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Structure of Relatively Biexact Group von Neumann Algebras

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Abstract: Using computations in the bidual of $\mathbb{B}(L^2M)$ we develop a new technique at the von Neumann algebra level to upgrade relative proper proximality to full proper proximality. This is used to structurally classify subalgebras of $L\Gamma$ where Γ is an infinite group that is biexact relative to a finite family of subgroups $\{\Lambda_i\}_{i\in I}$ such that each Λ_i is almost malnormal in Γ. This generalizes the result of Ding et al. (Properly proximal von Neumann algebras, 2022. arXiv:2204.00517) which classifies subalgebras of von Neumann algebras of biexact groups. By developing a combination with techniques from Popa's deformation-rigidity theory we obtain a new structural absorption theorem for free products and a generalized Kurosh type theorem in the setting of properly proximal von Neumann algebras.

1. Introduction

Recently the authors and Peterson in [13] developed the theory of small at infinity compactifications a la Ozawa [7], in the setting of tracial von Neumann algebras. At the foundation of this work lies the theory of operator M-bimodules and the several natural topologies that arise in this setting (see [16,25–27]). The small at infinity compactification is a canonical strong operator bimodule (in the sense of Magajna [25]) containing the compact operators. By using the noncommutative Grothendieck inequality (similar to Ozawa in [32]) it was seen that this strong operator bimodule coincides with $\mathbb{K}^{\infty,1}(M)$, the closure $\mathbb{K}^{\infty,1}(M)$ of $\mathbb{K}(L^2M)$ with respect to the $\|\cdot\|_{\infty,1}$-norm on $\mathbb{B}(L^2M)$ given by $\|T\|_{\infty,1} = \sup_{x\in M,\|x\|\leq 1}\|T\hat{x}\|_1$. The small at infinity compactification of a tracial von Neumann algebra M is then given by

$$\mathbb{S}(M) = \{T \in \mathbb{B}(L^2M) \mid [T, JxJ] \in \mathbb{K}^{\infty,1}(M), \text{ for all } x \in M\}.$$

It is easy to see that this operator M-system $\mathbb{S}(M)$ contains M and $\mathbb{K}(L^2M)$, and is an M-bimodule. The advantage of the strong operator bimodule perspective is that it to identify an operator $T \in \mathbb{S}(M)$ suffices to check that $[T, JxJ] \in \mathbb{K}^{\infty,1}(M)$ for all x in some weakly closed subset of M. This is what allows for the passage between the group and the von Neumann algebra settings. Using this technology [13] defined the notion of proper proximality for finite von Neumann algebras, extending the dynamical notion for groups [5]: A finite von Neumann algebra (M, τ) is properly proximal if there does not exist an M-central state φ on $\mathbb{S}(M)$ such that $\varphi_{|M} = \tau$. By identifying and studying this property in various examples, the authors of [13] obtained applications to the structure theory of II_1-factors. The goal of the present paper is to add to the list of applications.

The machinery underlying the results in this paper is built on is the notion of an M-boundary piece developed in [13], as an analogue of the group theoretic notion introduced in [5]. The motivation for considering this notion is that it allows for one to exploit the dynamics that is available only on certain locations of the Stone–Cech boundary of the group. For a group Γ, a boundary piece is a closed left and right invariant subset of $\beta(\Gamma)\backslash\Gamma$, whereas in the von Neumann algebra setting, it is denoted by $\mathbb{X}$ typically and is a certain hereditary C^*-subalgebra of $\mathbb{B}(L^2M)$ containing the compact operators (see Sect. 3.1). One then considers the small at infinity compactification relative to a boundary piece $\mathbb{S}_{\mathbb{X}}(M)$ where $\mathbb{K}^{\infty,1}(M)$ is replaced with $\mathbb{K}_{\mathbb{X}}^{\infty,1}(M)$, a suitable analogue for the boundary piece. Then one can define the notion of proper proximality relative to $\mathbb{X}$, demanding that there be no M-central state restricting to the trace on $\mathbb{S}_{\mathbb{X}}(M)$. The main example we will be working with is a boundary piece generated by a finite family of von Neumann subalgebras $\{M_i\}_{i=1}^n$ (see Example 3.1), which is adapted from the construction for a finite family of subgroups (see Example 3.3 in [5]).

In [12], the authors demonstrated an instance where relative proper proximality can be lifted to full proper proximality, i.e, when the boundary piece arises from subgroups that are almost malnormal[1] and not co-amenable (see Lemma 3.3 in [12]). The authors used this idea to classify proper proximality for wreath product groups. In this paper, we develop an analogue of this idea in the setting of von Neumann algebras (Theorem 1.1). In both cases, one has to work in the bidual of the small at infinity compactification for technical reasons, and this brings about an extra layer of subtlety especially in the von Neumann setting. More specifically we show that one can map the basic construction into the bidual version of the relative small at infinity compactification, provided the boundary piece arises from a mixing subalgebra. Composing with an appropriate state on this space, we get the link with relative amenability in the von Neumann setting. This upgrading theorem is the main new technical tool we develop in the present work:

Theorem 1.1. *Let M be a diffuse finite von Neumann algebra, $M_i \subset M$, $i = 1, \dots, n$ diffuse von Neumann subalgebras, and $A \subset pMp$ a von Neumann subalgebra, for some $p \in \mathcal{P}(M)$. Denoting by e_{M_i} the Jones projection of $M_i \subset M$, we further assume there exists a countable group $\mathcal{G} < \mathcal{U}(M)$ with $\mathcal{G}'' = M$ such that*

(1) *The family $\{uJvJe_{M_i}Jv^*Ju^* \mid u, v \in \mathcal{G}, i = 1, \dots, n\}$ consists of pairwise commuting projections;*
(2) *For each i, $M_i \subset M$ admits a Pimsner–Popa basis $\{m_k^i\}_{k\in\mathbb{N}} \subset M$ such that for any $u \in \mathcal{G}$ and $k \in \mathbb{N}$, we may find some $k_u \in \mathbb{N}$ and $u_k^i \in \mathcal{U}(M_i)$ such that $um_k^i = m_{k_u}^i u_k^i$, and elements in $\{m_k^i J m_\ell^i J e_{M_i} J(m_\ell^i)^* J(m_k^i)^* \mid k, \ell \in \mathbb{N}, i = 1, \dots, n\}$ are pairwise commuting.*

Assume that A is properly proximal relative to $\mathbb{X}$ inside M, where $\mathbb{X}$ is the boundary piece associated with $\{M_i\}_{i=1}^n$, and $M_i \subset M$ is mixing for each $i = 1, \dots, n$. Then

[1] A subgroup $H < G$ is almost malnormal if for all $g \in G\backslash H$, $gHg^{-1} \cap H$ is finite.

there exist projections $f_0 \in \mathcal{Z}(A)$ *and* $f_i \in \mathcal{Z}(A' \cap pMp)$, $1 \le i \le n$, *such that* Af_0 *is properly proximal and* Af_i *is amenable relative to* M_i *inside* M *for each* $1 \le i \le n$, *and* $\sum_{i=0}^{n} f_i = p$.

Remark 1.2. We point out that the above two conditions on the existence of such countable subgroup $\mathcal{G} < \mathcal{U}(M)$ are satisfied in the case of group von Neumann algebras, which is our main application. Indeed, when $M = L\Gamma$ and $M_i = L\Gamma_i$, where Γ is a countable discrete group and $\Gamma_i < \Gamma$ is an infinite subgroup for $i = 1, \ldots, n$, one may take $\mathcal{G} = \Gamma$ and for each i set $m_k^i = u_{t_k} \in \mathcal{U}(M)$, where $\{t_k\}_{k\in\mathbb{N}} \subset \Gamma$ is a transversal for Γ/Γ_i.

Using these ideas we are interested in classifying subalgebras of group von Neumann algebras arising from groups that are biexact relative to a family of subgroups (see e.g. [7, Chapter 15]). The first result of this kind was obtained in Theorem 7.2 of [13] where it was shown that every subalgebra of the von Neumann algebra of a biexact group either has an amenable direct summand or is properly proximal. As essentially observed there, what relative biexactness buys us is the relative proper proximality for any subalgebra, relative to the boundary piece arising from the subgroups. Combining this with our upgrading result above, we obtain our main result below which is a structure theorem for von Neumann subalgebras of group von Neumann algebras that are biexact relative to a family of subgroups where each subgroup is almost malnormal.

Theorem 1.3. *Let* Γ *be a countable group with a family of almost malnormal subgroups* $\{\Lambda_i\}_{i=1}^n$. *If* Γ *is biexact relative to* $\{\Lambda_i\}_{i=1}^n$, *then for any von Neumann subalgebra* $A \subset L\Gamma$, *there exists* $p \in \mathcal{Z}(A)$ *and projections* $p_j \in \mathcal{Z}((Ap)' \cap pL(\Gamma)p)$ *such that* $\bigvee_{j=1}^n p_j = p$ *and* Ap_i *is amenable relative to* $L\Lambda_i$ *inside* $L\Gamma$, *for each* $i = 1, \ldots, n$ *and* $Ap^{\perp}$ *is properly proximal.*

There are two natural instances where such a phenomenon (a countable group Γ with a family of almost malnormal subgroups $\{\Lambda_i\}_{i=1}^n$ where Γ is biexact relative to $\{\Lambda_i\}_{i=1}^n$) is observed: First is in the setting of free products, which we deal with in the present paper. Second is in the setting of wreath products, which is investigated in a follow-up work by the first author [11]. There is conjecturally a third setting of relative hyperbolicity, which we comment on in the end of the introduction.

Thanks to Bass–Serre theory [39] we have a complete understanding of subgroups of a free product of groups. As a result, one can derive results of the following nature: If $H < G_1 * G_2$ such that $|H \cap G_1| \ge 3$, then H is amenable only if $H < G_1$. This phenomenon is referred to as amenable absorption. Interestingly, the situation for von Neumann algebras is much more complicated. There is comparatively a very limited understanding of von Neumann subalgebras of free products. Whether every self adjoint operator in any finite von Neumann algebra is contained in a copy of the hyperfinite II_1-factor was itself an open problem for many years.[2] Popa settled it in the negative in [37] by discovering a surprising amenable absorption theorem for free product von Neumann algebras, thereby showing that a generator masa in $L\mathbb{F}_2$ is maximally amenable.

Popa's ideas been used to show maximal amenability in other situations (See for instance [6,9,34,41]). In the past decade there have been other new ideas that have been used to prove absorption theorems: Boutonnet–Carderi's approach [2] relies on elementary computations in a crossed-product C^*-algebra; Boutonnet–Houdayer [4] use the study of non normal conditional expectations; [17] used a free probabilistic approach

[2] This is a question of Kadison, Problem 7 from 'Problems on von Neumann algebras, Baton Rouge Conference'

to study absorption. Ozawa in [32] then gave a short proof of amenable absorption in tracial free products. There have also been a variety of important free product absorption results which are of a different flavor, and are structural in nature. See for example [22] and [10].

By applying our Theorem 1.3 in the setting of free products and using machinery from Popa's deformation-rigidity theory (specifically work of Ioana [21]), we obtain a generalized structural absorption theorem below:

Corollary 1.4. *Let (M_1, τ_1) and (M_2, τ_2) be such that $M_i \cong L\Gamma_i$ where Γ_i are countable exact groups and $M = M_1 * M_2$ be the tracial free product. Let $A \subset M$ be a von Neumann subalgebra with $A \cap M_1$ diffuse. If $A \subset M$ has no properly proximal direct summand, then $A \subset M_1$.*

Remark 1.5. Using results of the upcoming work [14], one can relax the assumption on M_i, from being infinite group von Neumann algebras of exact groups, to just that they are diffuse weakly exact von Neumann algebras. We do not comment more on this at the moment because for the sake of examples, the above setting already provides many.

The authors of [23] showed that there are examples of groups that are neither inner amenable nor properly proximal. All of these group von Neumann algebras fit into the setting of the above corollary. Note that Vaes constructed in [40] plenty of groups that are inner amenable, yet their group von Neumann algebras lack Property (Gamma). Hence our results give a strict generalization of (Gamma) absorption (see Houdayer's Theorem 4.1 in [18] and see also Theorem A in [17]) in these examples.

Remark 1.6. The above result is false if one considers amalgamated free products. For instance, take $M_1 = M_2 = L\mathbb{F}_2 \overline{\otimes} R$ and $A = (L\mathbb{Z} * L\mathbb{Z}) \overline{\otimes} R \subset M_1 *_R M_2$, where two copies of $L\mathbb{Z}$ in A are from M_1 and M_2, respectively.

Remark 1.7. Shortly before the posting of this paper, Drimbe announced a paper (see [15]) where he shows using Popa's deformation-rigidity theory that for any nonamenable inner amenable group Γ, if $L(\Gamma) \subset M_1 * M_2$, then $L(\Gamma)$ intertwines into M_i for some $i = 1, 2$. This in particular generalizes Corollary 1.4 in the case that $A \cong L\Gamma$ for some inner amenable group Γ, because he doesn't require any assumptions for M_i.

Our techniques also reveal the following new Kurosh type structure theorem for free products in the setting of proper proximality, (partially generalizing Corollary 8.1 in [15]). See also [19,22,31,35] for other important Kurosh type theorems.

Corollary 1.8. *Let $M = L\Gamma_1 * \cdots * L\Gamma_m = L\Lambda_1 * \cdots * L\Lambda_n$, where all groups Γ_i and Λ_j are countable exact nonamenable non-properly proximal i.c.c. groups. Then $m = n$ and after a permutation of indices $L\Gamma_i$ is unitarily conjugate to $L\Lambda_i$.*

We conclude by state the following folklore conjecture (also stated in [29]), which would provide another family of examples for applying Theorem 1.3. Indeed the peripheral subgroups below are almost malnormal (see Theorem 1.4 in [28]).

Conjecture 1. [29] If G is exact and hyperbolic relative to a family of peripheral subgroups $\{H_i\}_{i=1}^n$, then G is biexact relative to $\{H_i\}_{i=1}^n$.

2. Preliminaries

2.1. The basic construction and Pimsner–Popa orthogonal bases. Let M be a finite von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. The basic construction $\langle M, e_Q \rangle$ is defined as the von Neumann subalgebra of $\mathbb{B}(L^2M)$ generated by M and the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q)$. There is a semifinite faithful normal trace on $\langle M, e_Q \rangle$ satisfying $\text{Tr}(xe_Q y) = \tau(xy)$, for every $x, y \in M$.

Let $N \subset M$ be a von Neumann subalgebra. Then a Pimsner–Popa basis (see [36]) of M over N is a family of elements denoted $M/N = \{m_j\}_{j\in J} \subset M$ such that

(1) $E_N(m_j^* m_k) = \delta_{j,k} p_j$, where $p_j \in \mathcal{P}(N)$ is a projection.
(2) $L^2(M) = \bigoplus_{j\in J} m_j L^2(N)$ and every $x \in M$ has a unique decomposition $x = \sum_j m_j E_N(m_j^* x)$.

In the case that $N = L(\Lambda)$ and $M = L(\Gamma)$ where $\Lambda < \Gamma$, we can identify a Pimnser–Popa basis in M from a choice of coset-representatives i.e, $\Gamma = \bigsqcup_{k\geq 0} t_k \Lambda$, and $m_k := \lambda_{t_k} \in \mathcal{U}(L\Gamma)$: $M/N = \{u_j\}_{j\in J}$.

2.2. Popa's intertwining-by-bimodules.

Theorem 2.1. [38] *Let (M, τ) be a tracial von Neumann algebra and $P \subset pMp$, $Q \subset M$ be von Neumann subalgebras. Then the following are equivalent:*

(1) *There exist projections $p_0 \in P, q_0 \in Q$, a $*$-homomorphism $\theta : p_0 P p_0 \to q_0 Q q_0$ and a non-zero partial isometry $v \in q_0 M p_0$ such that $\theta(x)v = vx$, for all $x \in p_0 P p_0$.*
(2) *There is no sequence $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(x^* u_n y)\|_2 \to 0$, for all $x, y \in pM$.*

If one of these equivalent conditions holds, we write $P \prec_M Q$, and say that *a corner of P embeds into Q inside M.*

2.3. Relative amenability. Let $P \subset M$ and $Q \subset M$ be a von Neumann subalgebras. We say that P is *amenable relative to Q inside M* if there exists a sequence $\xi_n \in L^2(\langle M, e_Q \rangle)$ such that $\langle x\xi_n, \xi_n \rangle \to \tau(x)$, for every $x \in M$, and $\|y\xi_n - \xi_n y\|_2 \to 0$, for every $y \in P$. By [33], Theorem 2.1 P is amenable relative to Q inside M if and only if there exists a P-central state in the basic construction $\langle M, e_Q \rangle$ that is normal when restricted to M, and faithful on $\mathcal{Z}(P' \cap M)$.

2.4. Mixing subalgebras and free products of finite von Neumann algebras. Let M be a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Recall the inclusion $N \subset M$ is mixing if $L^2(M \ominus N)$ is mixing as an N-N bimodule, i.e., for any sequence $u_n \in \mathcal{U}(N)$ converging to 0 weakly, one has $\|E_N(xu_n y)\|_2 \to 0$ for any $x, y \in M \ominus N$. When M and N are both diffuse, we may replace sequence of unitaries with any sequence in N converging to 0 weakly [13, Theorem 5.9].

Remark 2.2. Let M be a diffuse finite von Neumann algebra and $N \subset M$ a diffuse von Neumann subalgebra. If $N \subset M$ is mixing, then it is easy to check that $e_N x JyJ e_N \in \mathbb{B}(L^2M)$ is a compact operator from M to L^2M assuming x or $y \in M \ominus N$.

Examples of mixing subalgebras include M_1 and $M_2 \subset M_1 * M_2$, where M_1 and M_2 are diffuse [24, Proposition 1.6] and $L\Lambda \subset L\Gamma$, where $\Lambda < \Gamma$ is almost malnormal (see Proposition 2.4 in [3]).

The following [20, Corollary 2.12] is crucial to the proof of Theorem 5.1.

Lemma 2.3 (Ioana). *Let M_1, M_2 be two diffuse tracial von Neumann algebras and $M = M_1 * M_2$ be the tracial free product. Let $A \subset M$ be a subalgebra such that A is amenable relative to M_1 in M. Then either $A \prec_M M_1$ or A is amenable.*

We also need the following case of the main result of [4]:

Theorem 2.4 (Boutonnet–Houdayer). *Let $M = M_1 * M_2$, where M_i are diffuse tracial von Neumann algebras. If $A \subset M$ is a von Neumann subalgebra that satisfies $A \cap M_1$ is diffuse and A is amenable relative to M_1 inside M, then $A \subset M_1$.*

3. Proper Proximality for von Neumann Algebras and Boundary Pieces

3.1. Boundary pieces from von Neumann subalgebras. Let M be a finite von Neumann algebra. An M-boundary piece is a hereditary C*-subalgebra $\mathbb{X} \subset \mathbb{B}(L^2M)$ such that $\mathbb{M}(\mathbb{X}) \cap M$ and $\mathbb{M}(\mathbb{X}) \cap JMJ$ are weakly dense in M and JMJ, respectively, where $\mathbb{M}(\mathbb{X})$ is the multiplier algebra of $\mathbb{X}$. To avoid pathological examples, we will always assume that $\mathbb{X} \neq \{0\}$, and it follows that $\mathbb{K}(L^2M) \subset \mathbb{X}$, by the assumption on $\mathbb{M}(\mathbb{X})$.

The main example of an M-boundary piece we use in this paper is one generated by von Neumann subalgebras. We recall some facts about hereditary C*-algebras for what follows (see e.g. [1, II.5]).

Let A be a C*-algebra. There is a one-to-one correspondence between the set of hereditary C*-subalgebras of A and the set of closed left ideals in A: given a hereditary C*-subalgebra $H \subset A$, $L_H := AH = \{ah \mid a \in A, h \in H\}$ is a closed left ideal; and for a closed left ideal $L \subset A$, $H_L = L \cap L^*$ is a hereditary C*-subalgebra of A. Given a subset of operators $\{b_i\}_{i\in I} \subset A$, the hereditary C*-subalgebra generated by $\{b_i\}_{i\in I}$ is $BAB = \{bab \mid b \in B_+, a \in A\}$, where B is the C*-subalgebra generated by $\{b_i\}_{i\in I}$.

Example 3.1. (Boundary piece generated by subalgebras) Let M be a finite von Neumann algebra. Suppose $M_i \subset M$, $i = 1, \dots, n$ are von Neumann subalgebras and denote by $e_{M_i} \in \mathbb{B}(L^2M)$ the orthogonal projection from L^2M onto the space L^2M_i. The M-boundary piece associated with the family of subalgebras $\{M_i\}_{i=1}^n$ is the hereditary C*-subalgebra of $\mathbb{B}(L^2M)$ generated by operators of the form $xJyJe_{M_i}$ with $x, y \in M$, $i = 1, \dots, n$, and it is clear that M and JMJ are contained in its multiplier algebra.

Lemma 3.2. *Let M be a finite von Neumann algebra and $M_i \subset M$, $i = 1, \dots, n$ von Neumann subalgebras such that the projections $\{e_{M_i}\}_{i=1}^n$ are pairwise commuting. Let $\mathbb{X}$ be the hereditary C*-subalgebra in $\mathbb{B}(L^2M)$ generated by $\{xJyJ(\vee_{i=1}^n e_{M_i}) \mid x, y \in M\}$ and $\mathbb{Y}$ the hereditary C*-subalgebra in $\mathbb{B}(L^2M)$ generated by $\{xJyJe_{M_i} \mid i = 1, \cdots n,\ x, y \in M\}$. Then $\mathbb{X} = \mathbb{Y}$.*

Proof. First note that $e_{M_i} \in \mathbb{X}$ for each i since $0 \leq e_{M_i} \leq \vee_{i=1}^n e_{M_i}$. We also have $\vee_{i=1}^n e_{M_i} \in \mathbb{Y}$. In fact, for each pair i, j, $e_{M_i} \wedge e_{M_j} \in \mathbb{Y}$ as $0 \leq e_{M_i} \wedge e_{M_j} \leq e_{M_i}$, and $e_{M_i} \vee e_{M_j} = e_{M_i} + e_{M_j} - e_{M_i} \wedge e_{M_j} \in \mathbb{Y}$ as $[e_{M_i}, e_{M_j}] = 0$. To see that $\mathbb{X} \subset \mathbb{Y}$, note that $L = \mathbb{B}(L^2M)\mathbb{X}$ is contained in $K = \mathbb{B}(L^2M)\mathbb{Y}$. Indeed, for any $x, y \in M$ and $T \in \mathbb{B}(L^2M)$, we have $T(\vee_{i=1}^n e_{M_i})xJyJ \in \mathbb{B}(L^2M)\mathbb{Y}xJyJ = \mathbb{B}(L^2M)\mathbb{Y}$ as M and JMJ are in the multiplier algebra of $\mathbb{Y}$. By a similar argument we see that $\mathbb{Y} \subset \mathbb{X}$. □

Fix an M-boundary piece $\mathbb{X}$ and let $\mathbb{K}^L_{\mathbb{X}}(M) \subset \mathbb{B}(L^2M)$ denote the $\|\cdot\|_{\infty,2}$ closure of the closed left ideal $\mathbb{B}(L^2M)\mathbb{X}$, i.e., $\mathbb{K}^L_{\mathbb{X}}(M) = \overline{\mathbb{B}(L^2 M)\mathbb{X}}^{\|\cdot\|_{\infty,2}}$, where $\|\cdot\|_{\infty,2}$ on $\mathbb{B}(L^2 M)$ is given by $\|T\|_{\infty,2} = \sup_{x\in(M)_1} \|T\hat{x}\|_2$ for $T \in \mathbb{B}(L^2M)$.

We let $\mathbb{K}_{\mathbb{X}}(M) = (\mathbb{K}^L_{\mathbb{X}}(M))^* \cap (\mathbb{K}^L_{\mathbb{X}}(M))$, which is a hereditary C*-subalgebra of $\mathbb{B}(L^2 M)$ with M and JMJ contained in $\mathbb{M}(\mathbb{K}^L_{\mathbb{X}}(M))$ [13, Sect. 3]. Denote by $\mathbb{K}^{\infty,1}_{\mathbb{X}}(M)$ the $\|\cdot\|_{\infty,1}$ closure of $\mathbb{K}_{\mathbb{X}}(M)$ in $\mathbb{B}(L^2 M)$, $\|T\|_{\infty,1} = \sup_{x,y\in(M)_1}\langle T\hat{x}, \hat{y}\rangle$ for $T \in \mathbb{B}(L^2 M)$ and it coincides with $\overline{\mathbb{X}}^{\|\cdot\|_{\infty,1}}$.

Now put $\mathbb{S}_{\mathbb{X}}(M) \subset \mathbb{B}(L^2M)$ to be

$$\mathbb{S}_{\mathbb{X}}(M) = \{T \in \mathbb{B}(L^2M) \mid [T, JxJ] \in \mathbb{K}^{\infty,1}_{\mathbb{X}}(M) \text{ for all } x \in M\},$$

which is an operator system that contains M. In the case when $\mathbb{X} = \mathbb{K}(L^2 M)$, we write $\mathbb{S}(M)$ instead of $\mathbb{S}_{\mathbb{K}(L^2 M)}(M)$.

Recall from [13, Theorem 6.2] that for a finite von Neumann subalgebra $N \subset M$ and an M-boundary piece $\mathbb{X}$, we say N is properly proximal relative to $\mathbb{X}$ in M if there is no N-central state φ on $\mathbb{S}_{\mathbb{X}}(M)$ that is normal on M. And we say M is properly proximal if M is properly proximal relative to $\mathbb{K}(L^2M)$ in M.

Remark 3.3. Let M and Q be finite von Neumann algebras, $\mathbb{X}$ an M-boundary piece, and $N \subset pMp$ be a von Neumann subalgebra, where $0 \neq p \in \mathcal{P}(M)$.

(1) Consider the u.c.p. map $\mathcal{E}_N := \mathrm{Ad}(e_N) \circ \mathrm{Ad}(pJpJ) : \mathbb{B}(L^2 M) \to \mathbb{B}(L^2N)$. Then by [13, Remark 6.3] that $\mathcal{E}_N(\mathbb{K}_{\mathbb{X}}(M)) \subset \mathbb{B}(L^2N)$ forms an N-boundary piece. And we say $\mathcal{E}_N(\mathbb{K}_{\mathbb{X}}(M))$ is the induced N-boundary piece, which will be denoted by $\mathbb{X}^N$.
(2) If N is properly proximal relative to $\mathbb{X}$ inside M, then zN is also properly proximal relative to $\mathbb{X}$ inside M for any $0 \neq z \in \mathcal{Z}(\mathcal{P}(N))$, since $\mathrm{Ad}(z) \circ \mathbb{S}_{\mathbb{X}}(M) \subset \mathbb{S}_{\mathbb{X}}(M)$.
(3) If N is properly proximal relative to $\mathbb{X}$ inside M, then N has no amenable direct summand. To see this, suppose qN is amenable for some $0 \neq q \in \mathcal{Z}(\mathcal{P}(N)$ and let φ be a qN-central state on $\mathbb{B}(L^2(qN))$. Consider $\mu := \varphi \circ \mathrm{Ad}(q) \circ \mathrm{Ad}(e_N) : \mathbb{B}(L^2M) \to \mathbb{C}$, and one checks that μ is a N-central state with $\mu_{|M}$ being normal.
(4) Notice that from the definition it follows that proper proximality is stable under taking direct sum. Thus we may take $f \in \mathcal{Z}(\mathcal{P}(Q))$ so that Qf is the maximal properly proximal direct summand of Q.

3.2. Bidual formulation of proper proximality. Given a finite von Neumann algebra M and a C*-subalgebra $A \subset \mathbb{B}(L^2M)$ such that M and JMJ are contained in $\mathbb{M}(A)$, we recall that $A^{M\sharp M}$ (resp. $A^{JMJ\sharp JMJ}$) denotes the space of $\varphi \in A^*$ such that for each $T \in A$ the map $M \times M \ni (a, b) \mapsto \varphi(aTb)$ (resp. $JMJ \times JMJ \ni (a, b) \mapsto \varphi(aTb)$) is separately normal in each variable and set $A^\sharp_J = A^{M\sharp M} \cap A^{JMJ\sharp JMJ}$. Moreover, we may view $(A^\sharp_J)^*$ as a von Neumann algebra in the following way, as shown in [13, Sect. 2]. Denote by $p_{\mathrm{nor}} \in \mathbb{B}(L^2M)^{**}$ the supremum of support projections of states in $\mathbb{B}(L^2M)^*$ that restrict to normal states on M and JMJ, so that M and JMJ may be viewed as von Neumann subalgebras of $p_{\mathrm{nor}}\mathbb{M}(A)^{**}p_{\mathrm{nor}}$. Note that p_{nor} lies in $\mathbb{M}(A)^{**}$ and $p_{\mathrm{nor}}\mathbb{M}(A)^{**}p_{\mathrm{nor}}$ is canonically identified with $(\mathbb{M}(A)^\sharp_J)^*$. Let $q_A \in \mathcal{P}(\mathbb{M}(A)^{**})$ be the central projection such that $q_A(\mathbb{M}(A)^{**}) = A^{**}$ and we may then identify $(A^\sharp_J)^*$ with $q_A p_{\mathrm{nor}}\mathbb{M}(A)^{**}p_{\mathrm{nor}} = p_{\mathrm{nor}}A^{**}p_{\mathrm{nor}}$, which is also a von Neumann algebra. Furthermore, if $B \subset A$ is another C*-subalgebra with M, $JMJ \subset \mathbb{M}(B)$, we may identify $(B_J^\sharp)^*$ with $q_B p_{\rm nor} A^{**} p_{\rm nor} q_B$, which is a non-unital subalgebra of $(A_J^\sharp)^*$.

We will need the following bidual characterization of properly proximal.

Lemma 3.4. [13, Lemma 8.5] *Let M be a separable tracial von Neumann algebra with an M-boundary piece $\mathbb{X}$. Then M is properly proximal relative to $\mathbb{X}$ if and only if there is no M-central state φ on*

$$\widetilde{\mathbb{S}}_{\mathbb{X}}(M) := \left\{T \in \left(\mathbb{B}(L^2M)_J^\sharp\right)^* \mid [T, a] \in \left(\mathbb{K}_{\mathbb{X}}(M)_J^\sharp\right)^* \text{ for all } a \in JMJ\right\}$$

such that $\varphi_{|M}$ is normal.

Using the above notations, we observe that we may identify $\tilde{\mathbb{S}}_{\mathbb{X}}(M)$ in the following way:

$$\begin{aligned}\widetilde{\mathbb{S}}_{\mathbb{X}}(M) &= \{T \in \left(\mathbb{B}(L^2M)_J^\sharp\right)^* | \, [T, a] \in \left(\mathbb{K}_{\mathbb{X}}(M)_J^\sharp\right)^*, \text{ for any } a \in JMJ\} \\ &= \{T \in p_{\rm nor}\mathbb{B}(L^2M)^{**} \\ &\quad p_{\rm nor} \mid [T, a] \in q_{\mathbb{X}} p_{\rm nor}\left(\mathbb{M}(\mathbb{K}_{\mathbb{X}}(M))\right)^{**} p_{\rm nor} q_{\mathbb{X}}, \text{ for any } a \in JMJ\},\end{aligned}$$

where $q_{\mathbb{X}}$ is the identity of $\mathbb{K}_{\mathbb{X}}(M)^{**} \subset \left(\mathbb{M}(\mathbb{K}_{\mathbb{X}}(M))\right)^{**}$. If we set $q_{\mathbb{K}} = q_{\mathbb{K}(L^2M)}$ to be the identity of $\mathbb{K}(L^2M)^{**} \subset \mathbb{B}(L^2M)^{**}$, then using the above description of $\tilde{\mathbb{S}}_{\mathbb{X}}(M)$, we have $q_{\mathbb{X}}^{\perp}\tilde{\mathbb{S}}_{\mathbb{X}}(M)q_{\mathbb{X}}^{\perp} \subset q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M)$, as $q_{\mathbb{X}}$ commutes with JMJ.

Remark 3.5. Recall that we may embed $\mathbb{B}(L^2M)$ into $(\mathbb{B}(L^2M)_J^\sharp)^*$ through the u.c.p. map $\iota_{\rm nor}$, which is given by $\iota_{\rm nor} = \mathrm{Ad}(p_{\rm nor}) \circ \iota$, where $\iota : \mathbb{B}(L^2M) \to \mathbb{B}(L^2M)^{**}$ is the canonical $*$-homomorphism into the universal envelope, and $p_{\rm nor}$ is the projection in $\mathbb{B}(L^2M)^{**}$ such that $p_{\rm nor}\mathbb{B}(L^2M)^{**}p_{\rm nor} = (\mathbb{B}(L^2M)_J^\sharp)^*$. Restricting $\iota_{\rm nor}$ to C*-subalgebra $A \subset \mathbb{B}(L^2M)$ satisfying $M, JMJ \subset \mathbb{M}(A)$ give rise to the embedding of A into $(A_J^\sharp)^*$, and $(\iota_{\rm nor})_{|M}$, $(\iota_{\rm nor})_{|JMJ}$ are faithful normal representations of M and JMJ, respectively. Furthermore, although $\iota_{\rm nor}$ is not a $*$-homomorphism, we still have $\mathrm{sp}Me_BM = \mathrm{sp}\{xe_By \mid x, y \in M\}$ is in the multiplicative domain of $\iota_{\rm nor}$, where $B \subset M$ is a von Neumann subalgebra and $e_B : L^2M \to L^2B$ is the Jones projection, by Lemma 3.8.

Lemma 3.6. *Let M be a finite von Neumann algebra and $\mathbb{X}$ an M-boundary piece. Let $\mathbb{X}_0 \subset \mathbb{K}_{\mathbb{X}}(M)$ be a* C*-*subalgebra and $\{e_n\}_{n\in I}$ an approximate unit of $\mathbb{X}_0$. If $\iota_{\rm nor}(\mathbb{X}_0)$ is weak* dense in $(\mathbb{K}_{\mathbb{X}}(M)_J^\sharp)^*$ and $\iota(e_n)$ commutes with $p_{\rm nor}$ for each $n \in I$, then* $\lim_n \iota_{\rm nor}(e_n) \in (\mathbb{K}_{\mathbb{X}}(M)_J^\sharp)^*$ *is the identity, where the limit is in the weak* topology.*

Proof. Let $e = \lim_n \iota_{\rm nor}(e_n) \in (\mathbb{K}_{\mathbb{X}}(M)_J^\sharp)^*$ be a weak* limit point and for any $T \in \mathbb{X}_0$, we have

$$e\iota_{\rm nor}(T) = \lim_n p_{\rm nor}\iota(e_n)\iota(T)p_{\rm nor} = \lim_n p_{\rm nor}\iota(e_nT)p_{\rm nor} = \iota_{\rm nor}(T),$$

and similarly $\iota_{\rm nor}(T)e = \iota_{\rm nor}(T)$. By density of $\iota_{\rm nor}(\mathbb{X}_0) \subset (\mathbb{K}_{\mathbb{X}}(M)_J^\sharp)^*$, we conclude that e is the identity in $(\mathbb{K}_{\mathbb{X}}(M)_J^\sharp)^*$. □

Lemma 3.7. *Let M be a finite von Neumann algebra and $M_d \subset M$, $d = 1, \ldots, n$ be von Neumann subalgebras such that $\{e_{M_d}\}$ are pairwise commuting. Set $e = \vee_{d=1}^{n} e_{M_d}$. Suppose $M_0 \subset M$ is a weakly dense* C*-*algebra. Denote by $\mathbb{X}$ the M-boundary piece associated with $\{M_d\}_{d=1}^{n}$ and $\mathbb{X}_0 \subset \mathbb{B}(L^2M)$ the norm closure of* $\mathrm{sp}\{x_1 J y_1 J T J y_2 J x_2 \mid x_i, y_i \in M_0, T \in e\mathbb{B}(L^2 M)e\}$. *Then $\iota_{\mathrm{nor}}(\mathbb{X}_0) \subset (\mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J})^*$ is weak* dense.*

Proof. Recall from Lemma 3.2 that $\mathbb{X}$ is the norm closure of $\mathrm{sp}\{x_1 J y_1 J S J y_2 J x_2 \mid x_i, y_i \in M, S \in e\mathbb{B}(L^2 M)e\}$. First we claim that for any $\varphi \in \mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J}$ and any $T = x_1 J y_1 J S J y_2 J x_2 \in \mathbb{X}$, with contractions $x_i, y_i \in M$ and $S \in e\mathbb{B}(L^2 M)e$, we may find a sequence $T_n \in \mathbb{X}_0$ such that $\varphi(T - T_n) \to 0$. Indeed, for each $i = 1, 2$, take sequences of contractions $x^i_\ell, y^i_\ell \in M_0$ such that $\lim_\ell \|x^i_\ell - x_i\|_2 = \lim_\ell \|y^i_\ell - y_i\|_2 = 0$.

Observe that for $T_{m,r,j,k} = x^1_m J y^1_r J S J y^2_j J x^2_k \in \mathbb{X}_0$ with $m, r, j, k \in \mathbb{N}$, we have

$$|\varphi(T - T_{m,r,j,k})| \le |\varphi((x_1 - x^1_m) J y_1 J S J y_2 J x_2)| + |\varphi(x^1_m J(y_1 - y^1_r) J S J y_2 J x_2)| + |\varphi(x^1_m J y^1_r J S J(y_2 - y^2_j) J x_2)| + |\varphi(x^1_m J y^1_r J S J y^2_j J(x_2 - x^2_k))|.$$

For any $n \in \mathbb{N}$, pick $m(n) \in \mathbb{N}$ such that $|\varphi((x_1 - x^1_{m(n)}) J y_1 J S J y_2 J x_2)| < 2^{-n}$. This is possible since $\varphi \in \mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J}$, which implies that $M \ni x \to \varphi(x J y_1 J S J y_2 J x_2) \in \mathbb{C}$ is a normal functional. Next we may pick $y^1_{r(n)}$ such that $|\varphi(J(y_1 - y^1_{r(n)}) J x^1_{m(n)} S J y_2 J x_2)| < 2^{-n}$, since $x^1_{m(n)}$ is already chosen. Repeating this process, we obtain $T_n = x^1_{m(n)} J y^1_{r(n)} J S J y^2_{j(n)} J x^2_{k(n)} \in \mathbb{X}_0$ with $|\varphi(T - T_n)| < 2^{2-n}$, which justifies the claim.

Moreover, as $\mathbb{X}$ is the norm closure of $\mathrm{sp}\{x_1 J y_1 J S J y_2 J x_2 \mid x_i, y_i \in M, S \in e\mathbb{B}(L^2 M)e\}$, we conclude that for any $\varphi \in \mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J}$ and any $T \in \mathbb{X}$, we may a sequence $T_n \in \mathbb{X}_0$ such that $|\varphi(T - T_n)| \to 0$. In other words, $\iota_{\mathrm{nor}}(\mathbb{X})$ is in the weak*-closure of $\iota_{\mathrm{nor}}(\mathbb{X}_0)$. We remark that the sequence $\{T_n\}$ is not necessarily uniformly bounded in norm.

Thus to show $\iota_{\mathrm{nor}}(\mathbb{X}_0) \subset (\mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J})^*$ is dense in the weak*-topology, it suffices to show $\iota_{\mathrm{nor}}(\mathbb{X}) \subset (\mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J})^*$ is dense in the weak*-topology. To this end, we show the unit ball of $\mathbb{X}$ is dense in the unit ball of $\mathbb{K}_{\mathbb{X}}(M)$ in $\|\cdot\|_{\infty,1}$, which then implies the weak*-density of $\iota_{\mathrm{nor}}(\mathbb{X}) \subset (\mathbb{K}_{\mathbb{X}}(M)^{\sharp}_{J})^*$ by [13, Proposition 3.1]. Let $T \in \mathbb{K}_{\mathbb{X}}(M) \subset \overline{\mathbb{X}}^{\|\cdot\|_{\infty,1}}$ be a contraction and $T_n \in \mathbb{X}$ such that $\|T - T_n\|_{\infty,1} \to 0$. By [13, Proposition 3.1], for each n there exists $a_n, b_n, c_n, d_n \in M$ and $z_n \in \mathbb{M}_2(\mathbb{B}(L^2M))$ such that $\lim_n(\|a_n\|_2^2 + \|b_n\|_2^2)^{1/2} = \lim_n \|z_n\| = \lim_n(\|c_n\|_2^2 + \|d_n\|_2^2)^{1/2} = 0$ and $T_n - T = \begin{pmatrix} J a_n J \\ b_n \end{pmatrix}^* z_n \begin{pmatrix} J c_n J \\ d_n \end{pmatrix}$. For some $N \in \mathbb{N}$, consider projections $e_n = J 1_{[0,1/N]}(a_n^* a_n + c_n^* c_n) J$ and $f_n = 1_{[0,1/N]}(b_n^* b_n + d_n^* d_n)$ Then $\|e_n J a_n^* J\|^2$, $\|J c_n J e_n\|^2$, $\|f_n b_n^*\|^2$, $\|d_n f_n\|^2 \le 1/N$ and hence $\|e_n f_n(T_n - T) f_n e_n\| \to 0$ as $n \to \infty$. Moreover, note that $1 - J e_n J \le N(a_n^* a_n + c_n^* c_n)$ and $\|a_n\|_2^2, \|c_n\|_2^2 \to 0$, and thus $e_n \to 1$ in the strong operator topology. Similarly, $f_n \to 1$ in the strong operator topology as well. Finally, we have

$$\|e_n f_n T_n f_n e_n - T\|_{\infty,1} \le \|e_n f_n(T_n - T) f_n e_n\| + \|e_n f_n T f_n e_n - T\|_{\infty,1} \to 0.$$

Since $e_n f_n T_n f_n e_n \in \mathbb{X}$, after renormalization we conclude that T may be approximated by a sequence of contractions in $\mathbb{X}$ in $\|\cdot\|_{\infty,1}$. □

Lemma 3.8. *Let M be a finite von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Let $e_N \in \mathbb{B}(L^2M)$ be the orthogonal projection onto L^2N. Then $\iota(e_N) \in \mathbb{B}(L^2 M)^{**}$ commutes with p_{nor}.*

Proof. Suppose $\mathbb{B}(L^2M)^{**} \subset \mathbb{B}(\mathcal{H})$ and notice that $\xi\mathcal{H}$ is in the range of p_{nor} if and only if $M \ni x \to \langle \iota(x)\xi, \xi\rangle$ and $JMJ \ni x \to \langle \iota(x)\xi, \xi\rangle$ are normal. For $\xi \in p_{\mathrm{nor}}\mathcal{H}$, we have $\varphi(x) := \langle \iota(x)\iota(e_B)\xi, \iota(e_N)\xi\rangle = \langle \iota(E_N(x))\xi, \xi\rangle$ is also normal for $x \in M$ and JMJ, which implies that $\iota(e_N)p_{\mathrm{nor}} = p_{\mathrm{nor}}\iota(e_N)p_{\mathrm{nor}}$. It follows that $\iota(e_N)$ and p_{nor} commutes. □

Lemma 3.9. *Let $N \subset M$ be a mixing von Neumann subalgebra admitting a Pimsner–Popa basis $\{m_k\}$ where $m_k \in M$. Let $\mathbb{X}_N$ be the associated boundary piece (see Example 3.1), and $q_{\mathbb{K}} \in (\mathbb{K}(L^2 M)^\sharp_J)^*$, $q_{\mathbb{X}_N} \in (\mathbb{K}_{\mathbb{X}_N}(M)^\sharp_J)^*$ be the respective identity elements. Then*

$$\sum_{k,l} q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_l J e_N J m_l^* J m_k^*) = q_{\mathbb{K}}^{\perp} q_{\mathbb{X}_N}.$$

Proof. Denote by $p_{k,l} = q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_l^* J e_N J m_l J m_k^*)$. Note that since $N \subset M$ is mixing, we have $e_N x J y J e_N - e_N E_N(x) J E_N(y) J e_N \in \mathbb{K}(M)$, i.e, is a compact operator when viewed as a bounded operator from the normed space M to $L^2(M)$. Indeed, we compute

$$\begin{aligned} & e_N x J y J e_N - e_N E_N(x) J E_N(y) J \\ &\quad = e_N(x - E_N(x)) J y J e_N + e_N E_N(x) J (y - E_N(y)) J e_N \\ &\quad = e_N(x - E_N(x)) J (y - E_N(y)) J e_N \\ &\qquad + e_N(x - E_N(x)) J E_N(y) J e_N + e_N E_N(x) J (y - E_N(y)) J e_N. \end{aligned}$$

Notice that $e_N(x - E_N(x)) J E_N(y) J e_N = e_N E_N(x) J (y - E_N(y)) J e_N = 0$ and thus

$$e_N x J y J e_N - e_N E_N(x) J E_N(y) J = e_N(x - E_N(x)) J (y - E_N(y)) J e_N \in \mathbb{K}(M).$$

It then follows that $\{p_{k,\ell}\}$ is a family of pairwise orthogonal projections, as

$$\begin{aligned} p_{k,l}\, p_{k',l'} &= q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_l J e_N J m_l^* m_{l'} J m_k^* m_{k'} e_N J m_{l'}^* J m_{k'}^*) \\ &= q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_l J e_N J q_l J q_k e_N J m_{l'}^* J m_{k'}^*)\delta_{k,k'}\delta_{l,l'} \\ &= q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_l^* J e_N J m_{l'}^* J m_{k'})\delta_{k,k'}\delta_{l,l'}, \end{aligned}$$

where $q_l \in \mathcal{P}(N)$ such that $q_l = E_N(m_l^* m_l)$ and automatically satisfies $m_l q_l = m_l$ (see Sect. 2.1).

Denote by $\mathbb{X}_0 \subset \mathbb{B}(L^2M)$ the hereditary C*-subalgebra generated by $xJyJe_N$ for x, y in the C*-algebra A generated by $\{m_k a\}_{a\in N, k\in\mathbb{N}}$. It is clear that $\mathbb{X}_0$ is an M-boundary piece and note that A is weakly dense (see Sect. 2.1, (2)) in M.

To see $\sum_{k,\ell} p_{k,\ell} = q_{\mathbb{K}}^{\perp} q_{\mathbb{X}_N}$, it suffices to show

$$\Big(\sum_{k',l'} p_{k',l'}\Big)\iota_{\mathrm{nor}}(m_k J m_\ell J a J b J e_N) = q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(m_k J m_\ell J a J b J e_N) \tag{1}$$

and

$$\iota_{\mathrm{nor}}(e_N J m_\ell J m_k a J b J)\Big(\sum_{k',l'} p_{k',l'}\Big) = q_{\mathbb{K}}^{\perp}\iota_{\mathrm{nor}}(e_N J m_\ell J m_k a J b J) \tag{2}$$

for all $a, b \in N$ and $k, l \in \mathbb{N}$. Indeed, every element in $\mathbb{X}_0$ can be written as a norm limit of linear spans consisting of elements of the from $x_1 J y_1 J T J y_2 J x_2$, where $x_i, y_i \in A$ and $T \in \mathbb{B}(L^2 N)$. Further we can assume $x_i = m_k a$ with $a \in N$ from density. Then we will get that for all $z \in \mathbb{X}_0$, $\sum_{k,l} q_{\mathbb{K}}^{\perp} \iota_{\text{nor}}(m_k J m_l^* J e_N J m_l J m_k^*) \iota_{\text{nor}}(z) = q_{\mathbb{K}}^{\perp} \iota_{\text{nor}}(z)$. Since by Lemma 3.7 we have $\mathbb{X}_0$ is weak* dense in $(\mathbb{K}_{\mathbb{X}_N}(M)_J^{\sharp})^*$, we conclude $\sum_{k,l} q_{\mathbb{K}}^{\perp} \iota_{\text{nor}}(m_k J m_l^* J e_N J m_l J m_k^*) = q_{\mathbb{K}}^{\perp} q_{\mathbb{X}_N}$.

Finally to see (1), notice that a direct computation shows that

$$p_{k',l'} \iota_{\text{nor}}(m_k J m_\ell J a J b J e_N) = \delta_{k,k'} \delta_{l,l'} q_{\mathbb{K}}^{\perp} \iota_{\text{nor}}(m_k J m_\ell J a J b J e_N).$$

Similarly, (2) holds as well. □

Lemma 3.10. *Let M be a finite von Neumann algebra and $M_i \subset M$, $i = 1, \ldots n$ be von Neumann subalgebras. Suppose $\mathcal{G} \subset \mathcal{U}(M)$ is a countable group such that $\mathcal{G}'' = M$ and $\{u J v J e_{M_i} J v^* J u^* \mid u, v \in \mathcal{G}, i = 1, \ldots, n\}$ is a family of pairwise commuting projections. Let $\mathbb{X}$ denote the boundary piece associated to $\{M_i\}_{i=1}^n$ as in Example 3.1. Let $\mathbb{X}_i$ denote the boundary pieces associated to M_i. Let q_i denote the identities of the von Neumann algebras $(\mathbb{K}_{\mathbb{X}_i}(M)_J^{\sharp})^*$ and $q_{\mathbb{X}}$ denote the identity of $(\mathbb{K}_{\mathbb{X}}(M)_J^{\sharp})^*$. Then we have that $q_{\mathbb{X}} = \vee_{i=1}^n q_i$.*

Proof. Recall from the beginning of this section that $(\mathbb{K}_{\mathbb{X}}(M)_J^{\sharp})^*$ is a von Neumann algebra, as M, JMJ are in the multiplier algebra of $\mathbb{M}(\mathbb{K}_{\mathbb{X}}(M))$. It is easy to see that $q_{\mathbb{X}} \geq q_i$ for each i.

Now we show that $q_{\mathbb{X}} \leq \vee_{i=1}^n q_i$. Let $F_n \subset \mathcal{G}$ be an increasing family of finite subsets such that $\cup F_n = \mathcal{G}$. Let $e_0 = \vee_i e_{M_i}$ and $e_n = \vee_{u,v \in F_n} u J v J (\vee_i e_{M_i}) J v^* J u^*$. Since $\{u J v J e_{M_i} J v^* J u^* \mid u, v \in \mathcal{G}, i = 1, \ldots, n\}$ is a family of pairwise commuting projections, we have $e_n \in \mathbb{X}$ by Lemma 3.2. If we denote by $\mathbb{X}_0 = \{x_1 J y_1 J T J y_2 J x_2 \mid x_1, x_2, y_1, y_2 \in M_0, T \in e_0 \mathbb{B}(L^2 M) e_0\}$, where M_0 is the C*-subalgebra generated by $\mathcal{G}$, then one checks that $\{e_n\}$ is an approximate unit of $\mathbb{X}_0$. In fact, note that for $u, v \in \mathcal{G}$, we have $e_0 J v J u e_n = (u J v J)(u^* J v^* J e_0 J v J u) e_n = e_0 J v J u$ whenever $u^* J v^* J e_0 J v J u \leq e_n$, i.e., whenever $u^*, v^* \in F_n$. Since M_0 is the C*-algebra generated by $\mathcal{G}$, this shows that for any $x, y \in M_0$, $\|(e_0 J y J x) e_n - e_0 J y J x\| \to 0$ as $n \to \infty$. Similarly, we also have $\|e_n (e_0 J y J x) - e_0 J y J x\| \to 0$ as $n \to \infty$, and hence $\{e_n\}$ is an approximate unit for $\mathbb{X}_0$.

We claim that $\iota_{\text{nor}}(e_n) \leq \vee_{i=1}^n q_i$. Note that $\vee_{u,v \in F_n} u J v J e_{M_i} J v^* J u^* \in \mathbb{X}_i$ and hence $\iota_{\text{nor}}(\vee_{u,v \in F_n} u J v J e_{M_i} J v^* J u^*) \leq q_i$. Furthermore, since

$$\iota_{\text{nor}}(e_n) = \iota_{\text{nor}}(\vee_{u,v \in F_n}(\vee_i u J v J e_{M_i} J v^* J u^*)) = \iota_{\text{nor}}(\vee_i(\vee_{u,v \in F_n} u J v J e_{M_i} J v^* J u^*)) \leq \vee_{i=1}^n q_i,$$

by Lemmas 3.6 and 3.7 we see that $q_{\mathbb{X}} = \lim_n \iota_{\text{nor}}(e_n) \leq \vee_{i=1}^n q_i$. □

3.3. Induced boundary pieces in the bidual.

Lemma 3.11. *Let M be a finite von Neumann algebra and $N \subset pMp$ a von Neumann subalgebra for some $0 \neq p \in \mathcal{P}(M)$. Set $E := \mathrm{Ad}(e_N) \circ \mathrm{Ad}(pJpJ) : \mathbb{B}(L^2 M) \to \mathbb{B}(L^2 N)$. Then its restriction $E_{|\mathbb{S}(M)}$ maps $\mathbb{S}(M)$ to $\mathbb{S}(N)$. Moreover, there exists a u.c.p. map $\tilde{E} : \tilde{\mathbb{S}}(M) \to \tilde{\mathbb{S}}(N)$ such that $\tilde{E}_{|M}$ agrees with the conditional expectation from M to N.*

Proof. To see $E_{|\mathbb{S}(M)} : \mathbb{S}(M) \to \mathbb{S}(N)$, note that $pJpJe_N J_N a J_N = JaJpJpJe_N$ for any $a \in N$. and $E : \mathbb{B}(L^2M) \to \mathbb{B}(L^2N)$ is $\|\cdot\|_{\infty,1}$-continuous. Thus for any $T \in \mathbb{S}(M)$ and any $a \in N$, we have

$$[E(T), J_N a J_N] = E([T, JaJ]) \in E(\overline{\mathbb{K}(M)}^{\|\cdot\|_{\infty,1}}) = \overline{\mathbb{K}(N)}^{\|\cdot\|_{\infty,1}} = \mathbb{K}^{\infty,1}(N),$$

i.e., $E(T) \in \mathbb{S}_(N)$.

Note that $E^* : \mathbb{B}(L^2N)^* \to \mathbb{B}(L^2M)^*$ maps $\mathbb{B}(L^2N)^\sharp_J$ to $\mathbb{B}(L^2M)^\sharp_J$ by [13, Lemma 5.3], and similarly $E^* : (\mathbb{K}(L^2N))^\sharp_J \to (\mathbb{K}(L^2M))^\sharp_J$. Therefore $\tilde{E} := (E^*{}_{|\mathbb{B}(L^2N)^\sharp_J})^* : (\mathbb{B}(L^2M)^\sharp_J)^* \to (\mathbb{B}(L^2N)^\sharp_J)^*$ and $\tilde{E}_{|(\mathbb{K}(L^2M))^\sharp_J)^*} : (\mathbb{K}(L^2M)^\sharp_J)^* \to (\mathbb{K}(L^2N)^\sharp_J)^*$. Hence we conclude that $\tilde{E} : \tilde{\mathbb{S}}(M) \to \tilde{\mathbb{S}}(N)$ with $\tilde{E}_{|M}$ agrees with the conditional expectation from M to N. □

3.4. Relative biexactness and relative proper proximality. Given a countable discrete group Γ, a boundary piece I is a $\Gamma \times \Gamma$ invariant closed ideal such that $c_0\Gamma \subset I \subset \ell^\infty\Gamma$ [5]. The small at infinity compactification of Γ relative to I is the spectrum of the C*-algebra $\mathbb{S}_I(\Gamma) = \{f \in \ell^\infty\Gamma \mid f - R_t f \in I, \text{ for any } t \in \Gamma\}$. Recall that Γ is said to be biexact relative to X if $\Gamma \curvearrowright \mathbb{S}_I(\Gamma)/I$ is topologically amenable [5], [7, Chapter 15], [30]. We remark that this is equivalent to $\Gamma \curvearrowright \mathbb{S}_I(\Gamma)$ is amenable. Indeed, first note that we have a Γ-equivariant unital embedding $\ell^\infty\Gamma \hookrightarrow I^{**}$ by taking $\{e_i\} \in I$ a Γ-asymptotically invariant approximate unit and consider a weak* limit point of $\phi_i : \ell^\infty\Gamma \ni f \to e_i f \in I \subset I^{**}$. Since we have $\Gamma \curvearrowright I^{**} \oplus (\mathbb{S}_I(\Gamma)/I)^{**} = \mathbb{S}_I(\Gamma)^{**}$ is amenable, and it follows that $\Gamma \curvearrowright \mathbb{S}_I(\Gamma)$ is an amenable action [8, Proposition 2.7].

The following is a general version of [13, Theorem 7.1], whose proof follows similarly. For the convenience of the reader we include the proof sketch below. A more general version of this is obtained in the upcoming work [14].

Theorem 3.12. *Let $M = L\Gamma$ where Γ is an nonamenable group that is biexact relative to a finite family of subgroups $\{\Lambda_i\}_{i\in I}$. Denote by $\mathbb{X}$ the M-boundary piece associated with $\{L\Lambda_i\}_{i\in I}$. If $A \subset pMp$ for some $0 \neq p \in \mathcal{P}(M)$ such that A has no amenable direct summands, then A is properly proximal relative to $\mathbb{X}^A$, where $\mathbb{X}^A$ is the induced A-boundary piece as in Remark 3.3).*

Proof. Consider the Γ-equivariant diagonal embedding $\ell^\infty(\Gamma) \subset \mathbb{B}(\ell^2\Gamma)$. Note that under this embedding $c_0(\Gamma, \{\Lambda_i\}_{i\in I})$ is sent to $\mathbb{X}$. Denote by $\mathbb{S}_{\mathbb{X}}(\Gamma) = \{f \in \ell^\infty(\Gamma) | f - fg \in c_0(\Gamma, \{\Lambda_i\}_{i\in I}), \forall g \in \Gamma\}$, the relative small at infinity compactification at the group level. Restricting this embedding to $\mathbb{S}_{\mathbb{X}}(\Gamma)$ then gives a Γ-equivariant embedding into $\mathbb{S}_{\mathbb{X}}(M)$. Therefore we obtain a $*$-homomorphism from $\mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma \to \mathbb{B}(\ell^2(\Gamma))$ whose image is contained in $\mathbb{S}_{\mathbb{X}}(M)$. Composing this with the map E from Lemma 3.11, we obtain a u.c.p map $\phi : \mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma \to \mathbb{S}_{\mathbb{X}^A}(A)$. By hypothesis we have a projection $p_0 \in \mathcal{Z}(A)$ and an Ap_0 bimodular u.c.p map $\Phi : \mathbb{S}_{\mathbb{X}^A}(A) \to Ap_0$. Further composing with this map we obtain a u.c.p map from $\widetilde{\phi} : \mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma \to Ap_0$.

Now set $\varphi : \mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma \to \mathbb{C}$, by $\varphi(x) := \frac{\langle x\widehat{p_0}, \widehat{p_0}\rangle}{\tau(p)}$. We then get a representation $\pi_\varphi : \mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma \to \mathcal{H}_\varphi$ and a state $\widetilde{\varphi} \in \mathbb{B}(\mathcal{H}_\varphi)_*$ such that $\varphi = \widetilde{\varphi} \circ \pi_\varphi$. Since $C_r^*(\Gamma)$ is weakly dense in M, we see by an argument of Boutonnet–Carderi (see Propositon 4.1 in [2]) that there is a projection $q \in (\pi_\varphi(\mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma))''$ such that $\widetilde{\varphi}(q) = 1$ and there exists a normal unital $*$-homomorphism $\iota : L(\Gamma) \to q\pi_\varphi(\mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma))''q$.

Since Γ is biexact relative to $\mathbb{X}$, we have that $\mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma$ is a nuclear C^*-algebra. Therefore there is a u.c.p map $\widetilde{\iota} : \mathbb{B}(\ell^2(\Gamma)) \to q(\mathbb{S}_{\mathbb{X}}(\Gamma) \rtimes_r \Gamma)''q$ extending ι. Now we see that $\widetilde{\varphi} \circ \widetilde{\iota}$ is an Ap_0 central state on $\mathbb{B}(\ell^2(\Gamma))$ showing that A has an amenable direct summand, which is a contradiction. □

In the case of general free products of finite von Neumann algebras $M = M_1 * M_2$ it ought to be the case that that if $A \subset M$ such that A has no amenable direct summand, then $A \subset M$ is properly proximal relative to the boundary piece generated by M_1 and M_2. However, currently we are only able to obtain this with an additional technical assumption that $M_i \cong L\Gamma_i$ where Γ_i are exact, so that $\Gamma_1 * \Gamma_2$ is biexact relative to $\{\Gamma_1, \Gamma_2\}$ [7, Proposition 15.3.12]. We record below a general result about subalgebras in free products which follows essentially from Theorem 9.1 in [13], however we do not get the boundary piece associated to the subalgebras M_i. We instead get the boundary piece associated to the word length:

Let M_1, M_2 be two finite von Neumann algebras and $M = M_1 * M_2$ be the tracial free product. Let $A \subset M$ be a nonamenable subalgebra. Consider the free product deformation from [22], i.e., $\tilde{M} = M * L\mathbb{F}_2$, $\theta_t = \mathrm{Ad}(u_1^t) * \mathrm{Ad}(u_2^t) \in \mathrm{Aut}(\tilde{M})$, with $u_1^t = \exp(it\alpha_1)$, $u_2^t = \exp(it\alpha_2)$, where α_1, α_2 are selfadjoint element in $L\mathbb{F}_2$ such that $\exp(i\alpha_1) = u_1$, $\exp(i\alpha_2) = u_2$ and u_1, u_2 are Haar unitaries in $L\mathbb{F}_2$. For $t > 0$, we have $E_M \circ \alpha_t = P_0 + \sum_{n=1}^{\infty} (\sin(\pi t)/\pi t)^{2n} P_n$ (see Sect. 2.5 in [20]), where P_n is the orthogonal projection to $\mathcal{H}_n = \oplus_{(i_1,\cdots,i_n)\in S_n} L^2(M_{i_1} \ominus \mathbb{C}) \otimes \cdots \otimes L^2(M_{i_n} \ominus \mathbb{C})$ and S_n is the set of alternating sequences of length n. Consider the hereditary C^*-algebra $\mathbb{X}_F$ generated by $\{P_n\}_{n\geq 0}$.

Proposition 3.13. *In the above setup, there exists a projection $p \in A$ such that Ap is amenable and $Ap^\perp$ is properly proximal relative to $\mathbb{X}_F$.*

Proof. It follows from the proof of [13, Proposition 9.1] that there exists an M-bimodular u.c.p. map $\phi : (M^{\mathrm{op}})' \cap \mathbb{B}(L^2\tilde{M} \ominus L^2 M) \to \tilde{\mathbb{S}}_{\mathbb{X}_F}(M)$. Moreover, since $L^2\tilde{M} \ominus L^2 M \cong L^2 M \overline{\otimes} \mathcal{K}$ as an M-M bimodule for some right M module $\mathcal{K}$ [20, Lemma 2.10], we may restrict ϕ to $\mathbb{B}(L^2 M) \otimes \mathrm{id}_{\mathcal{K}}$. Take $p \in \mathcal{Z}(A)$ to be the maximal projection such that Ap is amenable and $p \neq 1$ as A is nonamenable. If $Ap^\perp$ is not properly proximal relative to $\mathbb{X}$ inside M, i.e., there exists an A-central state φ on $\tilde{\mathbb{S}}_{\mathbb{X}_F}(M)$ which is normal when restricted to $p^\perp M p^\perp$. Then pick $q \in \mathcal{Z}(Ap^\perp)$ be the support projection of $(\varphi \circ \phi)_{|Ap^\perp}$ and we have Aq is amenable, which contradicts the maximality of p. □

4. The Upgrading Theorem

Proof of Theorem 1.1. First notice that since A is properly proximal relative to $\mathbb{X}$ inside M, it has no amenable direct summand by (3) of Remark 3.3. Let $f \in \mathcal{Z}(A)$ be the projection such that $Af^\perp$ is the maximal properly proximal direct summand of A by (4) of Remark 3.3, and we may assume $f \neq 0$ since otherwise A would be properly proximal. Therefore Af has no amenable direct summand, is properly proximal relative to $\mathbb{X}$ inside M by (2) of Remark 3.3 and has no properly proximal direct summand. It follows from Lemma 3.4 that there exists an Af-central state μ on $\tilde{\mathbb{S}}(Af)$ such that $\mu_{|Af}$ is normal. Moreover, by a maximal argument, we may assume $\mu_{|\mathcal{Z}(Af)}$ is faithful, as Af has no properly proximal direct summand.

Let $\tilde{E} : \tilde{\mathbb{S}}(M) \to \tilde{\mathbb{S}}(Af)$ be the u.c.p. map as in Lemma 3.11. Define a state $\varphi = \mu \circ \tilde{E} : \tilde{\mathbb{S}}(M) \to \mathbb{C}$, and it follows that φ is Af-central and $\varphi_{|fMf}$ is a faithful normal state.

Let $q_{\mathbb{K}}$ be the identity of the von Neumann algebra $(\mathbb{K}(L^2 M)^\sharp_J)^* \subset (\mathbb{B}(L^2 M)^\sharp_J)^*$, $q_{\mathbb{X}}$ the identity of von Neumann algebra $(\mathbb{K}_{\mathbb{X}}(M)^\sharp_J)^* \subset (\mathbb{B}(L^2 M)^\sharp_J)^*$. Note that $q_{\mathbb{K}} \le q_{\mathbb{X}}$ as $\mathbb{K}(L^2 M) \subset \mathbb{K}_{\mathbb{X}}(M)$.

First we analyze the support of φ. Observe that $\varphi(q_{\mathbb{K}}^\perp) = 1$. Indeed, if $\varphi(q_{\mathbb{K}}) > 0$, i.e., φ does not vanish on $(\mathbb{K}(L^2 M)^\sharp_J)^*$, then we may restrict φ to $\mathbb{B}(L^2 M)$, which embeds into $(\mathbb{K}(L^2 M)^\sharp_J)^*$ as a normal operator M-system [13, Sect. 8], and this shows that Af would have an amenable direct summand. Moreover, we have $\varphi(q_{\mathbb{X}}) = 1$. Indeed, if $\varphi(q_{\mathbb{X}}^\perp) > 0$, then

$$\frac{1}{\mu(q_{\mathbb{X}}^\perp)}\varphi \circ \mathrm{Ad}(q_{\mathbb{X}}^\perp) : \tilde{\mathbb{S}}_{\mathbb{X}}(M) \to \mathbb{C}$$

would be an Af-central that restricts to a normal state on fMf. Since $\mathbb{S}_{\mathbb{X}}(M)$ naturally embeds into $\widetilde{\mathbb{S}}_{\mathbb{X}}(M)$, this contradicts that Af is properly proximal relative to $\mathbb{X}$ inside M. Therefore we conclude that $\varphi(q_{\mathbb{X}} q_{\mathbb{K}}^\perp) = 1$.

For each $1 \le i \le n$, denote by $\mathbb{X}_i := \mathbb{X}_{M_i} \subset \mathbb{B}(L^2 M)$ the M-boundary piece associated with M_i and $q_i \in (\mathbb{K}_{\mathbb{X}_i}(M)^\sharp_J)^*$ the identity. Since $\vee_{i=1}^n q_i = q_{\mathbb{X}}$ by Lemma 3.10 and $[q_i, q_j] = 0$ by Lemma 3.9 and condition (2), we have $\varphi(q_j q_{\mathbb{K}}^\perp) > 0$ for some $1 \le j \le n$.

Claim: there exists a u.c.p. map $\phi : \langle M, e_{M_j}\rangle \to q_{\mathbb{K}}^\perp q_j \tilde{\mathbb{S}}(M) q_j$ such that $\phi(x) = q_{\mathbb{K}}^\perp q_j x$ for any $x \in M$.

Proof of the claim.. Denote by $\{m_k\}_{k\ge 0} \subset M$ a bounded Pimsner–Popa basis of M over M_i. For each $n \ge 0$, consider the u.c.p. map $\psi_n : \langle M, e_{M_j}\rangle \to \langle M, e_{M_j}\rangle$ given by

$$\psi_n(x) = (\sum_{k\le n} m_k e_{M_j} m_k^*) x (\sum_{\ell \le n} m_\ell e_{M_j} m_\ell^*),$$

and notice that ψ_n maps $\langle M, e_{M_j}\rangle$ into the $*$-subalgebra $A_0 := \mathrm{sp}\{m_k a e_{M_j} m_\ell^* \mid a \in M_j, k, \ell \ge 0\}$.

Recall notations from Remark 3.5. By Lemma 3.8, we have

$$\{\iota_{\mathrm{nor}}(J m_k J e_{M_j} J m_k^* J)\}_{k\ge 0} \subset (\mathbb{B}(L^2 M)^\sharp_J)^*$$

is a family of pairwise orthogonal projections. Set

$$e_j = \sum_{k\ge 0} \iota_{\mathrm{nor}}(J m_k J e_{M_j} J m_k^* J) \in (\mathbb{B}(L^2 M)^\sharp_J)^*$$

and define the map

$$\begin{aligned} \phi_0 : A_0 &\to q_{\mathbb{K}}^\perp (\mathbb{B}(L^2 M)^\sharp_J)^* \\ m_r a e_{M_j} m_\ell^* &\mapsto q_{\mathbb{K}}^\perp \iota_{\mathrm{nor}}(m_r a) e_j \iota_{\mathrm{nor}}(m_\ell^*). \end{aligned}$$

It is easy to check that ϕ_0 is well-defined. We then check that ϕ_0 is a $*$-homomorphism. It suffices to show that for any $x \in M$, we have

$$q_{\mathbb{K}}^\perp e_j \iota_{\mathrm{nor}}(x) e_j = q_{\mathbb{K}}^\perp \iota_{\mathrm{nor}}(E_{M_j}(x)) e_j. \tag{3}$$

Now we compute,

$$
\begin{aligned}
&q_{\mathbb{K}}^{\perp} e_j \iota_{\mathrm{nor}}(x) e_j \\
&\quad = q_{\mathbb{K}}^{\perp} \sum_{k,\ell\geq 0} \iota_{\mathrm{nor}}\big((Jm_k J e_{M_j} Jm_k^* J)x(Jm_\ell J e_{M_j} Jm_\ell^* J)\big) \\
&\quad = q_{\mathbb{K}}^{\perp} \sum_{k\geq 0} \iota_{\mathrm{nor}}\big((Jm_k J e_{M_j} Jm_k^* J)x(Jm_k J e_{M_j} Jm_k^* J)\big) \\
&\qquad + \sum_{k\neq \ell} \iota_{\mathrm{nor}}\big((Jm_k J e_{M_j} Jm_k^* J)x(Jm_\ell J e_{M_j} Jm_\ell^* J)\big).
\end{aligned}
$$

By Remark 2.2, we have $(Jm_k J e_{M_j} Jm_k^* J)(x - E_{M_j}(x))(Jm_\ell J e_{M_j} Jm_\ell^* J) \in \mathbb{B}(L^2 M)$ is a compact operator from

M to $L^2 M$ for $k \neq \ell$. Since $(Jm_k J e_{M_j} Jm_k^* J)E_{M_j}(x)(Jm_\ell J e_{M_j} Jm_\ell^* J) = 0$ if $\ell \neq k$, we have $\sum_{k\neq\ell} q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}(Jm_k J e_{M_j} Jm_k^* J x Jm_\ell J e_{M_j} Jm_\ell^* J) = 0$. Similarly, one checks that $q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}\big((Jm_k J e_{M_j} Jm_k^* J)x(Jm_k J e_{M_j} Jm_k^* J)\big) = q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}\big(E_{M_j}(x)(Jm_k J e_{M_j} Jm_k^* J)\big)$.

It then follows from (3) that ϕ_0 is a $*$-homomorphism. Now we verify that ϕ_0 is norm continuous.

Given $\sum_{i=1}^{d} m_{k_i} a_i e_{M_j} m_{\ell_i}^* \in A_0$, we may assume that $k_i \neq k_j$ and $\ell_i \neq \ell_j$ if $i \neq j$. Consider $P_k = q_{\mathbb{K}}^{\perp} \sum_{i=1}^{d} \iota_{\mathrm{nor}}(Jm_k J m_{\ell_i} e_{M_j} m_{\ell_i}^* Jm_k^* J)$ and

$Q_k = q_{\mathbb{K}}^{\perp} \sum_{i=1}^{d} \iota_{\mathrm{nor}}(Jm_k J m_{k_i} e_{M_j} m_{k_i}^* Jm_k^* J)$. We have P_k and Q_k are a projections and $P_k P_r = Q_k Q_r = 0$ if $k \neq r$ by Remark 2.2. And for the same reason, we have $\iota_{\mathrm{nor}}(e_{M_j} m_{\ell_i}^* Jm_k^* J)P_k = q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}(e_{M_j} m_{\ell_i}^* Jm_k^* J)$ as well as $\iota_{\mathrm{nor}}(e_{M_j} m_{k_i}^* Jm_k^* J)Q_k = q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}(e_{M_j} m_{k_i}^* Jm_k^* J)$ for each $1 \leq i \leq d$. Let $\mathcal{H}$ be the Hilbert space where $(\mathbb{B}(L^2 M)_J^{\sharp})^*$ is represented on. For $\xi, \eta \in (\mathcal{H})_1$, we compute

$$
\begin{aligned}
&|\langle \phi_0(\sum_{i=1}^{d} m_{k_i} a_i e_{M_j} m_{\ell_i}^*)\xi, \eta\rangle| \\
&\leq \sum_{k\geq 0} |\sum_{i=1}^{d} \langle q_{\mathbb{K}}^{\perp} \iota_{\mathrm{nor}}(e_{M_j} m_{\ell_i}^* Jm_k^* J)\xi, \iota_{\mathrm{nor}}(Jm_k J m_{k_i} e_{M_j} a_i)^* \eta\rangle| \\
&= \sum_{k\geq 0} |\sum_{i=1}^{d} \langle \iota_{\mathrm{nor}}(e_{M_j} m_{\ell_i}^* Jm_k^* J)P_k\xi, \iota_{\mathrm{nor}}(Jm_k J m_{k_i} e_{M_j} a_i)^* Q_k\eta\rangle| \\
&\leq \sum_{k\geq 0} \|\iota_{\mathrm{nor}}(Jm_k J(\sum_{i=1}^{d} m_{k_i} a_i e_{M_j} m_{\ell_i}^*)Jm_k^* J)\| \|P_k\xi\| \|Q_k\eta\| \\
&\leq (\sup_{k\in\mathbb{N}} \|m_k\|^2) \|\sum_{i=1}^{d} m_{k_i} a_i e_{M_j} m_{\ell_i}^*\| (\sum_{k\geq 0} \|P_k\xi\|^2)^{1/2} (\sum_{k\geq 0} \|Q_k\xi\|^2)^{1/2} \\
&\leq (\sup_{k\in\mathbb{N}} \|m_k\|^2) \|\sum_{i=1}^{d} m_{k_i} a_i e_{M_j} m_{\ell_i}^*\|.
\end{aligned}
$$

This shows that ϕ_0 is norm continuous as required.

Lastly we show that ϕ_0 maps into $q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M)$. As $\iota_{\text{nor}}(M)$ commutes with $\iota_{\text{nor}}(JMJ)$, it suffices to show $[e_j, \iota_{\text{nor}}(JuJ)] = 0$ for any $u \in \mathcal{G}$, which follows from condition (2). Indeed,

$$\iota_{\text{nor}}(JuJ)e_j\iota_{\text{nor}}(Ju^*J) = q_{\mathbb{K}}^{\perp}\Big(\sum_{k\geq 0}\iota_{\text{nor}}(Jum_kJe_{M_j}Jm_k^*u^*J)\Big),$$

while

$$Jum_kJe_{M_j}Jm_k^*u^*J = Jm_{k_u}u_kJe_{M_j}Ju_k^*m_{k_u}^*J = Jm_{k_u}Je_{M_j}Jm_{k_u}^*J,$$

and hence

$$\iota_{\text{nor}}(JuJ)e_j\iota_{\text{nor}}(Ju^*J) = q_{\mathbb{K}}^{\perp}\Big(\sum_{k\geq 0}\iota_{\text{nor}}(Jm_{k_u}Je_{M_j}Jm_{k_u}^*J)\Big) \leq e_j,$$

for any $u \in \mathcal{G}$. Since $\mathcal{G}$ is a group, we actually have $\iota_{\text{nor}}(JuJ)e_j\iota_{\text{nor}}(Ju^*J) = e_j$, as desired.

Combining all the above arugments, we may extend $\phi_0 : A \to q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M)$ to a $*$-homomorphism on A, where $A = \overline{A_0}^{\|\cdot\|}$ is a C*-algebra.

The next step is to define the map ϕ. For each $n \geq 0$, set $\phi_n := \phi_0 \circ \psi_n : \langle M, e_{M_j}\rangle \to q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M)$, which is c.p. and subunital by construction. We may then pick $\phi \in CB(\langle M, e_{M_j}\rangle, q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M))$ a weak* limit point of $\{\phi_n\}_{n\in\mathbb{N}}$, which exists as $q_{\mathbb{K}}^{\perp}\tilde{\mathbb{S}}(M)$ is a von Neumann algebra.

We claim that

$$\text{Ad}(q_j)\circ\phi : \langle M, e_{M_j}\rangle \to q_{\mathbb{K}}^{\perp}q_j\widetilde{\mathbb{S}}(M)q_j$$

is an M-bimodular u.c.p. map, which amounts to showing $\phi(x) = q_{\mathbb{K}}^{\perp}q_j\iota_{\text{nor}}(x)$ for any $x \in M$.

In fact, for any $x \in M$, we have

$$\begin{aligned}\phi(x) &= \lim_{n\to\infty}\phi_0\Big(\sum_{0\leq k,\ell\leq n}(m_kE_{M_j}(m_k^*xm_\ell)e_{M_j}m_\ell^*)\Big)\\ &= q_{\mathbb{K}}^{\perp}\lim_{n\to\infty}\sum_{0\leq k,\ell\leq n}\iota_{\text{nor}}(m_kE_{M_j}(m_k^*xm_\ell))e_j\iota_{\text{nor}}(m_\ell^*)\\ &= q_{\mathbb{K}}^{\perp}\lim_{n\to\infty}\sum_{0\leq k,\ell\leq n}\big(\iota_{\text{nor}}(m_k)e_j\iota_{\text{nor}}(m_k^*)\big)\iota_{\text{nor}}(x)\big(\iota_{\text{nor}}(m_\ell)e_j\iota_{\text{nor}}(m_\ell^*)\big),\end{aligned}$$

where the last equation follows from (3). Finally, note that $\{p_k\}_{k\geq 0}$ is a family of pairwise orthogonal projections by Remark 2.2, where

$$p_k := q_{\mathbb{K}}^{\perp}\iota_{\text{nor}}(m_k)e_j\iota_{\text{nor}}(m_k^*) = q_{\mathbb{K}}^{\perp}\sum_{r\geq 0}\iota_{\text{nor}}(Jm_rJm_ke_{M_j}m_k^*Jm_r^*J),$$

and $\sum_{k\geq 0}p_k = \sum_{k,r\geq 0}q_{\mathbb{K}}^{\perp}\iota_{\text{nor}}(Jm_rJm_ke_{M_j}m_k^*Jm_r^*J) = q_{\mathbb{K}}^{\perp}q_j$ by Lemma 3.9. Therefore, we conclude that $\phi(x) = q_{\mathbb{K}}^{\perp}q_j\iota_{\text{nor}}(x)$, as desired. □

Now consider $\nu = \varphi \circ \phi \in \langle M, e_{M_j} \rangle^*$ and notice that $\frac{1}{\varphi(q_{\mathbb{K}}^{\perp} q_j)} \nu$ is an Af-central state, which is a normal state when restricted to fMf. Let $f_j \in \mathcal{Z}((Af)' \cap fMf)$ be the support projection of $\nu_{|\mathcal{Z}((Af)' \cap fMf)}$ and then we have Af_j is amenable relative to M_j inside M [33, Theorem 2.1]. Apply the same argument for each i with $\varphi(q_{\mathbb{K}}^{\perp} q_i) > 0$, we then obtain projections $f_i \in fMf$ (possibly 0) such that Af_i is amenable relative to M_i inside M.

Finally, to show $\vee_{i=1}^n f_i = f$, note that $\varphi(q_i f_i) = \varphi(q_i)$ as

$$\varphi(q_i f_i^{\perp}) = \varphi(q_{\mathbb{K}}^{\perp} q_i f_i^{\perp}) = \varphi(\phi(f_i^{\perp})) = \nu(f_i^{\perp}) = 0.$$

Consequently we have

$$\varphi(\vee_{i=1}^n f_i) \geq \varphi(\vee_{i=1}^n q_i f_i) \geq \varphi(\vee_{i=1}^n q_i) = 1,$$

and hence $\vee_{i=1}^n f_i = f$ by the faithfulness of $\varphi_{|fMf}$. Since $f_i \in \mathcal{Z}((Af)' \cap fMf)$, we may rearrange these projections so that $\sum_{i=1}^n f_i = f$. □

5. Proofs of Main Theorems

Proof of Theorem 1.3. This follows from noticing that the Jones projections $e_{L\Lambda_i}$ pairwise commute, and then applying Theorems 1.1 and 3.12. □

Theorem 5.1. *Let (M_1, τ_1) and (M_2, τ_2) be such that $M_i \cong L\Gamma_i$ where Γ_i are countable exact groups and $M = M_1 * M_2$ be the tracial free product. Let $A \subset M$ be von Neumann subalgebra, then there exists projections $\{p_i\}_{i=1}^3 \in \mathcal{Z}(A' \cap M)$ such that $Ap_i \prec_M M_i$ for each $i = 1$ and 2, Ap_3 is amenable and $A(\vee_{i=1}^3 p_i)^{\perp}$ is properly proximal.*

Proof of Theorem 5.1. First note that the free products of the exact groups Γ_i is biexact relative to $\{\Gamma_1, \Gamma_2\}$ [7, Proposition 15.3.12] and $[e_{M_1}, e_{M_2}] = 0$. Then by Theorem 3.12, we may take f_1 and f_2 from Theorem 1.1 and let $p_i' \in \mathcal{Z}(Af_i)$ be the maximal projection such that Ap_i' is amenable for each $i = 1, 2$. Set $p_i = f_i - p_i'$ for $i = 1$ and 2, and $p_3 = p_1' + p_2'$ and the rest follows from Lemma 2.3. □

Proof of Corollary 1.4. Since $A \subset M$ has no properly proximal direct summand, it follows from Theorems 1.1 and 3.12 that there exists central projections f_1 and f_2 in $\mathcal{Z}(A' \cap M)$ such that Af_i is amenable relative to M_i inside M for each i, and $f_1 + f_2 = 1$.

If Af_2 is not amenable, then by Lemma 2.3 we have that $Af_2 \prec_M M_2$. However, since $A \cap M_1$ is diffuse, we may pick a sequence of trace zero unitaries $\{u_n\}$ in $A \cap M_1$ converging to 0. One then checks that $\|E_{M_2}(xu_n f_2 y)\|_2 \to 0$ for any $x, y \in M$, which is a contradiction. Therefore A is amenable relative to M_1 inside M. And then it follows from Theorem 2.4 that $A \subset M_1$. □

Proof of Corollary 1.8. Note that in the case of $A = L\Gamma_1$, we have $\mathcal{Z}(A' \cap M) = \mathbb{C}$ and hence Theorem 5.1 implies that either $L\Gamma_1 \prec_M L\Lambda_1$ or $L\Gamma_1 \prec_M L\Lambda_2 * \cdots * L\Lambda_m$. The same argument as in [15, Corollary 8.1] deduces the desired result. □

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