THE NONLOCAL STEFAN PROBLEM VIA A MARTINGALE TRANSPORT

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Abstract. We study the nonlocal Stefan problem, where the phase transition is described by a nonlocal diffusion as well as the change of enthalpy functions. By using a stochastic optimization approach introduced in [37], we construct global-time weak solutions and give a probabilistic interpretation for the solutions. An important ingredient in our analysis is a probabilistic interpretation of the enthalpy and temperature variables in terms of a particle system. Our approach in particular establishes the connection between the parabolic obstacle problem and the Stefan Problem for the nonlocal diffusions. For the melting problem, we show that our temperature-based solution coincides with the enthalpy-based ones studied in [3, 20, 19, 21], and obtain a new exponential convergence result.

1. Introduction

The one-phase Stefan problem is a phase transition model between liquid and solid that describes the melting of ice or the freezing of supercooled water [49, 41]. In this model the phase transition is described by a diffusion in the liquid phase and a change in the latent heat between two phases. Both effects are recorded in the *enthalpy* variable h. For the freezing problem, it can be written as

$$(St_1) \qquad \partial_t h + (-\Delta)^s \eta = 0, \quad h = \eta - 1 \text{ on } \{\eta > 0\},$$

where η represents the negative temperature of the fluid and $s \in (0,1]$ is fixed.

The melting problem can be written as

$$(St_2) \qquad \partial_t h + (-\Delta)^s \eta = 0, \quad h = \eta + 1 \text{ on } \{\eta > 0\}.$$

where η now represents the temperature of the fluid.

While the classical models consider the local diffusion $s = 1$, we allow the diffusion to be either local or nonlocal by setting $s \in (0,1]$. This extension is natural from the viewpoint of applications (for instance see [40, 54, 55]).

The nonlinear dynamics in the Stefan problem occurs at the interface $\Gamma = \partial\{\eta > 0\}$. Below we will regard the set $\{\eta > 0\}$ as the water/liquid region and $\{\eta = 0\}$ as the ice/solid region. We will see that the set $\{\eta = 0\}$ increases in time for (St_1), and decreases in time for (St_2). The challenge in studying either type of Stefan problem lies in the unknown regularity of both Γ and the enthalpy variable h across Γ.

While formally (St_1) and (St_2) fully describe the evolution of h and η, it is challenging to describe the evolution of the problem as a system, especially in the nonlocal case where h does not stay constant in the zero set of η; the reason being that the underlying particles could jump. The difficulty lies in the description of h on the interface $\partial\{\eta > 0\}$. For the nonlocal case, any type of regularity properties of Γ is unknown for $0 < s < 1$ for either (St_1) or (St_2), and likely Γ may be highly irregular due to the nonlocal diffusion present in the equation. For the local case, (St_1) is well-known to have non-unique evolution with jump-discontinuities [51, 15], though recently a physically natural notion of minimal-jump solutions was shown to generate unique solutions in one space dimension [22]. It is likely that the solutions have very little regularity for the interface [18, 42]. On the other hand, the local version of (St_2) is well-known to generate stable weak solutions with regularizing properties, extensively studied in the literature (see e.g. [2, 17, 29, 35, 41]). For the nonlocal case, most available result regarding (St_2) focus on well-posedness result for the enthalpy-version of the problem (see (St_h) below): see [3, 4, 19, 20, 21]. Later our analysis will in particular show that our temperature-based solution of (St_2) is equivalent to the enthalpy-based solution of (St_h), when $(h-\eta)(0,x)\in\{0,1\}$ (see Theorem 1.2 (c)). We also point out that our notion of weak solutions for (St_h) is the same as those studied in the aforementioned references.

Our goal in this paper is to characterize the enthalpy variable in both (St_1) and (St_2) with a particle dynamics approach, aiming for a better understanding of enthalpy behavior across the free boundary, in particular in the event of an irregular interface. More precisely, our main result is construction of global-in-time solutions of (St_1) and (St_2) based on the temperature variable η (see Theorems 1.2 and 1.3). As we will explain, this is a much more delicate task than the local case $s=1$, where h is constant in the zero set of η. For (St_2) our solutions are also the enthalpy-based weak solutions in the literature (Theorem 1.2), yet they come with stronger properties. For instance, using the new characterization of enthalpy for (St_2) and a variational approach to the associated particle dynamics, we show that the time integral of the temperature variable η in (St_2) solves a parabolic nonlocal obstacle problem (see (1.5) in Theorem 1.2). This connection is established for the first time in the nonlocal case $0<s<1$ (for the local case this is a classical and very useful fact, see [25, 38]). Our approach also yields a new quantitative convergence result for the enthalpy variable (Theorem 1.2 (d)), answering one of the open questions addressed in [21].

To achieve our result, we use a particle dynamics interpretation, which was considered in [34] and adapted by [37] to study the local Stefan problem. We will show that both (St_1) and (St_2) can be written in terms of the temperature variable as

$$\partial_t\eta+(-\Delta)^s\eta=-\rho,\quad \rho([0,\infty),\cdot)=\nu(\cdot). \tag{St}$$

Here, while η denotes the temperature or the distribution of active heat particles as in the aforementioned problems, ρ is a space-time measure that represents the distribution of *stopped particles* at time t, as the particles change their state from mobile to immobile: see Appendix A for the definition of $\rho([0,t),x)$, which can be formally considered as $\int_0^t\rho(s,x)ds$. The measure ν corresponds to the spatial capacity of the phase transition, or the final accumulated distribution of stopped particles, which represents the final outcome of the phase transition. The particles follow stochastic process X_t corresponding to the fractional Laplacian $(-\Delta)^s$, called $2s$-stable Lévy process, where particles can jump; see Definition 2.1. Extending the analysis of [37], we will rigorously justify this characterization of ρ and ν, using the particle dynamics based on an optimization problem we describe below; see (1.3) and Theorem 1.1.

To formulate (St_1) and (St_2) in terms of the variables in (St), we will verify that the enthalpy is given by the following formula:

$$h(t,x):=\begin{cases}\eta(t,x)+\rho([0,t),x)-\nu(x) & \text{for } (St_1),\\ \eta(t,x)+\rho([0,t),x) & \text{for } (St_2).\end{cases} \tag{1.1}$$

For (St_2), h denotes the distribution of total, active and stopped, heat particles. For (St_1), h also reflects the enthalpy loss in the active (liquid) phase. This formula is motivated by the fact that the stopped distribution ρ in (St) occurs where the moving particles of η are stopped by a space-time barrier that has a certain time-monotonicity (corresponding to (St_1) or (St_2)). As we will see in Section 6 such a barrier as well as the distribution ν will be uniquely determined by η, so that every term in the formula (1.1) is determined by η.

Unlike the local case of the Stefan problem, in the nonlocal case the particles jump, which generates a significant difference in the relation between η and h. In particular, while $h=$

$\eta \pm \chi_{\{\eta>0\}}$ for the local case $s=1$, for the nonlocal case $0<s<1$ the variables h and η have less explicit relation due to nonlocal effects. Our idea is to study properties of the rather singular measure ρ to show that h given by the formula (1.1) paired with η indeed solves (St_1)-(St_2). This is a delicate task we carry out in Section 5, with a careful analysis on the properties of ρ. We will discuss more about the differences of the local and non-local case in Section 1.1.

While there are parallel parts in our analysis of the optimization problem compared to the local case $s=1$ in [37], there are significant challenges that are new for the nonlocal case $0<s<1$. A fundamental difference lies in the associated stochastic process: while Brownian motion for $s=1$ has continuous sample paths, the symmetric $2s$-stable Lévy process for $0<s<1$ is a pure jump process with arbitrarily large jump sizes. As a consequence, our enthalpy function is no longer constant in the solid phase. This makes our particle-based definition of enthalpy more useful even for the stable melting problem (St_2). The jumps also bring interesting new challenges already in early parts of the analysis, as we will discuss in the next subsection: for instance the stopped particles ρ are no longer concentrated on Γ (see Figure 1). This leads to the lack of compactness for the existence property of our optimization problem $\mathcal{P}_f(\mu)$ (see Section 1.1 below for the definition), which we overcome with the idea of *compact active region*, see the discussion in Section 2. Combining this compactness with the results in [33] leads to the well-posedness of $\mathcal{P}_f(\mu)$.

Let us mention that our methods of constructing solutions to (St) do not rely on special properties of the $2s$-stable processes. The key properties of the stable process we exploit are that its sample path is càdlàg, i.e. right continuous with left limits, the infinitesimal generator is uniformly elliptic, and that the process is transient (see Assumption 1). In particular, we expect that our methods can be generalized to other various types of non-local operators and even mixed local and non-local diffusion, e.g. $(-\Delta)^s - \Delta$. It can also be easily extended to domains in Riemanninan manifolds.

1.1. **Summary of main ideas and results.** Throughout the paper, we denote by $(X_t)_{t\geq 0}$ the isotropic symmetric $2s$-stable Lévy process (see Definition 2.1 for a precise definition). In what is below much of the terminology and notions are from [37]. For the optimization problem to be stated below, we consider costs of the form

$$\mathcal{C}(\tau) := \mathbb{E}\left[\int_0^\tau L(s, X_s)ds\right], \tag{1.2}$$

where τ denotes a randomized stopping time (see Definition 2.2 for details) and $L(t,x)$ is a non-negative bounded function that is strictly monotone in time. We say that L is *of Type (I) (resp. Type (II))* if L is strictly increasing (resp. strictly decreasing) in time.

For a non-negative function $f \in L^\infty(\mathbb{R}^d)$ and a non-negative, bounded, compactly supported, and absolutely continuous measure μ on $\mathbb{R}^d$, we consider the following optimization problem introduced in [37]:

$$\mathcal{P}_f(\mu) := \inf_{(\tau,\nu)} \{\mathcal{C}(\tau) : X_0 \sim \mu,\ X_\tau \sim \nu \text{ and } \nu(\cdot) \leq f(\cdot)\}, \tag{1.3}$$

where $\nu \ll \text{Leb}$ and $\nu(\cdot)$ represents its Lebesgue density, and ν has an active compact region (see Definition 3.1). This is a free-target variant of the more classical problem, called the optimal Skorokhod problem in the literature, given as

$$\mathcal{P}_0(\mu,\nu) := \inf_{\tau \in \mathcal{T}_r(\mu,\nu)} \mathcal{C}(\tau), \tag{1.4}$$

where $\mathcal{T}_r(\mu,\nu)$ is the set of randomized stopping times τ that satisfies $X_0 \sim \mu$ and $X_\tau \sim \nu$, which stays less than or equal to $\tau_r := \inf\{t \geq 0 : X_t \notin B_r(0)\}$. Here, the randomized stopping time is a relaxed version of stopping time by introducing an external randomization, which will be precisely defined in Definition 2.2 later. The notion of a randomized stopping time has appeared in the literature [6, 47, 7].

Let us briefly describe the literature in the local case ($s = 1$), i.e. $(X_t)_{t\geq 0}$ is the Brownian motion. In this case, it was shown that the optimal stopping time of (1.4) can be determined by solving the local parabolic obstacle problem (see [30] for $d = 1$, and [34, 26] for $d > 1$). For more general processes, we refer to [33, 31, 48]. The connection between the free target problem $\mathcal{P}_f(\mu)$ and the Stefan problems has been established in [37] for the local case $s = 1$.

We extend the analysis of [37, 34] for the case $s = 1$ to construct solutions of the *nonlocal* Stefan problems, i.e. $0 < s < 1$, using the free target problem $\mathcal{P}_f(\mu)$. Below we will illustrate the similarity and novelty of the results in the nonlocal case.

As mentioned above, the presence of unbounded jumps in the stable processes has made the study of its Skorohord embedding problem $\mathcal{P}_0(\mu, \nu)$ in (1.4) more challenging compared to that of the Brownian motion. For instance, conditions under which one can find a stopping time τ with $\mathbb{E}[\tau] < \infty$ such that $X_0 \sim \mu$ and $X_\tau \sim \nu$ is well known for Brownian motion (see for instance [43] or [32, Theorem 1.5]). In the case of one dimensional Lévy processes, necessary and sufficient conditions for $\mathbb{E}[\tau] < \infty$ were derived in the recent paper [24]. On the other hand, by using potential theory, [31] established a connection for general Markov processes between the Skorohord embedding problem and the fractional parabolic obstacle problem for Type (I), though a similar analysis for Type (II) appears to be much more challenging.

Under a series of compactness assumptions, the Skorohord embedding problem $\mathcal{P}_0(\mu, \nu)$ was considered for a class of general processes in [33], using a dual problem approach. In particular, by restricting ourselves to target measures ν that have a *compact active region*, we are able to apply their results to characterize the optimal stopping time as a hitting time to a certain space-time barrier R (with some potential randomization at the initial time for Type (II)). We further proceed to show that the barrier is closed, with the aid of potential theory and elliptic regularity.

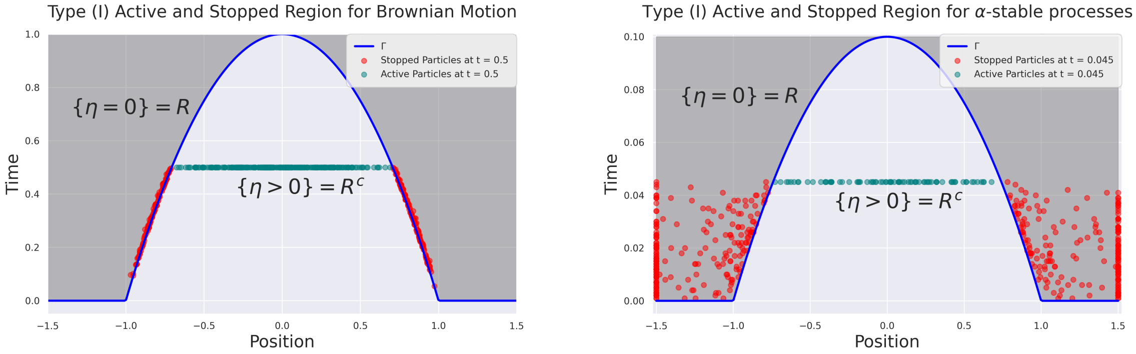
We first state the assumptions required for our results. Throughout this paper, we assume that Lagrangian L in the cost function, f, and s satisfy certain conditions (see Assumptions 1, 2, and 3 for details), and the initial measure μ is given by $\mu_0 dx$, where $\mu_0 \in L^\infty(\mathbb{R}^d)$ has a non-empty and compact support. Also with a slight abuse of notation, we interchangeably refer to a measure and its corresponding Lebesgue density. With these assumptions, we were able to obtain the following result, a nonlocal extension of the similar result in [37].

Theorem 1.1 (Theorem 3.11, Theorem 6.2, Theorem 6.6, and Theorem 6.7)**.** *Once it has an admissible solution, the variational problem $\mathcal{P}_f(\mu)$ in (1.3) has a unique optimal stopping time τ and an optimal target measure ν with bounded density. Moreover,*

(a) τ is the hitting time to a certain space-time barrier R (with a possible randomization on $\{\tau = 0\}$ for Type (II)).
(b) R is closed after a modification on a set of space-time measure zero.
(c) ν is independent of the choice or type of the cost.
(d) $\nu = f$ in the set $\{x \in \mathbb{R}^d : (t, x) \notin R$ for some $t > 0\}$.
(e) $\nu = \mu - (-\Delta)^s u$, where u solves the elliptic obstacle problem (6.5).

A key step for proving (b) in Theorem 1.1 is to characterize the set R as a space-time barrier generated by the potentials, which is a coincidence set of the parabolic obstacle problem for both Type (I) and Type (II), similar to the strategy taken by [37, 31].

This characterization of the barrier set would provide a connection between Stefan problems and parabolic obstacle problems as we will see below. In particular, the relation between parabolic obstacle problems for Type (II) and (1.4) appears to be new for $2s$-stable processes. For



(a) Brownian motion (b) α-stable process

FIGURE 1. Comparison of the Brownian motion (a) and α-stable process (b). Each dot represents $(X_{t\wedge\tau}, t\wedge\tau)$, where τ is the first entry time to the ice region R, and particles are normalized to the maximum norm of 1.5. Observe that ρ is supported on $\Gamma = \partial R$ for the Brownian motion, whereas ρ is supported on R for the α-stable processes.

Type (I), (b) still provides an improvement for the result of [31] where the barrier is shown to be only *finely* closed. The saturation property (d) is important: it will be used along with a characteristic function f to derive a solution of the Stefan problem.

To proceed with the PDE formulation of the problem, we use the *Eulerian variables* associated with initial data μ and a stopping time τ, introduced in [34]. Namely we denote η and ρ to be the distribution of respectively active particles and stopped particles associated to (μ, τ) (see Section 3.2, in particular Lemma 3.7, for the full definition). Let us mention that, in terms of the barrier set R in Theorem 1.1, $\{\eta > 0\}$ coincides with R^c (see Figure 1 and Lemma 4.11).

Let us first discuss our result for (St_2). As mentioned above, this problem has well-posed weak solutions for the enthalpy-based form of the equation:

$$(St_h) \qquad \partial_t h + (-\Delta)^s (h-1)_+ = 0.$$

For the notion of our weak solutions, we refer to Definition 5.3 - 5.4 for (St_2) and to Definition 5.14 for (St_h). We will show that our unique weak solution η for the temperature-based equation (St_2) generates the unique weak solution h of (St_h). Moreover, thanks to the probabilistic interpretation of h (Theorem 5.15), we provide a new exponential convergence result for (St_h).

Theorem 1.2 (Melting Stefan problem (St_2): Theorem 7.1, Theorem 7.2, and Theorem 5.15)**.** *Suppose that $\mu > 1$ on its support. Let (ν, τ) be the pair of optimal target measure and optimal stopping time for $\mathcal{P}_f(\mu)$ with $f \equiv 1$ and with a Type (II) cost. Let (η, ρ) be the Eulerian variables associated to (μ, τ), and define $h(t,x) := \eta(t,x) + \rho([0,t),x)$. Then the following statements hold:*

- *(a) The pair (h, η) is the unique weak solution to (St_2) with initial data $h(0,x) = \mu(x)$ and $\eta(0,x) = (\mu(x)-1)_+$, with the property that the set $\cap_{t>0}\{\eta(t,\cdot) > 0\}$ coincides with the support of μ.*
- *(b) $w(t,x) := \int_0^t \eta(s,x)ds$ is the unique weak solution to the parabolic obstacle problem*

$$(1.5) \qquad \min\{\partial_t w + (-\Delta)^s w + 1 - \mu, w\} = 0 \text{ in } (0,\infty)\times\mathbb{R}^d \text{ with } w(0,\cdot) = 0.$$

- *(c) h is the unique weak solution to (St_h) with initial data μ.*
- *(d) There exist constants $\gamma, C > 0$ depending only on μ, d and s such that for any $t \geq 0$,*

$$||h(t,\cdot) - \nu||_{L^1(\mathbb{R}^d)} \leq Ce^{-\gamma t}.$$

The key challenge that we face in the nonlocal case is to track the distribution of ρ, as mentioned earlier. See Section 5 for more discussion on this and a comparison to the local case. The connection to the parabolic obstacle problem (Theorem 7.1) is new for $0 < s < 1$: we hope that the regularity analysis of (1.5) provides a better understanding on (St_2), see Section 7 for the further discussion on this.

Note that our choice of initial data in Theorem 1.2 covers the general initial data considered in the literature with no initial *mushy region*, namely $\{x \in \mathbb{R}^d : 0 < (h-\eta)(0,x) < 1\} = \emptyset$. This restriction comes from our approach where we view h as a function of η and ρ, which does not account the initial mushy region.

For Type (I) cost, while we can address the general initial data μ, the initial domain (or the *initial trace*) $E := \bigcup_{t>0}\{\eta(t,\cdot) > 0\}$ cannot be arbitrarily chosen. Indeed we show that for a given initial data μ and a set G that contains $\{x \in \mathbb{R}^d : 0 < \mu(x) \leq 1\}$, there is a unique solution η of (St_1) that generates E and it matches G as its *insulated region*, namely

$$\Sigma(\eta) := \bigcap_{t>0}\{\eta(t,\cdot) > 0\} = G.$$

We will see that E is always larger than the support of μ, and strictly larger in most cases. Hence our solutions go through an instant expansion of its active region at initial time, from which the freezing process starts.

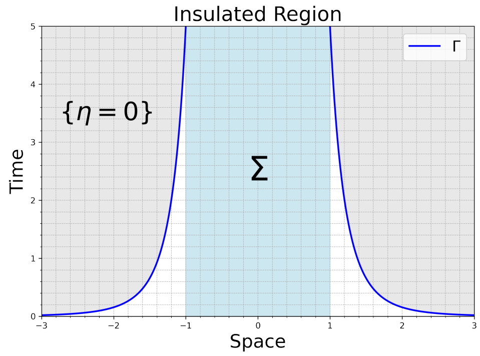


FIGURE 2. An illustration of the insulated region.

We now state our well-posedness result for the supercooled nonlocal Stefan problem (see Definition 5.3 -5.4 for the notion of our weak solutions (St_1)):

Theorem 1.3 (Freezing Stefan Problem (St_1): Theorem 7.5, Theorem 7.7, Theorem 7.8, and Corollary 7.9)**.** *Let G be a bounded open set in $\mathbb{R}^d$ that contains $\{x \in \mathbb{R}^d : 0 < \mu(x) \leq 1\}$. Let (η, ρ) be as in Theorem 1.2, but with a Type (I) cost and with $f := 1 - \chi_G$. Then the following statements hold.*

(a) The pair (h, η) is the unique weak solution of (St_1) satisfying $\eta(0,\cdot) = \mu$ and $\Sigma(\eta) = G$.

(b) The initial domain $E := \cup_{t>0}\{\eta(t,\cdot) > 0\}$ is given as a positivity set of the solution to the elliptic obstacle problem (6.5) with $f = 1 - \chi_G$. Also, E contains the support of μ.

(c) $w(t,x) := \int_t^\infty \eta(s,x)ds$ solves the parabolic obstacle problem

$$\min\{\partial_t w + (-\Delta)^s w + \nu, w\} = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^d \text{ with } w(0,\cdot) = (-\Delta)^{-s}(\mu - \nu),$$

where ν is the optimal target measure for $\mathcal{P}_f(\mu)$ with $f = 1 - \chi_G$.

(d) In particular, if μ is greater than 1 on its support, then one can choose G to be an empty set and $f \equiv 1$, for which (h, η) is the unique weak solution of (St_1) with initial data μ that vanishes in a finite time.

It is worth pointing out that G can be chosen without any topological assumption, whereas in the local case G must be simply connected. This is because, due to the nonlocal diffusion, particles can jump and stop anywhere outside of the active region.

While we do not pursue showing further stability of this class of solutions, we expect parallel results hold as in the local problem [37], where stability and comparison principle holds for the corresponding solutions with respect to perturbation of μ and Σ. Lastly we point out that, after completion of our manuscript, a global existence result for general initial measures $0 \leq \mu < 1$ and initial domains is recently achieved for the local Stefan problem [18]. Given this, it is likely that the corresponding result holds for our problem, but we do not explore it further here.

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2. Preliminaries

Let us start with reviewing preliminary results that we will use in this paper. We first define α-stable Lévy processes (see [1] for more details about Lévy processes).

Definition 2.1 (α-stable Lévy process)**.** *A continuous–time process $X = (X_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ is called a* Lévy process *if the following holds:*

- *Its sample paths are right-continuous and have left limits at every time.*
- *It has stationary and independent increments.*

*For $0 < \alpha < 2$, the Lévy process $X = (X_t)_{t\geq 0}$ with initial value $X_0 = 0$ is called α-*stable process *if it satisfies the self-similarity property*

$$X_t \sim t^{1/\alpha} X_1, \quad \forall t > 0.$$

(we say that $X \sim Y$ if X and Y have the same law). Furthermore, the α-stable process is called symmetric *if $X_t \sim -X_t$ for any $t > 0$.*

If $X = (X_t)_{t\geq 0}$ is a symmetric α-stable Lévy process, then there exists a constant $c > 0$ such that for any $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{i\theta X_t}] = e^{-ct|\theta|^\alpha}, \quad \forall t > 0. \tag{2.1}$$

We say that $X = (X_t)_{t\geq 0}$ is an *isotropic* symmetric α-stable Lévy process if $c = 1$. Notice that Brownian motion corresponds to the case $\alpha = 2$ and $c = \frac{1}{2}$.

Throughout the paper, we set the parameter $s \in (0,1)$ as $s := \alpha/2$. We work with the isotropic symmetric $2s$-stable Lévy process (from now on, we abbreviate it to the $2s$-stable process) $X = (X_t)_{t\geq 0}$, where s satisfies

$$\begin{cases} s \in (0, \frac{1}{2}) & \text{if } d = 1, \\ s \in (0, 1) & \text{else.} \end{cases} \tag{2.2}$$

The above assumption is due to the fact that X_t is a recurrent process if $d = 1$ and $s \geq 1/2$ (see [8]). We restrict ourselves to the regime where X_t is transient because we will use the associated potential (see Section 4) to analyze X_t and transisence is a typical assumption when one works with potentials [31, Remark 3.9].

Now we assume the same assumptions as in [33] about our probability space. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space which supports the $2s$-stable process $X = (X_t)_{t\geq 0}$ in $\mathbb{R}^d$. We assume that Ω is a Polish space and $X : \Omega \to D(\mathbb{R}^+; \mathbb{R}^d)$ (here we set $\mathbb{R}^+ := [0, \infty)$) is a continuous map onto the Skorokhod space of càdlàg paths (i.e. $\{X_t(\omega)\}_{t\in\mathbb{R}^+}$ is right continuous with left limits) with respect to the Skorokhod path space metric. We also assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and complete such that the process X is adapted to the filtration.

For our further connections with a nonlocal Stefan problem, we allow the law of the initial state X_0 to be any finite Radon measure on $\mathbb{R}^d$. First for any $x \in \mathbb{R}^d$, we denote by $(X_t^x)_{t\geq 0}$ the process initially starting from x (i.e. translation of the $2s$-stable process by x). Now for any finite Radon measure μ on $\mathbb{R}^d$, $X_0 \sim \mu$ means that the law of the process $(X_t)_{t\geq 0}$ is given by $\int \mathbb{P}((X_t^x)_{t\geq 0} \in \cdot) d\mu(x)$.

A random variable $\tau : \Omega \to \mathbb{R}^+$ is called a *stopping time* if $\tau^{-1}([0, t]) \in \mathcal{F}_t$ for any $t \geq 0$. Stopping times can be regarded as a rule that prescribes when each particle of the process stops moving. Also the particles are determined to stop based only on the information in the present or past.

It turns out that a relaxed notion of the stopping time, called a *randomized stopping time*, is useful to analyze the Skorokhod's embedding problem. Randomized stopping time assigns a probability distribution to each particle on when the particle should stop moving. Let $\mathcal{M}_1(\mathbb{R}^+)$ be the collection of probability measures on $\mathbb{R}^+$.

Definition 2.2 (Randomized stopping time)**.** *(See [6, 47, 7].) A randomized stopping time $\tau : \Omega \to \mathcal{M}_1(\mathbb{R}^+)$ is a measure-valued random variable such that for any $\omega \in \Omega$, $\tau_\omega := \tau(\omega)$ is a probability measure on $\mathbb{R}^+$ and for each $t \geq 0$,*

$$\omega \in \Omega \mapsto \tau_\omega([0, t]) \text{ is } \mathcal{F}_t\text{-measurable.}$$

We let $\mathcal{S}$ denote the collection of randomized stopping times.

For a randomized stopping time τ and $\varphi : \mathbb{R}^+ \to \mathbb{R}$, we interpret

$$\mathbb{E}[\varphi(\tau)] = \mathbb{E}\left[\int_{\mathbb{R}^+} \varphi(t)\tau(dt)\right].$$

2.1. **Fractional Subharmonic Ordering.** For $s \in (0, 1)$, the fractional Laplacian $(-\Delta)^s$ is a non-local generalization of the standard Laplacian. For Schwartz functions ψ of rapid decay,

$$(-\Delta)^s\psi(x) := C_{d,s}\text{P.V.} \int_{\mathbb{R}^d} \frac{\psi(x) - \psi(y)}{|x - y|^{d+2s}} dy = C_{d,s} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} \frac{\psi(x) - \psi(y)}{|x - y|^{d+2s}} dy, \tag{2.3}$$

where $|\cdot|$ denotes the ℓ^2-norm and the normalizing constant $C_{d,s}$ is chosen so that $(-\Delta)^s$ is a Fourier multiplier by $|\xi|^{2s}$ (see [23, Proposition 3.3] for details). We state the integration by parts formula for the fractional Laplacian: for Schwartz functions ψ and φ,

$$\int_{\mathbb{R}^d} \varphi \cdot (-\Delta)^s\psi = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}\varphi \cdot (-\Delta)^{\frac{s}{2}}\psi = \int_{\mathbb{R}^d} \psi \cdot (-\Delta)^s\varphi,$$

which can be easily deduced with an application of Plancherel's theorem.

The (negation of) fractional Laplacian $-(-\Delta)^s$ is the infinitesimal generator of the $2s$-stable process $X = (X_t)_{t\geq 0}$ (see [1]):

$$-(-\Delta)^s\psi(x) = \lim_{t\to 0} \frac{\mathbb{E}[\psi(X_t^x)] - \psi(x)}{t}.$$

Next, we define s-subharmonic functions. We denote by $C_b^2(\mathbb{R}^d)$ the collection of functions $f \in C^2(\mathbb{R}^d)$ such that $f, \nabla f, \nabla^2 f$ are all bounded.

Definition 2.3 (Fractional subharmonic)**.** *A function* $\varphi \in C_b^2(\mathbb{R}^d)$ *is called a* s-fractional subharmonic *function if* $(-\Delta)^s\varphi \leq 0$ *on* $\mathbb{R}^d$.

In addition, we define the ordering of measures with respect to the fractional Laplacian $(-\Delta)^s$. This notion of ordering is crucially related to the existence of a randomized stopping time in the Skorokhod embedding problem.

Definition 2.4 (Fractional Subharmonic Ordering)**.** *Two finite Radon measures* μ *and* ν *on* $\mathbb{R}^d$ *are in* s-fractional subharmonic ordering *if for any* s*-fractional subharmonic function* $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \varphi(x)\nu(dx).$$

We denote this relationship as $\mu \leq_{s\text{-SH}} \nu$. *Note that this in particular implies* $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$.

It is known that the fractional subharmonic ordering ensures the existence of a randomized stopping time (see [31, Theorem 4.3] and [48, Theorems 1 and 3]).

Proposition 2.5 (Theorems 1 and 3 in [48])**.** *Assume that* μ *and* ν *are finite Radon measures on* $\mathbb{R}^d$ *that satisfy* $\mu \leq_{s\text{-SH}} \nu$. *Then, there exists a randomized stopping time* τ *such that* $X_0 \sim \mu$ *and* $X_\tau \sim \nu$. *In addition,* τ *can be chosen so that it has a "minimal residual expectation". That is, if* σ *is any randomized stopping time such that* $X_0 \sim \mu$ *and* $X_\sigma \sim \nu$, *then*

$$\mathbb{E}[F(\tau)] \leq \mathbb{E}[F(\sigma)]$$

for any non-decreasing convex function $F : \mathbb{R}^+ \to \mathbb{R}$.

Conversely, the existence of a randomized stopping time τ with $\mathbb{E}[\tau] < \infty$ satisfying $X_0 \sim \mu$ and $X_\tau \sim \nu$ implies $\mu \leq_{s\text{-SH}} \nu$. To see this, by Itô's formula (see Proposition 2.6 below), for any s-fractional subharmonic function $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \varphi(x)\nu(dx).$$

This indeed implies that $\mu \leq_{s\text{-SH}} \nu$.

Proposition 2.6 (Itô's Formula)**.** *For any* $\varphi \in C_b^2(\mathbb{R}^d)$ *and a randomized stopping time* τ *with* $\mathbb{E}[\tau] < \infty$,

$$\mathbb{E}[\varphi(X_\tau)] = \mathbb{E}[\varphi(X_0)] + \mathbb{E}\left[\int_0^\tau -(-\Delta)^s\varphi(X_s)ds\right].$$

Remark 2.7. *We remark that the above Itô's formula is more commonly referred to as Dynkin's formula in the literature and that it holds for more general Feller processes (see* [39, Exercise 6.29] *or* [36, Lemma 19.21]*).*

3. OPTIMAL SKOROKHOD PROBLEM WITH COMPACT ACTIVE REGION

In this section, we introduce precise settings for the variational problem (1.4), where one of the notable new components lies in considering the particular class of stopping times τ. In the nonlocal case $0 < s < 1$, the randomized stopping time τ from Proposition 2.5 could satisfy $\mathbb{E}[\tau] = \infty$. We refer to [24] for discussions of necessary and sufficient conditions for the integrability of τ in the particular case $d = 1$. As we will see in Section 3.2, we need a stronger condition $\mathbb{E}[\exp(\gamma\tau)] < \infty$ ($\gamma > 0$ is some constant) to obtain enough compactness in the time variable of the space-time process $(t \wedge \tau, X_{t\wedge\tau})$.

These observations motivate us to introduce the following class of stopping times:

Definition 3.1 (Compact Active Region)**.** *For a randomized stopping time τ, we say that a pair (X, τ), which induces a process $(X_{t\wedge\tau})_{t\geq 0}$, has a* compact active region *if for some $r > 0$,*

$$\tau \leq \tau_r := \inf\{t \geq 0 : X_t \notin B_r(0)\}, \tag{3.1}$$

where $B_r(0) := \{x \in \mathbb{R}^d : |x| < r\}$. We interchangeably say that ν has a compact active region if there exists $r > 0$ and a randomized stopping time $\tau \leq \tau_r$ such that $X_\tau \sim \nu$. For such r, we say that (X, τ) or ν has a compact active region in $B_r(0)$.

Intuitively speaking, $\tau \leq \tau_r$ implies that particles are only allowed to actively move from inside $B_r(0)$, and even though they are allowed to jump to outside of the ball they must stop immediately once they arrived outside $B_r(0)$.

Remark 3.2 (Exponential Moments of τ_r)**.** *The condition $\tau \leq \tau_r$ implies that τ not only has a finite expectation but also has finite exponential moments, as long as μ is a finite measure. Indeed, by* [10, page 82], *if $\lambda > 0$ denotes the principal eigenvalue of the Dirichlet problem*

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } B_r(0), \\ u = 0 & \text{on } B_r(0)^c, \end{cases} \tag{3.2}$$

then

$$\lambda = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\tau_r > t | X_0 = x), \quad \forall x \in B_r(0). \tag{3.3}$$

Also, $\mathbb{P}(\tau_r = 0 | X_0 = x) = 1$ if $x \notin B_r(0)$. Therefore, for any $\gamma \in (0, \lambda)$,

$$\mathbb{E}[\exp(\gamma\tau_r)] = \int_{\mathbb{R}^d} \mathbb{E}[\exp(\gamma\tau_r)|X_0 = x]d\mu(x) \leq \int_{\mathbb{R}^d} \mathbb{E}[\exp(\gamma\tau_r)|X_0 = 0]d\mu(x) < \infty.$$

We refer the reader to Appendix B.1 for more information about λ and its relation to a Poincare's inequality for the fractional Laplacian.

Remark 3.3 (Fractional Subharmonic Ordering in $B_r(0)$)**.** *For $\mu \leq_{s\text{-SH}} \nu$, if ν has a compact active region in $B_r(0)$, then for any $\varphi \in C_b^2(B_r(0)) \cap C_b(\mathbb{R}^d)$ such that $(-\Delta)^s\varphi \leq 0$ in $B_r(0)$,*

$$\int_{\mathbb{R}^d} \varphi(x)\mu(dx) \leq \int_{\mathbb{R}^d} \varphi(x)\nu(dx),$$

due to Itô's Formula. This is a stronger condition than a fractional sub-harmonic ordering in the sense of Definition 2.4, since fractional sub-harmonic functions on $B_r(0)$ are not necessarily fractional sub-harmonic on $\mathbb{R}^d$.

Next, we show that the target measure ν associated with stopping times having a compact active region possesses a nice decay property. In the local case, the compact active region condition is equivalent to the target measure being compactly supported. Let us first consider the following example, showing that the distribution of X_{τ_r} (see (3.1) for the definition of τ_r) is not compactly supported, though it obviously has a compact active region $B_r(0)$.

Example 3.4. *For $s \in (0, 1)$ satisfying the condition (2.2), let $r > 0$ and $X = (X_t^x)_{t\geq 0}$ be a $2s$-stable process with $X_0 \sim \delta_x$ and $|x| < r$. Define the exit time $\tau_r^x := \inf\{t \geq 0 : X_t^x \notin B_r(0)\}$. Then the distribution of $X_{\tau_r^x}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, whose density is given by (see* [9]*)*

$$\nu_r^x(y) = C_{r,s,d} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^s \frac{1}{|x - y|^d} \chi_{\{|y|>r\}}. \tag{3.4}$$

In addition, if $X_0 \sim \mu$, where μ is bounded and has a compact support contained in $B_{r/2}(0)$, then the density ν_r of X_{τ_r} is given by

$$\nu_r(y) = \int_{\mathbb{R}^d} \nu_r^x(y)d\mu(x) \leq C \frac{1}{(|y|^2 - r^2)^s} \frac{1}{(|y| - \frac{r}{2})^d} \chi_{\{|y|>r\}} \in L^1(\mathbb{R}^d). \tag{3.5}$$

This in particular implies

$$\nu_r(y) = O(|y|^{-2s-d}), \quad |y| \to \infty. \tag{3.6}$$

Now, we state a decay property on target measures having a compact active region in $B_r(0)$.

Lemma 3.5. *Let $X = (X_t)_{t\geq 0}$ be a $2s$-stable process and τ be a randomized stopping time which has a compact active region in $B_r(0)$ (i.e. $\tau \leq \tau_r$). Let $X_\tau \sim \nu$, then $\nu \leq \nu_r$ on $B_r(0)^c$, where ν_r denotes the distribution of X_{τ_r} from Example 3.4.*

Proof. Let $A \subset B_r(0)^c$ be any Borel set. Under the event $X_\tau \in A$, by the definition of τ_r, we have $\tau \geq \tau_r$, which together with the condition $\tau \leq \tau_r$ implies $\tau = \tau_r$. Hence, $\{\omega : X_\tau \in A\} \subset \{\omega : X_{\tau_r} \in A\}$, which implies the desired bound. □

From now on, we work with measures that are absolutely continuous with respect to the Lebesgue measure and we will abuse notation by using the same symbol for both the measure and density. For $M, R > 0$ and a finite Radon measure μ on $\mathbb{R}^d$, define

$$\mathcal{A}_{\mu,M,R} := \{\nu \in \mathcal{M}(\mathbb{R}^d) : \mu \leq_{s\text{-SH}} \nu,\ \nu \text{ has a compact active region in } B_R(0) \text{ and } \nu \leq M\} \tag{3.7}$$

($\nu \leq M$ means that ν has a density w.r.t. Lebesgue measure, which is bounded by M).

Lemma 3.6. *Let $M > 0$ and μ be a compactly supported measure on $\mathbb{R}^d$ such that $\mu \in L^1(\mathbb{R}^d)$. Then, $\mathcal{A}_{\mu,M,R} \neq \emptyset$ for large enough $R > 0$ (depending only on M, the support of μ, and the L^1 norm of μ). In particular, there exists $\nu \in \mathcal{A}_{\mu,M,R}$ such that ν is supported on $B_R(0)^c$ for large enough $R > 0$.*

In the case of Brownian motion starting from the origin, one can construct an element of $\mathcal{A}_{\mu,M,R}$ using the fact that δ_0 is in subharmonic order with the uniform probability measure on any annuli centered at 0. We extend this argument to the case of $2s$-stable processes.

Proof. First let us consider the case when $\mu = \delta_x$ for some $x \in \mathbb{R}^d$. Let $R > 0$ such that $|x| \leq R/2$. By Itô's formula, for any s-fractional subharmonic function $\varphi \in C_b^2(\mathbb{R}^d)$ and $r \geq R$,

$$\varphi(x) \leq \mathbb{E}[\varphi(X_{\tau_r^x})] = \int_{\mathbb{R}^d} \varphi(y)\nu_r^x(y)dy \tag{3.8}$$

(recall that ν_r^x is defined in (3.4)). By averaging in r over $[R, 2R]$,

$$\varphi(x) \leq \frac{1}{R}\int_{\mathbb{R}^d}\int_R^{2R} \varphi(y)\nu_r^x(y)drdy. \tag{3.9}$$

Hence, using the formula (3.4), setting

$$\tilde{\nu}_R^x(y) := \frac{1}{R}\int_R^{2R} \nu_r^x(y)dr = \frac{C_{s,d}}{R}\int_R^{2R}\left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^s \frac{1}{|x-y|^d}\chi_{\{|y|>r\}}dr, \tag{3.10}$$

we have $\delta_x \leq_{s\text{-SH}} \tilde{\nu}_R^x$.

We show that for any $\varepsilon > 0$, $\tilde{\nu}_R^x \leq \varepsilon$ for large enough $R > 0$. Let us divide into the cases $|y| \geq 4R$ and $R \leq |y| \leq 4R$ (by (3.10), $\tilde{\nu}^x(y) = 0$ for $|y| < R$). First when $|y| \geq 4R$, using the fact $|x-y|^d \geq ||y|-|x||^d \geq 3^d R^d$ and $|y|^2 - r^2 \geq 12R^2$ for $r \in [R, 2R]$, we see from (3.10) that

$$\sup_{|y|\geq 4R} \tilde{\nu}_R^x(y) \leq C\frac{1}{R}\int_R^{2R} \frac{R^{2s}}{R^{2s+d}}dr = CR^{-d}. \tag{3.11}$$

In addition when $R \leq |y| \leq 4R$, using $|x-y|^d \geq ||y|-|x||^d \geq R^d/2^d$,

$$\begin{aligned}\tilde{\nu}_R^x(y) &\leq \frac{C}{R^{d+1}} \int_R^{2R} \frac{R^{2s}}{(|y|^2-r^2)^s} \chi_{\{|y|>r\}} dr \leq \frac{C}{R^{d+1}} \int_R^{|y|} \frac{R^{2s}}{(|y|-r)^s(|y|+r)^s} dr \\ &\leq \frac{C}{R^{d+1-s}} \int_R^{|y|} \frac{1}{(|y|-r)^s} dr = \frac{C}{R^{d+1-s}} (|y|-R)^{1-s} \leq CR^{-d}. \end{aligned} \tag{3.12}$$

Therefore, by (3.11) and (3.12), we conclude the following:

$$\text{For any } \varepsilon > 0, \quad \tilde{\nu}_R^x(y) \leq \varepsilon \text{ for } |x| \leq R/2, \text{ if } R \text{ is sufficiently large.} \tag{3.13}$$

We now extend this to general compactly supported finite measures μ. Let $R > 0$ be such that $\operatorname{supp}(\mu) \subset B_{R/2}(0)$. Then, by integrating (3.9) with respect to μ, we deduce $\mu \leq_{s\text{-SH}} \tilde{\nu}_R$ with

$$\tilde{\nu}_R(y) := \int_{\mathbb{R}^d} \tilde{\nu}_R^x(y) d\mu(x) = \frac{1}{R} \int_R^{2R} \int_{\mathbb{R}^d} \nu_r^x(y) d\mu(x) dr. \tag{3.14}$$

From (3.13), for any $\varepsilon > 0$, we have $\tilde{\nu}_R(y) \leq \varepsilon ||\mu||_{L^1}$ for large enough $R > 0$. This yields $\tilde{\nu}_R \leq M$ by taking small $\varepsilon > 0$.

Finally, we show that $\tilde{\nu}_R$ has a compact active region in $B_{2R}(0)$. Indeed, let $\tilde{\tau}$ be a randomized stopping time such that

$$\tilde{\tau} \sim \tau_U, \tag{3.15}$$

where U denotes the uniform distribution on $[R, 2R]$ independent of the process. Then we have $X_{\tilde{\tau}} \sim \tilde{\nu}_R$, since for any Borel $A \subset \mathbb{R}^d$,

$$\mathbb{P}(X_{\tilde{\tau}} \in A) = \frac{1}{R} \int_R^{2R} \mathbb{P}(X_{\tilde{\tau}} \in A \mid U = r) dr = \frac{1}{R} \int_R^{2R} \nu_r[A] dr \overset{(3.14)}{=} \tilde{\nu}_R[A].$$

Since $\tilde{\tau} \leq \tau_{2R}$ by (3.15), we conclude the proof.

□

3.1. Assumptions.

In this section, we state our assumptions in the optimal Skorokhod's embedding problem (1.4).

Assumption 1. *We assume that $s \in (0,1)$ for $d \geq 2$ and $s \in (0,1/2)$ for $d = 1$. This condition ensures that the $2s$-stable process $X = (X_t)_{t \geq 0}$ is transient.*

In addition, the measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ satisfy

1. *The source measure is non-trivial i.e. $\mu(\mathbb{R}^d) > 0$.*
2. *$\mu \leq_{s\text{-SH}} \nu$.*
3. *μ and ν are absolutely continuous with respect to the Lebesgue measure, whose densities satisfy $\mu \in L^\infty(\mathbb{R}^d)$ and $\nu \in L^\infty(\mathbb{R}^d)$.*
4. *There exists $r > 0$ such that μ is supported in $B_r(0)$ and ν has a compact active region in $B_r(0)$.*

By Proposition 2.5, there exists a randomized stopping time τ such that $X_0 \sim \mu$ and $X_\tau \sim \nu$. The last condition above implies that such τ satisfying $\tau \leq \tau_r$ exists. Then by Remark 3.2, τ has finite exponential moments.

The following is the assumption on our cost functional.

Assumption 2. *The Lagrangian L satisfies*

1. *$L \in C_b(\mathbb{R}^+ \times \mathbb{R}^d)$ is non-negative with $\partial_t L \in C_b(\mathbb{R}^+ \times \mathbb{R}^d)$.*
2. *L is strictly monotone in time. We say that L is of* Type (I) *if $t \mapsto L(t,x)$ is strictly increasing for any x, and is of* Type (II) *if it is strictly decreasing for any x.*

Finally, we induce a constraint on the upper bound f in the variational problem $\mathcal{P}_f(\mu)$ to ensure that the optimization set of $\mathcal{P}_f(\mu)$ is non-empty:

Assumption 3. *We assume that* $f \in L^\infty(\mathbb{R}^d)$ *and there exists* $\delta > 0$ *and a compact set* $K \subseteq \mathbb{R}^d$ *such that* $f(x) \geq \delta$ *for* $x \notin K$*.*

By Lemma 3.6, the optimization set for the variational problem $\mathcal{P}_f(\mu)$ is non-empty under Assumption 3.

For the pair of measures (μ, ν) and the Lagrangian L satisfying Assumptions 1 and 2, the existence and uniqueness of the optimal stopping time for the variational problem (1.4) follows from [33].

3.2. **Eulerian Variables.** In this section, we introduce *Eulerian variables* (η, ρ) that correspond to the optimal Skorokhod problem (1.4). These play a central role in connecting our problem with the PDE formulation later. The local case $s = 1$ was introduced in [34]. While our analysis largely follow the local case, we present the full proof for the nonlocal case due to the subtle differences of function spaces caused by jumps in the process.

Let (μ, ν) be a pair satisfying Assumption 1 and $X = (X_t)_{t \geq 0}$ be the $2s$-stable process with $X_0 \sim \mu$. From now on, we regard $r > 0$ from Assumption 1 as a fixed quantity and use the abbreviated notation

$$B := B_r(0).$$

Let $\overline{B}$ be the (topological) closure of B. Define

$$\mathcal{T}_r(\mu, \nu) := \{\tau \in \mathcal{S} : X_0 \sim \mu, X_\tau \sim \nu \text{ and } \tau \leq \tau_r\} \tag{3.16}$$

(recall that $\mathcal{S}$ denotes the set of randomized stopping times defined in Definition 2.2, and τ_r denotes the exit time defined in (3.1)). Note that $X_\tau \sim \nu$ means that

$$\mathbb{E}[g(X_\tau)] = \int_\Omega \int_{\mathbb{R}^+} g(X_t(\omega)) \tau_\omega(dt) \mathbb{P}(d\omega) = \int_{\mathbb{R}^d} g(y) \nu(dy), \quad \forall g \in C_b(\mathbb{R}^d).$$

Observe that $\mathcal{T}_r(\mu, \nu)$ is a convex set.

Next, we define function spaces as in [34]. For $\gamma > 0$, define

$$C_{-\gamma}(\mathbb{R}^+ \times \overline{B}) := \{w \in C(\mathbb{R}^+ \times \overline{B}) : \sup_{x \in \overline{B}} e^{-\gamma t} |w|(t, x) \to 0 \text{ as } t \to \infty\},$$

with the norm

$$||w||_{C_{-\gamma}(\mathbb{R}^+ \times \overline{B})} := \sup_{(t,x) \in \mathbb{R}^+ \times \overline{B}} |e^{-\gamma t} w(t, x)|.$$

In addition, let $C^{1,2}_{-\gamma}(\mathbb{R}^+ \times \overline{B})$ be the set of functions in $C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$ whose first-order time derivative and second-order spatial derivative belong to $C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$.

We also define some notation regarding the set of measures. For $\gamma > 0$, let

$$\mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{B}) := \{\mu \in \mathcal{M}(\mathbb{R}^+ \times \overline{B}) : \lim_{t \to \infty} e^{\gamma t} |\mu|(\overline{B}) = 0\}.$$

One can see that the dual space of $C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$ is $\mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{B})$.

In addition, we define time-dependent weighted Sobolev spaces. Let

$$L^2_\gamma(\mathbb{R}^+; H^s_0(B)) := \Big\{u : u(t, x) = 0 \text{ for } x \notin B \text{ and } \int_{\mathbb{R}^+} e^{\gamma t} [u(t, \cdot)]^2_{H^s(\mathbb{R}^d)} dt < \infty\Big\},$$

where $[\cdot]_{H^s(\mathbb{R}^d)}$ denotes the Gagliardo seminorm (see Appendix B for more details about H^s). Lastly, define

$$\mathcal{X} := \{f \in L^2_{-\gamma}(\mathbb{R}^+; H^s_0(B)) : \partial_t f \in L^2_{-\gamma}(\mathbb{R}^+; H^{-s}(B))\},$$

where $H^{-s}(B)$ denotes the dual space of $H^s_0(B)$. The associated norm is

$$||f||^2_{\mathcal{X}} := \int_{\mathbb{R}^+} e^{-\gamma t} \left[||f(t,\cdot)||^2_{H^s_0(B)} + ||\partial_t f(t,\cdot)||^2_{H^{-s}(B)} \right]. \tag{3.17}$$

From now on we assume that $\gamma \in (0,\lambda)$, where λ is defined in Remark 3.2.

We now introduce the notion of *Eulerian Variables* (η,ρ).

Lemma 3.7 (Eulerian variables). *For any $\tau \in \mathcal{T}_r(\mu,\nu)$, there exist $\eta \in \mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{B})$ and $\rho \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that for any $\varphi \in C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$ and $\psi \in C_c(\mathbb{R}^+ \times \mathbb{R}^d)$,*

$$\begin{cases} \mathbb{E}\left[\int_0^\tau \varphi(s,X_s)ds\right] = \iint_{\mathbb{R}^+\times\overline{B}} \varphi(s,x)\eta(ds,dx), \\ \mathbb{E}[\psi(\tau,X_\tau)] = \iint_{\mathbb{R}^+\times\mathbb{R}^d} \psi(s,x)\rho(ds,dx). \end{cases} \tag{3.18}$$

Also, η is absolutely continuous with respect to the space-time Lebesgue measure.

In addition for any test function $w \in C_c^\infty(\mathbb{R}^+ \times B)$,

$$\iint_{\mathbb{R}^+\times B} w(t,x)\rho(dt,dx) = \int_B w(0,x)\mu(dx) + \iint_{\mathbb{R}^+\times B} (\partial_t w(t,x) - (-\Delta)^s w(t,x))\,\eta(dt,dx). \tag{3.19}$$

Hence if $\tau > 0$ almost surely, then (η,ρ) solves as distributions

$$\begin{cases} \partial_t \eta + \rho = -(-\Delta)^s \eta & (t,x) \in (0,\infty)\times B, \\ \eta(t,x) = 0 & (t,x) \in (0,\infty)\times B^c, \\ \eta(0,x) = \mu(x) & x \in \mathbb{R}^d. \end{cases} \tag{3.20}$$

Proof. Since $\tau \le \tau_r$, τ has finite exponential moments (see Remark 3.2). Hence, $\varphi \mapsto \mathbb{E}\left[\int_0^\tau \varphi(s,X_s)ds\right]$ is a continuous linear functional on $C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$. Thus, by Riesz representation, there exists $\eta \in \mathcal{M}_\gamma(\mathbb{R}^+ \times \overline{B})$ such that for any $\varphi \in C_{-\gamma}(\mathbb{R}^+ \times \overline{B})$,

$$\mathbb{E}\left[\int_0^\tau \varphi(t,X_t)ds\right] = \iint_{\mathbb{R}^+\times\overline{B}} \varphi(t,x)\eta(dt,dx).$$

As $\tau \le \tau_r$, η is supported on $\mathbb{R}^+ \times \overline{B}$.

Now we show that η is absolutely continuous with respect to the space-time Lebesgue measure. Let $A \subset \mathbb{R}^+ \times \mathbb{R}^d$ be a null set, then by Fubini's theorem, its time section $A_t := \{x \in \mathbb{R}^d : (t,x) \in A\}$ satisfies $|A_t| = 0$ for t-a.e. Also (3.18) along with Fubini's theorem imply that

$$\eta[A] = \int_0^\infty \mathbb{P}[t<\tau,(t,X_t)\in A]dt = \int_0^\infty \mathbb{P}[t<\tau, X_t \in A_t]dt.$$

Since the law of X_t is absolutely continuous with respect to the spatial Lebesgue measure and $|A_t| = 0$, we have $0 = \mathbb{P}[X_t \in A_t] \ge \mathbb{P}[t<\tau, X_t\in A_t]$ for a.e. $t \ge 0$. Hence, $\eta[A] = 0$, which verifies the absolutely continuity.

Next, define $\rho \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ to be the law of (τ, X_τ), i.e. for any $\psi \in C_c(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\mathbb{E}[\psi(\tau,X_\tau)] = \iint_{\mathbb{R}^+\times\mathbb{R}^d} \psi(t,x)\rho(dt,dx).$$

Then, (3.19) is obtained as a direct consequence of Itô's formula:

$$\mathbb{E}\left[w(\tau,X_\tau)\right] = \int_{\mathbb{R}^d} w(0,x)\mu(dx) + \mathbb{E}\left[\int_0^\tau \left[\partial_t w(t,X_t) - (-\Delta)^s w(t,X_t)\right]dt\right]. \tag{3.21}$$

Finally to establish (3.20) under the condition $\tau > 0$, it remains to obtain (3.19) with a time interval $\mathbb{R}^+$ replaced by $(0,\infty)$, i.e. we rule out the possibility of integrals in (3.19) over $\mathbb{R}^+ \times \mathbb{R}^d$

having a mass at $t = 0$. Observe that

$$\iint_{\mathbb{R}^+\times\mathbb{R}^d} w(t,x)\rho(dt,dx) = \iint_{(0,\infty)\times\mathbb{R}^d} w(t,x)\rho(dt,dx) + \int_{\mathbb{R}^d} w(0,x)\rho(\{0\},dx). \tag{3.22}$$

Since $\tau > 0$ a.s., $\rho(\{0\}\times\mathbb{R}^d) = \mathbb{P}(\tau = 0) = 0$, implying that the last term above is zero. Recalling that η is absolutely continuous with respect to the space-time Lebesgue measure, from (3.19), we obtain

$$\iint_{(0,\infty)\times B} w(t,x)\rho(dt,dx) = \int_B w(0,x)\mu(dx) + \iint_{(0,\infty)\times B} \left(\partial_t w(t,x) - (-\Delta)^s w(t,x)\right)\eta(dt,dx).$$

This implies that $\eta(0,\cdot) = \mu(\cdot)$ in the sense of distributions, and thus we deduce (3.20).

□

Remark 3.8. *Note that* (3.18) *defines η and ρ respectively as the distribution of active and stopped particles associated with τ. If $\tau = 0$ with a positive probability, then the initial data in* (3.20) *has to be revised as $\mu(x) - \rho(\{0\},x)$. The instantly stopped distribution, $\rho(\{0\},\cdot)$, is defined in Definition A.1.*

Lemma 3.9 (Parabolic regularity)**.** *Let (η,ρ) solve* (3.20) *in the sense of distributions. Then for any $\gamma \in (0,\lambda)$, $(\eta,\rho) \in L^2_\gamma(\mathbb{R}^+; H^s_0(B)) \times \mathcal{X}^*$. Moreover there is $C = C(\gamma,d,s,B)$ such that*

$$||\eta||_{L^2_\gamma(\mathbb{R}^+;H^s_0(B))} \leq C||\mu||_{L^2(\mathbb{R}^d)} \text{ and } ||\eta(t,\cdot)||_{L^2(B)} \leq e^{-\gamma t}||\mu||_{L^2(\mathbb{R}^d)}.$$

Proof. As the above estimates are direct consequence of regularity theory of fractional heat equations, we only briefly sketch the proof (we refer to [34] in the case of heat equations). We first mollify (η,ρ) in space-time to construct smooth approximations $\{(\eta_\varepsilon,\rho_\varepsilon)\}_{\varepsilon>0}$. Then we have $\partial_t\eta_\varepsilon + \rho_\varepsilon = -(-\Delta)^s\eta_\varepsilon$ classically. Then we multiply this equation by $e^{\gamma t}\eta_\varepsilon$ and proceed with integration by parts. Afterwords, Poincare's inequality (see Appendix B.1) yields the desired bounds in the limit $\varepsilon \to 0$. □

Let us finalize this section by characterizing the optimal stopping time associated to (1.4) as a hitting time to some space-time barrier set. The result follows from [33], see Appendix C.

Definition 3.10 (Forward/backward Barrier)**.** *A set $R \subset \mathbb{R}^+ \times \mathbb{R}^d$ is called a* forward barrier *if $(t,x) \in R$, then $(t+\delta,x) \in R$ for any $\delta > 0$. Conversely, R is called a* backward barrier *if $(t,x) \in R$, then $(t-\delta,x) \in R$ for any $\delta \in [0,t]$.*

Theorem 3.11 (Optimal Stopping time as a Hitting time)**.** *Let $\mu,\nu \in \mathcal{M}(\mathbb{R}^d)$ satisfy Assumption 1 and the Lagrangian L satisfies Assumption 2. Then there is a unique optimal stopping time τ^* of* (1.4) *that has a compact active region in $B_r(0)$ (where r is from Assumption 1). Moreover the following holds for the optimal stopping time τ^*:*

1. *When L is Type (I), there exists a forward barrier $R_1^* = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d : t \geq s_1^*(x)\}$ with $s_1^* : \mathbb{R}^d \to \mathbb{R}^+$ measurable such that $\tau^* = \inf\{t \geq 0 : (t,X_t) \in R_1^*\}$. In particular, τ^* is a non-randomized stopping time (of the space-time process).*
2. *When L is Type (II), there exists a backward barrier $R_2^* = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d : t \leq s_2^*(x)\}$ with $s_2^* : \mathbb{R}^d \to \mathbb{R}^+$ measurable such that $\tau^* = \inf\{t \geq 0 : (t,X_t) \in R_2^*\}$ on $\{\tau^* > 0\}$. In particular τ^* is only randomized on $\{\tau^* = 0\}$.*

The proof of Theorem 3.11 will be presented in Appendix C.

4. Properties of the Barrier Set

In this section we show that the optimal barriers R from Theorem 3.11 are in fact closed in the usual topology on $\mathbb{R}^+ \times \mathbb{R}^d$. Although the barrier for Type (I) has been explored in both local and non-local contexts in [31], our approach yields stronger regularity results on R

by leveraging the elliptic regularity theory. While we use potential theoretic arguments similar to [37, 31], our proof strongly utilizes the probabilistic interpretation of the Eulerian variables.

From now on, we assume that $s \in (0,1)$ satisfies the condition (2.2).

4.1. **Potential Flow.** Define the Riesz kernel of order $2s$

$$N(y) := C_{s,d}\frac{1}{|y|^{d-2s}}, \tag{4.1}$$

where $C_{s,d} > 0$ is a normalizing constant (see [53]). Also for $m \in \mathcal{M}(\mathbb{R}^d)$, define the potential of m as

$$U_m(y) := \int_{\mathbb{R}^d} N(x-y)m(dy). \tag{4.2}$$

Now we show that the distributional fractional Laplacian of U_m is equal to m.

Lemma 4.1. *Assume that there exist $C, \varepsilon > 0$ such that $|m(x)| \le C/(1+|x|^{2s+\varepsilon})$ for all $x \in \mathbb{R}^d$. Then for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} (-\Delta)^s\varphi(x)\cdot U_m(x)dx = \int_{\mathbb{R}^d}\varphi(x)\cdot m(x)dx.$$

Proof. From our assumptions on m and φ, we have $((-\Delta)^s\varphi, U_m) \in L^1(\mathbb{R}^d)\times L^\infty(\mathbb{R}^d)$. Therefore, we have that $(-\Delta)^s\varphi(x)\cdot N(x-y)\cdot m(y) \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$. Thus Fubini's Theorem along with $N(x)$ in (4.1) being an even function implies that

$$\begin{aligned}\int_{\mathbb{R}^d}(-\Delta)^s\varphi(x)\cdot U_m(x)dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(-\Delta)^s\varphi(x)\cdot N(y-x)\cdot m(y)dxdy\\ &= \int_{\mathbb{R}^d}U_{(-\Delta)^s\varphi}(y)\cdot m(y)dy = \int_{\mathbb{R}^d}\varphi(y)\cdot m(y)dy.\end{aligned}$$

In the final equality we used $U_\psi(x) = (-\Delta)^{-s}\psi(x)$ for a smooth function ψ. ☐

In the next lemma, we state the spatial regularity of the potential U_m.

Lemma 4.2 (Spatial regularity)**.** *Assume that there exist $C, \varepsilon > 0$ such that $|m(x)| \le C/(1+|x|^{2s+\varepsilon})$ for all $x \in \mathbb{R}^d$. Then, $U_m \in C^\alpha(\mathbb{R}^d)$ for any $\alpha \neq 1$ with $0 < \alpha < 2s$ and satisfies*

$$||U_m||_{C^\alpha} \le C(||U_m||_{L^\infty} + ||m||_{L^\infty}).$$

Here for $\alpha \ge 1$, C^α denotes $C^{1,\alpha-1}$.

Proof. Note that $(-\Delta)^sU_m = m \in L^\infty(\mathbb{R}^d)$ due to Lemma 4.1 and the decay conditions on m implies $||U_m||_{L^\infty(\mathbb{R}^d)} \le C$. By the elliptic regularity estimate [52, Proposition 2.9]

$$||f||_{C^\alpha(\mathbb{R}^d)} \le C\left[||f||_{L^\infty(\mathbb{R}^d)} + ||(-\Delta)^sf||_{L^\infty(\mathbb{R}^d)}\right],$$

we conclude the proof. ☐

From now on, for measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ satisfying Assumption 1, we assume that $\tau \in \mathcal{S}$ is a randomized stopping time such that $X_0 \sim \mu$ and $X_\tau \sim \nu$. Let (η, ρ) be the Eulerian variables associated to τ. Then, the distribution of $X_{t\wedge\tau}$, which we call μ_t, can be decomposed into the active and stopped mass up to time t, namely

$$\mu_t(x) = \eta(t,x) + \rho([0,t),x), \tag{4.3}$$

where $\rho([0,t),\cdot)$ is defined in Definition A.1. In the next lemma, we rigorously verify this decomposition.

Lemma 4.3 (Characterization of μ_t)**.** *Let $\tau \in \mathcal{S}$ be such that $X_0 \sim \mu$ and $X_\tau \sim \nu$, and let (η, ρ) be the associated Eulerian variables. For $t \geq 0$, define μ_t to be the distribution of $X_{t\wedge\tau}$. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,*

$$\iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\mu_t(dx)dt = \iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\eta(t,x)dtdx + \iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\rho([0,t],x)dtdx.$$

In other words, (4.3) holds (t,x)-a.e.

Proof. By the definition of μ_t,

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \varphi(t,x)\mu_t(dx)dt &= \int_{\mathbb{R}^+} \mathbb{E}[\varphi(t, X_{t\wedge\tau})]dt \\ &= \mathbb{E}\left[\int_{\mathbb{R}^+} \varphi(t,X_t)\chi_{\{t<\tau\}}dt\right] + \int_{\mathbb{R}^+} \mathbb{E}[\varphi(t,X_\tau)\chi_{\{\tau\leq t\}}]dt.\end{aligned}$$

By the definition of (η,ρ) from Lemma 3.7, the above quantity is equal to

$$\begin{aligned}&\iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\eta(t,x)dtdx + \int_{\mathbb{R}^+}\left[\iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\chi_{\{s\leq t\}}\rho(ds,dx)\right]dt \\ &= \iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\eta(t,x)dtdx + \iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\rho([0,t],x)dtdx.\end{aligned}$$

In particular, this implies that $dt \otimes \mu_t(dx)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^+ \times \mathbb{R}^d$. Finally we point out that for any $x \in \mathbb{R}^d$, $\rho([0,t),x) = \rho([0,t],x)$ t-a.e. due to Lemma A.2, and thus we deduce (4.3) for (t,x)-a.e.

☐

Definition 4.4 (Potential Flow)**.** *For $\tau \in \mathcal{S}$ and $t \geq 0$, let μ_t be the distribution of $X_{t\wedge\tau}$. Then, $(U_{\mu_t})_{t\geq 0}$ (see (4.2)) is called the potential flow associated to τ.*

Lemma 4.5. *Let $\tau \in \mathcal{S}$ be such that $\tau \leq \tau_r = \inf\{t : X_t \notin B_r(0)\}$. Then, there exists $C > 0$ such that $\mu_t(x) \leq C(1+|x|^{d+2s})^{-1}$ for all $x \in \mathbb{R}^d$ and $t \geq 0$. In particular, $\sup_{t\geq 0} ||U_{\mu_t}||_{C^\alpha(\mathbb{R}^d)} < \infty$ for any $0 < \alpha < 2s$ with $\alpha \neq 1$.*

Proof. First recall from Assumption 1 that μ is compactly supported on $B := B_r(0)$. Let (η,ρ) be the Eulerian variables associated to τ, as given in Lemma 3.7. This then implies from (3.20) and the comparison principle imply that $\eta \leq u$, where u solves

$$\begin{cases} \partial_t u = -(-\Delta)^s u & (t,x) \in (0,\infty)\times B, \\ u(t,x) = 0 & (t,x) \in (0,\infty)\times B^c, \\ u(0,x) = \mu(x) & x \in B \end{cases}$$

(note that (3.20) is applicable even if $\tau = 0$ with a positive probability since by Remark 3.8, the initial data of η is at most μ). By the comparison principle, $0 \leq u \leq ||\mu||_{L^\infty}$ and u is compactly supported in $\overline{B}$. As $\eta \leq u$, it follows that $\eta \leq ||\mu||_{L^\infty}\chi_B$.

Next, Lemma 3.5, (3.6), and the fact $\nu \in L^\infty$ implies that

$$\rho([0,t),x) \leq \nu(x) \leq \frac{C}{1+|x|^{d+2s}}.$$

Thus Lemma 4.3 yields that $\mu_t(x) \leq C(1+|x|^{d+2s})^{-1}$. Therefore, as μ_t satisfies the condition in Lemma 4.2 uniformly in t, we conclude the proof. ☐

Parallel arguments to [37, Lemma 4.8] yield a time regularity for the potential flow:

Lemma 4.6. *For $\tau \in \mathcal{S}$ with $\mathbb{E}[\tau] < \infty$, let $(U_{\mu_t})_{t\geq 0}$ be the associated potential flow. Then there exists $C > 0$ such that for any $0 \leq t \leq t'$,*

$$-C(t'-t) \leq U_{\mu_{t'}}(x) - U_{\mu_t}(x) \leq 0,$$

in particular the potential flow is uniformly Lipschitz in time.

As a consequence of Lemmas 4.5 and 4.6, we obtain the following fact.

Lemma 4.7. *For $\tau \in \mathcal{S}$ with $\mathbb{E}[\tau] < \infty$, the associated potential flow $(U_{\mu_t})_{t\geq 0}$ satisfies*

(a) $\lim_{t\to 0+} U_{\mu_t} = U_\mu$ *uniformly.*
(b) $\lim_{t\to \infty} U_{\mu_t} = U_\nu$ *uniformly.*

We now introduce the notion of forward and backward barrier associated to the potential flow. This notion has been used in [30] for the one-dimension space and in [37, 31] for general dimensions.

Definition 4.8 (Barrier associated to potential flows)**.** *For $\tau \in \mathcal{S}$, let $(U_{\mu_t})_{t\geq 0}$ be the associated potential flow. The forward/backward barrier functions and regions are defined as*

$$s^{U,f}(x) := \inf\{t \geq 0 : U_{\mu_t}(x) = U_\nu(x)\}, \quad R^{U,f} := \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d : t \geq s^{U,f}(x)\}$$

and

$$s^{U,b}(x) := \sup\{t \geq 0 : U_{\mu_t}(x) = U_\mu(x)\}, \quad R^{U,b} := \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d : t \leq s^{U,b}(x)\}.$$

The corresponding stopping times are defined as

$$\tau^{U,f} := \inf\{t \geq 0 : U_{\mu_t}(X_t) = U_\nu(X_t)\}, \quad \tau^{U,b} := \inf\{t > 0 : U_{\mu_t}(X_t) = U_\mu(X_t)\}.$$

Equivalently, by the monotonicity of U_{μ_t},

$$\tau^{U,f} = \inf\{t \geq 0 : t \geq s^{U,f}(X_t)\}, \quad \tau^{U,b} = \inf\{t > 0 : t \leq s^{U,b}(X_t)\}.$$

By Lemmas 4.5 and 4.6, $s^{U,f}$ and $-s^{U,b}$ are lower semi-continuous. Hence, barriers $R^{U,f}$ and $R^{U,b}$ are closed in $\mathbb{R}^+ \times \mathbb{R}^d$.

4.2. **Potential analysis on the barrier.** We begin with our analysis on the barriers associated to potential flows. Throughout this section, we assume that $\tau \in \mathcal{S}$ satisfying $X_0 \sim \mu$ with $X_\tau \sim \nu$ is given by

$$\tau = \begin{cases} \inf\{t : t \geq s(X_t)\} & \text{(called Type (I))}, \\ \inf\{t : 0 < t \leq s(X_t)\} & \text{(called Type (II))}, \end{cases} \tag{4.4}$$

for some measurable function $s : \mathbb{R}^d \to \mathbb{R}^+$. The terminologies Type (I) and Type (II) are inspired by Theorem 3.11.

Our goal is to establish that the potential barrier induced by τ (in the sense of Definition 4.8) are essentially the same as the original barrier generated by a function s. By a parallel argument as in [37, Proposition 4.12], we obtain the following result:

Lemma 4.9. *Let τ be as given in (4.4). Then*

$$|\{x \in \mathbb{R}^d : s(x) < s^{U,f}(x)\}| = 0 \text{ for Type (I) and } |\{x \in \mathbb{R}^d : s(x) > s^{U,b}(x)\}| = 0 \text{ for Type (II).}$$

In particular, Lemma 4.9 says that the potential forward/backward barrier is the largest such barrier that embeds μ to ν. Using the short hand $\tau^U := \tau^{U,f}$ with $R^U := R^{U,f}$ for Type (I) and $\tau^U := \tau^{U,b}$ with $R^U := R^{U,b}$ for Type (II), we conclude the following:

Corollary 4.10. *Let τ be the optimizer of $\mathcal{P}_0(\mu,\nu)$ in (1.4) (assume additionally that $\tau > 0$ a.s. for costs of Type (II)). Then $\tau^U \leq \tau$ a.s.*

Proof. By Theorem 3.11, the optimizer τ is given by the space-time stopping time (4.4) for some measurable function $s:\mathbb{R}^d\to\mathbb{R}^+$. Let us first consider the Type (I) case. By Lemma 4.9 and recalling $X_\tau\sim\nu$ and $\nu\ll\mathrm{Leb}$, we have $s(X_\tau)\geq s^{U,f}(X_\tau)$ a.s. In addition, by Lemma C.1, the stopped particles are inside the barrier, i.e. $\tau\geq s(X_\tau)$ a.s. In summary, $\tau\geq s^{U,f}(X_\tau)$, implying that $\tau\geq\tau^{U,f}$ a.s.

The proof is similar for Type (II), except that when applying Lemma C.1, we need the condition $\tau>0$ a.s. ☐

With this result we can show that the active region is open:

Lemma 4.11. *Let τ be as given in (4.4) and (η,ρ) be the associated Eulerian variables. Denote by R^U the potential barrier in the sense of Definition 4.8. Then up to a zero-measure set (w.r.t. space-time Lebesgue measure),*

$$\{\eta>0\}=(R^U)^c.$$

In particular, recalling that R^U is closed, the positive set of η is open (up to a zero-measure set).

Proof. We follow the proof in [37]. Let

$$w(t,x):=\begin{cases}U_{\mu_t}(x)-U_\nu(x) & \text{for Type (I)},\\ U_\mu(x)-U_{\mu_t}(x) & \text{for Type (II)}.\end{cases}\tag{4.5}$$

Then by Lemma 4.6, $w\geq 0$. Also by definition of R^U, we have $(R^U)^c=\{w>0\}$. Hence, it suffices to prove that up to a zero-measure space-time set,

$$\{\eta>0\}=\{w>0\}.\tag{4.6}$$

For a test function $g\in C_c^\infty((0,\infty)\times\mathbb{R}^d)$, let $\varphi(t,\cdot):=N*g(t,\cdot)$ (recall that N denotes the Riesz kernel, see (4.1)), which satisfies $(-\Delta)^s\varphi(t,\cdot)=g(t,\cdot)$ and decays at infinity. By Lemmas 4.3 and A.3, in the case of Type (I),

$$\iint g(\partial_t w)dtdx=\iint\varphi(\partial_t\mu_t)dtdx=\iint\varphi(\partial_t\eta+\rho)dtdx=\iint-\varphi\cdot(-\Delta)^s\eta dtdx=\iint-g\eta dtdx.$$

By a similar computation for Type (II), along with the Lipschitz regularity of w (in time),

$$\partial_t w=\begin{cases}-\eta & \text{for Type (I)},\\ \eta & \text{for Type (II)}.\end{cases}\tag{4.7}$$

Integrating in time and using Lemma 4.7,

$$w(t,x)=\begin{cases}\int_t^\infty\eta(s,x)ds & \text{for Type (I)},\\ \int_0^t\eta(s,x)ds & \text{for Type (II)}.\end{cases}\tag{4.8}$$

We claim that this implies that $\{\eta>0\}\subset\{w>0\}$ up to a zero-measure set. Indeed, for Type (I), recalling $\{w(t,x)=0\}=\{t\geq s^U(x)\}$, we have $0=w(s^U(x),x)=\int_{s^U(x)}^\infty\eta(t,x)dt$ for every $x\in\mathbb{R}^d$, implying that $\eta(t,x)=0$ for a.e. $t\geq s^U(x)$. Hence, we verify the claim.

For the reverse direction, observe that by Lemma 4.9, $\rho[(R^U)^c]=0$. Hence η solves the fractional heat equation on the open set $(R^U)^c$ with a zero boundary data and initial data μ. By the strong maximum principle, $\eta>0$ on $(R^U)^c\cap\{t>0\}$. Therefore, we establish (4.6). ☐

Remark 4.12. *By Lemma 4.11, one can modify η on a set of space-time Lebesgue measure zero so that $\{\eta>0\}$ is open in $\mathbb{R}^+\times\mathbb{R}^d$ (in particular, $\{\eta>0\}=\{w>0\}$). From now on, we work with this modification of η.*

Let τ be the optimizer of $\mathcal{P}_0(\mu,\nu)$. Our goal is to show that in the case of Type (I) cost, $\tau=\tau^U$ a.s. Once this is verified, one can assume that the optimal stopping time of $\mathcal{P}_0(\mu,\nu)$ in Theorem 3.11 is τ^U, which allows us to abuse the notation $s=s^U$ and $R=R^U$.

Theorem 4.13. *Let τ optimize $\mathcal{P}_0(\mu,\nu)$ with a Type (I) cost. Then $\tau = \tau^U$ a.s.*

Proof. By Corollary 4.10, $\tau^U \le \tau$. In particular, as $\tau \le \tau_r = \inf\{t \ge 0 : X_t \notin B_r(0)\}$, we see that (X, τ^U) has an active compact region in $B_r(0)$.

Let (η^U, ρ^U) be the Eulerian variables associated to τ^U. We claim that $\eta^U[R^U] = 0$ and $\rho^U[R^U] = 1$. Indeed, by Fubini's Theorem,

$$\eta^U[R^U] = \int_0^\infty \mathbb{P}(s < \tau^U, (s, X_s) \in R^U) ds.$$

Since $(s, X_s) \notin R^U$ for $0 \le s < \tau^U$, we have $\eta^U[R^U] = 0$. Also, since R^U is closed and $t \mapsto X_t$ is right-continuous, $(\tau^U, X_{\tau^U}) \in R^U$ a.s., which implies $\rho^U[R^U] = 1$.

Next, let (η, ρ) be the Eulerian variables associated to τ from Lemma 3.7. By Lemma 4.11, $\eta[R^U] = 0$. In addition, by Lemma 4.9, $\rho[R^U] = 1$. Hence, by Theorem C.2, we obtain that $\rho = \rho^U$. In particular, as $\rho \sim (\tau, X_\tau)$ and $\rho^U \sim (\tau^U, X_{\tau^U})$, we conclude that $\tau \sim \tau^U$. Since $\tau^U \le \tau$ by Corollary 4.10, we deduce that $\tau = \tau^U$ a.s. ☐

Before we discuss the corresponding result for Type (II), we introduce the PDE characterization of the potential variable w. This result will be important later in Section 7 where we derive the connection between the nonlocal Stefan problem and the parabolic obstacle problem for Type (II) (see Theorems 7.2 and 7.5).

Theorem 4.14. *Let τ optimize $\mathcal{P}_0(\mu,\nu)$ with a Type (I) or (II) cost (assume that $\tau > 0$ a.s. in Type (II) case). Let w be defined in (4.5), where the potential flow is associated to τ. Then, w solves the following parabolic obstacle problem:*

$$\begin{cases} \min\{\partial_t w + (-\Delta)^s w + \nu, w\} = 0 \text{ with } w(0,\cdot) = U_\mu - U_\nu & \text{for Type (I)}, \\ \\ \min\{\partial_t w + (-\Delta)^s w + \nu - \mu, w\} = 0 \text{ with } w(0,\cdot) = 0 & \text{for Type (II)}. \end{cases} \tag{4.9}$$

Proof. We just consider the Type (II) case since parallel arguments work for Type (I) case. Let (η, ρ) be the Eulerian variables associated to the optimal stopping time τ. By taking the fractional Laplacian in (4.5), Lemma 4.3 and (4.7) gives

$$\partial_t w(t,x) + (-\Delta)^s w(t,x) = \mu(x) - \rho([0,t), x). \tag{4.10}$$

By Lemmas 4.2 and 4.6, w is continuous. Also, Lemma 4.11 (in particular, (4.6)) and Lemma 4.9 yields that if $w(t,x) > 0$ (equivalently, $t > s^U(x)$), then $\rho([0,t), x) = \nu(x)$. Hence,

$$\partial_t w + (-\Delta)^s w + \nu - \mu = 0 \quad \text{on } \{w > 0\}.$$

In addition, since $\nu(x) \ge \rho([0,t), x)$ for any $x \in \mathbb{R}^d$ and $t \ge 0$,

$$\partial_t w + (-\Delta)^s w + \nu - \mu \ge \partial_t w + (-\Delta)^s w + \rho([0,t), \cdot) - \mu = 0,$$

where we used (4.10) in the final equality. Thus we conclude that w solves (4.9).

☐

For Type (II), we present a direct probabilistic argument that does not rely on Theorem C.2, taking advantage of the nonlocal nature of diffusion. While the argument is more intuitive and simpler, it achieves a slightly weaker result, namely we only show that $s = s^U$ a.e. Because of this reason we will not pursue our method for Type (I), which appears to require more careful analysis. We first state a consequence of Theorem 4.14 for Type (II).

Corollary 4.15. *Let τ be the optimal stopping time for $\mathcal{P}_0(\mu,\nu)$ with a Type (II) cost, and assume that $\tau > 0$ a.s. For $t \ge 0$ and $\delta > 0$, assume that A is a Borel set in $\mathbb{R}^d$ of positive Lebesgue measure such that $A \subset \{x \in \mathbb{R}^d : (t+\delta, x) \in R^U\}$. Then, $\rho([t, t+\delta] \times A) > 0$.*

Proof. Let w be the potential for Type (II), defined in (4.5). Recalling that w is non-decreasing and Lipschitz in time (see Lemma 4.6), (4.7) implies that $\partial_t w = \eta$ a.e. As $\tau > 0$, $\eta(0, \cdot) = \mu(\cdot)$. Since μ is non-trivial and $\{\eta(t, \cdot) > 0\}$ is non-decreasing in t, we have $\partial_t w(t, \cdot) > 0$ on a set of positive measure for any $t > 0$.

If $(t + \delta, x) \in R^U$ (i.e. $w(t + \delta, x) = 0$), then $w(t, x) = 0$. Thus,

$$(-\Delta)^s w(t,x) = -C_{d,s} \int_{\mathbb{R}^d} \frac{w(t,y)}{|x-y|^{d+2s}} dy > -C_{d,s} \int_{\mathbb{R}^d} \frac{w(t+\delta,y)}{|x-y|^{d+2s}} dy = (-\Delta)^s w(t+\delta,x). \tag{4.11}$$

Note that $\partial_t w = \eta = 0$ a.e. on R^U and $\mu = 0$ a.e. on $\{x \in \mathbb{R}^d : (t, x) \in R^U \text{ for some } t\}$. Hence by (4.10),

$$-\rho([t, t + \delta), x) = (-\Delta)^s w(t + \delta, x) - (-\Delta)^s w(t, x) < 0,$$

where we used (4.11) in the last inequality. Now by integrating over A, we conclude the proof. ☐

Theorem 4.16. *Let μ, ν and τ be as given in Corollary 4.15 for a Type (II) cost. Then $R = R^U$ a.e. in $\mathbb{R}^+ \times \mathbb{R}^d$ and $s = s^U$ a.e. in $\mathbb{R}^d$.*

Proof. Note that $R = R^U$ a.e. implies that the barrier functions coincide, i.e. $s = s^U$ a.e. Hence it suffices to show $R = R^U$ a.e. Lemma 4.9 implies that $R \subset R^U$ a.e., and thus it remains to show that $|R^U \setminus R| = 0$. By Lemma C.1, ρ is supported on R, and thus $\rho(R^U \setminus R) = 0$.

Assume for the sake of a contradiction that $|R^U \setminus R| > 0$. Set $Z := R^U \setminus R$ and define the time section $Z_t := \{x \in \mathbb{R}^d : (t, x) \in Z\}$. By Fubini's theorem, there is $t > 0$ such that $|Z_t| > 0$. Since $R = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d : t \le s(x)\}$ and $\{t\} \times Z_t \notin R$, we have $t > s(x)$ for $x \in Z_t$. Thus, recalling $|Z_t| > 0$, there is n with $t > 1/n$ such that $Z_t^{(n)} := \{x \in Z_t : t - \frac{1}{n} > s(x)\}$ has a positive Lebesgue measure. Since $[t - \frac{1}{n}, t] \times Z_t^{(n)} \subset R^U \setminus R$ and $\rho(R^U \setminus R) = 0$, we have $\rho([t - \frac{1}{n}, t] \times Z_t^{(n)}) = 0$, contradicting Corollary 4.15. ☐

Now by combining Theorems 4.13 and 4.16 along with our assumption that $\nu \ll \text{Leb}$, we conclude the following.

Theorem 4.17. *Under the assumptions of Corollary 4.10, we have for both types that $s = s^U$ ν-a.e. For Type (I), we further have that $\tau = \tau^U$ a.s.*

We conclude this section by introducing the identity for $(-\Delta)^s w$, which will be useful when studying the associated Stefan problem in Section 5.

Proposition 4.18. *Let τ and w be as in Theorem 4.14 and let (η, ρ) be the Eulerian variables associated to τ. Then for any $t \ge 0$,*

$$-[(-\Delta)^s w(t, x)] \cdot \chi_{\{\eta=0\}}(t, x) = \rho((t, \infty), x) \cdot \chi_{\{\eta=0\}}(t, x) \tag{4.12}$$

for Type (I), and

$$-[(-\Delta)^s w(t, x)] \cdot \chi_{\{\eta=0\}}(t, x) = \rho((0, t), x) \cdot \chi_{\{\eta=0\}}(t, x) \tag{4.13}$$

for Type (II) (if $\tau > 0$ a.s.).

Proof. Using the representation of w in terms of the potential (4.5) and the decay property of μ_t from Lemma 4.5, we apply Lemma 4.1 together with Lemma 4.3 to obtain

$$(-\Delta)^s w(t,x) = \begin{cases} \mu_t(x) - \nu(x) = \eta(t,x) - \rho((t,\infty),x) & \text{for Type (I)}, \\ \mu(x) - \mu_t(x) = \mu(x) - \eta(t,x) - \rho([0,t),x) & \text{for Type (II)}. \end{cases} \tag{4.14}$$

We just consider the Type (II) case since parallel arguments work for Type (I) case. Since $\tau > 0$ a.s., $\rho([0, t), x) = \rho((0, t), x)$ x-a.e. for any $t > 0$. In addition, $\tau > 0$ a.s. implies that $\{\mu(\cdot) > 0\} \subset \{\eta(t, \cdot) > 0\}$ for any $t \ge 0$, since $\{\eta(t, \cdot) > 0\}$ is non-decreasing in t. This implies that $(-\Delta)^s w(t, x) \cdot \chi_{\{\eta=0\}}(t, x) = -\rho((0, t), x) \cdot \chi_{\{\eta=0\}}(t, x)$. ☐

5. Associated Stefan Problem

Suppose that τ is the optimal stopping time of the variational problem $\mathcal{P}_0(\mu,\nu)$ in (1.4), and let (η,ρ) be the associated Eulerian variables as in Lemma 3.7. In this section, we establish that η solves the nonlocal Stefan problem with an initial distribution μ and the weight ν. Before introducing the definition of solutions, let us present a heuristic discussion on the characterization of the enthalpy variable in terms of the Eulerian variables.

∘ *Review for the case $s=1$:* Let us briefly recall the proof for the local case $s=1$. In this case, (St_1) and (St_2) can be written as

$$\partial_t h - \Delta\eta = 0, \tag{5.1}$$

where the enthalpy h is given as a function of η by

$$h = \begin{cases} \eta - \chi_{\{\eta>0\}} & \text{for } (St_1),\\ \eta + \chi_{\{\eta>0\}} & \text{for } (St_2).\end{cases}$$

The discontinuity of h represents the unit amount of heat energy change associated with the phase transition. To illustrate our interpretation more clearly, we consider a generalized version of this problem, the equation (5.1) that represents a *weighted* rate of energy change ν, namely h is given by

$$h = \begin{cases} \eta - \nu\chi_{\{\eta>0\}} & \text{for } (St_{1,\nu}),\\ \eta + \nu\chi_{\{\eta>0\}} & \text{for } (St_{2,\nu})\end{cases}$$

(see below for the definition of ($St_{1,\nu}$) and ($St_{2,\nu}$)).

It was shown [37], following [34], that in the case of the Brownian motion, this can be solved with the aid of optimal Eulerian variables (η,ρ) of (St) associated to the target measure ν. A crucial step in the analysis is the particle interpretation of the enthalpy variable in terms of the distributions of particles at (t,x). Precisely, [37] deduced that the Eulerian variables generated by the cost of Type (I) and Type (II) yield the solutions to ($St_{1,\nu}$) and ($St_{2,\nu}$) respectively, with the enthalpy:

$$h(t,x) = \begin{cases} \eta(t,x) - \rho((t,\infty),x) & \text{for } (St_{1,\nu}),\\ \eta(t,x) + \rho([0,t),x) & \text{for } (St_{2,\nu}),\end{cases} \tag{5.2}$$

where $\rho((t,\infty),x) := \nu(x) - \rho([0,t],x)$. Note that this formula for the enthalpy is consistent with the enthalpy formula (1.1) since

$$\rho([0,t),x) - \nu(x) = \rho([0,t),x) - \rho([0,\infty),x) = -\rho([t,\infty),x) = -\rho((t,\infty),x),$$

where the final equality holds t-a.e. thanks to Lemma A.2. We remark that the choice of an open left end point on $\rho((t,\infty),x)$ and a closed right point on $\rho([0,t),x)$ is so that their values at time $t=0$ equals their limit as $t\to 0^+$, which will be used in our definition of weak solutions to the Stefan problem.

In the local case, it is crucial that ρ can be written in terms of η, for instance $\rho([0,t),x) = \nu\chi_{\{\eta>0\}}$ for Type (II). This allows us to write the Eulerian PDE solely in terms of η, which then leads to the characterization of the equation as the Stefan problem.

∘ *Heuristics for the nonlocal case.* Our task is to extend the characterization of the enthalpy (1.1) to the non-local case. We aim to represent ρ in terms of η using the Eulerian equation (3.20) and the forward/backward time-monotonicity of the barrier set $\{\eta=0\}$ for the Type (I)/(II) cases respectively. Our description of ρ should now reflect the fact that the stopped particles are no longer concentrated on the boundary, due to the jumps in the $2s$-stable process (see Figure 1). In other words ρ is supported on the whole $\{\eta=0\}$ instead of only on the free boundary.

We present the heuristic argument for the representation of ρ. We consider Type (II) with $\tau > 0$. As τ is the hitting time to the backward barrier R that is a.e. equal to $\{\eta = 0\}$ (see Theorem 3.11 (2)), we have

$$\rho([0,t),x) = \rho([0,\infty),x) = \nu(x) \text{ on } \{\eta > 0\}.$$

On the other hand, from Proposition 4.18, we have

$$-(-\Delta)^s w(t,x) = \rho([0,t),x) \text{ on } \{\eta = 0\} \text{ with } w(t,x) = \int_0^t \eta(a,x)da.$$

Putting these two identities together, we have that

$$\rho([0,t),x) = \nu(x)\chi_{\{\eta(t,x)>0\}} + \kappa_2(t,x)\chi_{\{\eta(t,x)=0\}} \tag{5.3}$$

where we define

$$\kappa_2(t,x) := -(-\Delta)^s w(t,x) = -\int_0^t (-\Delta)^s \eta(a,x)da. \tag{5.4}$$

It is crucial to note that in contrast to the local case, $\kappa_2 \neq 0$ on $\mathrm{Int}(\{\eta = 0\})$ unless η is identically zero. Together with (3.20) this, at least formally, show that our optimal Eulerian variables for the cost of Type (II) satisfy the nonlocal melting Stefan problem

$$\partial_t h + (-\Delta)^s \eta = 0, \qquad h = h_2(\eta) := \eta + \nu\chi_{\{\eta>0\}} + \kappa_2\chi_{\{\eta=0\}}. \tag{$St_{2,\nu}$}$$

Note that this expression of h is consistent with (1.1).

Similarly for Type (I) costs, at least formally, we have that

$$\rho((t,\infty),x) = \nu(x)\chi_{\{\eta(t,x)>0\}} + \kappa_1(t,x)\chi_{\{\eta(t,x)=0\}} \tag{5.5}$$

with

$$\kappa_1(t,x) := -(-\Delta)^s w(t,x) = -\int_t^\infty [(-\Delta)^s \eta(a,x)]da, \tag{5.6}$$

and establish that η solves the nonlocal freezing Stefan problem

$$\partial_t h + (-\Delta)^s \eta = 0, \qquad h = h_1(\eta) := \eta - \nu\chi_{\{\eta>0\}} - \kappa_1\chi_{\{\eta=0\}}, \tag{$St_{1,\nu}$}$$

thus verifying the formula (1.1).

Remark 5.1. *For η and its associated potential w, Proposition 4.18 establishes that for any $i = 1,2$ and $t \geq 0$, we have $0 \leq \kappa_i(t,\cdot)\chi_{\{\eta=0\}}(t,\cdot) \leq \nu(\cdot) \in L^1 \cap L^\infty$.*

Below we give rigorous justification of the above heuristic arguments. To this end we introduce a weak solution to ($St_{1,\nu}$) and ($St_{2,\nu}$) based on the above discussions. For $u \in L^1(\mathbb{R}^+;\mathbb{R}^d)$ for which there is a version $\tilde{u}$ of u (in $\mathbb{R}^+ \times \mathbb{R}^d$) such that $\{\tilde{u}(t,\cdot) > 0\}$ is non-decreasing or non-increasing in t and $\{\tilde{u} > 0\}$ is open in $\mathbb{R}^+ \times \mathbb{R}^d$, we define the *initial domain* of u as

$$E(u) := \lim_{t\to 0^+} \{u(t,\cdot) > 0\}. \tag{5.7}$$

In the following remark, we verify that the initial domain $E(u)$ is well-defined (up to zero-measure).

Remark 5.2. *We show that the initial domain is uniquely determined a.e. in $\mathbb{R}^d$. We only consider the case when $\{\tilde{u}(t,\cdot) > 0\}$ is non-increasing in t, since the non-decreasing case similarly follows. We claim that if u_1 and u_2 are two versions of u with $\{u_i(t,\cdot) > 0\}$ non-increasing in t and $\{u_i > 0\}$ open ($i = 1,2$), then for any $t > 0$,*

$$|\{u_1(t,\cdot) > 0\} \setminus \{u_2(t,\cdot) > 0\}| = 0. \tag{5.8}$$

Suppose that the above does not hold for some $t > 0$. Let $A := \{u_1(t,\cdot) > 0\} \cap \{u_2(t,\cdot) = 0\}$. As $\{u_1 > 0\}$ is open, for all $x \in A$, there is $r_x > 0$ such that $[t, t + r_x] \times \{x\} \subset \{u_1 > 0\}$. Setting $A_k := \{x \in A : r_x \geq 1/k\}$, there exists n such that $|A_n| > 0$. Then $|[t, t + 1/n] \times A_n| > 0$ and

by monotonicity of positivity sets, $[t, t+1/n] \times A_n \subset \{u_1 > 0\} \cap \{u_2 = 0\}$*. This contradicts the fact that both* u_1 *and* u_2 *are versions of* η*.*
As $\{\tilde{u}_i(t,\cdot) > 0\}$ *is non-increasing in* t*, the initial domain* $E(u_i)$ *can be written as*

$$E(u_i) = \bigcup_{n\in\mathbb{N}} \{u_i(1/n,\cdot) > 0\}.$$

Hence by (5.8)*, we deduce that* $E(u_1) = E(u_2)$ *up to zero-measure.*

We now define the notion of weak solutions to ($St_{1,\nu}$) and ($St_{2,\nu}$) for a non-negative initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and an initial domain E containing the support of u_0.

In the definition below, we introduce the variables u and v_i ($i = 1, 2$) for the definition of general weak solutions for weighted Stefan problems, with the understanding that they will later correspond to the Eulerian variable η and its associated potential w respectively, generated by $\mathcal{P}_0(\mu,\nu)$.

Definition 5.3 (Weighted Stefan Problem)**.** *Let* ν *and* u_0 *be two bounded and nonnegative functions on* $\mathbb{R}^d$*, and let* $E \subset \mathbb{R}^d$ *be a measurable set. Let* u *be a nonnegative function in* $L^1 \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$*, and define its time integrated version:*

$$v_1(t,x) := \int_t^\infty u(a,x)da \quad \text{and} \quad v_2(t,x) := \int_0^t u(a,x)da.$$

(1) *We say that* u *is a weak solution to* ($St_{1,\nu}$) *with initial data* (u_0, E) *if* $E(u) = E$ *and*
  (a) $\{u > 0\}$ *is open in* $\mathbb{R}^+ \times \mathbb{R}^d$*,*
  (b) $\{u(t,\cdot) > 0\}$ *is uniformly bounded and non-increasing in* t*,*
  (c) *There is a* $\kappa_1(t,x) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *such that* $0 \le (\kappa_1\chi_{\{u=0\}})(t,x) \le \nu(x)$ *and* $\kappa_1(t,\cdot) = -(-\Delta)^s v_1(t,\cdot)$ *as distributions for any* $t \ge 0$*. Namely, for* $t \ge 0$ *and* $\psi \in C_c^\infty(\mathbb{R}^d)$*,*

$$\int -(-\Delta)^s\psi(x)\cdot v_1(t,x)\,dx = \int \psi(x)\cdot\kappa_1(t,x)\,dx.$$

  (d) *For* $h = h_1(u) := u - \nu\chi_{\{u>0\}} - \kappa_1\chi_{\{u=0\}}$ *and for any* $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$*,*

$$\iint (-\partial_t\varphi\cdot h + [(-\Delta)^s\varphi]\,u)dtdx = \int \varphi(0,\cdot)(u_0 - \nu\chi_E + [(-\Delta)^s v_1](0,\cdot)\chi_{E^c})dx. \tag{5.9}$$

(2) *We say that* u *is a weak solution to* ($St_{2,\nu}$) *with initial data* (u_0, E) *if* $E(u) = E$ *and*
  (a') $\{u > 0\}$ *is open in* $\mathbb{R}^+ \times \mathbb{R}^d$*,*
  (b') $\{u(t,\cdot) > 0\}$ *uniformly bounded and non-decreasing in* t*,*
  (c') *There is a* $\kappa_2(t,x) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *such that* $0 \le (\kappa_2\chi_{\{u=0\}})(t,x) \le \nu(x)$ *and* $\kappa_2(t,\cdot) = -(-\Delta)^s v_2(t,\cdot)$ *as distributions for any* $t \ge 0$*.*
  (d') *For* $h = h_2(u) = u + \nu\chi_{\{u>0\}} + \kappa_2\chi_{\{u=0\}}$*, and for any* $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$*,*

$$\iint (-\partial_t\varphi\cdot h + [(-\Delta)^s\varphi]\,u)dtdx = \int \varphi(0,\cdot)(u_0 + \nu\chi_E)dx. \tag{5.10}$$

We say that (h_i, u) *solves* ($St_{i,\nu}$) *if* u *solves it with* $h_i = h_i(u)$ *as defined above, for* $i = 1, 2$*.*

Note that, by expanding out the definition of the enthalpy variable h, (5.9) reads as

$$\iint (-\partial_t\varphi\cdot(u - \nu\chi_{\{u>0\}} - \kappa_1\cdot\chi_{\{u=0\}}) + u\cdot(-\Delta)^s\varphi)dtdx$$
$$= \int \varphi(0,\cdot)(u_0 - \nu\chi_E + (-\Delta)^s v_1(0,\cdot)\cdot\chi_{E^c})dx,$$

and (5.10) reads as

$$\iint(-\partial_t\varphi\cdot(u+\nu\chi_{\{u>0\}}+\kappa_2\cdot\chi_{\{u=0\}})+u\cdot(-\Delta)^s\varphi)dtdx=\int\varphi(0,\cdot)(u_0+\nu\chi_E)dx.$$

Using the above definition of a solution to the weighted Stefan problem, we now define the solution to the Stefan problem (St_1) and (St_2).

Definition 5.4. *We say that u solves (St_1) (resp. (St_2)) if it solves ($St_{1,\nu}$) (resp. ($St_{2,\nu}$)) with some (non-negative) $\nu\le 1$ that is equal to 1 on $\{x\in\mathbb{R}^d : u(t,x)>0$ for some $t>0\}$. We also say that (h_i,u) solves (St_i) if u solves it with $h_i=h_i(u)$ (and with corresponding ν) as defined in Definition 5.3, for $i=1,2$.*

The condition $\nu\le 1$ reflects the fact that the maximal capacity of the inactive region to hold ice particles in our phase transition process in (St_1) or (St_2) equals one. Note that, in contrast to the local case [37], ν can not be chosen to be a characteristic function due to the jumps of the underlying process.

In Section 5.2, we show that our notion of solutions corresponds to those generated by the Eulerian variables (η,ρ) from Lemma 3.7 with a target measure ν. In particular, the solution is in the stronger regularity class, namely $\eta\in L^2(\mathbb{R}^+;H^s(\mathbb{R}^d))$ (see Lemma 3.9). In Section 5.3 we further show that for the equation (St_2), our notion of solution coincides with that of (St_h) which was considered in several recent papers [21, 19, 20]. This provides a new Sobolev regularity for solutions of (St_h), which was only known to be continuous [21].

In our proof with the Eulerian variables (η,ρ), the characterization of ρ in terms of η is a main step in showing that η is a weak solution to the weighted Stefan problems. Namely, we deduce (5.5) for Type (I) and (5.3) for Type (II).

5.1. Identification of ρ.

From now on, η denotes the modification of the Eulerian variable such that $\{\eta>0\}$ is open in $\mathbb{R}^+\times\mathbb{R}^d$ (see Remark 4.12).

Theorem 5.5 (ρ for Type (I))**.** *Let τ be the optimal stopping time for $\mathcal{P}_0(\mu,\nu)$ for a Type (I) cost and (η,ρ) be the associated Eulerian variables. Then, for any $\varphi\in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d)$,*

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^d}&\varphi(t,x)\rho(dt,dx)\\&=\iint_{\mathbb{R}^+\times\mathbb{R}^d}\Big(\partial_t\varphi\cdot\nu\chi_{\{\eta>0\}}+\partial_t\varphi\cdot\rho((t,\infty),x)\chi_{\{\eta=0\}}\Big)dtdx+\int_{\mathbb{R}^d}\varphi(0,x)\nu(x)dx.\end{aligned}\tag{5.11}$$

If $\tau>0$ a.s., then (5.5) holds (t,x)-a.e.

Proof. Using the disintegration formula in Definition A.1 (the map $x\mapsto\rho_x\in\mathcal{M}(\mathbb{R}^+)$ denotes the disintegration of ρ with respect to the spatial measure ν),

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^d}\varphi(t,x)\rho(dt,dx)&=\int_{\mathbb{R}^d}\int_0^\infty\varphi(t,x)\rho_x(dt)\nu(x)dx\\&=\int_{\mathbb{R}^d}\int_0^\infty\left[\int_0^t\partial_t\varphi(w,x)dw\right]\rho_x(dt)\nu(x)dx+\int_{\mathbb{R}^d}\int_0^\infty\varphi(0,x)\rho_x(dt)\nu(x)dx.\end{aligned}\tag{5.12}$$

By Fubini's theorem and noting that $\rho_x[\mathbb{R}^+]=1$ ν-a.e., (5.12) reduces to

$$\int_{\mathbb{R}^d}\int_0^\infty\partial_t\varphi(w,x)\left[\int_w^\infty\rho_x(dt)\right]\nu(x)dwdx+\int_{\mathbb{R}^d}\varphi(0,x)\nu(x)dx.\tag{5.13}$$

Observe that by Lemma A.2, $\int_w^\infty\rho_x(dt)=\rho_x((w,\infty))$ ν-a.e. By Theorems 3.11 and 4.17, there is $s^U:\mathbb{R}^d\to\mathbb{R}^+$ such that τ is the hitting time to the barrier $\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d : t\ge s^U(x)\}$.

By Lemma C.1, $\tau \geq s^U(X_\tau)$ a.s. This implies that for $t < s^U(x)$ (equivalently, $\eta(t,x) > 0$ by Lemma 4.11), $\rho_x((t,\infty)) = \mathbb{P}(\tau > t \mid X_\tau = x) = 1$. Therefore,

$$\iint \varphi(t,x)\rho(dt,dx) = \iint (\partial_t\varphi \cdot \nu\chi_{\{\eta>0\}} + \partial_t\varphi \cdot \nu\rho_x((t,\infty))\chi_{\{\eta=0\}})dtdx + \int \varphi(0,x)\nu(x)dx.$$

Again by the disintegration formula, we obtain (5.11).

Now because we verified (5.11), we will use it to prove (5.5). If $\tau > 0$ a.s. then $\rho(\{0\}\times\mathbb{R}^d) = 0$. Thus by Lemmas A.2 and A.3,

$$\iint \varphi(t,x)\rho(dt,dx) = -\iint \partial_t\varphi \cdot \rho((0,t),x)dtdx = -\iint (\partial_t\varphi)\left[\nu(x) - \rho((t,\infty),x)\right]dtdx. \tag{5.14}$$

Combining this with (5.11),

$$\iint \partial_t\varphi \cdot \rho((t,\infty),x)dtdx = \iint \left(\partial_t\varphi \cdot \nu\chi_{\{\eta>0\}} + \partial_t\varphi \cdot \rho((t,\infty),x)\chi_{\{\eta=0\}}\right)dtdx.$$

Note that by Proposition 4.18,

$$\rho((t,\infty),x)\chi_{\{\eta=0\}} = -[(-\Delta)^s w(t,x)]\chi_{\{\eta=0\}} = \kappa_1(t,x)\chi_{\{\eta=0\}}. \tag{5.15}$$

Hence, we deduce (5.5). □

Similarly we prove (5.3) for Type (II).

Theorem 5.6 (ρ for Type (II)). *Let τ be the optimal stopping time for $\mathcal{P}_0(\mu,\nu)$ with a Type (II) cost and (η,ρ) be the associated Eulerian variables. If $\tau > 0$ a.s., then for any $\varphi \in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d)$,*

$$\iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\rho(dt,dx) = -\iint_{\mathbb{R}^+\times\mathbb{R}^d} (\partial_t\varphi \cdot \nu\chi_{\{\eta>0\}} + \partial_t\varphi \cdot \rho((0,t),x)\chi_{\{\eta=0\}})dtdx. \tag{5.16}$$

In particular, (5.3) holds (t,x)-a.e.

Proof. Again by the disintegration formula,

$$\begin{aligned}\iint_{\mathbb{R}^+\times\mathbb{R}^d} \varphi(t,x)\rho(dt,dx) &= \int_{\mathbb{R}^d}\int_0^\infty \varphi(t,x)\rho_x(dt)\nu(x)dx \\ &= -\iint_{\mathbb{R}^+\times\mathbb{R}^d}\left[\int_t^\infty \partial_t\varphi(w,x)dw\right]\rho_x(dt)\nu(x)dx \\ &= -\int_{\mathbb{R}^d}\int_0^\infty \partial_t\varphi(w,x)\left[\int_0^w \rho_x(dt)\right]\nu(x)dwdx.\end{aligned}$$

Recall that by Lemma A.2 and $\tau > 0$ a.s. that $\int_0^w \rho_x(dt) = \rho_x((0,w))$ a.e. Now as $\tau > 0$ a.s., Lemma C.1 implies that $\tau \leq s(X_\tau)$ a.s. Since $s = s^U$ ν-a.e. by Theorem 4.17, we have $\tau \leq s(X_\tau) = s^U(X_\tau)$ a.s. Hence for $t > s^U(x)$ (equivalently, $\eta(t,x) > 0$ by Lemma 4.11), $\rho_x((0,t)) = 1$ ν-a.e., which implies (5.16).

This together with by Lemmas A.2 and A.3 imply

$$\iint_{\mathbb{R}^+\times\mathbb{R}^d} \partial_t\varphi \cdot \rho((0,t),x)dtdx = \iint_{\mathbb{R}^+\times\mathbb{R}^d} (\partial_t\varphi \cdot \nu(x)\chi_{\{\eta>0\}} + \partial_t\varphi \cdot \rho((0,t),x)\chi_{\{\eta=0\}})dtdx.$$

As in Theorem 5.5, $\rho((0,t),x)\chi_{\{\eta=0\}} = \kappa_2(t,x)\chi_{\{\eta=0\}}$, and thus we deduce (5.3). □

Theorem 5.5 implies that it suffices to verify the initial condition $\kappa_1(0,\cdot)\chi_{E^c} = \nu\chi_{E^c}$ to establish that the Eulerian variable η (with Type (I) cost) solves ($St_{1,\nu}$). Indeed, from (3.20) and (5.11), we have that for any $\varphi \in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d)$,

$$\iint -\partial_t\varphi(\eta - \nu\chi_{\{\eta>0\}} - \rho((t,\infty),x)\chi_{\{\eta=0\}}) + \iint (-\Delta)^s\varphi \cdot \eta = \int \varphi(0,x)(\mu(x) - \nu(x)).$$

By Proposition 4.18, this expression is equal to

$$\iint -\partial_t\varphi(\eta - \nu\chi_{\{\eta>0\}} + (-\Delta)^s w \cdot \chi_{\{\eta=0\}}) + \iint (-\Delta)^s\varphi \cdot \eta = \int \varphi(0,x)(\mu(x) - \nu(x)).$$

Hence in order to obtain that η is a weak solution to ($St_{1,\nu}$) with initial data μ and initial domain E, it remains to verify $\kappa_1(0,\cdot)\chi_{E^c} = \nu\chi_{E^c}$. We will prove this in Lemma 5.7.

Similarly Theorem 5.6 implies that it suffices to verify the initial condition $\nu\chi_E = 0$ to establish that the Eulerian variable η (with Type (II) cost) solves ($St_{2,\nu}$). Indeed, as above, Proposition 4.18 implies $\rho((0,t),x) \cdot \chi_{\{\eta(t,x)=0\}} = -(-\Delta)^s w \cdot \chi_{\{\eta(t,x)=0\}}$, and thus by (5.16) and (3.20), for any test function $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\iint -\partial_t\varphi(\eta + \nu\chi_{\{\eta>0\}} - (-\Delta)^s w \cdot \chi_{\{\eta=0\}}) + \iint (-\Delta)^s\varphi \cdot \eta = \int \varphi(0,x)\mu(x).$$

We will prove that $\nu\chi_E = 0$ in Lemma 5.8.

Now, we verify the initial condition. Recall that $E = E(\eta)$ denotes the initial domain of η, defined in (5.7).

Lemma 5.7 (Initial data for Type (I))**.** *Let τ be the optimal stopping time of $\mathcal{P}_0(\mu,\nu)$ for Type (I) and (η,ρ) be the associated Eulerian variables. Then $\kappa_1(0,\cdot) = \nu(\cdot)$ a.e. outside of E if $\tau > 0$ a.s.*

Proof. Since $\{\eta(t,\cdot) > 0\}$ is non-increasing in t, we have $\mathbb{R}^+ \times E^c \subset \{\eta = 0\}$. Also by Proposition 4.18, $-(-\Delta)^s w(t,x) = \rho((t,\infty),x)$ on $\mathbb{R}^+ \times E^c$. Noting that the condition $\tau > 0$ a.s. implies $\nu(\cdot) = \rho([0,\infty),\cdot) = \rho((0,\infty),\cdot)$, for $x \in E^c$, we have

$$\nu(x) = \rho((0,\infty),x) = -(-\Delta)^s w(0,x) = \kappa_1(0,x),$$

where the last identity follows from (5.6). ☐

Lemma 5.8 (Initial data for Type (II))**.** *With the same notation and assumptions as Theorem 5.6 for Type (II), we have $\nu = 0$ a.e. in E if $\tau > 0$ a.s.*

Proof. Let $s : \mathbb{R}^d \to \mathbb{R}^+$ be the barrier function associated to τ. Then since $\tau > 0$ a.s., by Lemma C.1, we have $\tau \le s(X_\tau)$. Also by Theorem 4.17, $s(X_\tau) = s^U(X_\tau)$ a.s. Hence, as $\tau \le s^U(X_\tau)$, we have $(\tau, X_\tau) \in \{w = 0\} = \{\eta = 0\}$ a.s., implying that $\eta(\tau, X_\tau) = 0$ a.s. Since $\eta(t,x) > 0$ for any $x \in E$ and $t \in \mathbb{R}^+$, we deduce that $X_\tau \notin E$ a.s. Recalling $X_\tau \sim \nu$, we conclude the proof. ☐

Observe that η satisfies the properties (c) and (c') in Definition 5.3 due to (4.5) and (4.8), along with Lemma 4.1 and Proposition 4.18. Then, applying (5.5) and (5.3), together with Theorem 5.5 and Lemma 5.7 for Type (I), and Theorem 5.6 with Lemma 5.8 for Type (II), allows us to conclude:

Theorem 5.9 (Solutions to the Fractional Stefan Problem)**.** *Let (η,ρ) be the Eulerian variables associated to the optimal stopping time τ of $\mathcal{P}_0(\mu,\nu)$, and assume that $\tau > 0$ a.s. Then*

- *For Type (I), η solves ($St_{1,\nu}$) with initial data $(\mu, E(\eta))$.*
- *For Type (II), η solves ($St_{2,\nu}$) with initial data $(\mu, E(\eta))$.*

We will later show that for some solutions of ($St_{1,\nu}$), the initial trace can be characterized as the positivity set of the elliptic obstacle problem (see Theorem 6.7) for the solutions we generate in Section 7. Also for ($St_{2,\nu}$), when μ has no mushy region, the initial trace of solutions we construct later will be shown to be the support of μ.

5.2. **Solutions to Weighted Stefan Problems.** In this section, we interpret the solution to the weighted Stefan problem as the Eulerian variables (η, ρ) from Lemma 3.7 associated to $\mathcal{P}_0(\mu, \nu)$ from (1.4). Assume that

$$0 \leq \nu \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d), \quad 0 \leq \eta_0 \in L^\infty(\mathbb{R}^d) \text{ of compact support.} \tag{5.17}$$

Theorem 5.10 (Consistency of ($St_{1,\nu}$))**.** *For ν and η_0 satisfying (5.17), let η be a weak solution to ($St_{1,\nu}$) with initial data (η_0, E). Define $s(x) := \sup\{t : \eta(t,x) > 0\}$, and let $\kappa_1 := -[(-\Delta)^s v_1]$ where $v_1(t,x) = \int_t^\infty \eta(a,x)da$ is from Definition 5.3. Now we define $\rho \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ as follows: for any $t > 0$,*

$$\rho((t,\infty),\cdot) := \nu\chi_{\{\eta(t,\cdot)>0\}}\chi_{\{s(\cdot)<\infty\}} + \kappa_1(t,\cdot)\chi_{\{\eta(t,\cdot)=0\}} \quad \textit{with } \rho(0,\cdot) \equiv 0. \tag{5.18}$$

Then, (η, ρ) is the optimal Eulerian variables from Lemma 3.7 for a cost of Type (I) between η_0 and $\tilde{\nu}$ whose density is given by

$$\tilde{\nu}(x) := \rho([0,\infty), x) = \nu(x)\chi_{\{0<s(x)<\infty\}} + \kappa_1(0,x)\chi_{\{s(x)=0\}}. \tag{5.19}$$

Proof. First, let us verify the second identity in (5.19). Since E is the collection of limit points of $\{\eta(t,\cdot) > 0\}$ as $t \to 0^+$ (see Definition 5.3), by the fact $\rho(\{0\} \times \mathbb{R}^d) = 0$ along with the dominated convergence theorem,

$$\rho([0,\infty) \times A) = \lim_{t \to 0^+} \rho((t,\infty) \times A) = \int_A \Big(\nu(x)\chi_E\chi_{\{s(x)<\infty\}} + \kappa_1(0,x)\chi_{E^c}\Big)\, dx.$$

This implies (5.19), since $E = \{x \in \mathbb{R}^d : s(x) > 0\}$.

Next note that, since the support of $\eta(t,\cdot)$ is non-increasing in time and E is compact, there is $r > 0$ such that $B_r(0)$ contains the support of $\eta(t,\cdot)$ for all $t \geq 0$. We show that (η, ρ) solve (3.20) in $(0,\infty) \times B_r(0)$. By the definition of ρ, for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times B_r(0))$,

$$\begin{aligned}
&\iint \partial_t\varphi \cdot \rho((t,\infty),x)dtdx \\
&\quad = \iint \partial_t\varphi \cdot (\nu\chi_{\{\eta>0\}}\chi_{\{s<\infty\}} + \kappa_1\chi_{\{\eta=0\}})dtdx \\
&\quad = \iint \partial_t\varphi \cdot (\nu\chi_{\{\eta>0\}} + \kappa_1\chi_{\{\eta=0\}})dtdx - \iint \partial_t\varphi \cdot \nu\chi_{\{\eta>0\}}\chi_{\{s=\infty\}}dtdx \\
&\quad = \iint \partial_t\varphi \cdot (\nu\chi_{\{\eta>0\}} + \kappa_1\chi_{\{\eta=0\}})dtdx + \int \varphi(0,\cdot)\nu\chi_{\{s=\infty\}}dx,
\end{aligned}$$

where the last equality follows from the following reason: As $\{\eta > 0\}$ is open in space-time and $\{\eta(t,\cdot) > 0\}$ is monotone in time (see Definition 5.3), the definition of $s(x)$ yields $\{\eta(t,x) > 0\} = \{t < s(x)\}$ and thus

$$\int \partial_t\varphi \cdot \chi_{\{\eta>0\}}dt = \int_0^{s(x)} \partial_t\varphi(t,x)dt = \varphi(s(x),x) - \varphi(0,x).$$

Observe that one can also write

$$\begin{aligned}
\iint \partial_t\varphi \cdot \rho((t,\infty),x)dtdx &= \iint \partial_t\varphi \cdot \tilde{\nu}dtdx - \iint \partial_t\varphi \cdot \rho((0,t],x)dtdx \\
&= -\int \varphi(0,\cdot)\tilde{\nu}dx + \iint \varphi\rho(dt,dx),
\end{aligned}$$

where the first equality is due to the definition of $\tilde{\nu}$ and the second equality follows from the integration by parts (Lemma A.3), along with the fact $\rho(\{0\} \times \mathbb{R}^d) = 0$.

Putting the above displays together,

$$\begin{aligned}\iint \varphi\rho(dt,dx) &= \iint \partial_t\varphi\cdot(\nu\chi_{\{\eta>0\}}+\kappa_1\chi_{\{\eta=0\}})dtdx+\int\varphi(0,\cdot)\nu\chi_{\{s=\infty\}}dx+\int\varphi(0,\cdot)\tilde{\nu}dx\\ &= \iint \partial_t\varphi\cdot(\nu\chi_{\{\eta>0\}}+\kappa_1\chi_{\{\eta=0\}})dtdx+\int\varphi(0,\cdot)\nu\chi_E dx+\int\varphi(0,\cdot)\kappa_1(0,\cdot)\chi_{E^c}dx, \end{aligned} \tag{5.20}$$

where the last identity follows from the fact that $E=\{s(\cdot)>0\}$ is a disjoint union of $E\cap\{s(\cdot)<\infty\}$ and $E\cap\{s(\cdot)=\infty\}=\{s(\cdot)=\infty\}$. Therefore, by the weak form of ($St_{1,\nu}$), namely (5.9), we conclude that (η,ρ) solves (3.20) with initial data η_0 in $(0,\infty)\times B_r(0)$.

Finally, we show that (η,ρ) is the Eulerian variable associated to the optimizer of $\mathcal{P}_0(\eta_0,\tilde{\nu})$ for a Type (I) cost. Let us define $\tau:=\inf\{t\geq 0: t\geq s(X_t)\}$. Since τ is bounded by the exit time of $B_r(0)$, it has finite moments. Let $(\overline{\eta},\overline{\rho})$ be the Eulerian variables associated to (η_0,τ). Then as $\{\eta=0\}$ is closed, $\overline{\eta}[\{\eta=0\}]=0$ and $\overline{\rho}[\{\eta=0\}]=1$. Hence by Theorem C.2, $\overline{\eta}=\eta$ and $\overline{\rho}=\rho$, concluding the proof. ☐

Remark 5.11. *The formula of ρ in Theorem 5.10 is consistent with the one in (5.5). This is because for the cost of Type (I), we have $\nu=0$ on $\{s=\infty\}$.*

For the consistency result of (St_2), the corresponding result to Theorem 5.10 is implied from Theorem 7.1 when there is no initial mushy region.

5.3. **Enthalpy variable.** In this section, we show that in the melting scenario, the weighted temperature-based Stefan problem ($St_{2,\nu}$) can be rewritten in terms of the associated enthalpy variable. This connects our approach to the several recent papers [21, 19, 20] which consider a particular case of the enthalpy equation, (St_h), to study fractional Stefan problems. Our approach furthermore connects (St_h) to the parabolic obstacle problem studied in [12, 46, 5, 11], extending the well-known connection in the local case to the non-local case for the first time.

Definition 5.12 (Enthalpy variable)**.** *Assume that ν is a bounded and nonnegative function in $\mathbb{R}^d$. Let η be the solution to ($St_{1,\nu}$) or ($St_{2,\nu}$) with initial domain E. Then the enthalpy variable h is defined as*

$$h(t,x):=\begin{cases}\eta(t,x)+\rho([0,t),x)-\nu(x) & \textit{for } (St_{1,\nu}),\\ \eta(t,x)+\tilde{\rho}([0,t),x)+\nu(x)\chi_E & \textit{for } (St_{2,\nu}),\end{cases}$$

where ρ for ($St_{1,\nu}$) is from (5.18) and $\tilde{\rho}$ for ($St_{2,\nu}$) is defined via

$$\tilde{\rho}([0,t),\cdot):=\nu(\cdot)\chi_{\{\eta(t,\cdot)>0\}}\chi_{E^c}+\kappa_2(t,\cdot)\chi_{\{\eta(t,\cdot)=0\}} \textit{ with } \tilde{\rho}(0,\cdot)\equiv 0,$$

where $\kappa_2:=[-(-\Delta)^s v_2]$ and $v_2(t,x):=\int_0^t\eta(a,x)da$. Our definition of $\tilde{\rho}$ is motivated from (5.3) and Lemma 5.8.

Remark 5.13. *For Type (II), η from Lemma 3.7 solves ($St_{2,\nu}$) due to Theorem 5.9. In this case (5.3) yields $\tilde{\rho}([0,t),\cdot)=\rho([0,t),\cdot)-\nu\chi_E$. Hence, our definition of the enthalpy coincides with (1.1) . We have chosen to remove the $\nu\chi_E$ term from ρ to emphasize its role as part of the initial enthalpy (see Theorem 5.15 below for details).*

Recall that (5.5) and (5.3) (for Type (I) and (II) respectively) allow us to rewrite h in terms of η when $\tau>0$ a.s. Below we show that the converse holds for the melting case ($St_{2,\nu}$) and in particular for (St_2). Namely in this case we can recast η in terms of the enthalpy variable, leading to an enthalpy-based formulation of ($St_{2,\nu}$). When $\nu\equiv 1$, our definition is equivalent to that of [21].

Definition 5.14 (Enthalpy form of ($St_{2,\nu}$))**.** *Let ν and h_0 be two bounded and nonnegative functions in $\mathbb{R}^d$. We say that $h \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is a weak solution to*

$$(St_{h,\nu}) \qquad \partial_t h = -(-\Delta)^s (h-\nu)_+ \quad \text{in } (0,\infty)\times\mathbb{R}^d, \quad h(0,\cdot) = h_0,$$

if for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} (-h\cdot\partial_t\varphi + (h-\nu)_+(-\Delta)^s\varphi)dxdt = \int_{\mathbb{R}^d} \varphi(0,\cdot)h_0 dx. \tag{5.21}$$

In particualr, if h is a weak solution to ($St_{h,\nu}$) with $\nu \equiv 1$, then we say that h is a weak solution to (St_h).

Note that in our definition of solutions to ($St_{2,\nu}$) and ($St_{h,\nu}$), we view ν as a target measure. In contrast, in the equations (St_2) and (St_h), we regard $\nu \equiv 1$ as a weight.

Theorem 5.15. *Let ν be a bounded and nonnegative function in $\mathbb{R}^d$.*

1. *Assume that η solves ($St_{2,\nu}$) with initial data (η_0, E). Then the enthalpy h from Definition 5.12 is a weak solution of ($St_{h,\nu}$) with $h_0 = \eta_0 + \nu\chi_E$.*
2. *Assume that η solves (St_2) (see Definition 5.4). Then the enthalpy h from Definition 5.12 is the unique weak solution of (St_h) with $h_0 = \eta_0 + \chi_E$, and η is the unique weak solution of (St_2) with initial data (η_0, E).*

Proof. Assume that η solves ($St_{2,\nu}$). Let us first show that $\eta(t,x) = (h(t,x) - \nu(x))_+$. To check this, fix $t > 0$. Then if $\eta(t,x) > 0$, then we have from Definition 5.12 that $\tilde{\rho}([0,t),x) = \nu(x)\chi_{E^c}(x)$ and thus $h(t,x) = \eta(t,x)+\nu(x)$. Next if $\eta(t,x) = 0$, then we have $h(t,x) = \kappa_2(t,x) \leq \nu(x)$ (see Definition 5.3), and thus $(h-\nu)_+(t,x) = 0$. Hence we have verified $\eta = (h-\nu)_+$, implying that for any test function $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, we have from (5.10) that

$$\int_0^\infty \int_{\mathbb{R}^d} (h(-\partial_t\varphi) + (h-\nu)_+(-\Delta)^s\varphi)dxdt = \int_{\mathbb{R}^d} \varphi(0,\cdot)(\eta_0 + \nu\chi_E)dx. \tag{5.22}$$

This shows that h is a weak solution to ($St_{h,\nu}$) with initial data $\eta_0 + \nu\chi_E$.

Next, we consider the equation (St_2). We claim that $(h-1)_+ = \eta$. Fix $t > 0$. Recalling Definition 5.4, $\nu(x) = 1$ when $\eta(t,x) > 0$, implying $h(t,x) = \eta(t,x) + 1$. Also since $\kappa_2(t,x) \leq \nu(x) \leq 1$, if $\eta(t,x) = 0$, then $(h(t,x)-1)_+ = (\kappa_2(t,x)-1)_+ = 0$. Hence we have verified $(h-1)_+ = \eta$. Since $E \subset \{x \in \mathbb{R}^d : \eta(t,x) > 0 \text{ for some } t > 0\}$, recalling Definition 5.4 again, we have $\nu\chi_E = \chi_E$. Thus by (5.10), for any test function φ,

$$\int_0^\infty \int_{\mathbb{R}^d} (h(-\partial_t\varphi) + (h-1)_+(-\Delta)^s\varphi)dxdt = \int_{\mathbb{R}^d} \varphi(0,\cdot)(\eta_0 + \chi_E)dx. \tag{5.23}$$

In particular, h is a weak solution to (St_h) with initial data $\eta_0 + \chi_E$. The uniqueness of weak solutions of (St_h) follows from [21, Theorem 2.3]. Lastly observe that this yields the uniqueness of $\eta = (h-1)_+$.

□

Based on our probabilistic characterization of h, we are able to prove a rate of convergence for h as $t \to \infty$ for Type (II). This type of result does not appear to be known for (St_2) in the literature.

Lemma 5.16. *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ satisfy Assumption 1. Let (η, ρ) be the Eulerian variables associated to τ, the optimal stopping time for $\mathcal{P}_0(\mu,\nu)$ with a Type (II) cost. There exist constants $\gamma = \gamma(r,d,s) > 0$ and $C = C(r,d,\mu,s) > 0$ (here, $r > 0$ is from (4) in Assumption 1) such that if $\tau > 0$ a.s., then for any $t \geq 0$,*

$$||h(t,\cdot) - \nu||_{L^1(\mathbb{R}^d)} \leq Ce^{-\gamma t},$$

where h is the enthalpy variable from Definition 5.12.

Proof. From Proposition 4.18, we have that $\kappa_2(t,x) = \rho((0,t),x)$, and thus by definition of h,

$$h(t,x) - \nu(x) = \eta(t,x) + \Big(\nu(x)\cdot\chi_{\{\eta>0\}}(t,x) + \rho((0,t),x)\cdot\chi_{\{\eta=0\}}(t,x) - \nu(x)\Big)\,. \tag{5.24}$$

Now from Lemma 3.7 and in particular (3.20), we have $\{x : \eta(t,x) > 0\} \subset B_r(0)$ for all $t \geq 0$. By Cauchy-Schwarz and Lemma 3.9, there exists $K = K(r,d), \gamma = \gamma(r,d,s) > 0$ such that

$$||\eta(t,\cdot)||_{L^1(\mathbb{R}^d)} \leq K||\eta(t,\cdot)||_{L^2(\mathbb{R}^d)} \leq Ke^{-\gamma t}||\mu||_{L^2(\mathbb{R}^d)}.$$

Using the triangle inequality in (5.24) along with the above inequality, we obtain

$$\begin{aligned} ||h(t,\cdot)-\nu||_{L^1(\mathbb{R}^d)} &\leq Ke^{-\gamma t}||\mu||_{L^2(\mathbb{R}^d)} + ||(\nu - \rho((0,t),\cdot))\chi_{\{\eta=0\}}||_{L^1(\mathbb{R}^d)} \\ &\leq Ke^{-\gamma t}||\mu||_{L^2(\mathbb{R}^d)} + ||\nu - \rho((0,t),\cdot)||_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Since $\nu(x) = \rho((0,\infty),x)$, we have $\nu(x) - \rho((0,t),x) = \rho([t,\infty),x)$. Hence we arrive at

$$||h(t,\cdot)-\nu||_{L^1(\mathbb{R}^d)} \leq Ke^{-\gamma t}||\mu||_{L^2(\mathbb{R}^d)} + \rho([t,\infty)\times\mathbb{R}^d). \tag{5.25}$$

Recalling that $\rho \sim (\tau, X_\tau)$ from Lemma 3.7, we have $\rho([t,\infty)\times\mathbb{R}^d) = \mathbb{P}[\tau \geq t]$. Since $\mathbb{E}[e^{\gamma\tau}] \leq \mathbb{E}[e^{\gamma\tau_r}] < \infty$ (see Remark 3.2), by Markov's inequality $\mathbb{P}[\tau \geq t] \leq e^{-\gamma t}\mathbb{E}[e^{\gamma\tau_r}]$. Putting these observations into (5.25), we see that

$$||h(t,\cdot)-\nu||_{L^1(\mathbb{R}^d)} \leq e^{-\gamma t}(K||\mu||_{L^2(\mathbb{R}^d)} + \mathbb{E}[e^{\gamma\tau_r}]),$$

concluding the proof. ☐

5.3.1. *A remark on Continuity of Enthalpy variable.* In the case of equation (St_h), it was shown [21] that for self-similar solutions, the enthalpy variable is continuous. This is rather surprising because in the local case, the enthalpy variable has a jump discontinuity across the free boundary. It would be thus interesting to consider regularity of the enthalpy variable in the nonlocal setting.

One possible way to obtain a regularity of the enthalpy variable is to use the newly obtained connection of ($St_{1,\nu}$) and ($St_{2,\nu}$) with the parabolic nonlocal obstacle problem (4.9). It was shown in [12, Theorem 2.1] that if $\nu - \mu$ is sufficiently smooth, that the solution w of (4.9) for Type (II) has a continuous $(-\Delta)^s w$. This is in contrast to the local case, where the optimal spatial regularity for w is $C^{1,1}$. Continuity of $(-\Delta)^s w$ implies from our enthalpy formula for Type (II) and Proposition 4.18 that the enthalpy for ($St_{2,\nu}$) is continuous. In addition, the free boundary regularity for (4.9) with Type (II) is well studied when $\nu - \mu$ is sufficiently smooth (see [46, 5, 11]).

These results unfortunately do not apply to (St_2) since we will see in Theorem 7.1 later that we require $\mu > 1$ in its support, which is equivalent to the absence of an initial mushy region. This is rather natural since one expects irregularity near the initial data. As in the local case, one would need a localization argument to show continuity of the enthalpy variable away from the initial time. It is unclear to the authors at the moment how such localization can be carried out for the parabolic nonlocal obstacle problem.

Definition 5.12 suggests that one can might be able to show the continuity of the enthalpy using probabilistic arguments. Let us focus on Type (I) case, where the optimal stopping time is given by $\tau = \inf\{t \geq 0 : t \geq s(X_t)\}$ for some measurable function $s : \mathbb{R}^d \to \mathbb{R}^+$ (see Theorem 3.11). By Lemma 4.11, $\{\eta(t,x) > 0\} = \{t < s(x)\}$. Thus, recalling $h = \eta - \nu\chi_{\{\eta>0\}} - \kappa_1\chi_{\{\eta=0\}}$, if η is continuous, then by using a version of κ_1 for Type (I), we have

$$\lim_{t\to s(x)^+} -h(t,x) = \nu(x)\mathbb{P}[\tau > s(x)|X_\tau = x] \text{ and } \lim_{t\to s(x)^-} -h(t,x) = \nu(x).$$

Hence, if $\mathbb{P}[\tau = s(x)|X_\tau = x] = 0$ and $s(x)$ is continuous, then $h(t,x)$ is continuous in time across the free boundary.

We expect that under mild assumptions on $s(x)$, such as all of its level sets having measure zero, we have $\mathbb{P}[\tau = s(x) \mid X_\tau = x] = 0$. This is because it seems unlikely for the particles to land on the graph of $s(x)$ under this assumption. There are some recent investigations in computing $\mathbb{P}[\tau = s(x) \mid X_\tau = x]$ for one dimensional Levy processes under some strong assumptions on $s(x)$ (see [14, 16]).

6. Optimal Target Problem

In this section, we consider the optimal target problem which will yield weak solutions of (St_1) and (St_2). Recall that $\mathcal{C}(\tau)$ denotes the cost defined in (1.2), i.e. $\mathcal{C}(\tau) = \mathbb{E}\left[\int_0^\tau L(s, X_s)ds\right]$, where the Lagrangian L satisfies Assumption 2. For the family of measures $\mathcal{A}_{\mu,M,R}$ given in (3.7), we associate the corresponding optimal stopping time:

$$\tilde{\mathcal{A}}_{\mu,M,R} := \{(\nu,\tau) : \nu \in \mathcal{A}_{\mu,M,R},\ \tau \text{ is an optimizer of } \mathcal{P}_0(\mu,\nu) \text{ in } (1.4)\}.$$

Throughout this section, we assume that the initial measure $\mu \in L^\infty(\mathbb{R}^d)$ is compactly supported and f satisfies Assumption 3.

Consider the constrained family of target measures and stopping times

$$\mathcal{A}_{R,f}(\mu) := \{(\nu,\tau) \in \bigcup_{M>0} \tilde{\mathcal{A}}_{\mu,M,R} \text{ with } \nu \leq f\},$$

and define

$$\mathcal{A}_f(\mu) := \bigcup_{R>0} \mathcal{A}_{R,f}(\mu).$$

Now, we consider the optimal target problem associated with f:

$$\mathcal{P}_f(\mu) := \inf\{\mathcal{C}(\tau) : (\nu,\tau) \in \mathcal{A}_f(\mu)\}. \tag{6.1}$$

It will also be useful to define the truncated problem

$$\mathcal{P}_{R,f}(\mu) := \inf\{\mathcal{C}(\tau) : (\nu,\tau) \in \mathcal{A}_{R,f}(\mu)\}.$$

By Lemma 3.6, $\mathcal{A}_{R,f}(\mu)$ is non-empty for sufficiently large $R > 0$, and thus $\mathcal{A}_f(\mu)$ is non-empty.

6.1. Existence and Uniqueness.

In this section, we establish the well-posedness of the variational problem $\mathcal{P}_f(\mu)$. To accomplish this, we first show the monotonicity property of the variational problem, a central ingredient for the analysis in the local case [37].

Theorem 6.1 (Monotonicity)**.** *Assume that $\mu_1 \leq \mu_2$ and $\nu_i \leq f$ for $i = 1,2$. Let τ_i be the optimal stopping time for $\mathcal{P}_0(\mu_i,\nu_i)$. Then, the following holds:*

(1) If (ν_1,τ_1) is optimal for $\mathcal{P}_f(\mu_1)$, then as stopping times from the initial distribution μ_1, we have that $\tau_1 \leq \tau_2$ a.s.

(2) If (ν_i,τ_i) is optimal for $\mathcal{P}_f(\mu_i)$ for $i = 1,2$, then $\nu_1 \leq \nu_2$ a.e.

If we further assume that $\tau_i \leq \tau_R = \inf\{t : X_t \notin B_R(0)\}$ ($R > 0$ is some constant) for $i = 1,2$, then the conclusions (1) *and* (2) *also hold for the version of $\mathcal{P}_{R,f}(\mu_i)$.*

Proof. The proof is analogous to that of [37, Theorem 7.1]. The only difference is that if R_i is the barrier associated to τ_i, then one can use Lemma C.1 to conclude that $(\tau_i, X_{\tau_i}) \in R_i$ (if $\tau_i > 0$ a.s. for Type (II)). ☐

Theorem 6.2 (Existence and Uniqueness of Optimizer)**.** *There exists an optimal pair (ν,τ) for $\mathcal{P}_f(\mu)$ with a $R = R(\mu,d,s,f) > 0$ such that $\tau \leq \tau_R$ almost surely. In addition, the optimizer is unique.*

Proof. We first show the existence of an optimizer for the truncated problem $\mathcal{P}_{R,f}(\mu)$ for large R. Since f satisfies Assumption 3, $\mathcal{A}_{R,f}(\mu) \neq \emptyset$ for large enough $R > 0$. Let $(\nu_n,\tau_n)_{n\geq 1}$ be the minimizing sequence for $\mathcal{P}_{R,f}(\mu)$. Since $\{\nu_n\}_{n\geq 1}$ is tight by Lemma 3.5, up to the subsequence, $(\nu_n)_{n\geq 1}$ converges weakly to some $\nu_{R,f}$. Denoting by $\tau_{R,f}$ the optimal stopping time for $\mathcal{P}_0(\mu,\nu_{R,f})$, we have $(\nu_{R,f},\tau_{R,f}) \in \mathcal{A}_{R,f}(\mu)$.

We claim that $\nu_{R,f}$ is the optimal target measure for $\mathcal{P}_{R,f}(\mu)$. Note that by Appendix C, $\mathcal{P}_0(\mu,\nu_n) = \mathcal{D}_0(\mu,\nu_n)$ ($\mathcal{D}_0$ denotes the dual problem of $\mathcal{P}_0$, see (C.1)). Also, since the map $\nu \mapsto \mathcal{D}_0(\mu,\nu)$ is convex, it is lower semi-continuous with respect to the weak topology. Thus,

$$\mathcal{P}_0(\mu,\nu_{R,f}) = \mathcal{D}_0(\mu,\nu_{R,f}) \le \liminf_{n\to\infty} \mathcal{D}_0(\mu,\nu_n) = \liminf_{n\to\infty} \mathcal{P}_0(\mu,\nu_n) = \liminf_{n\to\infty} \mathcal{C}(\tau_n),$$

verifying the claim. Hence, $(\nu_{R,f},\tau_{R,f})$ is optimal for $\mathcal{P}_{R,f}(\mu)$.

Observe that for $R' > R$, the optimal pair $(\nu_{R,f},\tau_{R,f})$ is admissible for $\mathcal{P}_{R',f}(\mu)$ as well. Thus, by Theorem 6.1, $\tau_{R',f} \le \tau_{R,f}$ a.s., implying that $\nu_{R'}$ is admissible for $\mathcal{P}_{R,f}(\mu)$ and $(\nu_{R',f},\tau_{R',f}) \in \mathcal{A}_{R,f}(\mu)$. Theorem 6.1 again yields that $\tau_{R,f} \le \tau_{R',f}$, thus $\tau_{R,f} = \tau_{R',f}$ a.s. and in particular $\nu_{R,f} = \nu_{R',f}$ a.e. This shows that $(\nu_{R,f},\tau_{R,f})$ is an optimal pair for $\mathcal{P}_f(\mu)$ as long as $R > 0$ is sufficiently large so that $\mathcal{A}_{R,f}(\mu) \neq \emptyset$.

The uniqueness of the optimal pair (ν,τ) follows from Theorem 6.1. ☐

By following the argument in [37, Theorem 7.6], we obtain the universality result for the optimal target measure.

Theorem 6.3 (Universality). *Let ν_i be the optimal target measure for $\mathcal{P}_f(\mu)$ with costs $\mathcal{C}_i$, $i = 1,2$, with different Lagrangians and of possibly different Types (I) or (II). Then $\nu_1 = \nu_2$ a.e.*

In addition, the monotocity result (Theorem 6.1) implies the following L^1 contraction:

Theorem 6.4 (L^1 Contraction). *Assume that ν_i is optimal for $\mathcal{P}_f(\mu_i)$, $i = 1,2$. Then,*

$$||(\nu_1-\nu_2)_+||_{L^1(\mathbb{R}^d)} \le ||(\mu_1-\mu_2)_+||_{L^1(\mathbb{R}^d)}.$$

Proof. This is a consequence of Theorem 6.1 (see [37, Theorem 7.8]). ☐

This also implies a BV bound when f is constant:

Theorem 6.5 (BV Bound). *Assume that f is constant and that ν is optimal for $\mathcal{P}_f(\mu)$. Then,*

$$||\nu||_{\mathrm{BV}} \le ||\mu||_{\mathrm{BV}}.$$

Proof. This follows from the L^1 contraction and that the problem is homogenous with respect to spatial shifts (see [37, Theorem 7.10]). ☐

6.2. **Identification the optimal target measure.** In this section, we show that the optimal target measure saturates the constraint for $\mathcal{P}_f(\mu)$.

Theorem 6.6 (Saturation on the Active Region). *Let (ν^*,τ^*) be the optimizer for $\mathcal{P}_f(\mu)$ and (η,ρ) be the associated Eulerian variables. Then $\nu^* = f$ a.e. in $\{x \in \mathbb{R}^d : \eta(t,x) > 0$ for some $t > 0\}$.*

Proof. Let us consider the Type (I) case since parallel arguments work for Type (II). Since $\nu^* \le f$, we have $\nu^* = f = 0$ when $f = 0$. Thus, it suffices to show that for any $t > 0$,

$$|\{\nu^* < f\} \cap \{f > 0\} \cap \{\eta(t,\cdot) > 0\}| = 0. \tag{6.2}$$

For $\delta > 0$, define $H_\delta := \{\nu^* \le f - \delta\} \cap \{f > 2\delta\}$. As $\{\nu^* < f\} \cap \{f > 0\} = \bigcup_{n\in\mathbb{N}} H_{1/n}$, it reduces to show that for any $t > 0$,

$$|H_\delta \cap \{\eta(t,\cdot) > 0\}| = 0. \tag{6.3}$$

Assume that $H_\delta \cap \{\eta(a,\cdot) > 0\}$ has a positive Lebesgue measure for some $a > 0$. Since the positive set of η is non-increasing in time,

$$\iint_{[0,a]\times H_\delta} \eta dtdx > 0.$$

By a characterization of η (Lemma 3.7) and Fubini's theorem,

$$\int_0^\infty \mathbb{P}[t < \tau^*, (t, X_t) \in [0,a] \times H_\delta] dt > 0.$$

Therefore, there exists $\bar{t} > 0$ such that

$$\mathbb{P}[\bar{t} < \tau^*, X_{\bar{t}} \in H_\delta] > 0. \tag{6.4}$$

One can deduce that this violates the optimality of τ^*. Indeed, if (6.4) holds, then one can construct a stopping time $\overline{\tau}$ which attains the strictly smaller value of $\mathcal{P}_f(\mu)$, by stopping some of the active particles at time $\bar{t}$. We refer to [37, Theorem 8.3] (Case (II)) for details in the case of Brownian motion. ☐

Given Theorem 6.6, one can characterize the optimal target measure ν for $\mathcal{P}_f(\mu)$ in terms of the non-local obstacle problem. Note that by Theorem 6.3, ν is the same for the cost of either Type (I) or Type (II).

Theorem 6.7. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be the unique continuous viscosity solution of*

$$\min\{(-\Delta)^s u + f - \mu, u\} = 0, \quad \lim_{|x|\to\infty} u(x) = 0 \text{ in } \mathbb{R}^d. \tag{6.5}$$

Then the optimal target measure ν for $\mathcal{P}_f(\mu)$ is given by $\nu = \mu - (-\Delta)^s u$ in $\mathbb{R}^d$. Also it satisfies that $\nu = f$ in $E := \{x \in \mathbb{R}^d : u(x) > 0\}$.

Proof. By Theorem 6.3, it suffices to only consider Type (I). Let $(U_{\mu_t})_{t\geq 0}$ be the potential flow associated to the optimizer (ν, τ) of $\mathcal{P}_f(\mu)$. By (4.6) and Theorem 6.6, $\nu = f$ a.e. in $\{x \in \mathbb{R}^d : w(t,x) > 0 \text{ for some } t > 0\}$ (recall that w is defined in (4.5)). Since w is continuous and non-increasing in time, setting

$$E' := \{x \in \mathbb{R}^d : w(0,x) > 0\}, \tag{6.6}$$

we obtain

$$\nu = f \quad \text{in } E'.$$

Hence $u(x) := w(0,x) = U_\mu(x) - U_\nu(x)$ satisfies

$$(-\Delta)^s u = \mu - \nu = \mu - f \quad \text{in } E'.$$

In addition, since $\nu \leq f$,

$$0 = (-\Delta)^s u + \nu - \mu \leq (-\Delta)^s u + f - \mu \quad \text{in } \mathbb{R}^d.$$

This shows that

$$\min\{(-\Delta)^s u + f - \mu, u\} = 0 \quad \text{in } \mathbb{R}^d.$$

Since ν has a compact active region, by Lemma 3.5, ν satisfies the decay condition (3.6). As μ is compactly supported, $u = U_\mu - U_\nu$ decays at infinity.

Hence we deduce that u satisfies (6.5). Due to the uniqueness of (6.5), which follows from the standard comparison principle (see [27]), we deduce $E = E'$ and conclude the proof. ☐

Remark 6.8. *Theorem 6.7 allows the classical variational problem to yield the optimal target measure ν for $\mathcal{P}_f(\mu)$. Assume that $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfies $(-\Delta)^s\varphi = f - \mu$ with $\lim_{|x|\to\infty} \varphi(x) = 0$. Then it is known that $v := u - \varphi$ minimizes the functional*

$$\int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}} w|^2$$

in $\{w \in H^s(\mathbb{R}^d) : w \geq \varphi\}$. (see for instance [27]).

Lemma 6.9. *Let (η, ρ) be the Eulerian variable associated to the optimizer of $\mathcal{P}_f(\mu)$ for Type (I). Then the initial domain $E(\eta)$ in (5.7) is same as the set E in Theorem 6.7 (up to zero-measure).*

Proof. Since $\{\eta(t,\cdot) > 0\}$ is non-increasing in t, we have $E(\eta) = \cup_{t>0}\{\eta(t,\cdot) > 0\}$. By (4.6), this set is same as $\cup_{t>0}\{w(t,\cdot) > 0\} = \{w(0,\cdot) > 0\}$ (up to zero-measure). This together with (6.6) imply that $E(\eta) = E$ (recall that $E = E'$ in the proof of Theorem 6.7). ☐

Finally, with the aid of our characterization of ν, we show that the conversion of particles from the distribution μ to ν is completed in a finite time for Type (I), if f is uniformly positive.

Theorem 6.10. *Further assuming that $f(x) \geq \delta$ for all $x \in \mathbb{R}^d$ ($\delta > 0$ is a constant), let τ be the optimal stopping time for $\mathcal{P}_f(\mu)$ for Type (I). Then for some constant $\overline{T} = \overline{T}(\delta, ||\mu||_{L^1}) < \infty$,*

$$\tau \leq \overline{T} \quad a.s.$$

Proof. By Lemma 6.9, denoting by $s : \mathbb{R}^d \to \mathbb{R}^+$ the barrier function associated to τ, we have $s = 0$ a.e. on E^c. Hence, by Theorems 4.17 and 6.6, we obtain the result by a parallel argument, similar to the one in [37, Theorem 8.6]. ☐

7. CONNECTION TO FRACTIONAL STEFAN PROBLEM

In this section, given any function f satisfying Assumption 3, we construct a solution to the (weighted) fractional Stefan problem ($St_{1,f}$) and ($St_{2,f}$), as defined in Definition 5.3;

$$(St_{1,f}) \qquad \partial_t h + (-\Delta)^s \eta = 0, \qquad h = h_1(\eta) := \eta - f\chi_{\{\eta>0\}} - \kappa_1 \chi_{\{\eta=0\}}$$

and

$$(St_{2,f}) \qquad \partial_t h + (-\Delta)^s \eta = 0, \qquad h = h_2(\eta) := \eta + f\chi_{\{\eta>0\}} + \kappa_2 \chi_{\{\eta=0\}}$$

with κ_1 and κ_2 defined in (5.6) and (5.4) respectively with the property $\kappa_i \leq f$ in $\{\eta = 0\}$.

We use the letter f to emphasize both its role as a weight and the role of the saturation result (Theorem 6.6) in constructing our solutions. Let us briefly outline the strategy. By Theorem 6.2, there exists a unique optimizer ν for $\mathcal{P}_f(\mu)$. Also by Theorem 6.6, $\nu = f$ on the active region, which allows us to apply Theorem 5.9 to obtain the result.

In the special case of $f \equiv 1$ we obtain solutions to the unweighted fractional Stefan problem (St_1) and (St_2). The results are parallel to the local case obtained in [37]. For (St_2), we obtain a unique solution with an active region which traces back to that of the initial data. While for (St_1), by allowing f to be a characteristic function, we will produce a unique solution in terms of the *insulated region* that has a prescribed initial data, but with an enlarged initial trace (see Theorem 7.7 later).

Let us first construct solutions to the weighted melting Stefan problem ($St_{2,f}$) when $\{0 < \mu \leq f\}$ is empty.

Theorem 7.1. *Let f satisfy Assumption 3 and further assume that $f > 0$ a.e. Consider $\mu = (f + \eta_0)\chi_\Sigma$, where $\eta_0 \in L^\infty(\mathbb{R}^d)$ is positive and Σ is a bounded Borel set in $\mathbb{R}^d$ having a positive Lebesgue measure. Let (ν, τ) be the optimizer for $\mathcal{P}_f(\mu)$ with a cost of Type (II), and let (η, ρ) be the associated Eulerian variables. Then η is a weak solution to ($St_{2,f}$) with initial data $(\eta_0\chi_\Sigma, \Sigma)$.*

In particular when $f \equiv 1$, η is the unique weak solution to (St_2) with initial data $(\eta_0\chi_\Sigma, \Sigma)$, and the corresponding enthalpy h from Definition 5.12 is the unique weak solution of (St_h) with initial data $h_0 = (\eta_0 + 1)\chi_\Sigma$. In addition, there exist constants $\gamma, C > 0$ depending only on μ, d and s such that for any $t \geq 0$,

$$||h(t,\cdot) - \nu||_{L^1(\mathbb{R}^d)} \leq Ce^{-\gamma t}. \tag{7.1}$$

Proof. Recall from Theorem 3.11 that the optimal stopping time τ is a hitting time to some backward barrier with a potential randomization on $\{\tau = 0\}$. Denoting by $s : \mathbb{R}^d \to \mathbb{R}^+$ the associated barrier function, we have $s = 0$ a.e. on Σ, since $\mu > f$ on Σ. In particular, the stopping time randomizes to stop the initial $f\chi_\Sigma$ particles. The remaining particles $\tilde{\mu} := \eta_0\chi_\Sigma$ are transported for a positive time to yield a final distribution $\tilde{\nu}$, the optimal target measure for $\mathcal{P}_{f(1-\chi_\Sigma)}(\tilde{\mu})$.

Let $\tilde{\tau} > 0$ be the stopping time τ restricted to the initial distribution $\tilde{\mu}$, and let $(\tilde{\eta}, \tilde{\rho})$ be the Eulerian variables associated to $(\tilde{\mu}, \tilde{\tau})$. Recalling that the initial particles $f\chi_\Sigma$ stop immediately, not affecting the active distribution η and thus $\tilde{\eta} = \eta$. Also, as $\tilde{\rho} + (f\chi_\Sigma)\delta_{\{t=0\}} \otimes dx = \rho$, the spatial marginal distribution of $\tilde{\rho}$ is $\tilde{\nu} := \nu - f\chi_\Sigma$.

Since $\tilde{\tau} > 0$ a.s., Theorem 5.9 implies that for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} (-\partial_t\varphi \cdot (\eta + \tilde{\nu}\chi_{\{\eta>0\}} + \kappa_2\chi_{\{\eta=0\}}) + (-\Delta)^s\varphi \cdot \eta)dxdt = \int_\Sigma \varphi(0,\cdot)\eta_0 dx$$

(κ_2 denotes the quantity (5.4), recall that $\tilde{\eta} = \eta$). Recall that the saturation result (Theorem 6.6) implies $\nu\chi_{\{\eta>0\}} = f\chi_{\{\eta>0\}}$. Since $\tilde{\nu} = \nu - f\chi_\Sigma$, it follows that $\tilde{\nu}\chi_{\{\eta>0\}} = f\chi_{\Sigma^c}\chi_{\{\eta>0\}}$. Therefore,

$$\int_0^\infty \int_{\mathbb{R}^d} (-\partial_t\varphi \cdot (\eta + f\chi_{\{\eta>0\}} + \kappa_2\chi_{\{\eta=0\}}) + (-\Delta)^s\varphi \cdot \eta)dxdt = \int_\Sigma \varphi(0,\cdot)\eta_0 dx + \mathcal{A}$$

with

$$\mathcal{A} := \int_0^\infty \int_\Sigma -\partial_t\varphi \cdot f\chi_{\{\eta>0\}}dxdt = \int_0^\infty \int_\Sigma -\partial_t\varphi \cdot f dxdt = \int_\Sigma \varphi(0,\cdot) f dx,$$

where the second equality is obtained from the fact that $\Sigma \subset \{\eta(t,\cdot) > 0\}$ for all $t > 0$.

Now, we verify that $E(\eta) = \Sigma$ a.e. Since $\Sigma \subset E(\eta)$ a.e., we aim to show the reverse inclusion. By Lemma 5.8, we have $\tilde{\nu}\chi_{E(\eta)} = 0$ a.e. Also by Theorem 6.6,

$$\tilde{\nu} = f \text{ a.e. on } \bigcup_{t>0} \{\eta(t,\cdot) > 0\} \setminus \Sigma.$$

Thus, as $f > 0$ a.e., we have that $E(\eta) \cap (\bigcup_{t>0}\{\eta(t,\cdot) > 0\} \setminus \Sigma)$ is a null set. Hence, as $E(\eta) \subset \bigcup_{t>0}\{\eta(t,\cdot) > 0\}$, we conclude that η solves ($St_{2,f}$) with initial data $(\eta_0\chi_\Sigma, \Sigma)$.

When $f \equiv 1$, ($St_{2,f}$) simplifies to (St_2) (see Definition 5.4). This implies η is a weak solution to (St_2). The uniqueness of η to (St_2) along with the uniqueness of h to (St_h) follow from Theorem 5.15.

Lastly we show (7.1). Recall that $\tilde{\tau} > 0$ a.s. and $(\eta, \tilde{\rho})$ are its associated Eulerian variables. By definition $\tilde{\nu} = \nu - \chi_\Sigma$ and $\tilde{\rho}([0,t), x) = \rho([0,t), x) - \chi_\Sigma(x)$, and thus $\tilde{h}(t,x) - \tilde{\nu}(x) = h(t,x) - \nu(x)$, where $\tilde{h}(t,x) := \eta(t,x) + \tilde{\rho}([0,t), x)$. Now by Theorem 6.2, there is an $R = R(\mu, d, s) > 0$ such that $\tau \le \tau_R$ almost surely and μ is supported on $B_R(0)$. So as $\tilde{\tau} > 0$ a.s., we may apply Lemma 5.16 on the pair $(\mu, \tilde{\nu})$ to deduce there are constants $\gamma = \gamma(\mu, d, s), C = C(\mu, d, s) > 0$ such that

$$||\tilde{h}(t,\cdot) - \tilde{\nu}(\cdot)||_{L^1(\mathbb{R}^d)} \le Ce^{-\gamma t}.$$

Recalling that $\tilde{h} - \tilde{\nu} = h - \nu$, we conclude the proof.

□

One can also show the Duvuat transform of our solution solves the parabolic obstacle problem:

Theorem 7.2 (Melting Obstacle Problem)**.** *Let f, μ and η be as given in Theorem 7.1. Then $w(t,x) := \int_0^t \eta(a,x)da$ solves the parabolic obstacle problem*

$$\min\{\partial_t w + (-\Delta)^s w + f - \mu, w\} = 0, \qquad w(0,\cdot) = 0. \tag{7.2}$$

Proof. By Theorem 4.14 and (4.8), it suffices to show that ν in (4.9) can be replaced with f. Indeed, the saturation result (Theorem 6.6) along with (4.6) imply that $\nu = f$ in $\{x \in \mathbb{R}^d : w(t,x) > 0 \text{ for some } t > 0\}$. As $\nu \le f$, we may apply the arguments in Theorem 6.7 to conclude the proof. □

Remark 7.3. *The regularity of* $(-\Delta)^s w$ *and the free boundary of* (7.2) *have been studied in the literature* [12, 5, 46, 28, 45] *when* $f-\mu$ *is smooth. For instance, if* $f-\mu$ *is smooth, then* $(-\Delta)^s w$ *is continuous for* $0<s<1$*, which would imply that the enthalpy is continuous. This would be an interesting contrast to the local case where the enthalpy jumps by* 1 *across the free boundary. Unfortunately this condition does not apply for us because our assumption in Theorem* 7.1*,* $\mu > f$ *in* $\{\mu>0\}$*, yields a discontinuous* μ *for most choices of* f*. While the discontinuity of* μ *only occurs at the initial free boundary, the local regularity result for* (7.2) *is not available at the moment.*

Next, we analyze the freezing problem (St_1).

Theorem 7.4. *Assume that* f *satisfies Assumption 3. Consider* $\mu = (f+\eta_0)\chi_\Sigma$*, where* $\eta_0 \in L^\infty(\mathbb{R}^d)$ *is positive and* Σ *is a Borel set in* $\mathbb{R}^d$ *having a positive Lebesgue measure. Let* (ν,τ) *be the optimizer for* $\mathcal{P}_f(\mu)$ *with a cost of Type (I), and let* (η,ρ) *be the associated Eulerian variables. Let* u *be a solution to* (6.5)*, and set* $E := \{u>0\}$*. Then* η *is a solution of* ($St_{1,f}$) *with initial data* (μ, E)*. In addition,* $\Sigma \subseteq E$*.*

Proof. Note that by Lemma 4.11 and Theorem 4.17, the initial trace of η is $\{s>0\}$, where $s:\mathbb{R}^d \to \mathbb{R}^+$ denotes the barrier function associated to the optimal stopping time τ. By Lemma 6.9, this set is equal to E a.e. We claim that $\Sigma \subseteq E$ a.e. Since τ is the hitting time to the epigraph of s, if $s=0$ on some $A\subseteq \Sigma$ with $|A|>0$, then the initial $\mu\chi_A$ particles stop immediately. Thus $\nu\chi_A \geq \mu\chi_A > f\chi_A$, which yields a contradiction since $\nu \leq f$. Therefore $\Sigma \subseteq E$.

Observe that by our choice of μ, the optimal stopping time τ satisfies $\tau>0$ a.s., since the above argument implies that $s>0$ a.e. on Σ. Also by the saturation result (Theorem 6.6), $\nu = f$ a.e. on $\{x\in\mathbb{R}^d : \eta(t,x)>0 \text{ for some } t>0\}$. Hence by Theorem 5.9 and Lemma 6.9, η solves ($St_{1,f}$) with initial data (μ,E). ☐

Next, we state a connection between ($St_{1,f}$) and the parabolic obstacle problem, which is a direct consequence of Theorem 4.14. Based on the instability of the supercooled problem, this indicates that w has a low regularity compared to the one for the melting case. At the level of the obstacle problem, this is due to the fact that $\partial_t w = -\eta \leq 0$. We refer to the examples constructed in [12, Remark 3.7] in the case $s=1/2$.

Theorem 7.5. *Let* f,μ *and* η *be as given in Theorem 7.4. Then* $w(t,x) := \int_t^\infty \eta(a,x)da$ *solves the parabolic obstacle problem*

$$\min\{\partial_t w + (-\Delta)^s w + \nu, w\} = 0, \qquad w(0,\cdot) = U_\mu - U_\nu. \tag{7.3}$$

Here ν *is the optimal target measure of* $\mathcal{P}_f(\mu)$ *and is also given by Theorem 6.7.*

Note that, in contrast to ($St_{2,f}$), the initial domain E depends on the choice of f in the case ($St_{1,f}$). Hence $f\equiv 1$ no longer corresponds to a unique solution of (St_1) with initial data μ. In fact uniqueness may not hold even with the given initial data and domain, see the example for the local case in [37]. We can however uniquely characterize solutions of (St_1) in terms of the *insulated region*, a region where the particles never freezes.

Definition 7.6 (Insulated Region)**.** *For a solution* η *of* (St_1)*, its insulated region is defined as*

$$\Sigma(\eta) := \bigcap_{t\geq 0}\{\eta(t,\cdot)>0\}. \tag{7.4}$$

By the argument of Remark 5.2, we see that if η_1 and η_2 are two versions of η (in $\mathbb{R}^+\times\mathbb{R}^d$) whose positive sets are open and non-increasing in time, then $\Sigma(\eta_1) = \Sigma(\eta_2)$ up to zero-measure.

Theorem 7.7. *Assume that* $\mu \in L^\infty(\mathbb{R}^d)$ *with a compact support which is not identically zero. Let* G *be a non-empty bounded open set in* $\mathbb{R}^d$ *that contains* $\{x \in \mathbb{R}^d : 0 < \mu(x) \le 1\}$*. Let* η *and* E *be given in Theorem 7.4 with* $f := 1 - \chi_G$*. Then,* η *solves* (St_1) *with initial data* (μ, E) *and an insulated region* G*.*

Proof. Our first goal is to show that G is the insulated region of η. Before verifying this, we show that for any time $t > 0$, $\eta(t, \cdot)$ is not identically zero. Since $\rho = 0$ on $\mathbb{R}^+ \times G$, by the Eulerian variable equation (3.20), η solves the fractional heat equation $\partial_t \eta = -(-\Delta)^s \eta$ on the open set $\mathbb{R}^+ \times G$ with initial data μ and the Dirichlet condition η. As $\mu(\cdot) = \eta(0, \cdot)$ is not identically zero, the maximum principle implies that $\eta(t, \cdot)$ is not identically zero for any $t \ge 0$.

Now we show that the insulated region of η is G. To see that $G \subseteq \Sigma(\eta)$, recall by (4.6) that $\{\eta > 0\} = \{w > 0\}$, where w solves (4.9). Since $\nu = 0$ on G,

$$\partial_t w + (-\Delta)^s w \ge 0 \quad \text{ on } (0, \infty) \times G.$$

As $\eta(t, \cdot)$ is not identically zero for any t, the same holds for w. Thus on its global minimum set $\{w = 0\}$, we have $\partial_t w + (-\Delta)^s w < 0$. Therefore, $w > 0$ on $(0, \infty) \times G$, which implies $G \subseteq \Sigma(\eta)$.

For the reverse inclusion, we first show that $\{\nu = 0\} \subseteq G$. To see this, recall from (4.9) that

$$\partial_t w + (-\Delta)^s w \ge 0 \quad \text{ on } (0, \infty) \times \{\nu = 0\}.$$

Hence using the parallel argument as above,

$$\{\nu = 0\} \subseteq \{x \in \mathbb{R}^d : w(t, x) > 0 \text{ for some } t > 0\} = \{x \in \mathbb{R}^d : \eta(t, x) > 0 \text{ for some } t > 0\}.$$

By the saturation result (Theorem 6.6), we have $\{\nu = 0\} \subseteq G$. As $\Sigma(\eta) \subseteq \{\nu = 0\}$, we deduce that $\Sigma(\eta) \subseteq G$. Therefore, we establish $\Sigma(\eta) = G$.

Finally, we verify that η solves (St_1). Observe that for any test function φ,

$$\int_0^\infty \int_G \partial_t \varphi \cdot \chi_{\{\eta > 0\}} dx dt = -\int_G \varphi(0, x) dx.$$

By Theorem 7.4, η is a solution to ($St_{1,f}$) with initial data (μ, E). Thus recalling $f := 1 - \chi_G$ and adding the above identity to the weak form of ($St_{1,f}$), setting $h := \eta - \chi_{\{\eta>0\}} - \kappa_1 \chi_{\{\eta=0\}}$,

$$\iint (-\partial_t \varphi \cdot h + (-\Delta)^s \varphi \cdot \eta) dt dx = \int \varphi(0, \cdot)(\mu - \chi_E - \kappa_1(0, \cdot)\chi_{E^c}) dx.$$

Hence we conclude the proof. ☐

Finally, we show that the initial data and the insulated region are enough to characterize (St_1). In particular, our choice of the initial trace coming from the elliptic obstacle problem is *necessary*.

Theorem 7.8. *Assume that* $\mu \in L^\infty(\mathbb{R}^d)$ *with a compact support. Then for any bounded measurable set* G *in* $\mathbb{R}^d$*, there is at most one solution to* (St_1) *with initial data* μ *and an insulated region* G*. If the solution exists (which we call* η*), then* η *is the Eulerian variable associated to the optimal stopping time of* $\mathcal{P}_f(\mu)$ *with* $f := 1 - \chi_G$*. In addition, the initial domain* $E = \{\eta(0^+, \cdot) > 0\}$ *is given by a positivity set of the obstacle solution solving* (6.5)*, and* E *contains the support of* μ*.*

Proof. By the consistency theorem (Theorem 5.10), we have that (η, ρ) (where ρ is defined in (5.18)) are the optimal Eulerian variables for a cost of Type (I) between μ and $\nu = \chi_{G^c \cap E} + \rho([0, \infty), \cdot)\chi_{E^c}$. We show that this is enough to characterize the target measure ν even on E^c.

First it follows from Section 4 that $w(t, x) := \int_t^\infty \eta(a, x) da$ solves the obstacle problem (4.9) for Type (I). In addition, we have $u(\cdot) := w(0, \cdot) = U_\mu - U_\nu$, and thus $\nu = \mu - (-\Delta)^s u$. Now let us show that u is uniquely specified by our assumptions.

Note that $G \subseteq E$, thus $\nu \leq 1 - \chi_G$ with an equality on E (recall $\rho([0,\infty),\cdot) \leq 1$ due to Definition 5.3). Hence, by the arguments in the proof of Theorem 6.7, we deduce that u is a unique continuous viscosity solution of (6.5) with $f = 1 - \chi_G$. In particular, any solution of (St_1) satisfying the assumptions in Theorem 7.8 induces a unique target measure ν.

Since the choice of ν is unique, a solution of (St_1) satisfying the assumptions in Theorem 7.8 is unique. This is because w solves the obstacle problem (4.9) for Type (I), which has a unique solution due to the comparison principle (see [27]).

Note that ν is the same target measure as $\mathcal{P}_{1-\chi_G}(\mu)$ due to Theorem 6.7, and then Lemma 6.9 implies that the initial trace is $E = \{u > 0\}$. Lastly, the fact that E contains the support of μ follows from Theorem 7.4. □

A special case is when G is an empty set and $f \equiv 1$. As a consequence of Theorems 6.10, 7.4 and 7.8, we establish the following corollary.

Corollary 7.9. *Let μ and η be as in Theorem 7.4 with $f \equiv 1$. Then η is the unique solution to (St_1) with initial data μ that vanishes in a finite time. The initial domain of η is a positivity set of the obstacle solution solving (6.5) with $f \equiv 1$.*

APPENDIX A. PRELIMINARIES ON MEASURES

For any Borel set A in the Euclidean space, we denote by $\mathcal{M}(A)$ (resp. $\mathcal{M}_1(A)$) the space of finite (signed) Radon measures (resp. probability measures) on A. As mentioned in the introduction, we are often interested in expressions of the form $\pi([0,t],x)$, where π is a space-time measure. We now give a precise definition of this expression:

Definition A.1 (Disintegration and Density)**.** *Assume that both $\pi \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ are non-negative. We say that π has a* spatial marginal f *if for any Borel set $A \subset \mathbb{R}^d$, $\pi([0,\infty) \times A) = \int_A f(x)dx$. This particularly implies that for any Borel set $I \subset \mathbb{R}^+$, the spatial measure $A \mapsto \pi(I \times A)$ is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^d$. We denote the associated density by $\pi(I,x)$.*

One can disintegrate the measure π into probability measures $\{\pi_x\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^+$ such that

$$\pi(I \times A) = \int_A \pi_x(I) f(x)dx, \tag{A.1}$$

where $I \subset \mathbb{R}^+$ is Borel. In other words, $\pi(I,x) = \pi_x(I)f(x)$ x-a.e.

We show that the choice of including or excluding the endpoints of the interval $[0,t]$ in $\pi([0,t],x)$ does not affect the expression in the above definition:

Lemma A.2. *Let π be as in Definition A.1. Set $\lambda_1(t,x) := \pi([0,t],x)$ and $\lambda_2(t,x) := \pi([0,t),x)$. Then $\lambda_i \in L^\infty(\mathbb{R}^+;\mathbb{R}^d)$ for $i = 1,2$. Also for any $x \in \mathbb{R}^d$, we have $\lambda_1(t,x) = \lambda_2(t,x)$ t-a.e.*

Proof. The L^∞ property follows from the fact that the spatial marginal of π is $f \in L^\infty(\mathbb{R}^d)$. The equality of λ_1 and λ_2 follows from the fact that for any fixed $x \in \mathbb{R}^d$, $\lambda_1(t,x)$ is monotone in t, and $\lambda_1(t,x) \neq \lambda_2(t,x)$ if and only if $\lambda_1(t,x)$ jumps at time t. As the set of times where monotone functions can jump is countable, we conclude the proof. □

Next, we state a general integration by parts formula for $\pi([0,t],x)$.

Lemma A.3. *Let π be as in Definition A.1 and $0 \leq a < b$. Then for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,*

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_a^b \partial_t\varphi(t,x)\cdot\pi([0,t],x)dtdx \\ &= \int_{\mathbb{R}^d}\varphi(b,x)\pi([0,b],x) - \int_{\mathbb{R}^d}\varphi(a,x)\pi([0,a],x) - \iint_{[a,b]\times\mathbb{R}^d}\varphi(t,x)\pi(dt,dx). \end{aligned} \tag{A.2}$$

Proof. We use the notation $\pi_x([0,t])$ from Definition A.1. Since $\pi([0,t],x) = \pi_x([0,t])f(x)$, by the integration by parts formula for the Riemann–Stieltjes integral, for each $x \in \mathbb{R}^d$,

$$\int_a^b \partial_t \varphi(t,x) \cdot \pi([0,t],x)dt = \varphi(b,x)\pi([0,b],x) - \varphi(a,x)\pi([0,a],x) - \int_a^b \varphi(t,x)f(x)\pi_x(dt).$$

Integrating this expression over $\mathbb{R}^d$ concludes the proof. □

APPENDIX B. FRACTIONAL SOBOLEV SPACE H^s AND OPTIMAL CONTROL

B.1. **Fractional Sobolev space H^s.** In this section, we briefly recall the fractional Sobolev spaces and the fractional Laplacian on $\mathbb{R}^d$. For $s \in (0,1)$, define the Gagliardo seminorm

$$[u]_{H^s(\mathbb{R}^d)} := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy.$$

With the aid of Fourier transform, Gagliardo seminorm can be written as

$$[u]_{H^s(\mathbb{R}^d)} = 2C_{d,s}^{-1}||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R}^d)}, \tag{B.1}$$

where $C_{d,s}$ denotes the normalization factor in the fractional Laplacian [23, Proposition 3.6].

Definition B.1. *We say that $u \in H^s(\mathbb{R}^d)$ if*

$$||u||^2_{H^s(\mathbb{R}^d)} := ||u||^2_{L^2(\mathbb{R}^d)} + [u]_{H^s(\mathbb{R}^d)} < \infty.$$

Also, given a bounded and open set Ω in $\mathbb{R}^d$, we say that $u \in H^s_0(\Omega)$ if $u = 0$ a.e. in Ω^c and $||u||_{H^s_0(\Omega)} := ||u||_{H^s(\mathbb{R}^d)} < \infty$.

We state the following Sobolev embedding theorem [23, Theorem 7.1].

Theorem B.2 (Sobolev embedding)**.** *Let Ω be a bounded and open set in $\mathbb{R}^d$ with a Lipschitz boundary. Suppose that $\mathcal{F} \subset L^2(\Omega)$ satisfies*

$$\sup_{f\in\mathcal{F}} ||f||_{L^2(\Omega)} < +\infty \quad \text{and} \quad \sup_{f\in\mathcal{F}} [f]_{H^s(\mathbb{R}^d)} < +\infty.$$

Then, $\mathcal{F}$ is pre-compact in $L^2(\Omega)$.

We introduce some relationships between the Gagliardo seminorm and the fractional Laplacian (see for instance [50, Proposition 9]):

Proposition B.3 (Poincare's Inequality)**.** *Let Ω be a bounded and open set in $\mathbb{R}^d$ with a Lipschitz boundary. Then, there is a constant $C > 0$ such that for any $u \in H^s_0(\Omega)$,*

$$||u||^2_{L^2(\Omega)} \leq C[u]_{H^s(\mathbb{R}^d)}.$$

The optimal constant C above is given by $C_{d,s}/(2\lambda)$, where $C_{d,s}$ is the normalization constant in (B.1) and λ is the principal eigenvalue of the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c. \end{cases} \tag{B.2}$$

In particular, (B.1) together with Poincaré's inequality imply that for any $u \in H^s_0(\Omega)$, the following fractional Poincaré inequality holds:

$$||u||^2_{L^2(\Omega)} \leq \frac{1}{\lambda}||(-\Delta)^{\frac{s}{2}}u||^2_{L^2(\mathbb{R}^d)}. \tag{B.3}$$

This immediately implies that the norm $||u||_{H^s_0(B)} := ||(-\Delta)^{\frac{s}{2}}u||_{L^2(\mathbb{R}^d)}$ is equivalent to the usual $H^s(\mathbb{R}^d)$ norm.

B.2. **Viscosity Solutions with respect to $(-\Delta)^s$.** In this section, we review some crucial properties of the fractional Laplacian and viscosity solutions.

Definition B.4 (Viscosity solution)**.** *Let $K > 0$ be a constant and Ω be a bounded and open set in $\mathbb{R}^d$. We say that a lower semi-continuous function ψ is a* viscosity super solution *to the equation $(-\Delta)^s\psi = -K$ in Ω if for any $x_0 \in \Omega$ and $\varphi \in C_b^2(\mathbb{R}^d)$ such that $\psi - \varphi$ has a global minimum at x_0,*

$$(-\Delta)^s\varphi(x_0) \geq -K.$$

We often say that ψ is a viscosity solution to $(-\Delta)^s u \geq -K$ if ψ is a viscosity super solution to $(-\Delta)^s u = -K$.

We show that viscosity solutions to $(-\Delta)^s u \geq -K$ are in fact also distributional solutions.

Lemma B.5. *Let $K > 0$ be a constant and Ω be a bounded and open set in $\mathbb{R}^d$. Assume that $(-\Delta)^s u \geq -K$ in Ω in the viscosity sense, $u(x) = 0$ for $x \notin \Omega$, and u is continuous. Then, for any non-negative function $\varphi \in C_c^\infty(\Omega)$,*

$$\int_{\mathbb{R}^d} u \cdot (-\Delta)^s\varphi \geq -K \int_{\mathbb{R}^d} \varphi.$$

Proof. For $\varepsilon > 0$, let u_ε be the inf-convolution of u, i.e.

$$u_\varepsilon(x) = \inf_{y\in\mathbb{R}^d} \left\{ u(y) + \frac{|x-y|^2}{2\varepsilon} \right\}.$$

Then, there is some constant c_ε with $\lim_{\varepsilon\to 0} c_\varepsilon = 0$ such that $(-\Delta)^s u_\varepsilon \geq -K - c_\varepsilon$ in Ω in the classical sense (see [13, Proposition 5.4]). By using a mollification, we may further assume that u_ε is smooth. Then, by integration by parts, for any non-negative $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^d} u_\varepsilon \cdot (-\Delta)^s\varphi = \int_{\mathbb{R}^d} (-\Delta)^s u_\varepsilon \cdot \varphi \geq (-K - c_\varepsilon) \int_{\mathbb{R}^d} \varphi.$$

The standard properties of the inf-convolution implies that $u_\varepsilon \to u$ locally uniformly. Since $(-\Delta)^s\varphi(y) = O(|y|^{-d-2s})$ as $y \to \infty$, by sending $\varepsilon \to 0$, we conclude the proof. ☐

APPENDIX C. OPTIMAL SKOROKHOD EMBEDDINGS

In this section, we consider the Skorokhod's embedding problem which involves the variational problem $\mathcal{P}_0(\mu,\nu)$ in (1.4). The variational problem (1.4) is a linear minimization problem over a convex set, and thus $\mathcal{P}_0(\mu,\nu)$ admits a dual problem. The associated dual problem is given by

$$\mathcal{D}_0(\mu,\nu) := \sup_{\substack{\psi\in C_c(\mathbb{R}^d) \\ G=(G_t)_{t\geq 0}\in\mathcal{K}^+_{-\gamma}}} \left\{ \int_{\mathbb{R}^d} \psi(z)\nu(dz) - \mathbb{E}[G_0] : G_t - \psi(X_t) \geq -\int_0^t L(s, X_s)ds \text{ for } t \leq \tau_r \right\}, \tag{C.1}$$

where $\mathcal{K}^+_{-\gamma}$ denotes the set of progressively measurable continuous supermartingales with respect to our $2s$-stable process with γ-exponential growth (see [34] for the local case).

Note that in the above variational problem (C.1), by the Doob-Meyer decomposition, we may assume that $G = (G_t)_{t\geq 0}$ is a martingale. In addition, we may further assume that

$$\psi(x) = 0, \quad \forall x \notin B, \tag{C.2}$$

where we recall $B = B_r(0)$ with r from Assumption 1. To see this, for $\psi \in C_c(\mathbb{R}^d)$, let $\hat{\psi}$ be a solution (the notions of weak solution and of viscosity solution agree, see [44, Remark 2.11] for explanations) to the equation

$$\begin{cases}(-\Delta)^s\hat{\psi}=0 & \text{in } B,\\ \hat{\psi}=\psi & \text{on } B^c.\end{cases}$$

Note that $\hat{\psi}\in C^\infty(B)\cap C_b(\mathbb{R}^d)$ due to [44], and thus by Ito's formula $\hat{\psi}(X_{t\wedge\tau_r})$ is martingale. Thus, if $(\psi,(G_t)_{t\geq 0})$ is an admissible pair in (C.1), then $(\psi-\hat{\psi},(G_t-\hat{\psi}(X_t))_{t\geq 0})$ is also an admissible pair with the same value on the action functional, since $\mu\leq_{s\text{-SH}}\nu$. Hence, one can assume (C.2) in the variational problem (C.1).

Therefore this implies that by [33, Theorem 2.3], for any μ and ν satisfying Assumption 1 and L satisfying Assumption 2,

$$\mathcal{P}_0(\mu,\nu)=\mathcal{D}_0(\mu,\nu). \tag{C.3}$$

We refer to [7, Theorem 1.2] and [34, Theorem A.1] for the Brownian motion case. In addition, by [33, Theorem 5.7], there exists an optimizer for $\mathcal{D}_0(\mu,\nu)$.

Now, we apply [33, Theorem 6.2] to establish Theorem 3.11.

Proof of Theorem 3.11. First, we verify the assumptions **(A0)**, **(A1)**, **(B0)**, **(B1)**, **(C0)**-**(C2)**, **(D0)** and **(D1)** in [33, Theorem 6.2] (under the additional disjointness assumption on μ and ν for a Type (II) cost).

Let $S_t:=\int_0^t L(s,X_s)ds$, then S_t satisfies Assumption **(A0)**, due to the continuity of $t\mapsto S_t$ which is a consequence of $L\in L^\infty$. To see **(A1)**, note that our compact active region assumption and non-negativity of L imply $S_t\leq S_{\tau_r}\in L^1$ (r is from Assumption 1), and thus S is uniformly integrable over stopping times.

To verify the condition **(B0)**, note that if $\psi\in C(\mathbb{R}^d)$ satisfies $\psi(x)=0$ for $x\notin B_r(0)$, then the réduite $\hat{\psi}$ of ψ on $B_r(0)$ is the viscosity solution to

$$\begin{cases}\min\{(-\Delta)^s\hat{\psi},\hat{\psi}-\psi\}=0 & \text{in } B_r(0),\\ \hat{\psi}(x)=\psi(x)=0 & \text{on } [B_r(0)]^c.\end{cases}$$

Also, this solution is known to be continuous [27]. Note that the condition **(B0)** is about the continuity of $\hat{\psi}$ for $\psi\in C_b(\mathbb{R}^d)$, but from our above argument of reducing the optimization set of $\mathcal{D}_0(\mu,\nu)$ to have $\psi(x)=0$ for $x\notin B_r(0)$, we only need to check **(B0)** for $\psi\in C_b(\mathbb{R}^d)$ with $\psi(x)=0$ for $x\notin B_r(0)$. Next, as $L\geq 0$, S_t is a submartingale, and thus **(B1)** holds.

Assumption **(C0)** and **(C1)** are satisfied due to Proposition B.3 and Lemma B.5 respectively. Also **(C2)** follows from the condition $L\in L^\infty$.

Finally Assumptions **(D0)** and **(D1)** are shown to be satisfied in [33] (if further μ and ν are disjoint in Type (II) case). This was proved in the example after the definition of **(D0)** and **(D1)** in [33].

Now with these assumptions having been verified, one can apply [33, Theorem 6.2] to obtain for costs of Type (I) that the optimal stopping time τ^* is given by

$$\tau^*=\inf\{t\geq 0:\varphi(t,X_t)=\psi(X_t)\}\leq\tau_r, \tag{C.4}$$

where ψ denotes the optimizer in a dual problem (C.1), which is bounded and lower semi-continuous that is zero outside $B_r(0)$, and

$$\varphi(t,x):=\sup_{\tau\in\mathcal{S},\tau\geq t}\mathbb{E}\left[\psi(X_\tau^{t,x})-\int_t^\tau L(a,X_a^{t,x})da\right], \tag{C.5}$$

where $X^{t,x}=(X_a^{t,x})_{a\geq t}$ denotes the $2s$-stable process started from x at time t.

We verify that $R_1:=\{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d:\varphi(t,x)=\psi(x)\}$ is a forward barrier in Type (I) case. Since L is strictly increasing in time, the dynamic programming principle [34, Proposition 4.2] implies that φ is non-increasing in time. In addition, by taking $\tau=t$ in (C.5), we have

$\varphi(t,x) \geq \psi(x)$. Therefore, it follows that if $\varphi(t,x) = \psi(x)$, then $\varphi(t+\delta,x) = \psi(x)$ for any $\delta > 0$, that is, R_1 is a forward barrier. In other words,

$$\tau^* = \inf\{t \geq 0 : t \geq s_1(X_t)\},$$

where $s_1(x) := \inf\{t \geq 0 : \varphi(t,x) = \psi(x)\}$.

One can argue similarly for Type (II) case, with the only exception being that μ and ν are required to be disjoint to apply the result [33, Theorem 6.2]. To achieve this, consider instead $\mathcal{P}_0(\tilde{\mu},\tilde{\nu})$ with $\tilde{\mu} := \mu - \mu_0$ and $\tilde{\nu} := \nu - \mu_0$, where μ_0 is the shared mass between μ and ν. It follows from optimality that if τ is the optimal stopping time for $\mathcal{P}_0(\mu,\nu)$, then on $\{\tau > 0\}$ we have that τ is the optimal stopping time for $\mathcal{P}_0(\tilde{\mu},\tilde{\nu})$. ☐

Next, we verify that the stopped particles are inside the barrier.

Lemma C.1. *Let τ be the optimizer of $\mathcal{P}_0(\mu,\nu)$ in (1.4) (assume additionally that $\tau > 0$ a.s. for Type (II)). Then if R denotes the associated barrier from Theorem 3.11, then $(\tau, X_\tau) \in R$ a.s. In other words, denoting by s the associated barrier function, we have $\tau \geq s(X_\tau)$ for Type (I) and $\tau \leq s(X_\tau)$ for Type (II).*

Proof. We only consider the Type (II) case since the argument is similar for Type (I) case. As $\tau > 0$ a.s., μ and ν are disjoint, since otherwise the optimizer τ would randomize at $t = 0$ to instantly stop the shared mass. Thus the optimal stopping time τ is given by (C.4). Then by [33, Theorem 2.4], $\varphi(\tau, X_\tau) = \psi(X_\tau)$ a.s., implying that $(\tau, X_\tau) \in R$ a.s. ☐

Finally a useful fact for this paper is the following uniqueness theorem for Eulerian variables for Type (I), which we will use to uniquely characterize the freezing Stefan problem. The statement is identical to [34, Lemma 4.5] which was originally proven for the Brownian motion.

Theorem C.2 (Uniqueness of Eulerian Variables for Type (I))**.** *Assume that R is a measurable forward barrier in $\mathbb{R}^+ \times \mathbb{R}^d$. Then, for any $\mu \in L^\infty(\mathbb{R}^d)$ with a compact support, there is at most one solution (η, ρ) to (3.20) satisfying $\eta(R) = 0$ and $\rho(R) = 1$.*

Proof. The argument is based on [34, Lemma 4.5] which deals with the case of Brownian motion. Let (η_i, ρ_i), $i = 1, 2$, be solutions to (3.20). Then, define $\eta := \eta_1 - \eta_2$ and $\rho := \rho_1 - \rho_2$. Consider the potential function φ

$$\varphi(t,\cdot) := N * (\eta(t,\cdot) + \rho([0,t),\cdot)), \tag{C.6}$$

where N denotes the Riesz potential (see (4.1)). Then by similar arguments as in Lemma 4.11,

$$\begin{cases} \partial_t \varphi = -\eta, \\ (-\Delta)^s \varphi(t,\cdot) = \eta(t,\cdot) + \rho([0,t),\cdot). \end{cases} \tag{C.7}$$

By taking φ as a test function in (3.20) for $i = 1, 2$,

$$\iint \varphi\rho(dt,dx) = \iint -2\eta^2 - \eta \cdot \rho([0,t),\cdot).$$

Note that since R is a measurable forward barrier, $\eta(t,x) \cdot \rho([0,t),x) = 0$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$.

Also by (C.7), $\varphi(t,x) = \lim_{s\to\infty} \varphi(s,x) =: \varphi_\infty(x)$ on R. Hence, as ρ is supported on R,

$$\iint \varphi\rho(dt,dx) = \iint \varphi_\infty \rho(dt,dx) = \int \varphi_\infty(x)\rho([0,\infty),dx) = \int |(-\Delta)^{s/2}\varphi_\infty(x)|^2,$$

where in the last equality we used the fact $(-\Delta)^s \varphi_\infty(x) = \rho([0,\infty),x)$. By the above displays,

$$\iint 2\eta^2 + \int |(-\Delta)^{s/2}\varphi_\infty|^2 = 0.$$

Therefore we obtain $\eta_1 = \eta_2$, which yields $\rho_1 = \rho_2$ as well. ☐

References

1. David Applebaum, *Lévy processes and stochastic calculus*, Cambridge university press, 2009.
2. I Athanasopoulos, L Caffarelli, Sandro Salsa, et al., *Regularity of the free boundary in parabolic phase-transition problems*, Acta Mathematica **176** (1996), 245–282.
3. Ioanni Athanasopoulos and Luis A Caffarelli, *Continuity of the temperature in boundary heat control problems*, Advances in Mathematics **224** (2010), no. 1, 293–315.
4. Ioannis Athanasopoulos, Luis Caffarelli, and Emmanouil Milakis, *The two-phase stefan problem with anomalous diffusion*, Advances in Mathematics **406** (2022), 108527.
5. Begoña Barrios, Alessio Figalli, and Xavier Ros-Oton, *Free boundary regularity in the parabolic fractional obstacle problem*, Communications on Pure and Applied Mathematics **71** (2018), no. 10, 2129–2159.
6. John R Baxter and Rafael V Chacon, *Compactness of stopping times*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete **40** (1977), 169–181.
7. Mathias Beiglböck, Alexander MG Cox, and Martin Huesmann, *Optimal transport and skorokhod embedding*, Inventiones mathematicae **208** (2017), 327–400.
8. Jean Bertoin, *Lévy processes*, vol. 121, Cambridge university press Cambridge, 1996.
9. Robert M Blumenthal, Ronald K Getoor, and DB Ray, *On the distribution of first hits for the symmetric stable processes*, Transactions of the American Mathematical Society **99** (1961), no. 3, 540–554.
10. Krzysztof Bogdan, Tomasz Byczkowski, Tadeusz Kulczycki, Michal Ryznar, Renming Song, and Zoran Vondracek, *Potential analysis of stable processes and its extensions*, Springer Science & Business Media, 2009.
11. Henrique Borrin and Diego Marcon, *An obstacle problem arising from american options pricing: regularity of solutions*, 2021, arXiv preprint arXiv:2107.03254.
12. Luis Caffarelli and Alessio Figalli, *Regularity of solutions to the parabolic fractional obstacle problem*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2013** (2013), no. 680, 191–233.
13. Luis Caffarelli and Luis Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **62** (2009), no. 5, 597–638.
14. Loïc Chaumont and Thomas Pellas, *Creeping of lévy processes through curves*, 2022, arXiv preprint arXiv:2205.06865.
15. Lincoln Chayes and Inwon C. Kim, *The supercooled stefan problem in one dimension*, Communications on Pure and Applied Analysis (CPAA) **11** (2012), no. 2, 845–859.
16. Zhiyi Chi, *On exact sampling of the first passage event of a lévy process with infinite lévy measure and bounded variation*, Stochastic Processes and their Applications **126** (2016), no. 4, 1124–1144.
17. Sunhi Choi and Inwon C Kim, *Regularity of one-phase stefan problem near lipschitz initial data*, American journal of mathematics **132** (2010), no. 6, 1693–1727.
18. Sunhi Choi, Inwon C Kim, and Young-Heon Kim, *Existence for the supercooled stefan problem in general dimensions*, arXiv preprint arXiv:2402.17154 (2024).
19. Félix del Teso, Jörgen Endal, and Espen R Jakobsen, *On distributional solutions of local and nonlocal problems of porous medium type*, Comptes Rendus Mathematique **355** (2017), no. 11, 1154–1160.
20. Félix Del Teso, Jörgen Endal, and Espen R Jakobsen, *Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type*, Advances in Mathematics **305** (2017), 78–143.
21. Félix del Teso, Jorgen Endal, and Juan Luis Vázquez, *The one-phase fractional stefan problem*, Mathematical Models and Methods in Applied Sciences **31** (2021), no. 01, 83–131.
22. François Delarue, Sergey Nadtochiy, and Mykhaylo Shkolnikov, *Global solutions to the supercooled stefan problem with blow-ups: regularity and uniqueness*, Probability and Mathematical Physics **3** (2022), no. 1, 171–213.
23. Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, *Hitchhiker's guide to the fractional sobolev spaces*, Bulletin des sciences mathématiques **136** (2012), no. 5, 521–573.
24. Leif Döring, Lukas Gonon, David J Prömel, and Oleg Reichmann, *On skorokhod embeddings and poisson equations*, Ann. Appl. Probab. **29** (2019), no. 4, 2302–2337.
25. Georges. Duvaut, *Résolution d'un problème de stefan (fusion d'un bloc de glace à zéro degrées)*, C. R. Acad. Sci. Paris **276** (1973), 1461–1463.
26. Samer Dweik, Nassif Ghoussoub, Young-Heon Kim, and Aaron Zeff Palmer, *Stochastic optimal transport with free end time*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **57** (2021), no. 2, 700 – 725.
27. Xavier Fernández-Real and Xavier Ros-Oton, *Integro-differential elliptic equations*, 2023.
28. Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra, *Regularity theory for nonlocal obstacle problems with critical and subcritical scaling*, 2023, arXiv preprint arXiv:2306.16008.
29. ______, *The singular set in the stefan problem*, 2023, Journal of the American Mathematical Society.
30. Paul Gassiat, Harald Oberhauser, and Gonçalo Dos Reis, *Root's barrier, viscosity solutions of obstacle problems and reflected fbsdes*, Stochastic Processes and their Applications **125** (2015), no. 12, 4601–4631.

31. Paul Gassiat, Harald Oberhauser, and Christina Z Zou, *A free boundary characterisation of the root barrier for markov processes*, Probability Theory and Related Fields **180** (2021), no. 1, 33–69.
32. Nassif Ghoussoub, Young-Heon Kim, and Tongseok Lim, *Optimal brownian stopping when the source and target are radially symmetric distributions*, SIAM Journal on Control and Optimization **58** (2020), no. 5, 2765–2789.
33. Nassif Ghoussoub, Young-Heon Kim, and Aaron Palmer, *Optimal stopping of stochastic transport minimizing submartingale costs*, Transactions of the American Mathematical Society **374** (2021), no. 10, 6963–6989.
34. Nassif Ghoussoub, Young-Heon Kim, and Aaron Zeff Palmer, *Pde methods for optimal skorokhod embeddings*, Calculus of Variations and Partial Differential Equations **58** (2019), no. 3, 1–31.
35. Mahir Hadžić and Steve Shkoller, *Global stability and decay for the classical stefan problem*, Communications on Pure and Applied Mathematics **68** (2015), no. 5, 689–757.
36. Olav Kallenberg and Olav Kallenberg, *Foundations of modern probability*, vol. 2, Springer, 1997.
37. Inwon Kim and Young-Heon Kim, *The stefan problem and free targets of optimal brownian martingale transport*, 2021, arXiv preprint arXiv:2110.03831.
38. Inwon Kim and Antoine Mellet, *Homogenization of one-phase stefan-type problems in periodic and random media*, Transactions of the American Mathematical Society **362** (2010), no. 8, 4161–4190.
39. Jean-François Le Gall, *Brownian motion, martingales, and stochastic calculus*, Springer, 2016.
40. JY Liu and MY Xu, *An exact solution to the moving boundary problem with fractional anomalous diffusion in drug release devices*, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik: Applied Mathematics and Mechanics **84** (2004), no. 1, 22–28.
41. Anvarbek Mukatovich Meirmanov, *The stefan problem*, vol. 3, Walter de Gruyter, 2011.
42. Sergey Nadtochiy, Mykhaylo Shkolnikov, and Xiling Zhang, *Scaling limits of external multi-particle dla on the plane and the supercooled stefan problem*, 2021, arXiv preprint arXiv:2102.09040.
43. Jan Obłój, *The Skorokhod embedding problem and its offspring*, Probability Surveys **1** (2004), 321 – 392.
44. Xavier Ros-Oton and Joaquim Serra, *The dirichlet problem for the fractional laplacian: regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées **101** (2014), no. 3, 275–302.
45. Xavier Ros-Oton, Clara Torres-Latorre, and Marvin Weidner, *Semiconvexity estimates for nonlinear integro-differential equations*, 2023, arXiv preprint arXiv:2306.16751.
46. Xavier Ros-Oton and Damià Torres-Latorre, *Optimal regularity for supercritical parabolic obstacle problems*, 2023, Communications on Pure and Applied Mathematics.
47. Hermann Rost, *The stopping distributions of a markov process*, Inventiones mathematicae **14** (1971), no. 1, 1–16.
48. ———, *Skorokhod stopping times of minimal variance*, Séminaire de Probabilités X Université de Strasbourg, Springer, 1976, pp. 194–208.
49. LI Rubinshtein, *The stefan problem*, vol. 27, American Mathematical Soc., 1971.
50. Raffaella Servadei, Enrico Valdinoci, et al., *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst **33** (2013), no. 5, 2105–2137.
51. Bernard Sherman, *A general one-phase stefan problem*, Quarterly of Applied Mathematics **28** (1970), no. 3, 377–382.
52. Luis Silvestre, *Regularity of the obstacle problem for a fractional power of the laplace operator*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **60** (2007), no. 1, 67–112.
53. Pablo Raúl Stinga, *User's guide to the fractional laplacian and the method of semigroups*, Handbook of fractional calculus with applications **2** (2019), 235–265.
54. Vaughan R Voller, *Fractional stefan problems*, International Journal of Heat and Mass Transfer **74** (2014), 269–277.
55. Yong Zhang, HongGuang Sun, Harold H Stowell, Mohsen Zayernouri, and Samantha E Hansen, *A review of applications of fractional calculus in earth system dynamics*, Chaos, Solitons & Fractals **102** (2017), 29–46.

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