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FRACTIONAL REGULARISATION OF THE CAUCHY PROBLEM FOR LAPLACE'S EQUATION AND APPLICATION IN SOME FREE BOUNDARY VALUE PROBLEMS

BARBARA KALTENBACHER AND WILLIAM RUNDELL

Abstract. In this paper we revisit the classical Cauchy problem for Laplace's equation as well as two further related problems in the light of regularisation of this highly ill-conditioned problem by replacing integer derivatives with fractional ones. We do so in the spirit of quasi-reversibility, replacing a classically severely ill-posed partial differential equations problem by a nearby well-posed or only mildly ill-posed one. In order to be able to make use of the known stabilising effect of one-dimensional fractional derivatives of Abel type we work in a particular rectangular (in higher space dimensions cylindrical) geometry. We start with the plain Cauchy problem of reconstructing the values of a harmonic function inside this domain from its Dirichlet and Neumann trace on part of the boundary (the cylinder base) and explore three options for doing this with fractional operators. The two other related problems are the recovery of a free boundary and then this together with simultaneous recovery of the impedance function in the boundary condition. Our main technique here will be Newton's method, combined with fractional regularisation of the plain Cauchy problem. The paper contains numerical reconstructions and convergence results for the devised methods.

Introduction

As its name suggests, the Cauchy Problem for Laplace's equation has a long history. By the early-middle of the nineteenth century it was known that prescribing the values u on the boundary $\partial\Omega$ of a domain Ω where $- \ u = 0$ held, allowed u to be determined uniquely within Ω. There was a similar statement for "flux" or the value of the normal derivative: the so-called Dirichlet and Neumann problems. These problems held great significance for an enormous range of applications evident at that time and provided solutions that depended continuously on the boundary measurements. It was also recognised that frequently a case would arise where part of the boundary is inaccessible and no measurements could be made there. In compensation one could measure both the value and the flux at the accessible part: the Cauchy problem. Since solutions of Laplace's equation can be considered as the real part of an underlying analytic function, analytic continuation still allowed uniqueness of the solution. However, the continuous dependence on the boundary data was lost; in fact in an extreme way.

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A famous reference to this state of affairs dates from the beginning of the twentieth century when Hadamard singled such problems out as being "incorrectly set" and hence unworthy of mathematical study, as they had "no physical significance." The backwards heat problem and the Cauchy problem were the prime exhibits [17–19].

By the middle of the twentieth century such problems had shown to have enormous physical significance and could not be ignored from any perspective. Methods had to be found to overcome the severe ill-conditioning. During this period the subject was extended to general inverse problems and included a vast range of situations for which the inverse map is an unbounded operator. Examples of still foundational papers from this period including applications are [11, 40–42].

One of the popular techniques dating from this period is the *method of quasi-reversibility* of Lattès and Lions [36]. In this approach the original partial differential equation was replaced by one in which the "incorrect" data allowed a well-posed recovery of its solution. This new equation contained a parameter that allowed for stable inversion for > 0 but, in addition (in a sense that had to be carefully defined), solutions of the regularising equation converged to that of the original as $\to 0$. It is now recognised that the original initial suggested choices of that time brought with them new problematic issues either because of additional unnatural boundary conditions required or an operator whose solutions behaved in a strongly different manner from the original that offset any regularising amelioration that it offered. Thus, the method came with a basic and significant challenge: finding a "closely-related" partial differential equation, depending on a parameter, that could use the data in a well-posed manner for > 0 and also be such that its solutions converged to those of the original equation as $\to 0$.

In other words, the central issue in using quasi-reversibility is in the choice of the regularising equation. Frequently, additional or modified initial/boundary conditions have to be imposed to maintain well-posed problem for this new equation. In the first case adding such conditions usually means taking a higher order operator and this has been a dominant theme for the Cauchy problem. A common choice is to use a bi-harmonic equation as the regularizer, that is, replace the Laplacian with $^2u + u = 0$. The fourth order operator suggested by Lattes and Lions then utilizes the Cauchy values $u(x,0)$ and $u_y(x,0)$ effectively but requires also derivative conditions $u_x(0,y)$ and $u(1,y)$ which are not known a priori. Despite the obvious hurdle this was their original approach in [36] and has remained a common formulation.

However, for numerical computation one needs a forwards problem solver and this can often be modified to suit the purpose. This was the approach taken in [9] where mixed finite elements were used and in [32] where finite differences and Carleman's estimates were used. As we will see our choice of a fractional order regularizing operator circumvents this issue.

In his various lectures, Hadamard noted, for example [19], that it is possible to write the solution in the form of an integral equation and remarked that this could lead to numerical methods. This has been taken up in various formulations that include iterative methods. A notable paper is [33] and the algorithm they put forward was later implemented numerically in [37]. Another, somewhat related, direction used the fact that in $\mathbb{R}^2$ solutions of Laplace's equation have an obvious extension to an analytic function and solutions could be sought by analytic continuation augmented by a suitable regularisation scheme. The above ideas lead naturally to converting the original problem to a moment problem and then obtaining regularisation by truncation. Examples here are [12, 22].

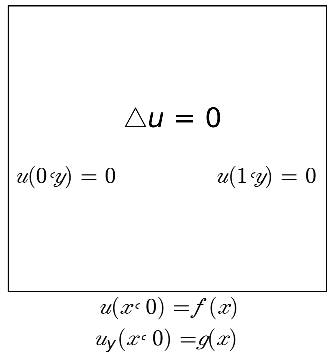
Interesting and important from applications is the case when the region is annular, for example a circular outer domain with a concentric, but not necessarily a circular inner one. A conformal mapping can achieve this and this approach has of course been followed. See, for example, [5]. This leads to the type of application where one not only doesn't know the boundary values on the interior curve but this curve itself has to be determined. Further examples with important analytic techniques can be found in [1, 8, 10]. We will return to this situation shortly.

Returning to the idea of a quasi-reversibility method and seeking a regularizing equation with many of the desired features and fewer drawbacks, here we will follow recent ideas for the backwards heat problem and replace the usual derivative in the "difficult" direction by a fractional derivative. In the parabolic case this was a time fractional derivative and one of the first papers taking this direction was [39]. It was later shown in [26] that the effectiveness of the method and the choice of the fractional exponent used strongly depended on both the final time T and the frequency components in the initial time function $u_0(x)$. This led to the current authors proposing a multi-level version with different fractional exponents depending on the frequencies in u_0, which of course had also to appear in the measured final time and were thus identifiable [28]. This "split frequency" method will also be used in the current paper.

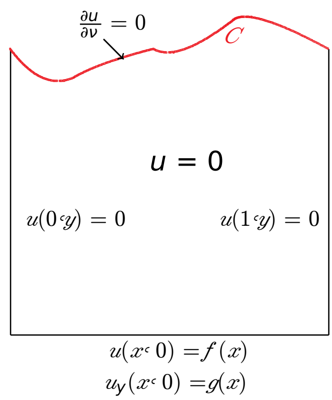
The advantage here is that such fractional operators arise naturally. The diffusion equation in (x, y) − coordinates results from a diffusive process in which the underlying stochastic process arises from sampling through a probability density function ψ. If ψ function has both finite mean and variance then it can be shown that the long term limit approaches Brownian motion resulting in classical derivatives. This can be viewed as a direct result of the central limit theorem. Allowing a finite mean but an infinite variance can lead directly to fractional derivatives [29].

However, while the use of time fractional derivatives and their behaviour in resulting partial differential equations is now well understood, the same cannot be said to the same degree for the case of space fractional derivatives. This statement notwithstanding it can identify the standard derivative as a limiting situation of one of fractional type. This makes such an operator a natural candidate for a quasi-reversibility operator.

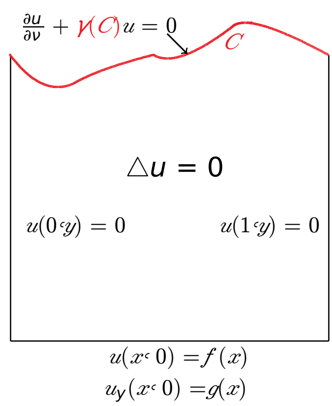
In our case if Ω is a rectangle with the top side inaccessible but data can be measured on the other three sides then a basic problem is to recover the solution $u(x,y)$ by also measuring the flux $\frac{\partial u}{\partial y}$ at, say, the bottom edge. In the usual language we have Dirichlet data on the two sides and Cauchy data on the bottom. We will consider this problem as Problem 1 in Section 1.



However there is a further possibility: the top side may be a curve (x) and we also do not know this curve – and want to do so. This is a classical example of a free-boundary value problem and a typical, and well-studied, example here is of corrosion to a partly inaccessible metal plate. This is our Problem 2 (cf. Section 2).



In addition, while the original top side was, say, a pure conductor or insulator with either $u = 0$ or $\frac{\partial u}{\partial \nu} = 0$ there, this has now to be re-modelled as an impedance condition where the impedance parameter is also likely unknown as a function of x. Recovery of both the boundary curve and the impedance coefficient is the topic of Problem 3 in Section 3.



Related to this are obstacle problems for elliptic problems in a domain Ω that seek to recover an interior object D from additional boundary data. This comes under this same classification, albeit with a different geometry. The boundary of D can be purely conductive, or purely insulating, or satisfy an impedance condition with a perhaps unknown parameter. The existing literature here is again extensive. We mention the survey by Isakov [23] which includes not only elliptic but also parabolic and hyperbolic equation-based problems. Other significant papers from this time period are [1, 4, 10, 32, 43]; a very recent overview on numerical methods for the Cauchy problem with further references can be found in [8].

For our purposes we wish to take advantage of the geometry described earlier where we are able, in some sense, to separate the variables and treat each of the differential operator's components in a distinct manner.

The structure of the paper is as follows. Each of the three Sections 1, 2, 3 first of all contains a formulation of the problem along with the derivation of a reconstruction method and numerical reconstruction results. In Section 1, this is a quasi-reversibility approach based on fractional derivatives; in Sections 2, 3 dealing with nonlinear problems, these are regularised Newton type methods. Sections 1, 3 also contain convergence results. In particular, in Section 3 we verify a range invariance condition on the forward operator that allows us to prove convergence of a regularised frozen Newton method.

1. Problem 1: The plain Cauchy problems

We start with the most fundamental of the three problems to be studied in this paper, namely the reconstruction of a harmonic function from its partial Cauchy data.

Problem 1.1. Given f, g in a domain Ω and the following elliptic boundary value problem on $\Omega \times (0, L)$,

$$
\begin{aligned}
&-\Delta u = -\Delta_x u - \partial_y^2 u = 0 && \text{in } \Omega \times (0, L),\\
&u(\cdot, y) = 0 && \text{on } \partial\Omega \times (0, L),\\
&u(x, 0) = f(x), \ \partial_y u(x, 0) = g(x) && x \in \Omega,
\end{aligned}
\tag{1}
$$

find $u(x, y)$ in the whole cylinder $\Omega \times (0, L)$.

This is a classical inverse problem going back to before Hadamard [19] and there exists a huge amount of literature on it. For a recent review and further references, see, e.g., [8].

We will here study possible regularization by replacing the partial derivative in the direction of data propagation (that is, the y direction) by a fractional one. Alternatively to this unidirectional way of using fractional operators one might think of fractional powers of the Laplacian. However, this does not appear to be viable for this purpose, because it would require a nonlocal version of the data as well, cf., e.g., [15], which cannot be extracted from the actually given data f, g.

To this end, we first of all derive a solution representation by expanding $u(\cdot, y)$, f, g in terms of the eigenfunctions φ_j (with corresponding eigenvalues λ_j) of $-\Delta_x$ on Ω with homogeneous Dirichlet boundary conditions

$$
u(x, y) = \sum_{j=1}^{\infty} u_j(y)\varphi_j(x), \quad f(x) = \sum_{j=1}^{\infty} f_j \varphi_j(x), \quad g(x) = \sum_{j=1}^{\infty} g_j \varphi_j(x),
$$

where for all $j \in \mathbb{N}$

$$u_j''(y) - \lambda_j u_j(y) = 0 \quad y \in (0,L), \quad u_j(0) = f_j, \quad u_j'(0) = g_j. \tag{2}$$

Thus

$$u(x,y) = \sum_{j=1}^{\infty} a_j \varphi_j(x), \tag{3}$$

where

$$\begin{aligned} a_j &= f_j \cosh(\sqrt{\lambda_j}\, y) + g_j \frac{\sinh(\sqrt{\lambda_j}\, y)}{\sqrt{\lambda_j}} \\ &= \frac{\sqrt{\lambda_j} f_j + g_j}{2\sqrt{\lambda_j}} \exp(\sqrt{\lambda_j}\, y) + \frac{\sqrt{\lambda_j} f_j - g_j}{2\sqrt{\lambda_j}} \exp(-\sqrt{\lambda_j}\, y). \end{aligned} \tag{4}$$

Here the negative Laplacian $-\Delta_x$ on Ω with homogeneous Dirichlet boundary conditions can obviously be replaced by an arbitrary symmetric positive definite operator $\mathcal{A}$ densely defined on a Hilbert space $\mathcal{H}$ (here $\mathcal{H} = L^2(\Omega)$), in particular by an elliptic differential operator with possibly x dependent coefficients on a d-dimensional Lipschitz domain Ω with homogeneous Dirichlet, Neumann or impedance boundary conditions. The corresponding eigensystem $(\lambda_j, \varphi_j)_{j\in\mathbb{N}}$ allows to define, e.g., the square root of the operator $\mathcal{A}$ by $\sqrt{\mathcal{A}}v = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\langle v, \varphi_j\rangle_H \varphi_j$ and the scale of Hilbert space norms

$$\|v\|_{\dot H^\sigma(\Omega)} := \Big(\sum_{j=1}^{\infty} \lambda_j^\sigma \langle v, \varphi_j\rangle_H^2\Big)^{1/2}. \tag{5}$$

1.1. **Regularisation by fractional differentiation.** Since the values of u have to be propagated in the y direction, starting from the data f, g at $y = 0$, the reason for ill-posedness (as is clearly visible in the exponential amplification of noise in this data, cf. (4)) results from the y-derivative in the PDE. We thus consider several options of regularising Problem 1.1 by replacing the second order derivative with respect to y by a fractional one, in the spirit of quasi-reversibility [3, 13, 36, 44–46]. We note in particular [29, Sections 8.3, 10.1] in the context of fractional derivatives.

In order to make use of integer (0th and 1st) order derivative data at $y = 0$, we use the Djrbashian-Caputo (rather than the Riemann-Liouville) version of the Abel fractional derivative. This has a left- and a right-sided version defined by

$$\begin{aligned} {}_0D_y^\beta v &= h_{2-\beta} * \partial_y^2 v, \quad \overline{{}_yD_L^\beta v}^{L} = h_{2-\beta} * \partial_y^2 \overline{v}^{L}, \\ h_{2-\beta}(y) &= \frac{1}{\Gamma(2-\beta) y^{\beta-1}}, \quad \overline{v}^{L}(y) = v(L-y) \end{aligned}$$

for $\beta \in (1,2)$, where $*$ denotes the (Laplace) convolution. Note that the Laplace transform of $h_{2-\beta}$ is given by $\hat h_{2-\beta}(s) = s^{\beta-2}$. Correspondingly, as solutions to initial value problems for fractional ODEs, Mittag-Leffler functions, as defined by (see, e.g., [29, Section 3.4])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad z \in \mathbb{C},$$

for $\alpha > 0$, and $\beta \in \mathbb{R}$ will play a role.

While using the spectral decomposition

$$u^{(\alpha)}(x,y) = \sum_{j=1}^{\infty} a_j^{(\alpha)}\,\varphi_j(x), \tag{6}$$

to approximate (3), (4) in the analysis, the computational implementation does not need the eigenvalues and eigenfunctions of $-\Delta_x$ but relies on the numerical solution of fractional PDEs, see (8), (12), (17), for which efficient methods are available, see, e.g., [2, 24, 35, 38].

1.1.1. *Left-sided Djrbashian-Caputo fractional derivative.* Replacing ∂_y^2 by ${}_0D_y^{2\alpha}$ with $2\alpha \approx 2$ amounts to considering, instead of (2), the fractional ODEs

$$ {}_0D_y^{2\alpha} u_j(y) - \lambda_j u_j(y) = 0 \quad y \in (0,L), \quad u_j(0) = f_j, \quad u_j'(0) = g_j,$$

whose solution by means of Mittag-Leffler functions (see, e.g., [29, Theorem 5.4]) yields

$$a_j^{(\alpha)} = f_j E_{2\alpha,1}(\lambda_j y^{2\alpha}) + g_j\, y E_{2\alpha,2}(\lambda_j y^{2\alpha}). \tag{7}$$

In view of the fact that $E_{2,1}(z) = \cosh(\sqrt{z})$ and $E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}$, this is consistent with (4). It corresponds to replacing (1) by

$$\begin{aligned} -\Delta_x u - {}_0D_y^{2\alpha} u &= 0 && \text{in } \Omega \times (0,L), \\ u(\cdot,y) &= 0 && \text{on } \partial\Omega \times (0,L), \\ u(x,0) = f(x), \ \partial_y u(x,0) &= g(x) && x \in \Omega. \end{aligned} \tag{8}$$

1.1.2. *Right-sided Djrbashian-Caputo fractional derivative.* Replacing ∂_y^2 by ${}_yD_L^{2\alpha}$ with $2\alpha \approx 2$ corresponds to replacing (2) by

$$ {}_yD_L^{2\alpha} u_j(y) - \lambda_j u_j(y) = 0 \quad y \in (0,L), \quad u_j(0) = f_j, \quad u_j'(0) = g_j$$

together with

$$u_j(L) = \tilde{a}_j, \quad u_j'(L) = \tilde{b}_j.$$

From the identity

$$ {}_yD_L^{2\alpha} u_j(L-\eta) = (h_{2-2\alpha} * w_j'')(\eta) \quad \text{for } w_j(\eta) = u_j(L-\eta) \text{ and } h_{2-2\alpha}(s) = s^{2\alpha-2},$$

we obtain the initial value problem

$$(h_{2-2\alpha} * w_j'')(\eta) - \lambda_j w_j(\eta) = 0 \quad \eta \in (0,L), \quad w_j(0) = \tilde{a}_j, \quad w_j'(0) = -\tilde{b}_j.$$

Taking Laplace transforms yields

$$s^{2\alpha}\hat{w}_j(s) - s^{2\alpha-1}\tilde{a}_j + s^{2\alpha-2}\tilde{b}_j - \lambda_j \hat{w}_j(s) = 0,$$

i.e.,

$$\hat{w}_j(s) = \frac{s^{2\alpha-1}}{s^{2\alpha}-\lambda_j}\tilde{a}_j - \frac{s^{2\alpha-2}}{s^{2\alpha}-\lambda_j}\tilde{b}_j, \tag{9}$$

and for the derivative

$$\widehat{w_j'}(s) = s\,\hat{w}_j(s) - \tilde{a}_j = \frac{\lambda_j}{s^{2\alpha}-\lambda_j}\tilde{a}_j - \frac{s^{2\alpha-1}}{s^{2\alpha}-\lambda_j}\tilde{b}_j. \tag{10}$$

From [29, Lemma 4.12] we obtain

$$\mathcal{L}(\eta^{k-1}E_{2\alpha,k}(\lambda\eta^{2\alpha}))(s) = \frac{s^{2\alpha-k}}{s^{2\alpha}-\lambda_j}, \quad k\in\{1,2\},$$
$$\mathcal{L}(-\eta^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda\eta^{2\alpha}))(s) = \frac{1}{s^{2\alpha}-\lambda_j}.$$

Inserting this into (9), (10) and evaluating at $\eta = L$ we obtain

$$f_j = E_{2\alpha,1}(\lambda_j L^{2\alpha})\tilde a_j - L E_{2\alpha,2}(\lambda_j L^{2\alpha})\tilde b_j$$
$$-g_j = -\lambda_j L^{2\alpha-1}E_{2\alpha,2\alpha}(\lambda_j L^{2\alpha})\tilde a_j - E_{2\alpha,1}(\lambda_j L^{2\alpha})\tilde b_j.$$

Resolving for $\tilde a_j$ and replacing L by y we get

$$a_j^{(\alpha)} = \frac{f_j E_{2\alpha,1}(\lambda_j y^{2\alpha}) + g_j y E_{2\alpha,2}(\lambda_j y^{2\alpha})}{\left(E_{2\alpha,1}(\lambda_j y^{2\alpha})\right)^2 - \lambda_j y^{2\alpha}E_{2\alpha,2\alpha}(\lambda_j y^{2\alpha})E_{2\alpha,2}(\lambda_j y^{2\alpha})}. \tag{11}$$

This corresponds to considering, in place of (1), the fractional problem

$$\begin{aligned} -\triangle_x u - {}_yD_L^{2\alpha} u &= 0 && \text{in } \Omega\times(0,L),\\ u(\cdot,y) &= 0 && \text{on } \partial\Omega\times(0,L),\\ u(x,0) = f(x),\ \partial_y u(x,0) &= g(x) && x\in\Omega. \end{aligned} \tag{12}$$

1.1.3. *Factorisation of the Laplacian.* An analysis of the two one-sided fractional approximations of ∂_y^2 does not seem to be possible since it would require a stability estimate for Mittag-Leffler functions with positive argument and index close to two. While convergence from below of the fractional to the integer derivative holds at any integer (thus also second) order, a stability estimate is not available. Therefore we look for a possibility to reduce the problem to one with first order y derivatives and treat the inverse problem similarly to a backwards heat problem to take advantage of recent work in this direction [28, 29]. One way to do so is to factorise the negative Laplacian so that the Cauchy problem becomes:
Given f, g in

$$\begin{aligned} -\triangle u = -\triangle_x u - \partial_y^2 u = (\partial_y - \sqrt{-\triangle_x})(-\partial_y - \sqrt{-\triangle_x})u &= 0 && \text{in } \Omega\times(0,L),\\ u(\cdot,y) &= 0 && \text{on } \partial\Omega\times(0,L),\\ u(x,0) = f(x),\ \partial_y u(x,0) &= g(x) && x\in\Omega, \end{aligned}$$

find $u(x,y)$. Recall that the square root of the Laplacian is defined spectrally as $\sqrt{-\triangle_x}\,v = \sum_{j=1}^\infty \sqrt{\lambda_j}\,\langle v,\varphi_j\rangle_{L^2(\Omega)}\varphi_j$.

More precisely, with $u_\pm = \frac12(u \pm \sqrt{-\triangle_x}^{-1}\partial_y u)$ we get the representation

$$u = u_+ + u_-,$$

where u_+, u_- can be obtained as solutions to the subproblems

$$\begin{aligned} \partial_y u_+ - \sqrt{-\triangle_x}\,u_+ &= 0 && \text{in } \Omega\times(0,L),\\ u_+(\cdot,y) &= 0 && \text{on } \partial\Omega\times(0,L),\\ u_+(x,0) = \tfrac12(f(x) + \sqrt{-\triangle_x}^{-1}g(x)) &=: u_{+0}(x) && x\in\Omega \end{aligned} \tag{13}$$

and

$$
(14)\qquad
\begin{aligned}
&\partial_y u_- + \sqrt{-\Delta_x}\, u_- = 0 && \text{in } \Omega \times (0,L),\\
&u_-(\cdot,y) = 0 && \text{on } \partial\Omega \times (0,L),\\
&u_-(x,0) = \tfrac{1}{2}\big(f(x) - \sqrt{-\Delta_x}^{\,-1} g(x)\big) && x \in \Omega.
\end{aligned}
$$

In fact, it is readily checked that if $u_\pm$ solve (13), (14) then $u = u_+ + u_-$ solves (1). The numerical solution of the initial value problem (14) and of the final value problem for the PDE in (13) can be stably and efficiently carried out combining an implicit time stepping scheme with methods recently developed for the solution of PDEs with fractional powers of the Laplacian. See, e.g., [6, 7, 20].

Since $\sqrt{-\Delta_x}$ is positive definite, the second equation (14) is well-posed, so there is no need to regularise. The first one (13) after the change of variables $t = L - y$, $w_+(x,t) = u_+(x,y)$ becomes a backwards heat equation (but with $\sqrt{-\Delta_x}$ in place of $-\Delta_x$)

$$
(15)\qquad
\begin{aligned}
&\partial_t w_+ + \sqrt{-\Delta_x}\, w_+ = 0 && \text{in } \Omega \times (0,L),\\
&w_+(\cdot,t) = 0 && \text{on } \partial\Omega \times (0,L),\\
&w_+(x,L) = \tfrac{1}{2}\big(f(x) + \sqrt{-\Delta_x}^{\,-1} g(x)\big) && x \in \Omega
\end{aligned}
$$

Regularising (15) by using in place of ∂_t a fractional "time" derivative ${}_0D_t^\alpha$ with $\alpha \approx 1, \alpha < 1$ (while leaving (14) unregularised) amounts to setting

$$
(16)\qquad a_j^\alpha = \frac{\sqrt{\lambda_j} f_j + g_j}{2\sqrt{\lambda_j}} \frac{1}{E_{\alpha,1}(-\sqrt{\lambda_j}\, y^\alpha)} + \frac{\sqrt{\lambda_j} f_j - g_j}{2\sqrt{\lambda_j}} \exp(-\sqrt{\lambda_j}\, y).
$$

This corresponds to replacing (1) by $u^\alpha = u_- + u_+^\alpha$, where $u_+^\alpha(x,y) = w_+^\alpha(x, L-y)$, with u_- solving (14) and w_+^α solving

$$
(17)\qquad
\begin{aligned}
&{}_0D_t^\alpha w_+^\alpha + \sqrt{-\Delta_x}\, w_+^\alpha = 0 && \text{in } \Omega \times (0,L),\\
&w_+^\alpha(\cdot,t) = 0 && \text{on } \partial\Omega \times (0,L),\\
&w_+^\alpha(x,L) = \tfrac{1}{2}\big(f(x) + \sqrt{-\Delta_x}^{\,-1} g(x)\big) && x \in \Omega.
\end{aligned}
$$

This approach can be refined by splitting the frequency range $(\lambda_j)_{j\in\mathbb{N}}$ into subsets $(\{\lambda_{K_i+1}, \ldots, \lambda_{K_{i+1}}\})_{i\in\mathbb{N}}$ and choosing the breakpoint K_i as well as the fractional order α_i for each of these subsets according to a discrepancy principle. For details, see [28].

1.2. **Reconstructions.** In this section we compare reconstructions with the three options (7), (11), (16) on a rectangular domain with $\Omega = (0,1)$. The latter was refined by the split frequency approach from [28] using the discrepancy principle for determining the breakpoints and differentiation orders. While this method is backed up by convergence theory (see Section 1.3), the same does not hold true for the options (7) and (11). Indeed, not even stability can be expected to hold from the known behaviour of the Mittag Leffler functions with positive argument, in particular for (7). This becomes visible in the reconstructions in Figure 1. The differentiation orders for (7), (11) were taken as the smallest (thus most stable) ones obtained in (16).

In this simple one-dimensional setting for Ω with known eigensystem, we can simply evaluate the formulas (7), (11) and (16). In a higher dimensional setting,

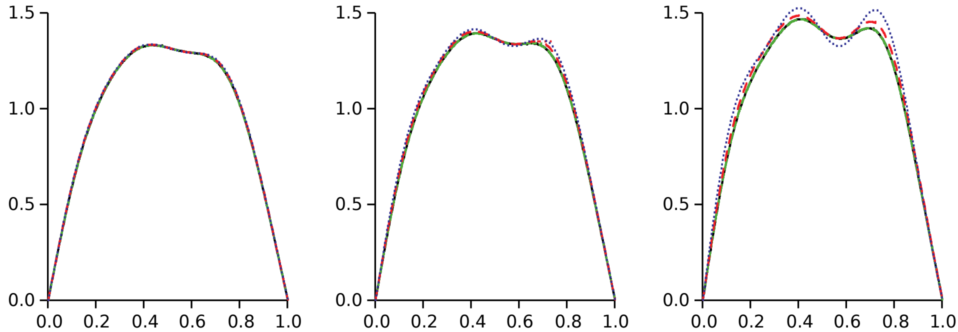


Figure 1. Reconstructions from formulas (7) (blue dotted), (11) (red, dashed) and (16) (green, irregularly dashed) as compared to the actual value (black, solid) from data with 1 per cent noise at different distances from the Cauchy data boundary. Left: $y=\frac{1}{3}$, Middle: $y=\frac{2}{3}$, Right: $y=1$.

one would numerically solve (8), (12) and (14), (17) by combining one of the by now well-established discretization methods for time-fractional derivatives such as the L^1 scheme or convolution quadrature (see, e.g., [25] for a recent overview) with a space discretization by, e.g., Galerkin finite elements; in case of (17), the latter requires an implementation of the square root of the Laplacian, see, e.g., [6, 7, 20].

The relative L^2 errors over the whole domain were 0.0275 for (7), 0.0135 for (11), $1.8597\cdot 10^{-4}$ for (16).

The split frequency factorised Laplace approach also worked well with much higher noise levels, as can be seen in Sections 2.1 and 3.1, where it is intrinsically used as part of the reconstruction algorithm. On the other hand, (7), (11) failed to provide reasonable reconstructions at higher noise levels, which is to be expected from a theoretical point of view.

1.3. **Convergence of the scheme (16).** We now provide a convergence analysis of the method from Section 1.1.3. As usual in regularisation methods, the total error can be decomposed into an approximation error and the propagated noise

$$\|u-u^{\alpha,\delta}\|_X \le \|u-u^{(\alpha)}\|_X + \|u^{(\alpha)}-u^{\alpha,\delta}\|_X. \tag{18}$$

Here $u^{\alpha,\delta}$ is the actual reconstruction from noisy data f^δ, g^δ leading to

$$u^\delta_{+0}=\tfrac{1}{2}\big(f^\delta(x)+\sqrt{-\triangle_x}^{\,-1}g^\delta(x)\big) \text{ with } \|u^\delta_{+0}-u_{+0}\|_Y\le\delta. \tag{19}$$

Using $X=L^2(0,L;\dot H^\sigma(\Omega))$ with $\dot H^\sigma$ as defined in (5) as the space for the sought-after function u, we can write the approximation error as

$$\begin{aligned}\|u-u^{(\alpha)}\|_{L^2(0,L;\dot H^\sigma(\Omega))} &= \Big(\int_0^L \sum_{j=1}^\infty \lambda_j^\sigma \big|u_j(y)-u_j^{(\alpha)}(y)\big|^2\,dy\Big)^{1/2}\\ &= \Big(\int_0^L \sum_{j=1}^\infty |u_{+,j}(y)|^2\lambda_j^\sigma \frac{1}{E_{\alpha,1}(-\sqrt{\lambda_j}\,y^\alpha)^2}\big(E_{\alpha,1}(-\sqrt{\lambda_j}\,y^\alpha)-\exp(-\sqrt{\lambda_j}\,y)\big)^2\,dy\Big)^{1/2}.\end{aligned} \tag{20}$$

where $u^\dagger_{+,j}(y) = u_{+0,j}\exp(\sqrt{\lambda_j}\, y)$, and the propagated noise term as

(21)
$$\begin{aligned}\|u^{(\alpha)} - u^{\alpha,\delta}\|_{L^2(0,L;\dot H^\sigma(\Omega))} &= \left(\int_0^L \sum_{j=1}^\infty \lambda_j^\sigma \left|u_j^{(\alpha)}(y) - u_j^{\alpha,\delta}(y)\right|^2 dy\right)^{1/2}\\ &= \left(\int_0^L \sum_{j=1}^\infty \left|u_{+0,j} - u_{+0,j}^\delta\right|^2 \lambda_j^\sigma \frac{1}{E_{\alpha,1}(-\sqrt{\lambda_j}\, y^\alpha)^2}\, dy\right)^{1/2}.\end{aligned}$$

In view of (15), convergence follows similarly to corresponding results on regularisation by backwards subdiffusion [29, Section 10.1.3] using two fundamental lemmas

- on stability, estimating $\frac{1}{E_{\alpha,1}(-\lambda y^\alpha)}$ (Lemma 1.1) and
- on convergence, estimating $E_{\alpha,1}(-\sqrt{\lambda_j}\, y^\alpha) - \exp(-\sqrt{\lambda_j}\, y)$ (Lemma 1.2)

that we here re-prove to track dependence of constants on the final "time" $\bar l$. This is important in view of the fact that as opposed to [29, Section 10.1.3], we consider a range of "final time values", that is, in our context, y values. In the following $\hat\lambda$ serves as a placeholder for $\sqrt{\lambda_j}$.

Lemma 1.1. *For all $\alpha \in (0,1)$,*

$$\frac{1}{E_{\alpha,1}(-\hat\lambda y^\alpha)} \le 1 + \Gamma(1-\alpha)\hat\lambda y^\alpha. \tag{22}$$

Proof. The bound (22) is an immediate consequence of the lower bound in [29, Theorem 3.25].

Lemma 1.2. *For any $\bar l > 0$, $\hat\lambda_1 \ge 0$, $\alpha_0 \in (0,1)$ and $p \in [1, \frac{1}{1-\alpha_0})$, there exists $\tilde C = \tilde C(\alpha_0, p, \bar l) = \sup_{\alpha\in[\alpha_0,1)} C(\alpha, p, \bar l) > 0$ with $C(\alpha, p, \bar l)$ as in (66), such that for any $\alpha \in [\alpha_0, 1)$ and for all $\hat\lambda > \hat\lambda_1$*

(23)
$$\|d_\alpha\|_{L^\infty(0,\bar l)} \le \tilde C \hat\lambda^{1/p}(1-\alpha), \qquad \|d_\alpha\|_{L^p(0,\bar l)} \le \tilde C(1-\alpha)$$
for the function $d_\alpha : [0,\bar l] \to \mathbb{R}$ defined by $d_\alpha(y) = E_{\alpha,1}(-\hat\lambda y^\alpha) - \exp(-\hat\lambda y)$.

Proof. See Appendix A.

The estimates from Lemma 1.2 become more straightforward if the values of y are constrained to a compact interval not containing zero, as relevant for Problem 3.1 in Section 3. This also allows us to derive L^∞ bounds on $E_{\alpha,1}(-\hat\lambda y^\alpha) - \exp(-\hat\lambda y)$, which would not be possible without bounding y away from zero, due to the singularities of the Mittag-Leffler functions there.

Lemma 1.3. *For any $0 < \underline{l} \le \bar l < \infty$, $\hat\lambda_1 \ge 0$, $\alpha_0 \in (0,1)$ and $p \in [1, \frac{1}{1-\alpha_0})$, there exist constants $\tilde C = \tilde C(\alpha_0, p, \bar l) > 0$ as in Lemma 1.2, $\hat C = \hat C(\alpha_0, \underline{l}, \bar l) > 0$, such that for any $\alpha \in [\alpha_0, 1)$ and for all $\hat\lambda > \hat\lambda_1$*

(24)
$$\|d_\alpha\|_{L^\infty(\underline{l},\bar l)} \le \tilde C(1-\alpha), \qquad \|\partial_y d_\alpha\|_{L^\infty(\underline{l},\bar l)} \le \hat C \hat\lambda(1-\alpha)$$
for the function $d_\alpha : [\underline{l},\bar l] \to \mathbb{R}$ defined by $d_\alpha(y) = E_{\alpha,1}(-\hat\lambda y^\alpha) - \exp(-\hat\lambda y)$.

Proof. See Appendix A.

Applying Lemmas 1.1, 1.2 in (18), (20), (21), we obtain the overall error estimate

$$
\begin{aligned}
\|u - u^{\alpha,\delta}\|_{L^2(0,L;\dot H^\sigma(\Omega))} \le{}& \|u - u^{(\alpha)}\|_{L^2(0,L;\dot H^\sigma(\Omega))} \\
&+ \|u^\delta_{+0} - u_{+0}\|_{\dot H^\sigma(\Omega)} + \frac{L^{\alpha+1/2}}{\sqrt{2\alpha+1}}\Gamma(1-\alpha)\,\|u^\delta_{+0} - u_{+0}\|_{\dot H^{\sigma+1}(\Omega)},
\end{aligned}
\tag{25}
$$

where we further estimate the approximation error

$$
\begin{aligned}
\|u - u^{(\alpha)}\|_{L^2(0,L;\dot H^\sigma(\Omega))} \le{}& \tilde C\,(1-\alpha)\,\|u^\dagger_+\|_{L^2(0,L;\dot H^{\sigma+1/p}(\Omega))} \\
&+ L^\alpha\,(1-\alpha)\Gamma(1-\alpha)\,\|u^\dagger_+\|_{L^2(0,L;\dot H^{\sigma+1+1/p}(\Omega))}.
\end{aligned}
$$

Under the assumption $u^\dagger_+ \in L^2(0,L;\dot H^{\sigma+1+1/p}(\Omega))$, from Lebesgue's Dominated Convergence Theorem and uniform boundedness of $(1-\alpha)\Gamma(1-\alpha)$ as $\alpha \to 1$, as well as convergence to zero of $E_{\alpha,1}(-\hat\lambda_j x^\alpha) - \exp(-\hat\lambda_j x)$ as $\alpha \to 1$, we obtain $\|u - u^{(\alpha)}\|_{L^2(0,L;\dot H^\sigma(\Omega))} \to 0$ as $\alpha \to 1$.

In view of the fact that the data space in (19) is typically $Y = L^2(\Omega)$, considering the propagated noise term, the $\dot H^\sigma$ and $\dot H^{\sigma+1}$ norms in estimate (25) reveal the fact that even when aiming for the lowest order reconstruction regularity $\sigma = 0$, the data needs to be smoothed.

Due to the infinite smoothing property of the forward operator, a method with infinite qualification is required for this purpose. We therefore use Landweber iteration for defining a smoothed version of the data $u^{\tilde\delta}_{+0} = v^{(i_*)}$ by

$$
v^{(i+1)} = v^{(i)} - A(v^{(i)} - u^\delta_{+0}), \qquad v^{(0)} = 0, \tag{26}
$$

where

$$
A = \mu(-\triangle_x)^{-\tilde\sigma} \tag{27}
$$

with $\tilde\sigma \in \{\sigma, \sigma+1\} \ge 1$ and $\mu > 0$ chosen so that $\|A\|_{L^2 \to L^2} \le 1$. In our implementation, we carried out the smoothing iteration for the most basic setting $\sigma = 0$ and $\tilde\sigma = 1$.

For convergence and convergence rates as the noise level tends to zero, we quote (for the proof, see the appendix of [28]) a bound in terms of $\|u_+(\cdot, l)\|_{L^2(\Omega)}$ for some fixed $l \in (0, L)$, where u_+ is the unstable component of the solution according to (13).

Lemma 1.4. *A choice of*

$$
i_* \sim l^{-2}\log\left(\frac{\|u_+(\cdot,l)\|_{L^2(\Omega)}}{\delta}\right) \tag{28}
$$

yields

$$
\begin{aligned}
&\|u_{+0} - u^{\tilde\delta}_{+0}\|_{L^2(\Omega)} \le C_1\delta, \\
&\|u_{+0} - u^{\tilde\delta}_{+0}\|_{\dot H^{\tilde\sigma}(\Omega)} \le C_2\, l^{-1}\,\delta\sqrt{\log\left(\frac{\|u_+(\cdot,l)\|_{L^2(\Omega)}}{\delta}\right)} =: \tilde\delta
\end{aligned}
\tag{29}
$$

for some $C_1, C_2 > 0$ independent of l and δ.

Thus, using $u^{\tilde\delta}_{+0}$ in place of u_{+0} in the reconstruction, we obtain the following convergence result.

Theorem 1.1. *Let the exact solution $u^\dagger$ of Problem 1.1 satisfy*

$$u^\dagger_+ \in L^2(0,L;\dot H^{\sigma+1+1/p}(\Omega))$$

for some $\sigma \geq 0$, $p > 1$ and let the noisy data satisfy (19) with smoothed data constructed as in Lemma 1.4. Further, assume that $\alpha = \alpha(\tilde\delta)$ is chosen such that $\alpha(\tilde\delta) \to 1$ and $\Gamma(1-\alpha(\tilde\delta))\tilde\delta \to 0$ as $\delta \to 0$. Then $\|u - u^{\alpha(\tilde\delta),\delta}\|_{L^2(0,L;\dot H^\sigma(\Omega))} \to 0$ as $\delta \to 0$.

Since $\Gamma(1-\alpha) \sim (1-\alpha)^{-1}$ as $\alpha \nearrow 1$, the condition $\Gamma(1-\alpha(\tilde\delta))\tilde\delta \to 0$ means that $\alpha(\tilde\delta)$ must not converge to unity too fast as the noise level vanishes – a well-known condition in the context of regularisation of ill-posed problems.

2. Problem 2: Reconstruction of an interface

Still considering a harmonic function and its Cauchy data on part of the boundary, the task is now to determine an interface characterised by vanishing Neumann, Dirichlet or impedance conditions. A by now classical reference for this problem is [1]. The regularised solution of Problem 1.1 will serve as a building block in the reconstruction scheme.

Problem 2.1. Given f, g in

$$\begin{aligned}
-\Delta u &= 0 && x \in \Omega,\ y \in (0,\ell(x)),\\
Bu(x,y) &= 0 && x \in \partial\Omega,\ y \in (0,\ell(x)),\\
u(x,0) = f(x),\ u_y(x,0) &= g(x) && x \in \Omega,
\end{aligned}$$

find $\ell : \Omega \to (0,L)$ such that one of the following three conditions (N) $\partial_\nu u = 0$ or (D) $u = 0$ or (I) $\partial_\nu u + \gamma u = 0$ holds on the interface defined by ℓ, that is,

$$\left.\begin{aligned}
&\text{(N)}\quad 0 = \partial_{\tilde\nu} u(x,\ell(x)) = u_y(x,\ell(x)) - \nabla_x \ell(x) \cdot \nabla_x u(x,\ell(x)) \quad \text{or}\\
&\text{(D)}\quad 0 = u(x,\ell(x)) \quad \text{or}\\
&\text{(I)}\quad 0 = \partial_{\tilde\nu} u(x,\ell(x)) + \sqrt{1+|\nabla_x \ell(x)|^2}\,\gamma(x)\,u(x,\ell(x))
\end{aligned}\right\} \quad x \in \Omega. \tag{30}$$

In (30) we use the non-normalised outer normal direction $\tilde\nu$ to arrive at a simpler expression. Here and below $\nabla_x u(x,\ell(x)) = (u_{x_i}(x,\ell(x)))_{i=1}^d$, while $\nabla_x[u(x,\ell(x))] = \big(\nabla_x u(x,\ell(x)) + u_y(x,\ell(x))\,\nabla_x \ell(x)\big)_{i=1}^d$, where $d \in \{1,2\}$ is the dimension of Ω. Moreover, we make the a priori assumption that the searched for domain is contained in the cylinder $\Omega \times (0,L)$.

The operator B determines the boundary conditions on the lateral boundary parts, which may also be of Dirichlet, Neumann, or impedance type.

To emphasise dependence on the parametrisation, we denote the domain as well as the fixed and variable parts of its boundary as follows

$$\begin{aligned}
D(\ell) &= \{(x,y) \in \Omega \times (0,L) : y \in (0,\ell(x))\},\\
\Gamma_0(\ell) &= \{(x,\ell(x)) : x \in \Omega\},\\
\Gamma_1 &= \Omega \times \{0\},\\
\Gamma_2(\ell) &= \{(x,y) : x \in \partial\Omega,\ y \in (0,\ell(x))\}
\end{aligned} \tag{31}$$

(note that $\Gamma_2()$ depends on only weakly via its endpoint set$(\partial\Omega)$). With this we can write the forward problem as

$$
(32)\qquad
\begin{aligned}
-\quad u &= 0 \text{ in} D(), \\
u &= f \text{ on } \Gamma_1, \\
Bu &= 0 \text{ on } \Gamma_2(), \\
&\left.\begin{array}{lll}
\text{(N)} & \partial_{\tilde\nu} u = 0 & \text{or} \\
\text{(D)} & u = 0 & \text{or} \\
\text{(I)} & \partial_\nu u + \gamma u = 0 &
\end{array}\right\} \text{ on } \Gamma_0().
\end{aligned}
$$

The full inverse problem naturally splits into two subproblems: The linear severely ill-posed Cauchy problem on $D(L)$ (which is our Problem 1.1) and a well-posed (or maybe mildly ill-posed) nonlinear problem of reconstructing the curve. Thus a straightforward approach for solving Problem 2.1 would be to first solve a regularised version of Problem 1.1 (e.g., in the way devised in Section 1) and then applying Newton's method to recover from (30).

However, we follow a combined approach, writing Problem 2.1 as an operator equation with the total forward operator F, and applying a Newton scheme, in which we make use of a regularised solution of Problem 1.1. Here the forward operator is defined by

$$F: \quad \to u_y(\le 0) - g \text{ where } u \text{ solves (32).}$$

Its linearisation in the direction $\underline{d}: \Omega \to \mathbb{R}$ of the parametrisation is given by $F\,()\underline{d} = \partial_y v \clubsuit_1$, where v solves

$$
(33)\qquad
\begin{aligned}
-\quad v &= 0 \text{ in} D(), \\
v &= 0 \text{ on } \Gamma_1, \\
Bv &= 0 \text{ on } \Gamma_2(), \\
&\left.\begin{array}{lll}
\text{(N)} & \partial_{\tilde\nu} v(x(x)) = \nabla_x \le(\underline{d}(x)\ \nabla_x u(x(x))) & \text{or} \\
\text{(D)} & v(x(x)) = -u_y(x(x))\underline{d}(x) & \text{or} \\
\text{(I)} & \partial_{\tilde\nu} v(x(x)) + \overline{1+\clubsuit\nabla_x(x)\ \clubsuit}\gamma(x) v(x(x)) = (G(u)\underline{d})(x) &
\end{array}\right\} \text{ on } \Gamma_0(),
\end{aligned}
$$

where in the Neumann case (N) we have used the PDE to find the representation given here and in the impedance case the boundary value function G is defined by

$$
\begin{aligned}
(G(u)\underline{d})(x) \;&:= \nabla_x\ \underline{d}(x)\ \nabla_x u(x(x)) \\
&- \gamma(x)\ \frac{\nabla_x \underline{d}(x)\ \le \nabla_x(x)}{\overline{1+\clubsuit\nabla_x(x)\ \clubsuit}}\, u(x(x)) + \overline{1+\clubsuit\nabla_x(x)\ \clubsuit}\, u_y(x(x))\underline{d}(x) .
\end{aligned}
$$

Note that this is obtained in a similar manner to the formula for the shape derivative [43, equation (3.1)], see also [21], and the identity $ds = \overline{1+\clubsuit\nabla_x(x)\ \clubsuit}\,dx$ for the arclength parametrisation s, but using $\tilde\nu$ in place of ν, as well as

$$x_h(x) = \begin{matrix} 0 \\ \underline{d}(x) \end{matrix}, \quad \tilde\nu(x) = \begin{matrix} -\nabla_x(x) \\ 1 \end{matrix}, \quad \nu(x) = \frac{1}{\overline{1+\clubsuit\nabla_x(x)\ \clubsuit}}\,\tilde\nu(x).$$

Thus, computation of a Newton step $\underline{d} = d^{(k)}$ starting from some iterate $^{(k)} =$ amounts to solving the system

(34)

$$
\begin{aligned}
-\ z &= 0 \text{ in } D(), \\
z &= f \text{ on } \Gamma_1, \\
Bz &= 0 \text{ on } \Gamma_2(), \\
z_y &= g \text{ on } \Gamma_1, \\
\text{(N)}\quad \partial_{\vec{\nu}} z(x(x)) &= \nabla_x \cdot (\underline{d}(x)\ \nabla_x u(x(x))) \quad \text{or} \\
\text{(D)}\quad z(x(x)) &= -u_y(x(x))\underline{d}(x) \quad \text{or} \\
\text{(I)}\quad \partial_{\vec{\nu}} z(x(x)) + \overline{1 + \nabla_x(x)}\ \gamma(x) z(x(x)) &= (G(u)\underline{d})(x)
\end{aligned}
\quad \text{on } \Gamma_0()
$$

(note that $z_y|_{\Gamma_1} - g = F() + F'()\underline{d}$).

If we solve (a regularised version, cf. Section 1, of) the Cauchy problem on the rectangular hold-all domain

$$
\begin{aligned}
-\ \hat{z} &= 0 \text{ in } D(L) = \Omega \times (0, L), \\
\hat{z} &= f \text{ on } \Gamma_1, \\
B\hat{z} &= 0 \text{ on } \Gamma_2(L) = \partial\Omega \times (0, L), \\
\partial_y \hat{z} - g &= 0 \text{ on } \Gamma_1
\end{aligned}
\tag{35}
$$

in advance, then by uniqueness of solutions to the Cauchy problem, z coincides with $\hat{z}$ on $D()$. Therefore, in each Newton step it only remains to compute $u^{(k)} = u$ from the well-posed mixed elliptic boundary value problem (32) with $= {}^{(k)}$ and update

$$^{(k+1)}(x) = {}^{(k)}(x) - \underline{d}(x),$$

where $\underline{d}$ is defined by (33) with $\hat{z}$ in place of z. In the cases (N) and (I) this requires solving a transport equation for $\underline{d}$.

We now provide the explicit formulas in the (altogether two dimensional) case of $\Omega = (0, M)$. In doing so, we also discuss some numerical aspects.

In the Neumann case, we have

(36)

$$
\begin{aligned}
\text{(N)}\quad & ^{(k+1)}(x) \\
&= {}^{(k)}(x) + \frac{1}{u_x^{(k)}(x, {}^{(k)}(x))} \int_0^x \partial_{\vec{\nu}} \hat{z}(\xi, {}^{(k)}(\xi))\, d\xi \\
&= {}^{(k)}(x) + \frac{1}{u_x^{(k)}(x, {}^{(k)}(x))} \int_0^x \left(\partial_y \hat{z}(\xi, {}^{(k)}(\xi)) - {}^{(k)\prime}(\xi)\partial_x \hat{z}(\xi, {}^{(k)}(\xi)) \right) d\xi.
\end{aligned}
$$

If B denotes the lateral Dirichlet trace, then this also needs to be regularised, since due to the identity $u(0, y) = 0 = u(M, y)$, the partial derivative $u_x(\cdot, y)$ has to vanish at least at one interior point x for each y. To avoid problems arising from division by zero, we thus solve a regularised version

$$
\begin{aligned}
({}^{(k+1)} - {}^{(k)}) = \operatorname{argmin}_{\underline{d}} & \int_0^M \left(\partial_{\vec{\nu}} z(x(x)) - \tfrac{d}{dx}[\underline{d}(x) u_x(x(x))] \right)^2 dx \\
& + \frac{1}{\rho_1} \int_0^M \underline{d}'(x)^2\, dx + \rho_2(\underline{d}(0)^2 + \underline{d}(M)^2)
\end{aligned}
$$

with a regularisation parameter $\frac{1}{\rho_1}$ and a penalisation parameter ρ_2 enforcing our assumption of $\underline{d}$ to be known at the boundary points.

In the Dirichlet case, the Newton step computes as

$$\text{(D)} \quad {}^{(k+1)}(x) = {}^{(k)}(x) - \frac{\text{ﬁ}(x, {}^{(k)}(x))}{u_y^{(k)}(x, {}^{(k)}(x))}, \tag{37}$$

a formula that remains valid for higher dimensional Ω. In case of lateral Dirichlet conditions $Bu = u = 0$ we have $u_y^{(k)}(0, (0)) = u_y^{(k)}(M, (M)) = 0$ and so would have to divide by numbers close to zero near the endpoints. This can be avoided by imposing Neumann conditions $Bu = \partial_\nu u = 0$ on the lateral boundary. Still, the problem is mildly ill-posed and thus needs to be regularised for the following reason. In view of the Implicit Function Theorem, the function, being implicitly defined by $u(x, (x)) = 0$, has the same order of differentiability as u. However, (37) contains an additional derivative of u as compared to. Obtaining a bound on $u_y^{(k)}$ in terms of ${}^{(k)}$ from elliptic regularity (cf., e.g., [16]) cannot be expected to be possible with the same level of differentiability.

In the impedance case, we have

$$G(u,)\underline{d} = \frac{d}{dx}[\mathfrak{a}[, u]\underline{d}] - \mathfrak{b}[, u]$$

with

$$\mathfrak{a}[, u](x) = \begin{cases} \frac{d}{dx} u(x, (x)) = u_x(x, (x)) & \text{if } (x) = 0, \\ \frac{1}{(x)} \frac{\gamma(x)}{\sqrt{1 + (x)^2}} u(x, (x)) + u_y(x, (x))) & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \mathfrak{b}[, u](x) &= -\sqrt{1 + (x)^2}\,\gamma(x) u_y(x, (x)) + \frac{d}{dx} \frac{(x)}{\sqrt{1 + (x)^2}}\gamma(x) u(x, (x)) \\ &= \frac{(x)}{\sqrt{1 + (x)^2}^3}\gamma(x) + \frac{(x)}{\sqrt{1 + (x)^2}}\gamma'(x) + \gamma(x)^2 \; u(x, (x)) \end{aligned} \tag{38}$$

and

$$\varphi(x) = \underline{d}(x)\; \mathfrak{a}[, u](x) \qquad a(x) = \frac{\mathfrak{b}[, u]}{\mathfrak{a}[, u]}(x),$$

$$b(x) = \partial_{\tilde\nu}\text{ﬁ}(x, {}^{(k)}(x)) + \sqrt{1 + (x)^2}\,\gamma(x)\text{ﬁ}(x, {}^{(k)}(x)).$$

Thus the Newton step amounts to solving $\frac{d}{dx}\varphi(x) - a(x)\varphi(x) = b(x)$, which yields

(39)

$$\begin{aligned} \text{(I)} \quad {}^{(k+1)}(x) = {}^{(k)}(x) - \frac{1}{\mathfrak{a}[{}^{(k)}, u^{(k)}](x)} \exp\; - \int_0^x \frac{\mathfrak{b}[{}^{(k)}, u^{(k)}]}{\mathfrak{a}[{}^{(k)}, u^{(k)}]}(s)\,ds \;\; \underline{d}(0)\; \mathfrak{a}[{}^{(k)}, u^{(k)}](0) \\ + \int_0^x b(s) \exp\; - \int_s^x \frac{\mathfrak{b}[{}^{(k)}, u^{(k)}]}{\mathfrak{a}[{}^{(k)}, u^{(k)}]}(t)\,dt \;\; ds. \end{aligned}$$

See Appendix B for details on the derivation of these formulas. Also here, due to the appearance of derivatives of u and, regularisation is needed.

Expanding on the impedance case, in Section 3, we will prove convergence of a regularised frozen Newton method for simultaneously recovering and γ.

Remark 2.1 (Uniqueness). In the Neumann case $\partial_{\tilde\nu} u = 0$ on $\Gamma_0()$, the linearisation $F'()$ is not injective since $F'()\underline{d} = 0$ only implies that $\underline{d}(x) u_x(x, (x))$ is constant.

There is nonuniqueness in the nonlinear inverse problem $F() = 0$ as well, as the counterexample $f(x) = \sin(\pi x/M)$, $g(x) = 0$, $u(x,y) = f(x)$ shows; all horizontal lines $(x) \equiv c$ for $c \in \mathbb{R}^+$ solve the inverse problem.

In the Dirichlet case $u = 0$ on $\Gamma_0()$, linearised uniqueness follows from the formula $z(x,(x)) = -u_y(x,(x))d(x)$ provided u_y does not vanish on an open subset Γ of $\Gamma_0()$. The latter can be excluded by Holmgren's theorem, since $-u = 0$, together with the conditions $u = 0, u_y = 0$ on $\Gamma_0()$, defines a noncharacteristic Cauchy problem and therefore would imply $u \equiv 0$ on $D()$, a contradiction to $f = 0$. Full uniqueness can be seen from the fact that if and $\tilde{}$ solve the inverse problem, then on the domain enclosed by these two curves (plus possibly some Γ_2 boundary part), u satisfies a homogeneous Dirichlet Laplace problem and therefore has to vanish identically. This, on the other hand, would yield a homogeneous Cauchy problem for u on the part $D(\min\{,\tilde{}\})$ that lies below both curves and thus imply that u vanishes identically there. Again we would then have a contradiction to $f = 0$.

This uniqueness proof would also work with Neumann or impedance instead of Dirichlet conditions on the lateral boundary Γ_2.

For uniqueness in the impedance case, see also [34, Theorem 2.2].

2.1. **Reconstructions.** Figure 2 shows reconstructions of (x) at 1 per cent noise in the flux data g (with respect to the L^2 norm). Here the actual curve was defined by

$$(x) = {}^\dagger(x) := L(0.8 + 0.1\cos(2\pi x)) \tag{40}$$

with $L = 0.1$, and the starting value was far from the actual curve (taken to be constant at $y = 0.2L$).

The left panel shows the case of Dirichlet conditions on the interface. No further progress in convergence took place after iteration 5. The lateral boundary conditions were of homogeneous Neumann type in order to avoid singularities near the corners.

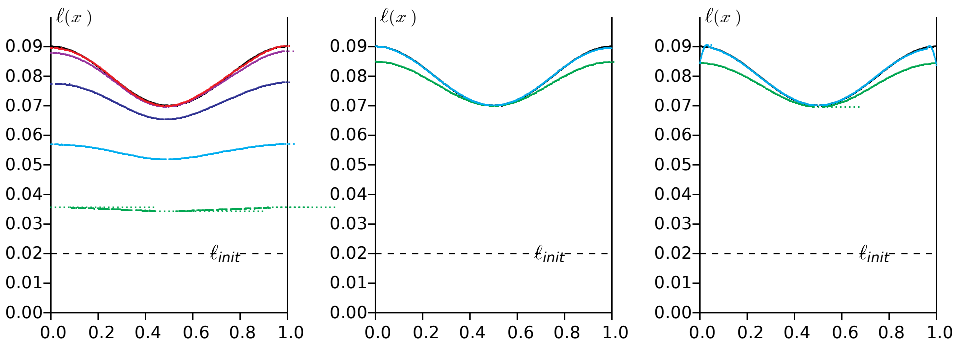


Figure 2. Recovery of (x) from data in with 1% noise: Dirichlet case (left), Neumann case (middle), impedance case (right)

In the middle and rightmost panels of Figure 2 we only show the first (green) and second (blue) iterations as the latter was effective convergence. In the leftmost panel we additionally show iteration 3 (navy blue), iteration 4 (purple) and iteration 5 (red); the latter being effective convergence.

The relative errors at different noise levels are given in the second column of Table 1.

Table 1. Relative errors for reconstructions of at several noise levels

	$- {}^{\dagger}{}_{L^2(\Omega)} / {}^{\dagger}{}_{L^2(\Omega)}$			
noise level	(D),$L=0.1$	(D),$L=0.5$	(N),$L=0.1$	(I),$L=0.1$
1%	0.0038	0.0158	0.0018	0.0077
2%	0.0084	0.0205	0.0093	0.0087
5%	0.0198	0.0380	0.0144	0.0110
10%	0.0394	0.0735	0.0563	0.0158

The same runs were also done with a larger distance L and the actual curve still according to (40). This resulted in the relative errors shown in column three of Table 1.

The middle and right panels of Figure 2 show reconstructions of (x) in the case of Neumann and impedance conditions ($\gamma = 0.1$) on the interface. The starting value was again relatively far from the actual curve and there was again 1% noise in the data. No further progress in convergence took place after iteration 2. For the relative errors at different noise levels, see the last two columns of Table 1.

3. Problem 3: Reconstruction of an interface and the impedance coefficient

Finally, we consider the problem of reconstructing both the interface characterized by a homogeneous impedance condition and the spatially varying impedance coefficient. Again, the solution of Problem 1.1 as devised in Section 1 will serve as a building block in the reconstruction procedure.

Problem 3.1. Given two pairs of Cauchy data (f_1, g_1), (f_2, g_2) in

$$\begin{aligned} - \; u_j &= 0 && x \in \Omega, \; y \in (0, (x)), \\ Bu_j(x, y) &= 0 && x \in \partial\Omega, \; y \in (0, (x)), \\ u_j(x, 0) = f_j(x), \; u_{jy}(x, 0) &= g_j(x) && x \in \Omega \end{aligned}$$

for $j = 1, 2$, find $: \Omega \to (0, L)$ and $\tilde{\gamma} : \Omega \to (0, \infty)$ such that

(41)
$$\begin{aligned} 0 = B_{,\tilde{\gamma}} u_j &:= \partial_{\tilde{\nu}} u_j(x, (x)) + \tilde{\gamma}(x) u_j(x, (x)) \\ &= u_{jy}(x, (x)) - \nabla_x (x) \cdot \nabla_x u_j(x, (x)) + \tilde{\gamma}(x) u_j(x, (x)) \qquad x \in \Omega, \end{aligned}$$

where $\tilde{\gamma}$ is the combined coefficient defined by $\tilde{\gamma}(x) = \sqrt{1 + |\nabla_x(x)|^2}\,\gamma(x)$.

The setting is as in Problem 2.1 otherwise and using the notation (31) we rewrite the forward problem as

$$\begin{aligned} - \; u_j &= 0 && \text{in } D(), \\ u_j &= f_j && \text{on } \Gamma_1, \\ Bu_j &= 0 && \text{on } \Gamma_2(), \\ B_{,\tilde{\gamma}} u_j &= 0 && \text{on } \Gamma_0(). \end{aligned} \tag{42}$$

Note that u_j actually satisfies the Poisson equation on the hold-all domain $D(L)$ with a fixed upper boundary defined by $L \geq$. We also point out that using the weak form of the forward problem

$$u - f_j \in H^1(D()) := \{w \in \dot{H}^1(\Omega) : Bw = 0 \text{ on } \Gamma_2(), \quad w = 0 \text{ on } \Gamma_1\}$$

and for all

$$w \in H^1(D()) : \int_\Omega \int_0^{(x)} (\nabla u \cdot \nabla w)(x, y)\,dy + \tilde{\gamma}(x)(u \cdot w)(x, (x)) \quad dx = 0,$$

no derivative of nor $\tilde{\gamma}$ is needed for computing u.

The forward operator $F = (F_1, F_2)$ is defined by

$$F_j : \quad \to u_{jy}(x, 0) - g_j(x) \text{ where } u_j \text{ solves (42)}, j \in \{1, 2\}.$$

Its linearisation is defined by $F_j'()[\underline{d}, \underline{d\gamma}] = v_{jy}|_{\Gamma_1}$, where v_j solves

$$\begin{aligned} - \quad v_j &= 0 && \text{in } D(), \\ v_j &= 0 && \text{on } \Gamma_1, \\ Bv_j &= 0 && \text{on } \Gamma_2(), \\ B_{,\tilde{\gamma}} v_j &= G(u_j, , \tilde{\gamma})(\underline{d}, \underline{d\gamma}) && \text{on } \Gamma_0(), \end{aligned}$$

cf., (41), where

$$\begin{aligned} (43) \qquad & G(u_j, , \tilde{\gamma})(\underline{d}, \underline{d\gamma})(x) \\ & := - u_{jyy}(x, (x)) - \nabla_x (x) \cdot \nabla_x u_{jy}(x, (x)) + \tilde{\gamma}(x) u_{jy}(x, (x)) \quad \underline{d} \\ & \quad + \nabla_x \underline{d}(x) \cdot \nabla_x u_j(x, (x)) - \underline{d\gamma}(x) u_j(x, (x)). \end{aligned}$$

Using the PDE we have the identity

$$\begin{aligned} & G(u_j, , \tilde{\gamma})(\underline{d}, \underline{d\gamma}) \\ & \quad = \nabla_x \quad \underline{d}(x) \quad \nabla_x u_j(x, (x)) - \underline{d}(x)\tilde{\gamma}(x) u_{jy}(x, (x)) - \underline{d\gamma}(x) u_j(x, (x)). \end{aligned}$$

Thus, computation of a Newton step $(\underline{d}, \underline{d\gamma}) = (\underline{d}^{(k)}, \underline{d\gamma}^{(k)})$ starting from some iterate $(^{(k)}, \tilde{\gamma}^{(k)}) = (, \tilde{\gamma})$ amounts to solving the system

$$(44) \qquad \left.\begin{aligned} - \quad z_j &= 0 && \text{in } D(), \\ z_j &= f_j && \text{on } \Gamma_1, \\ Bz_j &= 0 && \text{on } \Gamma_2(), \\ z_{jy} - g_j &= 0 && \text{on } \Gamma_1, \\ B_{,\tilde{\gamma}} z_j &= G(u_j, , \tilde{\gamma})(\underline{d}, \underline{d\gamma}) && \text{on } \Gamma_0() \end{aligned}\right\} \quad j \in \{1, 2\}$$

(note that $z_j = u_j + v_j$ and $z_{jy}|_{\Gamma_1} - g_j = F_j() + F_j'()(\underline{d}, \underline{d\gamma})$).

Using the regularised (according to Section 1) solutions $\tilde{z}_j$ of the Cauchy problem on the cylindrical hold-all domain

$$(45) \qquad \begin{aligned} - \quad \tilde{z}_j &= 0 \text{ in } D(L) = \Omega \times (0, L), \\ \tilde{z}_j &= f_j \text{ on } \Gamma_1, \\ B\tilde{z}_j &= 0 \text{ on } \Gamma_2(L) = \partial\Omega \times (0, L), \\ \tilde{z}_{jy} - g_j &= 0 \text{ on } \Gamma_1, \end{aligned}$$

this reduces to resolving the following system for $(\underline{d},\underline{d\gamma})$ on $\Gamma_0()$

$$
(46)\qquad
\begin{aligned}
G(u_1,\tilde{\gamma})(\underline{d},\underline{d\gamma}) &= B_{,\tilde{\gamma}}\phi_1,\\
G(u_2,\tilde{\gamma})(\underline{d},\underline{d\gamma}) &= B_{,\tilde{\gamma}}\phi_2.
\end{aligned}
$$

With the pre-computed regularised solutions ϕ_j of the Cauchy problem (45) one can therefore carry out a Newton step by computing $u_j^{(k)}=u_j$ from the well-posed mixed elliptic boundary value problem (42) with $=\ ^{(k)}$, $\tilde{\gamma}=\tilde{\gamma}^{(k)}$ and updating

$$
^{(k+1)}(x) = \ ^{(k)}(x) - \underline{d}(x), \qquad \tilde{\gamma}^{(k+1)}(x) = \tilde{\gamma}^{(k)}(x) - \underline{d\gamma}(x)
$$

with $\underline{d},\underline{d\gamma}$ solving (46).

To obtain more explicit expressions for $\underline{d}$ and $\underline{d\gamma}$ from (46), one can apply an elimination strategy, that is, multiply the boundary condition on $\Gamma_0()$ with $u_{j\pm1}(x)$ and subtract, to obtain

$$
\begin{aligned}
B_{,\tilde{\gamma}}v_1u_2 - B_{,\tilde{\gamma}}v_2u_1 &= u_2\nabla_x\cdot\left(\underline{d}\nabla_x u_1\right) - u_1\nabla_x\cdot\left(\underline{d}\nabla_x u_2\right) - \underline{d\tilde{\gamma}}\left(u_{1y}u_2 - u_{2y}u_1\right)\\
&= \nabla_x\cdot\left(\underline{d}\left(u_2\nabla_x u_1 - u_1\nabla_x u_2\right)\right) - \underline{d}\left(\nabla_x\left(u_{2y}\nabla_x u_1 - u_{1y}\nabla_x u_2\right) + \tilde{\gamma}\left(u_{1y}u_2 - u_{2y}u_1\right)\right)\\
&=: \nabla_x\cdot\left(\underline{d}W\right) - \underline{d}\tilde{\beta}
\end{aligned}
$$

with the Wronskian

$$
(47)\qquad W[\ ,u_1,u_2] := u_2()\ \nabla_x u_1() - u_1()\ \nabla_x u_2()
$$

and where we have skipped the arguments (x) of b_1, b_2, $\underline{d}$, $\underline{d\gamma}$, and $(x,(x))$ of u_1, u_2 and its derivatives for better readability.

In the (altogether two dimensional) case of $\Omega=(0,M)$, with the abbreviation $W^{(k)} := W[\ ^{(k)},u_1^{(k)},u_2^{(k)}]$, $\tilde{\beta}^{(k)} := \tilde{\beta}[\ ^{(k)},u_1^{(k)},u_2^{(k)}]$, this yields the explicit formulas

(48)

$$
\begin{aligned}
\underline{d}(x) &= \frac{1}{W^{(k)}(x)}\left(\exp\left(-\int_0^x \frac{\tilde{\beta}^{(k)}}{W^{(k)}}(s)\,ds\right)\underline{d}(0)\,W^{(k)}(0)\right.\\
&\qquad\left.+\int_0^x \tilde{b}(s)\exp\left(-\int_s^x \frac{\tilde{\beta}^{(k)}}{W^{(k)}}(t)\,dt\right)ds\right)\\
&\text{with}\quad \tilde{b}(x) = \left(\partial_{\tilde{\nu}}\phi_1(x,\ ^{(k)}(x)) + \tilde{\gamma}^{(k)}(x)\phi_1(x,\ ^{(k)}(x))\right)u_2^{(k)}(x,\ ^{(k)}(x))\\
&\qquad\qquad - \left(\partial_{\tilde{\nu}}\phi_2(x,\ ^{(k)}(x)) + \tilde{\gamma}^{(k)}(x)\phi_2(x,\ ^{(k)}(x))\right)u_1^{(k)}(x,\ ^{(k)}(x)),\\
\underline{d\gamma}(x) &= \frac{1}{u_1^{(k)}(x,\ ^{(k)}(x))}\left(\frac{d}{dx}\left(\underline{d}(x)u_{1x}^{(k)}(x,\ ^{(k)}(x))\right) - \underline{d}(x)\tilde{\gamma}^{(k)}(x)u_{1y}^{(k)}(x,(x)) - b_1(x)\right)\\
&= \frac{1}{u_1^{(k)}(x,\ ^{(k)}(x))}\left(\frac{d}{dx}\left(\underline{d}(x)W^{(k)}(x)\right)\right.\\
&\qquad\left.-\underline{d}(x)\left(\tilde{\gamma}^{(k)}(x)u_{1y}^{(k)}(x,(x)) + \frac{d}{dx}\frac{u_{1x}^{(k)}(x,\ ^{(k)}(x))}{W^{(k)}(x)}\right) - b_1(x)\right).
\end{aligned}
$$

A similar elimination procedure will be useful in the proof of linearised uniqueness in Section 3.2.2.

3.1. **Reconstructions.** In Figure 3 we show a simultaneous reconstruction of (x) and $\tilde{\gamma}(x)$ obtained from excitations with $f_1(x)=1+x+x^2$ and $f_2(x)=4x^2-3x^3$ (chosen to comply with the impedance conditions $Bu=\partial_\nu u+u=0$ at $x\in\{0,M\}$, $M=1$).

Assuming the boundary values of and $\tilde{\gamma}$ to be known, we used them in our (linear; in the concrete setting constant) starting guesses. The actual values are

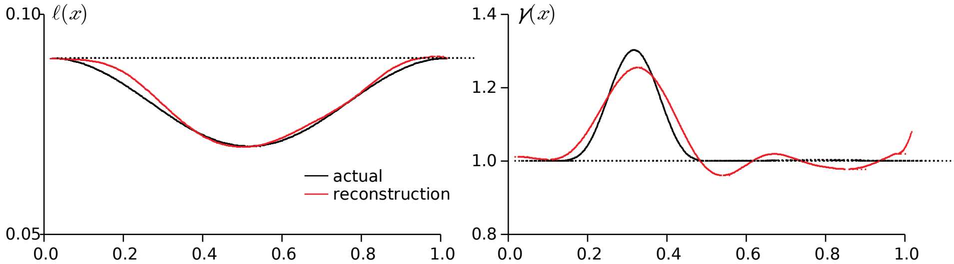


Figure 3. Simultaneous recovery of $\ell(x)$ and $\tilde\gamma(x)$

shown in black (solid), the reconstructed in red, the starting guesses as dashed lines.

The relative errors at different noise levels are listed in Table 2.

Table 2. Relative errors for reconstructions of ℓ and $\tilde\gamma$ at several noise levels

noise level	$\|\ell - \ell^\dagger\|_{L^2(\Omega)} / \|\ell^\dagger\|_{L^2(\Omega)}$	$\|\tilde\gamma - \tilde\gamma^\dagger\|_{L^2(\Omega)} / \|\tilde\gamma^\dagger\|_{L^2(\Omega)}$
1%	0.0145	0.0251
2%	0.0152	0.0263
5%	0.0191	0.0355
10%	0.0284	0.0587

While the numbers suggest approximately logarithmic convergence and this is in line with what is to be expected for such a severely ill-posed problem, our result in Section 3.2 is about convergence only. Proving rates would go beyond the scope of the objective of this paper.

3.2. **Convergence.** Proving convergence of iterative regularisation methods for nonlinear ill-posed problems always requires certain conditions on the nonlinearity of the forward operator. The one we will be able to verify here is range invariance of the linearised forward operator [27], see (52). These can be verified for a slightly modified formulation of the problem; in particular, instead of a reduced formulation involving the parameter-to-state map as used in Section 3.1, we consider the all-at-once formulation (involving a second copy of $\tilde\gamma$)

$$\begin{aligned} \tilde F_1(u_1, \ell, \tilde\gamma_1) &= 0, \\ \tilde F_2(u_2, \ell, \tilde\gamma_2) &= 0, \\ P(\tilde\gamma_1, \tilde\gamma_2) &= 0, \end{aligned} \tag{49}$$

where

$$\tilde F_j : (u_j, \ell, \tilde\gamma_j) \mapsto \begin{pmatrix} -\Delta u_j \\ u_j|_{\Gamma_1} - f_j \\ u_{j,\nu}|_{\Gamma_1} - g_j \\ B u_j|_{\Gamma_2(\ell)} \\ B_{\ell,\tilde\gamma} u_j|_{\Gamma_0(\ell)} \end{pmatrix}, \quad j \in \{1, 2\}, \qquad P(\tilde\gamma_1, \tilde\gamma_2) := \tilde\gamma_1 - \tilde\gamma_2 \tag{50}$$

with a given value ^{0} on the subset $\Sigma \subseteq \partial\Omega$ of the boundary. The first component of $\tilde{F}_j$ means that we consider the Poisson equation on the hold-all domain $D(L)$. The penalty term defined by P enforces both copies of $\tilde{\gamma}$ to coincide in the limit of the convergent Newton iterations.

3.2.1. *Range invariance of the forward operator.* The forward operator $\tilde{F}: \mathbb{X} \to \mathbb{Y}$ defined by $\tilde{F}(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) = (\tilde{F}_1(u_1, , \tilde{\gamma}_1), \tilde{F}_2(u_2, , \tilde{\gamma}_2))$ along with its linearisation

$$(51)\qquad \tilde{F}_j'(u_j, , \tilde{\gamma}_j)(\underline{du_j}, \underline{d}, \underline{d\gamma}_j) = \begin{pmatrix} -\ \underline{du_j} \\ \underline{du_j}\ _1 \\ \underline{du_j}_{y}\ _1 \\ B\underline{du_j}\ _{2(\ )} \\ B_{,\tilde{\gamma}_j}\ \underline{du_j}\ _{0(\ )} - G(u_j, , \tilde{\gamma})(\underline{d}, \underline{d\gamma}) \\ \underline{d}\ \end{pmatrix}$$

and the operator $r = (r_{u\,1}, r_{u\,2}, r, r_{\tilde{\gamma}\,1}, r_{\tilde{\gamma}\,2})$ defined so that $[B_{,\tilde{\gamma}_j} - B_{0,\tilde{\gamma}_{0,j}}]u_j + G(u_{0,j}, {}_0, \tilde{\gamma}_{0,j})(- {}_0), r_{\gamma\, j}(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2)) = 0$, that is,

$$\begin{aligned} r_{u\,j}(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) &= u_j - u_{0,j}, \\ r\,(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) &= - {}_0, \\ r_{\gamma\,j}(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) &= \frac{1}{u_{0,j}(\ {}_0)} \\ &\times\ - u_{0,j\,yy}(\ {}_0) - \nabla_x\ {}_0(x) \cdot \nabla_x u_{0,j\,y}(\ {}_0) + \tilde{\gamma}_0(x)u_{0,j\,y}(\ {}_0)\ (- {}_0) \\ &\quad + \nabla_x(- {}_0) \cdot \nabla_x u_{0,j}(\ {}_0) + u_{j\,y}() - \nabla_x(x) \cdot \nabla_x u_j() \\ &\quad + \tilde{\gamma}(x)u_j() - u_{j\,y}(\ {}_0) - \nabla_x\ {}_0(x) \cdot \nabla_x u_j(\ {}_0) + \tilde{\gamma}_0(x)u_j(\ {}_0) \end{aligned}$$

$j \in \{1, 2\}$ satisfies the differential range invariance condition

$$(52)\quad \text{for all}\ \xi \in U\ \exists r(\xi) \in \mathbb{X} := V^2 \times X \times (X^{\tilde{\gamma}})^2 :\ \tilde{F}(\xi) - \tilde{F}(\xi_0) = \tilde{F}'(\xi_0)r(\xi),$$

in a neighborhood U of the reference point $\xi_0 := (u_{0,1}, u_{0,2}, {}_0, \tilde{\gamma}_{0,1}, \tilde{\gamma}_{0,2})$. Here we use the abbreviation $u()$ for $(u())(x) = u(x, (x))$ and

$$\xi := (u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2).$$

Since the u_j do not necessarily need to satisfy the PDE, we have to use the representation (43) of G here.

Along with the identity (52), the convergence proof according to [27] requires an estimate on the difference between r and the shifted identity, which can be written as

$$\begin{aligned} &r(u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) - (u_1, u_2, , \tilde{\gamma}_1, \tilde{\gamma}_2) - (u_{0,1}, u_{0,2}, {}_0, \tilde{\gamma}_{0,1}, \tilde{\gamma}_{0,2}) \\ &= \left(0, 0, 0, \tfrac{1}{u_{0,1}(\ {}_0)}(I_1 + II_1 + III_1), \tfrac{1}{u_{0,2}(\ {}_0)}(I_2 + II_2 + III_2), 0\right)^T, \end{aligned}$$

with

$$\begin{aligned}
I_j &= \tilde\gamma_j\,(u_j\,() - u_{0,j}\,(\ _0) - \tilde\gamma_{0,j}\,u_{0,j\,y}\,(\ _0)(\ - \ _0) \\
&= \tilde\gamma_j\ \ (u_j - u_{0,j})() - (u_j - u_{0,j})(\ _0)\ \ + (\tilde\gamma_j - \tilde\gamma_{0,j})(u_{0,j}\,() - u_{0,j}\,(\ _0)) \\
&\qquad + \tilde\gamma_{0,j}\ \ u_{0,j}\,() - u_{0,j}\,(\ _0) - u_{0,j\,y}\,(\ _0)(\ - \ _0)\ , \\
II_j &= u_{0,j\,y}\,() - u_{0,j\,y}\,(\ _0) - u_{0,j\,yy}\,(\ _0)(\ - \ _0), \\
III_j &= -\nabla_x\ _{0,j} \le \nabla_x\ \ u_{0,j}\,() - u_{0,j}\,(\ _0) - u_{0,j\,y}\,(\ _0)(\ - \ _0) \\
&\qquad -\nabla_x\ \ \le \nabla_x\ \ (u_j - u_{0,j})() - (u_j - u_{0,j})(\ _0) \\
&\qquad -\nabla_x(\ - \ _0) \le \nabla_x\ \ u_{0,j}\,() - u_{0,j}\,(\ _0) + u_j\,(\ _0) - u_{0,j}\,(\ _0)\ .
\end{aligned}$$

This representation will allow us to estimate $r(\xi) - (\xi - \xi_0)$ in appropriate norms. The function spaces are supposed to be chosen according to

$$V \subseteq W^{2,\infty}(D(L)),\quad X \subseteq W^{1,p}(\Omega) \cap L^\infty(\Omega),\quad X^{\tilde\gamma} = L^p(\Omega).$$

The fact that X continuously embeds into $L^\infty(\Omega)$ allows us to guarantee $\ \le L$ for all with $\ - \ _0\ _X$ small enough. In this setting, using

$$\begin{aligned}
&|v(x,(x)) - v(x,\ _0(x))| = \int_0^1 v_y(x,\ _0(x) + s((x) - \ _0(x)))ds\ ((x) - \ _0(x)) \\
&\quad \le\ v_y\ _{L^\infty(D(\bar{\ }))}\ |(x) - \ _0(x)| \\
&|v(x,(x)) - v(x,\ _0(x)) - v_y(x,\ _0(x))| \\
&\quad = \int_0^1\int_0^1 v_{yy}(x,\ _0(x) + s\ s((x) - \ _0(x)))s\,ds\,ds\ \ ((x) - \ _0(x))^2 \\
&\quad \le\ v_{yy}\ _{L^\infty(D(\bar{\ }))}\ |(x) - \ _0(x)|^2,
\end{aligned}$$

we can further estimate

$$\begin{aligned}
\ I_j\ _{L^p(\Omega)} &\le \ \tilde\gamma_j\ _{L^p(\Omega)}\ \ u_{j\,y} - u_{0,j\,y}\ _{L^\infty(D(L))}\ \ \ - \ _0\ _{L^\infty(\Omega)} \\
&\quad + \ \tilde\gamma_j - \tilde\gamma_{0,j}\ _{L^p(\Omega)}\ \ u_{0,j\,y}\ _{L^\infty(D(L))}\ \ \ - \ _0\ _{L^\infty(\Omega)} \\
&\quad + \tfrac{1}{2}\ \tilde\gamma_{0,j}\ _{L^p(\Omega)}\ \ u_{0,j\,yy}\ _{L^\infty(D(L))}\ \ \ - \ _0\ ^2_{L^\infty(\Omega)}, \\
\ II_j\ _{L^p(\Omega)} &\le L^{1/p}\ \ \tfrac{1}{2}\ u_{0,j\,yyy}\ _{L^\infty(D(L))}\ \ \ - \ _0\ ^2_{L^\infty(\Omega)} \\
&\qquad + \ u_{j\,yy} - u_{0,j\,yy}\ _{L^\infty(D(L))}\ \ \ - \ _0\ _{L^\infty(\Omega)}\ , \\
\ III_j\ _{L^p(\Omega)} &\le \ \nabla_x\ _{0,j}\ _{L^p(\Omega)}\ \nabla_x u_{0,j\,yy}\ _{L^\infty(D(L))}\ \ \ - \ _0\ ^2_{L^\infty(\Omega)} \\
&\quad + \ \nabla_x\ \ _{L^p(\Omega)}\ \nabla_x u_{j\,y} - \nabla_x u_{0,j\,y}\ _{L^\infty(D(L))}\ \ \ - \ _0\ _{L^\infty(\Omega)} \\
&\quad + \ \nabla_x\ - \nabla_x\ _0\ _{L^p(\Omega)}\ \ \nabla_x u_{0,j\,y}\ _{L^\infty(D(L))}\ \ \ - \ _0\ _{L^\infty(\Omega)} \\
&\qquad + \ \nabla_x u_j\,(\ _0) - \nabla_x u_{0,j}\,(\ _0)\ _{L^\infty(\Omega)}\ .
\end{aligned}$$

Altogether, choosing $u_{0,j}$ to be smooth enough and bounded away from zero on $\ _0$

$$\frac{1}{u_{0,j}\,(\ _0)} \in L^\infty(\Omega),\quad u_{0,j\,y},\,u_{0,j\,yy} \in W^{1,\infty}(D(L)),\quad j \in \{1,2\}, \tag{53}$$

we have shown that

$$(54)\qquad \|r(\xi)-(\xi-\xi_0)\|_{V^2\times X\times(X^{\tilde\gamma})^2}\le C\|\xi-\xi_0\|^2_{V^2\times X\times(X^{\tilde\gamma})^2}$$

for some $C>0$. Referring to (53), note that in our all-at-once setting, $u_{0,j}$ does not necessarily need to satisfy a PDE, which (up to closeness to $u^\dagger$) allows for plenty of freedom in its choice. Analogously to, e.g., [30, 31] this provides us with the estimate

$$(55)\qquad \begin{aligned}\exists c_r\in(0,1)\,\forall\xi\in U(\subseteq V^2\times X\times(X^{\tilde\gamma})^2):\ &\|r(\xi)-r(\xi^\dagger)-(\xi-\xi^\dagger)\|_X\\ &\le c_r\|\xi-\xi^\dagger\|_X,\end{aligned}$$

where $\xi^\dagger=(u_1^\dagger,u_2^\dagger,{}^\dagger,\tilde\gamma_1^\dagger,\tilde\gamma_2^\dagger)$ is the actual solution.

3.2.2. *Linearised uniqueness.* Results on uniqueness of the nonlinear Problem 3.1 can be found in [4, 43]. In particular, linear independence of the functions g_1, g_2 is sufficient for determining both and $\tilde\gamma$ in (41).

Here we will show linearised uniqueness, as this is another ingredient of the convergence proof of Newton's method. More precisely, we show that the intersection of the nullspaces of the linearised forward operator $\tilde F'(u_{0,1},u_{0,2},{}_0,\tilde\gamma_{0,1},\tilde\gamma_{0,2})$ and the penalisation operator P is trivial. To this end, assume that $\tilde F'(u_{0,1},u_{0,2},{}_0,\tilde\gamma_{0,1},\tilde\gamma_{0,2})(\underline{du}_1,\underline{du}_2,\underline{d},\underline{d\gamma}_1,\underline{d\gamma}_2)=0$ and $P(\underline{d\gamma}_1,\underline{d\gamma}_2)=0$, where the latter simply means $\underline{d\gamma}_1=\underline{d\gamma}_2=:\underline{d\gamma}$. From the first four lines in (51) we conclude that $\underline{du}_j$ satisfies a homogeneous Cauchy problem and therefore has to vanish on $D(\ _j)$ for $j\in\{1,2\}$. Thus, $B_{,\tilde\gamma_j}\underline{du}_j=0$ and by an elimination procedure similar to the one that led to (48) but using the representation (43) of G we obtain

$$(56)\qquad 0=W[\,,u_1,u_2]\cdot\nabla_x\underline{d}+\beta[\,,u_1,u_2]\underline{d}\text{ in }\Omega,\qquad \underline{d}=0\text{ on }\Sigma$$

with W as in (47), $\beta[\,,u_1,u_2]=-\Big((u_{1\,yy}()-\nabla_x\cdot\nabla_x u_{1\,y}()+\tilde\gamma u_{1\,y}))u_2()+(u_{2\,yy}()-\nabla_x\cdot\nabla_x u_{2\,y}()+\tilde\gamma u_{2\,y}))u_1()\Big)$. Thus, under the assumption that the Wronskian W does not vanish and that Σ contains the inflow boundary

$$(57)\qquad \{x\in\partial\Omega:\ \nu^\Omega\cdot W<0\}\subseteq\Sigma,$$

with ν^Ω denoting the outward normal to $\partial\Omega$, uniqueness of the solution to the above boundary value problem (56) for a transport equation (that follows from the method of characteristics, see, e.g., [14, Chapter 3, Section 3.2]) yields $\underline{d}=0$ and therefore $\underline{d\gamma}=0$. Assuming that the Wronskian is bounded away from zero at the exact solution

$$(58)\qquad \exists c>0:\quad |W[{}^\dagger,u_1^\dagger,u_2^\dagger]|\ge c\quad\text{a.e. on }\Omega$$

by a continuity argument therefore allows us to conclude linearised uniqueness when linearising sufficiently close to the exact solution.

3.2.3. *Convergence of Newton type schemes.* We now combine the results from Sections 3.2.1, 3.2.2 to prove convergence of two Newton type schemes for solving Problem 3.1.

(a) Regularised frozen Newton with penalty.

We apply a frozen Newton method with conventional Tikhonov (and no fractional) regularisation but with penalty P as in [30, 31].

$$(59)\quad \xi^\delta_{n+1} = \xi^\delta_n + (K^*K + P^*P + \varepsilon_n I)^{-1}\left(K^*(h^\delta - \tilde F(\xi^\delta_n)) - P^*P\xi^\delta_n + \varepsilon_n(\xi_0 - \xi^\delta_n)\right),$$

where $K = \tilde F'(\xi_0)$ and K^* denotes the Hilbert space adjoint of $K : \mathbb{X} \to \mathbb{Y}$ and

$$(60)\quad \begin{aligned} \mathbb{X} &= H^{3+}(D(L))^2 \times H^s(\Omega) \times L^2(\Omega)^2, \\ \mathbb{Y} &= \left(L^2(D(L)) \times L^2(\Gamma_1)^2 \times L^2(\Gamma_2(L)) \times L^2(\Omega) \times L^2(\Sigma)\right)^2 \end{aligned}$$

with $s > d/2$, $s \geq 1$, where $d \in \{1, 2\}$ is the dimension of Ω. The regularisation parameters are taken from a geometric sequence $\varepsilon_n = \varepsilon_0\theta^n$ for some $\theta \in (0, 1)$, and the stopping index is chosen such that

$$(61)\quad \varepsilon_{n_*(\delta)} \to 0 \text{ and } \delta^2/\varepsilon_{n_*(\delta)-1} \to 0 \text{ as } \delta \to 0,$$

where δ is the noise level according to

$$(62)\quad \|(f_1^\delta, f_2^\delta, g_1^\delta, g_2^\delta) - (f_1, f_2, g_2, g_2)\|_{L^2(0,T;L^2(\Omega))} \leq \delta.$$

An application of [27, Theorem 2.2] together with our verification of range invariance (52) with (54) and linearised uniqueness yields the following convergence result.

Theorem 3.1. *Let $\xi_0 \in U := \mathcal{B}_\rho(\xi^\dagger)$ for some $\rho > 0$ sufficiently small, assume that (53), (57), (58), and $\mathcal{N}(K)^\perp \subseteq \mathcal{N}(P)$ hold and let the stopping index $n_* = n_*(\delta)$ be chosen according to (61).*

Then the iterates $(\xi^\delta_n)_{n\in\{1,\dots,n_*(\delta)\}}$ are well-defined by (59), remain in $\mathcal{B}_\rho(\xi^\dagger)$ and converge in $\mathbb{X}$ (defined as in (60)), $\|\xi^\delta_{n_*(\delta)} - \xi^\dagger\|_{\mathbb{X}} \to 0$ as $\delta \to 0$.

In the noise free case $\delta = 0$, $n_*(\delta) = \infty$ we have $\|\xi_n - \xi^\dagger\|_{\mathbb{X}} \to 0$ as $n \to \infty$.

(b) Frozen Newton with penalty, applied to fractionally regularised problem.

Replace $\tilde F_j$ in (50) by

$$(63)\quad \tilde F_j^\alpha : (u_{+j}, u_{-j}, \ldots, \tilde\gamma_j) \mapsto \begin{pmatrix} \partial_y^\alpha \overleftarrow{u}^L_{+j} - \sqrt{-\Delta_x}\,\overleftarrow{u}^L_{+j} \\ \partial_y u_{-j} - \sqrt{-\Delta_x}\, u_{-j} \\ u_{+j}|_{\Gamma_1} - \frac{1}{2}(f + \sqrt{-\Delta_x}^{-1} g) \\ u_{-j}|_{\Gamma_1} - \frac{1}{2}(f + \sqrt{-\Delta_x}^{-1} g) \\ (Bw_{+j}, Bu_{-j})|_{\Gamma_2(L)} \\ B_{\tilde\gamma}(u_{+j} + u_{-j})|_{\Gamma_0(\,)} \\ \cdot|_{\Sigma} - {}^0 \end{pmatrix}, \quad j \in \{1, 2\},$$

where $\partial_y^\alpha w$ is the fractional DC derivative with endpoint 0 and $\overleftarrow{u}^L(y) = u(L - y)$, cf. Section 1. Range invariance and linearised uniqueness can be derived analogously to the previous section (noting that the only nonlinear term is the one in the next to last line and coincides with the one in (50)) and therefore we can apply (59) with $\tilde F^\alpha$ in place of $\tilde F$ and conclude its convergence to a solution $(u^{\alpha,\dagger}_{+1}, u^{\alpha,\dagger}_{-1}, u^{\alpha,\dagger}_{+2}, u^{\alpha,\dagger}_{-2}, {}^{\alpha,\dagger}, \tilde\gamma_1^{\alpha,\dagger} = \tilde\gamma_2^{\alpha,\dagger})$ of (49) with $\tilde F_j^\alpha(u_{+j}, u_{-j}, \ldots, \tilde\gamma_j)$ in place of $\tilde F_j(u_j, \ldots, \tilde\gamma_j)$ for any fixed $\alpha \in (0, 1)$. With the abbreviation

$$\xi^\alpha := (u^\alpha_{+,1}, u^\alpha_{-,1}, u^\alpha_{+,2}, u^\alpha_{-,2}, {}^\alpha, \tilde\gamma_1^\alpha = \tilde\gamma_2^\alpha),$$

we thus have the following convergence result.

Corollary 3.1. *Let $\xi_0^\alpha \in U := \mathcal{B}_\rho(\xi^{\alpha,\dagger})$ for some $\rho > 0$ sufficiently small, assume that (53), (57), (58), and $\mathcal{N}(K)^\perp \subseteq \mathcal{N}(P)$ hold and let the stopping index $n_* = n_*(\delta)$ be chosen according to (61).*

Then the iterates $(\xi_n^{\alpha,\delta})_{n\in\{1,\dots,n_*(\delta)\}}$ are well-defined by (59) with $\tilde{F} := \tilde{F}^{\alpha(\delta)}$, remain in $\mathcal{B}_\rho(\xi^\dagger)$ and converge in $\mathbb{X}$ (defined as in (60) with $H^{3+}(D(L))^2$ replaced by $H^{3+}(D(L))^4$) $\|\xi_{n_*(\delta)}^{\alpha,\delta} - \xi^{\alpha,\dagger}\|_{\mathbb{X}} \to 0$ as $\delta \to 0$. In the noise free case $\delta = 0$, $n_*(\delta) = \infty$ we have $\|\xi_n^\alpha - \xi^{\alpha,\dagger}\|_{\mathbb{X}} \to 0$ as $n \to \infty$.

It remains to estimate the approximation error $(^{\alpha,\dagger} - ^{\dagger}, \tilde{\gamma}^{\alpha,\dagger} - \tilde{\gamma}^\dagger)$. In Section 1 we have seen that $u_j^\dagger = u_{+,j}^{1,\dagger} + u_{-,j}^{1,\dagger}$ where $u_{-,j}^{\alpha,\dagger} - u_{-,j}^{1,\dagger} = 0$ and

$$u_{+,j}^{\alpha,\dagger}(x,y) - u_{+,j}^{1,\dagger}(x,y) = \frac{1}{2}\sum_{i=1}^{\infty}\Big(f_{ij} + \frac{1}{\sqrt{\lambda_i}} g_{ij}\Big)\Big(\frac{1}{E_{\alpha,1}(-\sqrt{\lambda_i}\,y^\alpha)} - \exp(-\sqrt{\lambda_i}\,y)\Big)\varphi_i(x).$$

Moreover, subtracting the two identities $B_{^{\alpha,\dagger},\tilde{\gamma}^{\alpha,\dagger}}\, u_{+,j}^{\alpha,\dagger} = 0$, $B_{^\dagger,\tilde{\gamma}^\dagger}\, u_{+,j}^\dagger = 0$, we arrive at the following differential algebraic system for $\underline{d} = ^{\alpha,\dagger} - ^\dagger$, $d\underline{\gamma} = \tilde{\gamma}^{\alpha,\dagger} - \tilde{\gamma}^\dagger$

$$-\nabla_x u_{+,1}^\dagger(^\dagger)\cdot\nabla_x \underline{d} + d_1^\alpha\, \underline{d} + u_{+,1}^\dagger(^\dagger)\, d\underline{\gamma} = b_1^\alpha,$$
$$-\nabla_x u_{+,2}^\dagger(^\dagger)\cdot\nabla_x \underline{d} + d_2^\alpha\, \underline{d} + u_{+,2}^\dagger(^\dagger)\, d\underline{\gamma} = b_2^\alpha,$$

where

$$d_j^\alpha(x) = \int_0^1 I(u_{+,j,y}^{\alpha,\dagger}, ^{\alpha,\dagger}, \tilde{\gamma}^{\alpha,\dagger}; x, ^\dagger(x) + \theta(^{\alpha,\dagger}(x) - ^\dagger(x)))\,d\theta,$$
$$b_j^\alpha(x) = -I(u_{+,j}^{\alpha,\dagger} - u_{+,j}^\dagger, ^{\alpha,\dagger}, \tilde{\gamma}^{\alpha,\dagger}; x, ^{\alpha,\dagger}(x)),$$
$$I(u,,\tilde{\gamma}; x, y) = u_y(x,y) - \nabla_x(x)\cdot\nabla_x u(x,y) + \tilde{\gamma}u(x,y).$$

Assuming that the Wronskian

$$W^\dagger := u_{+,2}^\dagger(^\dagger)\,\nabla_x u_{+,1}^\dagger(^\dagger) - u_{+,1}^\dagger(^\dagger)\,\nabla_x u_{+,2}^\dagger(^\dagger) \tag{64}$$

and one of the factors $u_{+,j}^\dagger(^\dagger)$ of $d\underline{\gamma}$ are bounded away from zero,

$$\exists c > 0: \quad |W^\dagger| \geq c, \quad \max\{|u_{+,1}^\dagger(^\dagger)|, |u_{+,2}^\dagger(^\dagger)|\} \geq c \quad \text{a.e. on } \Omega, \tag{65}$$

we can conclude existence of a constant $C > 0$ independent of α (note that $W^\dagger$ and $u_{+,j}^\dagger$ do not depend on α) such that

$$\|^{\alpha,\dagger} - ^\dagger\|_{L^2(\Omega)} + \|\tilde{\gamma}^{\alpha,\dagger} - \tilde{\gamma}^\dagger\|_{L^2(\Omega)} \leq C\sum_{j=1}^{2}\|b_j^\alpha\|_{L^2(\Omega)} \leq \tilde{C}\sum_{j=1}^{2}\|u_{+,j}^{\alpha,\dagger} - u_{+,j}^\dagger\|_{C(\Omega\times(l,\underline{L}))}$$

and in the (altogether 2-dimensional) case $\Omega = (0, M)$ where the transport equation does not lead to a loss of regularity, even

$$\|^{\alpha,\dagger} - ^\dagger\|_{C^1(\Omega)} + \|\tilde{\gamma}^{\alpha,\dagger} - \tilde{\gamma}^\dagger\|_{C(\Omega)} \leq C\sum_{j=1}^{2}\|b_j^\alpha\|_{C(\Omega)} \leq \tilde{C}\sum_{j=1}^{2}\|u_{+,j}^{\alpha,\dagger} - u_{+,j}^\dagger\|_{C(\Omega\times(l,\underline{L}))}.$$

The latter can be estimated by means of Lemma 1.4 in Section 1.

Combining this with Corollary 3.1 we have the following convergence result.

Theorem 3.2. *Assume* (65) *with* $W^\dagger$ *according* (64) *and let* $\alpha(\delta) \to 0$ *as* $\delta \to 0$*. Moreover, let the assumptions of Corollary* 3.1 *hold for* $\alpha = \alpha(\delta)$*, and let* $(\xi_n^{\alpha(\delta),\delta})_{n\in\{1,\dots,n_*(\delta)\}}$ *be defined by* (59), (61) *with* $\tilde F := \tilde F^{\alpha(\delta)}$*. Then*

$$\begin{cases} \| {}_{n_*(\delta)}^{\alpha(\delta),\delta} - {}^\dagger \|_{L^2(\Omega)} + \| \tilde\gamma_{n_*(\delta)}^{\alpha(\delta),\delta} - \tilde\gamma^\dagger \|_{L^2(\Omega)} \to 0 \text{ as } \delta \to 0 & \text{if } \dim(\Omega) = 2, \\ \| {}_{n_*(\delta)}^{\alpha(\delta),\delta} - {}^\dagger \|_{H^1(\Omega)} + \| \tilde\gamma_{n_*(\delta)}^{\alpha(\delta),\delta} - \tilde\gamma^\dagger \|_{L^2(\Omega)} \to 0 \text{ as } \delta \to 0 & \text{if } \dim(\Omega) = 1. \end{cases}$$

Appendix A. Cauchy 1

Lemma A.1. *For any* $\hat\lambda_1 \geq 0$*,* $\alpha \in (0,1)$*,* $p \in (1, \frac{1}{1-\alpha})$*,* $\hat p := 1 + \frac{1}{p}$ *and the constant*

(66)
$$C(\alpha, p, \bar l) := \|E_{\alpha,\alpha/2}\|_{L^\infty(\mathbb{R}^+)} (\tilde C_0(\alpha,\alpha/2) + \tilde C_1(\alpha,\alpha/2)) \left(\frac{\hat p}{\alpha - \hat p}\right)^{\hat p} \max\{1, \sqrt{\bar l}\}^{\alpha - \hat p},$$

with $\tilde C_0$*,* $\tilde C_1$ *as in* (70)*, the following estimate holds*

$$\sup_{\hat\lambda > \hat\lambda_1} \|E_{\alpha,1}(-\hat\lambda \cdot^\alpha) - \cdot^{\alpha-1} E_{\alpha,\alpha}(-\hat\lambda \cdot^\alpha)\|_{L^p(0,\bar l)} \leq C(\alpha, p, \bar l)\,(1-\alpha).$$

Proof. Abbreviating

$$\mathfrak{g}_\mu(y) = \frac{1}{\Gamma(\mu)} y^{\mu-1}, \quad e_{\alpha,\beta}(y) = e_{\alpha,\beta}(y;\hat\lambda) = y^{\beta-1} E_{\alpha,\beta}(-\hat\lambda y^\alpha),$$

the quantity to be estimated is $w(y) := E_{\alpha,1}(-\hat\lambda y^\alpha) - y^{\alpha-1} E_{\alpha,\alpha}(-\hat\lambda y^\alpha) = e_{\alpha,1}(y) - e_{\alpha,\alpha}(y)$. Using the Laplace transform identities

(67)
$$(\mathcal{L}\mathfrak{g}_\mu)(\xi) = \xi^{-\mu}, \quad (\mathcal{L}e_{\alpha,\beta})(\xi) = \frac{\xi^{\alpha-\beta}}{\hat\lambda + \xi^\alpha},$$

we obtain, for some $\beta \in (\alpha - 1, \alpha)$ yet to be chosen,

$$\begin{aligned}(\mathcal{L}w)(\xi) &= \frac{\xi^{\alpha-1} - 1}{\hat\lambda + \xi^\alpha} = \frac{\xi^{\alpha-\beta}}{\hat\lambda + \xi^\alpha}(\xi^{\beta-1} - \xi^{\beta-\alpha}) \\ &= (\mathcal{L}e_{\alpha,\beta})(\xi)\, \big((\mathcal{L}\mathfrak{g}_{1-\beta})(\xi) - (\mathcal{L}\mathfrak{g}_{\alpha-\beta})(\xi)\big),\end{aligned}$$

hence, by Young's Convolution Inequality,

$$\|w\|_{L^p(0,\bar l)} = \|e_{\alpha,\beta} * (\mathfrak{g}_{1-\beta} - \mathfrak{g}_{\alpha-\beta})\|_{L^p(0,\bar l)} \leq \|e_{\alpha,\beta}\|_{L^q(0,\bar l)} \|\mathfrak{g}_{1-\beta} - \mathfrak{g}_{\alpha-\beta}\|_{L^r(0,\bar l)}$$

provided $\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}$.

For the first factor, under the condition

(68)
$$1 \leq q < \frac{1}{1-\beta},$$

that is necessary for integrability near zero, we get

(69)
$$\|e_{\alpha,\beta}\|_{L^q(0,\bar l)} \leq \|E_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^+)} \frac{\max\{1, \bar l\}^{(\beta-1)+1/q}}{((\beta-1)q+1)^{1/q}}.$$

The second factor can be estimated by applying the Mean Value Theorem to the function $\theta(y,\alpha,\beta) := \mathfrak{g}_{\alpha-\beta}(y)$ as follows

$$\mathfrak{g}_{1-\beta}(y) - \mathfrak{g}_{\alpha-\beta}(y) = \theta(y,1) - \theta(y,\alpha) = \frac{d}{d\alpha}\theta(y,\tilde\alpha,\beta)\,(1-\alpha) = \tilde\theta(y,\tilde\alpha,\beta)\,(1-\alpha),$$

where

$$\tilde\theta(y;\tilde\alpha,\beta) = y^{\tilde\alpha-\beta-1}\left(\tfrac{1}{\Gamma}(\tilde\alpha-\beta)\log(y) - \tfrac{\Gamma'}{\Gamma^2}(\tilde\alpha-\beta)\right)$$
$$= g_{\tilde\alpha-\beta}(y)\left(\log(y) - (\log\Gamma)'(\tilde\alpha-\beta)\right)$$

for some $\tilde\alpha\in(\alpha,1)$, with the digamma function $\psi = \frac{\Gamma'}{\Gamma} = (\log\circ\Gamma)'$, for which $\frac{\psi}{\Gamma} = \frac{\Gamma'}{\Gamma^2}$ is known to be an entire function as is also the reciprocal Gamma function, thus

$$\tilde C_0(\alpha,\beta) := \sup_{\tilde\alpha\in(\alpha,1)} \left|\tfrac{\psi}{\Gamma}(\tilde\alpha-\beta)\right| < \infty, \qquad \tilde C_1(\alpha,\beta) := \sup_{\tilde\alpha\in(\alpha,1)} \left|\tfrac{1}{\Gamma}(\tilde\alpha-\beta)\right| < \infty. \tag{70}$$

Integrability near $y=0$ of $(y^{\tilde\alpha-\beta-1})^r$ and of $(y^{\tilde\alpha-\beta-1}\log(y))^r$ holds iff

$$1 \le r < \frac{1}{1-\tilde\alpha+\beta} \tag{71}$$

and yields

$$\|g_{1-\beta} - g_{\alpha-\beta}\|_{L^r(0,\bar l)} \le \sup_{\tilde\alpha\in(\alpha,1)} \|\tilde\theta(\cdot;\tilde\alpha,\beta)\|_{L^r(0,\bar l)}\,(1-\alpha), \tag{72}$$

where

$$\|\tilde\theta(\cdot;\tilde\alpha,\beta)\|_{L^r(0,\bar l)} \le (\tilde C_0(\alpha,\beta) + \tilde C_1(\alpha,\beta))\,\frac{\max\{1,\bar l\}^{(\alpha-\beta-1)+1/r}}{((\alpha-\beta-1)r+1)^{1/r}}. \tag{73}$$

Conditions (68), (71) together with $\tilde\alpha\in(\alpha,1)$ are equivalent to

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1 > 1 - \tilde\alpha \text{ and } 1 < \beta + \frac{1}{q} < \tilde\alpha + \frac{1}{p}$$

which together with $\tilde\alpha\in(\alpha,1)$ leads to the assumption

$$\frac{1}{p} > 1 - \alpha$$

and the choice $\min\{0, 1-\frac{1}{q}\} < \beta < \alpha + \frac{1}{p} - \frac{1}{q}$.

To minimize the factors[1]

$$c_1(q,\beta) = ((\beta-1)q+1)^{-1/q}, \quad c_2(r,\alpha-\beta) = ((\alpha-\beta-1)r+1)^{-1/r}$$

in (69), (73) under the constraints (68), (71) and $\frac{1}{q}+\frac{1}{r} = 1+\frac{1}{p}$ we make the choice $\frac{1}{q} = \frac{1}{r} = \frac{1}{2}(1+\frac{1}{p})$, $\beta = \alpha/2$ that balances the competing pairs $q \leftrightarrow r$, $\beta \leftrightarrow \alpha-\beta$ and arrive at

$$c_1(q,\beta) = c_2(r,\alpha-\beta) = \left(\frac{1+\frac{1}{p}}{\alpha-1+\frac{1}{p}}\right)^{1+\frac{1}{p}}.$$

Proof of Lemma 1.2. To prove (23), we employ an energy estimate for the ode satisfied by $v(y) := E_{\alpha,1}(-\hat\lambda y^\alpha) - \exp(-\hat\lambda y) = u_{\alpha,\hat\lambda}(y) - u_{1,\hat\lambda}(y)$,

$$\partial_y v + \hat\lambda v = -(\partial_y^\alpha - \partial_y)u_{\alpha,\hat\lambda} =: \hat\lambda w,$$
$$\text{where } w = -\tfrac{1}{\hat\lambda}(\partial_y^\alpha - \partial_y)E_{\alpha,1}(-\hat\lambda y^\alpha) = E_{\alpha,1}(-\hat\lambda y^\alpha) - y^{\alpha-1}E_{\alpha,\alpha}(-\hat\lambda y^\alpha).$$

[1] We do not go for asymptotics with respect to $\bar l$ since we assume $\bar l$ to be moderately sized anyway.

Testing with $|v(\tau)|^{p-1}\operatorname{sign}(v(\tau))$, integrating from 0 to t, and applying Hölder's and Young's inequalities yields, after multiplication with p,

$$|v(y)|^p + \hat{\lambda} \int_0^y |v(\tau)|^p \, d\tau \leq \hat{\lambda} \int_0^y |w(\tau)|^p \, d\tau, \tag{74}$$

in particular

$$\|v\|_{L^\infty(0,\bar{l})} \leq \hat{\lambda}^{1/p} \|w\|_{L^p(0,\bar{l})}, \qquad \|v\|_{L^p(0,\bar{l})} \leq \|w\|_{L^p(0,\bar{l})}$$

for any $\bar{l} \in (0,\infty]$. The result then follows from Lemma A.1.

Proof of Lemma 1.3. For the second estimate, with $v = e_{\alpha,1} - e_{1,1}$ as in the proof of Lemma 1.2, we have to bound $\partial_y v = -\hat{\lambda}(e_{\alpha,\alpha} - e_{1,1})$, where

$$\begin{aligned}\mathcal{L}(e_{\alpha,\alpha} - e_{1,1})(\xi) = \frac{1}{\hat{\lambda}+\xi^\alpha} - \frac{1}{\hat{\lambda}+\xi} &= \frac{\xi - \xi^\alpha}{(\hat{\lambda}+\xi^\alpha)(\hat{\lambda}+\xi)} \\ &= \frac{\xi^{1-\gamma}}{\hat{\lambda}+\xi}\,\frac{\xi^{\alpha-\beta}}{\hat{\lambda}+\xi^\alpha}\,(\xi^{\beta+\gamma-\alpha} - \xi^{\beta+\gamma-1})\end{aligned}$$

for $\beta,\gamma > 0$ with $\beta+\gamma < \alpha$ yet to be chosen. In view of (67) we thus have

$$e_{\alpha,\alpha} - e_{1,1} = e_{1,\gamma} * e_{\alpha,\beta} * (\mathfrak{g}_{\alpha-\beta-\gamma} - \mathfrak{g}_{1-\beta-\gamma}) =: e_{1,\gamma} * e_{\alpha,\beta} * \underline{d\mathfrak{g}}.$$

Now, applying the elementary estimate

$$\begin{aligned}\|a * b\|_{L^\infty(\ell_2,\ell_3)} &= \sup_{y\in(\ell_2,\ell_3)} \left|\int_0^y a(y-z)b(z)\,dz\right| \\ &= \sup_{y\in(\ell_2,\ell_3)} \left|\int_0^{y-\ell_1} a(y-z)b(z)\,dz + \int_{y-\ell_1}^y a(y-z)b(z)\,dz\right| \\ &\leq \|a\|_{L^\infty(\ell_1,\ell_3)}\|b\|_{L^1(0,\ell_3-\ell_1)} + \|b\|_{L^\infty(\ell_2-\ell_1,\ell_3)}\|a\|_{L^1(0,\ell_1)}\end{aligned}$$

for $0 < \ell_1 < \ell_2 < \ell_3$, $a,b \in L^1(0,\ell_3)$, $a|_{(\ell_1,\ell_3)} \in L^\infty(\ell_1,\ell_3)$, $b|_{(\ell_2-\ell_1,\ell_3)} \in L^\infty(\ell_2-\ell_1,\ell_3)$, twice, namely with $a = e_{1,\gamma}$, $b = e_{\alpha,\beta}$, $\ell_1 = \underline{l}/3$, $\ell_2 = \underline{l}$, $\ell_3 = \bar{l}$ and with $a = e_{\alpha,\beta}$, $b = \underline{d\mathfrak{g}}$, $\ell_1 = \underline{l}/3$, $\ell_2 = 2\underline{l}/3$, $\ell_3 = \bar{l}$, along with Young's Convolution Inequality, we obtain

$$\begin{aligned}&\|e_{1,\gamma} * (e_{\alpha,\beta} * \underline{d\mathfrak{g}})\|_{L^\infty(\underline{l},\bar{l})} \\ &\quad\leq \|e_{1,\gamma}\|_{L^\infty(\underline{l}/3,\bar{l})}\|e_{\alpha,\beta} * \underline{d\mathfrak{g}}\|_{L^1(0,\bar{l})} + \|e_{\alpha,\beta} * \underline{d\mathfrak{g}}\|_{L^\infty(2\underline{l}/3,\bar{l})}\|e_{1,\gamma}\|_{L^1(0,\bar{l})} \\ &\quad\leq \|e_{1,\gamma}\|_{L^\infty(\underline{l}/3,\bar{l})}\|e_{\alpha,\beta}\|_{L^1(0,\bar{l})}\|\underline{d\mathfrak{g}}\|_{L^1(0,\bar{l})} \\ &\qquad + \left(\|e_{\alpha,\beta}\|_{L^\infty(\underline{l}/3,\bar{l})}\|\underline{d\mathfrak{g}}\|_{L^1(0,\bar{l}-\underline{l}/3)} + \|\underline{d\mathfrak{g}}\|_{L^\infty(\underline{l}/3,\bar{l})}\|e_{\alpha,\beta}\|_{L^1(0,\underline{l}/3)}\right)\|e_{1,\gamma}\|_{L^1(0,\bar{l})}.\end{aligned}$$

Using this with $\beta = \gamma = \alpha/3$ and (cf. (72))

$$\begin{aligned}\|\underline{d\mathfrak{g}}\|_{L^1(0,\bar{l})} &\leq \sup_{\tilde{\alpha}\in(\alpha,1)} \|\tilde{\theta}(\cdot;\tilde{\alpha},\alpha/3)\|_{L^1(0,\bar{l})}\,(1-\alpha), \\ \|\underline{d\mathfrak{g}}\|_{L^\infty(\underline{l}/3,\bar{l})} &\leq \sup_{\tilde{\alpha}\in(\alpha,1)} \|\tilde{\theta}(\cdot;\tilde{\alpha},\alpha/3)\|_{L^\infty(\underline{l}/3,\bar{l})}\,(1-\alpha),\end{aligned}$$

we arrive at the second estimate in (24) with $\hat{C}(\alpha_0, l, \bar{l}) = \sup_{\alpha\in(\alpha_0,1)} \check{C}(\alpha, l, \bar{l})$,

$$\check{C}(\alpha, l, \bar{l}) = e_{1,\alpha/3} \; {}_{L^\infty(l/3,\bar{l})} \; e_{\alpha,\alpha/3} \; {}_{L^1(0,\bar{l})} \sup_{\tilde{\alpha}\in(\alpha,1)} \tilde{\theta}(\leq\tilde{\alpha},\alpha,3) \; {}_{L^1(0,\bar{l})}$$
$$+ \; e_{1,\alpha/3} \; {}_{L^1(0,\bar{l})} \; e_{\alpha,\alpha/3} \; {}_{L^\infty(l/3,\bar{l})} \sup_{\tilde{\alpha}\in(\alpha,1)} \tilde{\theta}(\leq\tilde{\alpha},\alpha,3) \; {}_{L^1(0,\bar{l})}$$
$$+ \; e_{1,\alpha/3} \; {}_{L^1(0,\bar{l})} \; e_{\alpha,\alpha/3} \; {}_{L^1(0,l/3)} \sup_{\tilde{\alpha}\in(\alpha,1)} \tilde{\theta}(\leq\tilde{\alpha},\alpha,3) \; {}_{L^\infty(l/3,\bar{l})} \, .$$

The first estimate can be shown analogously.

Appendix B. Cauchy 2

In the impedance case, using the PDE, the right hand side of (33) can be written as

$$G(u, d) = d \; (x) \quad u_x(x, (x)) - \frac{(x)}{\sqrt{1+ (x)^2}} \gamma(x) u(x, (x))$$
$$+ d(x) \quad \frac{d}{dx} \; u_x(x, (x)) - \sqrt{1 + (x)^2}\, \gamma(x) u_y(x, (x))$$
$$= \frac{d}{dx} \; d(x) \quad u_x(x, (x)) - \frac{(x)}{\sqrt{1+ (x)^2}} \gamma(x) u(x, (x))$$
$$+ d(x) \quad - \sqrt{1 + (x)^2}\, \gamma(x) u_y(x, (x)) + \frac{d}{dx} \; \frac{(x)}{\sqrt{1+ (x)^2}} \gamma(x) u(x, (x))$$
$$= \frac{d}{dx} \varphi(x) - a(x)\varphi(x) = \frac{d}{dx}[d(x) \; \mathfrak{a}[u](x)] - d(x) \; \mathfrak{b}[u](x)$$

Here using the impedance conditions on u that yield

(75)

$$u_x(x, (x)) - \frac{(x)}{\sqrt{1+ (x)^2}} \gamma(x) u(x, (x))$$
$$= u_x(x, (x)) + \frac{(x)}{1+ (x)^2} \; u_y(x, (x)) - (x) u_x(x, (x))$$
$$= \frac{1}{1+ (x)^2} \; u_x(x, (x)) + (x) u_y(x, (x)) = \frac{\partial u(x, (x))}{1+ (x)^2} = \frac{1}{1+ (x)^2} \frac{d}{dx} u(x, (x))$$
$$= \begin{cases} \frac{d}{dx} u(x, (x)) = u_x(x, (x)) & \text{if } (x) = 0, \\ \frac{1}{(x)} \; \frac{\gamma(x)}{\sqrt{1+ (x)^2}} u(x, (x)) + u_y(x, (x))) & \text{else} \end{cases} =: \mathfrak{a}[u](x).$$

In our implementation we use the last expression that is based on

$$u_x(x, (x)) = \frac{1}{(x)} \quad \sqrt{1 + (x)^2}\, \gamma(x) u(x, (x)) + u_y(x, (x))) \qquad \text{if } (x) = 0,$$

since u_x is difficult to evaluate numerically unless the boundary is flat (case $(x) = 0$). Moreover,

$$- \sqrt{1 + (x)^2}\, \gamma(x) u_y(x, (x)) + \frac{d}{dx} \; \frac{(x)}{\sqrt{1+ (x)^2}} \gamma(x) u(x, (x))$$

(76)

$$= \frac{(x)}{\sqrt{1+ (x)^2}^{3}} \gamma(x) + \frac{(x)}{\sqrt{1+ (x)^2}} \gamma'(x) + \gamma(x)^2 \quad u(x, (x)) \qquad =: \mathfrak{b}[u](x),$$

we have

(77)
$$\varphi(x) = d(x) \quad \mathfrak{a}[u](x) \qquad a(x) = \frac{\mathfrak{b}[u]}{\mathfrak{a}[u]}(x)$$

with $\mathfrak{a}$, $\mathfrak{b}$ as defined in (75), (76).

Thus the Newton step in the impedance case reads as

$$\text{(I)}\quad {}^{(k+1)}(x) = {}^{(k)}(x) - \frac{1}{\mathfrak{a}[{}^{(k)}, u^{(k)}](x)} \exp\left(-\int_0^x \frac{\mathfrak{b}[{}^{(k)}, u^{(k)}]}{\mathfrak{a}[{}^{(k)}, u^{(k)}]}(s)\,ds\right) \underline{d}(0)\ \mathfrak{a}[{}^{(k)}, u^{(k)}](0)$$

$$+ \int_0^x b(s) \exp\left(-\int_s^x \frac{\mathfrak{b}[{}^{(k)}, u^{(k)}]}{\mathfrak{a}[{}^{(k)}, u^{(k)}]}(t)\,dt\right) ds,$$

$$\text{where}\ b(x) = \partial_\nu \mathfrak{u}(x, {}^{(k)}(x)) + \sqrt{1 + (x)^2}\,\gamma(x)\mathfrak{u}(x, {}^{(k)}(x)).$$

In particular, with Neumann conditions on the lateral boundary $B = \partial_x$ under the compatibility condition $(0) = 0$ we have $\mathfrak{a}[, u](0) = \partial_\tau(0, (0)) = u_x(0, (0)) = 0$ and therefore

$$\text{(I)}\quad {}^{(k+1)}(x) = {}^{(k)}(x) - \frac{1}{\mathfrak{a}[{}^{(k)}, u^{(k)}](x)} \int_0^x b(s) \exp\left(-\int_s^x \frac{\mathfrak{b}[{}^{(k)}, u^{(k)}]}{\mathfrak{a}[{}^{(k)}, u^{(k)}]}(t)\,dt\right) ds$$

$$= {}^{(k)}(x) - \underline{d}(x),$$

where the value at the left hand boundary point can be computed by means of l'Hospital's rule as (skipping the argument $[{}^{(k)}, u^{(k)}]$ for better readability)

$$\lim_{x\to 0} \underline{d}(x) = \lim_{x\to 0} \frac{\varphi(x)}{\mathfrak{a}(x)} = \lim_{x\to 0} \frac{\varphi'(x)}{\mathfrak{a}'(x)} = \lim_{x\to 0} \frac{b(x) - \frac{\mathfrak{b}}{\mathfrak{a}}(x)\varphi(x)}{\mathfrak{a}'(x)} = \lim_{x\to 0} \frac{b(x) - \mathfrak{b}(x)\underline{d}(x)}{\mathfrak{a}'(x)},$$

hence

$$\lim_{x\to 0} \underline{d}(x) = \lim_{x\to 0} \frac{1}{1 + \frac{\mathfrak{b}(x)}{\mathfrak{a}'(x)}} \frac{b(x)}{\mathfrak{a}'(x)} = \lim_{x\to 0} \frac{b(x)}{\mathfrak{a}'(x) + \mathfrak{b}(x)}$$

$$= \frac{\bar{z}_y(0, {}^{(k)}(0)) + \gamma(0)\bar{z}(0, {}^{(k)}(0))}{u_{xx}^{(k)}(0, {}^{(k)}(0)) + ({}^{(k)}\gamma(0) + \gamma(0)^2) u^{(k)}(0, {}^{(k)}(0))}.$$

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Department of Mathematics, Alpen-Adria-Universität Klagenfurt, Austria
Email address: barbara.kaltenbacher@aau.at

Department of Mathematics, Texas A&M University, Texas 77843
Email address: rundell@tamu.edu