xi^{k_1}(x_1^m) \oplus ... \oplus \xi^{k_j}(x_j^m)\right]$ for any sequence $(x_1^m, ..., x_j^m)$ of j-tuples of distinct points so that for all l, the sequence x_l^m converges to x.

A hyperconvex Frenet curve $(\xi^1, ..., \xi^{n-1})$ is entirely determined by ξ^1.

Example 3.2. *The standard example of a hyperconvex Frenet curve is the Veronese curve, described as follows. For $k > 1$, the vector space of homogeneous degree $k-1$ polynomials on $\mathbb{R}^2$ has dimension k and so is a model for $\mathbb{R}^k$. In these models, $\mathbb{RP}^{k-1}$ is the collection of homogeneous degree $k-1$ polynomials on $\mathbb{R}^2$, considered up to scaling.*

The Veronese embedding $\xi^1 : \mathbb{RP}^1 \to \mathbb{RP}^{n-1}$ *is given in these models by taking $(n-1)$-th powers of degree 1 homogeneous polynomials, i.e. $\xi^1([f]) = [f^{n-1}]$. In general, for $1 \leq k \leq n-1$, we define $\xi^k([f]) = \{[g] \in \mathbb{R}^k \mid f^k \text{ divides } g\}$. One may verify that $\xi = (\xi^1, ..., \xi^{n-1})$ is a hyperconvex Frenet curve. We call it the* Veronese curve.

An irreducible embedding $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ may be described explicitly in this polynomial model for $\mathbb{R}^k$. To do this, observe $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{R}^k$ by $Af = f \circ A^{-1}$. The induced map $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(n,\mathbb{R})$ has image in $\mathrm{PSL}(n,\mathbb{R})$ and so gives an embedding $\iota_n : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ that one may prove is irreducible. Note that the Veronese curve is equivariant with respect to ι_n. So Fuchsian representations in $\mathrm{PSL}(n,\mathbb{R})$ admit equivariant hyperconvex Frenet curves.

The relevant result to us here of Labourie and Guichard, that generalizes the above example and which serves as our working definition of a Hitchin representation, is:

Theorem 3.3 (Labourie [28] Theorem 1.4, Guichard [22] Théorème 1)**.** *A representation $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ is Hitchin if and only if there exists a ρ-equivariant hyperconvex Frenet curve.*

A fact that will be useful to us is that Hitchin representations $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{R})$ may always be lifted to $\mathrm{SL}(n,\mathbb{R})$. This was observed by Hitchin in [26], and also follows from e.g. [15], Corollary 2.3 and Theorem 4.1.

Though the definition of a hyperconvex Frenet curve is stated in terms of sums of ξ^k, work of Guichard [21] shows that intersections of ξ^k are also quite well-behaved, which is often the way in which we interact with the hyperconvex Frenet curve property.

Proposition 3.4 (Guichard [21] Lemme 6)**.** *Let $\xi = (\xi^1, ..., \xi^{n-1})$ be a hyperconvex Frenet curve. Then:*

(1) (General Position) If $n = \sum_{i=1}^{j} k_i$ and $x_1, ..., x_j \in \partial\Gamma$ are distinct, then

$$\bigcap_{i=1}^{j} \xi^{n-k_i}(x_i) = \{0\};$$

(2) (Dual Osculation) For any $x \in \partial\Gamma$ and $k_1, ..., k_j$ with $K = \sum_{l=1}^{j} k_l < n$ we have that for any sequence $(x_1^m, ..., x_j^m)$ of j-tuples of distinct points in $\partial\Gamma$ so that x_l^m converges to x for each l,

$$\xi^{n-K}(x) = \lim_{m\to\infty} \bigcap_{i=1}^{j} \xi^{k_i}(x_i^m).$$

3.2. **Properly Convex Foliated Projective Structures.** In this subsection, we recall some features of geodesic foliations on surfaces and collect the results of Guichard and Wienhard in [23] on $PSL(4,\mathbb{R})$ Hitchin representations that are relevant to us. Our notation and the content here follows [23]. The Fuchsian case is an instructive model, and is described in §3.2.3.

3.2.1. *Stable Foliations and Geodesic Foliations.* Fixing a hyperbolic metric on S identifies the geodesic foliations of $T^1\widetilde{S}$ and $T^1\mathbb{H}^2$, and identifies $\partial\Gamma$ with $\partial\mathbb{H}^2$. There is a well-known description of $T^1\mathbb{H}^2$ as orientation-compatible triples (t_+, t_0, t_-) of distinct points in $\partial\Gamma$. We denote the space of such triples $\partial\Gamma^{(3)+}$. One obtains this identification by associating to $(p, v) \in T^1S$ the endpoints at infinity of the geodesic ℓ determined by v as t_-, t_+, and the endpoint t_0 of the geodesic perpendicular to ℓ at p that makes (t_+, t_0, t_-) orientation-compatible (see Figure 1).

Under this identification, the leaves of the *stable foliation* $\overline{\mathcal{F}}$ of $T^1\mathbb{H}^2$ are the collections of elements of $\partial\Gamma^{(3)+}$ with fixed t_+ entry, and the leaves of the *geodesic foliation* $\overline{\mathcal{G}}$ are the collections of elements of $\partial\Gamma^{(3)+}$ with fixed t_- and t_+ entries. So the leaf spaces of $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are identified with $\partial\Gamma$ and $\partial\Gamma^{(2)} := \Gamma \times \Gamma - \{(x, x) \mid x \in \Gamma\}$. From its description above, the leaf space of $\overline{\mathcal{G}}$ is also identified with the collection of oriented geodesics in $\mathbb{H}^2$. In the following, we shall identify elements of $\partial\Gamma$ and $\partial\Gamma^{(2)}$ and the corresponding leaves of $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$.

We remark that both foliations $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are Γ-invariant, and so descend to foliations that we denote by $\mathcal{F}$ and $\mathcal{G}$, respectively, of T^1S. We also remark that $\mathcal{G}$ is the geodesic foliation of the reference hyperbolic metric. Because our identification between $T^1\widetilde{S}$ and $\partial\Gamma^{(3)+}$ is equivariant with respect to the natural actions of Γ, and the foliations on $\partial\Gamma^{(3)+}$ are independent of our reference metric, the topological type of the pair $(\mathcal{F}, \mathcal{G})$ is independent of the choice of hyperbolic metric.

3.2.2. *Domains of Discontinuity and Developing Maps.* The starting point for Guichard-Wienhard's theory of properly convex foliated projective structures are explicit parameterizations of domains of proper discontinuity in $\mathbb{RP}^3$ for $PSL(4,\mathbb{R})$ Hitchin in terms of hyperconvex Frenet curves ([23] §4). We recall this construction here.

Let $\rho : \Gamma \to PSL(4,\mathbb{R})$ be Hitchin with hyperconvex Frenet curve $\xi = (\xi^1, \xi^2, \xi^3)$. Following the notation of Guichard-Wienhard ([23] §4.1.2), define the two-argument map $\xi^1 : \partial\Gamma \times \partial\Gamma \to \mathbb{RP}^3$ by

$$\xi_t^1(t') = \begin{cases} \xi^3(t) \cap \xi^2(t') & t \neq t' \\ \xi^1(t) & t = t' \end{cases}.$$

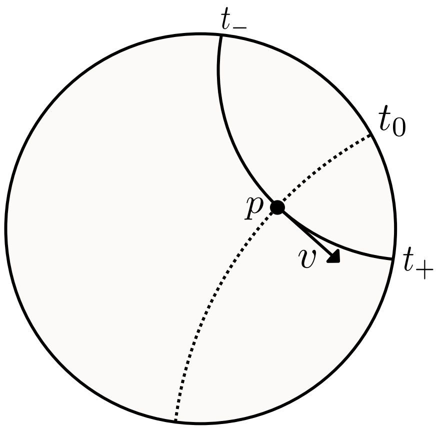

FIGURE 1. The unit tangent bundle $T^1\mathbb{H}^2$.

Then we can define a map ([23] §4.1.2, Eq. (7)) by

$$\begin{aligned} \mathrm{dev}: \quad & \partial\Gamma^{3+} \to \mathbb{RP}^3 \\ & (t_+, t_0, t_-) \mapsto \overline{\xi^1(t_+)\xi^1_{t_+}(t_-)} \cap \overline{\xi^1_{t_-}(t_+)\xi^1_{t_+}(t_0)}, \end{aligned}$$

where we denote the line in $\mathbb{RP}^3$ determined by two points a and b by $\overline{ab}$. Write $\Omega_\rho := \mathrm{dev}(\partial\Gamma^{(3)+})$. See Figure 2, and discussion below.

The following collects the features of this construction proved by Guichard-Wienhard in §4 of [23]. The most important points to us are (2), (3), (4), and (5). The assertions below all follow from further convexity features possessed by hyperconvex Frenet curves beyond their defining conditions (e.g. [21] Proposition 7).

Theorem 3.5 (Guichard-Wienhard [23], §4). *With notations as above,*

1. *The map* dev *is a homeomorphism of $\partial\Gamma^{(3)+}$ onto Ω_ρ, which is an open subset of $\mathbb{RP}^3$. It is equivariant with respect to the actions of Γ on $\partial\Gamma^{(3)+}$ and $\rho(\Gamma)$ on $\mathbb{RP}^3$.*
2. *The two argument map $(t, t') \mapsto \xi^1_t(t')$ is a continuous injection that is equivariant with respect to the actions of Γ on $\partial\Gamma \times \partial\Gamma$ and $\rho(\Gamma)$ on $\mathbb{RP}^3$. Its image is the boundary $\partial\Omega_\rho$ of Ω_ρ in $\mathbb{RP}^3$ and is a disjoint union of the lines $\xi^2(x)$ for $x \in \partial\Gamma$. That is, $\partial\Omega_\rho = \bigsqcup_{x\in\partial\Gamma} \xi^2(x)$.*
3. *The group $\rho(\Gamma)$ acts freely, properly discontinuously, and cocompactly on Ω_ρ, with quotient homeomorphic to T^1S.*
4. *For any $x \in \partial\Gamma$, the image $\xi^1_x(\partial\Gamma)$ of $\{x\} \times \partial\Gamma$ under the two-argument map ξ^1 is the boundary of* $\mathrm{dev}(x)$ *in $\xi^3(x)$. In the expression* $\mathrm{dev}(x)$ *we view x as a leaf of $\overline{\mathcal{F}}$, as in §3.2.1. The domain* $\mathrm{dev}(x)$ *is properly convex. The boundary of Ω_ρ is the disjoint union of the boundaries of these domains: $\partial\Omega_\rho = \bigsqcup_{x\in\partial\Gamma} \partial\mathrm{dev}(x)$, and the domains* $\mathrm{dev}(x)$ *foliate Ω_ρ.*
5. *For $(x, y) \in \partial\Gamma^{(2)}$, viewed as a leaf of $\overline{\mathcal{G}}$, the image* $\mathrm{dev}((x, y))$ *is the open line segment in* $\mathrm{dev}(x)$ *between $\xi^1_x(y)$ and $\xi^1(x)$. These segments foliate Ω_ρ.*
6. *A supporting line to $\partial\mathrm{dev}(x)$ at $\xi^2(y) \cap \xi^3(x)$ is $\xi^3(y) \cap \xi^3(x)$, and a supporting line to $\partial\mathrm{dev}(x)$ at $\xi^1(x)$ is $\xi^2(x)$.*

We will see in §3.3 that the domains $\mathrm{dev}(x)$ ($x \in \partial\Gamma$) are strictly convex and C^1, so that the supporting lines in Theorem 3.5.(6) give all supporting lines to the domains $\partial\mathrm{dev}(x)$. Figure 2, Right gives a sketch of $\partial\Omega_\rho$ and how a domain $\mathrm{dev}(x)$ sits in $\mathbb{RP}^3$.

Using the domains $\mathrm{dev}(x)$ above, we can make the main definition to our investigations:

Definition 3.6. *Given a Hitchin representation $\rho : \Gamma \to \mathrm{PSL}(4, \mathbb{R})$, for $x \in \partial\Gamma$ define $\mathfrak{s}_\rho(x) \in \mathfrak{C}$ to be the projective equivalence class of* $\mathrm{dev}(x)$. *We call $\mathfrak{s}_\rho$ the* leaf map *of ρ.*

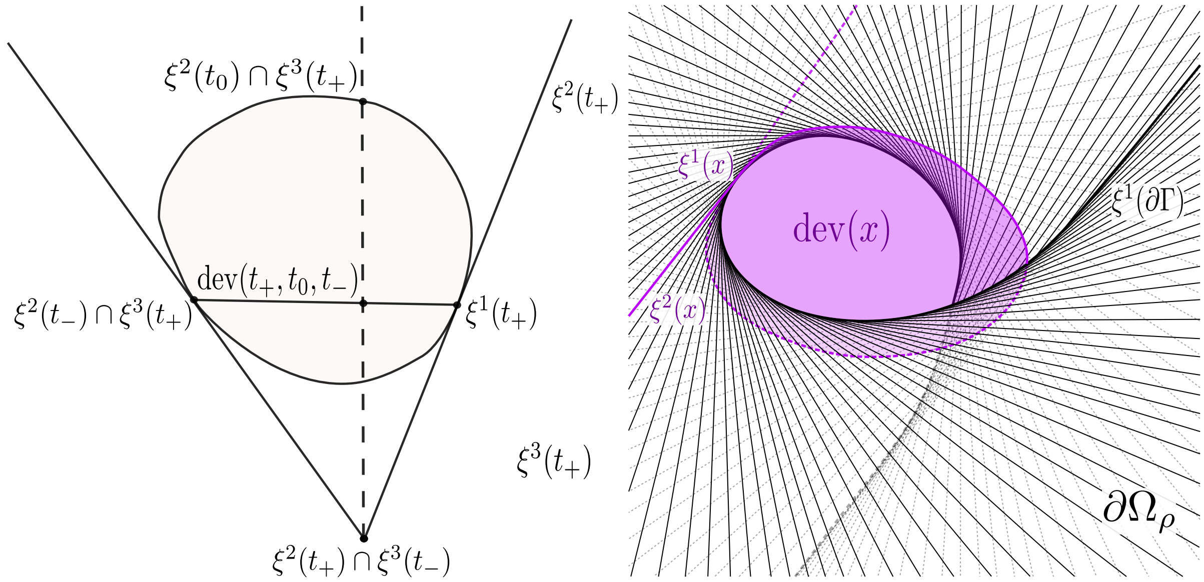

Figure 2. Left: the developing map in terms of the hyperconvex Frenet curve. Right: illustration of how the convex domains $\mathrm{dev}(x)$ sit inside of the domain of discontinuity Ω_ρ in an affine chart for $\mathbb{RP}^3$, emphasizing the ruling of $\partial\Omega_\rho$ by projective lines. Solid lines represent visible portions of $\partial\Omega_\rho$, and dotted line segments represent portions of $\partial\Omega_\rho$ that are not visible from the viewpoint of the illustration.

Let us briefly remark that Theorem 3.5.(3) implies that dev lifts to a developing map $\widetilde{\mathrm{dev}} : \widetilde{T^1S} \to \mathbb{RP}^3$ of a projective structure on T^1S with developing image Ω_ρ. Guichard and Wienhard then formalize the basic qualitative features of this projective structure induced by Theorem 3.5.(5) in the notion of a *properly convex foliated projective structure* on T^1S. The main result of [23] (Theorem 2.8) is a remarkable converse to Theorem 3.5, namely that *every* properly convex foliated projective structure on T^1S is equivalent to one of these examples by a projective equivalence that respects the foliations $\mathcal{G}$ and $\mathcal{F}$. This gives a correspondence between an appropriate moduli space of properly convex foliated projective structures and the Hitchin component $\mathrm{Hit}_4(S)$.

3.2.3. *The Fuchsian Case.* The case of Fuchsian representations in $\mathrm{PSL}(4, \mathbb{R})$ (described in §3.1.2 and §4.1 of [23]) is instructive. We briefly describe it. We use the notation of Example 3.2, so $\mathbb{RP}^3$ is viewed as the space of homogeneous degree 3 polynomials on $\mathbb{R}^2$, considered up to scaling.

Every $[f] \in \mathbb{RP}^3$ has three projective roots in $\mathbb{CP}^1$ counted with multiplicity, each of which is contained in $\mathbb{RP}^1$ or is one of a conjugate pair of non-real roots. On the other hand, polynomials on $\mathbb{C}$ are determined up to scale by their roots, so that $\mathbb{RP}^3$ parameterizes configurations of roots of real homogeneous degree 3 polynomials on $\mathbb{R}^2$.

There are four such combinatorial types, and the action of $\iota_4(\mathrm{PSL}(2, \mathbb{R}))$ on $\mathbb{RP}^3$ preserves types. The first are the polynomials that are a cube, which are exactly the image of ξ^1. The second are polynomials with a double root that are not cubes, which are exactly the points of the form $\xi^1_x(y)$ with $x \neq y$ in $\partial\Gamma$. The boundary $\partial\Omega_\rho$ is the union of these two types of points, i.e. of polynomials with a real root of multiplicity at least 2. We remark that $\partial\Omega_\rho$ may be explicitly computed in this case using discriminants: its intersection with an appropriate affine chart is the zero set of $F(x, y, z) = 18xyz - 4x^3z + x^2y^2 - 4y^3 - 27z^2$. This description is amenable to computer rendering, which when carried out shows the phenomena illustrated in Figures 2, Right and 3.

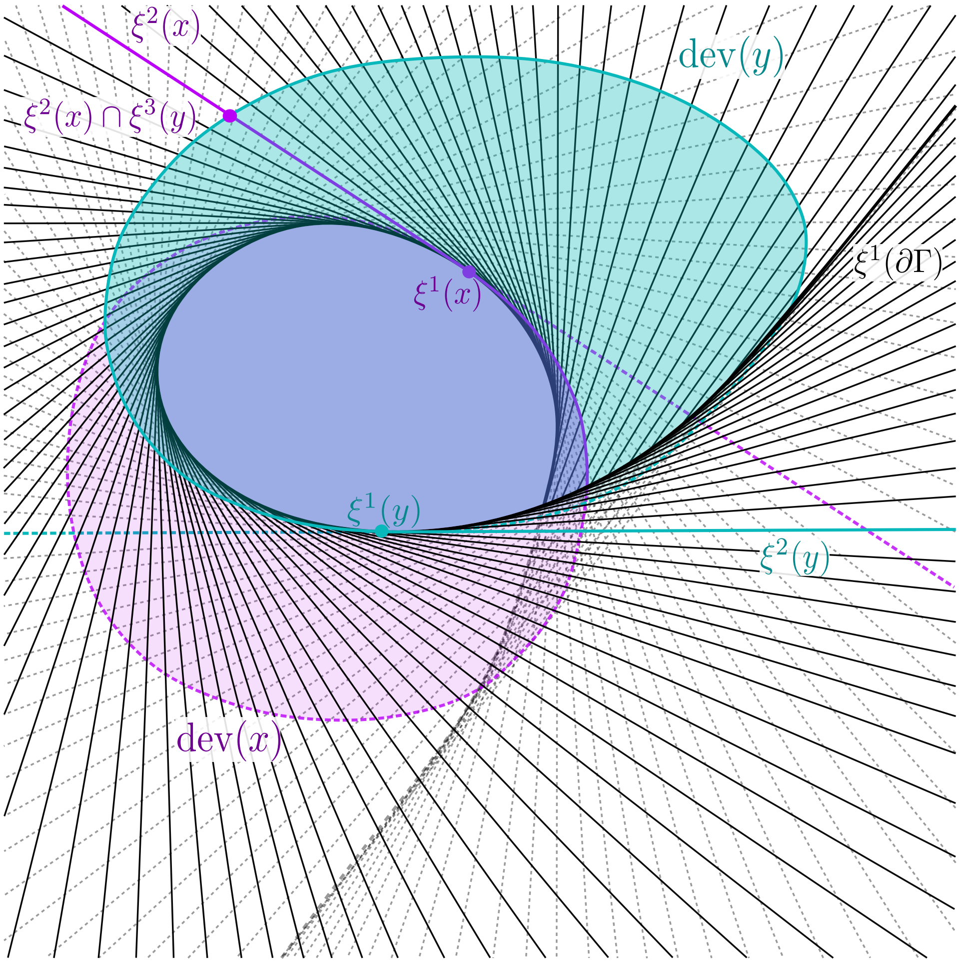

FIGURE 3. Sketch of the relative positions of two convex domains $\mathrm{dev}(x)$ and $\mathrm{dev}(y)$ in an affine chart for $\mathbb{RP}^3$. Note that the ruling of $\partial\Omega_\rho$ by lines gives a canonical maps between the boundaries of these convex domains.

The complement of $\partial\Omega_\rho$ has two connected components. The first, which we shall see is Ω_ρ, consists of polymomials with a conjugate pair of non-real zeroes. The second consists of polynomials with three distinct real zeroes.

The foliations and convexity appearing in Theorem 3.5 can be seen explicitly here. Let $\mathbb{R}^2$ have basis $\{x, y\}$. View the corresponding coordinate functions x and y as monomials on $\mathbb{R}^2$, and their projective classes as elements of $\mathbb{RP}^1$. Then the hyperplane $\xi^3([x])$ consists of the homogeneous degree 3 polynomials on $\mathbb{R}^2$ that are divided by x. So a basis for $\xi^3([x])$ is $\{x^3, x^2y, xy^2\}$. Work in the affine chart $\mathcal{A}$ for $\xi^3([x])$ that is associated to this basis and contains all polynomials with nonzero x^3-coordinate. Then the point $\xi^1(x)$ is $[1 : 0 : 0]$, the line $\mathcal{A} \cap \xi^2(x)$ is the horizontal axis $[1 : a : 0]$ $(a \in \mathbb{R})$, and $\partial\mathrm{dev}(x) \cap \mathcal{A}$ is the parabola given by $x(x + ay)^2 = [1 : 2a : a^2]$ $(a \in \mathbb{R})$. One may see that e.g. the point $x(x^2 + y^2) = [1 : 1 : 1]$ is in the convex region bounded by $\mathrm{dev}(x)$, so Ω_ρ consists of homogeneous polynomials with conjugate non-real roots.

3.3. **Two Remarks on Boundaries of Leaves.** In this subsection, we describe two basic geometric features of the leaves $\mathrm{dev}(x)$.

Our first observation is that the ruling of the boundary of Ω_ρ by $\xi^2(x)$ $(x \in \partial\Gamma)$ gives rise to natural identifications of boundaries of leaves $\partial\mathrm{dev}(x)$. Geometrically, any boundary point p of $\mathrm{dev}(x)$ is contained in exactly one $\xi^2(y)$ for $y \in \partial\Gamma$. Given another $x' \in \partial\Gamma$, the identification of boundaries maps p to the unique intersection of $\xi^2(y)$ with

$\partial\mathrm{dev}(x')$ (Figure 3). The identification of boundaries $\partial\mathrm{dev}(x) \to \partial\mathrm{dev}(x')$ is given by $\xi^1_x(t) \mapsto \xi^1_{x'}(t)$. These identifications vary continuously in x, x', and y as a consequence of continuity of $\xi^1_x(y)$, which follows from dual osculation in Proposition 3.4.

Our second observation concerns the structure of the boundary of $\partial\mathrm{dev}(x)$ for $x \in \partial\Gamma$: it is strictly convex and C^1. Strict convexity, in particular, is a tool that we use for some obstructions later.

Proposition 3.7 (Basic Regularity). *For all $x \in \partial\Gamma$, the leaf* $\mathrm{dev}(x)$ *is strictly convex and has C^1 boundary.*

Proof. To show that $\partial\mathrm{dev}(x)$ is C^1, we consider the collection $D^*_x \subset \mathrm{Gr}_2(\xi^3(x))$ of supporting lines to $\mathrm{dev}(x)$. From standard projective geometry (e.g. [19], Lemma 4.4.1), D^*_x is a topological circle. The path $\partial\Gamma \to D^*_x$ given by

$$y \mapsto \begin{cases} \xi^3(y) \cap \xi^3(x)) & y \neq x \\ \xi^2(x) & y = x \end{cases} \tag{3.1}$$

is a continuous injection of $\partial\Gamma \cong S^1$ into $D^*_x \cong S^1$, and so must be surjective. See Figure 2.

Since the map of Equation (3.1) surjects D^*_x, all supporting lines to $\mathrm{dev}(x)$ must be of the form $\xi^3(y) \cap \xi^3(x)$ or $\xi^2(x)$. As every point in $\partial\mathrm{dev}(x)$ is contained in exactly one such line by the General Position conclusion in Proposition 3.4, all boundary points of $\mathrm{dev}(x)$ have unique tangent lines. Because $\mathrm{dev}(x)$ is convex, this implies $\partial\mathrm{dev}(x)$ is C^1.

Strict convexity follows from the general position property of hyperconvex Frenet curves as follows. Supposing otherwise, $\partial\mathrm{dev}(x)$ must contain an interval I, contained in a line ℓ_I. For any $y \neq x \in \partial\Gamma$ so that $\xi^1_x(y)$ is in the interior of I, we must have $\xi^3(y) \cap \xi^3(x) = \ell_I$, as this is a supporting line to $\partial\mathrm{dev}(x)$ at a point in I. This is impossible by the general position property of hyperconvex Frenet curves, and proves strict convexity. $\square$

4. Proofs of the Main Theorems

In this section we prove our main theorems. The vast majority of the effort is spent showing $\mathfrak{s}_\rho$ is not constant unless the Hitchin representation ρ is Fuchsian. We begin by setting notation in §4.1. An outline of the structure of the core of our proofs is then given in §4.2, and the remainder of the paper is spent following this outline.

4.1. Notation, Conventions, and Definitions.

Let us begin by setting up notation to facilitate comparison of projective types of leaves.

The group $\mathrm{SL}(3, \mathbb{R})$ acts simply transitively on quadruples of points in general position in $\mathbb{RP}^2$. So, by fixing a point $t_0 \in \partial\Gamma$ and a continuously varying family of 4 points

$$\{(p_1(t), p_2(t), p_3(t), p_4(t)) \mid t \in \partial\Gamma\} \subset \mathbb{RP}^3$$

so that $p_i(t) \in \xi^3(t)$ $(i = 1, ..., 4)$ and the points $(p_1(t), p_2(t), p_3(t), p_4(t))$ are in general position within $\xi^3(t)$ for all $t \in \partial\Gamma$, we induce well-determined projective equivalences $\xi^3(t) \to \xi^3(t_0)$ for all $t \in \partial\Gamma$.

One way to produce such a normalization is to take 4 distinct points $x_1, ..., x_4 \in \partial\Gamma$ and let $p_i(t)$ $(i = 1, 2, 3, 4)$ be the unique point of intersection between $\xi^2(x_i)$ and $\partial\mathrm{dev}(t)$. The continuity of the points $p_i(t)$ results in such a normalization being continuous in the sense that the induced mappings from a reference $\mathbb{RP}^2$ with 4 fixed points in general position to $\xi^3(t) \subset \mathbb{RP}^3$ vary continuously.

Throughout the following, we shall once and for all fix such a normalization and view all domains $\mathrm{dev}(t)$ as subsets of $\mathbb{RP}^2 \cong \xi^3(t_0)$. When relevant, we will write the map $\xi^3(t) \to \xi^3(t_0)$ by $N_{t\to t_0}$. We denote $N_{t\to t_0}(\mathrm{dev}(t))$ by C_t. At times when not doing so would make notation extremely cumbersome, we abuse notation to suppress the normalization used to identify $\mathrm{dev}(t)$ and C_t.

Definition 4.1. *Given a Hitchin representation ρ, domains C_t as above, a subset $S \subset \partial\Gamma$, and a reference point $t_0 \in S$, a* projective equivalence of leaves over S *is a function $f : S \to \mathrm{Aut}(\xi^3(t_0))$ so that $f(t)C_{t_0} = C_t$ for all $t \in S$.*

Projective equivalences of leaves need not exist over a given subset $S \subset \partial\Gamma$. The leaf map $\mathfrak{s}_\rho$ is constant if and only if a family of projective equivalences over $\partial\Gamma$ exists. We do not assume continuity or any sort of regularity, measurability, or the like of projective equivalences over sets S unless explicitly noted.

At times, it will be useful to consider projective equivalences of leaves as two-argument maps between leaves seen as subsets of $\mathbb{RP}^3$, which the next bit of notation facilitates.

Definition 4.2. *Given a projective equivalence f of leaves over S and $t, t' \in S$, define the projective equivalence $f(t, t') : \mathrm{dev}(t) \to \mathrm{dev}(t')$ by*

$$f(t, t') = N^{-1}_{t' \to t_0} \circ f(t') \circ f(t)^{-1} \circ N_{t \to t_0}.$$

4.2. **Outline of Proof that non-Fuchsian Leaf Maps are Nonconstant.** Our proof assumes that $\mathfrak{s}_\rho$ is constant, so that there is a projective equivalence f over $\partial\Gamma$, and proves that ρ is Fuchsian through obtaining constraints on the eigenvalues of $\rho(\Gamma)$.

In order to get initial leverage for our arguments, we require some control on the automorphisms of individual leaves $\mathfrak{s}_\rho(x)$. The dichotomy we use to get this control is the closed subgroup Theorem, which in our setting implies that either for every $x \in \partial\Gamma$ every $\mathfrak{s}_\rho(x)$ has discrete projective automorphism group, or there is an $x \in \partial\Gamma$ so that $\mathrm{Aut}(\xi^3(t_0), C_x) \subset \mathrm{SL}(3, \mathbb{R})$ contains a 1-parameter subgroup.

The discrete case is the most involved. In it, we first show that though f may be everywhere discontinuous, we may modify f to obtain a *continuous* family $\widetilde{f}$ of projective equivalences over a nonempty open set $U \subset \partial\Gamma$, which can be enlarged using equivariance of leaf maps. The informal idea of the phenomenon underlying why this is possible is that all of the discontinuity of f comes from two sources: projective automorphisms of $\mathfrak{s}_\rho(x)$, and divergent families of projective equivalences A_t so that $A_t\overline{C_{t_0}}$ converges to $\overline{C_{t'}}$ in the Hausdorff topology for some t'. This is exploited by carefully choosing countable covers S_i of $\partial\Gamma$ so that f is well-behaved on each S_i, then applying the Baire category theorem to show some S_i is large enough to be useful.

Next, we use a "sliding" argument to show that if $\gamma \in \Gamma$ and there is a continuous family of projective equivalences g over an appropriate open set $U_\gamma \subset \partial\Gamma$, the logarithms of the eigenvalues of $\rho(\gamma)$ are evenly spaced.

Finally, we apply the eigenvalue constraints obtained from the condition that $\mathfrak{s}_\rho$ is constant to show that ρ must be Fuchsian. We do this by analyzing the constraints we have obtained on the Zariski closure of $\rho(\Gamma)$ and comparing this to Guichard's classification of Zariski closures. The main proposition here may be thought of as a simple case of a deep theorem of Benoist on Zariski-dense subgroups of linear groups [3]. We find that our eigenvalue constraints are impossible unless ρ is Fuchsian.

If one leaf has non-discrete automorphism group, the closed subgroup theorem forces this leaf to have extremely restricted structure, and in particular a rather smooth boundary. This, together with the closedness of the image of $\mathfrak{s}_\rho$, reduces to the case where every leaf is an ellipse. This is then handled similarly to the discrete case.

The discrete case is the topic of §4.3. Continuity is addressed in §4.3.1 and eigenvalue constraints in §4.3.2. In §4.3.3 we analyze the Zariski-closure of ρ and complete the discrete case. We show that leaf maps have closed image in §4.4. The non-discrete case is then completed in §4.5. We explain how Theorems 1.1 and 1.2 follow in §4.6.

4.3. **The Discrete Case.** In this subsection, we assume that the group $\mathrm{Aut}(\xi^3(t_0), C_x)$ of projective automorphisms of C_x is discrete for all $x \in \partial\Gamma$.

4.3.1. *Continuity.* We contend first with the poor separation of points in $\mathfrak{C}$. Some intuition from Benzécri's compactness theorem is that for a domain Ω with $\mathrm{Aut}(\Omega)$ discrete, projective equivalences of $\mathrm{Aut}(\Omega)$ and divergent sequences A_n so $A_n\Omega \to \Omega$ in $\mathfrak{C}$ should be the only possible discontinuities of a family of projective equivalences. The key observation of this paragraph is that in this setting, as in these two examples, all of the discontinuity of f comes from jumps of (locally) definite size.

It is useful to know that the domains C_t vary continuously in the Hausdorff topology.

Lemma 4.3 (Leaf Map Basics)**.** *Let $\rho \in \mathrm{Hit}_4(S)$. Then C_t is continuous in t, $\mathfrak{s}_\rho$ is continuous, and if $x \in \partial\Gamma$ we have $\mathfrak{s}_\rho(x) = \mathfrak{s}_\rho(\gamma x)$ for all $\gamma \in \Gamma$.*

Note that orbits of the action of Γ on $\partial\Gamma$ are dense, as this action is minimal. So for all $x \in \partial\Gamma$, the leaf map $\mathfrak{s}_\rho(x)$ is constant on the dense set Γx.

Proof. Observe that $\overline{C_t}$ varies continuously in the Hausdorff topology on domains in $\xi^3(t_0)$, since ∂C_t is parametrized by the continuous function $\partial\Gamma \to \xi^3(t_0)$ given by $N_{t\to t_0}(\xi_t(x))$ for $x \in \partial\Gamma$, and $\xi^1_t(x)$ depends continuously on t. So $\mathfrak{s}_\rho(t) = [C_t] \in \mathfrak{C}$ varies continuously.

For the other claim, if $\gamma \in \Gamma$ we have $\mathfrak{s}_\rho(\gamma x) = [\rho(\gamma)(\mathrm{dev}(x))]$, where $\rho(\gamma)|_{\xi^3(x)} : \xi^3(x) \to \xi^3(\gamma x)$ is induced by a linear map and hence a projective equivalence. □

We are now ready to prove the main proposition of this paragraph.

Proposition 4.4 (Modify to Continuity)**.** *Suppose that $\mathfrak{s}_\rho$ has countable image and every leaf $\mathfrak{s}_\rho(x)$ has discrete automorphism group. Then $\mathfrak{s}_\rho$ is constant and there is a continuous projective equivalence $\widetilde{f}$ of leaves over a non-empty open set $U \subset \partial\Gamma$.*

Proof. By hypothesis, we may write $\partial\Gamma = \bigsqcup_{m=1}^{\infty} D_m$ with D_m sets so that for all $m \in \mathbb{N}$ there is some projective equivalence of leaves f_m over D_m with respect to a reference point $s_m \in D_m$. To begin, let us fix a right-invariant metric d_P on $\mathrm{SL}(3,\mathbb{R})$ and a metric d_S on $\partial\Gamma$. Note that for all $s \in D_m$, we have $\mathrm{Aut}(\xi^3(t_0), C_s) = f_m(s)\mathrm{Aut}(\xi^3(t_0), C_{s_m})f_m(s)^{-1}$.

To proceed, we need locally uniform control in $f_m(s)$ on the separation of $\mathrm{Aut}(\xi^3(t_0), C_s)$ from the identity. To this end, we adopt the notation that for Λ a discrete subgroup of a Lie group G equipped with a right-invariant metric we set $\kappa(\Lambda) := \inf\{d(e,g) \mid g \in \Lambda - \{e\}\}$. Let us abbreviate conjugation by $\Psi_g : h \mapsto ghg^{-1}$. We obtain control through the following fact, which is a straightforward consequence of right-invariance of the metric on G. We include a proof for completeness and the convenience of the reader.

Lemma 4.5 (Discreteness is Conjugation-Stable)**.** *Let G be a Lie group and $\Lambda < G$ be a discrete subgroup. Then the function $\eta : g \mapsto \kappa(\Psi_g(\Lambda))$ is 2-Lipschitz.*

Proof. Let $x, g \in G$ be arbitrary. By right-invariance of d and the triangle inequality,

$$\begin{aligned}
|d(gxg^{-1}, e) - d(x,e)| &= |d(gx, g) - d(x,e)| \\
&= |d(gx, g) + d(gx, e) - d(gx, e) - d(x,e)| \\
&\leq |d(gx, g) - d(gx, e)| + |d(gx, e) - d(x, e)| \\
&\leq d(g, e) + d(gx, x) \\
&= 2d(g, e).
\end{aligned}$$

So for $g, g' \in G$, we have $|\eta(gg') - \eta(g')| = |\kappa(\Psi_g(\Psi_{g'}(\Lambda))) - \kappa(\Psi_{g'}(\Lambda))| \leq 2d(g,e) = 2d(gg', g')$, using the definition of η and the right-invariance of d.

□

Now let m be given, with reference point $s_m \in D_m$. By Lemma 4.5 (Discreteness is Conjugation-Stable), for each $g \in \mathrm{SL}(3,\mathbb{R})$, there exists a set K_g with the following properties:

(1) K_g is compact and contains g in its interior,
(2) Letting κ_g denote $\inf_{h \in K_g}(\kappa(\Psi_{h^{-1}}(\mathrm{Aut}(\xi^3(t_0), C_{s_m})))) = \inf_{h\in K_g}(\kappa(\mathrm{Aut}(\xi^3(t_0), hC_{s_m})))$, we have $\kappa_g > 0$,
(3) The map $K_g \times K_g \to \mathrm{SL}(3,\mathbb{R})$ given by $(h_1,h_2) \mapsto h_1 h_2^{-1}$ has image contained in the ball $B_{\kappa_g/2}(e)$.

Now let $\{K^m_{g_i}\}$ be a countable cover of $\mathrm{SL}(3,\mathbb{R})$ by such compact sets. Define $S^m_i \subset \partial\Gamma$ as $f_m^{-1}(K^m_{g_i})$. We show:

Claim. *The restriction of f_m to S^m_i is uniformly continuous.*

Proof of Claim. Fix $\epsilon > 0$. We must exhibit that there is some $\delta > 0$ so that if $d_S(t,t') < \delta$ and $f_m(t), f_m(t') \in K^m_{g_i}$, then $d_P(f_m(t), f_m(t')) < \epsilon$.

We first remark that the map $\overline{B_{\kappa_{g_i}/2}(e)} \times K^m_{g_i} \to \mathbb{R}$ given by $(A,h) \mapsto d_{\mathrm{Haus}}(h\overline{C_{s_m}}, Ah\overline{C_{s_m}})$ is continuous and has zero set exactly $\{e\} \times K^m_{g_i}$ by construction of κ_{g_i}. It follows from compactness that there is an $\epsilon' > 0$ so that if $h \in K^m_{g_i}$, $A \in \overline{B_{\kappa_{g_i}/2}(e)}$, and $d_{\mathrm{Haus}}(h\overline{C_{s_m}}, Ah\overline{C_{s_m}}) < \epsilon'$, then $A \in B_\epsilon(e)$.

As $\partial\Gamma$ is compact, the map $t \mapsto \overline{C_t}$ is uniformly continuous with respect to the Hausdorff topology on $\xi^3(t_0)$, hence there is a $\delta > 0$ so that if $d_S(t,t') < \delta$, then $d_{\mathrm{Haus}}(\overline{C_t}, \overline{C_{t'}}) < \epsilon'$. So if $d_S(t,t') < \delta$ and $t,t' \in S^m_i$, we have

$$\epsilon' > d_{\mathrm{Haus}}(\overline{C_t}, \overline{C_{t'}}) = d_{\mathrm{Haus}}(\overline{C_t}, f_m(t')f_m(t)^{-1}\overline{C_t}).$$

As $C_t = f_m(t)C_{s_m}$ with $f_m(t) \in K^m_{g_i}$ and $f_m(t')f_m(t)^{-1} \in B_{\kappa_{g_i}/2}(e)$, we have from our previous observation that $\epsilon > d_P(e, f_m(t')f_m(t)^{-1}) = d_P(f_m(t'), f_m(t))$ by right-invariance. □

The point of this claim to us is that for any i and m, there exists a continuous extension $\widetilde{f}^m_i$ of $f_m|_{S^m_i}$ to $\overline{S^m_i}$. Now observe that the two maps $\overline{S^m_i} \to \mathcal{C}(\xi^3(t_0))$ given by $t \mapsto C_t$ and $t \mapsto \widetilde{f}^m_i(t)C_{t_0}$ are continuous and agree on a dense subset of $\overline{S^m_i}$. Since $\mathcal{C}(\xi^3(t_0))$ is Hausdorff (§2), this shows $\widetilde{f}^m_i(t)C_{t_0} = C_t$ for all $t \in \overline{S^m_i}$. So $\widetilde{f}^m_i$ is a continuous projective equivalence of leaves over $\overline{S^m_i}$.

Now, as S^m_i cover $\partial\Gamma$ the collection $\{\overline{S^m_i}\}$ is a countable cover of $\partial\Gamma$ by closed sets. So by the Baire category theorem at least one $\overline{S^m_i}$ has non-empty interior. For any such i, m, setting $\widetilde{f} = \widetilde{f}^m_i$ yields the desired continuous family of projective equivalences of leaves over an open set U.

Having produced $\widetilde{f}$, we observe as remarked above that all C_x for $x \in U$ are projectively equivalent. Since the action of Γ on $\partial\Gamma$ is minimal and acts with North-South dynamics, it then follows that all C_x ($x \in \partial\Gamma$) are projectively equivalent. □

Using the action of Γ on $\partial\Gamma$, we may enlarge the open sets where we have continuous families of projective equivalences.

Corollary 4.6 (Enlarge Domains)**.** *Suppose $\mathfrak{s}_\rho$ is constant and every leaf $\mathfrak{s}_\rho(x)$ has discrete automorphism group. Let $\gamma \in \Gamma - \{e\}$ have attracting and repelling fixed-points $\gamma^+, \gamma^- \in \partial\Gamma$, respectively. Then there is a connected open set U containing γ^+ and γ^- and a continuous projective equivalence of leaves f over U.*

Proof. Proposition 4.4 (Modify to Continuity) produces an open set $U \subset \partial\Gamma$ and a continuous projective equivalence of leaves $\widetilde{f}$ over U. By equivariance of dev, for any

$\eta \in \Gamma$ we have

$$C_{\eta x} = N_{\eta x \to t_0}(\mathrm{dev}(\eta x)) = N_{\eta x \to t_0}(\rho(\eta)\mathrm{dev}(x)) = N_{\eta x \to t_0}(\rho(\eta)(N_{x \to t_0}^{-1}(C_x))).$$

So defining $f : \eta U \to \mathrm{SL}(3, \mathbb{R})$ by

$$\eta x \mapsto N_{\eta x \to t_0} \circ \rho(\eta) \circ N_{x \to t_0}^{-1} \circ \widetilde{f}(x)$$

gives a continuous projective equivalence of leaves over ηU. The corollary now follows from North-South dynamics of the action of Γ on $\partial\Gamma$. □

4.3.2. *Eigenvalue Constraints.* The goal of this paragraph is to constrain the Jordan canonical forms of $\rho(\gamma)$ for $\gamma \in \Gamma - \{e\}$. We accomplish this through obtaining eigenvalue constraints from an application of the continuity established in §4.3.1 and the closed subgroup Theorem. Throughout this paragraph, we suppress uses of normalization maps $N_{x \to t_0} : \mathrm{dev}(x) \to \xi^3(t_0)$ to make notation manageable.

To establish notation, for $\gamma \in \Gamma - \{e\}$ write the eigenvalues of $\rho(\gamma)$ as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ ordered with nonincreasing modulus, and denote $\log|\lambda_i|$ by ℓ_i for $i = 1, ..., 4$. The quadruple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is defined up to negation and $(\ell_1, \ell_2, \ell_3, \ell_4)$ is well-defined.

As ρ is Hitchin, $\rho(\gamma)$ is real-diagonalizable, $\ell_1 > \ell_2 > \ell_3 > \ell_4$, and all λ_i have the same sign (e.g. [37], discussion in proof of Theorem A). Denote the eigenlines corresponding to $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by e_1, e_2, e_3, e_4, respectively. We have $\xi^3(\gamma^+) = \mathrm{span}(e_1, e_2, e_3)$ and $\xi^3(\gamma^-) = \mathrm{span}(e_2, e_3, e_4)$. We show:

Proposition 4.7 (Jordan Form Constraints). *Suppose that ρ is a Hitchin representation, $\mathfrak{s}_\rho$ is constant, and $\mathfrak{s}_\rho(x)$ has discrete automorphism group for all $x \in \partial\Gamma$. Then for each $\gamma \in \Gamma - \{e\}$ there is a $\lambda > 1$ so that $\rho(\gamma)$ is conjugate to $\pm\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$.*

The main input is the following application of discreteness of automorphism groups of leaves. It shows that our continuous projective equivalences of leaves commute with ρ in an appropriate sense.

Lemma 4.8 (Commutativity Lemma). *Let $\gamma \in \Gamma - \{e\}$. If f is a continuous projective equivalence of leaves over a connected open set U containing γ^+ for some $\gamma \in \Gamma - \{e\}$, then for all $s \in U$ and $p \in \overline{\mathrm{dev}(\gamma^+)}$, we have*

$$\rho(\gamma)(p) = [f(\gamma s, \gamma^+) \circ \rho(\gamma) \circ f(\gamma^+, s)](p)$$

Proof. The maps $\{A_s\}_{s \in U}$ given by

$$\begin{aligned} A_s : \mathrm{dev}(\gamma^+) &\to \mathrm{dev}(\gamma^+) \\ p &\mapsto [f(\gamma s, \gamma^+) \circ \rho(\gamma) \circ f(\gamma^+, s)](p) \end{aligned}$$

are a continuous family of projective equivalences of $\mathrm{dev}(\gamma^+)$, and hence must be constant by discreteness of $\mathrm{Aut}(\xi^3(t_0), C_{\gamma^+})$. At $s = \gamma^+$ we have $A_s = \rho(\gamma)$. □

We now prove Proposition 4.7.

Proof of Proposition 4.7. Let $\gamma \in \Gamma - \{e\}$ be given. By Corollary 4.6 there is a connected open set U containing γ^+ and γ^- and a continuous projective equivalence of leaves f over U. Let $I \subset \partial\Gamma$ be a closed interval with endpoints γ^+ and γ^-.

Applying the Commutativity Lemma 4.7 with $s = \gamma^-$ shows that the restriction of $\rho(\gamma)$ to the open set $\mathrm{dev}(\gamma^+)$ coincides with

$$f(\gamma^-, \gamma^+) \circ \rho(\gamma) \circ f(\gamma^+, \gamma^-) = f(\gamma^+, \gamma^-)^{-1} \circ \rho(\gamma) \circ f(\gamma^-, \gamma^+).$$

Since projective equivalences are determined by their values on open sets and $f(\gamma^+, \gamma^-)$ is a projective equivalence, this implies the restrictions of $\rho(\gamma)$ to $\xi^3(\gamma^+)$ and $\xi^3(\gamma^-)$ are conjugate after rescaling.

So there is a $c \in \mathbb{R}^*$ so that $c(\lambda_1, \lambda_2, \lambda_3) = (\lambda_2, \lambda_3, \lambda_4)$, and hence $\lambda_1/\lambda_2 = \lambda_2/\lambda_3 = \lambda_3/\lambda_4 = c$. This, together with the restriction that all eigenvalues of $\rho(\gamma)$ have the same sign and $\rho(\gamma)$ has determinant 1 implies that $\rho(\gamma)$ is conjugate to $\pm\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ for some $\lambda > 1$. □

4.3.3. *Conclusion of Discrete Case.* Any Fuchsian representation ρ satisfies that for every $\gamma \in \Gamma - \{e\}$ the matrix $\rho(\gamma)$ ($\gamma \in \Gamma$) is conjugate to a matrix of the form $\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ for some $\lambda > 1$. To conclude, we must show that this property distinguishes Fuchsian representations. In particular, it is not possible for a non-Fuchsian representation to take values in a collection of distinct principal $\mathrm{PSL}(2, \mathbb{R})$ subgroups of $\mathrm{PSL}(4, \mathbb{R})$.

The classification of Zariski closures of Hitchin representations is useful to us here.[3] For a lift of ρ in the $\mathrm{PSL}(4, \mathbb{R})$ Hitchin component to $\mathrm{SL}(4, \mathbb{R})$, the classification states that the Zariski closure of $\rho(\Gamma)$ is conjugate to a principal $\mathrm{SL}(2, \mathbb{R})$ (in which case ρ is Fuchsian), is conjugate to $\mathrm{Sp}(4, \mathbb{R})$, or is $\mathrm{SL}(4, \mathbb{R})$.

Proposition 4.9 (Fuchsian from Eigenvalues)**.** *Suppose that ρ is lift of a $\mathrm{PSL}(4, \mathbb{R})$ Hitchin representation to $\mathrm{SL}(4, \mathbb{R})$ so that for all $\gamma \in \Gamma$, $\rho(\gamma)$ is conjugate to a matrix of the form $\pm\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ for some positive $\lambda = \lambda(\gamma) \in \mathbb{R} - \{0\}$. Then ρ is Fuchsian.*

Proof. By the classification of Zariski closures of Hitchin representations [41], it suffices to show that the Zariski closure of $\rho(\Gamma)$ is neither $\mathrm{SL}(4, \mathbb{R})$ nor conjugate to $\mathrm{Sp}(4, \mathbb{R})$.

We begin by recalling that if $a_1, ..., a_4$ are the eigenvalues of $A \in \mathrm{GL}(4, \mathbb{R})$, then the coefficients σ_i ($i = 0, ..., 3$) of the characteristic polynomial of A are the elementary symmetric polynomials in the variables $a_1, ..., a_4$, and are all polynomials in the entries of A. So let $F(a_1, a_2, a_3, a_4) = \prod_{i,j \in \{1,...,4\}} (a_i - a_j^3)$. Then F is a symmetric polynomial in $\{a_1, ..., a_4\}$, and so is an element of the polynomial ring $\mathbb{Z}[\sigma_0, ..., \sigma_3]$ by the fundamental theorem of symmetric polynomials. Consequently, F is a polynomial G in the entries of A. As all σ_i are conjugation-invariant, so is G.

Note, furthermore, that if A is conjugate to a matrix of the form $\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$, then $F(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ vanishes. So for a Hitchin representation ρ satisfying our hypotheses, the Zariski closure of $\rho(\Gamma)$ is contained in the vanishing locus of G.

On the other hand, for instance, the symplectic matrix $A = \mathrm{diag}(3, 2, 1/2, 1/3) \in \mathrm{Sp}(4, \mathbb{R})$ is not in the vanishing locus of G, as $F(3, 2, 1/2, 1/3) \neq 0$. As G is conjugation-invariant, this shows that the Zariski closure of $\rho(\Gamma)$ cannot contain any subgroup of $\mathrm{SL}(4, \mathbb{R})$ conjugate to $\mathrm{Sp}(4, \mathbb{R})$, which gives the claim. □

Remark. *Proposition 4.9 may also be proved using a deep theorem of Benoist on limit cones of Zariski-dense representations in linear groups ([3] Théorème 1.a.β). The proof above uses considerably more elementary tools than Benoist's theorem.*

Let us note that Propositions 4.9 and 4.7 are sufficient to rule out the case of discrete automorphism group:

Proposition 4.10. *Suppose that $\rho \in \mathrm{Hit}_4(S)$ and $\mathfrak{s}_\rho(\partial\Gamma)$ is countable. Then there is a leaf C_x of ρ for some $x \in \partial\Gamma$ so that $\mathrm{Aut}(\xi^3(x), C_x)$ is not discrete.*

Proof. Suppose otherwise, for contradiction. Take a $\mathrm{PSL}(4, \mathbb{R})$ Hitchin representation ρ with $\mathfrak{s}_\rho(\partial\Gamma)$ countable and so that every leaf C_x has discrete automorphism group. Then Proposition 4.4 (Modify to Continuity) shows $\mathfrak{s}_\rho$ is in fact constant and Proposition 4.7 shows that for every $\gamma \in \Gamma - \{e\}$ the matrix $\rho(\gamma)$ has Jordan canonical form

[3] The classification is due to Guichard in unpublished work, and also follows from recent results of Sambarino ([41], Corollary 1.5).

$\pm\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ for some $\lambda > 1$. Then Proposition 4.9 implies ρ is Fuchsian. This is impossible because for ρ Fuchsian, every leaf C_x is an ellipse. $\square$

4.4. **The Collection of all Leaves.** Following a suggestion of Benoist, we adapt an argument of Benzécri [6] (see also [19], proof of Theorem 4.5.6) to characterize the image of $\mathfrak{s}_\rho$. This will be used to give a short proof in the case of non-discrete automorphism group below. We maintain the notations of the previous section, notably the normalization maps $N_{t\to t_0} : \xi^3(t) \to \xi^3(t_0)$. We also adopt the notation that $\Pi : \mathcal{C}^* \to \mathcal{C}$ is the canonical projection given by forgetting pointings: $\Pi(\Omega, p) = \Omega$.

Proposition 4.11 (Benoist)**.** *Let $t \in \partial\Gamma$. Then $\mathfrak{s}_\rho(\partial\Gamma) = \mathrm{Cl}_{\mathfrak{C}}(\{[C_t]\})$.*

Proof. From the minimality of the action of Γ on $\partial\Gamma$, the continuity of $\mathfrak{s}_\rho$, and the observation that for $\gamma \in \Gamma$ and $t \in \partial\Gamma$ we have $[\mathfrak{s}_\rho(t)] = [\mathfrak{s}_\rho(\gamma t)]$, we see that $\mathfrak{s}_\rho(\partial\Gamma) \subset \mathrm{Cl}_{\mathfrak{C}}(\{C_t\})$. So it suffices to show that $\mathfrak{s}_\rho(\partial\Gamma)$ is closed in $\mathfrak{C}$.

We next describe the condition we shall verify in order to prove this. To show $\mathfrak{s}_\rho(\partial\Gamma)$ is closed in $\mathfrak{C}$ it suffices to show that the union of the $\mathrm{SL}(3,\mathbb{R})$-orbits of $\{C_t\}$ $(t \in \partial\Gamma)$ is closed in $\mathcal{C}$, which is equivalent to the closedness of the union of the $\mathrm{SL}(3,\mathbb{R})$-orbits of the preimages $\Pi^{-1}(\{gC_t\}) = \{gC_t\} \times (gC_t)$ $(t \in \partial\Gamma, g \in \mathrm{SL}(3,\mathbb{R}))$ in $\mathcal{C}^*$. This is in turn equivalent to showing the image $\mathfrak{L}$ of $\bigcup_{t\in\partial\Gamma} \Pi^{-1}(\{\mathfrak{s}_\rho(t)\})$ under the projection $\mathcal{Q}^* : \mathcal{C}^* \to \mathfrak{C}^*$ is closed. By Benzécri's compactness theorem, $\mathfrak{C}^*$ is a Hausdorff space and so compact sets in $\mathfrak{C}^*$ are closed. As $\mathfrak{C}^*$ is second-countable, it suffices to verify that $\mathfrak{L}$ is sequentially compact. This is what we shall prove.

Fix a compact set $K \subset \Omega_\rho$ so that $\rho(\Gamma)K = \Omega_\rho$. One verifies using compactness of K and $\partial\Gamma$ that the image of K after normalization is uniformly separated from the complement of the leaves in the sense that there is some $\delta > 0$, independent of $t \in \partial\Gamma$, so that if $p \in K \cap \xi^3(t)$ then $d_{t_0}(N_{t\to t_0}(p), \xi^3(t_0) - C_t) > \delta$.

So let $c_n \in \mathfrak{L}$ be a sequence. For all n, choose a leaf C_{t_n} and $p_n \in \mathrm{Int}(C_{t_n})$ so that $\mathcal{Q}^*((C_{t_n}, p_n)) = c_n$. Since $\rho(\Gamma)K = \Omega_\rho$, after applying projective equivalences arising from compositions of normalizations and the action of $\rho(\gamma)$ $(\gamma \in \Gamma)$ on Ω_ρ, we may arrange for $p_n \in K \cap C_{t_n}$. It follows from compactness of K and continuity features of our normalization that after taking a subsequence, there is some $t_\infty \in \partial\Gamma$ and $p_\infty \in C_{t_\infty} \cap K$ so that $\lim_{n\to\infty} (C_{t_n}, p_n) = (C_{t_\infty}, p_\infty)$ in $\mathcal{C}^*$. Hence $\mathcal{Q}^*(C_{t_\infty}, p_\infty) \in \mathfrak{L}$ is a limit point of c_n and so $\mathfrak{L}$ is compact, as desired. $\square$

4.5. **The Non-Discrete Case.** We show:

Proposition 4.12 (Only Ellipses)**.** *Suppose there is some $x \in \partial\Gamma$ so that $\mathfrak{s}_\rho(x)$ has non-discrete automorphism group. Then ρ is Fuchsian and $\mathfrak{s}_\rho(y)$ is the ellipse for all $y \in \partial\Gamma$.*

Proof. Let $x \in \partial\Gamma$ be so that $\mathrm{Aut}(\xi^3(t_0), C_x)$ is non-discrete. Then by the closed subgroup theorem, $\mathrm{Aut}(\xi^3(t_0), C_x)$ contains a one-parameter subgroup $H = \{A_t\}_{t\in\mathbb{R}}$. Then for any $p_0 \in \partial C_x$ the orbit Hp_0 is entirely contained in ∂C_x.

Since fixed-points of A_t for $t \neq 0$ are either isolated or contained in a line of fixed points and C_x is strictly convex, it follows that ∂C_x contains a nontrivial orbit $\mathcal{O}$ of H, which must be smooth. Note that $\mathcal{O}$ cannot have everywhere vanishing curvature, since then $\mathcal{O}$ would be a line segment and C_x is strictly convex. So ∂C_x must have a C^2 point of nonvanishing curvature. It is then a standard fact (e.g. [19] Ex. 4.5.2.3) that the ellipse $[O] \in \mathfrak{C}$ is contained in the $\mathfrak{C}$-closure of $\{[C_x]\}$.

By Proposition 4.11, there is some $y \in \partial\Gamma$ so $[C_y] = [O]$. Since the projective class $[O]$ of the ellipse is a closed point of $\mathfrak{C}$, by Lemma 4.3 (Leaf Map Basics) the preimage

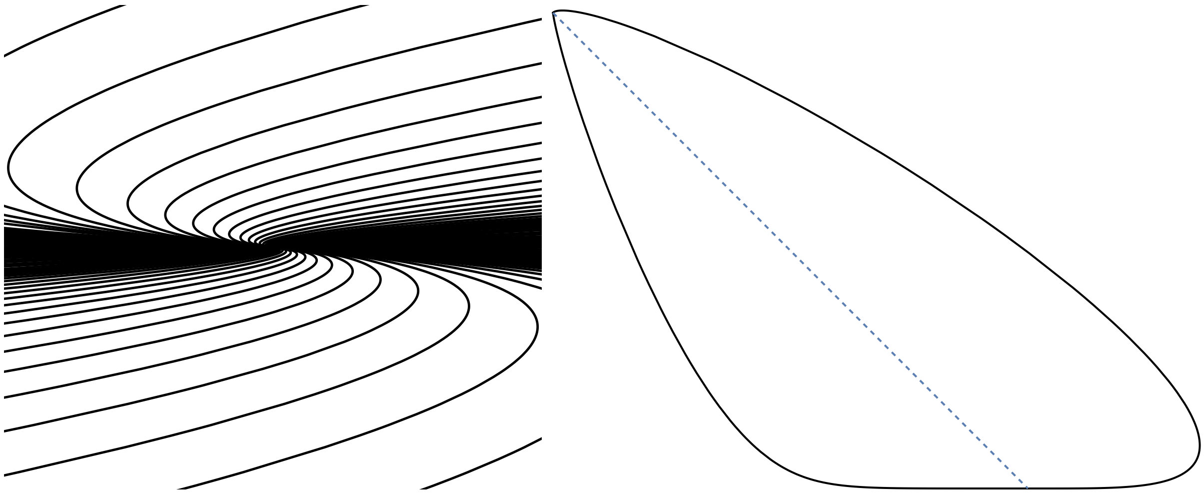

FIGURE 4. Some sample orbits of one-parameter subgroups of $\mathrm{SL}(3,\mathbb{R})$.

$\mathfrak{s}_\rho^{-1}(\{[O]\}) \subset \partial\Gamma$ is closed and contains a dense subset of $\partial\Gamma$, hence must be all of $\partial\Gamma$. So for all $t \in \partial\Gamma$, the leaf C_t is an ellipse.

Let $\gamma \in \Gamma - \{e\}$. Since C_{γ^+} and C_{γ^-} are ellipses and the restrictions of $\rho(\gamma)$ to $\xi^3(\gamma^+)$ and C_{γ^+} is diagonalizable, the restrictions $\rho(\gamma)|_{\xi^3(\gamma^+)}$ and $\rho(\gamma)|_{\xi^3(\gamma^-)}$ of $\rho(\gamma)$, must have Jordan forms that are scalar multiples of matrices of the form $\mathrm{diag}(\lambda_{\gamma^+}, 1, \lambda_{\gamma^+}^{-1})$ and $\mathrm{diag}(\lambda_{\gamma^-}, 1, \lambda_{\gamma^-}^{-1})$, respectively, for some $\lambda_{\gamma^+}, \lambda_{\gamma^-} \in \mathbb{R}^*$. The first constraint implies $\ell_1 - \ell_2 = \ell_2 - \ell_3$ and the second that $\ell_2 - \ell_3 = \ell_3 - \ell_4$. This implies $\rho(\gamma)$ has Jordan canonical form $\pm\mathrm{diag}(\lambda^3, \lambda, \lambda^{-1}, \lambda^{-3})$ for some $\lambda > 1$.

Now, Proposition 4.9 (Fuchsian from Eigenvalues) shows ρ is Fuchsian. □

4.6. **Deduction of Results.** We end by documenting how the results claimed in the introduction follow. We first note:

Theorem 4.13. *The leaf map $\mathfrak{s}_\rho$ is constant if and only if ρ is Fuchsian. If ρ is Fuchsian, then $\mathfrak{s}_\rho$ takes value the ellipse.*

Proof. The Fuchsian case is shown by Guichard-Wienhard in [23]. That $\mathfrak{s}_\rho$ is not constant if ρ is not Fuchsian follows from Propositions 4.10 and 4.12. □

The main theorems follow:

Proof of Theorem 1.1. The first equivalence is Theorem 4.13. The equivalence of (2) and (3) is given by the equivalence of constancy and countable image in Proposition 4.4. The parts of the equivalence of (4) with (1) pertaining to Fuchsian representations follow from standard facts about ellipses. That a closed point of $\mathfrak{C}$ or a divisible domain occuring as a leaf implies (2) follows from that divisible domains are closed points of $\mathfrak{C}$ ([6] §V.3 Proposition 3, see also [19] Theorem 4.5.6) together with Lemma 4.3 (Leaf Map Basics). Indeed, if $\mathfrak{s}_\rho(t)$ is a closed point of $\mathfrak{C}$, by continuity and Γ-invariance of $\mathfrak{s}_\rho$ (Lemma 4.3), $\mathfrak{s}_\rho^{-1}(\mathfrak{s}_\rho(t))$ is a closed, Γ-invariant subset of $\partial\Gamma$ and hence all of $\partial\Gamma$. That a leaf having non-discrete automorphism group implies ρ is Fuchsian follows from Proposition 4.12. □

Proof of Theorem 1.2. Combine Theorem 4.13 with Lemma 4.3 (Leaf Map Basics). □

Proof of Theorem 1.3. For ρ non-Fuchsian, that $\mathfrak{s}_\rho(\partial\Gamma)$ is a non-point closed subset of $\mathfrak{C}$ is Theorem 1.1 and Proposition 4.11. That $\mathfrak{s}_\rho(\partial\Gamma)$ is minimal among closed sets follows from the characterization in Proposition 4.11 that $\mathfrak{s}_\rho(\partial\Gamma)$ is the closure of any point in $\mathfrak{s}_\rho(\partial\Gamma)$. This proves the Theorem for $\mathfrak{C}(\mathbb{RP}^2)$.

The result for $\mathfrak{C}(\mathbb{RP}^n)$ for $n \geq 3$ reduces to the $n = 2$ case by a classical characterization of $\mathfrak{C}(\mathbb{RP}^n)$-closures of convex hulls due to Benzécri ([6] §V.3, Proposition 4), as follows. Let $\Omega \in \mathfrak{C}(\mathbb{RP}^n)$ be a convex domain so that the closure in $\mathfrak{C}(\mathbb{RP}^n)$ of $\{[\Omega]\}$ is a minimal closed subset of $\mathfrak{C}(\mathbb{RP}^n)$ that is not a point. Let Ω' be a convex domain in $\mathbb{RP}^{n+1}$ formed as the convex hull in an affine chart $\mathcal{A}$ of an inclusion of Ω in the intersection with $\mathcal{A}$ of a copy P of $\mathbb{RP}^n \subset \mathbb{RP}^{n+1}$ and a point $p \in \mathcal{A} - P$. Then Benzécri's proposition implies the closure of $\{[\Omega']\}$ in $\mathfrak{C}(\mathbb{RP}^{n+1})$ is a minimal closed subset in $\mathfrak{C}(\mathbb{RP}^{n+1})$ that is not a point. $\square$

Proof of Corollary 1.4. This follows from the standard fact (e.g. [19] Ex. 4.5.2.3) that if Ω is a properly convex domain in $\mathbb{RP}^2$ with a C^2 boundary point of nonvanishing curvature, then the $\mathfrak{C}$-closure of $\{[\Omega]\}$ contains the ellipse. If $\rho \in \mathrm{Hit}_4(S)$, then since $\mathfrak{s}_\rho$ has closed image, this implies that if any leaf $\mathfrak{s}_\rho$ contains a C^2 boundary point with nonzero curvature then there is a leaf $\mathfrak{s}_\rho(y)$ projectively equivalent to the ellipse. We conclude ρ is Fuchsian by Theorem 1.1. $\square$

The proof of Corollary 1.5 is similar to that of Corollary 1.4.

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