Journal of Optimization Theory and Applications (2024) 203:2328–2353
https://doi.org/10.1007/s10957-024-02466-7

Integral Resolvent and Proximal Mixtures

Minh N. Bùi[1] · **Patrick L. Combettes**[2]

Received: 8 November 2023 / Accepted: 23 May 2024 / Published online: 24 August 2024

Abstract

Using the theory of Hilbert direct integrals, we introduce and study a monotonicity-preserving operation, termed the integral resolvent mixture. It combines arbitrary families of monotone operators acting on different spaces and linear operators. As a special case, we investigate the resolvent expectation, an operation which combines monotone operators in such a way that the resulting resolvent is the Lebesgue expectation of the individual resolvents. Along the same lines, we introduce an operation that mixes arbitrary families of convex functions defined on different spaces and linear operators to create a composite convex function. Such constructs have so far been limited to finite families of operators and functions. The subdifferential of the integral proximal mixture is shown to be the integral resolvent mixture of the individual subdifferentials. Applications to the relaxation of systems of composite monotone inclusions are presented.

Keywords Hilbert direct integral · Integral proximal mixture · Integral resolvent mixture · Monotone operator · Proximal expectation · Resolvent expectation

1 Introduction

Monotone inclusions provide an effective template to model a wide spectrum of problems in optimization and nonlinear analysis [3, 7, 8, 18, 20, 26, 29]. The question of

Dedicated to the memory of Boris T. Polyak

Communicated by Stefan Ulbrich.

P. L. Combettes. The work of P. L. Combettes was supported by the National Science Foundation under Grant CCF-2211123.

✉ Patrick L. Combettes
plc@math.ncsu.edu

Minh N. Bùi
minh.bui@uni-graz.at

1 Department of Mathematics and Scientific Computing, NAWI Graz, University of Graz, 8010 Graz, Austria

2 Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

combining monotone and linear operators in a fashion that preserves monotonicity has been a recurrent topic; see, e.g., [5, 6, 9, 10, 19]. One such construct is the resolvent mixture [19], an operation that includes in particular the resolvent average [2]. It combines finitely many monotone and linear operators in such a way that the resolvent of the resulting operator is the sum of the individual linearly composed resolvents. Our objective is to extend this construct to arbitrary families of operators. Our analysis rests on the concept of Hilbert direct integrals of families of monotone operators proposed in [11]. Considering the case when the underlying operators are subdifferentials leads us to introduce the Hilbert direct integral of a family of convex functions, a notion that extends proximal mixtures of finite families and, in particular, the proximal average.

Our main contributions are the following.

- We introduce the notion of an integral resolvent mixture for arbitrary families of monotone operators acting on different spaces. This construction exploits the notion of Hilbert direct integrals of set-valued and linear operators from [11]. One of its salient features is that its resolvent is the Lebesgue integral of the linearly composed resolvents of the individual operators. A dual operation of integral resolvent comixture is also investigated.
- We introduce the notion of an integral proximal mixture for arbitrary families of functions defined on different spaces. Its proximity operator turns out to be the Lebesgue integral of the linearly composed proximity operators of the individual functions. A dual operation of integral proximal comixture is also investigated.
- As an instance of an integral resolvent mixture, we propose a notion of resolvent expectation for a family of maximally monotone operators and, likewise, of proximal expectation for a family of functions. These notions extend those of resolvent and proximal averages for finite families.
- We apply the above tools to the relaxation of systems of monotone inclusions involving linear operators. Applications fitting this framework are described and a proximal-type algorithm is proposed.

The paper is organized as follows. In Sect. 2, we set our notation and provide necessary theoretical tools. In Sect. 3, we study the integral resolvent mixture of a family of monotone operators. Section 4 is dedicated to the integral proximal mixture of a family of functions. In Sect. 5, we present an application to systems of monotone inclusions and discuss some special cases of interest arising in data analysis.

2 Notation and Background

We first present our notation, which follows [3].

Let $\mathcal{H}$ be a real Hilbert space with power set $2^{\mathcal{H}}$, identity operator $\mathrm{Id}_{\mathcal{H}}$, scalar product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$, associated norm $\|\cdot\|_{\mathcal{H}}$, and quadratic kernel $\mathcal{Q}_{\mathcal{H}} = \|\cdot\|_{\mathcal{H}}^2/2$.

Let C be a nonempty closed convex subset of $\mathcal{H}$. Then proj_C is the projection operator onto C and N_C is the normal cone operator of C.

Let $T\colon \mathcal{H} \to \mathcal{H}$ and $\tau \in \left]0, +\infty\right[$. Then T is nonexpansive if it is 1-Lipschitzian, τ-cocoercive if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Tx - Ty \rangle_{\mathcal{H}} \geqslant \tau \|Tx - Ty\|_{\mathcal{H}}^2, \tag{2.1}$$

and T is firmly nonexpansive if it is 1-cocoercive.

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. The graph of A is $\operatorname{gra} A = \{(x, x^*) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\}$, the inverse of A is the operator $A^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}$ with graph $\operatorname{gra} A^{-1} = \{(x^*, x) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\}$, the domain of A is $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$, the range of A is $\operatorname{ran} A = \bigcup_{x \in \operatorname{dom} A} Ax$, the set of zeros of A is $\operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$, the resolvent of A is $J_A = (\mathrm{Id}_{\mathcal{H}} + A)^{-1}$, and the Yosida approximation of A of index $\gamma \in]0, +\infty[$ is

$$^{\gamma}A = A \,\square\, \big(\gamma^{-1}\mathrm{Id}_{\mathcal{H}}\big) = \big(A^{-1} + \gamma \mathrm{Id}_{\mathcal{H}}\big)^{-1} = \frac{\mathrm{Id}_{\mathcal{H}} - J_{\gamma A}}{\gamma}. \tag{2.2}$$

Suppose that A is monotone. Then A is maximally monotone if any extension of $\operatorname{gra} A$ is no longer monotone in $\mathcal{H} \oplus \mathcal{H}$. In this case, $\operatorname{dom} J_A = \mathcal{H}$ and J_A is firmly nonexpansive.

Let $f\colon \mathcal{H} \to [-\infty, +\infty]$ and set $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\}$. The Moreau envelope of f is

$$f \,\square\, \mathscr{Q}_{\mathcal{H}}\colon \mathcal{H} \to [-\infty, +\infty]\colon x \mapsto \inf_{y \in \mathcal{H}} \big(f(y) + \mathscr{Q}_{\mathcal{H}}(x - y)\big) \tag{2.3}$$

and the conjugate of f is

$$f^*\colon \mathcal{H} \to [-\infty, +\infty]\colon x^* \mapsto \sup_{x \in \mathcal{H}} \big(\langle x \mid x^* \rangle_{\mathcal{H}} - f(x)\big). \tag{2.4}$$

Now suppose that $f \in \Gamma_0(\mathcal{H})$, that is, f is lower semicontinuous, convex, and such that $-\infty \notin f(\mathcal{H}) \neq \{+\infty\}$. The subdifferential of f is the maximally monotone operator

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x \mid x^* \rangle_{\mathcal{H}} + f(x) \leqslant f(y)\big\} \tag{2.5}$$

and the proximity operator $\operatorname{prox}_f = J_{\partial f}$ of f maps every $x \in \mathcal{H}$ to the unique minimizer of the function $\mathcal{H} \to]-\infty, +\infty]\colon y \mapsto f(y) + \mathscr{Q}_{\mathcal{H}}(x - y)$.

Finally, given a measure space $(\Omega, \mathcal{F}, \mu)$, the symbol $\forall^{\mu}$ means "for μ-almost every" [31].

Definition 2.1 ([19, Definition 1.1]) Let $\mathcal{H}$ and X be real Hilbert spaces, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, and let $L\colon \mathsf{X} \to \mathcal{H}$ be linear and bounded. The *resolvent composition* of A with L is the operator $L \diamond A\colon \mathsf{X} \to 2^{\mathsf{X}}$ given by

$$L \diamond A = \big(L^* \circ J_A \circ L\big)^{-1} - \mathrm{Id}_{\mathsf{X}} \tag{2.6}$$

and the *resolvent cocomposition* of A with L is $L \blacklozenge A = (L \diamond A^{-1})^{-1}$.

Definition 2.2 ([19, Definition 1.4]) Let $\mathcal{H}$ and X be real Hilbert spaces, let $f\colon \mathcal{H} \to [-\infty, +\infty]$, and let $L\colon \mathsf{X} \to \mathcal{H}$ be linear and bounded. The *proximal composition* of f with L is the function $L \diamond f\colon \mathsf{X} \to [-\infty, +\infty]$ given by

$$L \diamond f = \big((f^* \,\square\, \mathscr{Q}_{\mathcal{H}}) \circ L\big)^* - \mathscr{Q}_{\mathsf{X}}, \tag{2.7}$$

and the *proximal cocomposition* of f with L is $L \blacklozenge f = (L \diamond f^*)^*$.

Here are some notation and facts regarding integration in Hilbert spaces. Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space and let H be a separable real Hilbert space. For every $p \in [1, +\infty[$, set

$$\mathscr{L}^p\big(\Omega, \mathcal{F}, \mu; \mathsf{H}\big) = \left\{ x\colon \Omega \to \mathsf{H} \;\middle|\; x \text{ is } (\mathcal{F}, \mathcal{B}_{\mathsf{H}})\text{-measurable and } \int_\Omega \|x(\omega)\|_{\mathsf{H}}^p \,\mu(d\omega) < +\infty \right\}, \tag{2.8}$$

where $\mathcal{B}_{\mathsf{H}}$ is the Borel σ-algebra of H. The Lebesgue (also called Bochner) integral of a mapping $x \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathsf{H})$ is denoted by $\int_\Omega x(\omega)\mu(d\omega)$; see [31, Section V.§7] for background. We denote by $L^p(\Omega, \mathcal{F}, \mu; \mathsf{H})$ the space of equivalence classes of μ-a.e. equal mappings in $\mathscr{L}^p(\Omega, \mathcal{F}, \mu; \mathsf{H})$.

Lemma 2.3 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space, let H be a separable real Hilbert space, and let $x \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathsf{H})$. Then the following hold:*

(i) [31, Théorème 5.7.13] $\|\int_\Omega x(\omega)\mu(d\omega)\|_{\mathsf{H}} \leqslant \int_\Omega \|x(\omega)\|_{\mathsf{H}}\,\mu(d\omega)$.
(ii) Let $\mathsf{x}^* \in \mathsf{H}$. Then the function $\Omega \to \mathbb{R}\colon \omega \mapsto \langle x(\omega) \mid \mathsf{x}^*\rangle_{\mathsf{H}}$ is μ-integrable and

$$\int_\Omega \langle x(\omega) \mid \mathsf{x}^*\rangle_{\mathsf{H}}\,\mu(d\omega) = \left\langle \int_\Omega x(\omega)\mu(d\omega) \;\middle|\; \mathsf{x}^* \right\rangle_{\mathsf{H}}. \tag{2.9}$$

(iii) Suppose that μ is a probability measure. Then

$$\left\| \int_\Omega x(\omega)\mu(d\omega) \right\|_{\mathsf{H}}^2 \leqslant \int_\Omega \|x(\omega)\|_{\mathsf{H}}^2\,\mu(d\omega). \tag{2.10}$$

Proof (ii): Apply [31, Théorème 5.8.16] with the continuous linear functional $L = \langle \cdot \mid \mathsf{x}^*\rangle_{\mathsf{H}}$.

(iii): We derive from (i) and the Cauchy–Schwarz inequality that

$$\left\| \int_\Omega x(\omega)\mu(d\omega) \right\|_{\mathsf{H}}^2 \leqslant \left| \int_\Omega 1_\Omega(\omega)\|x(\omega)\|_{\mathsf{H}}\,\mu(d\omega) \right|^2 \leqslant \mu(\Omega) \int_\Omega \|x(\omega)\|_{\mathsf{H}}^2\,\mu(d\omega), \tag{2.11}$$

which concludes the proof. □

Notation 2.4 Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space, let X and H be separable real Hilbert spaces, and let $(\mathsf{T}_\omega)_{\omega\in\Omega}$ be a family of operators from X to H such that, for every $\mathsf{x} \in \mathsf{X}$, the mapping $\Omega \to \mathsf{H}\colon \omega \mapsto \mathsf{T}_\omega \mathsf{x}$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$-measurable. Let

$$\mathsf{D} = \left\{ \mathsf{x} \in \mathsf{X} \;\middle|\; \int_\Omega \|\mathsf{T}_\omega \mathsf{x}\|_{\mathsf{H}}\, \mu(d\omega) < +\infty \right\}. \tag{2.12}$$

Then

$$\int_\Omega \mathsf{T}_\omega \mu(d\omega)\colon \mathsf{D} \to \mathsf{H}\colon \mathsf{x} \mapsto \int_\Omega \mathsf{T}_\omega \mathsf{x}\, \mu(d\omega). \tag{2.13}$$

In particular, if μ is a probability measure, then

$$\mathsf{E}(\mathsf{T}_\omega)_{\omega\in\Omega} = \int_\Omega \mathsf{T}_\omega \mu(d\omega) \tag{2.14}$$

is the μ*-expectation* of the family $(\mathsf{T}_\omega)_{\omega\in\Omega}$.

The following setup describes the main functional setting employed in the paper. As in [11], it relies on the notion of a Hilbert direct integral of Hilbert spaces [25].

Assumption 2.5 Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space, let $(\mathsf{H}_\omega)_{\omega\in\Omega}$ be a family of real Hilbert spaces, and let $\prod_{\omega\in\Omega} \mathsf{H}_\omega$ be the usual real vector space of mappings x defined on Ω such that $(\forall \omega \in \Omega)\ x(\omega) \in \mathsf{H}_\omega$. Let $((\mathsf{H}_\omega)_{\omega\in\Omega}, \mathfrak{G})$ be an $\mathcal{F}$*-measurable vector field of real Hilbert spaces*, that is, $\mathfrak{G}$ is a vector subspace of $\prod_{\omega\in\Omega} \mathsf{H}_\omega$ which satisfies the following:

[A] For every $x \in \mathfrak{G}$, the function $\Omega \to \mathbb{R}\colon \omega \mapsto \|x(\omega)\|_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable.
[B] For every $x \in \prod_{\omega\in\Omega} \mathsf{H}_\omega$,

$$\Big[\ (\forall y \in \mathfrak{G})\ \ \Omega \to \mathbb{R}\colon \omega \mapsto \langle x(\omega) \mid y(\omega)\rangle_{\mathsf{H}_\omega} \text{ is } \mathcal{F}\text{-measurable}\ \Big] \quad \Rightarrow \quad x \in \mathfrak{G}. \tag{2.15}$$

[C] There exists a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $(\forall \omega \in \Omega)\ \overline{\operatorname{span}}\{e_n(\omega)\}_{n\in\mathbb{N}} = \mathsf{H}_\omega$.

Set

$$\mathfrak{H} = \left\{ x \in \mathfrak{G} \;\middle|\; \int_\Omega \|x(\omega)\|^2_{\mathsf{H}_\omega} \mu(d\omega) < +\infty \right\}, \tag{2.16}$$

and let $\mathcal{H}$ be the real Hilbert space of equivalence classes of μ-a.e. equal mappings in $\mathfrak{H}$ equipped with the scalar product

$$\langle \cdot \mid \cdot \rangle_{\mathcal{H}}\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}\colon (x, y) \mapsto \int_\Omega \langle x(\omega) \mid y(\omega)\rangle_{\mathsf{H}_\omega} \mu(d\omega), \tag{2.17}$$

where we adopt the common practice of designating by x both an equivalence class in $\mathcal{H}$ and a representative of it in $\mathfrak{H}$. We write

$$\mathcal{H} = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{H}_\omega \mu(d\omega) \tag{2.18}$$

and call $\mathcal{H}$ the *Hilbert direct integral of* $((\mathsf{H}_\omega)_{\omega\in\Omega}, \mathfrak{G})$ [25].

Here are some instances of Hilbert direct integrals [11].

Example 2.6 Let $p \in \mathbb{N} \smallsetminus \{0\}$, let $(\alpha_k)_{1\leqslant k\leqslant p}$ be a family in $]0, +\infty[$, let $(\mathsf{H}_k)_{1\leqslant k\leqslant p}$ be separable real Hilbert spaces, let $\mathfrak{G} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_p$ be the usual Cartesian product vector space, and set

$$\Omega = \{1, \ldots, p\}, \quad \mathcal{F} = 2^{\{1,\ldots,p\}}, \quad \text{and} \quad \big(\forall k \in \{1, \ldots, p\}\big) \;\; \mu\big(\{k\}\big) = \alpha_k. \tag{2.19}$$

Then $((\mathsf{H}_k)_{1\leqslant k\leqslant p}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and ${}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{H}_\omega \mu(d\omega)$ is the weighted Hilbert direct sum of $(\mathsf{H}_k)_{1\leqslant k\leqslant p}$, namely the Hilbert space obtained by equipping $\mathfrak{G}$ with the scalar product

$$\big((\mathsf{x}_k)_{1\leqslant k\leqslant p}, (\mathsf{y}_k)_{1\leqslant k\leqslant p}\big) \mapsto \sum_{k=1}^{p} \alpha_k \langle \mathsf{x}_k \mid \mathsf{y}_k \rangle_{\mathsf{H}_k}. \tag{2.20}$$

Example 2.7 Let $(\alpha_k)_{k\in\mathbb{N}}$ be a family in $]0, +\infty[$, let $(\mathsf{H}_k)_{k\in\mathbb{N}}$ be separable real Hilbert spaces, let $\mathfrak{G} = \prod_{k\in\mathbb{N}} \mathsf{H}_k$, and set

$$\Omega = \mathbb{N}, \quad \mathcal{F} = 2^{\mathbb{N}}, \quad \text{and} \quad (\forall k \in \mathbb{N}) \;\; \mu\big(\{k\}\big) = \alpha_k. \tag{2.21}$$

Then $((\mathsf{H}_k)_{k\in\mathbb{N}}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and ${}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{H}_\omega \mu(d\omega)$ is the Hilbert space obtained by equipping the vector space

$$\mathfrak{H} = \left\{ (\mathsf{x}_k)_{k\in\mathbb{N}} \in \mathfrak{G} \;\middle|\; \sum_{k\in\mathbb{N}} \alpha_k \|\mathsf{x}_k\|^2_{\mathsf{H}_k} < +\infty \right\} \tag{2.22}$$

with the scalar product

$$\big((\mathsf{x}_k)_{k\in\mathbb{N}}, (\mathsf{y}_k)_{k\in\mathbb{N}}\big) \mapsto \sum_{k\in\mathbb{N}} \alpha_k \langle \mathsf{x}_k \mid \mathsf{y}_k \rangle_{\mathsf{H}_k}. \tag{2.23}$$

Example 2.8 Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space, let H be a separable real Hilbert space, and set

$$\big[\; (\forall \omega \in \Omega) \;\; \mathsf{H}_\omega = \mathsf{H} \;\big] \quad \text{and} \quad \mathfrak{G} = \big\{ x \colon \Omega \to \mathsf{H} \mid x \text{ is } (\mathcal{F}, \mathcal{B}_{\mathsf{H}})\text{-measurable} \big\}. \tag{2.24}$$

Then $((\mathsf{H}_\omega)_{\omega\in\Omega}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and

$$^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{H}_\omega\mu(d\omega) = L^2\big(\Omega, \mathcal{F}, \mu; \mathsf{H}\big). \tag{2.25}$$

3 Integral Resolvent Mixtures

Our setting hinges on the following assumptions.

Assumption 3.1 Assumption 2.5 and the following are in force:

[A] For every $\omega \in \Omega$, $\mathsf{A}_\omega\colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ is maximally monotone.
[B] For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\mathsf{A}_\omega}x(\omega)$ lies in $\mathfrak{G}$.
[C] dom $^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega\mu(d\omega) \neq \varnothing$, where

$$^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega\mu(d\omega)\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ x^*(\omega) \in \mathsf{A}_\omega x(\omega)\big\} \tag{3.1}$$

is the *Hilbert direct integral of the operators* $(\mathsf{A}_\omega)_{\omega\in\Omega}$ *relative to* $\mathfrak{G}$ [11].

Assumption 3.2 Assumption 2.5 and the following are in force:

[A] X is a separable real Hilbert space.
[B] For every $\omega \in \Omega$, $\mathsf{L}_\omega\colon \mathsf{X} \to \mathsf{H}_\omega$ is linear and bounded.
[C] For every $\mathsf{x} \in \mathsf{X}$, the mapping $\mathfrak{e}_{\mathsf{L}}\mathsf{x}\colon \omega \mapsto \mathsf{L}_\omega\mathsf{x}$ lies in $\mathfrak{G}$.
[D] $0 < \int_\Omega \|\mathsf{L}_\omega\|^2\mu(d\omega) \leqslant 1$.

The main purpose of this section is to study the following objects which mix families of monotone and linear operators.

Definition 3.3 Suppose that Assumptions 3.1 and 3.2 are in force. The *integral resolvent mixture* of $(\mathsf{A}_\omega)_{\omega\in\Omega}$ and $(\mathsf{L}_\omega)_{\omega\in\Omega}$ is

$$\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega} = \left(\int_\Omega \big(\mathsf{L}_\omega^* \circ J_{\mathsf{A}_\omega} \circ \mathsf{L}_\omega\big)\mu(d\omega)\right)^{-1} - \mathrm{Id}_{\mathsf{X}}, \tag{3.2}$$

and the *integral resolvent comixture* of $(\mathsf{A}_\omega)_{\omega\in\Omega}$ and $(\mathsf{L}_\omega)_{\omega\in\Omega}$ is

$$\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega} = \Big(\overset{\diamond}{\mathsf{M}}\big(\mathsf{L}_\omega, \mathsf{A}_\omega^{-1}\big)_{\omega\in\Omega}\Big)^{-1}. \tag{3.3}$$

We start off with some properties of integrals of composite Lipschitzian operators.

Proposition 3.4 *Suppose that* Assumption 2.5 *is in force. Let* X *be a separable real Hilbert space, let* $\beta\colon \Omega \to\ ]0, +\infty[$ *be* $\mathcal{F}$*-measurable and such that* ess sup $\beta < +\infty$, *and for every* $\omega \in \Omega$, *let* $\mathsf{T}_\omega\colon \mathsf{H}_\omega \to \mathsf{H}_\omega$ *be* $\beta(\omega)$*-Lipschitzian and let* $\mathsf{L}_\omega\colon \mathsf{X} \to \mathsf{H}_\omega$ *be linear and bounded. Suppose that the following are satisfied:*

[A] For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \mathsf{T}_\omega x(\omega)$ lies in $\mathfrak{G}$.
[B] There exists $z \in \mathfrak{H}$ such that the mapping $\omega \mapsto \mathsf{T}_\omega z(\omega)$ lies in $\mathfrak{H}$.
[C] For every $\mathsf{x} \in \mathsf{X}$, the mapping $\mathfrak{e}_\mathsf{L}\mathsf{x}\colon \omega \mapsto \mathsf{L}_\omega \mathsf{x}$ lies in $\mathfrak{G}$.
[D] $\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) < +\infty$.

Set

$$\mathsf{T} = \int_\Omega \big(\mathsf{L}_\omega^* \circ \mathsf{T}_\omega \circ \mathsf{L}_\omega\big)\mu(d\omega) \quad \text{and} \quad \tau = \int_\Omega \|\mathsf{L}_\omega\|^2 \beta(\omega)\mu(d\omega). \tag{3.4}$$

Then the following hold:

(i) $\mathsf{T}\colon \mathsf{X} \to \mathsf{X}$ is well defined and τ-Lipschitzian.
(ii) Define $L\colon \mathsf{X} \to \mathcal{H}\colon \mathsf{x} \mapsto \mathfrak{e}_\mathsf{L}\mathsf{x}$ and $T = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{T}_\omega \mu(d\omega)$. Then L is well defined, linear, and bounded with $\|L\| \leqslant \sqrt{\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega)}$, and $\mathsf{T} = L^* \circ T \circ L$.
(iii) Suppose that, for every $\omega \in \Omega$, T_ω is $1/\beta(\omega)$-cocoercive and $\mathsf{L}_\omega \neq 0$. Then the following are satisfied:
  (a) T is $1/\tau$-cocoercive.
  (b) $\big\{\int_\Omega \mathsf{L}_\omega^*(\mathsf{T}_\omega x(\omega))\mu(d\omega) \mid x \in \mathfrak{H}\big\} \subset \overline{\operatorname{ran}}\,\mathsf{T}$.
  (c) $\operatorname{int}\big\{\int_\Omega \mathsf{L}_\omega^*(\mathsf{T}_\omega x(\omega))\mu(d\omega) \mid x \in \mathfrak{H}\big\} \subset \operatorname{ran}\mathsf{T}$.

Proof Observe that, by [11, Proposition 3.12(i)], the function $\Omega \to \mathbb{R}\colon \omega \mapsto \|\mathsf{L}_\omega\|$ is $\mathcal{F}$-measurable and, by [D],

$$\tau \leqslant (\operatorname{ess\,sup} \beta) \int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) < +\infty. \tag{3.5}$$

We set $(\forall \omega \in \Omega)\ \mathsf{R}_\omega = \mathsf{L}_\omega^* \circ \mathsf{T}_\omega \circ \mathsf{L}_\omega$.

(i): Let $\mathsf{x} \in \mathsf{X}$. It results from [11, Proposition 3.12(ii)] that the mapping $\omega \mapsto \mathsf{L}_\omega \mathsf{x}$ lies in $\mathfrak{H}$. In turn, [A] ensures that the mapping $\omega \mapsto \mathsf{T}_\omega(\mathsf{L}_\omega \mathsf{x})$ lies in $\mathfrak{G}$. Therefore, we deduce from [C] and [11, Lemma 2.2(i)] that, for every $\mathsf{y} \in \mathsf{X}$, the function $\Omega \to \mathbb{R}\colon \omega \mapsto \langle \mathsf{y} \mid \mathsf{R}_\omega \mathsf{x}\rangle_\mathsf{X} = \langle \mathsf{L}_\omega \mathsf{y} \mid \mathsf{T}_\omega(\mathsf{L}_\omega \mathsf{x})\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. Thus, since $(\Omega, \mathcal{F}, \mu)$ is a complete σ-finite measure space and X is separable, we infer from [31, Théorème 5.6.24] that the mapping $\Omega \to \mathsf{X}\colon \omega \mapsto \mathsf{R}_\omega \mathsf{x}$ is $(\mathcal{F}, \mathcal{B}_\mathsf{X})$-measurable. Next, since $\operatorname{ess\,sup} \beta < +\infty$, it follows from [A], [B], and [11, Proposition 3.4(i)] that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \mathsf{T}_\omega x(\omega)$ lies in $\mathfrak{H}$; in particular, $\int_\Omega \|\mathsf{T}_\omega 0\|_{\mathsf{H}_\omega}^2 \mu(d\omega) < +\infty$. Hence, because

$$\begin{aligned}(\forall \mathsf{y} \in \mathsf{X})(\forall \omega \in \Omega) \quad \|\mathsf{R}_\omega \mathsf{x} - \mathsf{R}_\omega \mathsf{y}\|_\mathsf{X} &\leqslant \|\mathsf{L}_\omega\|\, \|\mathsf{T}_\omega(\mathsf{L}_\omega \mathsf{x}) - \mathsf{T}_\omega(\mathsf{L}_\omega \mathsf{y})\|_{\mathsf{H}_\omega}\\ &\leqslant \|\mathsf{L}_\omega\| \beta(\omega) \|\mathsf{L}_\omega \mathsf{x} - \mathsf{L}_\omega \mathsf{y}\|_{\mathsf{H}_\omega}\\ &\leqslant \|\mathsf{L}_\omega\|^2 \beta(\omega) \|\mathsf{x} - \mathsf{y}\|_\mathsf{X},\end{aligned} \tag{3.6}$$

we derive from the triangle and Cauchy–Schwarz inequalities that

$$\int_\Omega \|\mathsf{R}_\omega \mathsf{x}\|_\mathsf{X}\, \mu(d\omega) \leqslant \int_\Omega \|\mathsf{R}_\omega \mathsf{x} - \mathsf{R}_\omega 0\|_\mathsf{X}\, \mu(d\omega) + \int_\Omega \|\mathsf{R}_\omega 0\|_\mathsf{X}\, \mu(d\omega)$$

$$\leqslant \tau\|\mathsf{x}\|_{\mathsf{X}} + \int_{\Omega} \|\mathsf{L}_\omega\| \, \|\mathsf{T}_\omega 0\|_{\mathsf{H}_\omega} \mu(d\omega)$$
$$\leqslant \tau\|\mathsf{x}\|_{\mathsf{X}} + \sqrt{\int_{\Omega} \|\mathsf{L}_\omega\|^2 \mu(d\omega)} \sqrt{\int_{\Omega} \|\mathsf{T}_\omega 0\|^2_{\mathsf{H}_\omega} \mu(d\omega)}$$
$$< +\infty. \tag{3.7}$$

Thus, [31, Théorème 5.7.21] implies that $\mathsf{T}\colon \mathsf{X} \to \mathsf{X}$ is well defined. Moreover, by virtue of (3.6) and Lemma 2.3(i), T is τ-Lipschitzian.

(ii): Thanks to [11, Items (ii) and (v) in Proposition 3.12], $L\colon \mathsf{X} \to \mathcal{H}$ is a well-defined bounded linear operator with adjoint

$$L^*\colon \mathcal{H} \to \mathsf{X}\colon x^* \mapsto \int_{\Omega} \mathsf{L}^*_\omega x^*(\omega)\mu(d\omega) \tag{3.8}$$

and $\|L\| \leqslant \sqrt{\int_{\Omega} \|\mathsf{L}_\omega\|^2 \mu(d\omega)}$. On the other hand, [11, Proposition 3.4(i)] asserts that $T\colon \mathcal{H} \to \mathcal{H}$ and that, for every $x \in \mathcal{H}$, a representative of Tx in $\mathfrak{H}$ is the mapping $\omega \mapsto \mathsf{T}_\omega x(\omega)$. Altogether, for every $\mathsf{x} \in \mathsf{X}$, because $\omega \mapsto \mathsf{L}_\omega \mathsf{x}$ is a representative of $L\mathsf{x}$ in $\mathfrak{H}$, we deduce that

$$L^*\big(T(L\mathsf{x})\big) = \int_{\Omega} \mathsf{L}^*_\omega\big(\mathsf{T}_\omega(\mathsf{L}_\omega \mathsf{x})\big)\mu(d\omega) = \mathsf{T}\mathsf{x}, \tag{3.9}$$

as announced.

(iii)(a): Take $\mathsf{x} \in \mathsf{X}$ and $\mathsf{y} \in \mathsf{X}$. Define an $\mathcal{F}$-measurable function on Ω by $\alpha\colon \Omega \to \,]0,+\infty[\colon \omega \mapsto \|\mathsf{L}_\omega\|^2\beta(\omega)/\tau$ and a probability measure P on $\mathcal{F}$ by $\mathsf{P}\colon \Xi \mapsto \int_{\Xi} \alpha(\omega)\mu(d\omega)$. Then we derive from items (ii) and (iii) of Lemma 2.3 together with [31, Théorème 5.10.13] that

$$\begin{aligned}
\langle \mathsf{x}-\mathsf{y} \mid \mathsf{T}\mathsf{x}-\mathsf{T}\mathsf{y}\rangle_{\mathsf{X}} &= \int_{\Omega} \big\langle \mathsf{x}-\mathsf{y} \,\big|\, \mathsf{L}^*_\omega\big(\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x})\big) - \mathsf{L}^*_\omega\big(\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{y})\big)\big\rangle_{\mathsf{X}}\, \mu(d\omega)\\
&= \int_{\Omega} \big\langle \mathsf{L}_\omega\mathsf{x}-\mathsf{L}_\omega\mathsf{y} \,\big|\, \mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x}) - \mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{y})\big\rangle_{\mathsf{H}_\omega} \mu(d\omega)\\
&\geqslant \int_{\Omega} \frac{1}{\beta(\omega)} \big\|\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x}) - \mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{y})\big\|^2_{\mathsf{H}_\omega} \mu(d\omega)\\
&\geqslant \int_{\Omega} \frac{1}{\|\mathsf{L}_\omega\|^2\beta(\omega)} \big\|\mathsf{L}^*_\omega\big(\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x})\big) - \mathsf{L}^*_\omega\big(\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{y})\big)\big\|^2_{\mathsf{X}}\, \mu(d\omega)\\
&= \frac{1}{\tau}\int_{\Omega} \Big\| \frac{1}{\alpha(\omega)}(\mathsf{R}_\omega\mathsf{x} - \mathsf{R}_\omega\mathsf{y})\Big\|^2_{\mathsf{X}} \mathsf{P}(d\omega)\\
&\geqslant \frac{1}{\tau}\Big\| \int_{\Omega} \frac{1}{\alpha(\omega)}(\mathsf{R}_\omega\mathsf{x} - \mathsf{R}_\omega\mathsf{y})\mathsf{P}(d\omega)\Big\|^2_{\mathsf{X}}\\
&= \frac{1}{\tau}\Big\| \int_{\Omega} \big(\mathsf{R}_\omega\mathsf{x} - \mathsf{R}_\omega\mathsf{y}\big)\mu(d\omega)\Big\|^2_{\mathsf{X}}
\end{aligned}$$

$$= \frac{1}{\tau}\|\mathsf{T}x - \mathsf{T}y\|_{\mathsf{X}}^2. \tag{3.10}$$

(iii)(b) and (iii)(c): Define L and T as in (ii), and recall that $\mathsf{T} = L^* \circ T \circ L$. In the light of [3, Corollary 20.28], (i) and (iii)(a) imply that T is maximally monotone. At the same time, we deduce from [11, Proposition 3.4(ii)] that $T\colon \mathcal{H} \to \mathcal{H}$ is cocoercive. Thus, it results from (3.8), [3, Example 25.20(i)], and [28, Theorem 5] that

$$\left\{\int_\Omega \mathsf{L}_\omega^*\big(\mathsf{T}_\omega x(\omega)\big)\mu(d\omega) \,\middle|\, x \in \mathfrak{H}\right\} = L^*(\operatorname{ran} T) \subset \overline{\operatorname{ran}}\big(L^* \circ T \circ L\big) = \overline{\operatorname{ran}}\,\mathsf{T} \tag{3.11}$$

and that

$$\operatorname{int}\left\{\int_\Omega \mathsf{L}_\omega^*\big(\mathsf{T}_\omega x(\omega)\big)\mu(d\omega) \,\middle|\, x \in \mathfrak{H}\right\} = \operatorname{int} L^*(\operatorname{ran} T) \subset \operatorname{ran}\big(L^* \circ T \circ L\big) = \operatorname{ran}\mathsf{T}, \tag{3.12}$$

which completes the proof. □

The main properties of integral resolvent mixtures can now be laid out.

Theorem 3.5 *Suppose that* Assumptions 3.1 and 3.2 *are in force. Set*

$$\mathsf{W} = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega} \quad \textit{and} \quad \mathsf{C} = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega}. \tag{3.13}$$

Then the following hold:

(i) $\mathsf{W}^{-1} = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega^{-1})_{\omega\in\Omega}$ *and* $\mathsf{C}^{-1} = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega^{-1})_{\omega\in\Omega}$.

(ii) W *and* C *are maximally monotone.*

(iii) $\mathsf{C} = (\mathrm{Id}_{\mathsf{X}} + \int_\Omega(\mathsf{L}_\omega^* \circ J_{\mathsf{A}_\omega} \circ \mathsf{L}_\omega - \mathsf{L}_\omega^* \circ \mathsf{L}_\omega)\mu(d\omega))^{-1} - \mathrm{Id}_{\mathsf{X}}$.

(iv) Suppose that μ is a probability measure and that, for every $\omega \in \Omega$, L_ω is an isometry. Then $\mathsf{W} = \mathsf{C}$.

(v) $J_{\mathsf{W}} = \int_\Omega(\mathsf{L}_\omega^* \circ J_{\mathsf{A}_\omega} \circ \mathsf{L}_\omega)\mu(d\omega)$.

(vi) $J_{\mathsf{C}} = \mathrm{Id}_{\mathsf{X}} + \int_\Omega(\mathsf{L}_\omega^* \circ J_{\mathsf{A}_\omega} \circ \mathsf{L}_\omega - \mathsf{L}_\omega^* \circ \mathsf{L}_\omega)\mu(d\omega)$.

(vii) $\mathsf{W} \,\square\, \mathrm{Id}_{\mathsf{X}} = \mathrm{Id}_{\mathsf{X}} - \int_\Omega(\mathsf{L}_\omega^* \circ (\mathsf{A}_\omega^{-1} \,\square\, \mathrm{Id}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega)\mu(d\omega)$.

(viii) $\mathsf{C} \,\square\, \mathrm{Id}_{\mathsf{X}} = \int_\Omega(\mathsf{L}_\omega^* \circ (\mathsf{A}_\omega \,\square\, \mathrm{Id}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega)\mu(d\omega)$.

(ix) $\operatorname{zer}\mathsf{C} = \operatorname{zer}\int_\Omega(\mathsf{L}_\omega^* \circ (\mathsf{A}_\omega \,\square\, \mathrm{Id}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega)\mu(d\omega)$.

(x) $\overline{\operatorname{dom}}\,\mathsf{W} = \overline{\{\int_\Omega \mathsf{L}_\omega^* x^*(\omega)\mu(d\omega) \mid x^* \in \mathcal{H} \text{ and } (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \operatorname{dom}\mathsf{A}_\omega\}}$.

(xi) $\overline{\operatorname{ran}}\,\mathsf{C} = \overline{\{\int_\Omega \mathsf{L}_\omega^* x^*(\omega)\mu(d\omega) \mid x^* \in \mathcal{H} \text{ and } (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \operatorname{ran}\mathsf{A}_\omega\}}$.

(xii) $\operatorname{int}\operatorname{dom}\mathsf{W} = \operatorname{int}\{\int_\Omega \mathsf{L}_\omega^*(J_{\mathsf{A}_\omega} x(\omega))\mu(d\omega) \mid x \in \mathcal{H}\}$.

(xiii) $\operatorname{int}\operatorname{ran}\mathsf{C} = \operatorname{int}\{\int_\Omega \mathsf{L}_\omega^*(J_{\mathsf{A}_\omega^{-1}} x(\omega))\mu(d\omega) \mid x \in \mathcal{H}\}$.

(xiv) Suppose that, for every $\omega \in \Omega$, A_ω is nonexpansive with $\operatorname{dom}\mathsf{A}_\omega = \mathsf{H}_\omega$. *Then* C *is nonexpansive.*

(xv) Let $\tau \in\,]0, +\infty[$, set $\delta = (\tau + 1)/\int_\Omega \|\mathsf{L}_\omega\|^2\mu(d\omega) - 1$, and suppose that, for every $\omega \in \Omega$, A_ω is τ-cocoercive with $\operatorname{dom}\mathsf{A}_\omega = \mathsf{H}_\omega$. *Then* C *is δ-cocoercive.*

Proof Set

$$A = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega \mu(d\omega). \tag{3.14}$$

Then [11, Theorem 3.8(i)] states that A is maximally monotone, and [11, Theorem 3.8(ii)(a)] asserts that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\mathsf{A}_\omega} x(\omega)$ lies in $\mathfrak{H}$ and

$$J_A = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} J_{\mathsf{A}_\omega} \mu(d\omega). \tag{3.15}$$

Therefore, by Assumption 3.2[D] and items (i) and (iii)(a) of Proposition 3.4,

$$\int_{\Omega} \big(\mathsf{L}_\omega^* \circ J_{\mathsf{A}_\omega} \circ \mathsf{L}_\omega\big)\mu(d\omega) \colon \mathsf{X} \to \mathsf{X} \text{ is a well-defined firmly nonexpansive operator,} \tag{3.16}$$

which confirms that W is well defined. Additionally, it follows from Proposition 3.4(ii) and Assumption 3.2[D] that

$$L \colon \mathsf{X} \to \mathcal{H} \colon \mathsf{x} \mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{x} \text{ is a well-defined bounded linear operator such that } \|L\| \leqslant 1, \tag{3.17}$$

and

$$(\forall \mathsf{x} \in \mathsf{X}) \quad L^*\big(J_A(L\mathsf{x})\big) = \int_{\Omega} \mathsf{L}_\omega^*\big(J_{\mathsf{A}_\omega}(\mathsf{L}_\omega \mathsf{x})\big)\mu(d\omega). \tag{3.18}$$

Moreover, the adjoint of L is given by [11, Proposition 3.12(v)]

$$L^* \colon \mathcal{H} \to \mathsf{X} \colon x^* \mapsto \int_{\Omega} \mathsf{L}_\omega^* x^*(\omega)\mu(d\omega). \tag{3.19}$$

Likewise, appealing to [11, Proposition 3.7], we deduce that C is well defined and

$$(\forall \mathsf{x} \in \mathsf{X}) \quad L^*\big(J_{A^{-1}}(L\mathsf{x})\big) = \int_{\Omega} \mathsf{L}_\omega^*\big(J_{\mathsf{A}_\omega^{-1}}(\mathsf{L}_\omega \mathsf{x})\big)\mu(d\omega). \tag{3.20}$$

Hence, by virtue of Definition 2.1,

$$\mathsf{W} = L \diamond A \quad \text{and} \quad \mathsf{C} = L \blacklozenge A. \tag{3.21}$$

(i): A consequence of (3.2) and (3.3).

(ii): In the light of [19, Theorem 4.5(i)–(ii)], the claim follows from (3.17) and (3.21).

(iii): A consequence of (3.18), (3.19), (3.21), and [19, Proposition 4.1(ii)].

(iv): By (2.17) and (3.17),

$$(\forall \mathsf{x} \in \mathsf{X}) \quad \|L\mathsf{x}\|_{\mathcal{H}}^2 = \int_\Omega \|\mathsf{L}_\omega \mathsf{x}\|_{\mathsf{H}_\omega}^2 \mu(d\omega) = \int_\Omega \|\mathsf{x}\|_{\mathsf{X}}^2 \,\mu(d\omega) = \mu(\Omega)\|\mathsf{x}\|_{\mathsf{X}}^2 = \|\mathsf{x}\|_{\mathsf{X}}^2, \tag{3.22}$$

which shows that L is an isometry. Consequently, the conclusion follows from (3.21) and [19, Proposition 4.1(iii)].

(v): An immediate consequence of (3.2).

(vi): An immediate consequence of (iii).

(vii): This follows from (3.21), [19, Proposition 4.1(xiv)], and (3.18).

(viii): This follows from (3.21), [19, Proposition 4.1(xv)], and (3.20).

(ix): Use (ii), (viii), and [3, Proposition 23.38].

(x): Set $U = \{x \in \mathcal{H} \mid (\forall^\mu \omega \in \Omega)\ x(\omega) \in \operatorname{dom} \mathsf{A}_\omega\}$. Then $\overline{U} = \overline{\operatorname{dom} A}$ [11, Theorem 3.8(iii)]. Hence, [19, Theorem 4.5(vi)] implies that

$$\overline{\operatorname{dom} \mathsf{W}} = \overline{L^*(\operatorname{dom} A)} = \overline{L^*\big(\overline{\operatorname{dom} A}\big)} = \overline{L^*(\overline{U})} = \overline{L^*(U)}. \tag{3.23}$$

This and (3.19) yield the desired identity.

(xi): Combine (3.3), (x), and the fact that $(\forall \omega \in \Omega)$ $\operatorname{dom} \mathsf{A}_\omega^{-1} = \operatorname{ran} \mathsf{A}_\omega$.

(xii): Use (3.2) and Proposition 3.4(iii)(c).

(xiii): A consequence of (3.3) and (xii).

(xiv): It follows from [11, Theorem 3.8(v)(a)] that

$$\text{for every } x \in \mathfrak{H}, \text{ the mapping } \omega \mapsto \operatorname{proj}_{\mathsf{A}_\omega x(\omega)} \mathsf{0} = \mathsf{A}_\omega x(\omega) \text{ lies in } \mathfrak{G}. \tag{3.24}$$

Hence, [11, Proposition 3.4(i)] implies that A is nonexpansive with $\operatorname{dom} A = \mathcal{H}$. Hence, it follows from [19, Proposition 4.9] and (3.21) that C is nonexpansive.

(xv): We argue as in (xiv) to deduce that $A\colon \mathcal{H} \to \mathcal{H}$ is τ-cocoercive. On the other hand, (3.17) ensures that $\|L\| < \sqrt{\tau + 1}$. Thus, it follows from (3.21), [19, Proposition 4.8], and Proposition 3.4(ii) that $\mathsf{C} = L \mathbin{\blacklozenge} A$ is cocoercive with constant $(\tau + 1)\|L\|^{-2} - 1 \geqslant \delta$. □

Remark 3.6 The motivation for calling $\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega}$ an integral resolvent mixture comes from Theorem 3.5(v).

Let us provide some examples of integral resolvent mixtures.

Example 3.7 Consider the setting of Example 2.6. Then (3.2) becomes

$$\overset{\diamond}{\mathsf{M}}(\mathsf{L}_k, \mathsf{A}_k)_{1\leqslant k\leqslant p} = \left(\sum_{k=1}^p \alpha_k \mathsf{L}_k^* \circ J_{\mathsf{A}_k} \circ \mathsf{L}_k\right)^{-1} - \mathrm{Id}_{\mathsf{X}}, \tag{3.25}$$

which is the *resolvent mixture* introduced in [19, Example 3.4].

Example 3.8 Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space and let $(\phi_\omega)_{\omega\in\Omega}$ be a family in $\Gamma_0(\mathbb{R})$ such that the function $\Omega \times \mathbb{R} \to]-\infty, +\infty]\colon (\omega, \mathsf{x}) \mapsto \phi_\omega(\mathsf{x})$ is $\mathcal{F}\otimes\mathcal{B}_{\mathbb{R}}$-measurable and $(\forall\omega \in \Omega)\ \phi_\omega \geqslant \phi_\omega(0) = 0$. Further, let X be a separable real Hilbert space and let $e \in L^2(\Omega, \mathcal{F}, \mu; \mathsf{X})$ be such that $0 < \int_\Omega \|e(\omega)\|_{\mathsf{X}}^2\,\mu(d\omega) \leqslant 1$. Set

$$(\forall\omega \in \Omega)\quad \mathsf{A}_\omega = \partial\phi_\omega \quad\text{and}\quad \mathsf{L}_\omega = \langle\cdot \mid e(\omega)\rangle_{\mathsf{X}}. \tag{3.26}$$

Then

$$\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega} = \left(\int_\Omega \big(\mathrm{prox}_{\phi_\omega}\langle\cdot \mid e(\omega)\rangle_{\mathsf{X}}\big)e(\omega)\mu(d\omega)\right)^{-1} - \mathrm{Id}_{\mathsf{X}}. \tag{3.27}$$

For instance, suppose that, for every $\omega \in \Omega$, ϕ_ω is the support function of a closed interval C_ω in $\mathbb{R}$ containing 0, with $\delta_\omega = \inf \mathsf{C}_\omega$ and $\rho_\omega = \sup \mathsf{C}_\omega$. Now set

$$\mathsf{W} = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{A}_\omega)_{\omega\in\Omega} \quad\text{and}\quad (\forall\mathsf{x} \in \mathsf{X})\quad \begin{cases}\Omega^-(\mathsf{x}) = \big\{\omega \in \Omega \mid \langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} > \rho_\omega\big\}\\ \Omega_-(\mathsf{x}) = \big\{\omega \in \Omega \mid \langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} < \delta_\omega\big\}.\end{cases} \tag{3.28}$$

Then

$$(\forall\mathsf{x} \in \mathsf{X})\quad J_{\mathsf{W}}\mathsf{x} = \int_{\Omega^-(\mathsf{x})} \big(\langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} - \rho_\omega\big)e(\omega)\mu(d\omega) + \int_{\Omega_-(\mathsf{x})} \big(\langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} - \delta_\omega\big)e(\omega)\mu(d\omega). \tag{3.29}$$

This process provides a representation of x which eliminates the contributions of the coefficients $\langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} \in [\delta_\omega, \rho_\omega]$. For instance, in the context of Example 2.7, if $(e(k))_{k\in\mathbb{N}}$ is an orthonormal basis and $\mathsf{C}_k = [-\rho_k, \rho_k]$, then J_{W} is known as a soft-thresholder and it has been used extensively in data analysis [21, 24].

Proof Let $\mathfrak{G} = \big\{x\colon \Omega \to \mathbb{R} \mid x \text{ is } \mathcal{F}\text{-measurable}\big\}$ and, for every $\omega \in \Omega$, let $\mathsf{H}_\omega = \mathbb{R}$. Then, in view of Example 2.8, Assumption 2.5 is satisfied and

$$\mathcal{H} = \int_\Omega^{\mathfrak{G}\,\oplus} \mathsf{H}_\omega\mu(d\omega) = L^2\big(\Omega, \mathcal{F}, \mu; \mathbb{R}\big). \tag{3.30}$$

Since $(\Omega, \mathcal{F}, \mu)$ is complete, we deduce from [30, Corollary 14.34 and Exercise 14.38] that, for every $x \in \mathfrak{G}$, the function $\Omega \to \mathbb{R}\colon \omega \mapsto \mathrm{prox}_{\phi_\omega} x(\omega)$ lies in $\mathfrak{G}$. Additionally, for every $\omega \in \Omega$, since $0 \in \mathrm{Argmin}\,\phi_\omega$, we get $0 \in \mathsf{A}_\omega 0$ and $J_{\mathsf{A}_\omega}0 = \mathrm{prox}_{\phi_\omega} 0 = 0$. Hence, the family $(\mathsf{A}_\omega)_{\omega\in\Omega}$ satisfies Assumption 3.1. Next, since $e\colon \Omega \to \mathsf{X}$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{X}})$-measurable, we deduce that, for every $\mathsf{x} \in \mathsf{X}$, the mapping $\Omega \to \mathbb{R}\colon \omega \mapsto \langle\mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} = \mathsf{L}_\omega\mathsf{x}$ lies in $\mathfrak{G}$. Further,

$$(\forall\omega \in \Omega)\quad \mathsf{L}_\omega^*\colon \mathbb{R} \to \mathsf{X}\colon \mathsf{x} \mapsto \mathsf{x}e(\omega) \tag{3.31}$$

and

$$\int_{\Omega} \|\mathsf{L}_\omega\|^2 \mu(d\omega) = \int_{\Omega} \|e(\omega)\|_{\mathsf{X}}^2 \, \mu(d\omega) = \|e\|_{\mathcal{H}}^2 \in \,]0, 1]. \tag{3.32}$$

This confirms that Assumption 3.2 is satisfied. Therefore, we obtain (3.27) by invoking (3.2). Next, let us establish (3.29). Take $\mathsf{x} \in \mathsf{X}$. Thanks to the $\mathcal{F}$-measurability of the function $\Omega \to \mathbb{R}\colon \omega \mapsto \operatorname{prox}_{\phi_\omega} \langle \mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}}$, we obtain

$$\begin{aligned} \Omega^-(\mathsf{x}) &= \big\{\omega \in \Omega \mid \langle \mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} - \operatorname{proj}_{\mathsf{C}_\omega} \langle \mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} > 0\big\} \\ &= \big\{\omega \in \Omega \mid \operatorname{prox}_{\phi_\omega} \langle \mathsf{x} \mid e(\omega)\rangle_{\mathsf{X}} > 0\big\} \\ &\in \mathcal{F}. \end{aligned} \tag{3.33}$$

Likewise, $\Omega_-(\mathsf{x}) \in \mathcal{F}$. On the other hand, by [3, Example 24.34],

$$(\forall \omega \in \Omega) \quad \operatorname{prox}_{\phi_\omega}\colon \mathbb{R} \to \mathbb{R}\colon \mathsf{x} \mapsto \begin{cases} \mathsf{x} - \rho_\omega, & \text{if } \mathsf{x} > \rho_\omega; \\ 0, & \text{if } \mathsf{x} \in \mathsf{C}_\omega; \\ \mathsf{x} - \delta_\omega, & \text{if } \mathsf{x} < \delta_\omega. \end{cases} \tag{3.34}$$

Therefore, we obtain (3.29) by using Theorem 3.5(v) and the fact that $(\forall \omega \in \Omega)$ $J_{\mathsf{A}_\omega} = \operatorname{prox}_{\phi_\omega}$. □

Next, we define the resolvent expectation of a family of maximally monotone operators.

Definition 3.9 Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space, let H be a separable real Hilbert space, and let $(\mathsf{A}_\omega)_{\omega\in\Omega}$ be a family of maximally monotone operators from H to 2^{H}. Suppose that, for every $\mathsf{x} \in \mathsf{H}$, the mapping $\Omega \to \mathsf{H}\colon \omega \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$-measurable and that $\int_{\Omega} \| J_{\mathsf{A}_\omega} 0\|_{\mathsf{H}}^2 \, \mu(d\omega) < +\infty$. Using the notation (2.14), the *resolvent expectation* of the family $(\mathsf{A}_\omega)_{\omega\in\Omega}$ is

$$\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega} = \big(\mathsf{E}(J_{\mathsf{A}_\omega})_{\omega\in\Omega}\big)^{-1} - \mathrm{Id}_{\mathsf{H}}. \tag{3.35}$$

Example 3.10 Consider the measure space $(\Omega, \mathcal{F}, \mu)$ of Example 2.6 with the additional assumption that $\sum_{k=1}^{p} \alpha_k = 1$. Let H be a separable real Hilbert space and let $(\mathsf{A}_k)_{1\leqslant k\leqslant p}$ be maximally monotone operators from H to 2^{H}. Then (3.35) becomes

$$\overset{\diamond}{\mathsf{E}}(\mathsf{A}_k)_{1\leqslant k\leqslant p} = \left(\sum_{k=1}^{p} \alpha_k J_{\mathsf{A}_k}\right)^{-1} - \mathrm{Id}_{\mathsf{H}}, \tag{3.36}$$

which is the *resolvent average* studied in [2].

Let us relate resolvent expectations to integral resolvent mixtures.

Proposition 3.11 *Consider the setting of* Example 2.8 *with the additional assumption that* μ *is a probability measure. Let* $(\mathsf{A}_\omega)_{\omega\in\Omega}$ *be a family of maximally monotone operators from* H *to* 2^{H} *such that, for every* $\mathsf{x} \in \mathsf{H}$*, the mapping* $\Omega \to \mathsf{H}\colon \omega \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ *is* $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$*-measurable and that* $\int_\Omega \|J_{\mathsf{A}_\omega}\mathsf{0}\|^2_{\mathsf{H}}\,\mu(d\omega) < +\infty$*. Then*

$$\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{A}_\omega)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{A}_\omega)_{\omega\in\Omega}. \tag{3.37}$$

Proof Appealing to [14, Lemma III.14] and the continuity of the operators $(J_{\mathsf{A}_\omega})_{\omega\in\Omega}$, we infer that the mapping $\Omega \times \mathsf{H} \to \mathsf{H}\colon (\omega, \mathsf{x}) \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ is $(\mathcal{F}\otimes\mathcal{B}_{\mathsf{H}}, \mathcal{B}_{\mathsf{H}})$-measurable. Thus, for every $x \in \mathfrak{H}$, the mapping $\Omega \to \mathsf{H}\colon \omega \mapsto J_{\mathsf{A}_\omega}x(\omega)$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$-measurable, i.e., it lies in $\mathfrak{G}$. On the other hand, letting $A = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{A}_\omega\mu(d\omega)$ and $r\colon \Omega \to \mathsf{H}\colon \omega \mapsto J_{\mathsf{A}_\omega}\mathsf{0}$ yields $-r \in Ar$, which implies that $\operatorname{dom} A \neq \varnothing$. Hence, it follows from (3.35) and (3.2) that $\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{A}_\omega)_{\omega\in\Omega}$, while the identity $\overset{\diamond}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{A}_\omega)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{A}_\omega)_{\omega\in\Omega}$ follows from Theorem 3.5(iv). □

By specializing Theorem 3.5 to the scenario of Proposition 3.11, we obtain at once the following properties of the resolvent expectation and, in particular, those of the resolvent average of finitely many operators studied in [2].

Corollary 3.12 *Consider the setting of* Definition 3.9*. Then the following hold:*

(i) $(\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega})^{-1} = \overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega^{-1})_{\omega\in\Omega}$.

(ii) $\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}$ *is maximally monotone.*

(iii) $J_{\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}} = \mathsf{E}(J_{\mathsf{A}_\omega})_{\omega\in\Omega}$.

(iv) $(\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}) \,\Box\, \mathrm{Id}_{\mathsf{H}} = \mathsf{E}(\mathsf{A}_\omega \,\Box\, \mathrm{Id}_{\mathsf{H}})_{\omega\in\Omega}$.

(v) $\overline{\operatorname{dom} \overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}} = \overline{\big\{\mathsf{E}x^* \mid x^* \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}) \text{ and } (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \operatorname{dom} \mathsf{A}_\omega\big\}}$.

(vi) $\overline{\operatorname{ran} \overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}} = \overline{\big\{\mathsf{E}x^* \mid x^* \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}) \text{ and } (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \operatorname{ran} \mathsf{A}_\omega\big\}}$.

(vii) $\operatorname{int} \operatorname{dom} \overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega} = \operatorname{int}\big\{\mathsf{E}(J_{\mathsf{A}_\omega}x(\omega))_{\omega\in\Omega} \mid x \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})\big\}$.

(viii) $\operatorname{int} \operatorname{ran} \overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega} = \operatorname{int}\big\{\mathsf{E}(J_{\mathsf{A}_\omega^{-1}}x(\omega))_{\omega\in\Omega} \mid x \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})\big\}$.

(ix) *Suppose that, for every* $\omega \in \Omega$, A_ω *is nonexpansive with* $\operatorname{dom} \mathsf{A}_\omega = \mathsf{H}_\omega$*. Then* $\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}$ *is nonexpansive.*

(x) *Let* $\tau \in\,]0, +\infty[$ *and suppose that, for every* $\omega \in \Omega$, A_ω *is* τ*-cocoercive with* $\operatorname{dom} \mathsf{A}_\omega = \mathsf{H}_\omega$*. Then* $\overset{\diamond}{\mathsf{E}}(\mathsf{A}_\omega)_{\omega\in\Omega}$ *is* τ*-cocoercive.*

4 Integral Proximal Mixtures

The integral proximal mixture will be cast in the following setting.

Assumption 4.1 Assumption 2.5 and the following are in force:

[A] For every $\omega \in \Omega$, $\mathsf{f}_\omega\colon \mathsf{H}_\omega \to\,]-\infty, +\infty]$ possesses a continuous affine minorant.

[B] There exists $r \in \mathfrak{H}$ such that the function $\omega \mapsto \mathsf{f}_\omega(r(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.
[C] There exists $r^* \in \mathfrak{H}$ such that the function $\omega \mapsto \mathsf{f}^*_\omega(r^*(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.
[D] For every $x^* \in \mathfrak{H}$, the mapping $\omega \mapsto \mathrm{prox}_{\mathsf{f}^*_\omega} x^*(\omega)$ lies in $\mathfrak{G}$.

Definition 4.2 Suppose that Assumptions 3.2 and 4.1 are in force. The *integral proximal mixture* of $(\mathsf{f}_\omega)_{\omega\in\Omega}$ and $(\mathsf{L}_\omega)_{\omega\in\Omega}$ is

$$\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \left(\int_\Omega \big((\mathsf{f}^*_\omega \,\square\, \mathscr{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega\big)\mu(d\omega)\right)^* - \mathscr{Q}_{\mathsf{X}}, \tag{4.1}$$

and the *integral proximal comixture* of $(\mathsf{f}_\omega)_{\omega\in\Omega}$ and $(\mathsf{L}_\omega)_{\omega\in\Omega}$ is

$$\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \left(\overset{\diamond}{\mathsf{M}}\big(\mathsf{L}_\omega, \mathsf{f}^*_\omega\big)_{\omega\in\Omega}\right)^*. \tag{4.2}$$

Item (viii) below connects Definitions 3.3 and 4.2.

Theorem 4.3 *Suppose that* Assumptions 3.2 and 4.1 *are in force. Then the following hold:*

(i) $\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{X})$.
(ii) $\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{X})$.
(iii) *Let* $\mathsf{x} \in \mathsf{X}$. *Then*

$$\big(\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}\big)(\mathsf{x}) = \min\left\{\int_\Omega \mathsf{f}^{**}_\omega\big(x(\omega)\big)\mu(d\omega) + \mathscr{Q}_{\mathcal{H}}(x) - \mathscr{Q}_{\mathsf{X}}(\mathsf{x}) \;\middle|\; x \in \mathcal{H} \text{ and } \int_\Omega \mathsf{L}^*_\omega x(\omega)\mu(d\omega) = \mathsf{x}\right\}.$$

(iv) $\overline{\mathrm{dom}\,\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}} = \overline{\left\{\int_\Omega \mathsf{L}^*_\omega x(\omega)\mu(d\omega) \mid x \in \mathcal{H} \text{ and } (\forall^\mu \omega \in \Omega)\ x(\omega) \in \mathrm{dom}\,\mathsf{f}^{**}_\omega\right\}}$.
(v) $(\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega})^* = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}^*_\omega)_{\omega\in\Omega} = (\mathscr{Q}_{\mathsf{X}} - \int_\Omega (\mathsf{f}^*_\omega \,\square\, \mathscr{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega\, \mu(d\omega))^* - \mathscr{Q}_{\mathsf{X}}$.
(vi) $(\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega})^* = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}^*_\omega)_{\omega\in\Omega}$.
(vii) $\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} \,\square\, \mathscr{Q}_{\mathsf{X}} + \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}^*_\omega)_{\omega\in\Omega} \,\square\, \mathscr{Q}_{\mathsf{X}} = \mathscr{Q}_{\mathsf{X}}$.
(viii) $\partial\, \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \partial \mathsf{f}^{**}_\omega)_{\omega\in\Omega}$.
(ix) $\mathrm{prox}_{\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}} = \int_\Omega (\mathsf{L}^*_\omega \circ \mathrm{prox}_{\mathsf{f}^{**}_\omega} \circ \mathsf{L}_\omega)\mu(d\omega)$.
(x) $\mathrm{prox}_{\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}} = \mathrm{Id}_{\mathsf{X}} - \int_\Omega (\mathsf{L}^*_\omega \circ \mathrm{prox}_{\mathsf{f}^*_\omega} \circ \mathsf{L}_\omega)\mu(d\omega)$.
(xi) $\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} \,\square\, \mathscr{Q}_{\mathsf{X}} = \int_\Omega (\mathsf{f}^{**}_\omega \,\square\, \mathscr{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega\, \mu(d\omega)$.
(xii) $\mathrm{Argmin}\, \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \mathrm{Argmin} \int_\Omega (\mathsf{f}^{**}_\omega \,\square\, \mathscr{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega\, \mu(d\omega)$.
(xiii) *Suppose that* μ *is a probability measure and that, for every* $\omega \in \Omega$, L_ω *is an isometry. Then* $\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}$.

Proof Set

$$\mathsf{g} = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} \ \text{ and } \ \mathsf{h} = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}. \tag{4.3}$$

In the light of [3, Propositions 13.12(ii) and 13.10(ii)], we infer from [A] and [B] of Assumption 4.1 that $(\forall\omega \in \Omega)$ $\mathsf{f}^*_\omega \in \Gamma_0(\mathsf{H}_\omega)$. Next, define $\varrho\colon \Omega \to \mathbb{R}\colon \omega \mapsto -\mathsf{f}_\omega(r(\omega))$. Then, by Assumption 4.1[B], $\varrho \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. Additionally, $(\forall\omega \in \Omega)$ $\mathsf{f}^*_\omega \geqslant \langle\cdot \mid r(\omega)\rangle + \varrho(\omega)$. Hence, we conclude that the family $(\mathsf{f}^*_\omega)_{\omega\in\Omega}$ satisfies the assumptions of [11, Theorem 4.7] and therefore that the family $(\partial\mathsf{f}^{**}_\omega)_{\omega\in\Omega}$ satisfies Assumption 3.1. Let us now check that the family $(\mathsf{f}^*_\omega)_{\omega\in\Omega}$ satisfies Assumption 4.1 by using the mapping r to fulfill [C]. To this end, we need to show that the function $\varphi\colon \omega \mapsto \mathsf{f}^{**}_\omega(r(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. First, it follows from [11, Theorem 4.7(ix)] that φ is $\mathcal{F}$-measurable. Further, since, for every $\omega \in \Omega$, $\mathsf{f}^{**}_\omega \leqslant \mathsf{f}_\omega$, Assumption 4.1[B] implies that φ is majorized by an integrable function. Finally, since

$$(\forall\omega \in \Omega) \quad \mathsf{f}^{**}_\omega \geqslant \langle\cdot \mid r^*(\omega)\rangle_{\mathsf{H}_\omega} - \mathsf{f}^*_\omega\big(r^*(\omega)\big), \tag{4.4}$$

Assumption 4.1[C] implies that φ is minorized by an integrable function. Next, we observe that [11, Theorem 4.7(i)–(ii)] assert that

$$g\colon \mathcal{H} \to \left]-\infty, +\infty\right]\colon x^* \mapsto \int_\Omega \mathsf{f}^*_\omega\big(x^*(\omega)\big)\mu(d\omega) \tag{4.5}$$

is a well-defined function in $\Gamma_0(\mathcal{H})$. Moreover, by [11, Theorem 4.7(viii)],

$$g \,\square\, \mathcal{Q}_{\mathcal{H}}\colon \mathcal{H} \to \mathbb{R}\colon x^* \mapsto \int_\Omega (\mathsf{f}^*_\omega \,\square\, \mathcal{Q}_{\mathsf{H}_\omega})\big(x^*(\omega)\big)\mu(d\omega) \tag{4.6}$$

and, by [11, Theorem 4.7(ix)],

$$g^*\colon \mathcal{H} \to \left]-\infty, +\infty\right]\colon x \mapsto \int_\Omega \mathsf{f}^{**}_\omega\big(x(\omega)\big)\mu(d\omega). \tag{4.7}$$

We also recall from (3.17) that

$$L\colon \mathsf{X} \to \mathcal{H}\colon \mathsf{x} \mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{x} \text{ is a well-defined bounded linear operator with } \|L\| \leqslant 1. \tag{4.8}$$

(i): We deduce from (4.1), Moreau's biconjugation theorem [3, Corollary 13.38], Definition 2.2, and [19, Example 3.6(ii)] that

$$\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \big((g \,\square\, \mathcal{Q}_{\mathcal{H}}) \circ L\big)^* - \mathcal{Q}_{\mathsf{X}} = L \diamond g^* \in \Gamma_0(\mathsf{X}). \tag{4.9}$$

(ii): It follows from (i), Definition 2.2, and [19, Example 3.10(i)] that

$$\mathsf{h} = \Big(\overset{\diamond}{\mathsf{M}}\big(\mathsf{L}_\omega, \mathsf{f}^*_\omega\big)_{\omega\in\Omega}\Big)^* = \big(L \diamond g^{**}\big)^* = L \blacklozenge g^* \in \Gamma_0(\mathsf{X}). \tag{4.10}$$

(iii): We derive from (4.9) and [3, Corollary 15.28(i) and Proposition 13.24(i)] that

$$(\forall \mathsf{x} \in \mathsf{X}) \quad \mathsf{g}(\mathsf{x}) = \min\big\{(g \,\square\, \mathcal{Q}_{\mathcal{H}})^*(x) - \mathcal{Q}_{\mathsf{X}}(\mathsf{x}) \mid x \in \mathcal{H} \text{ and } L^*x = \mathsf{x}\big\}$$
$$= \min\big\{g^*(x) + \mathcal{Q}_{\mathcal{H}}(x) - \mathcal{Q}_{\mathsf{X}}(\mathsf{x}) \mid x \in \mathcal{H} \text{ and } L^*x = \mathsf{x}\big\}. \tag{4.11}$$

Thus, (4.7) and (3.19) yield the announced identity.

(iv): Set $U = \big\{x \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ x(\omega) \in \operatorname{dom} \mathsf{f}_{\omega}^{**}\big\}$. Then [11, Theorem 4.7(v)] states that $\overline{\operatorname{dom}}\, g^* = \overline{U}$. Thus, it results from (4.9) and [19, Theorem 5.5(ii)] that

$$\overline{\operatorname{dom}}\,\mathsf{g} = \overline{\operatorname{dom}}(L \diamond g^*) = \overline{L^*(\operatorname{dom} g^*)} = \overline{L^*\big(\overline{\operatorname{dom}}\, g^*\big)} = \overline{L^*(\overline{U})} = \overline{L^*(U)}, \tag{4.12}$$

and the assertion follows from (3.19).

(v): It follows from (4.9), [19, Proposition 5.3(iv)], and (4.2) that

$$\mathsf{g}^* = \big(L \diamond g^*\big)^* = L \blacklozenge g^{**} = \overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega^*)_{\omega\in\Omega}. \tag{4.13}$$

At the same time, we derive from (4.9), [3, Proposition 13.29], and (4.6) that

$$\mathsf{g}^* = \big(\mathcal{Q}_{\mathsf{X}} - \big((g \,\square\, \mathcal{Q}_{\mathcal{H}}) \circ L\big)^{**}\big)^* - \mathcal{Q}_{\mathsf{X}}$$
$$= \left(\mathcal{Q}_{\mathsf{X}} - \int_{\Omega} (\mathsf{f}_\omega^* \,\square\, \mathcal{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega \, \mu(d\omega)\right)^* - \mathcal{Q}_{\mathsf{X}}. \tag{4.14}$$

(vi): Since $g \in \Gamma_0(\mathcal{H})$, we deduce from (4.10), [19, Proposition 5.3(v)], Moreau's biconjugation theorem, and (i) that

$$\mathsf{h}^* = \big(L \blacklozenge g^*\big)^* = L \diamond g^{**} = L \diamond g = \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega^*)_{\omega\in\Omega}. \tag{4.15}$$

(vii): Use (i), (v), and [3, Theorem 14.3(i)].

(viii): In view of (4.9), we derive from [3, Theorem 18.15], [11, Theorem 4.7(iv)], and (3.19) that

$$\partial \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega} = \Big(\nabla\big((g \,\square\, \mathcal{Q}_{\mathcal{H}}) \circ L\big)\Big)^{-1} - \mathrm{Id}_{\mathsf{X}}$$
$$= \Big(L^* \circ \big(\nabla(g \,\square\, \mathcal{Q}_{\mathcal{H}})\big) \circ L\Big)^{-1} - \mathrm{Id}_{\mathsf{X}}$$
$$= \left(\int_{\Omega} \big(\mathsf{L}_\omega^* \circ \operatorname{prox}_{\mathsf{f}_\omega^{**}} \circ \mathsf{L}_\omega\big)\mu(d\omega)\right)^{-1} - \mathrm{Id}_{\mathsf{X}} \tag{4.16}$$
$$= \overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \partial \mathsf{f}_\omega^{**})_{\omega\in\Omega}. \tag{4.17}$$

(ix): Use (4.16) and [3, Example 23.3].

(x): By [3, Proposition 13.16(iii)], $(\forall \omega \in \Omega)\ \mathsf{f}_\omega^{***} = \mathsf{f}_\omega^*$. Hence, it results from (ii), Moreau's decomposition [3, Theorem 14.3(ii)], (vi), and (ix) that

$$
\begin{aligned}
\operatorname{prox}_{\mathsf{h}} &= \mathrm{Id}_{\mathsf{X}} - \operatorname{prox}_{\mathsf{h}^*} \\
&= \mathrm{Id}_{\mathsf{X}} - \operatorname{prox}_{\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega^*)_{\omega\in\Omega}} \\
&= \mathrm{Id}_{\mathsf{X}} - \int_\Omega \big(\mathsf{L}_\omega^* \circ \operatorname{prox}_{\mathsf{f}_\omega^{***}} \circ \mathsf{L}_\omega\big)\mu(d\omega) \\
&= \mathrm{Id}_{\mathsf{X}} - \int_\Omega \big(\mathsf{L}_\omega^* \circ \operatorname{prox}_{\mathsf{f}_\omega^{*}} \circ \mathsf{L}_\omega\big)\mu(d\omega).
\end{aligned}
\tag{4.18}
$$

(xi): Because $g^* \in \Gamma_0(\mathcal{H})$, it results from (4.10), [19, Theorem 5.5(v)], [11, Theorem 4.7(viii)], and (4.8) that

$$
\mathsf{h} \,\square\, \mathscr{Q}_{\mathsf{X}} = \big(L \blacklozenge g^*\big) \,\square\, \mathscr{Q}_{\mathsf{X}} = (g^* \,\square\, \mathscr{Q}_{\mathcal{H}}) \circ L = \int_\Omega \Big(\big(\mathsf{f}_\omega^{**} \,\square\, \mathscr{Q}_{\mathsf{H}_\omega}\big) \circ \mathsf{L}_\omega\Big)\mu(d\omega).
\tag{4.19}
$$

(xii): Combine (xi) and [3, Proposition 17.5].

(xiii): In this case, L is an isometry and the conclusion follows from [19, Proposition 5.3(vii)], (4.9), and (4.10). □

Remark 4.4 The motivation for calling $\overset{\diamond}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}$ an integral proximal mixture comes from Theorem 4.3(ix).

Example 4.5 Consider the setting of Example 2.6. Then (4.1) becomes

$$
\overset{\diamond}{\mathsf{M}}(\mathsf{L}_k, \mathsf{f}_k)_{1\leqslant k\leqslant p} = \left(\sum_{k=1}^{p} \alpha_k (\mathsf{f}_k^* \,\square\, \mathscr{Q}_{\mathsf{H}_k}) \circ \mathsf{L}_k\right)^* - \mathscr{Q}_{\mathsf{X}},
\tag{4.20}
$$

which is the *proximal mixture* introduced in [19, Example 5.9].

Our next illustration concerns a new object: the proximal expectation of a family of functions.

Definition 4.6 Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space, let H be a separable real Hilbert space, and let $(\mathsf{f}_\omega)_{\omega\in\Omega}$ be a family of functions in $\Gamma_0(\mathsf{H})$ such that the function

$$
\Omega \times \mathsf{H} \to \left]-\infty, +\infty\right] \colon (\omega, \mathsf{x}) \mapsto \mathsf{f}_\omega(\mathsf{x})
\tag{4.21}
$$

is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{H}}$-measurable. Suppose that there exist $r \in \mathscr{L}^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ and $r^* \in \mathscr{L}^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ such that the functions $\omega \mapsto \mathsf{f}_\omega(r(\omega))$ and $\omega \mapsto \mathsf{f}_\omega^*(r^*(\omega))$ lie in $\mathscr{L}^1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R})$. Using the notation (2.14), the *proximal expectation* of the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ is

$$
\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} = \Big(\mathsf{E}\big(\mathsf{f}_\omega^* \,\square\, \mathscr{Q}_{\mathsf{H}}\big)_{\omega\in\Omega}\Big)^* - \mathscr{Q}_{\mathsf{H}}.
\tag{4.22}
$$

Proposition 4.7 *Consider the setting of* Example 2.8 *with the additional assumption that μ is a probability measure. Let $(\mathsf{f}_\omega)_{\omega\in\Omega}$ be a family of functions in $\Gamma_0(\mathsf{H})$ such that the function*

$$\Omega \times \mathsf{H} \to \left]-\infty, +\infty\right] \colon (\omega, \mathsf{x}) \mapsto \mathsf{f}_\omega(\mathsf{x}) \tag{4.23}$$

is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{H}}$-measurable. Suppose that there exist $r \in \mathscr{L}^2(\Omega, \mathcal{F}, \mu; \mathsf{H})$ and $r^* \in \mathscr{L}^2(\Omega, \mathcal{F}, \mu; \mathsf{H})$ such that the functions $\omega \mapsto \mathsf{f}_\omega(r(\omega))$ and $\omega \mapsto \mathsf{f}^*_\omega(r^*(\omega))$ lie in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. Then

$$\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{f}_\omega)_{\omega\in\Omega}. \tag{4.24}$$

Proof Note that

$$\mathfrak{H} = \mathscr{L}^2\big(\Omega, \mathcal{F}, \mu; \mathsf{H}\big). \tag{4.25}$$

Using the completeness of $(\Omega, \mathcal{F}, \mu)$, we derive from [1, Théorème 2.3], [14, Lemma III.14], and [3, Proposition 12.28] that, for every $x \in \mathfrak{H}$, the mapping $\Omega \to \mathsf{H} \colon \omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega} x(\omega)$ lies in $\mathfrak{G}$. Thus, for every $x^* \in \mathfrak{H}$, using [3, Theorem 14.3(ii)], we deduce that the mapping $\Omega \to \mathsf{H} \colon \omega \mapsto \operatorname{prox}_{\mathsf{f}^*_\omega} x^*(\omega) = x^*(\omega) - \operatorname{prox}_{\mathsf{f}_\omega} x^*(\omega)$ also lies in $\mathfrak{G}$. Hence, the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ satisfies Assumption 4.1. Thus, invoking Notation 2.4, we deduce from Theorem 4.3(xiii), (4.1), and (4.22) that

$$\begin{aligned}
\overset{\blacklozenge}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{f}_\omega)_{\omega\in\Omega} &= \overset{\diamond}{\mathsf{M}}(\mathrm{Id}_{\mathsf{H}}, \mathsf{f}_\omega)_{\omega\in\Omega} \\
&= \left(\int_\Omega \big((\mathsf{f}^*_\omega \,\square\, \mathcal{Q}_{\mathsf{H}}) \circ \mathrm{Id}_{\mathsf{H}}\big)\, \mu(d\omega)\right)^* - \mathcal{Q}_{\mathsf{H}} \\
&= \Big(\mathsf{E}\big(\mathsf{f}^*_\omega \,\square\, \mathcal{Q}_{\mathsf{H}}\big)_{\omega\in\Omega}\Big)^* - \mathcal{Q}_{\mathsf{H}} \\
&= \overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega},
\end{aligned} \tag{4.26}$$

as announced. □

Combining Theorem 4.3, Proposition 3.11, and Proposition 4.7 yields at once the following properties of the proximal expectation.

Proposition 4.8 *Consider the setting of* Definition 4.6. *Then the following hold:*

(i) $\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{H})$.

(ii) *Let* $\mathsf{x} \in \mathsf{H}$. *Then*

$$\begin{aligned}
&\big(\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}\big)(\mathsf{x}) \\
&= \min\left\{\int_\Omega \Big(\mathsf{f}_\omega\big(x(\omega)\big) + \mathcal{Q}_{\mathsf{H}}\big(x(\omega)\big)\Big)\mathsf{P}(d\omega) - \mathcal{Q}_{\mathsf{H}}(\mathsf{x}) \;\middle|\; x \in L^2\big(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}\big) \text{ and } \mathsf{E}x = \mathsf{x}\right\}.
\end{aligned} \tag{4.27}$$

(iii) $\overline{\operatorname{dom} \overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}} = \overline{\big\{\mathsf{E}x \mid x \in L^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H}) \textit{ and } (\forall^\mu \omega \in \Omega)\ x(\omega) \in \operatorname{dom} \mathsf{f}_\omega\big\}}$.

(iv) $(\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega})^* = \overset{\diamond}{\mathsf{E}}(\mathsf{f}^*_\omega)_{\omega\in\Omega} = (\mathsf{E}(\mathsf{f}_\omega \,\square\, \mathcal{Q}_\mathsf{H})_{\omega\in\Omega})^* - \mathcal{Q}_\mathsf{H}$.

(v) $\partial \overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{E}}(\partial\mathsf{f}_\omega)_{\omega\in\Omega}$.

(vi) $\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} \,\square\, \mathcal{Q}_\mathsf{H} = \mathsf{E}(\mathsf{f}_\omega \,\square\, \mathcal{Q}_\mathsf{H})_{\omega\in\Omega}$.

(vii) $\operatorname{prox}_{\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}} = \mathsf{E}(\operatorname{prox}_{\mathsf{f}_\omega})_{\omega\in\Omega}$.

(viii) $\operatorname{Argmin} \overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega} = \operatorname{Argmin} \mathsf{E}(\mathsf{f}_\omega \,\square\, \mathcal{Q}_\mathsf{H})_{\omega\in\Omega}$.

Remark 4.9 In Definition 4.6, consider the measure space $(\Omega, \mathcal{F}, \mu)$ of Example 2.6 with the additional assumption that $\sum_{k=1}^p \alpha_k = 1$. Then the proximal expectation becomes

$$\overset{\diamond}{\mathsf{E}}(\mathsf{f}_k)_{1\leqslant k\leqslant p} = \left(\sum_{k=1}^{p} \alpha_k \left(\mathsf{f}_k^* \,\square\, \mathcal{Q}_\mathsf{H}\right)\right)^* - \mathcal{Q}_\mathsf{H}. \tag{4.28}$$

(i) The function of (4.28) is the *proximal average* of the family $(\mathsf{f}_k)_{1\leqslant k\leqslant p}$. By specializing Proposition 4.8 to this setting, we recover properties of the proximal average found in [4].

(ii) Let $(\mathsf{f}_k)_{1\leqslant k\leqslant p}$ be functions in $\Gamma_0(\mathsf{H})$. In some data analysis applications (see, e.g., [16, 27, 32]), (4.28) has been used instead of the standard average $\sum_{k=1}^p \alpha_k \mathsf{f}_k$. The latter can be regarded as the empirical p-sample approximation to the true expectation $\mathsf{E}(\mathsf{f}_\omega)_{\omega\in\Omega}$ arising from a family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ in $\Gamma_0(\mathsf{H})$. Likewise, we can regard the proximal average (4.28) as the empirical approximation to the proximal expectation $\overset{\diamond}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}$.

Remark 4.10 The strategy described in Remark 4.9(ii) can be generalized as follows. Let μ be a probability measure. Then it may be appropriate in certain variational problems to replace the standard composite average $\int_\Omega (\mathsf{f}_\omega \circ \mathsf{L}_\omega)\mu(d\omega)$ by the integral proximal comixture $\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{f}_\omega)_{\omega\in\Omega}$ of Definition 4.2. The latter is easier to handle numerically as its proximity operator is explicitly given by Theorem 4.3(x) and it follows from Theorem 4.3(xii) that its set of minimizers coincides with that of the function $\int_\Omega ((\mathsf{f}_\omega \,\square\, \mathcal{Q}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega)\mu(d\omega)$.

5 Relaxation of Systems of Monotone Inclusions

We place our focus on the following general system of composite monotone inclusions.

Problem 5.1 Suppose that Assumptions 3.1 and 3.2 are in force and that $\mathsf{V} \neq \{0\}$ is a closed vector subspace of X. The task is to

$$\text{find } \mathsf{x} \in \mathsf{V} \text{ such that } (\forall^\mu \omega \in \Omega)\ 0 \in \mathsf{A}_\omega(\mathsf{L}_\omega \mathsf{x}). \tag{5.1}$$

The instantiations of Problem 5.1 found in [19, 22, 23] (see also [13, 15] for further special cases) correspond to the setting of Example 2.6 with finitely many inclusions, that is,

$$\text{find } \mathsf{x} \in \mathsf{V} \text{ such that } \big(\forall k \in \{1, \ldots, p\}\big) \;\; 0 \in \mathsf{A}_k(\mathsf{L}_k \mathsf{x}). \tag{5.2}$$

On the other hand, the instantiation of [12] corresponds to the setting of Example 2.8 where μ is a probability measure, $\mathsf{V} = \mathsf{H}$ and, for every $\omega \in \Omega$, A_ω is the normal cone operator of a nonempty closed convex subset C_ω of H and $\mathsf{L}_\omega = \mathrm{Id}_{\mathsf{H}}$, that is,

$$\text{find } \mathsf{x} \in \mathsf{H} \text{ such that } (\forall^\mu \omega \in \Omega) \;\; \mathsf{x} \in \mathsf{C}_\omega. \tag{5.3}$$

The last problem is known as the stochastic convex feasibility problem. Our formulation targets a much broader inclusion model than those.

Of interest to us are the scenarios in which Problem 5.1 has no solution and must be replaced by a relaxed one which furnishes meaningful solutions. We consider the following relaxation which corresponds, in the special case of Example 2.6, to that proposed in [23].

Problem 5.2 Suppose that Assumptions 3.1 and 3.2 are in force and that $\mathsf{V} \neq \{0\}$ is a closed vector subspace of X, and let $\gamma \in \left]0, +\infty\right[$. The task is to

$$\text{find } \mathsf{x} \in \mathsf{X} \text{ such that } 0 \in \Big(\mathrm{proj}_{\mathsf{V}} \diamond \overset{\bullet}{\mathsf{M}}(\mathsf{L}_\omega, \gamma \mathsf{A}_\omega)_{\omega \in \Omega}\Big)\mathsf{x}. \tag{5.4}$$

Let us examine the interplay between Problems 5.1 and 5.2.

Proposition 5.3 *Consider the settings of* Problems 5.1 and 5.2, *let* S_1 *and* S_2 *be their respective sets of solutions, and set* $\mathsf{W} = \mathrm{proj}_{\mathsf{V}} \diamond \overset{\bullet}{\mathsf{M}}(\mathsf{L}_\omega, \gamma \mathsf{A}_\omega)_{\omega \in \Omega}$. *Then the following hold:*

(i) W *is maximally monotone.*
(ii) $J_{\mathsf{W}} = \mathrm{proj}_{\mathsf{V}} \circ (\mathrm{Id}_{\mathsf{X}} + \int_\Omega (\mathsf{L}_\omega^* \circ (J_{\gamma \mathsf{A}_\omega} - \mathrm{Id}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega)\mu(d\omega)) \circ \mathrm{proj}_{\mathsf{V}}$.
(iii) S_1 *and* S_2 *are closed convex sets.*
(iv) Problem 5.2 is an exact relaxation of Problem 5.1 in the sense that $\mathsf{S}_1 \neq \varnothing \Rightarrow \mathsf{S}_2 = \mathsf{S}_1$.
(v) $\mathsf{S}_2 = \mathrm{zer}(N_{\mathsf{V}} + \int_\Omega (\mathsf{L}_\omega^* \circ ({}^\gamma \mathsf{A}_\omega) \circ \mathsf{L}_\omega)\mu(d\omega))$.

Proof Set $A = {}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathsf{A}_\omega \mu(d\omega)$ and

$$L \colon \mathsf{X} \to \mathcal{H} \colon \mathsf{x} \mapsto \mathfrak{e}_{\mathsf{L}} \mathsf{x}. \tag{5.5}$$

Then [11, Proposition 3.5] asserts that

$$J_{\gamma A} = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus J_{\gamma \mathsf{A}_\omega} \mu(d\omega) \quad \text{and} \quad {}^\gamma A = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus {}^\gamma \mathsf{A}_\omega \, \mu(d\omega). \tag{5.6}$$

In addition, it follows from [11, Proposition 3.12] that L is a well-defined bounded linear operator with adjoint

$$L^*\colon \mathcal{H} \to \mathsf{X}\colon x^* \mapsto \int_\Omega \mathsf{L}_\omega^* x^*(\omega)\mu(d\omega) \tag{5.7}$$

and such that $\|L\| \leqslant 1$. We also recall from (3.21) that

$$\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega, \gamma\mathsf{A}_\omega)_{\omega\in\Omega} = L \blacklozenge (\gamma A). \tag{5.8}$$

Further, (5.1) is equivalent to

$$\text{find } \mathsf{x} \in \mathsf{V} \text{ such that } 0 \in A(L\mathsf{x}) \tag{5.9}$$

and (5.4) is equivalent to

$$\text{find } \mathsf{x} \in \mathsf{X} \text{ such that } 0 \in \Big(\mathrm{proj}_{\mathsf{V}} \diamond \big(L \blacklozenge (\gamma A)\big)\Big)\mathsf{x}. \tag{5.10}$$

(i): We derive from [11, Theorem 3.8(i)] that A is maximally monotone and hence from [19, Theorem 4.5(ii)] that $L \blacklozenge (\gamma A)$ is likewise. In turn, [19, Theorem 4.5(i)] and (5.8) assert that $\mathsf{W} = \mathrm{proj}_{\mathsf{V}} \diamond (L \blacklozenge (\gamma A))$ is maximally monotone.

(ii): By invoking successively [19, Theorem 6.3(ii)], (5.6), and (5.7), we obtain

$$\begin{aligned} J_{\mathsf{W}} &= \mathrm{proj}_{\mathsf{V}} \circ \big(\mathrm{Id}_{\mathsf{X}} + L^* \circ (J_{\gamma A} - \mathrm{Id}_{\mathcal{H}}) \circ L\big) \circ \mathrm{proj}_{\mathsf{V}} \\ &= \mathrm{proj}_{\mathsf{V}} \circ \left(\mathrm{Id}_{\mathsf{X}} + \int_\Omega \big(\mathsf{L}_\omega^* \circ (J_{\gamma\mathsf{A}_\omega} - \mathrm{Id}_{\mathsf{H}_\omega}) \circ \mathsf{L}_\omega\big)\mu(d\omega)\right) \circ \mathrm{proj}_{\mathsf{V}}\,. \end{aligned} \tag{5.11}$$

(iii): Use (5.9), (5.10), and [19, Theorem 6.3(iii)].
(iv): Combine (5.10) and [19, Theorem 6.3(v)].
(v): Using (5.10), [19, Theorem 6.3(vi)], (5.6), and (5.7), we obtain

$$\mathsf{S}_2 = \mathrm{zer}\big(N_{\mathsf{V}} + L^* \circ ({}^{\gamma}A) \circ L\big) = \mathrm{zer}\left(N_{\mathsf{V}} + \int_\Omega \big(\mathsf{L}_\omega^* \circ ({}^{\gamma}\mathsf{A}_\omega) \circ \mathsf{L}_\omega\big)\mu(d\omega)\right), \tag{5.12}$$

which concludes the proof. □

We now present an algorithm to solve Problem 5.2.

Proposition 5.4 *Suppose that* Problem 5.2 *has a solution, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,2[$ *such that* $\sum_{n\in\mathbb{N}}\lambda_n(2-\lambda_n)=+\infty$, *and let* $\mathsf{x}_0\in\mathsf{V}$. *Iterate*

$$
\begin{array}{l}
\text{for } n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for } \mu\text{-almost every } \omega\in\Omega\\
\left\lfloor
\begin{array}{l}
y_n(\omega)=\mathsf{L}_\omega\mathsf{x}_n\\
q_n(\omega)=y_n(\omega)-J_{\gamma\mathsf{A}_\omega}y_n(\omega)
\end{array}
\right.\\
\mathsf{z}_n=\int_\Omega \mathsf{L}_\omega^*(q_n(\omega))\mu(d\omega)\\
\mathsf{x}_{n+1}=\mathsf{x}_n-\lambda_n\,\mathrm{proj}_\mathsf{V}\,\mathsf{z}_n.
\end{array}
\right.
\end{array}
\tag{5.13}
$$

Then $(\mathsf{x}_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to Problem* 5.2.

Proof Set $\mathsf{W}=\mathrm{proj}_\mathsf{V}\diamond\overset{\blacklozenge}{\mathsf{M}}(\mathsf{L}_\omega,\gamma\mathsf{A}_\omega)_{\omega\in\Omega}$ and recall from Proposition 5.3(ii) that

$$
J_\mathsf{W}=\mathrm{proj}_\mathsf{V}\circ\left(\mathrm{Id}_\mathsf{X}+\int_\Omega\big(\mathsf{L}_\omega^*\circ(J_{\gamma\mathsf{A}_\omega}-\mathrm{Id}_{\mathsf{H}_\omega})\circ\mathsf{L}_\omega\big)\mu(d\omega)\right)\circ\mathrm{proj}_\mathsf{V}\,. \tag{5.14}
$$

We derive from (5.13), (5.6), and (5.7) that $(\mathsf{x}_n)_{n\in\mathbb{N}}$ is generated by the proximal point algorithm

$$
(\forall n\in\mathbb{N})\quad \mathsf{x}_{n+1}=\mathsf{x}_n+\lambda_n(J_\mathsf{W}\mathsf{x}_n-\mathsf{x}_n). \tag{5.15}
$$

It then follows from [17, Lemma 2.2(vi)] that $(\mathsf{x}_n)_{n\in\mathbb{N}}$ converges weakly to a point in $\mathrm{zer}\,\mathsf{W}$, i.e., a solution to (5.4). □

Example 5.5 Let us specialize Problem 5.1 to the scenario in which

$$
\begin{aligned}
&(\forall\omega\in\Omega)\quad \mathsf{A}_\omega=\big(\mathrm{Id}_{\mathsf{H}_\omega}-\mathsf{T}_\omega+\mathsf{r}_\omega\big)^{-1}-\mathrm{Id}_{\mathsf{H}_\omega},\\
&\text{where}\quad\begin{cases}\mathsf{T}_\omega\colon\mathsf{H}_\omega\to\mathsf{H}_\omega \text{ is firmly nonexpansive}\\ \mathsf{r}_\omega\in\mathsf{H}_\omega.\end{cases}
\end{aligned}
\tag{5.16}
$$

Then (5.1) becomes

$$
\text{find } \mathsf{x}\in\mathsf{V} \text{ such that } (\forall^\mu\omega\in\Omega)\ \mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x})=\mathsf{r}_\omega. \tag{5.17}
$$

This model has been considered in [23] in the setting of Example 2.6. There, Ω is a finite set and each r_ω models the observation of an unknown signal $\mathsf{x}\in\mathsf{H}$ through a Wiener system, i.e., the concatenation of a nonlinear operator T_ω and a linear transformation L_ω. Our framework allows us to extend it to models with a continuum of observations. In this context, and (5.4) yields the relaxed problem

$$
\text{find } \mathsf{x}\in\mathsf{V} \text{ such that } \int_\Omega \mathsf{L}_\omega^*\big(\mathsf{T}_\omega(\mathsf{L}_\omega\mathsf{x})-\mathsf{r}_\omega\big)\mu(d\omega)\in\mathsf{V}^\perp. \tag{5.18}
$$

Furthermore, (5.13) becomes

$$\begin{array}{l}\text{for } n = 0, 1, \ldots \\ \left\lfloor \begin{array}{l} \text{for } \mu\text{-almost every } \omega \in \Omega \\ \left\lfloor \begin{array}{l} y_n(\omega) = \mathsf{L}_\omega \mathsf{x}_n \\ q_n(\omega) = \mathsf{T}_\omega y_n(\omega) - \mathsf{r}_\omega \end{array} \right. \\ \mathsf{z}_n = \int_\Omega \mathsf{L}_\omega^*(q_n(\omega))\mu(d\omega) \\ \mathsf{x}_{n+1} = \mathsf{x}_n - \lambda_n \operatorname{proj}_{\mathsf{V}} \mathsf{z}_n. \end{array} \right. \end{array} \tag{5.19}$$

References

1. Attouch, H.: Familles d'opérateurs maximaux monotones et mesurabilité. Ann. Mat. Pura Appl. **120**, 35–111 (1979)
2. Bartz, S., Bauschke, H.H., Moffat, S.M., Wang, X.: The resolvent average of monotone operators: dominant and recessive properties. SIAM J. Optim. **26**, 602–634 (2016)
3. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edn. Springer, New York (2017)
4. Bauschke, H.H., Goebel, R., Lucet, Y., Wang, X.: The proximal average: basic theory. SIAM J. Optim. **19**, 766–785 (2008)
5. Becker, S.R., Combettes, P.L.: An algorithm for splitting parallel sums of linearly composed monotone operators, with applications to signal recovery. J. Nonlinear Convex Anal. **15**, 137–159 (2014)
6. Boţ, R.I.: Conjugate Duality in Convex Optimization. Springer, Berlin (2010)
7. Brézis, H., Browder, F.: Partial differential equations in the 20th century. Adv. Math. **135**, 76–144 (1998)
8. Briceño-Arias, L.M., Kalise, D., Silva, F.J.: Proximal methods for stationary mean field games with local couplings. SIAM J. Control Optim. **56**, 801–836 (2018)
9. Briceño-Arias, L.M., Roldán, F.: Resolvent of the parallel composition and the proximity operator of the infimal postcomposition. Optim. Lett. **17**, 399–412 (2023)
10. Browder, F.E., Gupta, C.P.: Monotone operators and nonlinear integral equations of Hammerstein type. Bull. Amer. Math. Soc. **75**, 1347–1353 (1969)
11. Bùi, M. N., Combettes, P. L.: Hilbert direct integrals of monotone operators, Canad. J. Math., to appear
12. Butnariu, D., Flåm, S.D.: Strong convergence of expected-projection methods in Hilbert spaces. Numer. Funct. Anal. Optim. **16**, 601–636 (1995)
13. Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759–775 (2012)
14. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Springer, Berlin (1977)
15. Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
16. Cheung, Y.-M., Lou, J.: Proximal average approximated incremental gradient descent for composite penalty regularized empirical risk minimization. Machine Learn. **106**, 595–622 (2017)
17. Combettes, P.L.: Iterative construction of the resolvent of a sum of maximal monotone operators. J. Convex Anal. **16**, 727–748 (2009)
18. Combettes, P.L.: Monotone operator theory in convex optimization. Math. Program. **B170**, 177–206 (2018)
19. Combettes, P.L.: Resolvent and proximal compositions. Set-Valued Var. Anal. **31**, 22 (2023)
20. Combettes, P.L.: The geometry of monotone operator splitting methods. Acta Numer. **33**, 487–632 (2024)
21. Combettes, P.L., Wajs, V.R.: Signal recovery by proximal forward-backward splitting. Multiscale Model. Simul. **4**, 1168–1200 (2005)
22. Combettes, P.L., Woodstock, Z.C.: Reconstruction of functions from prescribed proximal points. J. Approx. Theory **268**, 105606 (2021)
23. Combettes, P.L., Woodstock, Z.C.: A variational inequality model for the construction of signals from inconsistent nonlinear equations. SIAM J. Imaging Sci. **15**, 84–109 (2022)

24. Daubechies, I., Defrise, M., De Mol, C.: An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. Comm. Pure Appl. Math. **57**, 1413–1457 (2004)
25. Dixmier, J.: Les Algèbres d'Opérateurs dans l'Espace Hilbertien (Algèbres de von Neumann), 2e éd. Gauthier–Villars, Paris (1969)
26. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
27. Hu, Z., Shaloudegi, K., Zhang, G., Yu, Y.: Federated learning meets multi-objective optimization. IEEE Trans. Network Sci. Eng. **9**, 2039–2051 (2022)
28. Pennanen, T.: On the range of monotone composite mappings. J. Nonlinear Convex Anal. **2**, 193–202 (2001)
29. Rockafellar, R.T.: Network Flows and Monotropic Optimization. Wiley, New York (1984)
30. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis, 3rd printing. Springer, New York (2009)
31. Schwartz, L.: Analyse III - Calcul Intégral. Hermann, Paris (1993)
32. Yu, Y.-L.: Better approximation and faster algorithm using the proximal average. In: Proc. Conf. Adv. Neural Inform. Process. Syst., pp. 458–466, (2013)