HUBER RINGS AND VALUATION SPECTRA

JOHN BERGDALL

Contents

Introduction

These notes expand on a four-hour lecture course given in Heidelberg in March 2023, as part of the "Spring School on non-Archimedean Geometry and Eigenvarieties". They are designed for graduate students and other learners. We introduce Huber rings and valuation theory alongside frequent examples. The notes are largely self-contained, though many details are given in exercises found following each lecture.

Context. Hensel developed the p-adic numbers and their analysis in the waning years of the 19th century. Tate's theory of rigid analytic spaces dates to the 1960's [Tat71]. The p-adic numbers allow for number theory modeled on power series expansions. Tate's theory models analytic geometry over the p-adic numbers, building spaces such as discs, annuli, and more, along with robust definitions of their rings of analytic functions. These models are applied to study problems in both geometry and number theory. One original motivation was uniformizing p-adic elliptic curves with split multiplicative reduction, now called Tate curves, via rigid analytic maps, in analogy with complex uniformization of (all) elliptic curves over the complex numbers.

Tate develops rigid analytic spaces using a class of rings, now called affinoid algebras, and their maximal ideal spectra. He equips these spaces with presheaves of functions, which he proves are actually sheaves. More precisely, the sheaves are sheaves only for a so-called Grothendieck topology. This major caveat is responsible for significant challenges in learning and using Tate's theory.

Three new theories were developed starting in the late 1980's:

(i) Raynaud's formal models [BL93a, BL93b, BLR95a, BLR95b].
(ii) Berkovich's analytic spaces [Ber90, Ber93].
(iii) Huber's adic spaces [Hub93, Hub94, Hub96].

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Conrad's four lectures at the 2007 Arizona Winter School [Con08] focused on Tate's theory, along with the work of Bosch–Lütkebohmert–Raynaud and Berkovich. Our lectures in Heidelberg, and those of our colleagues Hübner [Hüb24], Johansson [Joh24] and Heuer [Heu24], discuss Huber's theory and its applications.

Why write these notes now? First, interest in adic spaces has exploded since they became a pillar for Scholze's perfectoid spaces [Sch12, SW20]. For instance, the 2017 Arizona Winter School was dedicated to perfectoid spaces, and those lectures necessarily included only a rapid introduction to adic spaces [BCKW19]. Second, eigenvarieties are the Spring School's second topic. These are traditionally developed as rigid analytic spaces, in Tate's style, by Hida, Coleman and Mazur, Buzzard, and many more. Recent works [AIP18, JN19, Gul19], however, extend eigenvarieties to characteristic p local fields. All those works require Huber's perspective.

Motivation. The first two lectures focus on spaces of valuations, on which Huber's theory is based. As background, we recall Tate's spaces and how to shift into a valuative mindset. We work over $\mathbb{C}_p$, the p-adic complex numbers. The field $\mathbb{C}_p$ is complete for the p-adic norm $\|-\|_p$ *and* algebraically closed, which makes geometry more clear.

The fundamental ring in Tate's theory is the Tate algebra $\mathbb{C}_p\langle w\rangle$. It is defined as the ring of series

$$f = a_0 + a_1 w + a_2 w^2 + \cdots \in \mathbb{C}_p[\![w]\!]$$

with the property that

$$\lim_{i\to\infty} \|a_i\|_p = 0. \tag{0.0.1}$$

Tate models the closed unit disc over $\mathbb{C}_p$ as maximal ideals in $\mathbb{C}_p\langle w\rangle$. To explain, if $f \in \mathbb{C}_p\langle w\rangle$ and $\|\alpha\|_p \leq 1$, then $f(\alpha)$ converges by (0.0.1). The evaluation map

$$\mathbb{C}_p\langle w\rangle \xrightarrow{f\mapsto f(\alpha)} \mathbb{C}_p \tag{0.0.2}$$

therefore exists and has kernel $\langle w-\alpha\rangle \in \text{max-Spec}(\mathbb{C}_p\langle w\rangle)$. The **Weierstrass preparation theorem** implies all maximal ideals arise this way. This gives a bijection

$$\{\alpha \in \mathbb{C}_p \mid \|\alpha\|_p \leq 1\} \longleftrightarrow \text{max-Spec}(\mathbb{C}_p\langle w\rangle).$$

Maximal ideals also detect inequalities needed for analytic geometry. For instance, $\|\alpha\|_p \leq \frac{1}{p}$ if and only if $w-\alpha$ generates a maximal ideal in the larger ring

$$\begin{aligned}\mathbb{C}_p\langle \frac{w}{p}\rangle &= \{b_0 + b_1\frac{w}{p} + b_2(\frac{w}{p})^2 + \cdots \mid \lim_{i\to\infty}\|b_i\|_p = 0\}\\ &= \{a_0 + a_1 w + a_2 w^2 + \cdots \mid \lim_{i\to\infty}\|a_i\|_p p^{-i} = 0\}.\end{aligned}$$

The ring $\mathbb{C}_p\langle \frac{w}{p}\rangle \cong \mathbb{C}_p\langle w, v\rangle/\langle pw - v\rangle$ is an example of a $\mathbb{C}_p$-affinoid algebra. Rigid spaces are glued from affinoid spaces, which are the maximal ideal spectra of affinoid algebras, in analogy to how schemes are glued from the prime ideal spectral of rings.

This model for geometry faces a major technical issue. It is too easily disconnected. In Tate's theory, a disc of radius $p^{-s} < 1$ is given by

$$\{\|\alpha\|_p \leq p^{-s}\} \leftrightarrow \text{max-Spec}(\mathbb{C}_p\langle \frac{w}{p^s}\rangle).$$

These cover the open unit disc $\{\|\alpha\|_p < 1\}$. The boundary of the unit disc is

$$\{\|\alpha\|_p = 1\} \leftrightarrow \text{max-Spec}(\mathbb{C}_p\langle w, w^{-1}\rangle) = \text{max-Spec}(\mathbb{C}_p\langle w, v\rangle / \langle wv - 1\rangle).$$

Thus, the closed unit disc decomposes

$$\{\|\alpha\|_p \leq 1\} = \{\|\alpha\|_p = 1\} \cup \bigcup_{s>0} \{\|\alpha\|_p \leq p^{-s}\}, \tag{0.0.3}$$

with each piece being affinoid.

Why is this disconnection an issue? A naïve sheaf theory would produce $\mathbb{C}_p\langle w\rangle$ as the ring of functions on $\{\|\alpha\|_p \leq 1\}$. The disconnection (0.0.3) would then say that a series $\mathbb{C}_p\langle w\rangle$ can theoretically be defined by prescribing an analytic series on the disc's "boundary" $\{\|\alpha\|_p = 1\}$ and, independently, a compatible collection of series on disc $\{\|\alpha\|_p \leq p^{-s}\}$ with $s > 0$. So, a naïve sheaf theory would allow a single series that identically vanishes on the interior of the disc and not on the boundary. But, such a series is disallowed by the Weierstrass preparation theorem. Tate's solution was to use the language of Grothendieck topologies to disallow coverings such as (0.0.3).

Huber proposes a different model of p-adic geometry. (Berkovich's approach has the same origin.) Returning to the evaluation maps (0.0.2), if $\|\alpha\|_p \leq 1$, we define a (semi-)norm $|-|_\alpha$ on $\mathbb{C}_p\langle w\rangle$ by

$$|f|_\alpha = \|f(\alpha)\|_p. \tag{0.0.4}$$

The maximal ideal $\langle w - \alpha\rangle$ is equal to the **support** of $|-|_\alpha$, which is the set of f such that $|f|_\alpha = 0$. So, Tate's "points" are recovered directly from these norms. However, there are more norms on $\mathbb{C}_p\langle w\rangle$. For instance, there is the **Gauss norm**

$$\|a_0 + a_1 w + a_2 w^2 + \cdots\|_{\text{Gauss}} = \max_{i \geq 0} \|a_i\|_p. \tag{0.0.5}$$

This norm is special because it is a norm that gives $\mathbb{C}_p\langle w\rangle$ the structure of a complete topological ring. It is of a different nature than $|-|_\alpha$ in the sense its support is $\{0\}$.

Huber's theory goes even further. It considers more general objects called **valuations**. These satisfy the axioms of norms, except their target is not always the non-negative real numbers. Huber models rigid analytic geometry on spaces of *continuous* valuations on topological rings, such as $\mathbb{C}_p\langle w\rangle$.

Lecture contents. The first two lectures address Huber rings and their continuous valuation spectra. By the end of the second lecture, we explain:

(i) Huber's continuous valuation spectra and how localization theory detects subdiscs such as $\{\|\alpha\|_p \leq \frac{1}{p}\}$.
(ii) Huber's natural model for the closed unit disc will contain at least one point is that *not* naturally part of the "covering" seen in (0.0.3).

Point (i) supports comparing Huber's theory with Tate's. Point (ii) suggests Huber's theory treats coverings and analytic functions differently than Tate's.

Hübner's initial lecture on adic spectra [Hüb24] comes between our second and third lectures. The reader should read that first and then come back to our third lecture, where we detail further constructions with adic spectra.

Our fourth lecture is independent from those of our colleagues. In it, we analyze the closed unit disc, focusing on a systematic explanation of the point referred to in (ii) above.

References. Huber rings and their valuation theory are the topic of Huber's papers [Hub93, Hub94]. Other published references include Scholze's seminal work [Sch12] and Weinstein's lectures at the 2017 Arizona Winter School [BCKW19]. In addition, many have learned with the help of unpublished notes of Wedhorn [Wed19], Conrad [Con14], and Morel [Mor19]. These resources are all more ambitious than ours. We hope our lectures, in fact, introduce and complement more advanced texts.

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1. Huber rings

The primary goal of this lecture is defining Huber rings and giving their initial properties. We emphasize examples and material on bounded-ness and localization.

1.1. Definition.

Definition 1.1.1. *A* ***Huber ring*** *is a topological ring A that contains an open subring A_0 for which there is an ideal $I \subseteq A_0$ such that:*

(a) The topology on A_0 (and thus on A) is the I-adic topology.
(b) The ideal I is finitely generated.

Huber rings are called f-adic rings in the original work [Hub93]. The "-adic" refers to condition (a) in Definition 1.1.1, while the "f-" recognizes the finiteness assumption in part (b). The shift toward the term "Huber ring" follows the introduction of perfectoid spaces and derivative works. The main benefit of the new name is that the "f" in f-adic cannot be confused with an italicized "f", which frequently represents a mathematical object, such as a function.

In (a), saying A has the I-adic topology means a subset $U \ni 0$ is open if and only if $I^n \subseteq U$ for some n. The open sets around non-zero elements are determined by translation, since A is a topological ring. Note, however, that $I \subseteq A$ is an open additive subgroup in A. It has the structure of an ideal over A_0, only.

We call A_0 a **ring of definition** and $I \subseteq A_0$ an **ideal of definition**. The pair (A_0, I) is a **pair of definition**. Pairs of definition are auxiliary structures. They exist but need not be specified. Also, there is choice involved. For instance, if (A_0, I) is a pair of definition, then so is (A_0, I^n) for any $n \geq 1$.

1.2. Examples.

Example 1.2.1. Let A be any ring and I any finitely generated ideal in A. We equip A with the I-adic topology to make A a topological ring. Since I is finitely generated, A is a Huber ring with pair of definition (A, I).

As a specific case, let $A = \mathbb{Z}_p[\![w]\!]$. Unlike Tate's affinoid algebras, this is a power series ring without any convergence conditions. It is noetherian and local, with maximal ideal $\mathfrak{m} = \langle p, w\rangle$. The $\mathfrak{m}$-adic topology on A turns A into a Huber ring.

Example 1.2.2. The field of p-adic numbers $\mathbb{Q}_p$ is not covered by Example 1.2.1. Recall $\mathbb{Q}_p$ is a topological field with its topology defined by the p-adic norm $\|-\|_p$. A neighborhood basis of zero is given by the open balls $\{\|\alpha\|_p \leq p^{-n}\}$. Therefore,

$$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p \mid \|\alpha\|_p \leq 1\}$$

is open in $\mathbb{Q}_p$. Moreover, for $n \geq 0$ and $\alpha \in \mathbb{Z}_p$, we have

$$\alpha \in p^n\mathbb{Z}_p \iff \|\alpha\|_p \leq p^{-n}.$$

Therefore, the topology defined by $\|-\|_p$ on $\mathbb{Z}_p$ is the same as the topology induced by the (principal) ideal $p\mathbb{Z}_p$. We have shown $\mathbb{Q}_p$ is a Huber ring by identifying $(\mathbb{Z}_p, p\mathbb{Z}_p)$ as a pair of definition. It is even a Tate ring. See Section 1.3.

Example 1.2.3. A **non-Archimedean field** is a complete topological field K whose topology is defined by a non-trivial non-Archimedean norm

$$\|-\| : K \to \mathbb{R}_{\geq 0}.$$

Such K are Huber rings, just as for $K = \mathbb{Q}_p$. First, the subring

$$A_0 = \mathcal{O}_K := \{\alpha \in K \mid \|\alpha\| \leq 1\}$$

is open. Second, since $\|-\|$ is non-trivial, there exists $\alpha \in K^\times$ such that $\|\alpha\| \neq 1$. Define $\varpi = \alpha^{\pm 1}$, making sure $\|\varpi\| < 1$. Since $\|\varpi^n\| \to 0$ as $n \to +\infty$, the norm topology on A_0 is equivalent to the topology defined by the ideal $\varpi A_0 \subseteq A_0$. There is a wide range of choices for ϖ. Each such choice is called a **pseudo-uniformizer** for K. Examples of non-Archimedean fields include perfectoid fields discussed in Heuer's lectures [Heu24].

The field $K = \mathbb{C}_p$ is important to keep in mind. By definition, $\mathbb{C}_p$ is the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ with respect to the p-adic norm. The ring $A_0 = \mathcal{O}_{\mathbb{C}_p}$ is a local ring whose norm topology is defined by the principal ideal $p\mathcal{O}_{\mathbb{C}_p}$. This is *not* the topology defined by the maximal ideal

$$\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} = \{x \in \mathcal{O}_{\mathbb{C}_p} \mid \|x\|_p < 1\}.$$

Indeed, since $\mathbb{C}_p$ is algebraically closed we have $\mathfrak{m}^2_{\mathcal{O}_{\mathbb{C}_p}} = \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$. Therefore, the powers of $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$ do not shrink toward zero as the powers of $p\mathcal{O}_{\mathbb{C}_p}$ do.

Example 1.2.4. Let K be a non-Archimedean field with norm $\|-\|$. Define

$$A = K\langle w\rangle = \left\{ \sum_{i=0}^{\infty} a_i w^i \in K[\![w]\!] \mid \lim_{i\to\infty} \|a_i\| = 0 \right\}.$$

This is called a Tate algebra in one variable, in honor of its role in Tate's rigid analytic geometry. It is a topological ring, with topology defined by a norm $\|-\|_{\text{Gauss}}$

$$\|a_0 + a_1 w + a_2 w^2 + \cdots\|_{\text{Gauss}} = \max_{i\geq 0} \|a_i\|$$

that we first encountered in (0.0.5). This norm is called the **Gauss norm**, presumably because proofs that $\|-\|_{\text{Gauss}}$ is multiplicative resemble proofs of Gauss's lemma on irreducibility of integer polynomials over the rational field. The topological ring A is a Huber ring. A ring of definition is

$$A_0 = \mathcal{O}_K\langle w\rangle = \left\{\sum_{i=0}^{\infty} a_i w^i \in A \mid a_i \in \mathcal{O}_K \text{ for all } i\right\}$$
$$= \{f \in A \mid \|f\|_{\text{Gauss}} \leq 1\}.$$

An ideal of definition is

$$\varpi A_0 = \left\{\sum_{i=0}^{\infty} a_i w^i \in A \mid a_i \in \varpi\mathcal{O}_K \text{ for all } i\right\} = \{f \in A \mid \|f\|_{\text{Gauss}} < \|\varpi\|\}.$$

Here, ϖ is a pseudo-uniformizer for K as in Example 1.2.3. If $\|-\|$ is a discrete norm, then the inequality in the prior displayed equation becomes $\|f\|_{\text{Gauss}} < 1$.

Warning 1.2.5. In these examples, we have been careful to define a topological ring first and assert the Huber ring property second. More accurately, in each case we first defined a ring A with a topology. Second, we left as Exercise 1.1 to check each A was actually a topological ring. And, third, we identified a pair of definition (A_0, I).

Often you will find Huber rings A defined only using a pair of definition (A_0, I). After all, taking a topological ring A_0 and declaring $A_0 \subseteq A$ to be open is a valid way to describe a topology on a ring A. It just may not define a topological ring! This issue arises in practice, with rational localizations. So, we propose Exercises 1.2-1.3 as early work with topological rings.

1.3. **Example: Tate rings.** Tate rings are important enough that they get their own subsection. A **Tate ring** is a Huber ring A in which there exist a *unit* ϖ such that $\varpi^n \to 0$ as $n \to +\infty$. Such a ϖ is called a **pseudo-uniformizer**, borrowing from Example 1.2.3.

Let us examine the structure of a Tate ring A. Choose a pair of definition (A_0, I). Since $\varpi^n \to 0$ as $n \to +\infty$, there exists an $n \geq 1$ such that $\varpi^n \in I$. The element ϖ^n is still a pseudo-uniformizer. So, without loss of generality $\varpi \in I \subseteq A_0$. Now we claim:

(a) The topology on A_0 is the ϖA_0-adic topology.
(b) Algebraically, $A = A_0[\frac{1}{\varpi}]$.

Thus, any choice of a pseudo-uniformizer ϖ (in a given ring of definition) simultaneously controls the topological and algebraic structure of a Tate ring. For proof, ϖA_0 is an open A_0-ideal, since it is a multiplicative translate of the open subset $A_0 \subseteq A$. But also $\varpi \in I$ and so $\varpi A_0 \subseteq I$. This proves (a). For (b), consider any $f \in A$. Since multiplication by f is continuous on A, there exists n such that $f\varpi^n A_0 \subseteq A_0$, and therefore $f \in A_0[\frac{1}{\varpi}]$.

1.4. **Rings of definition.** Huber rings are topological rings. Pairs of definition (A_0, I) are auxiliary data. The goal of this section is illustrating the flexibility of this data.

Let A be a topological ring. A subset $X \subseteq A$ is called **bounded** if for any open neighborhood $U \ni 0$, there exists an open neighborhood $V \ni 0$ such that $XV \subseteq U$. Note that X, U, and V are *a priori* just subsets. The condition $XV \subseteq U$ means that if $x \in X$ and $v \in V$ then $xv \in U$. If U is an additive subgroup, which is often the case in the context of Huber rings, then the condition $XV \subseteq U$ is equivalent to $X \cdot V \subseteq U$ where $X \cdot V$ is the abelian group generated by XV.[1]

The next proposition classifies rings of definition from a topological perspective. It is repeatedly used in analyzing more refined structures in Section 1.5.

Proposition 1.4.1. *If A is a Huber ring, then a subring $A_0 \subseteq A$ is a ring of definition if and only if A_0 is open and bounded.*

Proof. Let A be a Huber ring and A_0 a ring of definition. First, A_0 is open by definition. Second, basic open neighborhoods of zero have the form $U = I^n$ with $I \subseteq A_0$ an ideal. Since $A_0 I^n \subseteq I^n$, we see that A_0 is bounded (taking $V = I^n$ as well).

We now argue the converse. Assume A_0 is open and bounded. Choose any pair of definition (B_0, J) for A. Since A_0 is open, it contains a power J^n of J. The pair (B_0, J^n) is also a pair of definition for A. Replacing J by J^n, we assume that $J \subseteq A_0$.

Now choose $f_1, \ldots, f_d \in J$ that generate J as a B_0-ideal. Define

$$I = \sum_{i=1}^{d} A_0 f_i \subseteq A_0.$$

We claim (A_0, I) is a pair of definition for A. Since I is a finitely generated A_0-ideal, we focus on showing the I-adic topology is the J-adic topology on A.

First, we show $J^2 \subseteq I$. Recall $J \subseteq A_0$. Then,

$$J^2 = J \cdot (\sum_{i=1}^{d} B_0 f_i) = \sum_{i=1}^{d} J f_i \subseteq \sum_{i=1}^{d} A_0 f_i = I.$$

Second, we show $I^n \subseteq J$ for some n. This is where we use that A_0 is bounded. Indeed, since A_0 is bounded and J is an ideal, we may choose n such that $A_0 \cdot J^n \subseteq J$. The A_0-ideal I^n is additively generated by elements $a f_1^{n_1} \cdots f_d^{n_d}$ where $n_1 + \cdots + n_d = n$ and $a \in A_0$. Since $f_1^{n_1} \cdots f_d^{n_d} \in J^n$, we conclude that $I^n \subseteq A_0 \cdot J^n \subseteq J$, as claimed. □

1.5. **Bounded conditions.** Let A be a topological ring. This subsection introduces power-bounded elements A° and topologically nilpotent elements $A^{\circ\circ}$.

1.5.1. *Power-bounded elements.* We say $f \in A$ is **power-bounded** if its powers

$$\{1, f, f^2, f^3, \ldots\}$$

form a bounded subset in A. It is traditional to use A° as notation:

$$A^\circ = \{f \in A \mid f \text{ is power-bounded}\}.$$

For instance, if $A = \mathbb{C}_p\langle w\rangle$, then $w \in A^\circ$, and in fact $A^\circ = \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. (See Exercise 1.4.)

Note that $f \in A^\circ$ demands that for any open subset $U \ni 0$, there is an open subset $V \ni 0$ for which $V \subseteq U$, and $fV \subseteq U$, and $f^2 V \subseteq U$, and so on. A finite number of these containments can be arranged, since multiplication by f is continuous on A, but the condition that $f \in A^\circ$ is more strict.

[1] The typographical difference between XV and $X \cdot V$ can cause problems while scanning. The "$X \cdot V$"-notation is used by Huber [Hub93]. It has stuck in many references. Beware!

It is clear that A° is closed for multiplication. It follows from Exercise 1.5 that A° is in fact a subring of A. If A is a Huber ring, it even equals the union of all rings of definition. The main step in the proof is the next lemma.

Lemma 1.5.2. *Let A be a Huber ring. If A_0 is a ring of definition and $f \in A^\circ$, then $A_0[f]$ is a ring of definition.*

Proof. Since A is a Huber ring, we only need to show $A_0[f]$ is open and bounded, by Proposition 1.4.1. Since $A_0 \subseteq A_0[f]$ already, the open-ness is clear. What about boundededness? Let $I \subseteq A_0$ be an ideal of definition. Since $f \in A^\circ$, there exists n such that $f^m I^n \subseteq I$ for all $m \geq 0$. Since I and I^n are A_0-ideals, we see that $A_0[f]I^n \subseteq I$. Then, $A_0[f]$ is bounded by Exercise 1.5. □

Proposition 1.5.3. *If A is a Huber ring, then*

$$A^\circ = \bigcup_{\substack{A_0 \subseteq A \\ \text{ring of def.}}} A_0.$$

So, A° is an open subring in A. It is also integrally closed in A.

Proof. Let A_0 a ring of definition. Since A_0 is bounded and closed under exponentiation, we see $A_0 \subseteq A^\circ$. Conversely, if $f \in A^\circ$ and A_0 is any ring of definition, then $f \in A_0[f]$. By Lemma 1.5.2, $A_0[f]$ is a ring of definition.

Having shown the equality in the proposition, we have that A° is open, and we already indicated why it is a subring. The primary observation to show A° is integrally closed is that the proof so far implies that if $f \in A$ is integral over A°, then f is integral over some ring of definition A_0. Given this, one checks that the finite A_0-algebra $A_0[f]$ is open and bounded. See Exercise 1.7. □

Proposition 1.5.3 *does not* claim A° is bounded. See Exercise 1.9.

1.5.4. *Topologically nilpotent elements.* An element $f \in A$ is called **topologically nilpotent** if for all open neighborhoods $U \ni 0$, we have $f^n \in U$ for $n \gg 0$. The sufficiently large "$\gg$" depends on f and U. As an example, a pseudo-uniformizer in a Tate ring is topologically nilpotent. The formal notation is

$$A^{\circ\circ} = \{f \in A \mid f \text{ is topologically nilpotent}\}.$$

Assume A is a Huber ring and A_0 is a ring of definition. If $f \in A^{\circ\circ}$ is topologically nilpotent, then the powers of f are in the bounded union $A_0 \cup \{1, \ldots, f^N\}$ for some N. Therefore, when A is a Huber ring we have $A^{\circ\circ} \subseteq A^\circ$. The analogue of Proposition 1.5.3 is the following result.

Proposition 1.5.5. *If A is a Huber ring, then*

$$A^{\circ\circ} = \bigcup_{\substack{I \subseteq A \\ \text{ideal of def.}}} I.$$

Moreover, $A^{\circ\circ}$ is a radical A°-ideal.

Proof. We show topologically nilpotent elements lie in ideals of definition. (The other containment is straightforward.) Suppose $f \in A^{\circ\circ}$. We just explained that $A^{\circ\circ} \subseteq A^\circ$, and so we may choose, by Proposition 1.5.3, a pair of definition (A_0, J) such that $f \in A_0$. Since J is open, there exists n such that $f^n \in J$. Now define $I = J + A_0 f \subseteq A_0$. Then, I is a finitely generated A_0-ideal because J is. We claim

(A_0, I) is a pair of definition. First, I is open since it contains J. Second, $f^n \in J$ and $fJ \subseteq J$, since $f \in A_0$. Therefore, $I^n \subseteq J$. We have shown the I-adic and J-adic topologies coincide, finishing the claim.

As in the case of power-bounded elements, we leave the auxiliary claim that $A^{\circ\circ}$ is a radical A°-ideal as Exercise 1.8. (A hint is provided.) □

Note, the proof makes clear that the notation "$I \subseteq A$ ideal of def." additionally indexes over all rings rings of definition A_0, not just the ideals of definition inside some fixed A_0.

1.6. **Rational localization.** We end our introduction to Huber rings by constructing **rational localizations**. These localizations are to Huber rings and adic spaces what ring-theoretic localizations are to all rings and schemes. That is, they are used to construct affine subspaces of adic spaces. The reader is invited to later meditate on this analogy in the context of the proof of the adic Nullstellensatz (Theorem 3.5.1).

Given a Huber ring A, a rational localization $A(\frac{g_1,\ldots,g_r}{s})$ is another Huber ring depending on elements $g_1, \ldots, g_r, s \in A$. Localizations appear at the start of [Hub94], where adic spaces are defined. See also Section 3.2 and Hübner's lectures [Hüb24]. You cannot localize with respect to all possible choices of elements. The criterion is that the A-ideal $\mathfrak{a} = As + Ag_1 + \cdots + Ag_r \subseteq A$ is *open*. Before we explain, here are examples:

(i) Suppose (A_0, I) is a pair of definition and $f_1, \ldots, f_d \in I$ are A_0-generators. Then, $\{g_1, \ldots, g_r\} = \{f_1, \ldots, f_d\}$ is valid with any s, since $I \subseteq \mathfrak{a}$.
(ii) Let A be a Tate ring. Then, the only open A-ideal is $\mathfrak{a} = A$. Therefore, the condition on $g_1, \ldots, g_r, s$ is that they generate A. See Exercise 1.10.
(iii) Suppose $A = \mathbb{Z}_p[\![w]\!]$ with the $\langle p, w\rangle$-adic topology. Then $\{g, s\} = \{w, p\}$ is a valid choice, but $\{g, s\} = \{p, p\}$ is not. See Exercise 1.11.
(iv) Localization on $\mathbb{C}_p\langle w\rangle$ is related to $\mathbb{C}_p\langle \frac{w}{p}\rangle$ in Section 1.7.

The topological ring $A(\frac{g_1,\ldots,g_r}{s})$ is defined in two steps, first algebraically and second topologically. We fix a pair of definition (A_0, I). This choice is ultimately immaterial, by the universal property in Theorem 1.6.2.

(RL-1) The underlying ring is $A(\frac{g_1,\ldots,g_r}{s}) = A[\frac{1}{s}]$.
(RL-2) Let $A' = A(\frac{g_1,\ldots,g_r}{s})$ and $A'_0 = A_0[\frac{g_1}{s}, \ldots, \frac{g_r}{s}] \subseteq A'$. We make A'_0 a topological ring by giving it the IA'_0-adic topology. We then give A' the unique topology where $A'_0 \subseteq A'$ is open.

By (RL-1) and (RL-2), we have a ring with a topology. But remember now Warning 1.2.5! We must justify that in fact we have defined a topological ring.

Lemma 1.6.1. *Let A be a Huber ring. Assume that $g_1, \ldots, g_r, s \in A$ generate an open A-ideal. Then $A(\frac{g_1,\ldots,g_r}{s})$ is a topological ring.*

The proof of Lemma 1.6.1 uses that I is finitely generated over A_0, which we have not *really* used until now. The only result where we explicitly used the property was Proposition 1.4.1. However, the issue there is preserving the finitely generated property while switching ideals of definition. The same result holds assuming only that A satisfies Definition 1.1.1(a). (We thank Kalyani Kansal for this observation.)

Proof of Lemma 1.6.1. The first step is a general simplification. Then, we write out the argument only in the case that A is a Tate ring. The main reason is to decrease the notations and to generate a proof that is simpler to recall.

For notation, define $A' = A(\frac{g_1,\dots,g_r}{s})$, $A_0' = A_0[\frac{g_1}{s},\dots,\frac{g_r}{s}]$ and $I' = IA_0'$. We give A_0' the I'-adic topology and declare $A_0' \subseteq A'$ open. If I is replaced by a different ideal of definition in A_0, the topology on A' does not change.

The topology on A' is built by declaring a topological ring $A_0' \subseteq A'$ to be open. By Exercise 1.2, we must only show multiplication by f' is continuous on A', for all $f' \in A'$. Since I' is generated by I over A_0', the explicit claim is that for all $f' \in A'$, there exists an n such that $f'I^n \subseteq A_0'$. If $f' = f \in A$, this is clear since A_0 is a topological ring for the I-adic topology. A general element of A' is $f' = f/s^N$ for some N. Therefore, only the case $f' = \frac{1}{s}$ is significant. Thus, we want $\frac{1}{s}I^n \subseteq A_0'$ for some n.

We now assume that A is a Tate ring. Choose a pseudo-uniformizer $\varpi \in A$ that belongs to A_0. By Section 1.3, we know $A = A_0[\frac{1}{\varpi}]$ and we may assume $I = \varpi A_0$. As mentioned in (ii) prior to the lemma, since A is a Tate ring, we are assuming that $g_1,\dots,g_r,s$ generate the *unit* ideal. So, ϖ may be expressed as

$$\varpi = a_0 s + a_1 g_1 + \cdots + a_r g_r \quad (a_i \in A). \tag{1.6.1}$$

Since $A = A_0[\frac{1}{\varpi}]$, there is a positive integer n such that $a_i\varpi^{n-1} \in A_0$ for all i. By (1.6.1), we then have that

$$\varpi^n \in A_0 s + A_0 g_1 + \cdots + A_0 g_r. \tag{1.6.2}$$

And we are done now because

$$\frac{1}{s}I^n = \frac{1}{s}\varpi^n A_0 \subseteq A_0[\frac{g_1}{s},\dots,\frac{g_r}{s}] = A_0'$$

In general, the open-ness of $As + Ag_1 + \cdots + Ag_r$ leads to expressions similar to (1.6.1), with ϖ is replaced by any one of a finite number of A_0-generators of an ideal of definition I. The conclusion, analogous to (1.6.2), is that $A_0 s + A_0 g_1 + \cdots + A_0 g_r$ is open. Fill in the details and finish the argument as Exercise 1.12. ☐

Theorem 1.6.2 (Rational localization)**.** *Let A be a Huber ring and assume that $g_1,\dots,g_r,s \in A$ generate an open ideal of A.*

(a) For (A_0, I) a fixed pair of definition, $A(\frac{g_1,\dots,g_r}{s})$ is a Huber ring.
(b) The natural map $A \to A(\frac{g_1,\dots,g_r}{s})$ is initial among continuous morphisms $A \to B$, for B a Huber ring, for which the image of s is invertible and the image of each g_i/s is power-bounded.

Proof. We proved in Lemma 1.6.1 that $A' = A(\frac{g_1,\dots,g_r}{s})$ is a topological ring. By construction it has a ring of definition $A_0' = A_0[\frac{g_1}{s},\dots,\frac{g_r}{s}]$ with finitely generated ideal of definition IA_0'. This proves that A' is a Huber ring, possibly depending on the choice of (A_0, I). That choice disappears once we prove the universal property (b), since the property itself makes no reference to (A_0, I).

For (b), the localization map $\iota : A \to A' = A[\frac{1}{s}]$ is continuous, since $I \subseteq \iota^{-1}(IA_0')$. Moreover, s is a unit in A' and $\frac{g_j}{s}$ is power-bounded, since it lies in the ring of definition A_0' of A' (recall Proposition 1.5.3). Suppose $\varphi : A \to B$ is given as in (b). Since $\varphi(s)$ is invertible in B, there is a natural factorization

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \iota}\downarrow & \nearrow_{\psi} & \\ A' & & \end{array}$$

at the level of A-algebras. We must check ψ is continuous. Fix an open neighborhood $U \ni 0$ in B, which we assume is an additive subgroup. Since $\frac{\varphi(g_i)}{\varphi(s)}$ is power-bounded in B for all i, there exists an open neighborhood $V \ni 0$ for which

$$\psi(\frac{g_i}{s})^m V = (\frac{\varphi(g_i)}{\varphi(s)})^m V \subseteq U$$

for all $m \geq 0$ and all i. Since φ is continuous, $I^n \subseteq \varphi^{-1}(V)$ for some n. Since $(I')^n = I^n A_0'$ is spanned by elements of the form $f(\frac{g_i}{s})^m \in A'$ for $f \in I^n$ and U is an additive subgroup, it follows that $(I')^n \subseteq \psi^{-1}(U)$. □

We make two complementary remarks. First, the proof clarifies that the universal property is valid as long as B possesses a neighborhood basis of zero consisting of additive subgroups (B is a "non-Archimedean ring"). Second, some authors assume in Theorem 1.6.2 that $g_1, \ldots, g_r$ generate an open ideal. There is practically no difference, since the definitions (RL-1) and (RL-2) make $A(\frac{g_1,\ldots,g_r}{s}) = A(\frac{g_1,\ldots,g_r,s}{s})$.

1.7. **Example: Localizing $\mathbb{C}_p\langle w\rangle$.** The final section of this lecture connects rational localization on $\mathbb{C}_p\langle w\rangle$ to the $\mathbb{C}_p$-affinoid algebra $\mathbb{C}_p\langle \frac{w}{p}\rangle$. The discussion is simplified by focusing first on $\mathbb{C}_p[w]$.

Consider $A = \mathbb{C}_p[w]$ as a topological ring with the topology induced from the Gauss norm. That is, if $f = a_0 + a_1 w + \cdots + a_n w^n$, then

$$\|f\|_{\text{Gauss}} = \max_{i \geq 0} \|a_i\|_p.$$

Just like $\mathbb{C}_p\langle w\rangle$, we see A is a Tate ring with pseudo-uniformizer $p \in A$. A ring of definition is $A_0 = \mathcal{O}_{\mathbb{C}_p}[w]$ and an ideal of definition is $p\mathcal{O}_{\mathbb{C}_p}[w]$.

Now localize with $g = w$ and $s = p$. The hypothesis of Theorem 1.6.2 is satisfied because s is a unit in A. As a ring,

$$A(\frac{w}{p}) = \mathbb{C}_p[w][\frac{1}{p}] = \mathbb{C}_p[w].$$

So, $A = A(\frac{w}{p})$, still. But the topology is new! The original topology has a basis around zero given by $p^n\mathcal{O}_{\mathbb{C}_p}[w]$. The new topology has a basis around zero given by

$$p^n A_0[\frac{w}{p}] = p^n \mathcal{O}_{\mathbb{C}_p}[w, \frac{w}{p}] = p^n \mathcal{O}_{\mathbb{C}_p}[\frac{w}{p}]. \tag{1.7.1}$$

Here is a concrete difference. In the Gauss norm topology, w *is* power bounded but *not* topologically nilpotent since $w^n \notin p\mathcal{O}_{\mathbb{C}_p}[w]$ for any n. Yet, in $A(\frac{w}{p})$ we have

$$w^n = p^n(\frac{w}{p})^n \in p^n A_0[\frac{w}{p}]$$

Therefore, w *is* topologically nilpotent in $A(\frac{w}{p})$.

We chose A to be the polynomial ring so that (1.7.1) was most clear. The connection to affinoid algebras is via completion. The main point is that $\mathbb{C}_p\langle w\rangle$ is the completion of $\mathbb{C}_p[w]$ for the Gauss norm. As Exercise 1.13, the reader can check that the topology on $A(\frac{w}{p}) = \mathbb{C}_p[w]$ is the *same* as the topology induced by the norm

$$|f|_{\frac{1}{p}} := \max_{i \geq 0} \|a_i\|_p p^{-i},$$

and $\mathbb{C}_p\langle \frac{w}{p} \rangle$ is the completion of $\mathbb{C}_p[w]$ for $|-|_{\frac{1}{p}}$. (The completions here are all with respect to norms. We revisit more general completions of Huber rings in Section 3.4.)

Section 1 Exercises.

Exercise 1.1. Prove that the rings with topology described in Examples 1.2.1-1.2.4 are all topological rings.

Exercise 1.2. Let A be a ring and $A_0 \subseteq A$ a subring and I an A_0-ideal. Consider A_0 as a topological ring with the I-adic topology as in Example 1.2.1. Define a topology on A by declaring $A_0 \subseteq A$ to be open. Show that the following are equivalent:

(i) A is a topological ring.
(ii) For all $f \in A$, multiplication by f is continuous on A.
(iii) For all $f \in A$, there exists an $n \gg 0$ such that $fI^n \subseteq A_0$.

Exercise 1.3. These examples have the same flavor (with similar solutions). In each, we give a ring A containing a topological ring A_0. The exercise is to check that declaring $A_0 \subseteq A$ to be open will *not* make A into a topological ring.

(a) Let $A_0 = \mathcal{O}_{\mathbb{C}_p}$ with the $\mathfrak{m}$-adic topology. Show that declaring $\mathcal{O}_{\mathbb{C}_p} \subseteq \mathbb{C}_p$ to be open will not make $\mathbb{C}_p$ a topological field.
(b) Consider $\mathbb{Z}_p[\![w]\!]$ with the $\langle p, w\rangle$-adic topology. Show that if one declares $\mathbb{Z}_p[\![w]\!] \subseteq \mathbb{Z}_p[\![w]\!][\frac{1}{p}]$ is open, then $\mathbb{Z}_p[\![w]\!][\frac{1}{p}]$ will not be a topological ring.

Exercise 1.4. Let $A = \mathbb{C}_p\langle w\rangle$.

(a) Show that w is power-bounded but not topologically nilpotent.
(b) Show that, in fact, $A^\circ = \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$, and

$$A^{\circ\circ} = \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}\langle w\rangle = \left\{ \sum_{i=0}^{\infty} a_i w^i \in A \mid a_i \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \text{ for all } i \right\}.$$

Exercise 1.5. Let A be a topological ring and X and Y subsets of A.

(a) Show that if $X \subseteq Y$ and Y is bounded, then X is bounded.
(b) Show that if A is a Huber ring and X and Y are bounded, then so is $X \cdot Y$.
(c) Suppose A is a Huber ring. Show that $X \subseteq A$ is bounded if and only if for any ideal of definition I, there exists an $n \geq 1$ such that $XI^n \subseteq I$.

Exercise 1.6. Show that if A is a Huber ring and A_0 and A_0' are rings of definition, then there exists a ring of definition A_0'' containing both.

Exercise 1.7. Show that if A is a Huber ring, then A° is integrally closed in A.

<u>**Hint**</u>. A sketch is given at the end of the proof of Proposition 1.5.3.

Exercise 1.8. Let A be a Huber ring.

(a) Show that $f \in A$ is topologically nilpotent if and only if there exists an ideal of definition $I \subseteq A$ and an integer $n \geq 1$ such that $f^n \in I$.
(b) Show that $A^{\circ\circ} \subseteq A^\circ$ is a radical A°-ideal.

Hint. In (a), there exists n such that $f^n \in I$ and m such that $f^j I^m \subseteq I$ for all $j \leq n-1$. Then $f^N \in I$ for $N \geq n(m+1)$. For (b), the tricky part of the ideal property is showing $A^{\circ\circ}$ is closed for addition. For that, it may be helpful to note that if f, g are topologically nilpotent, then they lie in a common ring of definition A_0 (Exercise 1.6).

Exercise 1.9. Let $A = \mathbb{Q}_p[\varepsilon] = \mathbb{Q}_p \oplus \mathbb{Q}_p\varepsilon$ with $\varepsilon^2 = 0$ and the p-adic topology induced on each factor. Show that A is a Huber ring and

$$A^\circ = \mathbb{Z}_p \oplus \mathbb{Q}_p\varepsilon.$$

Conclude that A° need not be a ring of definition.

Exercise 1.10. Let A be a Tate ring.

(a) Show that any ideal of definition for A contains a pseudo-uniformizer.
(b) Show that if $\mathfrak{a}$ is an *open* ideal of A, then $\mathfrak{a} = A$.

Exercise 1.11. Let $A = \mathbb{Z}_p[\![w]\!]$ with the $\mathfrak{m}$-adic topology where $\mathfrak{m} = \langle w, p\rangle$. This is a Huber ring as in Example 1.2.1.

(a) Show that $g_1 = p$ and $s = p$ is invalid for the hypotheses in Theorem 1.6.2.
(b) Try to define $A(\frac{p}{p})$ as in (RL-1) and (RL-2). Confirm that you *do not* get a topological ring.
(c) Re-affirm directly that your objection disappears for the ring $A(\frac{w}{p})$.

Exercise 1.12. Let A be a Huber ring. Suppose $g_1, \ldots, g_r, s \in A$ generate an open ideal. Show that $A(\frac{g_1,\ldots,g_r}{s})$ is a topological ring, as promised in Lemma 1.6.1.

Hint. A hint is given at the end of the proof of Lemma 1.6.1 in the text.

Exercise 1.13. Show that the topology on $\mathbb{Q}_p[w](\frac{w}{p})$ in Section 1.7 is the topology induced on $\mathbb{Q}_p[w]$ given by the norm

$$|a_0 + a_1 w + \cdots + a_d w^d| = \max_i \|a_i\|_p p^{-i}.$$

2. Valuation theory

The second lecture introduces valuation theory. The primary goal is discussing continuous valuations on topological rings. One highlight is a simple criterion (Proposition 2.7.1) for a valuation on a Huber ring to be continuous. The final discussion will focus on the continuous valuation spectrum for the Tate algebra $\mathbb{C}_p\langle w\rangle$.

2.1. **Definition.** The symbol Γ refers to a **totally ordered abelian group**. We write the group operation multiplicatively. By definition, Γ is an abelian group with an order relation $\leq$ satisfying the following axioms.

(a) For all $\gamma_1, \gamma_2 \in \Gamma$, either $\gamma_1 \leq \gamma_2$ or $\gamma_2 \leq \gamma_1$, and both occur if and only if $\gamma_1 = \gamma_2$ ("totally" ordered).
(b) If $\gamma \in \Gamma$, then $\gamma' \mapsto \gamma\gamma'$ preserves $\leq$.

From the first axiom, it makes sense to write $\gamma_1 < \gamma_2$ provided $\gamma_1 \leq \gamma_2$ and $\gamma_1 \neq \gamma_2$. We always extend Γ to the totally ordered monoid $\Gamma \cup \{0\}$. The multiplicative structure is given by $0 \cdot \gamma = 0 = \gamma \cdot 0$, for all $\gamma \in \Gamma$. The order extends by $0 < \gamma$ for all $\gamma \in \Gamma$.

Definition 2.1.1. *Let A be a ring. A **valuation** on A is a function $|-| : A \to \Gamma \cup \{0\}$ such that $|0| = 0$ and $|1| = 1$, and for all $f, g \in A$ we have*

$$|fg| = |f||g| \qquad (|-| \text{ is multiplicative});$$
$$|f+g| \leq \max\{|f|, |g|\} \qquad (\text{the ultrametric triangle inequality}).$$

Note, we are using valuation as a term generalizing an ultrametric norm, as opposed to something like the p-adic valuation $v_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$. This is a common choice of language in this research area. In If we were to restrict to valuations valued in $\mathbb{R}_{>0}$, we might prefer the terminology semi-norm. ("Semi-" because we allow for $|f| = 0$ even if $f \neq 0$.) The terminology "valuation" seems to be preferred because Γ may in fact not embed into $\mathbb{R}_{>0}$. Finally, confusion may arise since we often consider a ring A topologized by a norm. To avoid confusion, we reserve $|-|$ for a valuation on A, while $\|-\|$ will denote a fixed norm on A.

2.2. Examples.

Example 2.2.1. Let A be an integral domain. The **trivial valuation** $|-|_{\mathrm{triv}}$ is

$$|a|_{\mathrm{triv}} = \begin{cases} 1 & \text{if } a \neq 0; \\ 0 & \text{if } a = 0. \end{cases}$$

Therefore, if A is any ring and $\mathfrak{p} \subseteq A$ is a prime ideal, we have the trivial valuation $A \twoheadrightarrow A/\mathfrak{p} \xrightarrow{|-|_{\mathrm{triv}}} \{0,1\}$ modulo $\mathfrak{p}$.

Example 2.2.2. Let $A = \mathbb{Q}$. Some valuations on A are given by:

- The trivial valuation $|-| = |-|_{\mathrm{triv}}$.
- The valuation $|-| = \|-\|_\infty$ given by the Archimedean norm $\mathbb{Q} \xrightarrow{\|-\|_\infty} \mathbb{R}_{\geq 0}$.
- The valuation $|-| = \|-\|_\ell$ given by an ℓ-adic norm $\mathbb{Q} \xrightarrow{\|-\|_\ell} \mathbb{R}_{\geq 0}$ for a prime ℓ.

Ostrowski's theorem states these are the only valuations on $\mathbb{Q}$, up to equivalence. (See Section 2.5 below for "equivalence").

Example 2.2.3. Let $A = \mathbb{Q}_p$. We already know about $|-|_{\mathrm{triv}}$ and $\|-\|_p$. But, there are more. A theorem sometimes attributed to Chevalley states that if L/K is a field extension, then a valuation on K extends to a valuation on L. See [Bou98, Chapter VI, §3, no. 3, Proposition 5], for instance. So, each valuation on $\mathbb{Q}$ extends to a valuation on $\mathbb{Q}_p$ (in many ways). Note, this algebraic phenomenon requires enlarging the target group Γ. On $\mathbb{Q}_p$, the valuations $|-| = |-|_{\mathrm{triv}}$ and $|-| = \|-\|_p$ are distinguished (up to equivalence) as the ones that take value in a cyclic group. The p-adic norm is distinguished as the only one that is continuous for the p-adic topology on $\mathbb{Q}_p$. (Compare with Exercise 2.6, after reading a bit further.)

Example 2.2.4. Let $A = \mathbb{C}_p\langle w \rangle$ be the one-variable Tate algebra over $\mathbb{C}_p$. For $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, we have a valuation

$$f \overset{|-|_\alpha}{\longmapsto} \|f(\alpha)\|_p$$

on A. These valuations were considered in our motivation (page 3). There is also the valuation $|-|_1 := \|-\|_{\mathrm{Gauss}}$ that defines the topology on A. It is given by

$$f = a_0 + a_1 w + a_2 w^2 + \cdots \rightsquigarrow |f|_1 = \max_{i \geq 0} \|a_i\|_p.$$

We switch notation to $|-|_1$, so that we can easily change "1" to another real number r such that $0 < r \leq 1$. That is, we define

$$|f|_r = \max_{i \geq 0} \|a_i\|_p r^i.$$

Each $|-|_\alpha$ is semi-norm, while each $|-|_r$ is a norm. A more exotic example is given in Section 2.4 below.

2.3. **Continuous valuations.** In the examples above, $\mathbb{Q}_p$ and $\mathbb{C}_p\langle w \rangle$ are topological rings. Huber rings are also topological. So, let us explain continuity for valuations. Assume $A \xrightarrow{|-|} \Gamma \cup \{0\}$ is a valuation. The **value group** $\Gamma_{|-|}$ is the subgroup of Γ generated by the *non-zero* $|f|$ for $f \in A$. If $A = K$ is a field, then $\Gamma_{|-|} = |K^\times|$. In general, it is the *smallest* subgroup of Γ through which $|-|$ factors.

Now suppose A is a topological ring. We say that $|-|$ is a **continuous valuation** on A if for all $\gamma \in \Gamma_{|-|}$, the subset

$$U_\gamma = \{f \in A \mid |f| < \gamma\} \subseteq A \tag{2.3.1}$$

is open in A. (The U_γ are open *subgroups* even.) In the examples above:

(i) Let A be a topological ring and $\mathfrak{p}$ a prime ideal. The trivial valuation modulo $\mathfrak{p}$ is continuous if and only if $\mathfrak{p} \subseteq A$ is open.
(ii) The p-adic norm is continuous on $\mathbb{Q}_p$. The trivial norm is not.
(iii) The valuations $|-|_\alpha$ and $|-|_r$ on $\mathbb{C}_p\langle w \rangle$ are continuous.

What might a reasonable discontinuous valuation look like? Consider $\mathbb{C}_p[w] \subseteq \mathbb{C}_p\langle w \rangle$ with the topology induced from the Gauss norm. Then,

$$|a_0 + a_1 w + \cdots + a_n w^n|_r = \max_{i=0,\ldots,n} \|a_i\|_p r^i$$

is a valuation on $\mathbb{C}_p[w]$ for all $0 \leq r < \infty$. However, it is continuous if and only if $r \leq 1$. See Exercise 2.1.

2.4. **An exotic example.** As promised in Example 2.2.4, we give an extended example of a valuation on $\mathbb{C}_p\langle w \rangle$ that is not a (semi-)norm. Define $\Gamma = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ with the "read left to right" ordering. The technical term is **lexicographic**. In symbols,

$$a < c \implies (a,b) < (c,d) \qquad (\text{any } b, d),$$

and

$$e < b \implies (a,e) < (a,b).$$

The right-hand portion of Figure 2.4.1 illustrates the order relation on Γ.

Now pick a real number $0 < \varepsilon < 1$. We define $|-|_{1^-}$ on $\mathbb{C}_p\langle w \rangle$ by

$$|a_0 + a_1 w + a_2 w^2 + \cdots|_{1^-} = \max_{i \geq 0} (\|a_i\|_p, \varepsilon^i) \in \Gamma.$$

What does $|-|_{1^-}$ measure? Write $|f|_{1^-} = (a,b)$. Then, $a \geq \|a_i\|_p$ for all i by definition of the lexicographic order. Since $a = \|a_i\|_p$ for some i as well, we see

$$a = \max_{i \geq 0} \|a_i\|_p = |f|_1. \tag{2.4.1}$$

So, the first coordinate of $|f|_{1^-}$ is the Gauss norm of f. What about the second? Suppose that $i_1 < i_2 < \cdots < i_n$ are the indices where $\|a_{i_j}\|_p = |f|_1$. Then, in Γ, we have

$$(\|a_{i_n}\|_p, \varepsilon^{i_n}) < \cdots < (\|a_{i_2}\|_p, \varepsilon^{i_2}) < (\|a_{i_1}\|_p, \varepsilon^{i_1}) = |f|_{1^-}.$$

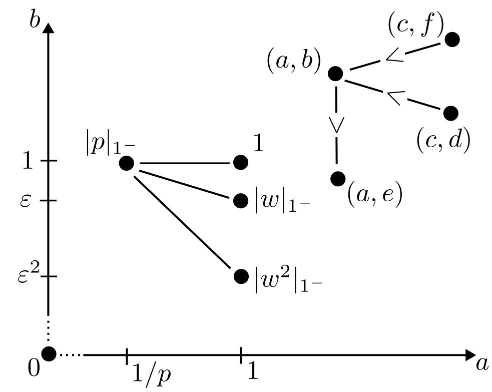


FIG. 2.4.1. A visualization of $\Gamma \cup \{0\}$ for $\Gamma = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. The right-hand portion of the figure illustrates the generic order relation. The left-hand portion illustrates the position of p versus w under the valuation $|-|_{1^-}$.

Thus, $|f|_{1^-}$ records, in the second coordinate, the index of least degree whose coefficient realizes the Gauss norm. We propose Exercise 2.2 to interpret $|-|_{1^-}$ as combining the Gauss norm with the order of vanishing of polynomials modulo p.

The calculation shows $|-|_{1^-}$ is continuous on $\mathbb{C}_p\langle w\rangle$. Indeed, if $a, b > 0$ are real numbers, we have shown that within $\mathbb{C}_p\langle w\rangle$ we have

$$U_a^{|-|_1} \subseteq U_{(a,b)}^{|-|_{1^-}}.$$

(The superscripts indicate the valuation to which the "U_γ"-notation is being applied.) Each $U_a^{|-|_1}$ is open, since $|-|_1$ defines the topology on $\mathbb{C}_p\langle w\rangle$. Thus each $U_{(a,b)}^{|-|_{1^-}}$ is open as well.

Finally, $|-|_{1^-}$ is written with a "1^-" to promote intuition that the coordinate w is measured infinitesimally below 1. Indeed, $|p|_{1^-} = (1/p, 1)$ and $|w|_{1^-} = (1, \varepsilon)$. Therefore,

$$|p|_{1^-} < |w|_{1^-}^n < 1 \qquad \text{(for all } n\text{)}. \tag{2.4.2}$$

It is as if w as squeezed between $1/\sqrt[n]{p}$ and 1 for all n. See Figure 2.4.1 again. We will give another sense in which $|-|_{1^-}$ is near to 1 in Section 2.8.

2.5. **Valuation spectra.** Valuation spectra are defined as equivalence classes of valuations. Let A be any ring with valuations $|-|_1$ and $|-|_2$. We say that $|-|_1$ is **equivalent** to $|-|_2$ if for all $f, g \in A$ we have

$$|f|_1 \leq |g|_1 \iff |f|_2 \leq |g|_2.$$

Note that $|-|_1$ and $|-|_2$ may take values in different abstract groups. When they are equivalent, there is an *isomorphism* of value groups turning one valuation into the other. This and another interpretation of equivalence are suggested as Exercise 2.5.

One impact of equivalence for valuation is that it turns seemingly true statements into actually true ones. For instance, "there is only one continuous valuation on $\mathbb{Q}_p$" is only true if it is understood up to equivalence. (See Exercise 2.6.) After all,

$\| - \|_p^2$ and $\| - \|_p$ are distinct continuous valuations. Another example of equivalence is that $| - |_{1^-}$ depends on the choice of the parameter ε only up to equivalence.

Now assume A is a topological ring. Its **continuous valuation spectrum** is

$$\mathrm{Cont}(A) = \{\text{continuous valuations on } A\}/(\text{valuation equivalence}).$$

We suggest as Exercise 2.7 showing that whether or not a valuation $|-|$ is continuous depends only on $| - |$ up to equivalence. Therefore, there is a natural inclusion $\mathrm{Cont}(A) \subseteq \mathrm{Cont}(A_{\mathrm{disc}})$, where A_{disc} is A with the discrete topology. The larger space

$$\mathrm{Spv}(A) := \mathrm{Cont}(A_{\mathrm{disc}})$$

is called the **valuation spectrum**. No continuity qualification is imposed on $\mathrm{Spv}(A)$. For a non-zero ring, the valuation spectrum is always non-empty, since one always has the trivial valuation modulo a prime ideal. The non-emptiness of $\mathrm{Cont}(A)$ is more subtle. See Section 4.8 for a related discussion. In the remainder of this subsection and Section 2.6, we make formal constructions on $\mathrm{Spv}(A)$. The continuous valuations return in Section 2.7.

We will use x to denote an element of $\mathrm{Spv}(A)$. Let $| - |$ be a choice of representative for the class x. If $f \in A$, we define notation

$$|f(x)| := |f|. \tag{2.5.1}$$

Since the target group of $| - |$ is not well-defined, (2.5.1) has no clear meaning. One route, mentioned above, is solving Exercise 2.5 to see $|f|$ is well-defined up to ordered group isomorphism on value groups. The route we will take is to play more loosely and agree to only use the notation (2.5.1) in situations where only the class x, and not the choice of $| - |$, matters.

For instance, if $x \in \mathrm{Spv}(A)$, then its **support** is defined to be

$$\mathrm{supp}(x) = \{f \in A \mid |f(x)| = 0\}.$$

The support depends only on the equivalence class x because "$|f| = 0$" is the same as "$|f| \leq |0|$". It is a prime ideal, and if A is topological and $x \in \mathrm{Cont}(A)$, then it is closed. Prove these facts as Exercise 2.11.

The notation (2.5.1) is also used in equipping $\mathrm{Spv}(A)$ with a topology. For $g, s \in A$ we define

$$U(\frac{g}{s}) = \{x \in \mathrm{Spv}(A) \mid |g(x)| \leq |s(x)| \neq 0\}. \tag{2.5.2}$$

Weinstein observes that this "blends features of the Zariski topology on schemes with the topology on rigid spaces" ([BCKW19, p. 6]). Indeed, $x \in U(\frac{g}{s})$ implies both that $s \notin \mathfrak{p} = \mathrm{supp}(x)$ (a Zariski condition) and $|g(x)| \leq |s(x)|$ (a rigid condition). A basic open set for the topology on $\mathrm{Spv}(A)$ is, by definition, a finite intersection

$$\begin{aligned} U(\frac{g_1, \dots, g_r}{s}) &:= \bigcap_{i=1}^{r} U(\frac{g_i}{s}) \\ &= \{x \in \mathrm{Spv}(A) \mid |g_i(x)| \leq |s(x)| \neq 0 \text{ for all } i\}. \end{aligned}$$

If A is a topological ring, we equip $\mathrm{Cont}(A) \subseteq \mathrm{Spv}(A)$ with the induced topology by these basic opens.

We have two warnings before going further. First, be careful about cancellation. The set $U(\frac{s}{s})$ *has* a condition:

$$U(\frac{s}{s}) = \{x \in \mathrm{Spv}(A) \mid |s(x)| \neq 0\}. \tag{2.5.3}$$

Second, we can form $U(\frac{g_1,\ldots,g_r}{s})$ for any choice of $g_1,\ldots,g_r,s$. If A is a topological ring then a **rational subset** is one of the form

$$U(\frac{g_1,\ldots,g_r}{s})$$

where $g_1,\ldots,g_r,s \in A$ are chosen to generate an open A-ideal. This is the same condition required for rational localization of Huber rings. We return to this in Section 3.2.

2.6. **The support map.** The function $x \mapsto \mathrm{supp}(x)$ defines a *continuous* function

$$\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A).$$

To see this, note that basic open sets in $\mathrm{Spec}(A)$ take the form

$$D(s) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid s \notin \mathfrak{p}\}$$

for $s \in A$. Thus, $\mathrm{supp}^{-1}(D(s)) = U(\frac{s}{s})$.

We next describe the fibers of the support map. Let $|-|$ be a valuation on A and $\mathfrak{p}$ its support. If $f \in A$ and $f \notin \mathfrak{p}$, then the strong triangle inequality implies $|f+g| = |f|$ for all $g \in \mathfrak{p}$. The same holds if $f \in \mathfrak{p}$, since $\mathfrak{p}$ is an additive subgroup. This shows $|f|$ depends only on $f \bmod \mathfrak{p} \in A/\mathfrak{p}$ and $|-|$ factors through $A/\mathfrak{p}$, on which it defines a valuation with support the zero ideal. In particular, $|-|$ extends to a valuation on the fraction field $\mathrm{Frac}(A/\mathfrak{p})$. In summary, there is always a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{|-|} & \Gamma \cup \{0\} \\ \downarrow & \overset{|-|}{\nearrow} & \uparrow {\scriptstyle |-|} \\ A/\mathfrak{p} & \hookrightarrow & \mathrm{Frac}(A/\mathfrak{p}). \end{array} \tag{2.6.1}$$

The factorization only depends on $|-|$ up to equivalence, in the sense that (2.6.1) induces a well-defined map

$$\mathrm{Spv}(A) \supseteq \mathrm{supp}^{-1}(\mathfrak{p}) \to \mathrm{Spv}(\mathrm{Frac}(A/\mathfrak{p})) \tag{2.6.2}$$

for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$. In fact, (2.6.2) gives a homeomorphism

$$\mathrm{supp}^{-1}(\mathfrak{p}) \xrightarrow{\simeq} \mathrm{Spv}(\mathrm{Frac}(A/\mathfrak{p})).$$

The support map therefore allows us to treat $\mathrm{Spv}(A)$ as families of valuations *over residue fields* of $\mathrm{Spec}(A)$.

The target of (2.6.2) is the valuation spectrum of a field. This is a space classically understood through algebra. Let K be any field. For $x \in \mathrm{Spv}(K)$, the subring

$$A_x = \{\alpha \in K \mid |\alpha(x)| \leq 1\} \subseteq K$$

is called the **valuation ring** of x. Each A_x is a valuation ring in the sense of commutative algebra (see [Bou98, Chapter VI, §1, no. 2]). That is, if $\alpha \in K$, then either $\alpha \in A_x$ or $\alpha^{-1} \in A_x$. Any valuation ring is a local ring. In this case, one can directly check the set of non-units in A_x is equal to those $\alpha \in A_x$ such that

$|\alpha(x)| < 1$, and that this set forms an A_x-ideal, written $\mathfrak{m}_x$. According to Exercises 2.12-2.14, the association

$$\text{Spv}(K) \xrightarrow{x \mapsto A_x} \{\text{valuation subrings } A \subseteq K\} \tag{2.6.3}$$

is a bijection. We will use this bijection in our analysis of the closed unit disc in Section 4.

Finally, the inclusion $\text{Cont}(A) \subseteq \text{Spv}(A)$ endows $\text{Cont}(A)$ with a topology as well. The support map is, by definition, continuous when restricted to $\text{Cont}(A)$. However, the above analysis is purely algebraic. One of Huber's preliminary results on continuous valuation spectra ([Hub93, Theorem 3.1]) is an analysis of the support fibers in $\text{Cont}(A)$ when A is a Huber ring. We will not study that result. Instead, in the next section we will directly analyze continuity of valuations on Huber rings, rather than analyzing the space $\text{Cont}(A)$ itself.

The reader may want practice manipulating $\text{Cont}(-)$. Exercise 2.8 is recommended, as are Exercises 2.9 and 2.10, where the important concept of an **adic morphism** of Huber rings is explained.

2.7. **Continuous valuations on Huber rings.** We now focus on Huber rings. The primary goal is explaining which valuations on Huber rings are continuous.

Let Γ be a totally ordered abelian group. We say $\gamma \in \Gamma \cup \{0\}$ is **co-final** in Γ if, for all $\delta \in \Gamma$ we have $\gamma^n < \delta$ for $n \gg 0$. This is similar to topological nilpotence. Note that $\gamma = 0$ is automatically co-final. It is also true that every co-final γ must be less than 1. Indeed, if $1 \leq \gamma$ then $1 \leq \gamma^n$ for all $n \geq 0$. If $\Gamma \subseteq \mathbb{R}_{>0}$, then the co-final γ are indeed just the γ such that $\gamma < 1$.

Now suppose that $|-| : A \to \Gamma \cup \{0\}$ is a valuation. Recall the value group $\Gamma_{|-|}$ is the smallest subgroup of Γ containing the non-zero $|f|$ for $f \in A$. For valuations on Huber rings, we have the following continuity criterion.

Proposition 2.7.1 (Continuity criterion)**.** *Let A be a Huber ring. Let*

$$|-| : A \to \Gamma \cup \{0\}$$

be a valuation and $\Gamma_{|-|}$ be its value group. The following conditions are equivalent:

(i) $|-|$ is continuous.

(ii) If f is topologically nilpotent, then $|f|$ is co-final in $\Gamma_{|-|}$.

(iii) Suppose (A_0, I) is a pair of definition and write $I = A_0 f_1 + \cdots + A_0 f_d$. Then, for each i we have $|f_i|$ is co-final in $\Gamma_{|-|}$ and $|f f_i| < 1$ for all $f \in A_0$.

In particular, with notation as in (iii), suppose $|-|$ is a valuation, each $|f_i|$ is co-final in $\Gamma_{|-|}$, and some $|f_i| \neq 0$. Then,

$$|-| \textit{ is continuous} \iff |f| < \frac{1}{\max(|f_1|, \ldots, |f_d|)} \quad \textit{for all } f \in A_0.$$

Proof. First suppose $|-|$ is continuous. If $\gamma \in \Gamma_{|-|}$, then the set

$$U_\gamma = \{f \in A \mid |f| < \gamma\}$$

is open in A. So, if $f \in A$ is topologically nilpotent, then $f^n \in U_\gamma$ for all $n \gg 0$. In symbols, $|f|^n < \gamma$ for all $n \gg 0$. So, $|f|$ is co-final in $\Gamma_{|-|}$. This shows (i) implies (ii).

Now assume that (ii) holds. Fix the notation (A_0, I) as in (iii). The elements of I are among the topologically nilpotent elements in A. Therefore, (ii) implies each

$|f_i|$ is co-final in $\Gamma_{|-|}$. In fact, each $g \in I$ has co-final value $|g|$, which implies the weaker conclusion that if $g \in I$ then $|g| < 1$. Therefore, if $f \in A_0$ and $i = 1, \ldots, d$, then $|ff_i| < 1$ because $g = ff_i \in I$. We have shown (ii) implies (iii).

Finally, we prove that (iii) implies (i) by a direct argument. Suppose that $\gamma \in \Gamma_{|-|}$. By assumption in (iii), $|f_i|$ is co-final in $\Gamma_{|-|}$ for $i = 1, \ldots, d$. There are only d-many f_i, so there exists $n \geq 0$ such that $|f_i|^n < \gamma$ for all i at once. We now claim

$$I^{nd+1} \subseteq U_\gamma. \tag{2.7.1}$$

If proven, then U_γ is open in A. Since γ was arbitrary, we have proven (iii) implies (i).

We now show (2.7.1). The A_0-ideal I^{nd+1} is generated as an abelian group by elements of the form

$$g = ff_1^{m_1} \cdots f_d^{m_d} \qquad f \in A_0, \;\; m_1 + \cdots + m_d = nd + 1. \tag{2.7.2}$$

Since U_γ is an additive subgroup of A, it is enough to show $g \in U_\gamma$ for such g. Now, in (2.7.2) we have d-many m's and they sum to $nd + 1$. So, $m_i \geq n + 1$ for some i. Since $f \in A_0$ we have $|ff_i| < 1$ by (iii). Since $|f_j| < 1$ for all j, in any case, we can write $g = f' f_i^n$ where $|f'| < 1$. Therefore, $|g| < \gamma$, completing the proof that (2.7.1) holds. □

We have two reasons for explaining Proposition 2.7.1. The primary reason is that we will use the criterion to check valuations are continuous in Sections 3-4. The secondary reason is that the criterion is only implicitly presented in Huber's paper [Hub93]. It is more explicit in the notes by Conrad [Con14], Morel [Mor19], and Wedhorn [Wed19]. See especially [Con14, Corollary 9.3.3]. However, in those sources, the criterion is established alongside arguments showing $\mathrm{Cont}(A)$ is a spectral space if A is a Huber ring. In particular, the other references all appeal to the theorem [Hub93, Theorem 3.1] referenced at the end of the prior section. While learning this material, we have found it helpful to have a direct argument toward Proposition 2.7.1, which allows for a nearly instant check on whether a valuation is continuous.

2.8. **Example:** $\mathrm{Cont}(\mathbb{C}_p\langle w\rangle)$**.** In the final section of our second lecture, we revisit continuous valuations on $\mathbb{C}_p\langle w\rangle$. We explain an instance of specialization in $\mathrm{Cont}(\mathbb{C}_p\langle w\rangle)$, illustrate the support map, and introduce Huber's model for the closed unit disc.

Let $A = \mathbb{C}_p\langle w\rangle$, which is a topological ring with the topology endowed from the Gauss norm $|-|_1 = \|-\|_{\mathrm{Gauss}}$. The valuations on A listed in Example 2.2.4 are continuous. We re-list them.

(i) If $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, then $|f(x_\alpha)| = \|f(\alpha)\|_p$ defines a point $x_\alpha \in \mathrm{Cont}(A)$.
(ii) If $r \leq 1$, the r-Gauss norm $|f(x_r)| = |f|_r$ defines a point $x_r \in \mathrm{Cont}(A)$.
(iii) We also have $x_{1^-} \in \mathrm{Cont}(A)$ given by $|f(x_{1^-})| = |f|_{1^-}$, where on non-zero f we have

$$|a_0 + a_1 w + a_2 w^2 + \cdots|_{1^-} = \max_{i\geq 0}(\|a_i\|_p, \varepsilon^i) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0},$$

for some choice of $0 < \varepsilon < 1$. See Section 2.4.

We previously argued in (2.4.1) in Section 2.4 that $|-|_1$ and $|-|_{1^-}$ are related by a commuting diagram

(2.8.1)
$$\begin{array}{ccc} & A & \\ {\scriptstyle |-|_{1^-}} \swarrow & & \searrow {\scriptstyle |-|_1} \\ \mathbb{R}_{>0} \times \mathbb{R}_{>0} \cup \{0\} & \xrightarrow[(a,b)\mapsto a]{} & \mathbb{R}_{>0} \cup \{0\}. \end{array}$$

Here, the horizontal projector $(a, b) \mapsto a$ is a morphism of totally ordered abelian groups. So, along with continuity of $|-|_{1^-}$, we deduce that if $g, s \in A$, then

$$|g|_{1^-} \leq |s|_{1^-} \neq 0 \implies |g|_1 \leq |s|_1 \neq 0.$$

In terms of the topology on $\mathrm{Cont}(A)$, we have

$$|-|_{1^-} \in U(\frac{g}{s}) \implies |-|_1 \in U(\frac{g}{s}).$$

Therefore, each open in $\mathrm{Cont}(A)$ that contains x_{1^-} also contains x_1. Said another way, x_{1^-} lies in the closure of x_1 within $\mathrm{Cont}(A)$. This gives a new sense to the intuition that x_{1^-} is infinitesimally close to x_1.

Next, we analyze supports. It is clear that $\mathrm{supp}(x_\alpha) = \langle w - \alpha \rangle$. As Exercise 2.15, the reader can check that $x = x_\alpha$ is the *only* point of $\mathrm{Cont}(A)$ with this property. Since $\langle w - \alpha \rangle \in \mathrm{Spec}(A)$ is a closed point, so is $x_\alpha = \mathrm{supp}^{-1}(\langle w - \alpha \rangle) \in \mathrm{Cont}(A)$. The closedness is one way the x_α are distinguished from the Gauss point.

The remaining points of $\mathrm{Cont}(A)$ have generic support $\{0\}$. We analyze these points more closely in Section 4. The point x_{1^-} is closed, while x_r is non-closed whenever $r = \|\alpha\|_p$ lies in the value group of $\mathbb{C}_p$. Since x_{1^-} and x_1 have the same support *and* x_{1^-} lies in the closure of x_1, we call x_{1^-} a **vertical specialization** of x_1. See Exercise 2.18 for details on this terminology. A cartoon is drawn in Figure 2.8.1.

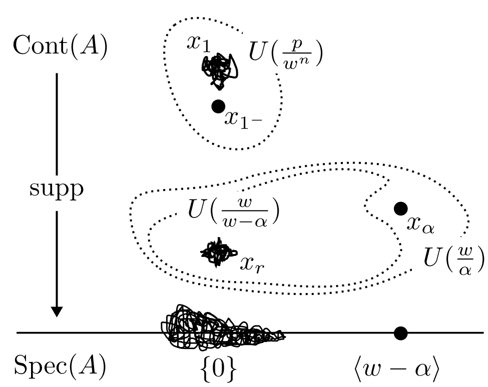


FIG. 2.8.1. A visualization of the support map for $A = \mathbb{C}_p\langle w \rangle$. The element $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ is meant to have $\|\alpha\|_p = r$, and n is chosen so large that $\frac{1}{p} > r^n$.

We also examine the continuity criterion. We may choose $\mathcal{O}_{\mathbb{C}_p}\langle w \rangle$ as a ring of definition and p as a (principal) generator of an ideal of definition. Let $|-|$ be a valuation on A. Then, $|p| \neq 0$ since p is a unit. The continuity criterion Proposition 2.7.1 implies that $|-|$ is continuous if and only if $|p|$ is co-final in $\Gamma_{|-|}$ and

$$|f| < |p|^{-1} \tag{2.8.2}$$

for all $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. This gives a second argument for the continuity of $|-|_{1^-}$. Indeed, comparing with Figure 2.4.1, we see the following.

(a) The element $|p|_{1^-} = (\frac{1}{p}, 1)$ is co-final in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$.
(b) If $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$, then $|f|_{1^-} \leq |f|_1 \leq 1 < |p^{-1}|_{1^-}$.

Note, our continuity check here shows that $|f|_{1^-} \leq 1$ for $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. This is strictly stronger than the requirement (2.8.2). Indeed, it suggests a modification of x_{1^-} that makes w infinitesimally larger than 1 rather than smaller. We achieve this by defining

$$|a_0 + a_1 w + a_2 w^2 + \cdots|_{1^+} = \max_{i \geq 0}(\|a_i\|_p, \varepsilon^{-i}).$$

(Still $0 < \varepsilon < 1$.) The value $|f|_{1^+}$ measures the Gauss norm in the first coordinate and the *largest* index coefficient realizing the Gauss norm in the second coordinate. It is still a valuation and still continuous because

$$1 < |w^n|_{1^+} = (1, \varepsilon^{-n}) < (p, 1) = |p^{-1}|_{1^+}. \tag{2.8.3}$$

We therefore have a new point $x_{1^+} \in \mathrm{Cont}(A)$. It lies in the closure of x_1 as before.

Finally, Huber's model for the **closed unit disc** is given by

$$D = \{x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle) \mid |w(x)| \leq 1\}.$$

From the definitions, we see D contains x_α for each $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, it contains x_r for all $0 < r \leq 1$, and it contains x_{1^-}. There is a strict containment $D \subsetneq \mathrm{Cont}(\mathbb{C}_p\langle w\rangle)$ because $x_{1^+} \notin D$ by (2.8.3). One of the primary results in Huber's [Hub93] is the following theorem. Here we specialize to the context of D. See Theorem 3.1.1 later, as well.

Theorem 2.8.1 (Huber)**.** *The topological space D is quasi-compact and quasi-separated. Moreover, if $g_1, \ldots, g_r, s \in \mathbb{C}_p\langle w\rangle$ generate the unit ideal, then the rational subset $U(\frac{g_1,\ldots,g_r}{s})$ is also quasi-compact.*

Note that the hypothesis in Theorem 2.8.1 is the same that appears in the rational localization Theorem 1.6.2. Compare with Section 3.2.

To illustrate Huber's theorem, let us return to the issue of disconnection raised on page 3. The closed unit disc D certainly separates into $D = V_1 \cup V_{<1}$ where

$$V_1 = \{x \in D \mid |w(x)| = 1\} \quad \text{and} \quad V_{<1} = \{x \in D \mid |w(x)| < 1\}.$$

Since $|p(x)| \neq 0$ and $|w(x)| < 1$ on $V_{<1}$, it would *seem* that $V_{<1}$ can be written as a union

$$V_{<1} = \bigcup_{0<n} U(\frac{w^n}{p}) = \{x \in D \mid |w(x)|^n < |p(x)| \text{ for } n \gg 0\}. \tag{2.8.4}$$

However, the point $x_{1^-} \in D$ lies in a gap between $V_{<1}$ and the union. This is good! If there were not a gap, then Huber's theorem would be contradicted by V_1 together with the "cover" of $V_{<1}$ alleged in (2.8.4).

Section 2 Exercises.

Exercise 2.1. Let $\mathbb{C}_p[w]$ be given the topology induced from the Gauss norm. For $0 \leq r < \infty$, define

$$|a_0 + a_1 w + \cdots + a_m w^m|_r = \max_n \|a_n\| r^n.$$

(a) Show that $|-|_r$ is a valuation on $\mathbb{C}_p[w]$.
(b) Show that $|-|_r$ is continuous if and only if $r \leq 1$.

Exercise 2.2. Let $f \in \mathbb{C}_p\langle w\rangle$. Assume $a = |f|_1 = \|\alpha\|_p$ for some $\alpha \in \mathbb{C}_p$. Therefore, $\alpha^{-1}f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle^\times$ has a non-zero reduction $\overline{f} := \alpha^{-1}f \bmod \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \in \overline{\mathbb{F}}_p[w]$. Show that $|f|_{1^-} = (a, \varepsilon^n)$ where n is the order of vanishing of $\overline{f}$ at $w = 0$.

Exercise 2.3. Let $|-| : A \to \Gamma \cup \{0\}$ be a valuation on a topological ring A. For $\gamma \in \Gamma_{|-|}$, define

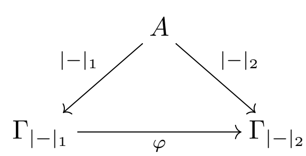
$$\overline{U}_\gamma = \{f \in A \mid |f| \leq \gamma\}.$$

(a) Show that if $|-|$ is continuous, then each $\overline{U}_\gamma$ is open.
(b) Show that if $|-|$ is non-trivial and each $\overline{U}_\gamma$ is open, then $|-|$ is continuous.

Exercise 2.4. Let A be a topological ring. Show that $|-|_{\text{triv}}$ is continuous on A if and only if A is discrete.

Exercise 2.5. Let A be a ring and $|-|_1, |-|_2$ two valuations on A. Show that the following conditions are equivalent:

(i) $|-|_1$ is equivalent to $|-|_2$.
(ii) $\mathrm{supp}(|-|_1) = \mathfrak{p} = \mathrm{supp}(|-|_2)$ and the induced valuations on $\mathrm{Frac}(A/\mathfrak{p})$ have the same valuation rings.
(iii) There exists an isomorphism $\varphi : \Gamma_{|-|_1} \xrightarrow{\simeq} \Gamma_{|-|_2}$ of totally ordered abelian groups making the following diagram commute



Exercise 2.6. Consider the p-adic numbers $\mathbb{Q}_p$.

(a) Show that if $|-| : \mathbb{Q}_p \to \Gamma \cup \{0\}$ with Γ cyclic, then $|-|$ is equivalent to either $|-|_{\text{triv}}$ or $\|-\|_p$.
(b) Show that any continuous valuation on $\mathbb{Q}_p$ is equivalent to $\|-\|_p$.
(c) Show that (b) also holds for $\mathbb{C}_p$ instead of $\mathbb{Q}_p$.

Exercise 2.7. Let $|-|_1$ and $|-|_2$ be equivalent valuations on a topological ring. Show that $|-|_1$ is continuous if and only if $|-|_2$ is continuous.

Exercise 2.8. Let $\varphi : A \to B$ be a map of rings.

(a) Show that the induced map $\mathrm{Spv}(B) \to \mathrm{Spv}(A)$ is continuous.
(b) Show that A and B are topological rings and φ is continuous, then the induced map $\mathrm{Cont}(B) \to \mathrm{Cont}(A)$ is well-defined (and continuous).

Exercise 2.9. Let A and B be Huber rings and $\varphi : A \to B$ a ring homomorphism. We call φ an **adic morphism** if there exists a pair of definition (A_0, I) for A and a ring of definition B_0 for B such that $\varphi(A_0) \subseteq B_0$ and $(B_0, \varphi(I)B_0)$ is a pair of definition for B.

(a) Show that if φ is an adic morphism, then φ is continuous.
(b) Show that if A is a Tate ring and φ is continuous, then B is a Tate ring and φ is adic.
(c) Suppose $g_1, \ldots, g_r, s \in A$ generate an open A-ideal. Show that the localization map $A \to A(\frac{g_1,\ldots,g_r}{s})$ is adic.
(d) Suppose φ is continuous. Show that φ is adic if and only if for *any* rings of definitions $A_0 \subseteq A$ and $B_0 \subseteq B$, if $\varphi(A_0) \subseteq B_0$ and $I \subseteq A_0$ is an ideal of definition, then $\varphi(I)B_0 \subseteq B_0$ is an ideal of definition.

Exercise 2.10. Let A and B be Huber rings and $\varphi : A \to B$ an adic morphism. Let $\psi : \mathrm{Cont}(B) \to \mathrm{Cont}(A)$ be the induced map. Show that if $U \subseteq \mathrm{Cont}(A)$ is a rational subset, then $\psi^{-1}(U) \subseteq \mathrm{Cont}(B)$ is also a rational subset.

Exercise 2.11. Let A be a ring.

(a) Show that if $x \in \mathrm{Spv}(A)$, then $\mathrm{supp}(x)$ is a prime ideal in A.
(b) Show that if $x, y \in \mathrm{Spv}(A)$ and $y \in \overline{\{x\}}$, then $\mathrm{supp}(x) \subseteq \mathrm{supp}(y)$.
(c) Assume A is a topological ring and $x \in \mathrm{Cont}(A)$. Show that $\mathrm{supp}(x)$ is closed in A.

Exercise 2.12. Let K be a field. A **valuation subring** of K is a subring $A \subseteq K$ such that if $\alpha \neq 0$ in K then either $\alpha \in A$ or $\alpha^{-1} \in A$. Let $x \in \mathrm{Spv}(K)$. Define

$$A_x = \{\alpha \in K \mid |\alpha(x)| \leq 1\}.$$

(a) Show that A_x is a valuation subring of K.
(b) Show directly that $\mathfrak{m}_x = \{\alpha \in A_x \mid |\alpha(x)| < 1\}$ is an ideal in A_x, and it consists of all the non-units in A_x. Conclude that A_x is a local ring.
(c) Show that $y \in \overline{\{x\}}$ within $\mathrm{Spv}(K)$ if and only if $A_y \subseteq A_x$.

Exercise 2.13. Let K be a field and $A \subseteq K$ a valuation subring. Set $\Gamma_A = K^\times / A^\times$.

(a) Show that Γ_A is a totally ordered abelian group under the ordering

$$\alpha A^\times \leq \beta A^\times \iff \alpha A \subseteq \beta A. \tag{2.8.5}$$

(b) Show that the function

$$|\alpha(x_A)| = \begin{cases} \alpha A^\times & \text{if } \alpha \neq 0; \\ 0 & \text{if } \alpha = 0, \end{cases}$$

defines a valuation $x_A \in \mathrm{Spv}(K)$.

Exercise 2.14. Show that the maps $x \mapsto A_x$ and $A \mapsto x_A$ from Exercises 2.12 and 2.13 are inverse bijections between $\mathrm{Spv}(K)$ and the set of valuation subrings $A \subseteq K$.

Exercise 2.15. Show that if $x \in \mathrm{Cont}(\mathbb{C}_p\langle w \rangle)$ and $\mathrm{supp}(x) = \langle w - \alpha \rangle$ for some $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, then $x = x_\alpha$.

<u>**Hint**</u>. Use the criterion (ii) in Exercise 2.5 and Exercise 2.6(c).

Exercise 2.16. Let Γ be a totally ordered abelian group. A subgroup Δ is **convex** in Γ if, for all $\delta_1, \delta_2 \in \Delta$ and $\gamma \in \Gamma$ such that $\delta_1 \leq \gamma \leq \delta_2$, then $\gamma \in \Delta$.

Assume now that Δ is convex in Γ. Define $\leq$ on Γ/Δ by

$$\gamma\Delta \leq \gamma'\Delta \iff \gamma \leq \gamma'\delta \text{ for some } \delta \in \Delta.$$

(a) Show that $\leq$ defines a total order on Γ/Δ and $\Gamma \to \Gamma/\Delta$ is a morphism of totally ordered abelian groups.
(b) Suppose that the exact sequence

$$1 \to \Delta \xrightarrow{\iota} \Gamma \xrightarrow{\pi} \Gamma/\Delta \to 1$$

is split as totally ordered abelian groups. Let $s : \Gamma/\Delta \to \Gamma$ be a splitting. Show that the group isomorphism

$$\Gamma/\Delta \times \Delta \xrightarrow{(s,\iota)} \Gamma$$

is an isomorphism of totally ordered abelian groups when the product group is given the *lexicographic* order.

Hint. In (a), convexity is only used to show the "strongly symmetric" property of an order relation — in this case to prove that $\gamma\Delta = \gamma'\Delta$ if and only if both $\gamma\Delta \leq \gamma'\Delta$ and $\gamma'\Delta \leq \gamma\Delta$.

Exercise 2.17. Let $\varphi : \Gamma \to \Gamma'$ be a morphism of totally ordered abelian groups.
(a) Show that $\ker(\varphi) \subseteq \Gamma$ is a convex subgroup.
(b) Show that $\operatorname{im}(\varphi) \subseteq \Gamma'$ is a totally ordered abelian group.
(c) Show that $\varphi : \Gamma/\ker(\varphi) \xrightarrow{\simeq} \operatorname{im}(\varphi)$ as totally ordered abelian groups.

Exercise 2.18. Let A be a ring. For $x, y \in \operatorname{Spv}(A)$, we say y is a **vertical specialization** of x if
(i) $y \in \overline{\{x\}}$, and
(ii) $\operatorname{supp}(y) = \operatorname{supp}(x)$.

We also say that x is a **vertical generization** of y.
(a) Show that if x is a vertical generization of y, then there is a natural quotient map $\Gamma_y \twoheadrightarrow \Gamma_x$ of totally ordered abelian groups.
(b) If $y \in \operatorname{Spv}(A)$, show that

$$\{\text{vertical generizations of } y\} \to \{\Delta \subseteq \Gamma_y \text{ convex subgroups}\}$$
$$x \mapsto \ker(\Gamma_y \twoheadrightarrow \Gamma_x)$$

is a bijection.
(c) Suppose A is a topological ring and $y \in \operatorname{Cont}(A)$. Show that if x is a vertical generization then either $x \in \operatorname{Cont}(A)$ or x is the trivial valuation modulo $\operatorname{supp}(y)$.

Hint. If $\operatorname{supp}(x) = \operatorname{supp}(y)$, then x and y can be viewed as valuations on the same field. Then, use Exercises 2.12-2.14 for (a). For (c), use Exercise 2.3.

3. Constructions with adic spectra

Let A be a Huber ring. This lecture focuses on the adic spectrum

$$\operatorname{Spa}(A, A^+) = \{x \in \operatorname{Cont}(A) \mid |f(x)| \leq 1 \text{ for all } f \in A^+\}, \tag{3.0.1}$$

which is the topic of Hübner's initial lecture [Hüb24].

The ring A^+ is a **ring of integral elements**, which means $A^+ \subseteq A^\circ$ and A^+ is open and integrally closed in A. Proposition 1.5.3 shows A° is always a ring of integral elements. A pair (A, A^+) is called a **Huber pair**. We make $X = \operatorname{Spa}(A, A^+)$ a topological space via the inclusion $X \subseteq \operatorname{Cont}(A)$.

Our initial goal in Section 3.1 is to discuss the bounds imposed on adic spectra. In Sections 3.2-3.4, we describe how localization, tensor products, and completions impact Huber pairs. Then, in Section 3.5, we prove that

$$A^+ = \{f \in A \mid |f(x)| \leq 1 \text{ for all } x \in X\} \tag{3.0.2}$$

when $X = \mathrm{Spa}(A, A^+)$.

The bulk of our energy is spent on (3.0.2). It reminds one of the statement "$I(V(I)) = I$ if I is a radical ideal", which one encounters as the Nullstellensatz in algebraic geometry. For this reason, Conrad even refers to (3.0.2) as an *adic Nullstellensatz* in [Con18, Theorem 2.25]. We observe here, in further support of this name, that the first step in the proof we present involves localization in a way reminiscent of the "Rabinowitsch trick" used in proofs of the Nullstellensatz.

Presenting (3.0.2) seems near optimal, in terms of satisfaction, for a proof using only what we explained in Sections 1 and 2. It provides a chance to use rational localization of Huber rings and we also get to introduce a technique called horizontal specialization, which complements the vertical specializations described in Section 2.8 and Exercise 2.18.

One topic we will not address is the role of the containment $A^+ \subseteq A^\circ$. It is not required for (3.0.2). The containment is crucial in Huber's study [Hub93, Proposition 3.6] of whether or not adic spectra are empty, where he proves that $\mathrm{Spa}(A, A^+) = \emptyset$ if and only if $\{0\}$ is dense in A. (Recall $\mathrm{Spec}(A) = \emptyset$ if and only if $A = \{0\}$.)

3.1. **Bounds on adic spectra.** Our initial goal is clarifying how adic spectra generalize the closed unit disc

$$D = \{x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle) \mid |w(x)| \leq 1\}.$$

In defining D, we impose a bound $|w(x)| \leq 1$ on only $w \in \mathbb{C}_p\langle w\rangle$, while (3.0.1) imposes bounds on all of A^+. There *is* a difference in the style of definition. Examining the definition of rational subsets, imposing bounds on single functions, or a short list of them, is natural. Instead of $\mathrm{Spa}(A, A^+)$, one might consider the more basic object

$$\mathrm{Spa}(A, \Sigma) = \{x \in \mathrm{Cont}(A) \mid |f(x)| \leq 1 \text{ for all } f \in \Sigma\}, \tag{3.1.1}$$

where $\Sigma \subseteq A$ is any subset.

There are two basic remarks. First, sets defined as (3.1.1) present technical challenges. Writing proofs would require learning how to track Σ while applying algebro-topological constructions to A. It would be similar to tracking functions that define a projective algebraic variety rather than an ideal sheaf. Second, there is no loss of generality in focusing only on $\Sigma = A^+$ where A^+ is open and integrally closed in A. Indeed, for all Σ, there exists an open and integrally closed ring $A^+ \supseteq \Sigma$ such that if $|f(x)| \leq 1$ for all $f \in \Sigma$, then $|f(x)| \leq 1$ for all $f \in A^+$. See Exercises 3.1-3.2. Therefore, if $\Sigma \subseteq A^\circ$, then

$$\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, A^+)$$

where A^+ is a ring of integral elements. In the case $A = \mathbb{C}_p\langle w\rangle$ and $\Sigma = \{w\}$, the relevant ring is even $A^+ = A^\circ = \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. Therefore,

$$D = \{x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle) \mid |w(x)| \leq 1\} = \mathrm{Spa}(\mathbb{C}_p\langle w\rangle, \mathcal{O}_{\mathbb{C}_p}\langle w\rangle). \tag{3.1.2}$$

See Exercise 3.3.

For the record, a general version of Huber's theorem Theorem 2.8.1 is:

Theorem 3.1.1 (Huber, [Hub93, Theorem 3.5])**.** *Let* (A, A^+) *be a Huber pair. Then,* $\mathrm{Spa}(A, A^+)$ *is quasi-compact and quasi-separated. If* $g_1, \ldots, g_r, s \in A$ *generates an open* A*-ideal, then the rational subset* $U(\frac{g_1,\ldots,g_r}{s})$ *is quasi-compact as well.*

Technically, if you look at Huber's theorem you will find that $\mathrm{Spa}(A, A^+)$ is a **spectral space** and that each rational subset is constructible. Spectral space were defined by Hochster [Hoc69]. The theorem we stated is just a portion of Huber's theorem, since spectral spaces are, in particular, quasi-compact and quasi-separated and, in addition, constructible subsets of quasi-compact spaces are quasi-compact.

Instead of proving this theorem, we move on to explain how constructions with Huber rings extend to constructions with Huber pairs.

3.2. **Construction: rational localization.** Suppose (A, A^+) is a Huber pair and $g_1, \ldots, g_r, s \in A$ generate an open A-ideal. In Section 1.6 we defined the rational localization

$$B = A(\frac{g_1, \ldots, g_r}{s}).$$

The underlying ring is $B = A[\frac{1}{s}]$. Suppose (A_0, I) is a pair of definition. A ring of definition for B is equal to $B_0 = A_0[\frac{g_1}{s}, \ldots, \frac{g_r}{s}]$, with ideal of definition $J = IB_0$. The Huber ring B is independent of the choice of (A_0, I). Considering A^+, we assume without loss of generality that $A_0 \subseteq A^+$. (See Exercise 3.4.) Then, we define

$$B^+ = \text{integral closure of } A^+[\frac{g_1}{s}, \ldots, \frac{g_r}{s}] \text{ within } B.$$

We claim B^+ is a ring of integral elements for B. First, $A_0 \subseteq A^+$ and so $B_0 \subseteq B^+$. Therefore, B^+ is open in B. Second, B^+ is integrally closed in B by construction. Finally, we claim that $B^+ \subseteq B^\circ$. To see this, start by noting that for each i, we have $\frac{g_i}{s} \in B_0$, and $B_0 \subseteq B^\circ$ by Proposition 1.5.3. By construction of B, if $A_0' \subseteq A$ is *any* ring of definition, then the image of A_0' in B is contained in some ring of definition B_0'. Therefore, *loc. cit.* implies the image of A° in B is contained in B°. Since $A^+ \subseteq A^\circ$, we have shown B° contains $A^+[\frac{g_1}{s}, \ldots, \frac{g_r}{s}]$. Finally, $B^+ \subseteq B^\circ$ because B° is itself integrally closed by Proposition 1.5.3, once again. (The integral closure step is generally required. See Exercise 3.5.)

Rational localizations are related to rational subsets, as we now explain. Suppose that $(A, A^+) \to (B, B^+)$ is the natural morphism of Huber pairs implicit in the prior paragraph. In the middle of [Hub94, Lemma 1.5(ii)], it is proven that we in fact have a natural commuting diagram

$$\begin{array}{ccc} \mathrm{Spa}(B, B^+) & \longrightarrow & \mathrm{Spa}(A, A^+) \\ & \overset{\cong}{\searrow} & \uparrow \\ & & U(\frac{g_1,\ldots,g_r}{s}) \end{array} \qquad (3.2.1)$$

where the diagonal arrow is a homeomorphism. More precisely, the rational subsets in $\mathrm{Spa}(B, B^+)$ correspond bijectively with the rational subsets contained in U, via the diagonal arrow. How difficult is the proof? Based on Sections 1 and 2, we *could* prove that (3.2.1) exists, the diagonal arrow is a bijection, and that the pre-image of a rational subset in $\mathrm{Spa}(A, A^+)$ is rational in $\mathrm{Spa}(B, B^+)$. See Exercise 3.6-3.7. It is more difficult to show a rational subset in $\mathrm{Spa}(B, B^+)$ maps onto a rational subset in U. The issue is that rational subsets of $\mathrm{Spa}(B, B^+)$ are built from elements generating an open B-ideal. After clearing denominators, there is no reason for them to generate an open A-ideal. The proofs we know rely on knowing *a priori* that U is quasi-compact, which is part of Theorem 3.1.1. The idea is explained in Exercise 3.8.

3.3. **Construction: tensor products (of Tate rings).** In this section, we explain tensor products $B \otimes_A C$ when A is a Tate ring. We extend the construction to **Tate–Huber pairs** (Huber pairs with first entry a Tate ring).

Suppose that A is a Tate ring and B and C are Huber rings with continuous ring morphisms $A \xrightarrow{\varphi} B$ and $A \xrightarrow{\psi} C$. Then, B and C are also Tate rings because the image of a pseudo-uniformizer for A under a continuous ring morphism is a pseudo-uniformizer in the target. See Exercise 2.9.

We form $R := B \otimes_A C$ algebraically, and now we make it a topological ring. Choose rings of definitions $B_0 \subseteq B$ and $C_0 \subseteq C$. Since $\varphi^{-1}(B_0) \cap \psi^{-1}(C_0)$ is an open subring, it contains a ring of definition A_0 (Exercise 3.4). We define

$$R_0 := \mathrm{img}(B_0 \otimes_{A_0} C_0 \to B \otimes_A C).$$

Now choose $\varpi \in A_0$, a pseudo-uniformizer for A. We equip R_0 with the ϖR_0-adic topology, making R_0 into a topological ring. We make R a ring with a topology by declaring R_0 is open in R. We leave it as Exercise 3.9 that this makes R into a topological ring, which is indeed a Tate ring. We dealt with a similar situation in Section 1.6 with rational localizations. A universal property confirms that the definition of R does not depend on the choices made. Namely, the A-algebra maps

$$\mathrm{id} \otimes 1 : B \to R$$
$$1 \otimes \mathrm{id} : C \to R$$

are *continuous*, and they are initial with respect to pairs of continuous A-algebras maps $B \to S$ and $C \to S$.

Extending to Tate–Huber pairs goes like this. If we start with $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ and $(A, A^+) \xrightarrow{\psi} (C, C^+)$, we may define

$$R^+ = \text{ the integral closure of } \mathrm{img}(B^+ \otimes_{A^+} C^+ \to B \otimes_A C) \subseteq R.$$

This makes R^+ integrally closed in R. In the construction, we could have assumed $B_0 \subseteq B^+$ and $C_0 \subseteq C^+$ from the start, and so $A_0 \subseteq A^+$. Thus R^+ is open. It is left as Exercise 3.10 that the image of $B^\circ \otimes_{A^\circ} C^\circ$ is contained in R°, from which the containment $R^+ \subseteq R^\circ$ follows.

The difficulty when A is not a Tate ring is defining the topological ring structure on R. To do this in general one imposes the condition that $A \to B$ and $A \to C$ are adic morphisms of Huber rings, as in Exercise 2.9. In another direction, an anonymous referee points out that future students may learn how to apply Clausen and Scholze's theory of condensed mathematics and analytic rings to streamline the construction of tensor products.

3.4. **Construction: completions.** The goal here is defining the completion of a Huber ring. The delicate point is that completions are topologically defined with respect to the underlying abelian group, so the algebraic structure of ring needs to be constructed by hand.

Let A be a Huber ring and (A_0, I) a pair of definition. The additive subgroups $I, I^2, I^3, \ldots$ are a neighborhood basis of zero in A. They are also ideals in A_0. We can form the ring-theoretic completion

$$\widehat{A_0} = \varprojlim_n A_0/I^n,$$

which becomes a complete topological ring. Recall, topologically, the quotient ring A_0/I^n is discrete and then $\widehat{A_0}$ is given the subspace topology via the inclusion

$$\widehat{A_0} \hookrightarrow \prod_{n=1}^{\infty} A_0/I^n.$$

Since the ideal I is *finitely generated*, it is a theorem (see [Sta23, Tag 05GG]) that this topology on $\widehat{A_0}$ coincides with the topology defined by the ideal $I\widehat{A_0} \subseteq \widehat{A_0}$, and

$$\widehat{A_0} \cong \varprojlim_n \widehat{A_0}/I^n\widehat{A_0}.$$

Replacing A_0 by A, the only choice we have for forming the completion is

$$\widehat{A} = \varprojlim_n A/I^n \hookrightarrow \prod_{n=1}^{\infty} A/I^n. \tag{3.4.1}$$

This is a complete topological group. The individual factors A/I^n are not rings, but $\widehat{A}$ can be given the structure of a topological ring in three steps.

(i) It is clear that $\widehat{A}$ is an $\widehat{A_0}$-module. But, one can also show $\widehat{A_0} \subseteq \widehat{A}$ is open and $\widehat{A_0}$ acts by continuous module operations on $\widehat{A}$.

(ii) There is a natural A-module structure on $\widehat{A}$. For $f \in A$, the continuity of the multiplication by f on A makes $f : A/I^{n+r} \to A/I^{r+1}$ well-defined for some $n \geq 1$ depending on f, uniform in r. A module structure is thus induced on the projective limit over r.

(iii) Since A is dense in $\widehat{A}$ and $\widehat{A_0}$ is open, there is an additive decomposition $\widehat{A} = A + \widehat{A_0}$. One defines the structure of a ring on $\widehat{A}$ by using the previous module structures and then forcing the distributive law to hold.

The details are outlined as Exercise 3.11.

Since $\widehat{A_0}$ is open in $\widehat{A}$ and the topology on $\widehat{A_0}$ is the $I\widehat{A_0}$-adic topology, we conclude that $\widehat{A}$ is a Huber ring. The continuous morphism $A \to \widehat{A}$ is initial for maps $A \to B$ with B complete. So, $\widehat{A}$ is independent of the initial choice of pair of definition (A_0, I).

If (A, A^+) is a Huber pair, we can go back to the start and assume that $A_0 \subseteq A^+$. Then, we form the completion

$$\widehat{A^+} = \varprojlim_n A^+/I^n.$$

This defines an open subring of $\widehat{A}$. Unlike the constructions in Sections 3.2-3.3, the ring $\widehat{A^+}$ is already integrally closed and so we get a Huber pair $(\widehat{A}, \widehat{A^+})$. In addition one can confirm directly that $\widehat{(A^\circ)} = (\widehat{A})^\circ$, so the property "$A^+ = A^\circ$" is preserved by completions. See Exercise 3.12.

Finally, since $\widehat{A}$ is the completion of A, the natural map $\mathrm{Cont}(\widehat{A}) \to \mathrm{Cont}(A)$ is a bijection. It induces, when (A, A^+) is a Huber pair, a canonical map

$$\mathrm{Spa}(\widehat{A}, \widehat{A^+}) \to \mathrm{Spa}(A, A^+) \tag{3.4.2}$$

that is also bijective. Huber proves (3.4.2) is a homeomorphism [Hub93, Proposition 3.9]. The difficulties are similar to those we discussed with rational localization already.

3.5. **Identifying** A^+**.** The remaining goal in this lecture is showing that rings of integral elements are intrinsic to adic spectra, just as radical ideals are intrinsic to closed subsets of affine schemes.

Theorem 3.5.1 (The adic Nullstellensatz)**.** *Let A be a Huber ring. Suppose that $A^+ \subseteq A$ is open and integrally closed. Then,*

$$A^+ = \{f \in A \mid |f(x)| \leq 1 \textit{ for all } x \in \mathrm{Spa}(A, A^+)\}. \tag{3.5.1}$$

By definition, A^+ is contained in the right-hand side of (3.5.1). To prove the theorem, we need to prove that if $f \in A$ but $f \notin A^+$, then there exists a *continuous* valuation x on A such that $|f(x)| > 1$ and $|g(x)| \leq 1$ for all $g \in A^+$.

The argument occurs in three steps. In the first step, we reduce to the case where f is a unit in A. This is where we use that A^+ is integrally closed. In the second step, we construct a candidate valuation x_0, without imposing a continuity condition. The construction is pure algebra, including a brutal extension of a valuation from one field to a larger field. The extension is so uncontrolled that arguing directly for continuity is hopeless. Therefore, in the third step, we replace x_0 by a continuous valuation x, while preserving the bounds imposed on f and $g \in A^+$. This is where the openness of A^+ is used. The replacement step is based on a process called **horizontal specialization**. Properties of horizontal specialization will be given as exercises, but note that it is a fundamental technique in Huber's papers. Seeing the proof of Theorem 3.5.1 may inspire the reader to study original sources more carefully.

The proof we give of Theorem 3.5.1 is essentially the same as in [Hub93, Lemma 3.3(i)] and [Con14, Theorem 10.3.6]. The main difference is that, in the third step, we argue for continuity directly from Proposition 2.7.1, whereas other proofs refer to a result [Hub93, Theorem 3.1] that recognizes $\mathrm{Cont}(A)$ within $\mathrm{Spv}(A)$.

For Sections 3.6, 3.7, and 3.8, we reserve A for a fixed Huber ring, A^+ for an open and integrally closed subring, and f an element of A such that $f \notin A^+$.

3.6. **Nullstellensatz: The reduction step.** We seek $x \in X = \mathrm{Spa}(A, A^+)$ with $|f(x)| > 1$. In principle, we can limit our search to the open subset $U(\frac{1}{f}) = \{x \in X \mid 1 \leq |f(x)|\}$. In terms of rings, we focus on $B = A(\frac{1}{f})$ and B^+, which we define to be the integral closure of $A^+[\frac{1}{f}]$ in $A(\frac{1}{f})$. Note B is a Huber ring, since $\{1, f\}$ generate the unit ideal in A. By the argument in Section 3.2, $B^+ \subseteq B$ is open and integrally closed.

We claim that $f \notin B^+$. Indeed, if $f \in B^+$, then f is integral over $A^+[\frac{1}{f}]$. Clearing denominators in $A[\frac{1}{f}]$, we find a polynomial relation $f^m + gf^{m-1} + \cdots = 0$ in A, with coefficients $g \in A^+$. This is impossible because A^+ is integrally closed and $f \notin A^+$. So, $f \notin B^+$.

Finally, if $x \in \mathrm{Spa}(B, B^+)$ and $|f(x)| > 1$, then its image in $\mathrm{Spa}(A, A^+)$ satisfies the same inequality. Replacing (A, A^+) with (B, B^+), we will now assume that

(i) f is a unit in A, and
(ii) $\frac{1}{f} \in A^+$ but $f \notin A^+$.

3.7. **Nullstellensatz: The algebraic argument.** This part of the argument is pure algebra. We assume (i) and (ii). We will construct $x_0 \in \mathrm{Spv}(A)$ such that $|g(x_0)| \leq 1$ for all $g \in A^+$ while $|f(x_0)| > 1$.

By (ii), the element $\frac{1}{f} \in A^+$ is not a unit. Choose a prime $\mathfrak{p} \subseteq A^+$ with $\frac{1}{f} \in \mathfrak{p}$. By (i), f *is* a unit in A. Therefore $\frac{1}{f}$ is definitely not nilpotent in $A^+_{\mathfrak{p}}$. So, choose a minimal prime $\mathfrak{q}$ in A^+ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\frac{1}{f} \notin \mathfrak{q}$. We consider then the localizations $A^+_{\mathfrak{q}} \subseteq A_{\mathfrak{q}}$. A prime ideal of (the non-zero ring) $A_{\mathfrak{q}}$ contracts to a prime $\mathfrak{Q}$ of A such that $\mathfrak{Q} \cap A^+ \subseteq \mathfrak{q}$. Equality holds since $\mathfrak{q}$ is minimal among primes in A^+. We now have a ring extension $A^+/\mathfrak{q} \subseteq A/\mathfrak{Q}$ that gives rise to a field extension

$$K^+ = \mathrm{Frac}(A^+/\mathfrak{q}) \subseteq \mathrm{Frac}(A/\mathfrak{Q}) = K. \tag{3.7.1}$$

We now reference commutative algebra and valuation theory. Focusing first just on K^+, [Mat89, Theorem 10.2] implies that we may construct a valuation subring $R^+ \subseteq K^+$ such that

$$A^+/\mathfrak{q} \subseteq R^+ \subseteq K^+ \text{ and } \mathfrak{m}_{R^+} \cap A^+/\mathfrak{q} = \mathfrak{p}/\mathfrak{q}.$$

As explained in Section 2.6 and Exercises 2.12-2.14, there is a unique $x^+ \in \mathrm{Spv}(K^+)$ with $A_{x^+} = R^+$. We view $x^+ \in \mathrm{Spv}(K^+) \cong \mathrm{supp}^{-1}(\mathfrak{q}) \subseteq \mathrm{Spv}(A^+)$. We then observe:

(a) Since $A^+/\mathfrak{q} \subseteq A^+_x$, we have $|g(x^+)| \leq 1$ for all $g \in A^+$.
(b) Since $\frac{1}{f} \notin \mathfrak{q}$, we have $0 < |\frac{1}{f}(x^+)|$. Yet, $\frac{1}{f}$ mod $\mathfrak{q} \in \mathfrak{m}_{R^+}$ and so $|\frac{1}{f}(x^+)| < 1$.

Bringing K into the discussion, Chevalley's theorem [Bou98, Chapter VI, §3, no. 3, Proposition 5] says the inclusion $K^+ \subseteq K$ induces a *surjection* $\mathrm{Spv}(K) \twoheadrightarrow \mathrm{Spv}(K^+)$. Therefore, we choose

$$x_0 \in \mathrm{Spv}(K) \cong \mathrm{supp}^{-1}(\mathfrak{Q}) \subseteq \mathrm{Spv}(A)$$

that lifts x^+. If $g \in A^+$, then $|g(x_0)| = |g(x^+)| \leq 1$ by (a). Since $f \in A^\times$, we have $|f(x_0)| > 1$ by (b). This completes the construction of x_0.

3.8. **Nullstellensatz: The specialization maneuver.** So far, we have $x_0 \in \mathrm{Spv}(A)$ such that $|f(x_0)| > 1$, while $|g(x_0)| \leq 1$ for all $g \in A^+$. Now we replace $x_0 \in \mathrm{Spv}(A)$ by $x \in \mathrm{Cont}(A)$ without altering the constraints. The rest of the argument relies on A^+ being open in A.

As preparation, we examine how close x_0 is to being continuous. Let Γ_0 be the value group of x_0. By Proposition 2.7.1, the continuity of x_0 depends on whether or not $|h(x_0)|$ is co-final in Γ_0 for $h \in A^{\circ\circ}$. Consider $s \in A$ with $|s(x_0)| \neq 0$ and $h \in A^{\circ\circ}$. Since A^+ is open in A and h is topologically nilpotent we have $h^n sf \in A^+$ as $n \to \infty$. So, $|h^n sf(x_0)| \leq 1$ as $n \to \infty$. Since $|s(x_0)| \neq 0$ and $|f(x_0)| > 1$ we see

$$|h(x_0)|^n \leq \frac{1}{|s(x_0)||f(x_0)|} < \frac{1}{|s(x_0)|} \quad (n \gg 0). \tag{3.8.1}$$

Strictly speaking, this *does not* show $|h(x_0)|$ is co-final in Γ_0, but it is close and we will end up using the estimate (3.8.1). We now adjust x_0 in three steps.

(I) Let $\Gamma_1 \subseteq \Gamma_0$ be the subgroup generated by all $|s(x_0)| \geq 1$ for $s \in A$. The general element of Γ_1 is
$$\frac{|t(x_0)|}{|s(x_0)|}$$
where that $s, t \in A$ and $|s(x_0)|, |t(x_0)| \geq 1$.

(II) Let $\overline{\Gamma}_1 \subseteq \Gamma_0$ be the convex closure of Γ_1. This is the subgroup of elements in Γ_0 that lie between two elements of Γ_1. See Exercise 3.13. If $\delta \in \overline{\Gamma}_1$, then (I) implies there exists $s, t \in A$ with $|s(x_0)|$, $|t(x_0)| \geq 1$ such that
$$\frac{1}{|s(x_0)|} \leq \frac{|t(x_0)|}{|s(x_0)|} \leq \delta. \tag{3.8.2}$$

(III) We now define $x \in \mathrm{Spv}(A)$. For $s \in A$, set
$$|s(x)| = \begin{cases} |s(x_0)| & \text{if } |s(x_0)| \in \overline{\Gamma}_1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.8.3}$$
We leave as Exercise 3.14 that this is a valuation on A. In fact, it is the most extreme case of a process called **horizontal specialization**. The same formula defines a valuation if $\overline{\Gamma}_1$ is replaced by any convex subgroup $\Delta \subseteq \Gamma_0$ containing Γ_1.

We now argue that $|g(x)| \leq 1$ for $g \in A^+$ and $|f(x)| > 1$ and that x is continuous.

1. In (3.8.3), we see $|s(x)| \leq |s(x_0)|$ for all $s \in A$. Given $|g(x_0)| \leq 1$ for $g \in A^+$, we therefore also have $|g(x)| \leq 1$ for $g \in A^+$.
2. On the other hand, $|f(x_0)| \in \Gamma_1 \subseteq \overline{\Gamma}_1$. So, $|f(x)| = |f(x_0)| > 1$.
3. Finally, suppose $h \in A^{\circ\circ}$ and $\delta \in \overline{\Gamma}_1$ is arbitrary. By (3.8.1) and (3.8.2) we may choose $s \in A$ such that $|s(x_0)| \neq 0$ and
$$|h(x)|^n \leq |h(x_0)|^n < \frac{1}{|s(x_0)|} \leq \delta \qquad (n \gg 0).$$
So, $|h(x)|$ is co-final in $\overline{\Gamma}_1$, and x is continuous by Proposition 2.7.1.

Section 3 Exercises.

Exercise 3.1. Let A be a Huber ring.
(a) Show that if $x \in \mathrm{Cont}(A)$ and $f \in A^{\circ\circ}$, then $|f(x)| < 1$.
(b) Show that if A^+ is a ring of integral elements, then $A^{\circ\circ} \subseteq A^+$.

Exercise 3.2. Let A be a Huber ring and $\Sigma \subseteq A$ any subset. Define
$$\mathrm{Spa}(A, \Sigma) = \{x \in \mathrm{Cont}(A) \mid |f(x)| \leq 1 \text{ for all } f \in \Sigma\}.$$
Let A^+ be the integral closure in A of the subring generated by Σ and $A^{\circ\circ}$.
(a) Show that A^+ is open and integrally closed.
(b) Show that $\mathrm{Spa}(A, \Sigma) = \mathrm{Spa}(A, A^+)$.

Exercise 3.3. Show that
$$\mathrm{Spa}(\mathbb{C}_p\langle w\rangle, \mathcal{O}_{\mathbb{C}_p}\langle w\rangle) = \{x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle) \mid |w(x)| \leq 1\}.$$

Exercise 3.4. Let A be a Huber ring and $B \subseteq A$ an open subring. Show that there exists a ring of definition A_0 of A that is contained in B.

Exercise 3.5. Let (A, A^+) be a Huber pair and (B, B^+) its rational localization with respect to $g_1, \ldots, g_r, s$. Show by example that $A^+[\frac{g_1}{s}, \ldots, \frac{g_d}{s}] \neq B^+$, possibly.

Exercise 3.6. Let A be a Huber ring and assume $g_1, \dots, g_r, s \in A$ generate an open A-ideal. Let $B = A(\frac{g_1,\dots,g_r}{s})$. Given a ring of integral elements A^+, define the corresponding Huber pair (B, B^+) as in Section 3.2.

(a) Show that the natural map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ factors through the rational subset $U(\frac{g_1,\dots,g_r}{s}) \subseteq \mathrm{Spa}(A, A^+)$.
(b) Show that the natural map $\mathrm{Spa}(B, B^+) \to U(\frac{g_1,\dots,g_r}{s})$ is a bijection.
(c) Show that the preimage of a rational subset in $\mathrm{Spa}(A, A^+)$ is a rational subset in $\mathrm{Spa}(B, B^+)$.

Exercise 3.7. Let A be a Huber ring and assume that $g_1, \dots, g_r, s \in A$ generate an open A-ideal. Let $B = A(\frac{g_1,\dots,g_r}{s})$. Define

$$U = U(\frac{g_1, \dots, g_r}{s}) = \{x \in \mathrm{Cont}(A) \mid |g_j(x)| \leq |s(x)| \neq 0 \text{ for all } j\}.$$

Show that the natural map $\mathrm{Cont}(B) \to \mathrm{Cont}(A)$ does not always factor through U.

Exercise 3.8. Let A be a Huber ring and I an ideal of definition. Assume U is quasi-compact in $\mathrm{Spa}(A, A^+)$ and $s \in A$ such that $|s(x)| \neq 0$ on U.

(a) Show that if $x \in U$, then there exists n such that $|f(x)| \leq |s(x)|$ for all $f \in I^n$.
(b) Suppose that $g_1, \dots, g_r$ is any list of elements of A. Show that there exists elements $f_1, \dots, f_d \in I$ such that the ideal generated by $g_1, \dots, g_r, f_1, \dots, f_d$ is open in A and
$$U(\frac{g_1, \dots, g_r}{s}) = U(\frac{g_1, \dots, g_r, f_1, \dots, f_d}{s}).$$
(c) Show that the map in part (a) of Exercise 3.6 maps rational subsets to rational subsets.

Exercise 3.9. Let A, B, and C be Tate rings and assume there are continuous ring morphisms $A \to B$ and $A \to C$.

(a) Show that $R = B \otimes_A C$ with the topology defined in Section 3.3 is a topological ring.
(b) Show that R is a Tate ring.
(c) Verify the universal property of the tensor product R with respect to pairs of continuous map $B \to S$ and $C \to S$.

Hint. See Exercise 1.2 for part (a).

Exercise 3.10. Let $A \to B$ be a continuous morphism of Tate rings.

(a) Show that the natural map $A^\circ \to B$ factors through B°.
(b) Suppose in addition that $A \to C$ is a continuous morphism of Tate rings. Show that the natural map $B^\circ \otimes_{A^\circ} C^\circ \to B \otimes_A C$ factors through $(B \otimes_A C)^\circ$.

Exercise 3.11. Let A be a Huber ring and (A_0, I) a pair of definition. Define

$$\widehat{A} = \varprojlim_n A/I^n \supseteq \widehat{A_0} = \varprojlim_n A_0/I^n.$$

(a) Show that $\widehat{A_0} \subseteq \widehat{A}$ is open.
(b) Show that if $\widetilde{f} \in \widehat{A_0}$, then multiplication by $\widetilde{f}$ is continuous on $\widehat{A}$.
(c) If $f \in A$ and $\widetilde{g} = (g_j) \in \widehat{A}$, show there exists n such that $(fg_n, fg_{n+1}, \dots)$ has well-defined image in $\widehat{A}$. Show that $\widetilde{g} \mapsto f\widetilde{g}$ is continuous on $\widehat{A}$.

(d) Show that if $\widetilde{g} \in \widehat{A}$ then $\widetilde{g} = \widetilde{g}_0 + f$ for some $\widetilde{g}_0 \in \widehat{A_0}$ and $f \in A$.
(e) Given $\widetilde{g} = \widetilde{g}_0 + f \in \widehat{A}$ and $\widetilde{h} = \widetilde{h}_0 + k \in \widehat{A}$ as in part (d), show

$$\widetilde{g}\widetilde{h} = \widetilde{g}_0\widetilde{h}_0 + f\widetilde{h}_0 + \widetilde{g}_0 k + fk$$

is well-defined in $\widehat{A}$, and it makes $\widehat{A}$ a topological ring.

Exercise 3.12. Let (A, A^+) be a Huber pair.

(a) Show that the completion $\widehat{A^+}$ is integrally closed in $\widehat{A}$.
(b) Let $B = \widehat{A}$. Show that $\widehat{A^\circ} = B^\circ$.

Exercise 3.13. Suppose that Γ is a totally ordered abelian group. Convex subgroups of Γ were defined in Exercise 2.16. This exercise shows that if $\Delta \subseteq \Gamma$, then Δ is always contained in a smallest convex subgroup $\overline{\Delta}$ called the **convex closure** of Δ within Γ. Namely, define

$$\overline{\Delta} = \{\gamma \in \Gamma \mid \delta_1 \leq \gamma \leq \delta_2 \text{ for some } \delta_1, \delta_2 \in \Delta\}.$$

(a) Show that $\overline{\Delta}$ is a subgroup of Γ.
(b) Show that $\overline{\Delta}$ is a convex.
(c) Show that

$$\overline{\Delta} = \bigcap_{\substack{\Gamma \supseteq \Delta_0 \supseteq \Delta \\ \Delta_0 \text{ convex}}} \Delta_0.$$

Exercise 3.14. Let A be a ring and $|-| : A \to \Gamma \cup \{0\}$ a valuation. Assume that $\Delta \subseteq \Gamma_{|-|}$ is a *convex* subgroup that contains $|f|$ for all $f \in A$ such that $|f| \geq 1$.

(a) Show that

$$|f|_\Delta = \begin{cases} |f| & \text{if } |f| \in \Delta; \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation on A.
(b) Show that $|-|_\Delta$ lies in the closure of $|-|$ within $\operatorname{Spv}(A)$.

4. The closed unit disc

In this lecture, we analyze the closed unit disc

$$D = \operatorname{Spa}(\mathbb{C}_p\langle w\rangle, \mathcal{O}_{\mathbb{C}_p}\langle w\rangle).$$

The reader looking for a complete analysis can consult [Con14, Section 11]. Our focus will be more narrow and perhaps prepare the learner for a more detailed treatment.

First, we broadly discuss how to distinguish points in D. We have studied D in Example 2.2.4 and Sections 2.4 and 2.8. We revisit those discussions. Second, we focus on the Gauss point $x_1 \in D$. We saw in Section 2.8 that x_{1^-} lies in the closure, but we will systematically produce many similar points. The main theorem (Theorem 4.7.3) exactly describes the closure of x_1 within D.

4.1. **The Huber ring $\mathbb{C}_p\langle w\rangle$.** We begin by reviewing $\mathbb{C}_p\langle w\rangle$ as a Huber ring. First, recall $\mathbb{C}_p$ is complete and algebraically closed. Its ring of integers is

$$\mathcal{O}_{\mathbb{C}_p} = \{\alpha \in \mathbb{C}_p \mid \|\alpha\|_p \le 1\}.$$

The maximal ideal $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$ consists of those α with $\|\alpha\|_p < 1$. The residue field is

$$\overline{\mathbb{F}}_p \cong \overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p} \cong \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}.$$

Second, the one-variable Tate algebra $\mathbb{C}_p\langle w\rangle$ is a principal ideal domain. Up to scalar, the irreducible elements are the linear polynomials $w - \alpha$ with $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. Thus,

$$\mathrm{Spec}(\mathbb{C}_p\langle w\rangle) = \{\{0\}\} \cup \{\langle w - \alpha\rangle \mid \alpha \in \mathbb{C}_p\}.$$

This is a consequence of the **Weierstrass preparation theorem**, which says that any $f \in \mathbb{C}_p\langle w\rangle$ factors uniquely as $f = p^\mu P u$ where μ is an integer and $P \in \mathbb{C}_p[w]$ is a monic polynomial and $u \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle^\times$.

The topology on $\mathbb{C}_p\langle w\rangle$ is the one induced by the Gauss norm. Thus $\mathbb{C}_p\langle w\rangle$ is a Tate ring with pseudo-uniformizer p. The following is a specific instance of the continuity criterion for valuations on $\mathbb{C}_p\langle w\rangle$.

Proposition 4.1.1. *Let $x \in \mathrm{Spv}(\mathbb{C}_p\langle w\rangle)$ with value group Γ_x. The following are equivalent:*

(i) The point x belongs to D.
(ii) We have $|p(x)|$ is co-final in Γ_x and $|f(x)| \le 1$ for all $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$.

Proof. By Proposition 2.7.1, if $x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle)$, then $|p(x)|$ is co-final in Γ_x. If x also lies in D, then of course $|f(x)| \le 1$ for all $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. Therefore (i) implies (ii).

Now suppose x is a valuation and (ii) holds. First, since $|p(x)|$ is co-final in Γ_x, we have $|p(x)| < 1$. Second, if $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$, then $|f(x)| \le 1$ by assumption and so

$$|f(x)| \le 1 < |p(x)|^{-1}.$$

Thus $x \in \mathrm{Cont}(\mathbb{C}_p\langle w\rangle)$ by Proposition 2.7.1. Of course, once x is continuous, it lies in D by assumption. Therefore (ii) implies (i). □

4.2. **Classical and "disc" points.** Proposition 4.1.1 simplifies checking whether valuations on $\mathbb{C}_p\langle w\rangle$ lie in D. Or, at least, it practically reduces a supposition "$x \in D$" to simply checking x defines a valuation. Let us tally several points of D, called classical and (nested, perhaps) disc points.

4.2.1. *Classical points.* Let $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. Then there is a point $x_\alpha \in D$ given by

$$|f(x_\alpha)| = \|f(\alpha)\|_p$$

for all $f \in \mathbb{C}_p\langle w\rangle$. This defines a valuation since the p-adic norm is a valuation. It is the *only* $x \in D$ with $\mathrm{supp}(x) = \langle w - \alpha\rangle$. See Exercise 2.15.

4.2.2. *Disc points.* Suppose $0 < r \le 1$ and $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. We define

$$D_r(\alpha) = \{\alpha' \in \mathcal{O}_{\mathbb{C}_p} \mid \|\alpha - \alpha'\|_p \le r\}.$$

This is the closed disc of radius r centered at α. If $f \in \mathbb{C}_p\langle w\rangle$, it has a series expansion

$$f = b_0 + b_1(w - \alpha) + b_2(w - \alpha)^2 + \cdots$$

with $\lim_{i\to\infty} b_i = 0$. We define $x_{\alpha,r} \in D$ according to

$$|f(x_{\alpha,r})| = \max_{i\geq 0} \|b_i\|_p r^i \in \mathbb{R}_{\geq 0}.$$

Given $x_{\alpha,r}$ is a valuation, it lies in D by Proposition 4.1.1. Indeed $|p(x_{\alpha,r})| = \frac{1}{p}$ is co-final in $\mathbb{R}_{>0}$ and if $b_i \in \mathcal{O}_{\mathbb{C}_p}$ for all i, then $|f(x_{\alpha,r})| \leq 1$, clearly. We leave as Exercise 4.1 to check that $x_{\alpha,r}$ is a valuation and

$$|f(x_{\alpha,r})| = \sup_{\alpha' \in D_r(\alpha)} \|f(\alpha')\|_p.$$

So, $x_{\alpha,r}$ depends only on $D_r(\alpha)$, rather than α. We write $x_D = x_{\alpha,r}$ if $D = D_r(\alpha)$.

4.2.3. *Nested discs.* Suppose that $D_\bullet : D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ is a sequence of discs in $\mathcal{O}_{\mathbb{C}_p}$. Then, we define

$$|f(x_{D_\bullet})| = \inf_{i\geq 0} |f(x_{D_i})|.$$

Admitting this defines a valuation, it is continuous by Proposition 4.1.1. Discussing why this is a valuation is not a priority in this lecture. See [Con14, Section 11.3], instead. We at least point out there are three possible behaviors:

- The intersection $\bigcap_{i\geq 0} D_i$ may be a single point $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. Then, $x_{D_\bullet} = x_\alpha$.
- The intersection $\bigcap_{i\geq 0} D_i$ may be another disc D. Then, $x_{D_\bullet} = x_D$.
- The intersection $\bigcap_{i\geq 0} D_i$ may be empty! This option is available because $\mathbb{C}_p$ is a metric field that is *not* **spherically complete**. In that case, $x_{D_\bullet}$ is a valuation we have not yet constructed, but on which we will not dwell.

4.3. **Review of valuation rings.** Points in D are classified, in part, by their valuation rings and residue fields. So, we review the constructions from Section 2.6 and Exercises 2.12-2.14.

Let K be a field. A subring $A \subseteq K$ is a valuation ring if, given $\alpha \in K^\times$, either α or α^{-1} belong to A. If A is a valuation ring, the non-zero principal fractional ideals $\{\alpha A \mid \alpha \in K^\times\}$ are totally ordered by inclusion. In fact, $\alpha \mapsto \alpha A$ defines a bijection

$$K^\times/A^\times \leftrightarrow \{\alpha A \mid \alpha \in K^\times\}.$$

Therefore, the group $\Gamma_A = K^\times/A^\times$ is naturally a totally ordered abelian group. In terms of cosets, this order is $\alpha A^\times \leq \beta A^\times$ if and only if $\alpha\beta^{-1} \in A$.

The natural function $K \to \Gamma_A \cup \{0\}$ defines a valuation on K. We write x_A for its equivalence class. One of the exercises mentioned is to show $x_A \leftrightarrow A$ defines a bijection between $\mathrm{Spv}(K)$ and the set of valuation subrings of K. The inverse is

$$x \mapsto A_x = \{\alpha \in K \mid |\alpha(x)| \leq 1\}.$$

The ring A_x has a maximal ideal $\mathfrak{m}_x$. Its residue field is $A_x/\mathfrak{m}_x$.

4.4. **High-level classification of points in D.** Points in D are often classified into "types" called Types 1-5. See [Sch12, Example 2.20] or [BCKW19, p. 7-8]. Here, we describe the classification without proof, augmented by listing auxiliary data. For each $x \in D$, we look at its support $\mathrm{supp}(x)$, its value group Γ_x, and its residue field $A_x/\mathfrak{m}_x$.

Warning 4.4.1. The residue fields $A_x/\mathfrak{m}_x$ are all characteristic p fields. They are *different* than any kind of residue field gotten by viewing $x \in D$ as a point in a $\mathbb{C}_p$-rigid analytic variety. Those geometric residue fields arise from the structure sheaf over D. They are all characteristic zero fields.

We will additionally indicate whether $x \in D$ is closed, and then we will finally list the Type. The result is compiled in Table 4.4.1. All but the bottom row of the table has been explained in Section 4.2.

Name	supp(x)	Γ_x	$A_x/\mathfrak{m}_x$	Closed?	Type
x_α	$\langle w-\alpha\rangle$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	Closed	1
$x_{\alpha,r}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p(t)$	Non-closed	2
$x_{\alpha,r}$ $(r \notin p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}}r^{\mathbb{Z}} \subseteq \mathbb{R}^\times$	$\overline{\mathbb{F}}_p$	Closed	3
$x_{D_\bullet}$ $(\cap D_i = \emptyset)$	$\{0\}$	$p^{\mathbb{Q}}$	$\overline{\mathbb{F}}_p$	Closed	4
$x^\lambda_{\alpha,r}$ $(r \in p^{\mathbb{Q}})$	$\{0\}$	$p^{\mathbb{Q}} \times (\frac{1}{2})^{\mathbb{Z}}$	$\overline{\mathbb{F}}_p$	Closed	5

Table 4.4.1. Classification of points in the closed unit disc D.

The classical points x_α are the Type 1 points. The second and third rows list the disc points $x_{\alpha,r}$, but the data is separated according to whether or not $r \in p^{\mathbb{Q}} = \|\mathbb{C}_p^\times\|_p$. When r lies in $p^{\mathbb{Q}}$, the point $x_{\alpha,r}$ is not closed in D. We have already seen this phenomenon at the Gauss point $x_{0,1} = x_1$. In this case, we also see the valuation ring residue field is a transcendental extension of $\overline{\mathbb{F}}_p$. The field generator t is, essentially, the reduction of the coordinate function on the boundary of the rational disc. These phenomena do not occur for disc points of radius not in the value group of $\mathbb{C}_p$. For those points, instead, the value group is larger than that of $\mathbb{C}_p$. The fourth row shows the data for the nested disc points that are neither true disc points nor classical points.

The final row indicates a type of point that we have not yet seen. These points are parametrized by $\lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$. They comprise the non-trivial points in the closure of $x_{\alpha,r}$ whenever $r \in p^{\mathbb{Q}}$. Their value group is a product group with the lexicographic order. Two examples of such points are $x^0_{0,1} = x_{1^-}$ and $x^\infty_{0,1} = x_{1^+}$ from Section 2.8, with the caveat that if $(\alpha, r) = (0,1)$, then we do not allow $\lambda = \infty$ in Table 4.4.1, since $x_{1^+} \notin D$. (See Exercise 4.7 for these calculations.)

The goal of Sections 4.6 and 4.7 is to explain the final row, to make precise the construction of $x^\lambda_{0,1}$ for $\lambda \in \mathbb{A}^1(\overline{\mathbb{F}}_p)$. We will discover these points while simultaneously showing they form the non-trivial points in the closure of the Gauss point within D. Before that, we sketch a cartoon of D that is meant to *suggest* the $x^\lambda_{0,1}$'s exist.

4.5. **A schematic drawing, focused on the Gauss point.** We pause to draw a cartoon. We will sketch what D looks like near the Gauss point $x = x_{0,1}$. Versions of this picture are drawn in other places, for instance [Con14, Section 11.3] or [Sch12, Example 2.20]. We have no intention of giving mathematical meaning to these drawings. That is why we have abstract algebra!

There are four steps to create our drawing. They are shown sequentially in Figure 4.5.1 and explained in writing now.

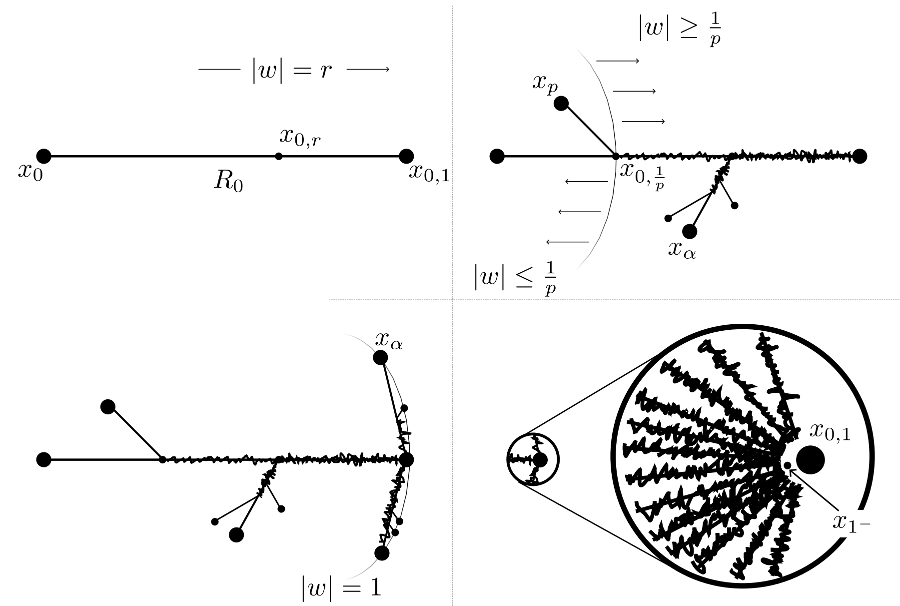


FIG. 4.5.1. Drawing of the closed unit disc D from the perspective of the Gauss point.

First, we plot the Gauss point $x_{0,1}$ and the classical point x_0, which we perceive as the origin of D. For now, both points are drawn as simple black dots, even though $x_{0,1}$ is non-closed in D. We connect these points with a ray R_0. The points of the ray are the disc points $x_{0,r}$ with $0 \leq r \leq 1$. The horizontal scale r measures $|w|$ over D.

Second, we add the classical point x_p. As explained in Section 4.2, disc points depend only on physical discs. Since $D_1(0) = D_1(p)$, we see that $x_{0,1} = x_{p,1}$. There is no new Gauss point to consider from x_p's perspective. Analogous to the first step, we draw the ray connecting x_p to the Gauss point. The two rays overlap because

$$x_{0,r} = x_{p,r} \iff \frac{1}{p} \leq r.$$

Overlap is drawn as a thicker, fuzzier, line. The process can be repeated for x_α with $\alpha \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$. For each x_α, its ray to the Gauss point intersects R_0 starting at $x_{0,\|\alpha\|_p}$.

Third, if $\alpha \in \mathcal{O}_{\mathbb{C}_p}^\times$, then $D_r(\alpha) \neq D_r(0)$ for all $r < 1$. Therefore, the ray from x_α to the Gauss point intersects R_0 *only* at $x_{0,1}$. The corresponding points x_α somehow lie on the boundary of D. Of course, different x_α's can have intersecting rays. Indeed, the fundamental principle we are repeatedly using is that if $\alpha, \alpha' \in \mathcal{O}_{\mathbb{C}_p}$, then

$$\begin{aligned} \alpha \equiv \alpha' \bmod \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} &\iff D_r(\alpha) = D_r(\alpha') \text{ for some } r < 1 \\ &\iff R_\alpha \cap R_{\alpha'} \supsetneq \{x_{0,1}\}. \end{aligned}$$

(Here R_α is the ray connecting x_α to the Gauss point.)

Finally, we zoom in on the Gauss point. We find rays parametrized by

$$\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \cong \overline{\mathbb{F}}_p = \mathbb{A}^1(\overline{\mathbb{F}}_p).$$

Zooming in with higher and higher magnification will eventually omit *any* given classical point and *any* given disc point not equal to the Gauss point. But it will never omit x_{1^-}, which lies in the closure of the Gauss point by Section 2.8. To turn this picture into mathematics, we are going to explain how to place x_{1^-} in a $\mathbb{A}^1(\overline{\mathbb{F}}_p)$-parametrized set of points, forming the non-trivial points in the closure of the Gauss point.

4.6. **The key valuation ring construction.** This section complements the review of valuation rings in Section 4.3. Let K be a field, and let $A \subseteq K$ be a valuation subring with maximal ideal $\mathfrak{m}$. To keep notations clear, let $L = A/\mathfrak{m}$. Denote by $\pi : A \to L$ the natural quotient map.

Suppose $B \subseteq A$ is also a valuation subring of K. Then, $\mathfrak{m} \subseteq B$. Indeed, if $x \in \mathfrak{m}$ is non-zero then $x^{-1} \notin A$ and therefore $x^{-1} \notin B$. Since B is a valuation ring, we conclude $x \in B$ as claimed. Thus, $\overline{B} = B/\mathfrak{m}$ makes sense and is a valuation subring of L. We leave it as Exercise 4.2 to check the converse, i.e. that if $\overline{B} \subseteq L$ is a valuation subring of L then $\pi^{-1}(B) \subseteq A$ is a valuation subring of K. Admitting that, we get inverse bijections

$$\{\text{valuation rings } B \subseteq A\} \overset{\simeq}{\longleftrightarrow} \{\text{valuation subrings } \overline{B} \subseteq L\} \tag{4.6.1}$$
$$B \mapsto B/\mathfrak{m}$$
$$\pi^{-1}(\overline{B}) \leftarrow\!\shortmid \overline{B}.$$

Suppose we have a pair $B \leftrightarrow \overline{B}$ under (4.6.1). Write $x \in \mathrm{Spv}(K)$ for the valuation corresponding to A, write $y \in \mathrm{Spv}(K)$ for the one corresponding to B, and $\lambda \in \mathrm{Spv}(L)$ for the one determined by $\overline{B}$. As Exercise 4.2, check the following statement as well:

(a) The three value groups are arranged in a natural exact sequence

$$1 \to \Gamma_\lambda \to \Gamma_y \to \Gamma_x \to 1. \tag{4.6.2}$$

(b) Within $\mathrm{Spv}(K)$ we have the specialization relation $y \in \overline{\{x\}}$.

Example 4.6.1. We will use (4.6.1) to construct $y \in \mathrm{Spv}(K)$ from $\lambda \in \mathrm{Spv}(L)$. The relevant L is $L \cong \overline{\mathbb{F}}_p(w)$, so let us describe $\mathrm{Spv}(L)$ in that case.

Let $\lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$. For $f \in \overline{\mathbb{F}}_p(w)$, write $\mathrm{ord}_\lambda f$ for the order of vanishing of f at $w = \lambda$.[2] We thus get a valuation $|-|_\lambda : \overline{\mathbb{F}}_p(w) \to \mathbb{R}_{\geq 0}$ by

$$|f|_\lambda = (\frac{1}{2})^{\mathrm{ord}_\lambda f}.$$

The value group is $\Gamma_\lambda = (\frac{1}{2})^{\mathbb{Z}}$. Ostrowski's theorem is that the non-trivial elements of $\mathrm{Spv}(\overline{\mathbb{F}}_p(w))$ are represented by the $|-|_\lambda$'s. The proof is recalled as Exercise 4.3.

[2]Recall, $\mathrm{ord}_\infty f = \deg(f)$.

4.7. **The closure of the Gauss point.** We can now describe the closure of the Gauss point in D. We begin with two lemmas.

Lemma 4.7.1. *Let A be a Tate ring. For $x, y \in \mathrm{Cont}(A)$, the following are equivalent:*

(i) The point y lies in the closure $\overline{\{x\}}$.
(ii) We have $\mathrm{supp}(x) = \mathrm{supp}(y)$ and $A_y \subseteq A_x$.

Proof. We will show that if $y \in \overline{\{x\}}$, then $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$. This is where the Tate condition on A is used. The rest of (i) implies (ii), and all of (ii) implies (i) are left as Exercise 4.4. (The Tate condition can also be weakened. See Exercise 4.6.)

Since A is a Tate ring, we may choose a pseudo-uniformizer $\varpi \in A$. Since y is a continuous valuation, $|\varpi(y)|$ is co-final in the value group Γ_y. Therefore,

$$\begin{aligned} f \in \mathrm{supp}(y) &\iff |f(y)| = 0 \\ &\iff |f(y)| \leq |\varpi^n(y)| \qquad \text{(for all } n\text{)}. \end{aligned}$$

The same equivalences hold if x replaces y. Yet, if $y \in \overline{\{x\}}$ then, by definition,

$$|f(y)| \leq |\varpi^n(y)| \implies |f(x)| \leq |\varpi^n(x)|,$$

for all n. Therefore, $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$. □

For the remainder of this section, we let $x = x_{0,1} \in D$. Let $K = \mathrm{Frac}(\mathbb{C}_p\langle w\rangle)$ and let $A_x \subseteq K$ be the valuation ring of x, with maximal ideal $\mathfrak{m}_x$ and residue field L_x. Our second lemma determines the field L_x. Note that

$$\mathcal{O}_{\mathbb{C}_p}\langle w\rangle \subseteq A_x = \{f \in K \mid |f(x)| \leq 1\}.$$

This containment is strict because $\frac{1}{w}$ belongs to A_x but not $\mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. More generally, A_x contains $(w-\alpha)^{-1}$ for any $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, and none of those elements lie in $\mathcal{O}_{\mathbb{C}_p}\langle w\rangle$. These examples are the only essential difference at the level of residue fields.

Lemma 4.7.2. *The inclusion $\mathcal{O}_{\mathbb{C}_p}\langle w\rangle \subseteq A_x$ induces an isomorphism $\overline{\mathbb{F}}_p(w) \cong L_x$.*

Proof. Note that $\mathfrak{m}_x \cap \mathcal{O}_{\mathbb{C}_p}\langle w\rangle = \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}\langle w\rangle$. Therefore,

$$\overline{\mathbb{F}}_p[w] \cong \mathcal{O}_{\mathbb{C}_p}[w]/\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}[w] \cong \mathcal{O}_{\mathbb{C}_p}\langle w\rangle/\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}\langle w\rangle \subseteq L_x,$$

which extends to $\overline{\mathbb{F}}_p(w) \subseteq L_x$. We claim *this* inclusion is an equality.

To start, by Weierstrass preparation any fraction $f/g \in K$ can be expressed as

$$\frac{f}{g} = \frac{P}{Q}u,$$

where $P, Q \in \mathbb{C}_p[w]$ and $u \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle^\times$. We may multiply Q by a non-zero scalar so that $Q \in \mathcal{O}_{\mathbb{C}_p}[w]$ but not $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}[w]$. Then, scale P by the same factor.

Suppose now $f/g \in A_x$. Since $u \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle^\times$, we have $|u(x)| = 1$. Therefore,

$$|P(x)| \leq |Q(x)|.$$

Since $Q \in \mathcal{O}_{\mathbb{C}_p}[w]$, we also have $P \in \mathcal{O}_{\mathbb{C}_p}[w]$ by definition of x. Thus $P/Q \bmod \mathfrak{m}_x \in \overline{\mathbb{F}}_p(w)$. Since $u \bmod \mathfrak{m}_x \in \overline{\mathbb{F}}_p[w]$, we find $f/g \bmod \mathfrak{m}_x \in \overline{\mathbb{F}}_p(w)$, as claimed. □

It *seems* the discussion in this section and the previous one gives bijections

$$\begin{aligned}
\mathbb{P}^1(\overline{\mathbb{F}}_p) &\leftrightarrow \{\overline{B} \subseteq \overline{\mathbb{F}}_p(w) \text{ a valuation subring}\} && \text{(Example 4.6.1)}\\
&\leftrightarrow \{\overline{B} \subseteq L_x \text{ a valuation subring}\} && \text{(Lemma 4.7.2)}\\
&\leftrightarrow \{A_y \subseteq A_x \text{ a valuation subring of } K\} && \text{(Section 4.6)}\\
&\leftrightarrow y \in \overline{\{x\}} && \text{(Lemma 4.7.1).}
\end{aligned}$$

However, there is one caveat. For $\lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ we get a point $x_{0,1}^{\lambda} \in \mathrm{Spv}(\mathbb{C}_p\langle w\rangle)$ via the first three bijections, and $x_{0,1}^{\lambda} \in \overline{\{x\}}$ within the valuation spectrum, by the proof of Lemma 4.7.1. However, we need to show $x_{0,1}^{\lambda}$ is actually continuous!

Theorem 4.7.3.

(a) *Each* $x_{0,1}^{\lambda}$ *is a continuous valuation on* $\mathbb{C}_p\langle w\rangle$.
(b) *If* $\lambda \neq \infty$, *then* $x_{0,1}^{\lambda} \in D$.
(c) *Within* D *we have* $\overline{\{x_{0,1}\}} = \{x_{0,1}\} \cup \{x_{0,1}^{\lambda} \mid \lambda \in \mathbb{A}^1(\overline{\mathbb{F}}_p)\}$.

Proof. The majority of the proof is an explicit analysis in support of (a).

Let $\pi : A_x \to L_x$ be the natural projection map. Note $\overline{\mathbb{F}}_p(w) \cong L_x$ by Lemma 4.7.2 and, in this identification, $\pi(\mathcal{O}_{\mathbb{C}_p}) \subseteq \overline{\mathbb{F}}_p$. For $\lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$ we let $\overline{B}_\lambda \subseteq L_x$ be the valuation ring arising from λ in Example 4.6.1. By direct examination, $\overline{\mathbb{F}}_p \subseteq \overline{B}_\lambda$ and therefore $\mathcal{O}_{\mathbb{C}_p} \subseteq \pi^{-1}(\overline{B}_\lambda)$.

The value group of x is identified as $K^\times/A_x^\times$ by Section 4.3. Let $y = x_{0,1}^{\lambda}$ and $A_y \subseteq A_x$ be its valuation ring. The prior paragraph shows that $\mathcal{O}_{\mathbb{C}_p} \subseteq A_y$. Therefore, the value group $K^\times/A_y^\times$ fits into a diagram

$$\begin{array}{ccc}
\mathbb{C}_p^\times & \xrightarrow{\;y\;} & K^\times/A_y^\times \\
\big\downarrow & \nearrow & \big\downarrow \\
\mathbb{C}_p^\times/\mathcal{O}_{\mathbb{C}_p}^\times & \xrightarrow[x]{\simeq} & K^\times/A_x^\times .
\end{array}$$

The diagonal arrow is injective because the bottom arrow is. Thus, the exact sequence

$$1 \to \Gamma_\lambda \to K^\times/A_y^\times \to K^\times/A_x^\times \to 1$$

in (4.6.2) is split as totally ordered abelian groups. From Example 4.6.1, the kernel is $\Gamma_\lambda \cong (\frac{1}{2})^{\mathbb{Z}}$. Moreover, part (b) of Exercise 2.16 implies the ordering on

$$\Gamma_y \cong \Gamma_x \times (\frac{1}{2})^{\mathbb{Z}}$$

is the *lexicographic order*. The analysis in Section 2.8 now shows y defines a continuous valuation. This completes the proof of (a).

For (b) we need to see that $\mathcal{O}_{\mathbb{C}_p}\langle w\rangle \subseteq A_y$ if and only if $\lambda \neq \infty$. By the bijection in Section 4.6, it is equivalent to see that $\overline{\mathbb{F}}_p[w] \subseteq \overline{B}_\lambda$ if and only if $\lambda \neq \infty$. But that is clear by definition of the λ-adic valuation on $\overline{\mathbb{F}}_p(w)$. Finally, (c) follows from (a) and (b), together with the overall discussion preceding the theorem statement. □

The reader can work as Exercise 4.7 that the splitting used to prove (a) gives rise to identifications $x_{0,1}^{0} = x_{1^-}$ and $x_{0,1}^{\infty} = x_{1^+}$ as in Section 2.8. We also outline, as Exercise 4.8, the adjustments required to adapt the process to other points of D. Namely, for $r < 1$ and $\alpha \in \mathcal{O}_{\mathbb{C}_p}$, we see in Section 4.4 that $x_{\alpha,r}$ is a non-closed point when r lies in the value group of $\mathbb{C}_p$. In fact, the closure will be in bijection with $\mathbb{P}^1(\overline{\mathbb{F}}_p)$. (The exception in Theorem 4.7.3(b) disappears.)

The analysis of *all* the points in D is incomplete, since we did not say anything at all about Type 3 or 4 points in Table 4.4.1. The reader who wants all the details can see [Con14, Section 11]. Or, note that if $\mathbb{C}_p$ is replaced by a non-Archimedean field whose value group is $\mathbb{R}_{>0}$ and which is spherically complete, then the only points that exist are Types 1, 2, and 5. Type 3 points become Type 2 and Type 4 points become Type 1. From this perspective, the introduction of the Type 5 points is really the heart of the adic unit disc beyond the classical and rational disc points.

4.8. **Final comments.** What might the reader look at next? In Lemma 4.7.1, we realized every point in the closure of the Gauss point in D has generic support $\{0\}$. This is a more general phenomenon, occurring at **analytic** points on adic spectra. We introduce this notion in Exercises 4.5 and 4.6.

The reader would do well to understand the notion of analytic points, next. The Lemma 4.7.1 we proved implicitly uses that every point on an adic specturm of a Tate ring is analytic. Analytic points are used more generally to analyze adic spectra. A good target theorem for a learner would be [Hub93, Proposition 3.6] on whether $\mathrm{Spa}(A, A^+)$ is empty or not. We discussed this on page 26. It is plausible to unwind the argument of that result using the tools outlined in these notes (cf. [Con14, Section 11.6] and [Mor19, Section III.4.4]). In doing so, the reader will need to follow the construction of certain spaces $\mathrm{Spv}(A, I)$ of "valuations with support conditions" introduced by Huber. The benefit of doing so would be that these spaces with support conditions, and specialization arguments as in Section 3.8, are crucially applied in proving Huber's Theorem 3.1.1 on the geometric structure of $\mathrm{Spa}(A, A^+)$.

The other option, hopefully one the reader has already begun, is plowing ahead with the sheaf theory on adic spaces and perfectoid spaces outlined in the sibling lectures [Hüb24, Joh24, Heu24]. It is completely plausible for users of Huber's theory of adic spaces to never truly need to study the proof of Theorem 3.1.1, as long as their intuition is guided by enough examples (as Section 4 tries to illustrate).

Section 4 Exercises.

Exercise 4.1. Let $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ and r a real number with $0 < r \leq 1$. For $f \in \mathbb{C}_p\langle w \rangle$ we write

$$f = b_0 + b_1(w-\alpha) + b_2(w-\alpha)^2 + \cdots$$

with $b_i \in \mathbb{C}_p$ converging to zero as $i \to \infty$. Let

$$|f(x_{\alpha,r})| = \max_{i \geq 0} \|b_i\| r^i.$$

(a) Show that $x_{\alpha,r}$ defines a (continuous) valuation on $\mathbb{C}_p\langle w \rangle$.

(b) Show that

$$f \mapsto \sup_{\alpha' \in D_r(\alpha)} \|f(\alpha')\|_p$$

is continuous on $\mathbb{C}_p\langle w \rangle$.

(c) Show that

$$|f(x_{\alpha,r})| = \sup_{\alpha' \in D_r(\alpha)} \|f(\alpha')\|_p.$$

Hint. Once (a) and (b) are shown, (c) can be checked directly on polynomials. It may be helpful to study the case $r \in \|\mathbb{C}_p^\times\|_p = p^{\mathbb{Q}}$ first and do the general case as a limiting process.

Exercise 4.2. Let K be a field and $A \subseteq K$ a valuation ring with maximal ideal $\mathfrak{m}$. Let $L = A/\mathfrak{m}$ be the residue field of A and $\pi : A \to L$ be the natural projection.

(a) Show that if $\overline{B} \subseteq L$ is a valuation subring, then $B = \pi^{-1}(\overline{B}) \subseteq A$ is a valuation subring of K as well.

(b) By Exercise 2.13, the valuation rings A, B, and $\overline{B}$ correspond to valuations on K, K, and L. Show that the corresponding value groups sit in a natural exact sequence

$$1 \to \Gamma_{\overline{B}} \to \Gamma_B \to \Gamma_A \to 1.$$

(c) Let $x_A, x_B \in \mathrm{Spv}(K)$ be the valuations corresponding to A and B, respectively. Show that $x_B \in \overline{\{x_A\}}$ within $\mathrm{Spv}(K)$.

Exercise 4.3. Suppose that F is an algebraically closed field. For $\lambda \in \mathbb{P}^1(F)$ define $|-|_\lambda : F(w) \to \mathbb{R}_{\geq 0}$ by

$$|f|_\lambda = (\frac{1}{2})^{\mathrm{ord}_{w=\lambda}(f)}.$$

(Note that $|f|_\infty = 2^{-\deg(f)}$, for $f \in F[w]$.)

(a) Show that $|-|_\lambda$ is a valuation for all λ.

Now suppose $F(w) \xrightarrow{|-|} \Gamma \cup \{0\}$ is any valuation and $|\alpha| \leq 1$ for all $\alpha \in F$. (This condition on the scalars is automatic if $F = \overline{\mathbb{F}}_p$ because every non-zero element of $\overline{\mathbb{F}}_p$ is a root of unity.)

(b) If $|w| > 1$, show that $|a_0 + a_1 w + \cdots + a_n w^n| = |w|^n$ for all $a_0, \ldots, a_n \in F$ with $a_n \neq 0$. Conclude that $|-|$ is equivalent to $|-|_\infty$.

(c) Now suppose $|w| \leq 1$.
- Show that $\mathfrak{p} = \{f \in F[w] \mid |f(w)| < 1\}$ is a prime ideal in $F[w]$.
- Show that if $\mathfrak{p} = \{0\}$, then $|-|$ is equivalent to $|-|_{\mathrm{triv}}$.
- Show that if $\mathfrak{p} = \langle w - \lambda \rangle$, then $|-|$ is equivalent to $|-|_\lambda$.

Exercise 4.4. Let A be any ring. Let $x, y \in \mathrm{Spv}(A)$.

(a) Suppose that $y \in \overline{\{x\}}$. Show that $\mathrm{supp}(x) \subseteq \mathrm{supp}(y)$.

(b) Show that if $\mathrm{supp}(x) = \mathrm{supp}(y)$ and $A_y \subseteq A_x$, then $y \in \overline{\{x\}}$.

Exercise 4.5. Let A be a Huber ring and $x \in \mathrm{Cont}(A)$. Recall, $\mathrm{supp}(x) \subseteq A$ is always a closed prime ideal. We call x **analytic** if $\mathrm{supp}(x)$ is *not* open in A. Show that the following are equivalent for $x \in \mathrm{Cont}(A)$:

(i) The point x is analytic.

(ii) There exists $f \in A^{\circ\circ}$ such that $|f(x)| \neq 0$.

(iii) For any ideal of definition I there exists $f \in I$ such that $|f(x)| \neq 0$.

Exercise 4.6. Let A be a topological ring. Show that if $x \in \mathrm{Cont}(A)$ is analytic and $y \in \mathrm{Cont}(A)$ lies in $\overline{\{x\}}$, then $\mathrm{supp}(y) = \mathrm{supp}(x)$. Therefore, if x is analytic, then its only specializations are vertical. (See Exercise 2.18.)

Exercise 4.7. Let $\lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)$. Consider $x_{0,1}^\lambda$ constructed in Section 4.7. Show that

$$\mathbb{C}_p\langle w \rangle \xrightarrow{x_{0,1}^\lambda} \mathbb{R}_{>0} \times \mathbb{R}_{>0} \cup \{0\}$$

can be defined by the following recipe:

- First, $|p(x_{0,1}^{\lambda})| = (\frac{1}{p}, 1)$.
- Second, if $f \in \mathcal{O}_{\mathbb{C}_p}\langle w\rangle$ and $\bar{f} \in \overline{\mathbb{F}}_p[w]$ is its reduction modulo $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$, then

$$|f(x_{0,1}^{\lambda})| = (|f|_1, |\bar{f}|_{\lambda}).$$

Conclude that $x_{0,1}^{0} = x_{1^-}$ and $x_{0,1}^{\infty} = x_{1^+}$.

Exercise 4.8. Let $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ and $\beta \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$. Let $r = \|\beta\|_p < 1$. Let $x = x_{\alpha,r}$ be the disc point given in Section 4.2.

(a) Show that $t = \frac{w-\alpha}{\beta} \in A_x$ and there is a natural isomorphism $\overline{\mathbb{F}}_p(t) \cong A_x/\mathfrak{m}_x$.
(b) Show that the containment $\mathcal{O}_{\mathbb{C}_p}\langle w\rangle \subseteq A_x$ has image $\overline{\mathbb{F}}_p$ in $A_x/\mathfrak{m}_x$.
(c) Show that inside D, the closure of x is given by

$$\overline{\{x_{\alpha,r}\}} = \{x_{\alpha,r}\} \cup \{x_{\alpha,r}^{\lambda} \mid \lambda \in \mathbb{P}^1(\overline{\mathbb{F}}_p)\},$$

for points $x_{\alpha,r}^{\lambda}$ constructed via the mechanism of Section 4.6.

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University of Arkansas, Department of Mathematical Sciences, 850 W Dickson Street, Fayetteville, AR., USA.

Email address: `bergdall@uark.edu`