# AN INVERSE SPECTRAL PROBLEM FOR NON-COMPACT HANKEL OPERATORS WITH SIMPLE SPECTRUM

*By*

PATRICK GÉRARD, ALEXANDER PUSHNITSKI AND SERGEI TREIL*

**Abstract.** We consider an inverse spectral problem for a class of non-compact Hankel operators $H$ such that the modulus of $H$ (restricted onto the orthogonal complement to its kernel) has simple spectrum. Similarly to the case of compact operators, we prove a uniqueness result, i.e., we prove that a Hankel operator from our class is uniquely determined by the spectral data. In other words, the spectral map, which maps a Hankel operator to the spectral data, is injective. Further, in contrast to the compact case, we prove the failure of surjectivity of the spectral map, i.e., we prove that not all spectral data from a certain natural set correspond to Hankel operators. We make some progress in describing the image of the spectral map. We also give applications to the cubic Szegő equation. In particular, we prove that not all solutions with initial data in BMOA are almost periodic; this is in a sharp contrast to the known result for initial data in VMOA.

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# 1 Introduction

**1.1 Overview.** In the mid-1980s, Khrushchev and Peller [16], motivated by the spectral theory of stationary Gaussian processes, asked to describe all non-negative self-adjoint operators that are unitarily equivalent to the modulus of a Hankel operator $\Gamma$ (i.e., to the operator $|\Gamma| := (\Gamma^*\Gamma)^{1/2}$).

This problem was actively studied from the mid 1980s to early 1990s, see [31, 32, 34, 25], until the final result was obtained by Treil [33]: any positive semi-definite self-adjoint operator that is non-invertible and whose kernel is either trivial or infinite-dimensional is unitarily equivalent to the modulus of a Hankel operator. This gives a complete solution to the problem, since it is easy to see that any Hankel operator is not invertible and cannot have a finite-dimensional kernel.

Later, motivated by problems in control theory, Megretskii, Peller and Treil started to investigate of the analogous problem for self-adjoint Hankel operators. The question was to describe all possible types of spectral measures and the multiplicity functions, corresponding to self-adjoint Hankel operators.

A complete solution to this problem was given in [21] (also see [21] for the history of the problem). The answer was slightly more complicated than for the modulus of a Hankel operator: besides the obvious properties of non-invertibility and the absence of a finite-dimensional kernel, some “almost symmetry” property of the spectral multiplicity function was also required.

In both of the these problems, the spectral datum[1] (i.e., the type of the spectral measure and the multiplicity function) does not determine the corresponding Hankel operator uniquely: in fact, with the exception of trivial cases, there are infinitely many self-adjoint Hankel operators with the same spectral datum.

[1]we use the convention singular: datum, plural: data.

In the early 2010s, the interest in inverse spectral problems for Hankel operators was renewed due to the work of Gérard and Grellier [5, 6] on the cubic Szegő equation. This is a totally non-dispersive evolution equation which is completely integrable and possesses a Lax pair, which involves a Hankel operator (see Section 10.1 for the details). Motivated by this, in [8] Gérard and Grellier developed a new type of direct and inverse spectral theory for compact Hankel operators. The Hankel operators appearing in this theory are generally not self-adjoint, and the language of anti-linear operators gives a convenient way to represent the spectral datum in this case.

Another new feature of this theory is that the spectral datum was constructed from the pair of Hankel operators $\Gamma$ and $\Gamma S$, where $S$ is the shift operator in the Hardy space $H^2$. In this case there is a bijection between compact Hankel operators and the corresponding spectral data, and the class of spectral data sets corresponding to the compact operators can be explicitly described. In this construction, the evolution of the spectral datum under the cubic Szegő equation is very simple, which makes the bijectivity very desirable.

The next natural step in this line of research is the study of the direct and inverse spectral problem for non-compact Hankel operators. For a few years, the work of two of the authors (Gérard and Pushnitski) was motivated by the conjecture that the bijective spectral map of [8] admits a natural extension to the non-compact case; some preliminary steps in this direction were made in [11]. One of the aims of the present paper is to show that this conjecture is false, in some precise sense to be explained below. For a suitable class of Hankel operators (which includes many non-compact ones), we construct a natural extension of the spectral map of [8] and show that it is injective, but not surjective. We also give an application to the cubic Szegő equation, corresponding to non-compact Hankel operators. We show that in general, solutions to this equation with the initial data in BMOA are NOT almost-periodic, in contrast with the case of the initial data in VMOA.

An important new component of the present work is the functional model for contractions (=operators of norm $\leq 1$) on a Hilbert space. A key ingredient to proving that a given spectral datum corresponds to some Hankel operator is checking that a certain contraction, constructed from the spectral datum, is asymptotically stable. (A contraction $T$ is called **asymptotically stable** if $T^n \to 0$ in the strong operator topology as $n \to \infty$.) In the compact case, it turns out that the asymptotic stability always holds. In the non-compact case, we show that the asymptotic stability sometimes holds and sometimes doesn't, depending on some spectral properties of the contraction. Here we use some latest advances [19] from the theory of the Clark model. A more precise discussion is postponed to Section 2.

**1.2 The structure of the paper.** In this section we introduce Hankel operators, describe the direct spectral problem and the spectral data, and present our first main result: uniqueness. Proofs are postponed to Section 3. In Section 2, we discuss the problem of surjectivity of the spectral map and informally describe our main results concerning the failure of surjectivity. In Section 4 we collect without proof some operator theoretic background (which mainly concerns the spectral theory of contractions on Hilbert spaces and the Clark model) that is required for the construction of the rest of the paper. Sections 5–8 are the core of the paper; here we state and prove our main results concerning the failure of surjectivity and the description of the image of the spectral map. In Section 9 we describe the special case of self-adjoint Hankel operators. In Section 10 we give an application to the cubic Szegő equation. Some technical parts of proofs are postponed to Appendices.

**1.3 Notation.** For a Hilbert space $X$, we denote the inner product of elements $f, g \in X$ by $\langle f, g\rangle_X$; we omit the subscript $X$ if there is no danger of confusion. For a bounded self-adjoint operator $A$ in $X$ and for $v \in X$, we denote by

$$\langle v\rangle_A := \operatorname{clos}\operatorname{span}\{A^n v, n = 0, 1, 2, \dots\}$$

the cyclic subpace of $A$ generated by $v$. We recall that $A$ is said to have simple spectrum if $X = \langle v\rangle_A$ for some element $v \in X$; any such element is called **cyclic** for $A$. We denote by $\rho_v^A$ the spectral measure of $A$ corresponding to $v$, i.e.,

$$\langle f(A)v, v\rangle = \int_{\mathbb{R}} f(s)d\rho_v^A(s) \tag{1.1}$$

for any continuous function $f$.

We denote by $\mathbf{S}_p$, $p > 0$, the standard Schatten class of compact operators; in particular, $\mathbf{S}_1$ is trace class and $\mathbf{S}_2$ is the Hilbert–Schmidt class.

For a finite measure $\rho$ on $\mathbb{R}$, we denote $L^2(\rho) \equiv L^2(\mathbb{R}, d\rho)$, and we usually use the letter $s$ to denote the independent variable in $\mathbb{R}$. We denote by $\mathbb{1} \in L^2(\rho)$ the function identically equal to one.

We denote by $H^2 = H^2(\mathbb{T})$ the standard Hardy space of functions on the unit circle $\mathbb{T}$,

$$f = f(z) = \sum_{j=0}^{\infty} \widehat{f}_j z^j, \quad |z| = 1, \quad \sum_{j=0}^{\infty} |\widehat{f}_j|^2 < \infty;$$

the above series converges in $L^2(\mathbb{T})$. Note that this series also converges uniformly on compact subsets of $\mathbb{D}$, so $f$ can be interpreted as an analytic function in the unit disc $\mathbb{D}$. The values of $f$ on $\mathbb{T}$ can be found as the **non-tangential boundary**

**values** of this analytic function; according to classical results these non-tangential limits exist a.e. on $\mathbb{T}$. Note also that the set $H^\infty = H^\infty(\mathbb{D})$ of all bounded analytic function is a subset of $H^2$.

We denote by $\{z^m\}_{m=0}^\infty$ the standard basis in the Hardy space $H^2$; in particular, we denote by $z^0$ the element of $H^2$ identically equal to one (as notation $\mathbb{1}$ is already taken). The Szegő projection $P$ is the orthogonal projection onto $H^2$ in $L^2(\mathbb{T})$,

$$P : \sum_{k=-\infty}^{\infty} \widehat{f}_k z^k \mapsto \sum_{k=0}^{\infty} \widehat{f}_k z^k.$$

Recall that the shift operator $S$ on $H^2$ is the multiplication by $z$, $Sf = zf(z), f \in H^2$, and its adjoint (the backward shift) $S^*$ is given by

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

We refer, e.g., to [26, Appendix 2] for the definition of the classes BMOA($\mathbb{T}$) and VMOA($\mathbb{T}$).

We shall denote by $A_{\rm ac}$ the, a.c., part of a self-adjoint operator $A$ and by $\simeq$ the unitary equivalence between operators. For a linear operator $A$, we denote by $\overline{\operatorname{Ran} A}$ the closure of the range of $A$.

**1.4 Hankel operators $\Gamma_u$.** A **Hankel matrix** is an infinite matrix of the form $\{\gamma_{j+k}\}_{j,k=0}^\infty$, i.e., the entries must depend on the sum of indices. A **Hankel operator** is a bounded operator in the Hardy space $H^2$, whose matrix in the standard basis $\{z^k\}_{k=0}^\infty$ is a Hankel matrix. An equivalent alternative definition is that a Hankel operator is a bounded operator $\Gamma$ in $H^2$ such that the commutation relation

$$\Gamma S = S^* \Gamma, \tag{1.2}$$

is satisfied, where $S$ is the shift operator in $H^2$.

For a Hankel operator $\Gamma$ one can define its **analytic symbol** $u$ as

$$u(z) := \Gamma z^0 = \sum_{k=0}^{\infty} \gamma_k z^k.$$

In this paper we will skip the word analytic and use the term symbol for $u$. We will also use the notation $\Gamma_u$ to indicate the Hankel operator with the symbol $u$. It is a well-known fact [26, Theorem 1.1.2] that the operator $\Gamma_u$ is bounded if and only if the symbol $u$ belongs to the class BMOA($\mathbb{T}$) of the functions of bounded mean oscillation. On the other hand, we have $u = \Gamma_u z^0 \in H^2$; it will be important for us to consider the symbol $u$ as an element of $H^2$.

One can give a more "analytic" formula for the Hankel operator $\Gamma_u$. Namely, denote by $J$ the involution in $L^2(\mathbb{T})$,

$$Jf(z) = f(\overline{z}).$$

Then for $u \in$ BMOA, the Hankel operator $\Gamma_u$ with the matrix $\{\widehat{u}_{j+k}\}_{j,k=0}^{\infty}$ is defined by

$$\Gamma_u f = P(uJf),$$

initially on the set of polynomials $f \in H^2$.

**1.5 Anti-linear Hankel operators $H_u$.** Clearly, Hankel matrices $\{\gamma_{j+k}\}_{j,k=0}^{\infty}$ are symmetric (with respect to transposition). This can be expressed as the statement that Hankel operators belong to the class of so-called **complex symmetric operators**. Namely, let us denote by $\mathbf{C}$ the anti-linear (a.k.a. conjugate-linear) involution in $H^2$,

$$\mathbf{C}f(z) = \overline{f(\overline{z})}; \tag{1.3}$$

in other words, for $f(z) = \sum_{k=0}^{\infty} a_k z^k$ we have $\mathbf{C}f(z) = \sum_{k=0}^{\infty} \overline{a}_k z^k$. Then the symmetry of Hankel matrices means that Hankel operators satisfy the identity

$$\Gamma_u \mathbf{C} = \mathbf{C}\Gamma_u^*, \tag{1.4}$$

which is exactly the definition of the so-called $\mathbf{C}$-symmetric operators, cf. [4].

As it is customary in the theory of complex symmetric operators, it will be convenient to deal with the **anti-linear** version of Hankel operators:

$$H_u f = \Gamma_u \mathbf{C} f = P(u\overline{f}), \quad f \in H^2.$$

Through the rest of the paper, we focus on anti-linear Hankel operators $H_u$; one exception is the discussion of the self-adjoint case, when it is more convenient to talk about the linear version $\Gamma_u$. Since $\mathbf{C}$ satisfies

$$\langle \mathbf{C}f, g\rangle = \langle \mathbf{C}g, f\rangle, \quad f, g \in H^2,$$

from the symmetry property (1.4) it follows that Hankel operators (in fact, all complex symmetric operators) satisfy the identity

$$\langle H_u f, g\rangle = \langle H_u g, f\rangle, \quad f, g \in H^2. \tag{1.5}$$

Note that for the anti-linear Hankel operator $H_u$ we have

$$H_u^2 = \Gamma_u \mathbf{C} \Gamma_u \mathbf{C} = \Gamma_u \mathbf{C}^2 \Gamma_u^* = \Gamma_u \Gamma_u^*;$$

thus $H_u^2$ is linear, self-adjoint and positive semi-definite. Furthermore, since the conjugation $\mathbf{C}$ commutes with the shift $S$, it follows from (1.2) that the anti-linear Hankel operators also satisfy the commutation relation

$$H_u S = S^* H_u, \tag{1.6}$$

and that any bounded anti-linear operator $H_u$ on $H^2$ satisfying this commutation relation is a Hankel operator.

By (1.6), the kernel of $H_u$ is an $S$-invariant subspace of $H^2$. It follows that $\operatorname{Ker} H_u$ is either trivial or infinite-dimensional. Furthermore, $\overline{\operatorname{Ran}} H_u$ is an invariant subspace for $S^*$.

One ot the advantages of working with the anti-linear Hankel operators $H_u$ instead of their linear counterparts $\Gamma_u$ is that $\overline{\operatorname{Ran}} H_u = (\operatorname{Ker} H_u)^\perp$. Indeed,

$$\overline{\operatorname{Ran}} H_u = \overline{\operatorname{Ran}} \Gamma_u \mathbf{C} = \overline{\operatorname{Ran}} \Gamma_u = (\operatorname{Ker} \Gamma_u^*)^\perp = (\operatorname{Ker} \mathbf{C}\Gamma_u^*)^\perp,$$

and the desired identity follows since $\mathbf{C}\Gamma_u^* = \Gamma_u \mathbf{C} = H_u$.

We will denote by $H_u^{\mathrm{e}}$ the essential part of the Hankel operator $H_u$,

$$H_u^{\mathrm{e}} := H_u|_{\overline{\operatorname{Ran}} H_u}.$$

The subspace $\overline{\operatorname{Ran}} H_u$ is invariant for $H_u$, and for any element $f \in H^2$ we have

$$H_u f = H_u^{\mathrm{e}} P_{\overline{\operatorname{Ran}} H_u} f,$$

where $P_{\overline{\operatorname{Ran}} H_u}$ is the orthogonal projection onto $\overline{\operatorname{Ran}} H_u$.

**1.6 The truncated operators $\widetilde{\Gamma}_u$ and $\widetilde{H}_u$.** Along with the Hankel operators $\Gamma_u$ and $H_u$ we will consider their truncated versions

$$\widetilde{\Gamma}_u = \Gamma_u S = S^* \Gamma_u = \Gamma_{S^* u}, \quad \widetilde{H}_u = H_u S = S^* H_u = H_{S^* u}.$$

Note that $\widetilde{\Gamma}_u$ is also a Hankel operator (with symbol $S^* u$), and its matrix is obtained from the matrix of $\Gamma_u$ by removing the first row (or the first column).

As it turns out, under the assumptions discussed below, the spectral invariants of the Hankel operators $H_u$ and $\widetilde{H}_u$, described in Proposition 1.2 below, uniquely determine the symbol $u$.

We recall that the shift operator satisfies the identities

$$S^* S = I, \quad SS^* = I - \langle \cdot, z^0 \rangle z^0,$$

where $\langle\cdot, z^0\rangle z^0$ is the rank one projection onto constant functions in $H^2$. From here and from the definition of $\widetilde{H}_u$ we get the rank one identity

$$\widetilde{H}_u^2 = H_u^2 - \langle\,\cdot\,, u\rangle u. \tag{1.7}$$

This identity is key to the whole inverse spectral theory of Hankel operators.

Similarly to $H_u^{\mathrm{e}}$, we denote by $\widetilde{H}_u^{\mathrm{e}}$ the essential part of $\widetilde{H}_u$, viz.

$$\widetilde{H}_u^{\mathrm{e}} := \widetilde{H}_u|_{\overline{\mathrm{Ran}H_u}}; \tag{1.8}$$

since $\overline{\mathrm{Ran}}H_u$ is an invariant subspace for both $H_u$ and $S^*$, it is also an invariant subspace for $\widetilde{H}_u$. We should emphasize that unlike $H_u^{\mathrm{e}}$, the operator $\widetilde{H}_u^{\mathrm{e}}$ can have a non-trivial (one-dimensional) kernel. The rank one identity (1.7) translates to

$$(\widetilde{H}_u^{\mathrm{e}})^2 = (H_u^{\mathrm{e}})^2 - \langle\,\cdot\,, u\rangle u. \tag{1.9}$$

**1.7 The simplicity of the spectrum.** Our main assumption on $H_u$ and $\widetilde{H}_u$ in this paper is

$$(H_u^{\mathrm{e}})^2 \quad \text{and} \quad (\widetilde{H}_u^{\mathrm{e}})^2 \quad \text{have simple spectra.} \tag{1.10}$$

We will denote by $\mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ the set of all $u \in \mathrm{BMOA}(\mathbb{T})$ satisfying (1.10).

**Remark.** On the one hand, it is very easy to construct examples of Hankel operators that do not satisfy this assumption: it suffices to consider self-adjoint Hankel operators with eigenvalues with multiplicity $> 1$. On the other hand, there is one important particular case when the simplicity condition (1.10) holds true. This case is most conveniently described in terms of the linear realization of Hankel operators. By [11, Theorem 2.4], if both $\Gamma_u$ and $\Gamma_{S^*u}$ are positive semi-definite, then the simplicity condition (1.10) holds.

Our first auxiliary result (proved in Section 3) is

**Theorem 1.1.** *Let $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$, i.e.,* (1.10) *holds. Then $u$ is a cyclic element for both $(H_u^{\mathrm{e}})^2$ and $(\widetilde{H}_u^{\mathrm{e}})^2$, i.e.,*

$$\langle u\rangle_{H_u^2} = \langle u\rangle_{\widetilde{H}_u^2} = \overline{\mathrm{Ran}}H_u.$$

**Remark.** In general, $\overline{\mathrm{Ran}}\widetilde{H}_u \neq \langle u\rangle_{\widetilde{H}_u^2}$. For example, if $u = 1$, then $\widetilde{H}_u = 0$ and so $\{0\} = \overline{\mathrm{Ran}}\widetilde{H}_u \neq \langle u\rangle_{\widetilde{H}_u^2} = \mathrm{span}(u)$.

**1.8 Anti-linear operators with simple spectrum of modulus.** Here we discuss a "spectral theorem" for a class of anti-linear operators that have properties mirroring those of Hankel operators. Let $A$ be a bounded anti-linear operator in a Hilbert space $X$, satisfying the identity (cf. (1.5))

$$\langle Af, g\rangle = \langle Ag, f\rangle \tag{1.11}$$

for any elements $f$ and $g$ in the Hilbert space; we will call such operators **symmetric anti-linear operators.** Then

$$\langle A^2 f, f\rangle = \langle Af, Af\rangle \geq 0,$$

and so $A^2$ is a (linear) positive semi-definite operator.

Recall that for a linear operator $T$ its **modulus** $|T|$ is defined as $|T| := (T^*T)^{1/2}$; the operator $T^*T$ is positive semi-definite, so its non-negative square root is well defined. Similarly, for an anti-linear operator $A$ satisfying (1.11) the operator $A^2$ is positive semi-definite, so the non-negative square root is well defined, and we set $|A| := (A^2)^{1/2}$; this is a linear positive semi-definite operator.

Let us assume that $A^2$ has a simple spectrum with a cyclic element $v$. Then trivially, $v$ is also a cyclic vector for $|A| := (A^2)^{1/2}$. Let $\rho = \rho_v^{|A|}$ be the scalar spectral measure for $|A|$ corresponding to the vector $v$, see (1.1). Note that $\rho$ is a finite measure with $\operatorname{supp}\rho \subset [0, \infty)$.

The spectral theorem for self-adjoint operators says that the operator $|A|$ is unitarily equivalent to the multiplication by the independent variable $s$ in $L^2(\rho)$, and the corresponding unitary operator $U : L^2(\rho) \to X$ intertwining $|A|$ and the multiplication operator is given by

$$Uf = f(|A|)v \tag{1.12}$$

(defined initially on polynomials $f$ and extended by continuity).

The statement below can be regarded as a substitute for polar decomposition of linear operators.

**Proposition 1.2** (Spectral Theorem for symmetric anti-linear operators)**.** *Let $A$ be a bounded symmetric anti-linear operator in a Hilbert space. Assume that $|A|$ has a simple spectrum with a cyclic element $v$, and let $\rho = \rho_v^{|A|}$. Then there exists a unimodular Borel function $\psi$ such that the operator $A$ is unitariy equivalent to its model $\mathcal{A}$ in $L^2(\rho)$,*

$$\mathcal{A}f(s) = s\psi(s)\overline{f(s)}, \quad f \in L^2(\rho), \tag{1.13}$$

*where the unitary operator $U : L^2(\rho) \to X$, $AU = U\mathcal{A}$ is given by* (1.12).

The proof is given in Section 3.

**Remark 1.3.** It will be seen from the proof of the proposition that the function $\psi$ is uniquely defined as an element of $L^\infty(\rho_0)$, where $\rho_0$ is the restriction of the measure $\rho$ to $(0, \infty)$. Note that $\rho_0$ differs from $\rho$ if and only if $\rho$ has an atom at 0. On the other hand, it is clear that the value $\psi(0)$ is of no importance for the action of $\mathcal{A}$.

**Remark.** One can see from the definition (1.12) of the unitary operator $U$ that

$$X = \{f(|A|)v \ : f \in L^2(\rho)\};$$

while the operators $f(|A|)$ can be unbounded, the vector $v$ is always in the domain of $f(|A|)$ for $f \in L^2(\rho)$. Thus, we can rewrite the representation (1.13) for the model $\mathcal{A}$ as an abstract representation for $A$,

$$Af(|A|)v = |A|\psi(|A|)\overline{f}(|A|)v. \tag{1.14}$$

**1.9 Direct spectral problem: spectral measures and unimodular functions.** Let $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$, i.e., (1.10) is satisfied. Let us apply Proposition 1.2 to the anti-linear operators $H_u^{\mathrm{e}}$ and $\widetilde{H}_u^{\mathrm{e}}$; we will use the same cyclic vector $v = u$ in both cases.

For the operator $H_u^{\mathrm{e}}$ we get its spectral measure $\rho = \rho_u^{|H_u^{\mathrm{e}}|}$; note that since $u \in \operatorname{Ran} H_u$, we have $\rho_u^{|H_u^{\mathrm{e}}|} = \rho_u^{|H_u|}$; we will use the notation $\rho_u^{|H_u|}$ for typographical reasons. We also get the unitary operator $U : L^2(\rho) \to \overline{\operatorname{Ran}} H_u^2$ given by (1.12) with $A = H_u^{\mathrm{e}}$ and $v = u$,

$$Uf = f(|H_u^{\mathrm{e}}|)u, \quad f \in L^2(\rho), \tag{1.15}$$

so

$$U^*|H_u^{\mathrm{e}}|U = \mathcal{M},$$

where $\mathcal{M}$ is the operator of multiplication by the independent variable $s$ in $L^2(\rho)$. By Proposition 1.2 we have

$$[U^* H_u^{\mathrm{e}} U]f(s) = \overline{\Psi_u(s)}s\overline{f(s)}, \quad f \in L^2(\rho) \tag{1.16}$$

where $\Psi_u$ is a complex-valued unimodular Borel function; we write $\overline{\Psi}_u$ rather than $\Psi_u$ in the above formula for consistency of notation with [7].

Similarly, defining the spectral measure $\widetilde{\rho} = \rho_u^{|\widetilde{H}_u|}$ (again, it coincides with the spectral measure of the operator $|\widetilde{H}_u^{\mathrm{e}}|$) and the unitary operator $\widetilde{U} : L^2(\widetilde{\rho}) \to \overline{\operatorname{Ran}} H_u^2$ by

$$\widetilde{U}f = f(|\widetilde{H}_u^{\mathrm{e}}|)u, \quad f \in L^2(\widetilde{\rho}),$$

we get that

$$[\widetilde{U}^* \widetilde{H}_u^{\mathrm{e}} \widetilde{U}] f(s) = \widetilde{\Psi}_u(s) s \overline{f(s)}, \quad f \in L^2(\widetilde{\rho}), \tag{1.17}$$

where $\widetilde{\Psi}_u$ is a Borel unimodular function.

To summarize: we have two measures $\rho$, $\widetilde{\rho}$ and two unimodular functions $\Psi_u$ and $\widetilde{\Psi}_u$ as spectral characteristics of the Hankel operator $H_u$.

Since the measure $\rho$ does not have an atom at 0, by Remark 1.3 the function $\Psi_u$ is unique as an element of $L^\infty(\rho)$. However, the measure $\widetilde{\rho}$ can have an atom at 0, so we can only say that $\widetilde{\Psi}_u$ is unique as an element of $L^\infty(\widetilde{\rho}_0)$, where $\widetilde{\rho}_0$ is the restriction of $\widetilde{\rho}$ to $(0, \infty)$. Also, one can see from (1.17) that the value $\widetilde{\Psi}_u(0)$ does not matter for the action of $\widetilde{H}_u^{\mathrm{e}}$, so we can assume that $\widetilde{\Psi}_u$ is unique in $L^\infty(\widetilde{\rho})$.

**1.10 Remarks about the measures $\rho$ and $\widetilde{\rho}$.** The measure $\rho$ must satisfy

$$\int_0^\infty \frac{d\rho(s)}{s^2} \le 1. \tag{1.18}$$

Indeed, we know that

$$u = H_u z^0 = H_u^{\mathrm{e}} P_{\overline{\operatorname{Ran} H_u}} z^0, \quad U^* u = \mathbb{1},$$

so the representation (1.16) implies that $U^*$ maps the vector $P_{\overline{\operatorname{Ran} H_u}} z^0$ to the function $q \in L^2(\rho)$, $q(s) = \overline{\Psi_u(s)}/s$. Since $\|P_{\overline{\operatorname{Ran} H_u}} z^0\|_{H^2} \le \|z^0\|_{H^2} = 1$, we conclude that $\|q\|_{L^2(\rho)} \le 1$, which is exactly the estimate (1.18).

The measures $\rho$ and $\widetilde{\rho}$ are not independent, and $\widetilde{\rho}$ is uniquely defined by $\rho$. To explain this, we introduce two important operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ in $L^2(\rho)$ that will play a key role in our construction below. We have already defined $\mathcal{M}$ in the previous subsection; this is the multiplication operator by the independent variable $s$ in $L^2(\rho)$. Now consider the operator

$$\mathcal{M}^2 - \langle \cdot, \mathbb{1} \rangle \mathbb{1} = \mathcal{M}(I - \langle \cdot, q_0 \rangle q_0)\mathcal{M},$$

where $q_0(s) = 1/s$. The inequality (1.18) implies that $\|q_0\|_{L^2(\rho)} \le 1$, so the above operator is trivially non-negative. Let us consider its (non-negative) square root

$$\widetilde{\mathcal{M}} := (\mathcal{M}^2 - \langle \cdot, \mathbb{1} \rangle \mathbb{1})^{1/2}. \tag{1.19}$$

The definition of $\widetilde{\mathcal{M}}$ can be equivalently rewritten as

$$\widetilde{\mathcal{M}}^2 = \mathcal{M}^2 - \langle \cdot, \mathbb{1} \rangle \mathbb{1},$$

which mirrors the rank one identity (1.9).

We can easily see that the unitary equivalence $U$ maps the triple $(\mathcal{M}, \widetilde{\mathcal{M}}, \mathbb{1})$ to the triple $(|H_u^{\mathrm{e}}|, |\widetilde{H}_u^{\mathrm{e}}|, u)$, so $\widetilde{\rho}$ is the spectral measure of the operator $\widetilde{\mathcal{M}}$ with respect to the vector $\mathbb{1} \in L^2(\rho)$. Thus $\widetilde{\rho}$ is uniquely determined by $\rho$.

**1.11 The spectral data and Uniqueness.** To conclude, with each Hankel operator $H_u$ with $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ we associate the following **spectral datum:**

(i) The measure $\rho$ with bounded support on $(0, \infty)$ satisfying the normalization (1.18) (and the measure $\widetilde{\rho}$ on $[0, \infty)$, uniquely defined by $\rho$ as described above in Section 1.10).

(ii) Two unimodular functions $\Psi_u \in L^\infty(\rho)$ and $\widetilde{\Psi}_u \in L^\infty(\widetilde{\rho}_0)$, where $\widetilde{\rho}_0 := \widetilde{\rho}|_{(0,\infty)}$; the functions $\Psi_u$ and $\widetilde{\Psi}_u$ are unique as vectors in the corresponding $L^\infty$ spaces.

So, formally speaking the spectral datum for $u$ (equivalently $H_u$) is given by the triple

$$\Lambda(u) := (\rho, \Psi_u, \widetilde{\Psi}_u).$$

We do not include the measure $\widetilde{\rho}$ in the spectral data because $\widetilde{\rho}$ is determined by $\rho$, as explained in the previous subsection.

Our first main result is

**Theorem 1.4** (Uniqueness)**.** *Any symbol $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ is uniquely determined by the spectral datum $\Lambda(u)$, i.e., the spectral map*

$$\mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T}) \ni u \mapsto \Lambda(u) = (\rho, \Psi_u, \widetilde{\Psi}_u) \tag{1.20}$$

*is injective.*

Moreover, we will give an explicit formula for the symbol $u$ in terms of the spectral datum, see (2.8) and (2.9) below. The proof of Theorem 1.4 is given in Section 3.

Recall that $\operatorname{Ker} H_u$ is either trivial or infinite-dimensional. It turns out that one can easily distinguish between these two cases by looking at the spectral data.

**Theorem 1.5** (Triviality of kernel)**.** *For $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$, we have $\operatorname{Ker} H_u = \{0\}$ if and only if*

$$\int \frac{d\rho(s)}{s^2} = 1 \quad \text{and} \quad \int \frac{d\rho(s)}{s^4} = \infty. \tag{1.21}$$

This theorem was proved in [8, Theorem 4]. More precisely, in [8], it was stated for compact $H_u$ and in slightly different terms, but the idea of the proof remains the same. For the case of self-adjoint Hankel operators it also appeared earlier in [21, Theorem III.2.1]; a similar dynamical systems approach also works in the general case.

For completeness we give a proof in the Appendix B. We note that the first condition in (1.21) is equivalent to $z^0 \in \overline{\operatorname{Ran} H_u}$, and the second one is equivalent to $z^0 \notin \operatorname{Ran} H_u$, see the proof.

**1.12 The self-adjoint case.** Here we discuss the interesting special case when the linear Hankel operator $\Gamma_u$ is self-adjoint. Evidently, $\Gamma_u$ is self-adjoint if and only if all Fourier coefficients $\widehat{u}_j$ are real; if $\Gamma_u$ is self-adjoint, then so is $\widetilde{\Gamma}_u$.

**Theorem 1.6.** *Let $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$; then $\Gamma_u$ is self-adjoint if and only if both $\Psi_u$ and $\widetilde{\Psi}_u$ are functions with values $\pm 1$.*

Moreover, in the self-adjoint case formulas (1.16), (1.17) for the action of $H_u$ and $\widetilde{H}_u$ can be interpreted as polar decompositions of $\Gamma_u$ and $\widetilde{\Gamma}_u$. In order to state this precisely, we recall the relevant key definitions and facts.

For a bounded operator $T$ on a Hilbert space, there exists a unique partial isometry $\Phi$ with the initial subspace $\overline{\mathrm{Ran}T^*}$ and the final subspace $\overline{\mathrm{Ran}T}$ such that the **polar decomposition** $T = \Phi|T|$ holds, where $|T| = \sqrt{T^*T}$. If $T$ is self-adjoint, then $\Phi$ is also self-adjoint and commutes with $|T|$. Furthermore, if the spectrum of $|T|$ is simple, then one can write $\Phi = \varphi(|T|)$, where $\varphi$ is a Borel function with values $\pm 1$. The function $\varphi$ is uniquely defined up to values on sets of measure zero with respect to the spectral measure of $|T|$. One can also write $\Phi = \varphi(|T|)$ if $T$ has a multi-dimensional kernel but the spectrum of the restriction $|T||_{\overline{\mathrm{Ran}T}}$ is simple; in this case one must set $\varphi(0) = 0$.

We apply this to the case $T = \Gamma_u$ or $T = \widetilde{\Gamma}_u$; note that in this case $|\Gamma_u| = |H_u|$ and $|\widetilde{\Gamma}_u| = |\widetilde{H}_u|$.

**Theorem 1.7.** *Let $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ be such that then $\Gamma_u$ is self-adjoint. Then the polar decompositions of $\Gamma_u$ and $\widetilde{\Gamma}_u$ can be written as*

$$\Gamma_u = \Psi_u(|\Gamma_u|)|\Gamma_u|, \quad \widetilde{\Gamma}_u = \widetilde{\Psi}_u(|\widetilde{\Gamma}_u|)|\widetilde{\Gamma}_u|, \tag{1.22}$$

*where one should set $\Psi_u(0) = \widetilde{\Psi}_u(0) = 0$ in case of non-trivial kernels.*

The proofs of the above two theorems are given in Section 3.

**1.13 What can be said about the case of non-trivial spectral multiplicity?** We conclude this section with remarks on the case when the simplicity assumption (1.10) is not satisfied. What would be the natural choice for the spectral datum in this case?

This question was answered in [7] for the case of compact Hankel operators $H_u$. Observe that in this case, the measure $\rho$ is purely atomic, supported on the set of singular values of $H_u$. The spectral datum is still the triple $(\rho, \Psi_u, \widetilde{\Psi}_u)$, but the functions $\Psi_u$ and $\widetilde{\Psi}_u$ (defined on the set of singular values of $H_u$ and $\widetilde{H}_u$ respectively) are no longer scalar-valued but take values in the set of all finite Blaschke products.

In [7] it is proved that the spectral map, defined in a suitable way, is injective and surjective.

Another case was considered in [12]: all Hankel operators $H_u$ such that the spectrum of $|H_u|$ is finite. In a similar spirit, the spectral datum is the triple $(\rho, \Psi_u, \widetilde{\Psi}_u)$, where $\Psi_u$ and $\widetilde{\Psi}_u$ are functions from the spectrum of $|H_u|$ and $|\widetilde{H}_u|$ into the set of all inner functions, and the spectral map was proved to be injective and surjective.

As for the general case, in [18] an abstract approach to the inverse spectral problem for general Hankel operators was considered. The abstract spectral datum there is similar in spirit to what is presented here, but the values of functions $\Psi_u$ and $\widetilde{\Psi}_u$ are unitary operators. In addition, a special anti-linear conjugation $\mathbf{J}$, commuting with both $|H_u|$ and $|\widetilde{H}_u|$ (which is implicit in this paper), is also a part of the spectral datum. The spectral map is injective, if one treats the spectral data as natural equivalence classes. And similarly to the present paper, the abstract spectral datum corresponds to a Hankel operator if and only if an appropriately constructed operator is asymptotically stable.

For the case of compact operators, the (non-trivial) translation from the language used in [18] to the description in [12] was provided in [18].

It is likely that the constructions of [7] and [12] can be combined to give a description of a spectral map in the case when $|H_u|$ has only a point spectrum. It could also be possible to use the ideas from [18] to extend the result to the case of a purely singular spectrum.

However, the fundamental question of transparent representation of the spectral data in the general case when $|H_u|$ has non-trivial absolutely continuous spectrum and non-trivial multiplicity remains a mystery.

## 2 The problem of surjectivity

**2.1 The abstract spectral data and the problem of surjectivity.** First let us discuss

**Question.** What is the natural target space for the spectral map (1.20)?

Below we describe the set of triples $(\rho, \Psi, \widetilde{\Psi})$, that we call the abstract spectral data, that plays the role of the target space.

Let $\rho$ be a finite Borel measure with a bounded support on $(0, \infty)$, satisfying the normalization condition (1.18). We then define the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ in $L^2(\rho)$ exactly as explained in Section 1.10, i.e., $\mathcal{M}$ is the multiplication by the independent variable and

$$\widetilde{\mathcal{M}} := (\mathcal{M}^2 - \langle\,\cdot\,, \mathbb{1}\rangle\mathbb{1})^{1/2}.$$

Let $\widetilde{\rho}$ be the spectral measure of $\widetilde{\mathcal{M}}$, corresponding to the vector $\mathbb{1}$. Picking two unimodular functions $\Psi \in L^\infty(\rho)$ and $\widetilde{\Psi} \in L^\infty(\widetilde{\rho}_0)$ (where $\widetilde{\rho}_0 := \widetilde{\rho}|_{(0,\infty)}$), we get the triple

$$\Lambda = (\rho, \Psi, \widetilde{\Psi}),$$

which we will call the **abstract spectral datum**: the word abstract here emphasizes the fact that this datum a priori does not have to come from a Hankel operator. The set of all abstract spectral data is the natural target space for the spectral map (1.20).

We arrive at the main problem addressed in this paper:

**Question.** Is the spectral map (1.20) surjective?

In other words, does every abstract spectral datum come from a Hankel operator?

For several years, the authors of this paper believed that the answer is "yes". For example, as it was proved in [8], the answer is affirmative in the case of compact Hankel operators: in this case the measure $\rho$ is a purely atomic measure with 0 being the only possible accumulation point of its support.

The other case is the so-called **double positive** case, treated in [11], where both operators $\Gamma_u$ and $\widetilde{\Gamma}_u$ are non-negative self-adjoint operators. It was shown in [11, Theorem 2.4] that in this case the simplicity condition (1.10) is satisfied. In this case both unimodular functions $\Psi_u$ and $\widetilde{\Psi}_u$ are identically equal to 1. It was also shown in [11] that in this case any abstract spectral datum (i.e. any measure $\rho$ satisfying the normalization condition (1.18)) comes from a self-adjoint Hankel operator $\Gamma_u$.

**2.2 Informal description of main results.** In order to simplify our discussion, we introduce the following notation. For an abstract spectral datum $\Lambda_* = (\rho, \Psi, \widetilde{\Psi})$, we will write $\Lambda_* \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$, if $\Lambda_*$ is in the range of the spectral map (1.20), i.e., if $\Lambda_*$ is the spectral datum of some Hankel operator $H_u$.

Here we informally describe our main results.

- The spectral map (1.20) is NOT surjective, i.e., there are abstract spectral data with $\Lambda_* \notin \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

We do not have a simple easy-to-check criterion for an abstract spectral datum to be in $\Lambda(\mathrm{BMOA}_{\mathrm{simp}})$, but we come close to it.

- For an abstract spectral datum $\Lambda_*$, we have $\Lambda_* \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$ if and only if a certain contraction $\Sigma^*$, constructed from $\Lambda_*$, is asymptotically stable (i.e., $(\Sigma^*)^n \to 0$ strongly as $n \to \infty$). See Theorem 2.1 for the precise statement.

The asymptotic stability of $\Sigma^*$ is not easy to check. However, in many cases we can reduce it to a more explicit condition.

- Under some mild additional assumptions (e.g., $\Psi$ and $\widetilde{\Psi}$ are Hölder continuous at 0), we have $(\rho, \Psi, \widetilde{\Psi}) \in \Lambda(\mathrm{BMOA_{simp}})$ if and only if the unitary operator
$$\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$$
has a purely singular spectrum. Here $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the operators in $L^2(\rho)$ defined in the previous subsection.
- Using the previous result, we construct a wide range of examples of spectral data that are (or are not) in $\Lambda(\mathrm{BMOA_{simp}})$.

**2.3 Introducing the model** $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$**.** Let $u \in \mathrm{BMOA_{simp}}(\mathbb{T})$. Restricting the identity $\widetilde{H}_u = S^*H_u$ to the $S^*$-invariant subspace $\overline{\mathrm{Ran} H_u}$ we write

$$\widetilde{H}_u^{\mathrm{e}} = \left(S^*|_{\overline{\mathrm{Ran} H_u}}\right) H_u^{\mathrm{e}}. \tag{2.1}$$

Recall also the rank one identity (1.9). Let us map these identities to $L^2(\rho)$, where $\rho = \rho_u^{|H_u|}$, by using the unitary operator $U$ defined in (1.15). In order to do this, let us define the anti-linear operators $\mathcal{H}$, $\widetilde{\mathcal{H}}$ and the (linear) contraction $\Sigma$ in $L^2(\rho)$ by

$$\begin{aligned} \mathcal{H} &= U^*H_u^{\mathrm{e}}U, \\ \widetilde{\mathcal{H}} &= U^*\widetilde{H}_u^{\mathrm{e}}U, \\ \Sigma^* &= U^*\left(S^*|_{\overline{\mathrm{Ran} H_u}}\right)U, \qquad \Sigma := (\Sigma^*)^*. \end{aligned} \tag{2.2}$$

Multiplying (2.1) and (1.9) by $U^*$ on the left and by $U$ on the right, we obtain the identities

$$\widetilde{\mathcal{H}} = \Sigma^*\mathcal{H}, \tag{2.3}$$

$$\widetilde{\mathcal{H}}^2 = \mathcal{H}^2 - \langle \cdot, \mathbb{1}\rangle \mathbb{1} \tag{2.4}$$

in $L^2(\rho)$.

The triple $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$ is our model for $(H_u^{\mathrm{e}}, \widetilde{H}_u^{\mathrm{e}}, S^*|_{\overline{\mathrm{Ran} H_u}})$; this model plays a central role in our construction.

Rewriting (1.16), (1.17) in terms of the model operators $\mathcal{H}$, $\widetilde{\mathcal{H}}$, we obtain

$$\mathcal{H}f = \mathcal{M}\overline{\Psi}(\mathcal{M})\overline{f}, \quad f \in L^2(\rho), \tag{2.5}$$

$$\widetilde{\mathcal{H}}f = \widetilde{\mathcal{M}}\widetilde{\Psi}(\widetilde{\mathcal{M}})\overline{f}, \quad f \in L^2(\rho), \tag{2.6}$$

where $\Psi = \Psi_u$, $\widetilde{\Psi} = \widetilde{\Psi}_u$ and the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are as discussed in Section 1.10.

**2.4 Model coming from abstract spectral data.** One can also set up a triple $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$ starting from an abstract spectral datum $\Lambda = (\rho, \Psi, \widetilde{\Psi})$. We define the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ as described in Section 2.1 and define the anti-linear operators $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ by (2.5) and (2.6). In order to define $\Sigma$, we first note that $\widetilde{\mathcal{M}}^2 \leq \mathcal{M}^2$, i.e.,

$$\|\widetilde{\mathcal{M}}f\| \leq \|\mathcal{M}f\|, \quad \forall f \in L^2(\rho),$$

and therefore (see Douglas' lemma in Section 4) the operator $\widetilde{\mathcal{M}}\mathcal{M}^{-1}$, defined initially on the dense set $\operatorname{Ran}\mathcal{M}$, extends to $L^2(\rho)$ as a contraction. We then define the contraction

$$\Sigma^* = \widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}\mathcal{M}^{-1}\Psi(\mathcal{M}) \tag{2.7}$$

and set $\Sigma = (\Sigma^*)^*$. With these definitions, the key identities (2.3) and (2.4) are satisfied.

**2.5 Surjectivity: reduction to the asymptotic stability of $\Sigma^*$.** Let $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be an abstract spectral datum, and let $\Sigma^*$ be as defined in (2.7). Our first main result concerning surjectivity is

**Theorem 2.1.** *The triple $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ is the spectral datum for some Hankel operator $H_u$ with $u \in \mathrm{BMOA}_{\mathrm{simp}}$ if and only if $\Sigma^*$ is asymptotically stable (i.e., $\Sigma^{*n} \to 0$ in the strong operator topology).*

**2.6 Explicit formula for the symbol.** One can give an explicit formula for the symbol $u$ in terms of the corresponding operator $\Sigma^*$, defined via the spectral data of $u$. The following statement is logically part of the uniqueness Theorem 1.4, but we place it here because it was convenient for us to state the uniqueness theorem before describing the model $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$.

**Theorem 2.2.** *Let $u \in \mathrm{BMOA}_{\mathrm{simp}}$, and let $\Sigma^*$ be defined by* (2.2)*. Then $u$ can be found through the explicit formula*

$$\widehat{u}_k = \langle (\Sigma^*)^k \mathbb{1}, q \rangle_{L^2(\rho)}, \quad k \geq 0, \tag{2.8}$$

*or equivalently*

$$u(z) = \langle (I - z\Sigma^*)^{-1} \mathbb{1}, q \rangle_{L^2(\rho)}, \quad z \in \mathbb{D}, \tag{2.9}$$

*where $q \in L^2(\rho)$, $q(s) := \overline{\Psi(s)}/s$.*

## 3 Proofs of preliminary results

**3.1 Proof of the "spectral theorem" (Proposition 1.2).** Let $\mathcal{A}$ be the anti-linear operator in $L^2(\rho)$, defined by $\mathcal{A} := U^*AU$, where $U$ is the unitary operator defined by (1.12). Then, since $A$ trivially commutes with $|A|^2 = A^2$ and $U^*|A|U = \mathcal{M}$, where $\mathcal{M}$ is the multiplication by the independent variable $s$ in $L^2(\rho)$, we conclude that $\mathcal{A}$ commutes with $\mathcal{M}^2$ and so with $\mathcal{M}$.

Denote by $\mathfrak{C}$ the standard conjugation acting on functions on $\mathbb{R}$,

$$\mathfrak{C}f(s) = \overline{f(s)},$$

and define the linear operator $\mathcal{B}$ in $L^2(\rho)$ as $\mathcal{B} := \mathfrak{C}\mathcal{A}$. Since $\mathfrak{C}$ commutes with $\mathcal{M}$, we find that $\mathcal{B}$ also commutes with $\mathcal{M}$. Therefore, $\mathcal{B}$ is the multiplication by a function $g \in L^\infty(\rho)$,

$$\mathcal{B}f = gf \quad \forall f \in L^2(\rho);$$

note that $g$ as an element of $L^\infty(\rho)$ is unique.

For any $f \in L^2(\rho)$

$$\|gf\|^2_{L^2(\rho)} = \|\mathcal{B}f\|^2_{L^2(\rho)} = \|\mathcal{A}f\|^2_{L^2(\rho)} = \langle \mathcal{A}^2 f, f\rangle = \langle \mathcal{M}^2 f, f\rangle = \int_{\mathbb{R}} s^2 |f(s)|^2 \mathrm{d}\rho(s);$$

the second equality holds because $\mathfrak{C}$ preserves the norm. So we conclude that $|g(s)|^2 = s^2$ $\rho$-a.e., therefore it can be represented as

$$g(s) = s\overline{\psi(s)},$$

where $\psi$ is a unimodular function, i.e., $|\psi(s)| = 1$ $\rho$-a.e. (the reason for the complex conjugation is purely notational, and will be clear in a moment).

Using the fact that $\mathcal{A} = \mathfrak{C}\mathcal{B}$ we conclude that

$$\mathcal{A}f(s) = \overline{g(s)f(s)} = \overline{s\overline{\psi(s)}f(s)} = s\psi(s)\overline{f(s)},$$

which is exactly the conclusion of the proposition. □

**3.2 Cyclicity of $u$: preliminaries.** To prove Theorem 1.1 we start with a trivial observation.

**Lemma 3.1.** *Let $R = R^*$ be a bounded self-adjoint operator, and let $R_\alpha := R + \alpha\langle\,\cdot\,, p\rangle p$, $\alpha \in \mathbb{R}$, be its rank one perturbation. Then*

(i) *There holds $\langle p\rangle_R = \langle p\rangle_{R_\alpha}$.*

(ii) *If both $R$ and $R_\alpha$ have a simple spectrum, then there exists a vector $v_2 \in \langle p\rangle_R^\perp$ such that the vector $v = p + v_2$ a cyclic for both $R$ and $R_\alpha$.*

**Proof.** The first statement is easy: by induction we find

$$R_\alpha^n p \in \operatorname{span}\{R^k p : 0 \le k \le n\},$$

which implies the inclusion $\langle p\rangle_{R_\alpha} \subset \langle p\rangle_R$ for all $\alpha \in \mathbb{R}$. Since $R = R_\alpha - \alpha\langle\,\cdot\,, p\rangle p$, the converse inclusion follows.

By the statement (i), the subspace $\langle p\rangle_R$ is an invariant subspace for both $R$ and $R_\alpha$; since both operators are self-adjoint it is in fact reducing for both. Furthemore, the action of the operators $R$ and $R_\alpha$ coincide on $\langle p\rangle_R^\perp$. Now it remains to take $v = p + v_2$, where $v_2$ is a cyclic vector for $R\,|\langle p\rangle_R^\perp$. □

In the proof of Theorem 1.1 we use the model $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$ introduced in Section 2.3. Note, however, that in Section 2.3 we have used the fact that $u$ is cyclic for $H_u^{\mathrm{e}}$ and $\widetilde{H}_u^{\mathrm{e}}$. In order to avoid a circular argument, here we start the proof by setting up a slight modification of the same model, using another cyclic vector, which exists by Lemma 3.1.

Thus, let $v = u + v_2 \in \overline{\operatorname{Ran}} H_u$ be the cyclic vector for both $(H_u^{\mathrm{e}})^2$ and $(\widetilde{H}_u^{\mathrm{e}})^2$, which exists by statement (ii) (with $p = u$) of Lemma 3.1. Let $\rho = \rho_v^{|H_u|}$ be the spectral measure of $|H_u^{\mathrm{e}}|$ corresponding to $v$. Let $U$ be the unitary operator given by (1.12) with $A = H_u^{\mathrm{e}}$,

$$Uf := f(|H_u^{\mathrm{e}}|)v, \quad f \in L^2(\rho).$$

As in Section 2.3, we define the operators $\mathcal{H}$, $\widetilde{\mathcal{H}}$ and $\Sigma$ in $L^2(\rho)$ by

$$\begin{aligned} \mathcal{H} &= U^* H_u U, \\ \widetilde{\mathcal{H}} &= U^* \widetilde{H}_u^{\mathrm{e}} U, \\ \Sigma^* &= U^* \left( S^*|_{\overline{\operatorname{Ran}} H_u} \right) U, \\ \Sigma &:= (\Sigma^*)^*. \end{aligned}$$

For these operators, from the definition of $\widetilde{H}_u$ and from the rank one identity (1.9) we obtain

$$\widetilde{\mathcal{H}} = \Sigma^* \mathcal{H} = \mathcal{H}\Sigma, \tag{3.1}$$

$$\widetilde{\mathcal{H}}^2 = \mathcal{H}^2 - \langle\,\cdot\,, p\rangle p, \tag{3.2}$$

where $p = U^* u$. Note that $U^* v = \mathbb{1}$, so $U^* u = \chi_E$ for some Borel set $E \subset \sigma(|H_u|)$; what will be essential here is that both $U^* v$ and $U^* u$ are real-valued.

**3.3 Proof of Theorem 1.1.** Step 1. The action of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ on $L^2(\rho)$.

By construction, $v$ is a cyclic element for both $|H_u^{\mathrm{e}}|$ and $|\widetilde{H}_u^{\mathrm{e}}|$, thus $\mathbb{1} = U^* v$ is a cyclic element for both $|\mathcal{H}|$ and $|\widetilde{\mathcal{H}}|$. By Proposition 1.2, we find

$$\mathcal{H} f(|\mathcal{H}|)\mathbb{1} = |\mathcal{H}|\psi(|\mathcal{H}|)\overline{f}(|\mathcal{H}|)\mathbb{1},$$
$$\widetilde{\mathcal{H}} f(|\widetilde{\mathcal{H}}|)\mathbb{1} = |\widetilde{\mathcal{H}}|\widetilde{\psi}(|\widetilde{\mathcal{H}}|)\overline{f}(|\widetilde{\mathcal{H}}|)\mathbb{1}$$

for some unimodular functions $\psi$ and $\widetilde{\psi}$ and for all admissible $f$ (i.e., $f \in L^2(\rho)$ for the first identity and $f(|\widetilde{\mathcal{H}}|)\mathbb{1} \in L^2(\rho)$ for the second one).

Step 2. Conjugations on $L^2(\rho)$.

Since $|\mathcal{H}|$ coincides with the operator $\mathcal{M}$ of multiplication by the independent variable in $L^2(\rho)$, we find that

$$\overline{g} = \overline{f}(|\mathcal{H}|)\mathbb{1}, \quad \text{if } g = f(|\mathcal{H}|)\mathbb{1}$$

for any admissible $f$. Further, if $f(x) = x^{2n}$, using the fact that $p$ is real-valued, from (3.2) we find that

$$\overline{g} = \overline{f}(|\widetilde{\mathcal{H}}|)\mathbb{1}, \quad \text{if } g = f(|\widetilde{\mathcal{H}}|)\mathbb{1}. \tag{3.3}$$

Taking linear combinations and using an approximation argument, we obtain (3.3) for all admissible $f$. To conclude, combining with the previous step, we find that

$$\mathcal{H} g = |\mathcal{H}|\psi(|\mathcal{H}|)\overline{g}, \quad \widetilde{\mathcal{H}} g = |\widetilde{\mathcal{H}}|\widetilde{\psi}(|\widetilde{\mathcal{H}}|)\overline{g}$$

for all $g \in L^2(\rho)$.

Step 3. The action of $\Sigma^*$ in $L^2(\rho)$.

Recall that $\langle p\rangle_{\mathcal{H}^2}^{\perp}$ is an invariant (in fact, reducing) subspace for both operators $|\mathcal{H}|$ and $|\widetilde{\mathcal{H}}|$ and the actions of these operators coincide on this subspace. Thus, for all $g \in \langle p\rangle_{\mathcal{H}^2}^{\perp}$ we have

$$\mathcal{H} g = |\mathcal{H}|\psi(|\mathcal{H}|)\overline{g}, \quad \widetilde{\mathcal{H}} g = |\mathcal{H}|\widetilde{\psi}(|\mathcal{H}|)\overline{g}.$$

By (3.1), we find that $\langle p\rangle_{\mathcal{H}^2}^{\perp}$ is an invariant subspace for $\Sigma^*$ and the action of $\Sigma^*$ on this subspace reduces to the multiplication by a unimodular function:

$$\Sigma^* g = \widetilde{\psi}(|\mathcal{H}|)\overline{\psi}(|\mathcal{H}|)g, \quad g \in \langle p\rangle_{\mathcal{H}^2}^{\perp}.$$

It follows that

$$\|(\Sigma^*)^n g\|_{L^2(\rho)} = \|g\|_{L^2(\rho)}, \quad g \in \langle p\rangle_{\mathcal{H}^2}^{\perp}$$

for all $n \geq 0$. On the other hand, $\Sigma^*$ is unitarily equivalent to the restriction of $S^*$ onto its invariant subspace $\overline{\operatorname{Ran} H_u}$, and we know that $(S^*)^n \to 0$ in the strong operator topology as $n \to \infty$. It follows that $(\Sigma^*)^n \to 0$ in the strong operator topology. We have arrived at a contradiction. $\square$

**3.4 Proof of Theorems 1.4 (uniqueness) and 2.2 (formula for $u$).** Throughout this section, we fix $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ and the corresponding Hankel operator $H_u$, and set $\rho = \rho_u^{|H_u|}$. We use the model $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma)$ of Section 2.3.

We first recall that by the definition (1.15) of $U$, we have $U^* u = \mathbb{1}$. Further, as discussed in Section 1.10, we have $u = H_u^{\mathrm{e}} P_{\overline{\mathrm{Ran} H_u}} z^0$ and

$$U^* P_{\overline{\mathrm{Ran} H_u}} z^0 = q, \qquad q(s) = \overline{\Psi_u(s)}/s.$$

Thus, for $k \geq 0$ we have

$$\begin{aligned} \widehat{u}_k &= \langle u, z^k \rangle = \langle u, S^k z^0 \rangle = \langle (S^*)^k u, z^0 \rangle = \langle (S^*|_{\overline{\mathrm{Ran} H_u}})^k u, P_{\overline{\mathrm{Ran} H_u}} z^0 \rangle \\ &= \langle (\Sigma^*)^k U^* u, U^* P_{\overline{\mathrm{Ran} H_u}} z^0 \rangle = \langle (\Sigma^*)^k \mathbb{1}, q \rangle. \end{aligned}$$

Since all objects in the right-hand side are defined in terms of the spectral datum $\Lambda(u) = (\rho, \Psi_u, \widetilde{\Psi}_u)$, the injectivity of the map $u \mapsto \Lambda(u)$ is proved. The proof of Theorem 1.4 is complete. □

Multiplying both sides of (3.4) by $z^k$ and summing over $k \geq 0$ we get an explicit formula for $u$,

$$u(z) = \langle (I - z\Sigma^*)^{-1} \mathbb{1}, q \rangle_{L^2(\rho)}, \qquad z \in \mathbb{D}.$$

The proof of Theorem 2.2 is complete. □

**3.5 Self-adjoint case: proof of Theorems 1.6 and 1.7.** First let us assume that $\Gamma_u$ is self-adjoint, i.e., that all coefficients $\widehat{u}_j$ are real. We will prove that both $\Psi_u$ and $\widetilde{\Psi}_u$ take values $\pm 1$ and the polar decomposition (1.22) holds.

Let us rewrite (1.14) for $A = H_u|_{\overline{\mathrm{Ran} H_u}}$, $v = u$ in terms of the linear realization $\Gamma_u$. Since $\Gamma_u = \Gamma_u^*$, we have $|H_u| = |\Gamma_u^*| = |\Gamma_u|$, and $H_u = \mathbf{C}\Gamma_u = \Gamma_u \mathbf{C}$ (so $\Gamma_u$ commutes with the conjugation $\mathbf{C}$ defined in (1.3)).

Noticing that $\overline{\mathrm{Ran}\Gamma_u} = \overline{\mathrm{Ran} H_u}$ is $\Gamma_u$-invariant, and defining

$$\Gamma_u^{\mathrm{e}} := \Gamma_u|_{\overline{\mathrm{Ran}\Gamma_u}} = \Gamma_u|_{\overline{\mathrm{Ran} H_u}},$$

we can rewrite (1.14) for $A = H_u^{\mathrm{e}} := H_u|_{\overline{\mathrm{Ran} H_u}}$, with $v = u$ as

$$\Gamma_u^{\mathrm{e}} \mathbf{C} f(|\Gamma_u^{\mathrm{e}}|) u = |\Gamma_u^{\mathrm{e}}| \overline{\Psi}_u(|\Gamma_u^{\mathrm{e}}|) \overline{f}(|\Gamma_u|) u, \quad f \in L^2(\rho). \tag{3.4}$$

We have $\mathbf{C}u = u$ and therefore $\mathbf{C} f(|\Gamma_u^{\mathrm{e}}|) u = \overline{f}(|\Gamma_u^{\mathrm{e}}|) u$ (by a standard approximation from polynomials $f$). Using this, we can rewrite (3.4) as

$$\Gamma_u^{\mathrm{e}} \overline{f}(|\Gamma_u^{\mathrm{e}}|) u = |\Gamma_u^{\mathrm{e}}| \overline{\Psi}_u(|\Gamma_u^{\mathrm{e}}|) \overline{f}(|\Gamma_u^{\mathrm{e}}|) u, \quad f \in L^2(\rho),$$

so

$$\Gamma_u^{\mathrm{e}} = |\Gamma_u^{\mathrm{e}}| \overline{\Psi}_u(|\Gamma_u^{\mathrm{e}}|) = \overline{\Psi}_u(|\Gamma_u^{\mathrm{e}}|) |\Gamma_u^{\mathrm{e}}|.$$

The last identity gives the polar decomposition of $\Gamma_u^{\mathrm{e}}$, and since it is self-adjoint, the operator $\Psi_u(|\Gamma_u^{\mathrm{e}}|)$ is a self-adjoint unitary operator, so $\Psi_u$ takes values $\pm 1$. If we assign $\Psi_u(0) := 0$, we get the polar decomposition

$$\Gamma_u = |\Gamma_u|\overline{\Psi}_u(|\Gamma_u|) = \overline{\Psi}_u(|\Gamma_u|)|\Gamma_u|,$$

where $\Psi_u(|\Gamma_u|)$ is a self-adjoint partial isometry, $\operatorname{Ker}\Psi_u(|\Gamma_u|) = \operatorname{Ker}\Gamma_u$.

A similar argument can be applied to $\widetilde{\Psi}_u$. If $\operatorname{Ker}\widetilde{\Gamma}_u^{\mathrm{e}} = \{0\}$, the reasoning is exactly the same; if $\operatorname{Ker}\widetilde{\Gamma}_u^{\mathrm{e}} \neq \{0\}$ (which may happen), a slight modification is needed. Namely, we need first to consider the polar decomposition of $\widetilde{\Gamma}_u|_{\overline{\operatorname{Ran}\widetilde{\Gamma}_u}}$. Noticing that $\widetilde{\rho}_0$ is the spectral measure of the operator $|\widetilde{\Gamma}_u|\,|_{\overline{\operatorname{Ran}\widetilde{\Gamma}_u}}$ with respect to the vector $\widetilde{u} := P_{\overline{\operatorname{Ran}\widetilde{\Gamma}_u}} u$, we then can write, assigning $\widetilde{\Psi}_u(0) := 0$, that

$$\widetilde{\Gamma}_u = |\widetilde{\Gamma}_u|\widetilde{\Psi}_u(|\widetilde{\Gamma}_u|) = \widetilde{\Psi}_u(|\widetilde{\Gamma}_u|)|\widetilde{\Gamma}_u|.$$

So $\widetilde{\Psi}_u$ takes values $\pm 1$ and the polar decomposition for $\widetilde{\Gamma}_u$ has the required form (1.22).

Finally, assume that both $\Psi$ and $\widetilde{\Psi}$ are real-valued, and let us prove that the Fourier coefficients $\widehat{u}_m$ are real for all $m$. We use formula (3.4). Denote $A = \mathcal{M}^{-1}\Psi(\mathcal{M})$ and $B = \widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}$. By our assumptions, both $A$ and $B$ are self-adjoint, $A$ may be unbounded, but $BA$ is bounded (extends to a bounded operator from a dense set). The operator

$$(\Sigma^*)^m\widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}} = (BA)^m B$$

is self-adjoint for all $m \geq 0$. Since $\mathbb{1} = \Psi(\mathcal{M})\mathcal{M}q$, we can write

$$\widehat{u}_m = \langle(\Sigma^*)^{m-1}\widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}q, q\rangle_{L^2(\rho)} = \langle(BA)^{m-1}Bq, q\rangle_{L^2(\rho)},$$

and since $(BA)^{m-1}B$ is self-adjoint, $\widehat{u}_m$ is real for all $m \geq 1$. The proof of Theorems 1.6 and 1.7 is complete. □

## 4 Operator theoretic background

In the following sections, we will use some more specialized operator theoretic material, related mainly to the functional model for contractions. In this section, we collect without proof the corresponding background facts.

**4.1 Douglas Lemma.** We will need the following simple fact.

**Lemma 4.1.** *Let $A$, $B$ be operators in a Hilbert space such that*

$$\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$$

*and*

$$B^* B \le A^* A.$$

*Then the operator $BA^{-1}$, defined on a dense set* $\operatorname{Ran} A$*, extends to a contraction $T$. The adjoint $T^*$ is given by the formula $T^* = (A^*)^{-1} B^*$; note that the boundedness of $T$ implies that* $\operatorname{Ran} B^* \subset \operatorname{Dom}(A^*)^{-1}$*, so the above expression is defined on the whole space.*

In this paper we will often apply this lemma to self-adjoint operators $\mathcal{M}$, $\widetilde{\mathcal{M}}$, $\operatorname{Ker} \mathcal{M} = \{0\}$, such that $\widetilde{\mathcal{M}}^2 \le \mathcal{M}^2$, to define contractions $\widetilde{\mathcal{M}}\mathcal{M}^{-1}$, $\mathcal{M}^{-1}\widetilde{\mathcal{M}}$.

**4.2 Inner functions, model spaces and the compressed shift.** A non-constant function $\theta \in H^2(\mathbb{T})$ is called **inner**, if $|\theta| = 1$ a.e. on the unit circle. For an inner function $\theta$, the **model space** $K_\theta$ is the subspace of $H^2(\mathbb{T})$, defined by

$$K_\theta = H^2(\mathbb{T}) \cap (\theta H^2(\mathbb{T}))^\perp.$$

We refer to [23, 3] for background on model spaces.

Beurling's theorem states that if a non-trivial subspace $K \subsetneq H^2(\mathbb{T})$ is invariant for the backward shift $S^*$, then $K = K_\theta$ for some inner $\theta$.

Let $\theta$ be an inner function and let $P_\theta$ be the orthogonal projection onto $K_\theta$ in $H^2(\mathbb{T})$. The operator $S_\theta = P_\theta S$ on $K_\theta$ is called the **compressed shift**. Since $K_\theta$ is an invariant subspace for $S^*$, we have $S_\theta^* f = S^* f$ for $f \in K_\theta$. It is not difficult to compute that

$$I - S_\theta S_\theta^* = \langle \cdot, P_\theta \mathbb{1} \rangle P_\theta \mathbb{1}, \quad I - S_\theta^* S_\theta = \langle \cdot, S^*\theta \rangle S^*\theta, \tag{4.1}$$

where $P_\theta \mathbb{1} = 1 - \overline{\theta(0)}\theta$.

**4.3 Contractions in a Hilbert space.** Let $T$ be a contraction in a Hilbert space. The **defect spaces** of $T$ and $T^*$ are defined as

$$\mathcal{D}_T := \overline{\operatorname{Ran}}(I - T^*T), \quad \mathcal{D}_{T^*} := \overline{\operatorname{Ran}}(I - TT^*),$$

and the **defect indices** of $T$ are the (ordered) pair of numbers

$$(\partial_T, \partial_{T^*}), \quad \partial_T = \dim \mathcal{D}_T, \quad \partial_{T^*} = \dim \mathcal{D}_{T^*}.$$

In particular, the shift operator $S$ has the defect indices $(0, 1)$ and the compressed shift $S_\theta$ (for any inner $\theta$) has the defect indices $(1, 1)$, see (4.1).

A contraction $T$ is called **completely non-unitary** (c.n.u.), if $T$ is not unitary on any of its invariant subspaces. The following result is known as Langer's lemma (see, e.g., [24, Lemma 1.2.6]).

**Lemma 4.2.** *Let $T$ be a contraction in a Hilbert space $X$. Then $X$ can be represented as an orthogonal sum $X = X_{\mathrm{u}} \oplus X_{\mathrm{cnu}}$, such that*

$$T = \begin{pmatrix} T_{\mathrm{u}} & 0 \\ 0 & T_{\mathrm{cnu}} \end{pmatrix} \quad \text{in } X_{\mathrm{u}} \oplus X_{\mathrm{cnu}},$$

*where $T_{\mathrm{u}}$ is unitary and $T_{\mathrm{cnu}}$ is completely non-unitary.*

### 4.4 Contractions with defect indices $(0, 1)$ and $(1, 1)$.

**Theorem 4.3.** *Let $T$ be a c.n.u. contraction with defect indices* $(0, 1)$. *Then $T$ is unitarily equivalent to the forward shift operator $S$. In this case* $\operatorname{Re} T$ *has a purely a.c. spectrum* $[-1, 1]$ *of multiplicity one.*

The first part follows from the Kolmogorov–Wold decomposition, see [22, Theorem I.1.1]. For the second part, we note that the matrix of $2 \operatorname{Re} S$ in the standard basis in $H^2(\mathbb{T})$ is the Jacobi matrix

$$2 \operatorname{Re} S = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and it is well-known that the spectrum of this matrix is purely a.c., coincides with the interval $[-2, 2]$ and has multiplicity one (see, e.g., [30, Section 1.1.3]).

The following statement will be crucial in our construction.

**Theorem 4.4.** *Let $T$ be a c.n.u. contraction with defect indices* $(1, 1)$. *Then the following statements are equivalent:*

(i) $T^{*n} \to 0$ *strongly as* $n \to \infty$.
(ii) $T^n \to 0$ *strongly as* $n \to \infty$.
(iii) *The operator* $\operatorname{Re} T$ *has a purely singular spectrum.*
(iv) *The operator $T$ is unitarily equivalent to the compressed shift operator $S_\theta$ for some inner function $\theta$.*

We discuss the proof in Appendix C.

**4.5 Dilations of contractions.** Let $T$ be a contraction on a Hilbert space $X$. Further, let $Y$ be another Hilbert space such that $X$ is a subspace of $Y$, let $P_X$ be the orthogonal projection onto $X$ in $Y$ and let $V$ be a bounded operator in $Y$. Then $V$ is called a **dilation** of $T$, if for any $n \geq 0$ we have

$$T^n f = P_X V^n f, \quad \forall f \in X. \tag{4.2}$$

**Theorem 4.5** ([22, Theorem II.6.4]). *Let $T$ be a c.n.u. contraction. Then there exists a dilation $V$ of $T$ such that $V$ is a unitary operator with a purely a.c. spectrum.*

In fact, any *minimal* unitary dilation (this means that the span of $V^n X$ for $n \geq 0$ is dense in $Y$) of a c.n.u. contraction has a purely a.c. spectrum; see [22] for details.

**4.6 Trace class perturbations.**

**Theorem 4.6** (Kato–Rosenblum). *Let $A$ and $B$ be self-adjoint (or unitary) operators in a Hilbert space $X$ such that the difference $A - B$ is trace class. Then the absolutely continuous parts of $A$ and $B$ are unitarily equivalent.*

The following generalization of the Kato–Rosenblum theorem was found by Ismagilov in [15]; see also [14, 29] for different proofs.

**Theorem 4.7** (Ismagilov). *Let $A$ and $B$ be bounded self-adjoint operators such that $AB$ is trace class. Then the a.c. parts of the operators $A + B$ and $A \oplus B$ are unitarily equivalent.*

We will also need the following result on trace class perturbations, due to M. G. Krein [17]. (Much more precise results in terms of the class of $f$ are now available; see, e.g., [27]).

**Theorem 4.8.** *Let $A$ and $B$ be bounded self-adjoint operators in a Hilbert space $X$ such that the difference $A - B$ is trace class. Let $f$ be a differentiable function on $\mathbb{R}$ such that the derivative $f'$ is a Fourier transform of a finite complex-valued measure on $\mathbb{R}$. Then $f(A) - f(B)$ is also trace class.*

**4.7 Spectral measures.** We recall that for a unitary operator in a Hilbert space, its spectral measure is a projection-valued measure on $\mathbb{T}$ and for a self-adjoint operator, its spectral measure is a projection-valued measure on $\mathbb{R}$. Furthermore, if $U$ is unitary and $\operatorname{Re} U = (U + U^*)/2$, then the spectral measure of $\operatorname{Re} U$ on $\mathbb{R}$ is the pushforward of the spectral measure of $U$ on $\mathbb{T}$ by the map $z \mapsto (z + \overline{z})/2$. From here we obtain the following simple conclusion, which we will use throughout the paper.

**Proposition 4.9.** *The spectrum of a unitary operator $U$ is purely a.c. (resp., purely singular) if and only if the spectrum of the self-adjoint operator* $\operatorname{Re} U$ *is purely a.c. (resp., purely singular).*

## 5 Reduction to asymptotic stability: proof of Theorem 2.1

**5.1 The "only if" part.** We use the model $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma^*)$ of Section 2.3. If the triple $(\rho, \Psi, \widetilde{\Psi})$ is the spectral datum for some Hankel operator $H_u$, then by the definition (2.2), the operator $\Sigma^*$ is unitarily equivalent to the restriction of the backward shift $S^*$ to the $S^*$-invariant subspace $\overline{\operatorname{Ran} H_u}$ (this subspace may coincide with the whole space $H^2(\mathbb{T})$).

The operator $S^*$ is asymptotically stable, and so its restriction to any invariant subspace is also asymptotically stable. We conclude that $\Sigma^*$ is asymptotically stable.

In the rest of this section, we prove the "if" part; this requires several steps. Throughout the proof, we use the model of Section 2.4.

**5.2 The "if" part: checking the commutation relations.** Let us show that the model operators $\mathcal{H}$, $\widetilde{\mathcal{H}}$, $\Sigma^*$ (defined in (2.5), (2.6), (2.7)) satisfy the relations

$$\widetilde{\mathcal{H}} = \Sigma^* \mathcal{H} = \mathcal{H}\Sigma. \tag{5.1}$$

Using the fact that $\Psi$ is unimodular, we have

$$\begin{aligned}\Sigma^*\mathcal{H}f &= \widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}\mathcal{M}^{-1}\Psi(\mathcal{M})\overline{\Psi}(\mathcal{M})\mathcal{M}\overline{f} \\ &= \widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}\overline{f} = \widetilde{\mathcal{H}}f.\end{aligned}$$

For any bounded Borel function $\Phi$ we have

$$\overline{(\Phi(\mathcal{M})f)} = \overline{\Phi}(\mathcal{M})\overline{f} \quad \text{and} \quad \overline{(\Phi(\widetilde{\mathcal{M}})f)} = \overline{\Phi}(\widetilde{\mathcal{M}})\overline{f}$$

and therefore

$$\overline{(\Sigma f)} = \Psi(\mathcal{M})\mathcal{M}^{-1}\widetilde{\mathcal{M}}\widetilde{\Psi}(\widetilde{\mathcal{M}})\overline{f}.$$

It follows that

$$\mathcal{H}\Sigma f = \overline{\Psi}(\mathcal{M})\mathcal{M}\overline{\Sigma f} = \widetilde{\mathcal{M}}\widetilde{\Psi}(\mathcal{M})\overline{f} = \widetilde{\mathcal{H}}f.$$

We have checked (5.1).

**5.3 The "if" part: setting up the unitary equivalence.** Define the operator $\mathcal{U} : L^2(\rho) \to H^2(\mathbb{T})$ as

$$\mathcal{U}f(z) := \sum_{k=0}^{\infty} \langle (\Sigma^*)^k f, q\rangle z^k = \sum_{k=0}^{\infty} \langle f, \Sigma^k q\rangle z^k, \quad z \in \mathbb{T}, \tag{5.2}$$

where $q(s) = \overline{\Psi(s)}/s$. Below we check that $\mathcal{U}$ is an isometry.

It follows from the definition (2.7) of $\Sigma^*$ that

$$\begin{aligned} \Sigma\Sigma^* &= \Psi(\mathcal{M})^*\mathcal{M}^{-1}\widetilde{\mathcal{M}}^2\mathcal{M}^{-1}\Psi(\mathcal{M}) = \Psi(\mathcal{M})^*\mathcal{M}^{-1}(\mathcal{M}^2 - \langle \cdot, \mathbb{1}\rangle \mathbb{1})\mathcal{M}^{-1}\Psi(\mathcal{M}) \\ &= I - \langle \cdot, q\rangle q. \end{aligned} \tag{5.3}$$

From here it follows that

$$\|f\|^2 - \|\Sigma^* f\|^2 = |\langle f, q\rangle|^2.$$

Applying this identity to $(\Sigma^*)^k f$ and summing over $k$ from 0 to $n-1$ we get that for any $f \in L^2(\rho)$ and any $n \in \mathbb{N}$,

$$\|f\|^2 - \|(\Sigma^*)^n f\|^2 = \sum_{k=0}^{n-1} |\langle f, \Sigma^k q\rangle|^2.$$

Here comes the crucial point in the proof: by the asymptotic stability of $\Sigma^*$, we have that $\|(\Sigma^*)^n f\|^2 \to 0$ as $n \to \infty$, and so

$$\sum_{m=0}^{\infty} |\langle f, \Sigma^m q\rangle|^2 = \|f\|^2,$$

i.e., the map $\mathcal{U} : L^2(\rho) \to H^2(\mathbb{T})$, defined in (5.2), is an isometry.

**5.4 The "if" part: defining the Hankel operator.** Define the operators $A$ and $\widetilde{A}$ on $H^2(\mathbb{T})$ by

$$A := \mathcal{U}\mathcal{H}\mathcal{U}^*, \quad \widetilde{A} := \mathcal{U}\widetilde{\mathcal{H}}\mathcal{U}^*.$$

We would like to check that $A$ and $\widetilde{A}$ are Hankel operators. First we show that $\mathcal{U}$ intertwines $S^*$ and $\Sigma^*$. By the definition (5.2) of the map $\mathcal{U}$, we have

$$\mathcal{U}\Sigma^* f(z) = \sum_{k=0}^{\infty} \langle \Sigma^* f, \Sigma^k q\rangle z^k = \sum_{k=0}^{\infty} \langle f, \Sigma^{k+1} q\rangle z^k = S^*\mathcal{U}f(z),$$

and so we find that

$$\mathcal{U}\Sigma^* = S^*\mathcal{U} \tag{5.4}$$

and by taking adjoints

$$\Sigma\mathcal{U}^* = \mathcal{U}^* S. \tag{5.5}$$

Note that (5.4) implies that $\operatorname{Ran}\mathcal{U}$ is a $S^*$-invariant subspace of $H^2(\mathbb{T})$.

Using (5.5) and (5.1), we find

$$AS = \mathcal{U}\mathcal{H}\mathcal{U}^* S = \mathcal{U}\mathcal{H}\Sigma\mathcal{U}^* = \mathcal{U}\widetilde{\mathcal{H}}\mathcal{U}^* = \widetilde{A}.$$

Similarly,

$$S^*A = S^*\mathcal{U}\mathcal{H}\mathcal{U}^* = \mathcal{U}\Sigma^*\mathcal{H}\mathcal{U}^* = \mathcal{U}\widetilde{\mathcal{H}}\mathcal{U}^* = \widetilde{A}.$$

Therefore $A$ satisfies the commutation relation

$$AS = S^*A = \widetilde{A},$$

and so $A$ is a Hankel operator. Setting $u := Az^0$, we can write $A = H_u$ and then $\widetilde{A} = H_u S = \widetilde{H}_u$. It remains to prove that $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ and that the spectral datum of $u$ coincides with the abstract spectral datum $(\rho, \Psi, \widetilde{\Psi})$.

**5.5 The "if" part: concluding the proof.** Denote by $U$ the operator $\mathcal{U}$ with the target space restricted to $\operatorname{Ran}\mathcal{U}$, so $U$ is a unitary operator. Here we use the same notation as for the map (1.15); as we shall soon see, this is indeed the same map in disguise.

Since $\overline{\operatorname{Ran}}\mathcal{H} = L^2(\rho)$, from the definition $H_u = \mathcal{U}\mathcal{H}\mathcal{U}^*$ we find that

$$\overline{\operatorname{Ran}} H_u = \operatorname{Ran}\mathcal{U}.$$

Thus, in our new notation we find

$$H_u^{\mathrm{e}} = U\mathcal{H}U^*, \quad \widetilde{H}_u^{\mathrm{e}} = U\widetilde{\mathcal{H}}U^*. \tag{5.6}$$

Let us check that $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$. By the definition of $H_u$, it is a bounded operator and therefore $u \in \mathrm{BMOA}(\mathbb{T})$. Next, from (5.6) we find

$$(H_u^{\mathrm{e}})^2 = U\mathcal{H}^2U^* = U\mathcal{M}^2U^*, \quad (\widetilde{H}_u^{\mathrm{e}})^2 = U\widetilde{\mathcal{H}}^2U^* = U\widetilde{\mathcal{M}}^2U^*;$$

recall that here $\mathcal{M}$ is the multiplication by the independent variable in $L^2(\rho)$ and $\widetilde{\mathcal{M}}$ is defined by (1.19). It is obvious that $\mathcal{M}^2$ has a simple spectrum with the cyclic element $\mathbb{1}$. By Lemma 3.1(i), the same is true for $\widetilde{\mathcal{M}}^2$. Thus, the simplicity of spectrum condition (1.10) is satisfied and so $u \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$.

Our next step is to check the identity $U^*u = \mathbb{1}$. We first note that by the definition (5.2) of $\mathcal{U}$, for any $f \in L^2(\rho)$ we have

$$\langle \mathcal{U}f, z^0\rangle_{H^2} = \langle f, q\rangle_{L^2(\rho)},$$

and therefore $\mathcal{U}^* z^0 = q$. Further, we have

$$u = H_u z^0 = \mathcal{U}\mathcal{H}\mathcal{U}^* z^0 = \mathcal{U}\mathcal{H}q.$$

Recalling formula (2.5) for the action of $\mathcal{H}$, we find that

$$\mathcal{H}q(s) = s\overline{\Psi}(s)\overline{q(s)} = s\overline{\Psi}(s)\overline{\overline{\Psi}(s)/s} = 1,$$

and so we conclude that $u = \mathcal{U}\mathbb{1} = U\mathbb{1}$ and therefore $U^* u = \mathbb{1}$.

Finally, we check that the map $U$ coincides with the map defined by (1.15). For $f \in L^2(\rho)$, we find

$$f(|H_u^{\mathrm{e}}|)u = Uf(|\mathcal{H}|)U^* u = Uf(\mathcal{M})\mathbb{1} = Uf,$$

as required.

We conclude that for the Hankel operator $H_u$ and the map $U$, satisfying (1.15), we have the identities (5.6), where $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ correspond to our abstract spectral datum $\Lambda = (\rho, \Psi, \widetilde{\Psi})$. This means that the abstract spectral datum $\Lambda$ coincides with the spectral datum $\Lambda(u)$.

# 6 Initial results about asymptotic stability

In this section, as a warm-up, we present some easy initial results on asymptotic stability. In what follows, $(\rho, \Psi, \widetilde{\Psi})$ is an abstract spectral datum. We recall that this means that $\rho$ is a finite Borel measure with a bounded support on $(0, \infty)$, satisfying the normalization condition (1.18), and $\Psi \in L^\infty(\rho)$ and $\widetilde{\Psi} \in L^\infty(\widetilde{\rho}_0)$ are unimodular complex-valued functions.

**6.1 The operator $\Sigma_0^*$ is asymptotically stable.** Let the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ in $L^2(\rho)$ be as defined in Section 2.1. Recall that the operator $\Sigma_0^*$ in $L^2(\rho)$ was defined by $\Sigma_0^* := \widetilde{\mathcal{M}}\mathcal{M}^{-1}$. Our purpose here is to prove

**Theorem 6.1.** *The operator $\Sigma_0^*$ is asymptotically stable.*

By Theorem 2.1, this implies that any spectral datum of the form $(\rho, \mathbb{1}, \mathbb{1})$ is in $\Lambda(\mathrm{BMOA}_{\mathrm{simp}})$; this was one of the main results of [11].

In order to prove Theorem 6.1, we consider the **symmetrization** $\mathfrak{S}_0$ of $\Sigma_0^*$

$$\mathfrak{S}_0 := \mathcal{M}^{-1/2}\Sigma_0^*\mathcal{M}^{1/2} = \mathcal{M}^{-1/2}\widetilde{\mathcal{M}}\mathcal{M}^{-1/2}.$$

Note that $\|\mathfrak{S}_0\| \leq 1$. Indeed, from $0 \leq \widetilde{\mathcal{M}}^2 \leq \mathcal{M}^2$ by the Heinz inequality we find $\widetilde{\mathcal{M}}^{1/2} \leq \mathcal{M}^{1/2}$, and therefore by Douglas' Lemma (Lemma 4.1) the operator $Q := \widetilde{\mathcal{M}}^{1/2}\mathcal{M}^{-1/2}$ extends from a dense set to a contraction, and its adjoint is given by $Q^* = \mathcal{M}^{-1/2}\widetilde{\mathcal{M}}^{1/2}$. Thus

$$\mathfrak{S}_0 = Q^*Q, \tag{6.1}$$

so $\mathfrak{S}_0$ is a contraction.

**Lemma 6.2.** *The operator $\mathfrak{S}_0$ is asymptotically stable.*

**Proof.** By (6.1), the operator $\mathfrak{S}_0$ is self-adjoint and $0 \le \mathfrak{S}_0 \le I$. So in order to prove the asymptotic stability of $\mathfrak{S}_0$, it is sufficient to show that 1 is not an eigenvalue of $\mathfrak{S}_0$. Let us prove this. We have

$$\widetilde{\mathcal{M}} = \mathcal{M}^{1/2}\mathfrak{S}_0\mathcal{M}^{1/2},$$

and therefore

$$\widetilde{\mathcal{M}}^2 = \mathcal{M}^{1/2}\mathfrak{S}_0\mathcal{M}\mathfrak{S}_0\mathcal{M}^{1/2}.$$

On the other hand,

$$\widetilde{\mathcal{M}}^2 = \mathcal{M}^2 - \langle\,\cdot\,,\mathbb{1}\rangle\mathbb{1} = \mathcal{M}^{1/2}(\mathcal{M} - \langle\,\cdot\,,b\rangle b),\mathcal{M}^{1/2},$$

where $b = \mathcal{M}^{-1/2}\mathbb{1}$, i.e., $b(s) = s^{-1/2}$.

Comparing these two representations for $\widetilde{\mathcal{M}}^2$ and using the fact that

$$\operatorname{Ker}\mathcal{M}^{1/2} = \{0\}$$

we find

$$\mathfrak{S}_0\mathcal{M}\mathfrak{S}_0 = \mathcal{M} - \langle\,\cdot\,,b\rangle b. \tag{6.2}$$

Suppose $f \in \operatorname{Ker}(\mathfrak{S}_0 - I)$, i.e., $\mathfrak{S}_0 f = f$. Evaluating the quadratic form of the last identity on $f$, we find

$$\langle\mathcal{M}f, f\rangle = \langle\mathcal{M}f, f\rangle - |\langle f, b\rangle|^2,$$

and so $f \perp b$. Substituting $f \perp b$ back into (6.2), we get

$$\mathfrak{S}_0\mathcal{M}f = \mathcal{M}f,$$

and so $\mathcal{M}f \in \operatorname{Ker}(\mathfrak{S}_0 - I)$.

Thus, $\operatorname{Ker}(\mathfrak{S}_0 - I)$ is an invariant subspace of $\mathcal{M}$ which is orthogonal to $b$. Since $b$ is a cyclic element for $\mathcal{M}$, it follows that $\operatorname{Ker}(\mathfrak{S}_0 - I) = \{0\}$. □

**Corollary 6.3.** *The operator $Q = \widetilde{\mathcal{M}}^{1/2}\mathcal{M}^{-1/2}$ is a strict contraction, i.e.,*

$$\|Qx\| < \|x\| \quad \forall x \neq 0.$$

**Proof.** By construction, $Q$ is a contraction, $\|Qx\| \le \|x\|$ for all $x$. Assume that $\|Qx\| = \|x\|$ for some $x \neq 0$. Since $\mathfrak{S}_0 = Q^*Q$, we conclude that $\langle \mathfrak{S}_0 x, x\rangle = \|Qx\|^2 = \|x\|^2$. But $\|\mathfrak{S}_0\| \le 1$, so $\mathfrak{S}_0 x = x$, which contradicts Lemma 6.2. □

**Lemma 6.4.** *Let bounded operators A, B, K satisfy*

$$KA = BK, \tag{6.3}$$

*and let* $\|B\| \le 1$. *Assume that* $\operatorname{Ran} K$ *is dense, and that the operator A is asymptotically stable. Then B is also asymptotically stable.*

**Proof.** Iterating (6.3) we get that

$$KA^n = B^n K, \qquad n \in \mathbb{N}.$$

Since $A$ is asymptotically stable, we see that for all $x \in \operatorname{Ran} K$

$$\|B^n x\| \to 0 \quad \text{as } n \to \infty. \tag{6.4}$$

But $\|B^n\| \le 1$, so operators $B^n$ are uniformly bounded. Since $\operatorname{Ran} K$ is dense, the $\varepsilon/3$-Theorem says that (6.4) holds for all $x$, i.e. $B$ is asymptotically stable. □

**Proof of Theorem 6.1.** From the definition of $\mathfrak{S}_0$ we see that

$$\mathcal{M}^{1/2}\mathfrak{S}_0 = \Sigma_0^* \mathcal{M}^{1/2},$$

and $\operatorname{Ran} \mathcal{M}^{1/2}$ is dense in $L^2(\rho)$. Now we apply Lemma 6.4 with $K = \mathcal{M}^{1/2}$, $A = \mathfrak{S}_0$ and $B = \Sigma_0^*$. □

**6.2 Self-adjoint Hankel operators and positivity.** In this subsection we discuss the self-adjoint case, when both operators $\Gamma$ and $\widetilde{\Gamma} = \Gamma S$ are self-adjoint. According to Theorem 1.6, in terms of the spectral data, this corresponds to the case when both unimodular functions $\Psi$ and $\widetilde{\Psi}$ are real-valued.

**Theorem 6.5.** *Let* $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ *be an abstract spectral datum such that* $\Psi$ *and* $\widetilde{\Psi}$ *are real-valued and one of them is identically equal to* 1. *Then* $\Sigma^*$ *is asymptotically stable, i.e.,* $\Lambda \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

This theorem gives us a complete description of the spectral data in the case of self-adjoint Hankel operators, when one of the operators $\Gamma$, $\widetilde{\Gamma}$ is non-negative.

**Proof of Theorem 6.5.** Let us introduce the **symmetrization** $\mathfrak{S}^*$ of $\Sigma^*$,

$$\mathfrak{S}^* := \mathcal{M}^{-1/2}\Sigma^*\mathcal{M}^{1/2} = Q^*\widetilde{\Psi}(\widetilde{\mathcal{M}})Q\Psi(\mathcal{M}),$$

where $Q = \widetilde{\mathcal{M}}^{1/2}\mathcal{M}^{-1/2}$ is as in Corollary 6.3. Since $\Psi$ and $\widetilde{\Psi}$ are real-valued, the operators $\Psi(\mathcal{M})$ and $\widetilde{\Psi}(\widetilde{\mathcal{M}})$ are self-adjoint. Let us prove that $\mathfrak{S}^*$ is asymptotically stable.

If $\Psi \equiv 1$, we have $\mathfrak{S}^* = Q^*\widetilde{\Psi}(\widetilde{\mathcal{M}})Q$, so $\mathfrak{S}^*$ is self-adjoint. The fact that $Q$ is a strict contraction (see Corollary 6.3) implies that $\pm 1$ are not eigenvalues of $\mathfrak{S}^*$, so $\mathfrak{S}^*$ is asymptotically stable.

If $\widetilde{\Psi} \equiv 1$, we get that $\mathfrak{S}^* = Q^*Q\Psi(\mathcal{M})$. This operator is not self-adjoint, but

$$(\mathfrak{S}^*)^n = Q^*(Q\Psi(\mathcal{M})Q^*)^{n-1}Q\Psi(Q), \tag{6.5}$$

and the operator $Q\Psi(\mathcal{M})Q^*$ is self-adjoint. Since $Q^*$ is a strict contraction, the points $\pm 1$ are not the eigenvalues of $Q\Psi(\mathcal{M})Q^*$, so $Q\Psi(\mathcal{M})Q^*$ is asymptotically stable. Identity (6.5) together with Lemma 6.4 shows that $\mathfrak{S}^*$ is asymptotically stable as well.

Finally, we have

$$\mathcal{M}^{1/2}\mathfrak{S}^* = \Sigma^*\mathcal{M}^{1/2},$$

so by Lemma 6.4 with $K = \mathcal{M}^{1/2}$ the asymptotic stability of $\mathfrak{S}^*$ implies the asymptotic stability of $\Sigma^*$. □

## 7 Asymptotic stability and singular spectrum

In this section we present one of our key results which related the asymptotic stability of $\Sigma^*$ to its spectral properties. As in the previous section, below $(\rho, \Psi, \widetilde{\Psi})$ is an abstract spectral datum, and $\Sigma^*$ is the operator in $L^2(\rho)$ defined in (2.7).

**7.1 Defect indices of $\Sigma^*$.** In what follows, the consideration of $\Sigma^*$ will proceed in two slightly different ways depending on the defect indices of $\Sigma^*$. In the following lemma we describe these two possible cases.

**Lemma 7.1.** *Let $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be an abstract spectral datum, and let $\Sigma^*$ be the operator* (2.7) *constructed from it.*

(i) *If*

$$\int_0^\infty \frac{d\rho(s)}{s^2} = 1 \quad \textit{and} \quad \int_0^\infty \frac{d\rho(s)}{s^4} = \infty, \tag{7.1}$$

*then the defect indices of $\Sigma^*$ are $(1, 0)$, so $\Sigma$ is an isometry.*

(ii) *If* (7.1) *fails, i.e., if we have either*

$$\int_0^\infty \frac{d\rho(s)}{s^2} = 1 \quad \textit{and} \quad \int_0^\infty \frac{d\rho(s)}{s^4} < \infty, \tag{7.2}$$

*or*

$$\int_0^\infty \frac{d\rho(s)}{s^2} < 1, \tag{7.3}$$

*then the defect indices of* $\Sigma^*$ *are* $(1, 1)$.

**Proof.** We have $\Sigma^* = \widetilde{\Psi}(\widetilde{\mathcal{M}})\Sigma_0^*\Psi(\mathcal{M})$, where $\Sigma_0^* := \widetilde{\mathcal{M}}\mathcal{M}^{-1}$. The operators $\Psi(\mathcal{M})$ and $\widetilde{\Psi}(\widetilde{\mathcal{M}})$ are unitary, and so the defect indices of $\Sigma^*$ and $\Sigma_0^*$ coincide. Thus, it suffices to consider the defect indices of $\Sigma_0^*$.

By Theorem 6.1, the operator $\Sigma_0^*$ is asymptotically stable. By Theorem 2.1, this means that the triple $(\rho, \mathbb{1}, \mathbb{1})$ is the spectral datum of some Hankel operator $H_u$.

(i) Suppose (7.1) is satisfied. Note that (7.1) is identical to (1.21), and so by Theorem 1.5, we have $\operatorname{Ker} H_u = \{0\}$, and therefore (see (2.2)) the operator $\Sigma_0^*$ is unitarily equivalent to the backward shift $S^*$, and so the defect indices of $\Sigma^*$ are $(1, 0)$.

(ii) Suppose (7.1) fails. Then again by Theorem 1.5, the kernel of $H_u$ is non-trivial and so $\Sigma_0^*$ is unitarily equivalent to the restriction of $S^*$ to the subspace $\overline{\operatorname{Ran}} H_u$. By Beurling's theorem, this subspace is a model space $K_\theta := H^2 \ominus \theta H^2$ for some inner function $\theta$ and so $\Sigma_0^*$ is unitarily equivalent to $S_\theta^*$, where $S_\theta$ is the compressed shift on $K_\theta$. It follows (see (4.1)) that the defect indices of $\Sigma_0^*$ are $(1, 1)$. □

**7.2 Asymptotic stability and singular spectrum.** Recall that the a.c. spectrum of a self-adjoint or unitary operator is said to equal a Borel set $E$ if the a.c. part of the spectral measure is mutually absolutely continuous with the Lebesgue measure restricted to $E$.

**Theorem 7.2.** *Let the triple* $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ *be an abstract spectral datum.*

(i) *Assume that* (7.1) *holds, i.e., that* $\Sigma^*$ *has defect indices* $(1, 0)$. *Then* $\Sigma^*$ *is asymptotically stable iff the a.c. spectrum of* $\operatorname{Re}\Sigma$ *is* $[-1, 1]$ *with multiplicity one.*

(ii) *Assume that* (7.1) *does not hold, i.e., that* $\Sigma^*$ *has defect indices* $(1, 1)$. *Then* $\Sigma^*$ *is asymptotically stable iff the a.c. part of* $\operatorname{Re}\Sigma$ *is empty.*

Before proceeding to the proof, we need a lemma. This lemma is one of the central points of our argument. Below we refer to the unitary and c.n.u. parts of a contraction according to Langer's lemma, see Lemma 4.2.

**Lemma 7.3.** *Let* $\Sigma^*$ *be the operator constructed from an abstract spectral datum. Then the unitary part of* $\Sigma$ *is either purely absolutely continuous or absent.*

**Proof.** We use the model $(\mathcal{H}, \widetilde{\mathcal{H}}, \Sigma)$ as described in Section 2.4. Let us write

$$L^2(\rho) = X_{\rm sing} \oplus X_{\rm r},$$

where $X_{\rm sing}$ is the singular subspace of the unitary part of $\Sigma$, and $X_{\rm r}$ is the "remainder" part, i.e., the sum of the completely non-unitary subspace of $\Sigma$ and the absolutely continuous subspace of the unitary part of $\Sigma$. Our aim is to show that $X_{\rm sing} = \{0\}$.

**Step 1. The spectral measures associated with $\Sigma_{\rm r}$.** By construction, $\Sigma_{\rm r}$ is an orthogonal sum of a unitary part $\Sigma_{\rm u}$ with the purely a.c. spectrum and a completely non-unitary part $\Sigma_{\rm cnu}$.

For any $f \in X_{\rm r}$ and any polynomial $\varphi$ of $z$, we have

$$\|\varphi(\Sigma_{\rm r})f\|^2 = \|\varphi(\Sigma_{\rm u})f_{\rm u}\|^2 + \|\varphi(\Sigma_{\rm cnu})f_{\rm cnu}\|^2, \tag{7.4}$$

where $f_{\rm u}$ and $f_{\rm cnu}$ are the projections of $f$ onto the corresponding subspaces. We can write

$$\|\varphi(\Sigma_{\rm u})f_{\rm u}\|^2 = \int_{\mathbb{T}} |\varphi(z)|^2 d\mu_f^{\rm u}(z),$$

where $\mu_f^{\rm u}$ is the spectral measure of $\Sigma_{\rm u}$, associated with the vector $f_{\rm u}$. By construction, this measure is absolutely continuous.

Now let us consider the second term in the r.h.s. of (7.4). Since $\Sigma_{\rm cnu}$ is a c.n.u. contraction, we can consider its minimal unitary dilation $V$, which has a purely a.c. spectrum, see Theorem 4.5. Taking linear combinations of (4.2), we obtain

$$\varphi(\Sigma_{\rm cnu})f_{\rm cnu} = P_{\rm cnu}\varphi(V)f_{\rm cnu},$$

where $P_{\rm cnu}$ is the orthogonal projection onto the c.n.u. subspace of $\Sigma$. This yields

$$\|\varphi(\Sigma_{\rm cnu})f_{\rm cnu}\|^2 = \|P_{\rm cnu}\varphi(V)f_{\rm cnu}\|^2 \leq \|\varphi(V)f_{\rm cnu}\|^2 = \int_{\mathbb{T}} |\varphi(z)|^2 d\mu_f^{\rm cnu}(z),$$

where $\mu_f^{\rm cnu}$ is the spectral measure of $V$ associated with the vector $f_{\rm cnu}$. By Theorem 4.5, the measure $\mu_f^{\rm cnu}$ is purely a.c.

Summarizing, we can write

$$\|\varphi(\Sigma_{\rm r})f\|^2 \leq \int_{\mathbb{T}} |\varphi(z)|^2 d\mu_f(z), \tag{7.5}$$

where $\mu_f = \mu_f^{\rm u} + \mu_f^{\rm cnu}$ is an absolutely continuous measure on $\mathbb{T}$.

**Step 2. A commutation relation.** We have

$$\Sigma = \begin{pmatrix} \Sigma_{\rm sing} & 0 \\ 0 & \Sigma_{\rm r} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

with respect to our decomposition $L^2(\rho) = X_{\text{sing}} \oplus X_{\text{r}}$. Iterating the commutation relation (5.1), we find

$$\Sigma^{*n}\mathcal{H} = \mathcal{H}\Sigma^n.$$

In our orthogonal decomposition, we can write this relation as

$$\begin{pmatrix} \Sigma_{\text{sing}}^{*n} & 0 \\ 0 & \Sigma_{\text{r}}^{*n} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \Sigma_{\text{sing}}^{n} & 0 \\ 0 & \Sigma_{\text{r}}^{n} \end{pmatrix}.$$

If we write this as a system of four equations, one of them will read

$$\Sigma_{\text{sing}}^{*n} h_{12} = h_{12}\Sigma_{\text{r}}^{n}.$$

Taking a linear combination of these equations and taking into account the anti-linearity of $h_{12}$, we obtain

$$\varphi(\Sigma_{\text{sing}})^* h_{12} = h_{12}\varphi(\Sigma_{\text{r}}) \tag{7.6}$$

for any analytic polynomial $\varphi(z) = \sum_{k=0}^{n} a_k z^k$.

**Step 3.** $\mathcal{H}$ **is diagonal.** Let us choose a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of analytic polynomials such that:

(i) $\|\varphi_n\|_{H^\infty} \le 1$;
(ii) $\varphi_n(z) \to 0$ for a.e. $z \in \mathbb{T}$ (with respect to the Lebesgue measure);
(iii) $\liminf_{n\to\infty}|\varphi_n(z)| \ge c > 0$ for a.e. $z \in \mathbb{T}$ with respect to the (singular) spectral measure of $\Sigma_{\text{sing}}$.

The existence of such polynomials $\varphi_n$ is given by the following lemma.

**Lemma 7.4.** *Let $\nu$ be a singular (regular, Borel) measure on the unit circle $\mathbb{T}$. There exists a sequence of analytic polynomials $\varphi_n$, satisfying properties (i)–(iii) above.*

**Proof.** Let $E$ be the set of Lebesgue measure zero ($|E| = 0$), supporting $\nu$, i.e., such that $\nu(\mathbb{T} \setminus E) = 0$. By the regularity of $\nu$ and the Lebesgue measure there exist increasing sequences of compacts $K_n \subset E$, $F_n \subset \mathbb{T} \setminus E$ such that

$$\lim_{n\to\infty} \nu(K_n) = \mu(E), \qquad \lim_{n\to\infty} |\mathbb{T} \setminus F_n| = 0.$$

Since $\operatorname{dist}(F_n, K_n) > 0$ for all $n$, one can choose continuous functions $f_n : \mathbb{T} \to [0, 1]$ such that $f_n|_{K_n} \equiv 1, f_n|_{F_n} \equiv 0$.

Using the Weierstrass approximation theorem, let us choose trigonometric polynomials $p_n = \sum_{k=-N_n}^{N_n} a_k z^k$ such that $\|f_n - p_n\|_{L^\infty(T)} \le 2^{-n}$ for all $n \ge 1$. Then the analytic polynomials $\varphi_n(z) := z^{N_n} p_n(z)/2$ give the desired sequence. □

We continue the proof of Lemma 7.3. Let us substitute $\varphi_n$ into (7.5). Since $\mu_f$ is absolutely continuous, by conditions (i) and (ii) and the dominated convergence theorem we find

$$\|\varphi_n(\Sigma_r)f\|^2 \leq \int_{\mathbb{T}} |\varphi_n(z)|^2 d\mu_f(z) \to 0, \quad n \to \infty,$$

i.e., $\varphi_n(\Sigma_r) \to 0$ strongly. On the other hand, condition (iii) and Fatou's Lemma imply that for any element $f \in X_{sing}$ we have

$$\liminf_{n\to\infty} \|\varphi_n(\Sigma_{sing})^* f\|^2 = \liminf_{n\to\infty} \int |\varphi_n|^2 d\nu_f \geq \int c^2 d\nu_f = c^2\|f\|^2;$$

here $\nu_f$ is the spectral measure of $\Sigma_{sing}$ associated with the vector $f$.

Substituting $\varphi_n$ into (7.6) and letting $n \to \infty$ we then conclude that $h_{12} = 0$. Using the symmetry condition

$$\langle \mathcal{H}f, g\rangle = \langle \mathcal{H}g, f\rangle,$$

from here it is easy to see that $h_{21} = 0$, and so $\mathcal{H}$ is diagonal in the orthogonal decomposition $X_{sing} \oplus X_r$. Thus, $\mathcal{H}^2 = \mathcal{M}^2$ is also diagonal in this decomposition, and in particular $X_{sing}$ is an invariant subspace for $\mathcal{M}^2$.

**Step 4. Concluding the proof.** By (5.3), we know that $\Sigma$ satisfies

$$\Sigma\Sigma^* = I - \langle \cdot, q\rangle q, \quad q(s) = \overline{\Psi}(s)/s.$$

Since $\Sigma\Sigma^* = I \oplus \Sigma_r\Sigma_r^*$, we conclude that $q \in X_r$. On the other hand, $q(s) \neq 0$ $\rho$-a.e. and so $q$ is a cyclic element for $\mathcal{M}^2$. We find that $X_{sing}$ is an invariant subspace for $\mathcal{M}^2$, orthogonal to its cyclic element $q$. Thus, $X_{sing} = \{0\}$. The proof is complete. □

**Proof of Theorem 7.2.** By Langer's lemma (Lemma 4.2) we have

$$\text{Re}\,\Sigma = \text{Re}\,\Sigma_u \oplus \text{Re}\,\Sigma_{cnu}, \tag{7.7}$$

where $\Sigma_u$ is unitary and $\Sigma_{cnu}$ is completely non-unitary.

(i) Suppose (7.1) holds and so $\Sigma^*$ has defect indices $(1, 0)$.

First suppose that $\Sigma^*$ is asymptotically stable. Then the unitary part of $\Sigma$ is absent, and so $\Sigma$ is completely non-unitary. Then, by Theorem 4.3, $\Sigma$ is unitarily equivalent to the shift operator $S$ and so Re $\Sigma$ has a purely a.c. spectrum $[-1, 1]$ with multiplicity one, as required.

Conversely, suppose that the a.c. spectrum of Re $\Sigma$ is $[-1, 1]$ with multiplicity one. By Lemma 7.3 and Proposition 4.9, the spectrum of Re $\Sigma_u$ is purely a.c. Next, applying Theorem 4.3 again, we find that $\Sigma_{cnu}$ is unitarily equivalent to the shift operator $S$ and so $\operatorname{Re}\Sigma_{\mathrm{cnu}}$ has a purely a.c. spectrum $[-1,1]$ with multiplicity one. Denoting the unitary equivalence by $\simeq$, we can rewrite (7.7) for the a.c. parts as

$$\operatorname{Re} S \simeq \operatorname{Re}\Sigma_{\mathrm{u}} \oplus \operatorname{Re} S.$$

Considering the multiplicity functions of the spectrum on both sides, we see that the term $\operatorname{Re}\Sigma_{\mathrm{u}}$ must be absent from this expression. Thus, $\Sigma^* = \Sigma^*_{\mathrm{cnu}} \simeq S^*$, and so $\Sigma^*$ is asymptotically stable.

(ii) Suppose (7.1) fails and so $\Sigma$ has defect indices $(1,1)$.

First suppose that $\Sigma^*$ is asymptotically stable. Then the unitary part of $\Sigma$ is absent, and so $\Sigma$ is completely non-unitary and by Theorem 4.4 the operator $\operatorname{Re}\Sigma$ has a purely singular spectrum, as required.

Conversely, suppose that the a.c. spectrum of $\operatorname{Re}\Sigma$ in $[-1,1]$ is absent. By Lemma 7.3, the spectrum of $\operatorname{Re}\Sigma_{\mathrm{u}}$ is purely a.c.; thus, the unitary part of $\Sigma_u$ is absent, and so $\Sigma$ is c.n.u. Applying Theorem 4.4 again, we find that $\Sigma^*$ is asymptotically stable, as required. □

## 8 Reduction to spectral properties of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$

In this section we show that under some additional assumptions the operator $\Sigma^*$ is asymptotically stable if and only if the spectrum of the unitary operator $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is purely singular. While at first glance this condition does not look much simpler than the conditions in Theorem 7.2, it will allow us to easily construct examples and counterexamples.

Below $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ is an abstract spectral datum and $\mathcal{M}$, $\widetilde{\mathcal{M}}$ are the operators in $L^2(\rho)$ constructed as in Section 2.1. We recall that the operator $\Sigma_0^*$ in $L^2(\rho)$ was defined by $\Sigma_0^* := \widetilde{\mathcal{M}}\mathcal{M}^{-1}$.

**8.1 Reduction is possible if $I - \Sigma_0 \in \mathbf{S}_1$.** We start with the easiest case, when the difference $I - \Sigma_0$ is trace class.

**Theorem 8.1.** *Let the abstract spectral datum $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be such that $I - \Sigma_0 \in \mathbf{S}_1$. Then $\Sigma^*$ is asymptotically stable if and only if the unitary operator*

$$\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$$

*has a purely singular spectrum.*

**Remark 8.2.** By Lemma 7.1, the operator $\Sigma_0^*$ has defect indices either $(1, 0)$ or $(1, 1)$. By Theorem 6.1, the operator $\Sigma_0^*$ is asymptotically stable, and hence it is c.n.u. By Theorems 4.3 and 4.4, we see that there are two possibilities:

(i) $\Sigma_0^*$ has defect indices $(1, 0)$, and then it is unitarily equivalent to $S^*$, where $S$ is the shift operator in $H^2$;

(ii) $\Sigma_0^*$ has defect indices $(1, 1)$, and then it is unitarily equivalent to $S_\theta^*$, where $S_\theta$ is the compressed shift operator $S_\theta$ in a model space $K_\theta$ for some inner function $\theta$.

Observe that condition $I - \Sigma_0 \in \mathbf{S}_1$ is incompatible with (i), because $I - S$ is not a trace class operator. So the assumption $I - \Sigma_0 \in \mathbf{S}_1$ necessitates that we have (ii).

**Proof of Theorem 8.1.** We have

$$\begin{aligned} \Psi(\mathcal{M})^*\widetilde{\Psi}(\widetilde{\mathcal{M}})^* - \Sigma &= \Psi(\mathcal{M})^*\widetilde{\Psi}(\widetilde{\mathcal{M}})^* - \Psi(\mathcal{M})^*\Sigma_0\widetilde{\Psi}(\widetilde{\mathcal{M}})^* \\ &= \Psi(\mathcal{M})^*(I - \Sigma_0)\widetilde{\Psi}(\widetilde{\mathcal{M}})^* \in \mathbf{S}_1 \end{aligned}$$

and so, taking real parts,

$$\operatorname{Re}(\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})) - \operatorname{Re}\Sigma \in \mathbf{S}_1.$$

Applying Proposition 4.9 and the Kato–Rosenblum Theorem, we find that the spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is purely singular if and only if the spectrum of $\operatorname{Re}\Sigma$ is purely singular.

Finally, as discussed in Remark 8.2, the operator $\Sigma_0$ has defect indices $(1, 1)$, and so $\Sigma$ has the same defect indices. Thus Theorem 7.2(ii) applies and so the spectrum of $\operatorname{Re}\Sigma$ is purely singular if and only if $\Sigma^*$ is asymptotically stable. □

### 8.2 Sufficient conditions for $I - \Sigma_0 \in \mathbf{S}_1$.

The previous theorem leads to the natural question: how to characterize abstract spectral data which correspond to the case $I - \Sigma_0 \in \mathbf{S}_1$? We give some sufficient conditions that guarantee this inclusion. We start with the simplest condition.

**Lemma 8.3.** *Let the abstract spectral datum* $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ *be such that* $\operatorname{supp}\rho$ *is separated away from* $0$*; then* $I - \Sigma_0 \in \mathbf{S}_1$.

**Proof.** The idea is to apply Theorem 4.8 to the operators $\mathcal{M}^2$ and $\widetilde{\mathcal{M}}^2 = \mathcal{M}^2 - \langle\,\cdot\,, \mathbb{1}\rangle\mathbb{1} \geq 0$ and the function $\varphi(s) = \sqrt{s}$. The function $\varphi$ is not sufficiently smooth to comply with the hypothesis of Theorem 4.8, but we can modify it so that the resulting function is in $C_0^\infty(\mathbb{R})$.

Indeed, by assumptions $\sigma(\mathcal{M}^2) \subset [a, R]$ with some $0 < a < R < \infty$; since $\widetilde{\mathcal{M}}^2$ is a rank one perturbation of $\mathcal{M}^2$, we find that $\sigma(\widetilde{\mathcal{M}}^2) \subset \{\lambda_0\} \cup [a, R]$, with some eigenvalue $\lambda_0 \geq 0$. It is clear that we can modify $\varphi$ outside the set $\{\lambda_0\} \cup [a, R]$ such that the resulting function $\widetilde{\varphi}$ is in $C_0^\infty(\mathbb{R})$. Thus, Theorem 4.8 applies to $\widetilde{\varphi}$ and we get

$$\widetilde{\varphi}(\widetilde{\mathcal{M}}^2) - \widetilde{\varphi}(\mathcal{M}^2) = \varphi(\widetilde{\mathcal{M}}^2) - \varphi(\mathcal{M}^2) = \widetilde{\mathcal{M}} - \mathcal{M} \in \mathbf{S}_1.$$

By assumption, the operator $\mathcal{M}$ is invertible, so left multiplying $\mathcal{M} - \widetilde{\mathcal{M}}$ by $\mathcal{M}^{-1}$ and recalling that $\Sigma_0 = \mathcal{M}^{-1}\widetilde{\mathcal{M}}$, we get the conclusion of the lemma. □

Next, we give a slightly more precise sufficient condition. As discussed in Remark 8.2, if $I - \Sigma_0 \in \mathbf{S}_1$, then condition (7.1) is not satisfied, which means that either (7.2) or (7.3) holds. The following lemma says that under conditions that are slightly stronger than (7.2) or (7.3), we have $I - \Sigma_0 \in \mathbf{S}_1$.

**Lemma 8.4.** *Assume that for some $\varepsilon > 0$, we have either*

$$\int_0^\infty \frac{d\rho(s)}{s^2} = 1, \quad \int_0^\infty \frac{d\rho(s)}{s^{4+\varepsilon}} < \infty \tag{8.1}$$

*or*

$$\int_0^\infty \frac{d\rho(s)}{s^2} < 1, \quad \int_0^\infty \frac{d\rho(s)}{s^{2+\varepsilon}} < \infty. \tag{8.2}$$

*Then $I - \Sigma_0$ is trace class.*

The proof is elementary but a little technical; it is given in Appendix A.

### 8.3 Reduction is possible if $\Psi$ and $\widetilde{\Psi}$ are Hölder at 0.

Finally, we turn to the case when $I - \Sigma_0$ is not necessarily trace class. We give a more precise condition, whose proof is based on the application of Ismagilov's Theorem.

**Theorem 8.5.** *Let the abstract spectral datum $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be such that the limits $\Psi(0_+)$ and $\widetilde{\Psi}(0_+)$ exist and that for some $\varepsilon > 0$ we have*

$$\sup_{t>0} t^{-\varepsilon}|\Psi(t) - \Psi(0_+)| < \infty, \quad \sup_{t>0} t^{-\varepsilon}|\widetilde{\Psi}(t) - \widetilde{\Psi}(0_+)| < \infty.$$

*Then $\Sigma^*$ is asymptotically stable if and only if the unitary operator*

$$\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$$

*has a purely singular spectrum.*

We give the proof in Appendix A.

**8.4 Open question.** The previous Theorem naturally leads to the following question.

**Open question.** For a general abstract spectral datum $\Lambda$, is it true that $\Sigma^*$ is asymptotically stable if and only if the spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is purely singular?

At first glance, reduction to $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ does not seem very useful since in general it is not an easy task to decide if this operator has a purely singular spectrum. But in concrete situations this allows us to give convenient sufficient conditions for $\Sigma^*$ to be asymptotically stable, i.e. (see Theorem 2.1) for a spectral datum $\Lambda$ to be in $\Lambda(\mathrm{BMOA_{simp}})$. Most importantly, it also allows to construct counterexamples.

**8.5 Positive results.**

**Theorem 8.6.** *Let* $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ *be an abstract spectral datum. Let* $\Psi_\#$ *be a differentiable unimodular complex valued function on* $[0, \infty)$ *such that its derivative admits the representation*

$$\frac{d}{ds}\Psi_\#(s) = s \int_{-\infty}^{\infty} e^{is^2 t} d\mu(t)$$

*with some finite complex-valued measure* $\mu$ *on* $\mathbb{R}$. *If* $(\rho, \Psi, \widetilde{\Psi}) \in \Lambda(\mathrm{BMOA_{simp}})$, *then* $(\rho, \overline{\Psi_\#}\Psi, \Psi_\#\widetilde{\Psi}) \in \Lambda(\mathrm{BMOA_{simp}})$*; in particular,* $(\rho, \overline{\Psi_\#}, \Psi_\#) \in \Lambda(\mathrm{BMOA_{simp}})$.

**Proof.** By our assumptions on $\Psi_\#$, the function $s \mapsto \Psi_\#(\sqrt{s})$ satisfies the hypothesis of Theorem 4.8. Since $\widetilde{\mathcal{M}}^2 - \mathcal{M}^2$ is a rank one operator, it follows that

$$\Psi_\#(\widetilde{\mathcal{M}}) - \Psi_\#(\mathcal{M}) = \Psi_\#(\sqrt{\widetilde{\mathcal{M}}^2}) - \Psi_\#(\sqrt{\mathcal{M}^2}) \in \mathbf{S}_1,$$

and so, left-multiplying by $\Psi_\#(\mathcal{M})^*$, we find

$$\Psi_\#(\mathcal{M})^*\Psi_\#(\widetilde{\mathcal{M}}) - I \in \mathbf{S}_1.$$

Next, denote

$$\Sigma^* = \widetilde{\Psi}(\widetilde{\mathcal{M}})\,\Sigma_0^*\Psi(\mathcal{M}), \qquad \Sigma_\#^* = \Psi_\#(\widetilde{\mathcal{M}})\,\Sigma^*\Psi_\#(\mathcal{M})^*;$$

here the operator $\Sigma$ corresponds to the spectral datum $(\rho, \Psi, \widetilde{\Psi})$ and $\Sigma_\#$ corresponds to the spectral datum $(\rho, \overline{\Psi_\#}\Psi, \Psi_\#\widetilde{\Psi})$. We have

$$\begin{aligned}\Sigma_\#^* &= \Psi_\#(\mathcal{M})(\Psi_\#(\mathcal{M})^*\Psi_\#(\widetilde{\mathcal{M}}))\,\Sigma^*\Psi_\#(\mathcal{M})^* \\ &= \Psi_\#(\mathcal{M})\,\Sigma^*\Psi_\#(\mathcal{M})^* + \text{trace class operator}.\end{aligned}$$

By taking real parts, it follows that

$$\operatorname{Re}\Sigma_\# = \Psi_\#(\mathcal{M})(\operatorname{Re}\Sigma^*)\Psi_\#(\mathcal{M})^* + \text{trace class operator}.$$

Thus, using the Kato-Rosenblum theorem, we find that $\operatorname{Re}\Sigma_\#$ satisfies the hypothesis of Theorem 7.2 if and only if $\operatorname{Re}\Sigma$ satisfies them. Thus, $\Sigma_\#^*$ is asymptotically stable if and only if $\Sigma^*$ is. Finally, if $\Psi = \widetilde{\Psi} = 1$, then by Theorem 6.1 we have $(\rho, 1, 1) \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$, and therefore $(\rho, \overline{\Psi_\#}, \Psi_\#) \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$. □

**8.6 Counterexamples.** If one of the functions $\Psi$ or $\widetilde{\Psi}$ is constant, then the problem of spectral analysis of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ simplifies significantly and reduces to the spectral analysis of a multiplication operator. Recall that the spectral type of a multiplication operator is easy to determine. Namely, if $\mathcal{M}$ is the multiplication by the independent variable $s$ in $L^2(\rho)$, then a spectral measure (of maximal spectral type) of the operator $\Psi(\mathcal{M})$ is the pushforward of $\rho$ by $\Psi$; we denote this pushforward measure by $\rho \circ (\Psi^{-1})$.

We immediately get the following generalization of Theorem 6.5.

**Theorem 8.7.** *Let $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be an abstract spectral datum such that $I - \Sigma_0 \in \mathbf{S}_1$ and suppose that one of the two measures $\rho \circ (\Psi^{-1})$, $\widetilde{\rho} \circ (\widetilde{\Psi}^{-1})$ is supported at a single point. Then $\Lambda \in \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$ if and only if the other measure is purely singular.*

**Proof.** Suppose that $\widetilde{\rho} \circ (\widetilde{\Psi}^{-1})$ is supported at a point $\zeta$, where $|\zeta| = 1$. Then

$$\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M}) = \zeta\Psi(\mathcal{M}).$$

Thus, the spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is singular if and only if the measure $\rho \circ (\Psi^{-1})$ is singular. It remains to apply Theorem 8.1.

The case when $\rho \circ (\Psi^{-1})$ is supported at a point is considered in the same way. □

Finally, for definiteness, we give a concrete example of a spectral data $\Lambda$ that is not in $\Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

**Corollary 8.8.** *Let $\Lambda = (\rho, \Psi, \widetilde{\Psi})$ be an abstract spectral datum, where the measure $\rho$ is absolutely continuous with $\operatorname{supp}\rho = [a, b]$, $0 < a < b < \infty$, and $\Psi(s) = e^{is}$, $\widetilde{\Psi}(s) = 1$. Then $\Lambda \notin \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$, i.e. $\Lambda$ does not correspond to any Hankel operator.*

**Remark.** It is known (see [1, Propositions 9.1.11, 9.1.12]) that for a finite measure $\mu$ without atoms on $\mathbb{T}$, there exists a Borel measurable (and even continuous) function $F : \mathbb{T} \to [0, 1]$ such that the measure $\mu \circ (F^{-1})$ is the Lebesgue measure on $[0, 1]$. Using this fact, for any given $\rho$ without atoms one can always construct $\Psi$ such that $(\rho, \Psi, 1) \notin \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

## 9 The self-adjoint case

**9.1 A counterexample for self-adjoint Hankel operators.** In this section we consider the question of surjectivity of the spectral map in the case of self-adjoint Hankel operators $\Gamma_u$. By Theorem 1.6, in this case the spectral datum $(\rho, \Psi, \widetilde{\Psi})$ satisfies the additional constraint that $\Psi$ and $\widetilde{\Psi}$ take values $\pm 1$. It is reasonable to ask whether all abstract spectral datum with this additional constraint are in $\Lambda(\mathrm{BMOA}_{\mathrm{simp}})$. It turns out that the answer to this is negative. However, the corresponding counterexample is more subtle and based on a deep result [28] of perturbation theory.

Let $\rho$ be an absolutely continuous measure on an interval $(a, b)$, $0 < a < b < \infty$, $d\rho(x) = w(x)dx$, where $w$ is a strictly positive Hölder continuous function on $(a, b)$. Multiplying $\rho$ by an appropriate positive constant we can ensure that the normalization condition (1.18) is satisfied.

Take any $s_0 \in (a, b)$, and define

$$\Psi(s) = \widetilde{\Psi}(s) = \begin{cases} -1, & s < s_0, \\ 1, & s \geq s_0. \end{cases}$$

**Theorem 9.1.** *Under the above assumptions the operator*

$$\Sigma^* = \widetilde{\Psi}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}\mathcal{M}^{-1}\Psi(\mathcal{M})$$

*is not asymptotically stable, and so (by Theorem 2.1)* $(\rho, \Psi, \widetilde{\Psi}) \notin \Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

In the rest of this section, we present the proof.

**9.2 Overview of the proof.** Since $\operatorname{supp} \rho$ is separated from 0, the condition $\Sigma_0 - I \in \mathbf{S}_1$ is easily seen to be satisfied, see Lemma 8.3. Then by Theorem 8.1, in order to show that $\Sigma^*$ is not asymptotically stable it is sufficient to show that the absolutely continuous spectrum of operator $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is non-empty. Denoting by $E_A$ the (projection-valued) spectral measure of a self-adjoint operator $A$, we can write

$$\begin{aligned}\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M}) &= (I - 2E_{\widetilde{\mathcal{M}}}((-\infty, s_0)))(I - 2E_{\mathcal{M}}((-\infty, s_0))) \\ &= (I - 2E_{\widetilde{\mathcal{M}}^2}((-\infty, s_0^2)))(I - 2E_{\mathcal{M}^2}((-\infty, s_0^2))),\end{aligned}$$

and so the question reduces to investigating the geometry of the ranges of the two spectral projections $E_{\mathcal{M}^2}((-\infty, s_0^2))$ and $E_{\widetilde{\mathcal{M}}^2}((-\infty, s_0^2))$. This question has been studied in [28] in the general framework of scattering theory. We recall the relevant results of [28] in the next subsection. They assert that in our case the a.c. spectrum of the product

$$E_{\mathcal{M}}((-\infty, s_0))E_{\widetilde{\mathcal{M}}}([s_0, \infty))E_{\mathcal{M}}((-\infty, s_0))$$

is non-empty. From here, using some general results on the geometry of two subspaces in a Hilbert space (we use Halmos' paper [13]) it is not difficult to derive that the a.c. spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is also non-empty.

**9.3 Products of spectral projections.** Here we briefly recall some of the results of [28], adapted to the particular case at hand. Let $A_0$ and $A_1$ be bounded (for simplicity) self-adjoint operators in a Hilbert space $X$, such that the difference $A_1 - A_0$ is a (negative) rank one operator:

$$A_1 - A_0 = -\langle \cdot , \omega\rangle\omega,$$

where $\omega$ is a non-zero element in $X$.

Assume that the operators $A_0$ and $A_1$ have a purely absolutely continuous spectrum on an interval $(\alpha, \beta)$. Assume also that the derivatives $F_0'(s)$ and $F_1'(s)$, where

$$F_0(s) := \langle E_{A_0}(-\infty, s)\omega, \omega\rangle, \quad F_1(s) := \langle E_{A_1}(-\infty, s)\omega, \omega\rangle,$$

exist for $s \in (\alpha, \beta)$ and are Hölder continuous functions of $s$. Define

$$\varkappa(s) = \pi^2(F_0'(s))^{1/2}F_1'(s)(F_0'(s))^{1/2}, \quad s \in (\alpha, \beta). \tag{9.1}$$

The following fact was proved in [28], see Lemma 3.2(ii) there.

**Lemma 9.2.** *Under the above assumptions the absolutely continuous part of the operator*

$$E_{A_0}((-\infty, s))E_{A_1}([s, \infty))E_{A_0}((-\infty, s)) \tag{9.2}$$

*is unitarily equivalent to the operator of multiplication by $x$ in $L^2([0, \varkappa(s)], \mathrm{d}x)$.*

In Section 9.5 below we take $A_0 = \mathcal{M}^2$, $A_1 = \widetilde{\mathcal{M}}^2$, $\omega = \mathbb{1} \in L^2(\rho)$ and show that the above hypotheses are satisfied and $\varkappa(s) > 0$, and so the operator (9.2) has a non-trivial absolutely continuous part. From there we will deduce that the operator $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ has a non-trivial absolutely continuous part. In order to do this, we will use some Hilbert space geometry; this is discussed in the next subsection.

**Remark.** The focus of [28] was the connection between the a.c. spectrum of combinations of spectral projections (9.2) (and other similar ones) and the eigenvalues of the scattering matrix of the pair of operators $A_0, A_1$. In the case at hand (when $A_1 - A_0$ is a rank one operator), the scattering matrix is simply a unimodular function on the a.c. spectrum of $A_0$, and it can be expressed directly in terms of $\varkappa(s)$. In any case, Lemma 9.2, which was an intermediate step in [28], suffices for our purposes, and so we are not discussing the scattering matrix here.

**9.4 Pairs of projections and the a.c. spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$.** Let us start with a brief discussion of some of the construction of Halmos' beautiful paper [13]. Let $P$ and $Q$ be two orthogonal projections in a Hilbert space. Following Halmos, we will say that $P$ and $Q$ are in **generic position**, if each of the four subspaces

$$\operatorname{Ran} P \cap \operatorname{Ran} Q, \quad \operatorname{Ker} P \cap \operatorname{Ker} Q, \quad \operatorname{Ran} P \cap \operatorname{Ker} Q, \quad \operatorname{Ker} P \cap \operatorname{Ran} Q \tag{9.3}$$

are trivial.

**Theorem 9.3** ([13])**.** *Let $P$, $Q$ be two orthogonal projections in a generic position. Then there exist self-adjoint positive semi-definite commuting contractions $\mathcal{S}$ and $\mathcal{C}$, with $\mathcal{S}^2 + \mathcal{C}^2 = I$ and* $\operatorname{Ker}\mathcal{S} = \operatorname{Ker}\mathcal{C} = \{0\}$, *such that the pair $P$, $Q$ is unitarily equivalent to the pair*

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{C}^2 & \mathcal{C}\mathcal{S} \\ \mathcal{C}\mathcal{S} & \mathcal{S}^2 \end{pmatrix}.$$

We can now put this together with Lemma 9.2.

**Lemma 9.4.** *Assume, in the hypothesis of Lemma 9.2, that for some $s \in (\alpha, \beta)$ we have $\varkappa(s) > 0$. Then the a.c. spectrum of the unitary operator*

$$(I - 2E_{A_0}((-\infty, s)))(I - 2E_{A_1}((-\infty, s)))$$

*coincides with the arc on the unit circle*

$$\{1 - 2\sigma^2 + i\sigma\sqrt{1 - \sigma^2} : \ \sigma \in [-\varkappa(s), \varkappa(s)]\}. \tag{9.4}$$

*In particular, this a.c. spectrum is non-empty.*

**Proof.** Denote

$$P = E_{A_0}(-\infty, s), \quad Q = E_{A_1}(-\infty, s).$$

These projections are not necessarily in generic position, but for our purposes it is sufficient to consider their generic parts. Namely, let us write our Hilbert space $X$ as

$$X = X_0 \oplus X_{\mathrm{gen}},$$

where $X_0$ is the orthogonal sum of the four subspaces (9.3). It is easy to see that each of these four subspaces is invariant for both $P$ and $Q$, and therefore $X_{\rm gen}$ is also invariant for both $P$ and $Q$. Furthermore, the pair

$$P_{\rm gen} := P|_{X_{\rm gen}}, \quad Q_{\rm gen} := Q|_{X_{\rm gen}}$$

is in a generic position. Thus, according to Theorem 9.3, we can write

$$P_{\rm gen} = U \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} U^*, \quad Q_{\rm gen} = U \begin{pmatrix} \mathcal{C}^2 & \mathcal{C}\mathcal{S} \\ \mathcal{C}\mathcal{S} & \mathcal{S}^2 \end{pmatrix} U^*, \tag{9.5}$$

where $U$ is a unitary operator.

Since the restriction of $P$ and $Q$ onto each of the four subspaces (9.3) is either 0 or $I$, the absolutely continuous parts of the the operators $P(I-Q)P$ and $(I-2Q)(I-2P)$ coincide with the absolutely continuous parts of their generic counterparts $P_{\rm gen}(1-Q_{\rm gen})P_{\rm gen}$ and $(I-2Q_{\rm gen})(I-2P_{\rm gen})$ respectively.

One can see from (9.5) that the operator $\mathcal{S}^2$ is unitarily equivalent to the operator

$$P_{\rm gen}(1-Q_{\rm gen})P_{\rm gen}|_{X_{\rm gen}},$$

so the absolutely continuous part of $\mathcal{S}^2$ is unitarily equivalent to the absolutely continuous part of $P(I-Q)P$, which is described by Lemma 9.2. So, the absolutely continuous part of $\mathcal{S}^2$ is unitarily equivalent to the multiplication by the independent variable $x$ in $L^2([0,\varkappa(s)], \mathrm{d}x)$. Alternatively: the absolutely continuous part of $\mathcal{S}$ is unitarily equivalent to the multiplication by $x$ in $L^2([0,\sqrt{\varkappa(s)}], \mathrm{d}x)$.

On the other hand, according to our model (9.5), we have

$$U^*(I-2Q_{\rm gen})(I-2P_{\rm gen})U = \begin{pmatrix} -I+2\mathcal{C}^2 & -2\mathcal{C}\mathcal{S} \\ 2\mathcal{C}\mathcal{S} & I-2\mathcal{S}^2 \end{pmatrix} = \begin{pmatrix} I-2\mathcal{S}^2 & -2\mathcal{C}\mathcal{S} \\ 2\mathcal{C}\mathcal{S} & I-2\mathcal{S}^2 \end{pmatrix},$$

where we have used the identity $\mathcal{S}^2 + \mathcal{C}^2 = I$ at the last step. The numerical matrix

$$B(\mathsf{s}) := \begin{pmatrix} 1-2\mathsf{s}^2 & -2\mathsf{cs} \\ 2\mathsf{cs} & 1-2\mathsf{s}^2 \end{pmatrix}, \quad 0 < \mathsf{s} < 1, \ \mathsf{c} = \sqrt{1-\mathsf{s}^2} > 0$$

has eigenvalues $\lambda_\pm(\mathsf{s}) := 1-2\mathsf{s}^2 \pm 2i\mathsf{s}\sqrt{1-\mathsf{s}^2}$, and therefore $B(\mathsf{s})$ can be decomposed as

$$B(\mathsf{s}) = V(\mathsf{s}) \begin{pmatrix} \lambda_+(\mathsf{s}) & 0 \\ 0 & \lambda_-(\mathsf{s}) \end{pmatrix} \mathsf{V}(\mathsf{s})^*$$

where $V(\mathsf{s}) = (v_{j,k}(\mathsf{s}))_{j,k=1}^2$ is a unitary $2\times 2$ matrix. The matrix $V(\mathsf{s})$ can be explicitly computed, and can be chosen so the function $\mathsf{s} \mapsto V(\mathsf{s})$ is continuous (and so measurable) on the interval (0, 1). Therefore

$$U^*(I-2Q_{\rm gen})(I-2P_{\rm gen})U = B(\mathcal{S}) = V(\mathcal{S})\begin{pmatrix}\lambda_+(\mathcal{S}) & 0\\ 0 & \lambda_-(\mathcal{S})\end{pmatrix}V(\mathcal{S})^*,$$

where $V(\mathcal{S}) = (v_{j,k}(\mathcal{S}))_{j,k=1}^2$. So $(I-2Q_{\rm gen})(I-2P_{\rm gen})$ is unitarily equivalent to the direct sum $\lambda_+(\mathcal{S})\oplus\lambda_-(\mathcal{S})$. From here we see that the a.c. spectrum of $(I-2Q)(I-2P)$ is given by the arc (9.4), and in particular it is non-empty. □

**9.5 Proof of Theorem 9.1.** As mentioned above, we take $A_0 = \mathcal{M}^2$, $A_1 = \widetilde{\mathcal{M}}^2$, $\omega = \mathbb{1} \in X = L^2(\rho)$, $(\alpha,\beta) = (a^2,b^2)$ in Lemma 9.2; we need to check that the hypothesis of this Lemma is satisfied and $\varkappa(s) > 0$. We have

$$F_0(s) = \langle E_{\mathcal{M}^2}((-\infty,s))\mathbb{1},\mathbb{1}\rangle = \int_{-\infty}^{\sqrt{s}} d\rho(s') = \int_{-\infty}^{\sqrt{s}} w(s')ds',$$

and so the derivative

$$F_0'(s) = \frac{d}{ds}\int_{-\infty}^{\sqrt{s}} w(s')ds' = \frac{1}{2\sqrt{s}}w(\sqrt{s})$$

exists and (by our assumptions on $w$) is Hölder continuous on $(a^2,b^2)$. Let us consider $F_1(s)$. We use the standard rank one identity (which follows from the resolvent identity)

$$T_1(z) = \frac{T_0(z)}{1-T_0(z)},$$

where

$$T_0(z) = \langle(\mathcal{M}^2-z)^{-1}\mathbb{1},\mathbb{1}\rangle,\quad T_1(z) = \langle(\widetilde{\mathcal{M}}^2-z)^{-1}\mathbb{1},\mathbb{1}\rangle.$$

By definition, the operator $T_0(z)$ is the Cauchy transform of a Hölder continuous function, and therefore $T_0(x+i0)$ is Hölder continuous on $(\alpha,\beta)$. Note that in our case $\operatorname{Im} T(x+i0) > 0$ on the interval $(\alpha,\beta)$, and so $1-T_0(x+i0)\neq 0$ on this interval. Further, we have

$$\operatorname{Im} T_1(x+i0) = \frac{\operatorname{Im} T_0(x+i0)}{|1-T_0(x+i0)|^2},$$

and so the density $F_1'(s) = \operatorname{Im} T_1(s+i0)$ is also Hölder continuous and non-vanishing on $(\alpha,\beta)$.

We have checked the hypotheses of Lemma 9.2 and we have established that $\varkappa(s) > 0$ in (9.1). Using Lemma 9.4, we find that the a.c. spectrum of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$ is non-empty. The proof of Theorem 9.1 is complete. □

## 10 Applications to the cubic Szegő equation

**10.1 The cubic Szegő equation.** The cubic Szegő equation is the Hamiltonian evolution equation

$$i\partial_t u = P(|u|^2 u), \tag{10.1}$$

where $P$ is the Szegő projection, i.e., the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2$. Here $u = u(t, z)$, $t \in \mathbb{R}$, $z \in \mathbb{T}$, and the projection $P$ is taken in variable $z$. In this section for typographical reasons we omit the variable $z$, and will be using $u(t)$ instead of more formal $u(t, \cdot)$.

This equation was introduced in [5] where it has been proved to be wellposed on the intersection of $H^2(\mathbb{T})$ with the Sobolev space $W^{s,2}(\mathbb{T})$, for every $s \geq \frac{1}{2}$. More recently, the wellposedness was extended to $\mathrm{BMOA}(\mathbb{T})$ in [10]. In this case, since $\mathrm{BMO}(\mathbb{T}) \subset \bigcap_{p<\infty} L^p(\mathbb{T})$, the right-hand side of (10.1) is in $H^2(\mathbb{T})$, so (10.1) can be interpreted as an ODE with $H^2(\mathbb{T})$-valued functions.

An important property of this equation is that it admits a Lax pair structure involving Hankel operators $H_u$ and $\widetilde{H}_u$, which stimulated the study of the spectral map $\Lambda$, starting with functions $u$ in $H^2(\mathbb{T}) \cap W^{\frac{1}{2},2}(\mathbb{T})$ and $\mathrm{VMOA}(\mathbb{T})$; see [8, 7].

Our first result is the following description of the action of the Szegő dynamics on the set $\Lambda(\mathrm{BMOA}_{\mathrm{simp}})$.

**Theorem 10.1.** *Let* $u_0 \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ *with* $\Lambda(u_0) = (\rho, \Psi_0, \widetilde{\Psi}_0)$. *Denote by u the solution of* (10.1) *such that* $u(0) = u_0$. *Then, for every* $t \in \mathbb{R}$, *we have* $u(t) \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ *and*

$$\Lambda(u(t)) = (\rho, \mathrm{e}^{-its^2}\Psi_0(s), \mathrm{e}^{its^2}\widetilde{\Psi}_0(s)).$$

This result is consistent with Theorem 8.6 with $\Psi_\#(s) = e^{its^2}$. The proof is given in Sections 10.2–10.4.

An important issue in the study of the cubic Szegő equation is the long time behavior of its solutions. Firstly let us discuss the boundedness of trajectories. Using the Lax pair structure, one can prove that trajectories are bounded in $W^{\frac{1}{2},2}(\mathbb{T})$ and in $\mathrm{BMOA}(\mathbb{T})$. However, in [7], it is proved that trajectories are generically unbounded in $W^{s,2}(\mathbb{T})$ for every $s > \frac{1}{2}$.

Secondly comes the problem of almost periodicity of the trajectories in spaces where they are bounded. Let us recall that a function $F = F(x)$ on the real line with values in a Banach space $X$ is called **almost periodic** if it can be approximated (in $C(\mathbb{R}; X)$) by finite linear combinations of functions of the form $F(x) = e^{iax}\psi$, where $a \in \mathbb{R}$ and $\psi \in X$.

In [7], it is proven that every trajectory in $W^{\frac{1}{2},2}(\mathbb{T})$ is almost periodic. Using the same method, a similar result holds for trajectories in VMOA($\mathbb{T}$). It is therefore natural to ask whether this almost periodicity holds for every trajectory in BMOA($\mathbb{T}$). The following theorem shows that the dynamics are much richer in this case.

**Theorem 10.2.** *Let* $u_0 \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$ *with* $\Lambda(u_0) = (\rho, \Psi_0, \widetilde{\Psi}_0)$. *Denote by* $u = u(t)$ *the solution of* (10.1) *such that* $u(0) = u_0$. *Then, if* $\rho$ *is not a pure point measure, then the Fourier coefficient* $\widehat{u}_0(t)$ *of* $u(t)$ *is not almost periodic, and therefore* $u = u(t)$ *(as a function with values in* BMOA*) is not almost periodic.*

The proof is given in Section 10.5.

We conclude by discussing the role of the simplicity condition (1.10). In [7], the action of the Szegő dynamics on the spectral data was described for all compact operators $H_u$ (without the simplicity assumption). In fact, the formula is exactly the same as in Theorem 10.1, where $\Psi_0(s)$ and $\widetilde{\Psi}_0(s)$ are functions with values in the set of Blaschke products. The multiplicity of singular values seems to play a role in the phenomenon of weak turbulence (i.e., growth of high Sobolev norms) of solutions to the Szegő equations. More precisely, in [7], using the vicinity of solutions with multiple spectrum, the authors construct a $G_\delta$-dense set of initial conditions such that the corresponding solutions are weakly turbulent.

**10.2 The action of $H_{u(t)}$ and $\widetilde{H}_{u(t)}$ for smooth initial data** Here we make the first step towards the proof of Theorem 10.1: we describe the evolution under the cubic Szegő equation for smooth initial data.

**Lemma 10.3.** *Assume the hypothesis of Theorem 10.1 and assume in addition that* $u_0$ *is smooth:* $u_0 \in C^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. *Then, using our notation* (1.1) *for spectral measures, we have*

$$\rho_{u(t)}^{|H_{u(t)}|} = \rho_{u_0}^{|H_{u_0}|}, \qquad \rho_{u(t)}^{|\widetilde{H}_{u(t)}|} = \rho_{u_0}^{|\widetilde{H}_{u_0}|} \tag{10.2}$$

*for all* $t > 0$. *Furthermore, for any continuous* $f$ *and any* $t > 0$ *we have*

$$H_{u(t)} f(|H_{u(t)}|) u(t) = |H_{u(t)}| \overline{f}(|H_{u(t)}|) \overline{\Psi}_{u_0}(|H_{u(t)}|) \mathrm{e}^{itH_{u(t)}^2} u(t), \tag{10.3}$$

$$\widetilde{H}_{u(t)} f(|\widetilde{H}_{u(t)}|) u(t) = |\widetilde{H}_{u(t)}| \overline{f}(|\widetilde{H}_{u(t)}|) \widetilde{\Psi}_{u_0}(|\widetilde{H}_{u(t)}|) \mathrm{e}^{it\widetilde{H}_{u(t)}^2} u(t). \tag{10.4}$$

**Proof.** Throughout the proof, we write $u$ in place of $u(t)$ if there is no danger of confusion. For $u_0 \in C^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, we borrow from [9] the following Lax pair identities,

$$\frac{dH_u}{dt} = [B_u, H_u], \qquad B_u := \frac{i}{2}H_u^2 - iT_{|u|^2},$$
$$\frac{d\widetilde{H}_u}{dt} = [\widetilde{B}_u, \widetilde{H}_u], \qquad \widetilde{B}_u := \frac{i}{2}\widetilde{H}_u^2 - iT_{|u|^2},$$

where $T_a$ denotes the **Toeplitz operator** with symbol $a \in L^\infty(\mathbb{T})$, $T_a : H^2 \to H^2$,

$$T_a f = P(af), \quad f \in H^2.$$

Then we define $W = W(t)$, $\widetilde{W} = \widetilde{W}(t)$ to be the solutions of the following linear ODEs on the set of bounded linear operators on $H^2(\mathbb{T})$,

$$\frac{dW}{dt} = B_u W, \quad \frac{d\widetilde{W}}{dt} = \widetilde{B}_u \widetilde{W}, \quad W(0) = \widetilde{W}(0) = I.$$

One easily checks that $W(t)$ and $\widetilde{W}(t)$ are unitary operators and

$$H_{u(t)} = W(t)H_{u_0}W(t)^*, \quad \widetilde{H}_{u(t)} = \widetilde{W}(t)\widetilde{H}_{u_0}\widetilde{W}(t)^*. \tag{10.5}$$

Consequently,

$$|H_{u(t)}| = W(t)|H_{u_0}|W(t)^*, \quad |\widetilde{H}_{u(t)}| = \widetilde{W}(t)|\widetilde{H}_{u_0}|\widetilde{W}(t)^*.$$

Next, let us identify $W(t)^*z^0$, $W(t)^*u(t)$, $\widetilde{W}(t)^*u(t)$. We begin with $W(t)^*z^0$:

$$\frac{d}{dt}W(t)^*z^0 = -W(t)^*B_u z^0,$$

with

$$B_u z^0 = \frac{i}{2}H_u^2 z^0 - iT_{|u|^2}z^0 = -\frac{i}{2}H_u^2 z^0.$$

Hence

$$\frac{d}{dt}W(t)^*z^0 = \frac{i}{2}W(t)^*H_u^2 z^0 = \frac{i}{2}H_{u_0}^2 W(t)^*z^0.$$

This yields

$$W(t)^*z^0 = \mathrm{e}^{i\frac{t}{2}H_{u_0}^2}z^0.$$

Consequently,

$$W(t)^*u(t) = W(t)^*H_{u(t)}z^0 = H_{u_0}W(t)^*z^0 = H_{u_0}\mathrm{e}^{i\frac{t}{2}H_{u_0}^2}z^0,$$

and therefore, using the anti-linearity of $H_{u_0}$,

$$W(t)^*u(t) = \mathrm{e}^{-i\frac{t}{2}H_{u_0}^2}u_0. \tag{10.6}$$

On the other hand,

$$\frac{d}{dt}W(t)^*\widetilde{W}(t) = -W(t)^*B_{u(t)}\widetilde{W}(t) + W(t)^*\widetilde{B}_{u(t)}\widetilde{W}(t) = W(t)^*(\widetilde{B}_{u(t)} - B_{u(t)})\widetilde{W}(t)$$
$$= \frac{i}{2}W(t)^*(\widetilde{H}^2_{u(t)} - H^2_{u(t)})\widetilde{W}(t) = \frac{i}{2}(W(t)^*\widetilde{W}(t)\widetilde{H}^2_{u_0} - H^2_{u_0}W(t)^*\widetilde{W}(t)).$$

We infer

$$W(t)^*\widetilde{W}(t) = \mathrm{e}^{-i\frac{t}{2}H^2_{u_0}}\mathrm{e}^{i\frac{t}{2}\widetilde{H}^2_{u_0}},$$

and consequently

$$\widetilde{W}(t)^*u(t) = \mathrm{e}^{-i\frac{t}{2}\widetilde{H}^2_{u_0}}\mathrm{e}^{i\frac{t}{2}H^2_{u_0}}W(t)^*u(t) = \mathrm{e}^{-i\frac{t}{2}\widetilde{H}^2_{u_0}}\mathrm{e}^{i\frac{t}{2}H^2_{u_0}}\mathrm{e}^{-i\frac{t}{2}H^2_{u_0}}u_0 = \mathrm{e}^{-i\frac{t}{2}\widetilde{H}^2_{u_0}}u_0. \tag{10.7}$$

Next, using (10.6) and (10.5),

$$\begin{aligned}\langle f(|H_{u(t)}|)u(t), u(t)\rangle &= \langle W(t)^*f(|H_{u(t)}|)u(t), W(t)^*u(t)\rangle \\ &= \langle f(|H_{u_0}|)\,\mathrm{e}^{-i\frac{t}{2}H^2_{u_0}}u_0, \mathrm{e}^{-i\frac{t}{2}H^2_{u_0}}u_0\rangle \\ &= \langle f(|H_{u_0}|)u_0, u_0\rangle\end{aligned}$$

and we obtain the first one of the identities (10.2). The second one is obtained in a similar way.

Next, since $u_0 \in \mathrm{BMOA}_{\mathrm{simp}}(\mathbb{T})$, for every continuous function $f$ we have (cf. (1.14))

$$\begin{aligned}H_{u_0}f(|H_{u_0}|)u_0 &= |H_{u_0}|\overline{\Psi}_{u_0}(|H_{u_0}|)\overline{f}(|H_{u_0}|)u_0,\\ \widetilde{H}_{u_0}f(|\widetilde{H}_{u_0}|)u_0 &= |\widetilde{H}_{u_0}|\widetilde{\Psi}_{u_0}(|\widetilde{H}_{u_0}|)\overline{f}(|\widetilde{H}_{u_0}|)u_0.\end{aligned}$$

Then, using (10.6), (10.7) and (10.5),

$$\begin{aligned}W(t)^*H_{u(t)}f(|H_{u(t)}|)u(t) &= H_{u_0}f(|H_{u_0}|)\mathrm{e}^{-i\frac{t}{2}H^2_{u_0}}u_0\\ &= |H_{u_0}|\overline{\Psi}_{u_0}(|H_{u_0}|)\overline{f}(|H_{u_0}|)\mathrm{e}^{i\frac{t}{2}H^2_{u_0}}u_0\\ &= W(t)^*|H_{u(t)}|\overline{f}(|H_{u(t)}|)\overline{\Psi}_{u_0}(|H_{u(t)}|)\mathrm{e}^{itH^2_{u(t)}}u(t),\\ \widetilde{W}(t)^*\widetilde{H}_{u(t)}f(|\widetilde{H}_{u(t)}|)u(t) &= \widetilde{H}_{u_0}f(|\widetilde{H}_{u_0}|)\mathrm{e}^{-i\frac{t}{2}\widetilde{H}^2_{u_0}}u_0\\ &= |\widetilde{H}_{u_0}|\widetilde{\Psi}_{u_0}(|\widetilde{H}_{u_0}|)\overline{f}(|\widetilde{H}_{u_0}|)\mathrm{e}^{i\frac{t}{2}\widetilde{H}^2_{u_0}}u_0\\ &= \widetilde{W}(t)^*|\widetilde{H}_{u(t)}|\overline{f}(|\widetilde{H}_{u(t)}|)\widetilde{\Psi}_{u_0}(|\widetilde{H}_{u(t)}|)\mathrm{e}^{it\widetilde{H}^2_{u(t)}}u(t),\end{aligned}$$

and finally we arrive at (10.3) and (10.4). □

**10.3 Approximation argument.** As the second step, we extend identities (10.2), (10.3) and (10.4) to the general case of initial data $u_0 \in \mathrm{BMOA}_{\mathrm{simp}}$. The new difficulty here is that the operator $T_{|u(t)|^2}$ is unbounded, hence the unitary operators $W(t)$ and $\widetilde{W}(t)$ are more difficult to define. Therefore we prefer to use an approximation argument.

**Lemma 10.4.** *Let* $u \in \mathrm{BMOA}_{\mathrm{simp}}$*; then there exists a sequence of polynomial functions* $u_n \in \mathrm{BMOA}_{\mathrm{simp}}$ *converging to* $u$ *strongly in* $H^2(\mathbb{T})$ *with a uniform bound in the* BMOA *norm.*

**Proof. Step 1. Approximation by polynomial functions.** Take $0 \le r_n \nearrow 1$, and define $u_n(z) := u(r_n z)$, $z \in \mathbb{T}$. Clearly $u_n \to u$ strongly in $H^2(\mathbb{T})$, and writing $H_u$ in a matrix form (with respect to the standard basis in $H^2$), it is easy to see that

$$\|H_{u_n}\| \le \|H_u\|.$$

Observe that one of the equivalent norms on BMOA is given by $\|u\|_{\mathrm{BMOA}} = \|H_u\|$. It follows that

$$\sup_n \|u_n\|_{\mathrm{BMOA}} \le \|u\|_{\mathrm{BMOA}}.$$

Functions $u_n$ are analytic in the closed unit disc $\overline{\mathbb{D}}$, so they can be approximated by polynomial functions uniformly in $\overline{\mathbb{D}}$. Since the norm in $C(\overline{\mathbb{D}})$ is stronger than the norm of BMOA, we obtain approximations by polynomial functions in $H^2$ with the uniform bound on the BMOA norm.

**Step 2. Approximation by polynomial functions in** $\mathrm{BMOA}_{\mathrm{simp}}$**.** Denote by $\mathscr{P}_N$ the vector space of polynomial functions of degree at most $N$, and by $\mathscr{P}_{N,\mathrm{simp}}$ the subset of $u \in \mathscr{P}_N$ satisfying the simplicity of spectrum condition (1.10). To complete the proof of the lemma, it suffices to show that $\mathscr{P}_{N,\mathrm{simp}}$ is dense in $\mathscr{P}_N$. Given $u \in \mathscr{P}_N$, we observe that the range of $H_u$ is contained in $\mathscr{P}_N$. It follows that $u \in \mathscr{P}_{N,\mathrm{simp}}$ whenever the $(N+1)$ vectors $u, H_u^2(u), \dots, H_u^{2N}(u)$ are linearly independent, or equivalently whenever the Gram determinant

$$G_N(u) := \det\langle H_u^{2k}(u), H_u^{2\ell}(u)\rangle_{0\le k,\ell\le N}$$

is not zero. Since $G_N(u)$ is a polynomial function of the real parts and of the imaginary parts of the Fourier coefficients of $u$, the set $\{G_N \ne 0\}$ is either empty or a dense open subset of $\mathscr{P}_N$. Therefore we are reduced to proving that $\mathscr{P}_{N,\mathrm{simp}}$ is not empty. Consider $u(z) := z^{N-1} + z^N$. The matrix of the linear Hankel operator

$\Gamma_u = \Gamma_u^*$ in the basis $\{z^k, k = 0, \dots, N\}$ of $\mathscr{P}_N \supset \operatorname{Ran} \Gamma_u$ is

$$\begin{pmatrix} 0 & 0 & \cdot & 1 & 1 \\ 0 & \cdot & 1 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & \cdot & 0 \\ 1 & 0 & \cdot & 0 & 0 \end{pmatrix}.$$

Consequently, $\Gamma_u$ and so $H_u$ are injective on $\mathscr{P}_N$. Moreover, since

$$H_u^2 = \Gamma_u \mathbf{C}\mathbf{C}\Gamma_u^* = \Gamma_u \Gamma_u^* = \Gamma_u^2,$$

one can check that the matrix of $H_u^2$ in the same basis is three-diagonal, viz.

$$\begin{aligned} H_u^2(z^0) &= 2z^0 + z, \\ H_u^2(z^k) &= z^{k-1} + 2z^k + z^{k+1}, \quad \text{if } 1 \le k \le N-1, \\ H_u^2(z^N) &= z^{N-1} + z^N. \end{aligned}$$

From these formulae, we infer, via an induction argument on $k$, that there exist real numbers $c_{k,j}$ such that $H_u^{2k}(z^0) = z^k + \sum_{j<k} c_{k,j} z^j$ for $k = 0, \dots, N$. We conclude that the vectors $H_u^{2k}(z^0)$, $k = 0, \dots, N$, are linearly independent, and, applying $H_u$, that $H_u^{2k}(u)$, $k = 0, \dots, N$, are linearly independent, or that $u \in \mathscr{P}_{N,\mathrm{simp}}$. □

Before proceeding, for the purposes of clarity we state a (well-known) simple fact as a lemma.

**Lemma 10.5.** *Let $u, u_n \in \mathrm{BMOA}$, $n \in \mathbb{N}$, with $\sup_n \|u_n\|_{\mathrm{BMOA}} < \infty$, and assume that $\|u_n - u\|_{H^2} \to 0$ as $n \to \infty$. Then we have the strong convergence $H_{u_n} \to H_u$, $\widetilde{H}_{u_n} \to \widetilde{H}_u$ and $f(|H_{u_n}|) \to f(|H_u|)$, $f(|\widetilde{H}_{u_n}|) \to f(|\widetilde{H}_u|)$ for any continuous function $f$.*

**Proof.** For any $m \ge 0$ we have

$$H_u z^m = H_u S^m z^0 = (S^*)^m H_u z^0 = (S^*)^m u$$

and therefore $\|H_{u_n} z^m - H_u z^m\|_{H^2} \to 0$ as $n \to 0$. It follows that $\|H_{u_n} p - H_u p\|_{H^2} \to 0$ for all polynomials $p$. The uniform bound on $\|u_n\|_{\mathrm{BMOA}}$ is equivalent to the uniform bound on the operator norms of $H_{u_n}$, and so by the "$\varepsilon/3$-argument" we conclude that $H_{u_n} \to H_u$ strongly. It follows that $H_{u_n}^2 \to H_u^2$ strongly, and therefore $f(H_{u_n}^2) \to f(H_u^2)$ for any continuous function $f$. Since $|H_u| = \sqrt{H_u^2}$, we also obtain $f(|H_{u_n}|) \to f(|H_u|)$ for any continuous $f$.

Finally, since $\widetilde{H}_u = H_{S^* u}$ and $\|S^* u_n \to S^* u\|_{H^2} \to 0$ with the uniform bound on the BMO norms of $S^* u_n$, we obtain the corresponding statements for $\widetilde{H}_{u_n}$. □

We also quote a corollary of the main result of [10] on the continuous dependence of the solution to the cubic Szegő equation on the initial data.

**Proposition 10.6** ([10, Theorem 1])**.** *Suppose* $u_0, u_{0,n} \in \mathrm{BMOA}$*,* $n \geq 1$ *are such that* $\|u_0 - u_{0,n}\|_{H^2} \to 0$ *and* $\sup_n \|u_{0,n}\|_{\mathrm{BMOA}} < \infty$*. Let* $u(t)$*,* $u_n(t)$ *be the solutions to* (10.1) *with the initial data* $u(0) = u_0$*,* $u_n(0) = u_{0,n}$*. Then for any* $t > 0$*, we have* $\|u_n(t) - u(t)\|_{H^2} \to 0$ *as* $n \to \infty$*. Furthermore, the* BMO *norm is preserved by the Szegő dynamics, i.e.,*

$$\|u(t)\|_{\mathrm{BMOA}} = \|u_0\|_{\mathrm{BMOA}}, \quad t > 0.$$

Now we are ready to extend Lemma 10.3 to non-smooth initial data.

**Lemma 10.7.** *Assume the hypothesis of Theorem 10.1. Then for any continuous* $f$ *relations* (10.2)*,* (10.3) *and* (10.4) *hold true.*

**Proof.** Using Lemma 10.4, for a given $u_0 \in \mathrm{BMOA}_{\mathrm{simp}}$ we construct a sequence of polynomial functions $u_{0,n} \in \mathrm{BMOA}_{\mathrm{simp}}$ converging to $u_0$ in $H^2(\mathbb{T})$ and uniformly bounded in BMO norm. For each $u_{0,n}$, the conclusion of Lemma 10.3 holds. Our purpose is to pass to the limit $n \to \infty$ in (10.2), (10.3) and (10.4).

**Step 1. Convergence of measures: passing to the limit in** (10.2)**.** For each $n$, we write (10.2) in the weak form as

$$\begin{aligned} &\langle f(|H_{u_n(t)}|)u_n(t), u_n(t)\rangle = \langle f(|H_{u_n(0)}|)u_n(0), u_n(0)\rangle, \\ &\langle f(|\widetilde{H}_{u_n(t)}|)u_n(t), u_n(t)\rangle = \langle f(|\widetilde{H}_{u_n(0)}|)u_n(0), u_n(0)\rangle \end{aligned} \tag{10.8}$$

for any continuous function $f$. By Lemma 10.5, we can pass to the limit $n \to \infty$ in the right-hand side. Similarly, by Proposition 10.6 combined with Lemma 10.5, we can pass to the limit in the left-hand side. We obtain the desired relations (10.2), expressed in the weak form.

**Direction of further proof.** For every $n$, we have the identities (10.3) and (10.4):

$$H_{u_n(t)} f(|H_{u_n(t)}|)u_n(t) = |H_{u_n(t)}|\overline{f}(|H_{u_n(t)}|)\overline{\Psi}_{u_n(0)}(|H_{u_n(t)}|)e^{itH^2_{u_n(t)}}u_n(t), \tag{10.9}$$

$$\widetilde{H}_{u_n(t)} f(|\widetilde{H}_{u_n(t)}|)u_n(t) = |\widetilde{H}_{u_n(t)}|\overline{f}(|\widetilde{H}_{u_n(t)}|)\widetilde{\Psi}_{u_n(0)}(|\widetilde{H}_{u_n(t)}|)e^{it\widetilde{H}^2_{u_n(t)}}u_n(t). \tag{10.10}$$

Our aim is to pass to the limit here as $n \to \infty$.

In order to motivate the next step, let us make the following remark. Assume that $\Psi_{u_n(0)}$ was a continuous function independent of $n$. Then we could pass to the limit in (10.9) by Lemma 10.5. Unfortunately, this assumption is not true and so we need to use a roundabout argument; we will pass to the limit in the right-hand sides of (10.9) and (10.10) by considering the weak forms of these identities. But first we need to establish the weak convergence of spectral measures multiplied by the factors $\overline{\Psi}_{u_n(0)}$ and $\widetilde{\Psi}_{u_n(0)}$ appearing in the right-hand sides.

**Step 2. Convergence of measures multiplied by $\overline{\Psi}$, $\widetilde{\Psi}$.** At $t = 0$ by Lemma 10.5 we have for every continuous function $f$

$$H_{u_n(0)}f(|H_{u_n(0)}|)u_n(0) \to H_{u(0)}f(|H_{u(0)}|)u(0).$$

Since both $u_n(0)$ and $u(0)$ are in $\mathrm{BMOA}_{\mathrm{simp}}$, we can write this as

$$\overline{\Psi}_{u_n(0)}(|H_{u_n(0)}|)\overline{f}(|H_{u_n(0)}|)u_n(0) \to \overline{\Psi}_{u(0)}(|H_{u(0)}|)\overline{f}(|H_{u(0)}|)u(0), \tag{10.11}$$

and similarly we obtain

$$\widetilde{\Psi}_{u_n(0)}(|\widetilde{H}_{u_n(0)}|)\overline{f}(|\widetilde{H}_{u_n(0)}|)u_n(0) \to \widetilde{\Psi}_{u(0)}(|\widetilde{H}_{u(0)}|)\overline{f}(|\widetilde{H}_{u(0)}|)u(0).$$

Taking the inner product of (10.11) with $u_n(0)$ and observing that

$$\|u_n(0) - u(0)\|_{H^2} \to 0,$$

we find

$$\int_0^\infty \overline{\Psi}_{u_n(0)}(s)\overline{f}(s)\,d\rho_n(s) \to \int_0^\infty \overline{\Psi}_{u(0)}(s)\overline{f}(s)\,d\rho(s), \tag{10.12}$$

where $\rho_n = \rho_{u_n(0)}^{|H_{u_n(0)}|}$ and $\rho = \rho_{u(0)}^{|H_{u(0)}|}$. Similarly, we obtain

$$\int_0^\infty \widetilde{\Psi}_{u_n(0)}(s)\overline{f}(s)\,d\widetilde{\rho}^n(s) \to \int_0^\infty \widetilde{\Psi}_{u(0)}(s)\overline{f}(s)\,d\widetilde{\rho}(s),$$

where $\widetilde{\rho}_n = \rho_{u_n(0)}^{|\widetilde{H}_{u_n(0)}|}$ and $\widetilde{\rho} = \rho_{u(0)}^{|\widetilde{H}_{u(0)}|}$.

**Step 3. Passing to the limit in** (10.9)**,** (10.10)**.** We will pass to the limit in (10.9); the second identity (10.10) can be treated similarly. Fix $t > 0$ and denote

$$v(t) := H_{u(t)}f(|H_{u(t)}|)u(t), \quad w(t) := |H_{u(t)}|\overline{f}(|H_{u(t)}|)\overline{\Psi}_{u(0)}(|H_{u(t)}|)\mathrm{e}^{itH^2_{u(t)}}u(t);$$

our aim is to prove that $v(t) = w(t)$. By Proposition 10.6 and Lemma 10.5, we have

$$v_n(t) := H_{u_n(t)}f(|H_{u_n(t)}|)u_n(t) \to H_{u(t)}f(|H_{u(t)}|)u(t) = v(t)$$

in $H^2$, and therefore

$$\langle v_n(t), u_n(t)\rangle \to \langle v(t), u(t)\rangle.$$

On the other hand, by (10.9) and (10.8),

$$\begin{aligned}\langle v_n(t), u_n(t)\rangle &= \langle |H_{u_n(t)}|\overline{f}(|H_{u_n(t)}|)\overline{\Psi}_{u_n(0)}(|H_{u_n(t)}|)e^{itH^2_{u_n(t)}}u_n(t), u_n(t)\rangle \\ &= \langle |H_{u_n(0)}|\overline{f}(|H_{u_n(0)}|)\overline{\Psi}_{u_n(0)}(|H_{u_n(0)}|)e^{itH^2_{u_n(0)}}u_n(0), u_n(0)\rangle \\ &= \int_0^\infty s\overline{\Psi}_{u_n(0)}(s)\overline{f}(s)e^{its^2}\,d\rho_n(s).\end{aligned}$$

Using (10.12), followed by (10.2) (which was established at the first step of the proof), we find

$$\langle v_n(t), u_n(t)\rangle \to \int_0^\infty s\overline{\Psi}_{u(0)}(s)\overline{f}(s)e^{its^2}d\rho(s) = \langle w(t), u(t)\rangle .$$

Putting this together, we obtain

$$\langle v(t), u(t)\rangle = \langle w(t), u(t)\rangle .$$

Changing $f$ into $fg$, the above identity implies that the orthogonal projection of $v(t)$ onto $\langle u(t)\rangle_{H^2_{u(t)}}$ equals $w(t)$. Since $v(t)$ and $w(t)$ have the same norm, we conclude that these two vectors are equal. □

**10.4 The simplicity of spectrum; concluding the proof of Theorem 10.1.** It remains to prove that $u(t) \in \mathrm{BMOA}_{\mathrm{simp}}$ for every $t \in \mathbb{R}$. This is a consequence of the following lemma.

**Lemma 10.8.** *Let $u \in \mathrm{BMOA}(\mathbb{T})$ be such that*

$$H_u(\langle u\rangle_{H^2_u}) \subset \langle u\rangle_{H^2_u}, \quad \widetilde{H}_u(\langle u\rangle_{\widetilde{H}^2_u}) \subset \langle u\rangle_{\widetilde{H}^2_u}. \tag{10.13}$$

*Then $u \in \mathrm{BMOA}_{\mathrm{simp}}$.*

**Proof.** Recall that, since $\widetilde{H}^2_u = H^2_u - \langle \cdot, u\rangle u$, we have $\langle u\rangle_{\widetilde{H}^2_u} = \langle u\rangle_{H^2_u} = \langle u\rangle$. Denote

$$Z := \overline{\mathrm{Ran}} H_u \cap \langle u\rangle^\perp;$$

our aim is to prove that $Z = \{0\}$. By definition, we have $H_u(Z) \subset Z$ and $\widetilde{H}_u(Z) \subset Z$. Moreover every $h \in Z$ can be written as

$$h = \lim_{n\to\infty} H_u h_n, \quad h_n := H_u(H^2_u + \tfrac{1}{n})^{-1} h \in Z.$$

Consequently, $S^*h = \lim_{n\to\infty} \widetilde{H}_u h_n \in Z$ and

$$\langle h, z^0\rangle = \lim_{n\to\infty} \langle H_u h_n, z^0\rangle = \lim_{n\to\infty} \langle H_u z^0, h_n\rangle = \lim_{n\to\infty} \langle u, h_n\rangle = 0.$$

We conclude that $S^*(Z) \subset Z$ and $Z \perp z^0$, hence $Z \perp z^n$ for every $n$, and finally $Z = \{0\}$. □

**Proof of Theorem 10.1.** By Lemma 10.7, we have the inclusions (10.13) for $u = u(t)$. It follows that $u(t) \in \mathrm{BMOA}_{\mathrm{simp}}$. The first relation in (10.2) shows that the measure $\rho = \rho_{u(t)}^{|H_{u(t)}|}$ is independent of $t$. Relations (10.3) and (10.4) show that the dynamics of the unimodular functions $\Psi$ and $\widetilde{\Psi}$ is as claimed in the statement of the theorem. □

**10.5 Proof of Theorem 10.2.** By Theorem 2.2 (see (2.8)), we have

$$\widehat{u}_k(t) = \langle (\Sigma(t)^*)^k \mathbb{1}, q(t) \rangle_{L^2(\rho)},$$

where $\Sigma(t)$ is given by (2.7) with functions $\Psi$, $\widetilde{\Psi}$ replaced by

$$\Psi_{u(t)}(s) = e^{-its^2} \Psi_0(s) \quad \text{and} \quad \widetilde{\Psi}_{u(t)}(s) = e^{its^2} \widetilde{\Psi}_0(s)$$

respectively, and the function $q(t) = q(t, \cdot)$ is given by

$$q(t, s) = \overline{\Psi_{u(t)}(s)}/s = e^{its^2} \overline{\Psi_0(s)}/s.$$

In particular, we get for $k = 0$ that

$$\widehat{u}_0(t) = \langle \mathbb{1}, q(t) \rangle_{L^2(\rho)} = \int_{\mathbb{R}} e^{-its^2} \frac{\Psi_0(s)}{s} d\rho(s).$$

That means the function $\widehat{u}_0(t)$ is the Fourier transform of the image (pushforward) of the complex measure (of bounded variation)

$$\frac{\Psi_0(s)}{s} \, d\rho(s)$$

under the map $s \mapsto s^2$. Therefore, Theorem 10.2 follows from the following lemma.

**Lemma 10.9.** *Let $\mu$ be a complex Borel measure on $\mathbb{R}$ of bounded variation such that the Fourier transform*

$$\widehat{\mu}(t) = \int_{\mathbb{R}} \mathrm{e}^{-it\lambda} \, d\mu(\lambda)$$

*is an almost periodic function. Then $\mu$ is pure point.*

**Proof.** We decompose $\mu$ as the sum of a pure point measure and a diffuse measure

$$\mu = \sum_{j=1}^{\infty} a_j \delta(\lambda - \lambda_j) + \mu_d,$$

where $\sum_{j=1}^{\infty} |a_j| < \infty$ and $\mu_d(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{R}$. Then

$$\widehat{\mu}(t) = \sum_{j=1}^{\infty} a_j \mathrm{e}^{-i\lambda_j t} + \widehat{\mu}_d(t),$$

and the almost periodicity of $\widehat{\mu}$ implies the almost periodicity of $\widehat{\mu}_d$. For every $T > 0$, the Fubini theorem yields

$$\frac{1}{2T} \int_{-T}^{T} |\widehat{\mu}_d(t)|^2 \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin T(\lambda - \lambda')}{T(\lambda - \lambda')} d\mu_d(\lambda) d\overline{\mu}_d(\lambda').$$

As $T \to +\infty$, the integrand in the right-hand side tends to 0 for every $\lambda \neq \lambda'$. Since $\mu_d$ does not see points, $\mu_d \otimes \overline{\mu}_d$ does not see the diagonal. Therefore the dominated convergence theorem implies that the right-hand side tends to 0. The almost periodic function $\widehat{\mu}_d$ satisfies

$$\frac{1}{2T}\int_{-T}^{T} |\widehat{\mu}_d(t)|^2\, dt \to 0$$

as $T \to +\infty$, hence it is identically 0. From the injectivity of the Fourier transformation, this implies $\mu_d = 0$, hence $\mu$ is pure point. □

## Appendix A Proofs of the reductions to the spectral properties of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$

Here we give the proofs of two technical statements: Lemma 8.4 and Theorem 8.5.

### A.1 Sufficient conditions for $\Sigma_0 - I \in \mathbf{S}_1$ in terms of $\rho$.

**Proof of Lemma 8.4.** We start with the formula

$$\mathcal{M} = \frac{2}{\pi}\int_0^\infty \mathcal{M}^2(\mathcal{M}^2 + t^2 I)^{-1}\, dt = \frac{2}{\pi}\int_0^\infty (I - t^2(\mathcal{M}^2 + t^2 I)^{-1}) dt;$$

if $\mathcal{M}$ is a non-negative real number, this is a trivial identity, and if $\mathcal{M}$ is a positive semi-definite self-adjoint operator, it suffices to combine the scalar identity with the spectral representation of $\mathcal{M}$. Of course, the same identity holds for $\widetilde{\mathcal{M}}$.

The operator $\widetilde{\mathcal{M}}^2 = \mathcal{M}^2 - \langle \cdot, \mathbb{1}\rangle \mathbb{1}$ is a rank one perturbation of $\mathcal{M}$, so by the standard resolvent identity

$$(\widetilde{\mathcal{M}}^2 + t^2 I)^{-1} = (\mathcal{M}^2 + t^2 I)^{-1} + \frac{1}{\Delta(-t^2)}\langle \cdot, (\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}\rangle (\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1},$$

where $\Delta$ is the perturbation determinant,

$$\Delta(-t^2) = 1 - \langle (\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}, \mathbb{1}\rangle = 1 - \int_0^\infty \frac{d\rho(s)}{s^2 + t^2},$$

we get that

$$\widetilde{\mathcal{M}} = \mathcal{M} - \frac{2}{\pi}\int_0^\infty \frac{t^2}{\Delta(-t^2)}\langle \cdot, (\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}\rangle (\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}\, dt. \tag{A.1}$$

Recall that $\Sigma_0^* = \widetilde{\mathcal{M}}\mathcal{M}^{-1}$. Multiplying (A.1) by $\mathcal{M}^{-1}$ on the right, we find that

$I - \Sigma_0^*$ can be represented as an integral of rank one operators:

$$\begin{aligned} I - \Sigma_0^* &= \frac{2}{\pi}\int_0^\infty \frac{t^2}{\Delta(-t^2)}\langle \cdot\,, \mathcal{M}^{-1}(\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}\rangle(\mathcal{M}^2 + t^2 I)^{-1}\mathbb{1}\, dt \\ &= \frac{2}{\pi}\int_0^\infty \frac{t^2}{\Delta(-t^2)}\langle \cdot\,, a_t\rangle b_t\, dt, \\ a_t(s) &:= s^{-1}(s^2+t^2)^{-1}, \quad b_t(s) := (s^2+t^2)^{-1}. \end{aligned} \tag{A.2}$$

First assume (8.2). Then

$$\Delta(-t^2) \geq \Delta(0) = 1 - \int_0^\infty \frac{d\rho(s)}{s^2} > 0.$$

We estimate the norms of $a_t$ and $b_t$ as follows:

$$\begin{aligned} \|a_t\|^2 &= \int_0^\infty s^{-2}(s^2+t^2)^{-2} d\rho(s) \leq t^{-4}\int_0^\infty s^{-2} d\rho(s) = Ct^{-4}, \quad t > 0, \\ \|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2} d\rho(s) \leq t^{-2+\varepsilon}\int_0^\infty s^{-2-\varepsilon} d\rho(s) = Ct^{-2+\varepsilon}, \quad 0 < t < 1, \\ \|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2} d\rho(s) \leq t^{-4}\int_0^\infty d\rho(s) = Ct^{-4}, \quad t > 1. \end{aligned}$$

Then

$$\|I - \Sigma_0^*\|_{\mathbf{S}_1} \leq C\int_0^\infty t^2\|a_t\|\|b_t\| dt \leq C\int_0^1 t^2 t^{-2} t^{-1+\varepsilon/2} dt + C\int_1^\infty t^2 t^{-4} dt < \infty.$$

Next, assume (8.1). Then

$$\begin{aligned} \Delta(-t^2) &= \int_0^\infty \frac{d\rho(s)}{s^2} - \int_0^\infty \frac{d\rho(s)}{s^2+t^2} \\ &= t^2\int_0^\infty \frac{d\rho(s)}{s^2(s^2+t^2)} \geq t^2\int_0^\infty s^{-4} d\rho(s) = ct^2. \end{aligned} \tag{A.3}$$

We estimate the norms of $a_t$ and $b_t$ as follows:

$$\begin{aligned} \|a_t\|^2 &= \int_0^\infty s^{-2}(s^2+t^2)^{-2} d\rho(s) \leq t^{-2+\varepsilon}\int_0^\infty s^{-4-\varepsilon} d\rho(s) = Ct^{-2+\varepsilon}, \quad t > 0, \\ \|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2} d\rho(s) \leq \int_0^\infty s^{-4} d\rho(s) = C, \quad 0 < t < 1, \\ \|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2} d\rho(s) \leq t^{-4}\int_0^\infty d\rho(s) = Ct^{-4}, \quad t > 1. \end{aligned}$$

Then

$$\|I - \Sigma_0^*\|_{\mathbf{S}_1} \leq C\int_0^\infty \|a_t\|\|b_t\| dt \leq C\int_0^1 t^{-1+\varepsilon/2} dt + C\int_1^\infty t^{-1+\varepsilon/2} t^{-2} dt < \infty.$$

The proof is complete. □

**A.2 Trace class inclusions for $(I - \Sigma_0^*)\mathcal{M}^\varepsilon$.** In this subsection we prove preliminary statements that will be used below in the proof of Theorem 8.5.

**Lemma A.1.** *For any $\varepsilon > 0$, the operator $(I - \Sigma_0^*)\mathcal{M}^\varepsilon$ is trace class.*

**Proof.** We may assume $0 < \varepsilon < 1$. As in (A.2), we represent $(I - \Sigma_0^*)\mathcal{M}^\varepsilon$ as an integral of rank one operators:

$$(I - \Sigma_0^*)\mathcal{M}^\varepsilon = \frac{2}{\pi}\int_0^\infty \frac{t^2}{\Delta(-t^2)}\langle \cdot\,, a_t\rangle b_t\, dt,$$
$$a_t(s) = s^{-1+\varepsilon}(s^2+t^2)^{-1}, \quad b_t(s) = (s^2+t^2)^{-1}.$$

First assume that

$$\int_0^\infty \frac{d\rho(s)}{s^2} < 1.$$

Then $\Delta(-t^2) \geq \Delta(0) > 0$. We estimate the norms of $a_t$ and $b_t$ as follows:

$$\begin{aligned}
\|a_t\|^2 &= \int_0^\infty s^{-2+2\varepsilon}(s^2+t^2)^{-2}d\rho(s) \leq (t^2)^{-2+\varepsilon}\int_0^\infty s^{-2+2\varepsilon}(s^2+t^2)^{-\varepsilon}d\rho(s)\\
&\leq t^{-4+2\varepsilon}\int_0^\infty s^{-2+2\varepsilon}s^{-2\varepsilon}d\rho(s) = Ct^{-4+2\varepsilon}, \quad t > 0,\\
\|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2}d\rho(s) \leq t^{-2}\int_0^\infty s^{-2}d\rho(s) = Ct^{-2}, \quad 0 < t < 1,\\
\|b_t\|^2 &= \int_0^\infty (s^2+t^2)^{-2}d\rho(s) \leq t^{-4}\int_0^\infty d\rho(s) = Ct^{-4}, \quad t > 1.
\end{aligned}$$

Then

$$\begin{aligned}
\|(I - \Sigma_0^*)\mathcal{M}^\varepsilon\|_{\mathbf{S}_1} &\leq \int_0^\infty t^2\|a_t\|\|b_t\|dt\\
&\leq C\int_0^1 t^2t^{-2+\varepsilon}t^{-1}dt + C\int_1^\infty t^2t^{-2+\varepsilon}t^{-2}dt < \infty.
\end{aligned}$$

Now consider the case

$$\int_0^\infty \frac{d\rho(s)}{s^2} = 1.$$

For $t > 1$, as in (A.3), we have $\Delta(-t^2) \geq ct^2$ and the above estimates for $\|a_t\|$ and $\|b_t\|$ will do. For $0 < t < 1$ we need to be more careful. We write

$$t^{-2}\Delta(-t^2) = \int_0^t \frac{d\rho(s)}{s^2(s^2+t^2)} + \int_t^\infty \frac{d\rho(s)}{s^2(s^2+t^2)} \geq \frac{1}{2t^2}\int_0^t \frac{d\rho(s)}{s^2} + \frac{1}{2}\int_t^\infty \frac{d\rho(s)}{s^4}.$$

Next,

$$\begin{aligned}
\|b_t\|^2 &= \int_0^\infty \frac{d\rho(s)}{(s^2+t^2)^2} = \int_0^t \frac{d\rho(s)}{(s^2+t^2)^2} + \int_t^\infty \frac{d\rho(s)}{(s^2+t^2)^2}\\
&\leq \frac{1}{t^2}\int_0^t \frac{d\rho(s)}{s^2} + \int_t^\infty \frac{d\rho(s)}{s^4}
\end{aligned}$$

and similarly

$$\begin{aligned}\|a_t\|^2 &= \int_0^\infty \frac{s^{2\varepsilon}d\rho(s)}{s^2(s^2+t^2)^2} = \int_0^t \frac{s^{2\varepsilon}d\rho(s)}{s^2(s^2+t^2)^2} + \int_t^\infty \frac{s^{2\varepsilon}d\rho(s)}{s^2(s^2+t^2)^2}\\ &\le \frac{1}{t^{4-2\varepsilon}}\int_0^t \frac{d\rho(s)}{s^2} + \frac{1}{t^{2-2\varepsilon}}\int_t^\infty \frac{d\rho(s)}{s^4}\\ &= t^{-2+2\varepsilon}\left(\frac{1}{t^2}\int_0^t \frac{d\rho(s)}{s^2} + \int_t^\infty \frac{d\rho(s)}{s^4}\right).\end{aligned}$$

Integrating, we find

$$\begin{aligned}\|(I-\Sigma_0^*)\mathcal{M}^\varepsilon\|_{\mathbf{S}_1} &\le \frac{2}{\pi}\int_0^1 \frac{t^2}{\Delta(-t^2)}\|a_t\|\|b_t\|dt + C\int_1^\infty \|a_t\|\|b_t\|dt\\ &\le 2\int_0^1 t^{-1+\varepsilon}dt + C\int_1^\infty t^{-2+\varepsilon}t^{-2}dt < \infty.\end{aligned}$$

The proof is complete. □

**Lemma A.2.** *For any $\varepsilon > 0$, the operators $\mathcal{M}^\varepsilon(I-\Sigma_0^*)$, $\widetilde{\mathcal{M}}^\varepsilon(I-\Sigma_0^*)$, $(I-\Sigma_0^*)\widetilde{\mathcal{M}}^\varepsilon$ are trace class.*

**Proof.** Taking adjoints in the previous lemma, we find $\mathcal{M}^\varepsilon(I-\Sigma_0) \in \mathbf{S}_1$ for any $\varepsilon > 0$. Since

$$\Sigma_0\Sigma_0^* = I - \langle\,\cdot\,, q_0\rangle q_0, \qquad q_0(s) = 1/s,$$

we have

$$\mathcal{M}^\varepsilon(I-\Sigma_0^*) = \mathcal{M}^\varepsilon(\Sigma_0 - I)\Sigma_0^* + \text{rank one operator},$$

and so $\mathcal{M}^\varepsilon(I-\Sigma_0^*)$ is trace class.

Next, from $\widetilde{\mathcal{M}}^2 \le \mathcal{M}^2$ by Heinz inequality we have $\widetilde{\mathcal{M}}^{2\varepsilon} \le \mathcal{M}^{2\varepsilon}$ for any $0 < \varepsilon < 1$, and so by Lemma 4.1, $\widetilde{\mathcal{M}}^\varepsilon\mathcal{M}^{-\varepsilon}$ is a bounded operator. Therefore the operators

$$\widetilde{\mathcal{M}}^\varepsilon(I-\Sigma_0^*) = (\widetilde{\mathcal{M}}^\varepsilon\mathcal{M}^{-\varepsilon})(\mathcal{M}^\varepsilon(I-\Sigma_0^*))$$

and

$$\widetilde{\mathcal{M}}^\varepsilon(I-\Sigma_0) = (\widetilde{\mathcal{M}}^\varepsilon\mathcal{M}^{-\varepsilon})(\mathcal{M}^\varepsilon(I-\Sigma_0))$$

are trace class. □

**A.3 Proof of Theorem 8.5 (reduction to a.c. part of $\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M})$).** We denote

$$c = \widetilde{\Psi}(0_+)\Psi(0_+), \qquad W = \widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M}).$$

First we prove a lemma.

**Lemma A.3.** *Under the hypothesis of Theorem 8.5, we have*

$$\Sigma^* = c\Sigma_0^* - cI + W + \textit{trace class operator}, \tag{A.4}$$

*and the products*

$$(\Sigma_0^* - I)(W - cI), \quad (W - cI)(\Sigma_0^* - I), \quad (\Sigma_0 - I)(W - cI), \quad (W - cI)(\Sigma_0 - I) \tag{A.5}$$

*are trace class.*

**Proof.** First let us prove that the operators

$$(\Psi(\mathcal{M}) - \Psi(0_+)I)(I - \Sigma_0^*), \quad (I - \Sigma_0^*)(\Psi(\mathcal{M}) - \Psi(0_+)I),$$
$$(\widetilde{\Psi}(\widetilde{\mathcal{M}}) - \widetilde{\Psi}(0_+)I)(I - \Sigma_0^*), \quad (I - \Sigma_0^*)(\widetilde{\Psi}(\widetilde{\mathcal{M}}) - \widetilde{\Psi}(0_+)I)$$

are trace class. The first two inclusions follow from Lemmas A.1 and A.2 by writing

$$\Psi(\mathcal{M}) - \Psi(0_+)I = \mathcal{M}^\varepsilon \varphi(\mathcal{M}) = \varphi(\mathcal{M})\mathcal{M}^\varepsilon,$$

where

$$\varphi(t) = t^{-\varepsilon}(\Psi(t) - \Psi(0_+)), \quad \varphi \in L^\infty.$$

The second two inclusions are obtained in the same way from Lemma A.2.

Now consider the four operator products (A.5). For the first one, we have

$$\begin{aligned}
&(\Sigma_0^* - I)(\widetilde{\Psi}(\widetilde{\mathcal{M}})\Psi(\mathcal{M}) - \widetilde{\Psi}(0_+)\Psi(0_+)I) \\
&\quad = (\Sigma_0^* - I)(\widetilde{\Psi}(\widetilde{\mathcal{M}}) - \widetilde{\Psi}(0_+)I)\Psi(\mathcal{M}) + \widetilde{\Psi}(0_+)(\Sigma_0^* - I)(\Psi(\mathcal{M}) - \Psi(0_+)I),
\end{aligned}$$

where the right-hand side is trace class by the first part of the proof. The other three operators are considered in the same way.

Let us prove (A.4). We have

$$\begin{aligned}
\Sigma^* &= \widetilde{\Psi}(\widetilde{\mathcal{M}})\Sigma_0^*\Psi(\mathcal{M}) \\
&= \widetilde{\Psi}(\widetilde{\mathcal{M}})(\Sigma_0^* - I)\Psi(\mathcal{M}) + W \\
&= \widetilde{\Psi}(0_+)(\Sigma_0^* - I)\Psi(\mathcal{M}) + (\widetilde{\Psi}(\widetilde{\mathcal{M}}) - \widetilde{\Psi}(0_+))(\Sigma_0^* - I)\Psi(\mathcal{M}) + W \\
&= \widetilde{\Psi}(0_+)(\Sigma_0^* - I)\Psi(0_+) + \widetilde{\Psi}(0_+)(\Sigma_0^* - I)(\Psi(\mathcal{M}) - \Psi(0_+)) \\
&\quad + (\widetilde{\Psi}(\widetilde{\mathcal{M}}) - \widetilde{\Psi}(0_+))(\Sigma_0^* - I)\Psi(\mathcal{M}) + W \\
&= c(\Sigma_0^* - I) + W + \text{trace class operator},
\end{aligned}$$

where we have used the first part of the proof at the last step. □

**Proof of Theorem 8.5.** Now let us give the proof of Theorem 8.5. We shall denote by $A_{\mathrm{ac}}$ the a.c. part of a self-adjoint operator $A$ and by $\simeq$ the unitary equivalence between operators.

From Lemma A.3 it follows that

$$\mathrm{Re}(\Sigma^* - cI) = \mathrm{Re}(c\Sigma_0^* - cI) + \mathrm{Re}(W - cI) + \text{trace class operator}$$

and

$$\mathrm{Re}(c\Sigma_0^* - cI)\,\mathrm{Re}(W - cI) \in \mathbf{S}_1, \quad \mathrm{Re}(W - cI)\,\mathrm{Re}(c\Sigma_0^* - cI) \in \mathbf{S}_1.$$

Applying Ismagilov's theorem and the Kato-Rosenblum theorem (see Section 4), we find

$$(\mathrm{Re}(\Sigma^* - cI))_{\mathrm{ac}} \simeq (\mathrm{Re}(c\Sigma_0^* - cI))_{\mathrm{ac}} \oplus (\mathrm{Re}(W - cI))_{\mathrm{ac}}.$$

Shifting all operators here by $\mathrm{Re}\, c$, this simplifies to

$$(\mathrm{Re}\,\Sigma^*)_{\mathrm{ac}} \simeq (\mathrm{Re}(c\Sigma_0^*))_{\mathrm{ac}} \oplus (\mathrm{Re}\,W)_{\mathrm{ac}}. \tag{A.6}$$

This is our key formula. The rest of the proof proceeds slightly differently, depending on the defect indices of $\Sigma^*$.

**The case of defect indices** $(1,1)$**.** Recall that in this case by Theorem 4.4 (applied to $c\Sigma_0^*$), $\mathrm{Re}(c\Sigma_0^*)$ has a purely singular spectrum. By (A.6), it follows that

$$(\mathrm{Re}\,\Sigma^*)_{\mathrm{ac}} \simeq (\mathrm{Re}\,W)_{\mathrm{ac}}.$$

Now by Theorem 7.2(ii), $\Sigma^*$ is asymptotically stable iff the spectrum of $\mathrm{Re}\,W$ is singular. Applying Proposition 4.9, we see that this is true iff the spectrum of $W$ is singular. The proof in this case is complete.

**The case of defect indices** $(1,0)$**.** In this case, the proof is similar but we have to look at the multiplicity of the a.c. spectrum.

Here $\Sigma_0^* \simeq S^*$ and so $\mathrm{Re}(c\Sigma_0^*) \simeq \mathrm{Re}(cS^*)$, where $\mathrm{Re}(cS^*)$ (which is a Jacobi matrix) has a purely a.c. spectrum $[-1,1]$ of multiplicity one. From (A.6) we find

$$(\mathrm{Re}\,\Sigma^*)_{\mathrm{ac}} \simeq (\mathrm{Re}(cS^*))_{\mathrm{ac}} \oplus (\mathrm{Re}\,W)_{\mathrm{ac}}. \tag{A.7}$$

Looking at the multiplicity function of the a.c. spectrum and applying Theorem 7.2(i), we find that $\Sigma^*$ is asymptotically stable if and only if the second term in (A.7) disappears, i.e., if and only if the spectrum of $W$ is singular. The proof is complete. □

## Appendix B Proof of Theorem 1.5

Denote for brevity $R = \operatorname{Ran} H_u$. We first prove that $\operatorname{Ker} H_u = \{0\}$ is equivalent to $z^0 \in \overline{R} \setminus R$.

Assume that $\operatorname{Ker} H_u = \{0\}$. Then $\overline{R} = H^2$ and so, of course, $z^0 \in \overline{R}$; we need to prove that $z^0 \notin R$. Suppose $z^0 \in R$; then $z^0 = H_u w$ for some $w \in H^2$. Denote $\psi = zw$; then $H_u \psi = H_u S w = S^* H_u w = 0$, and so $\psi \in \operatorname{Ker} H_u$, which contradicts our assumption.

Assume that $\operatorname{Ker} H_u \neq \{0\}$. Suppose $z^0 \in \overline{R}$; we need to check that $z^0 \in R$. By Beurling's theorem, $\operatorname{Ker} H_u = \varphi H^2$ for some inner function $\varphi$. Since $z^0 \in \overline{R}$, we have $z^0 \perp \operatorname{Ker} H_u = \varphi H^2$ and so $z^0 \perp \varphi$. Then $\varphi = Sw$ for some inner function $w$. We have $0 = H_u \varphi = H_u S w = S^* H_u w$, so $H_u w$ is a constant function. This constant function is non-zero, because otherwise we would have $w \in \operatorname{Ker} H_u = \varphi H^2 = zwH^2$, which is impossible. Thus, normalizing $w$ if necessary, we find that $z^0 = H_u w$, and so $z^0 \in R$.

Next, we prove that $z^0 \in \overline{R}$ is equivalent to the first condition in (1.21). Indeed, $z^0 \in \overline{R}$ is equivalent to $\int_0^\infty d\rho_{z^0}^{|H_u|}(s) = 1$. Since

$$d\rho_u^{|H_u|}(s) = d\rho_{H_u z^0}^{|H_u|}(s) = s^2 d\rho_{z^0}^{|H_u|}(s),$$

this is equivalent to $\int_0^\infty s^{-2} d\rho_u^{|H_u|}(s) = 1$, which is the first condition in (1.21).

Finally, suppose $z^0 \in \overline{R}$; let us prove that $z^0 \notin R$ is equivalent to the second condition in (1.21). If $z^0 = H_u w$ with $w \in H^2$, then

$$d\rho_{z^0}^{|H_u|}(s) = d\rho_{H_u w}^{|H_u|}(s) = s^2 d\rho_w^{|H_u|}(s),$$

and so

$$\int_0^\infty s^{-4} d\rho_u^{|H_u|}(s) = \int_0^\infty s^{-2} d\rho_{z^0}^{|H_u|}(s) = \int_0^\infty d\rho_w^{|H_u|}(s) < \infty.$$

Conversely, if $\int_0^\infty s^{-4} d\rho_u^{|H_u|}(s) < \infty$, then $u = H_u^2 w$ for some $w \in H^2$. It follows that

$$H_u(z^0 - H_u w) = u - H_u^2 w = 0,$$

and so $z^0 - H_u w \in \operatorname{Ker} H_u$. Since by assumption $z^0 \in \overline{R}$, we have $z^0 - H_u w = 0$, so $z^0 \in R$. The proof of Theorem 1.5 is complete. □

## Appendix C Proof of Theorem 4.4

While it is probably possible to give an "elementary" proof of Theorem 4.4, bypassing the Sz.-Nagy–Foiaş functional model, we prefer a more "high brow" approach, since it highlights a lot of interesting connections.

**C.1 Functional model for c.n.u. contractions with defect indices** $(1, 1)$**.** Let us recall some known facts about the Sz.-Nagy–Foiaş functional model for contractions, focussing on the case of defect indices $(1, 1)$.

Any c.n.u. contraction $T$ is unitarily equivalent to its **functional model**, which is completely determined by the so-called characteristic function $\theta$ of the operator $T$. This characteristic function $\theta$ is generally an operator-valued one; but in the case of defect indices $(1, 1)$ it is a scalar-valued **strictly contractive** (i.e., $|\theta(0)| < 1$) analytic function in the unit disc $\mathbb{D}$.

If the characteristic function $\theta$ of $T$ is an inner function, then the model space for $T$ is the space $K_\theta$ defined above in Section 4.2, and the operator $T$ is unitarily equivalent to the compressed shift $S_\theta$.

If $\theta$ is not inner, then the model is more complicated; in particular, in this case the model space consists of vector-valued functions with values in $L^2$. However, we do not need the complete description of the model here: we only need the following well-known fact.

**Proposition C.1.** *Let $T$ be a c.n.u. contraction with defect indices* $(1, 1)$*, and let $\theta$ be its characteristic function.*

(i) *If $\theta$ is inner, then both $T$ and $T^*$ are asymptotically stable.*

(ii) *If $\theta$ is not inner, then neither $T$ nor $T^*$ are asymptotically stable.*

This proposition follows, for example, from [22, Proposition VI.3.5]. In this proposition $T \in C_{\cdot 0}$ means that $T^*$ is asymptotically stable, and $T \in C_{0\cdot}$ means that $T$ is asymptotically stable. Note that for scalar-valued functions the notion of inner and $*$-inner functions coincide.

Of course, part (i) of Proposition C.1 follows directly from the fact that both the compressed shift $S_\theta$ and its adjoint $S_\theta^*$ are asymptotically stable; this is an easy exercise.

**C.2 Rank one unitary extensions and characteristic function.** For a contraction $T$ with defect indices $(1, 1)$ there exists a rank one perturbation $K$ such that the operator $V = T + K$ (which we will call a **rank one unitary extension** of $T$) is unitary.

To construct such $V$, it suffices to notice that $T$ acts unitarily from $(\mathcal{D}_{T^*})^\perp$ onto $(\mathcal{D}_T)^\perp$, and therefore it maps the one-dimensional defect space $\mathcal{D}_{T^*}$ onto the defect space $\mathcal{D}_T$. Replacing the action of $T$ on $\mathcal{D}_{T^*}$ by a unitary operator from $\mathcal{D}_{T^*}$ to $\mathcal{D}_T$ yields the desired rank one unitary extension $V$. Clearly, such an extension is not unique and any two such extensions differ by a rank one operator.

For a unitary operator $V$ in a Hilbert space, a subspace $E$ is called **$*$-cyclic** if the linear span of the set $\{V^n E : n \in \mathbb{Z}\}$ is dense in our Hilbert space. If $V = T + K$ is a rank one unitary extension of a contraction $T$ with defect indices $(1, 1)$, then we can say that $T = V - K$ is a rank one perturbation of the unitary operator $V$.

Let $b \in \operatorname{Ran} K$ be a unit norm vector. It is a simple exercise (see, e.g., [19, Section 1] or [20, Section 1]) to show that $T$ can be represented as

$$T = V + (\gamma - 1)\langle \cdot , V^* b\rangle b \tag{C.1}$$

with $\gamma \in \mathbb{D}$.

It is also not hard to see that if $\operatorname{span}\{b\} = \operatorname{Ran} K$ is $*$-cyclic for $V$ and $|\gamma| < 1$, then $T$ is c.n.u. For the formal proof see [20, Lemma 1.4], where a more general case of finite rank perturbations was treated. We mention that in our case the matrix $\Gamma$ from [20] reduces to a scalar $|\gamma| < 1$.

On the other hand, if $\operatorname{Ran} K$ is not a $*$-cyclic subspace for $V$, then trivially $T$ is not c.n.u. Indeed, in this case the subspace $(\operatorname{span}\{V^n b : n \in \mathbb{Z}\})^\perp$ is a reducing subspace for both $V$ and $T$, and $T$ coincides with $V$ there.

Combining these facts, we get the following statement.

**Proposition C.2.** *Let $T$ be a c.n.u. contraction with defect indices* $(1, 1)$*, and let $V = T + K$ be a rank one unitary extension of $T$. Then* $\operatorname{Ran} K$ *is a $*$-cyclic subspace for $V$.*

Finally, the following well-known statement relates the spectral properties of a rank one unitary extension and the characteristic function of a c.n.u. contraction.

**Proposition C.3.** *Let $T$ be a c.n.u. contraction with defect indices* $(1, 1)$*, and let $V = T + K$ be a rank one unitary extension of $V$. Then the characteristic function $\theta$ of $T$ is inner if and only if $V$ has a purely singular spectrum.*

**Remark.** The choice of the extension $V$ is not important. Indeed, if $V_1$ and $V_2$ are two such extensions, then $V_1 - V_2$ is a rank one operator, and so by the Kato–Rosenblum theorem, $V_1$ has a purely singular spectrum if and only if $V_2$ has the same property.

**Proof of Proposition C.3.** D. Clark [2] has described all rank one unitary extensions of the compressed shift $S_\theta$; in particular, he showed that all these extensions have a purely singular spectrum. This proves that if the characteristic function $\theta$ is inner, then all rank one unitary extensions have a purely singular spectrum.

To prove the converse, we compute the characteristic function of the operator $T = T_\gamma$ given by (C.1). For the case $\gamma = 0$ the characteristic function $\theta = \theta_0$ is given by the relation

$$\frac{1+\theta_0(z)}{1-\theta_0(z)} = \int_{\mathbb{T}} \frac{1+z\overline{\xi}}{1-z\overline{\xi}} d\rho_f^V(\xi), \tag{C.2}$$

where $\rho_f^V$ is the spectral measure of $V$, corresponding to the unit vector $b$. For $\gamma \neq 0$ the corresponding characteristic functions $\theta = \theta_\gamma$ can be computed as a linear fractional transformation of $\theta_0$,

$$\theta_\gamma = \frac{\theta_0 - \gamma}{1 - \overline{\gamma}\theta_0}, \tag{C.3}$$

see [19, Section 2.4] for the details. One can see immediately from (C.2) that $\theta_0$ is inner if and only if the measure $\rho_f^V$ is purely singular. Identity (C.3) implies that the same holds for all $\theta_\gamma$. □

**C.3 Proof of Theorem 4.4.** It is convenient to introduce one more equivalent condition:

(v) *Any rank one unitary perturbation $V$ of $T$ has purely singular spectrum.*

The statement (iv) means that the characteristic function of $T$ is inner, see Section C.1.

Equivalence of (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iv) follows from Proposition C.1.

Equivalence (v) $\Longleftrightarrow$ (iv) follows from Proposition C.3.

To show that (v) $\Longleftrightarrow$ (iii), let us notice that $\operatorname{Re} V$ and $V$ have a purely singular spectrum simultaneously, see Proposition 4.9. But $\operatorname{Re} T$ is a finite rank perturbation of $\operatorname{Re} V$, so the Kato–Rosenblum Theorem implies the desired equivalence.

The proof of Theorem 4.4 is complete. □

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*Patrick Gérard*
Université Paris-Saclay
Laboratoire de Mathématiques d'Orsay
CNRS, UMR 8628, France
email: patrick.gerard@universite-paris-saclay.fr

*Alexander Pushnitski*
Department of Mathematics
King's College London
Strand, London, WC2R 2LS, UK
email: alexander.pushnitski@kcl.ac.uk

*Sergei Treil*
Department of Mathematics
Brown University
Providence, RI 02912, USA
email: treil@math.brown.edu

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