REGULARITY OF HELE-SHAW FLOW WITH SOURCE AND DRIFT

INWON KIM AND YUMING PAUL ZHANG

Abstract. In this paper we study the regularity property of Hele-Shaw flow, where source and drift are present in the evolution. More specifically we consider Hölder continuous source and Lipschitz continuous drift. We show that if the free boundary of the solution is locally close to a Lipschitz graph, then it is indeed Lipschitz, given that the Lipschitz constant is small. When there is no drift, our result establishes $C^{1,\gamma}$ regularity of the free boundary by combining our result with the obstacle problem theory. In general, when the source and drift are both smooth, we prove that the solution is non-degenerate, indicating higher regularity of the free boundary.

1. Introduction

Let $\vec{b} : \mathbb{R}^d \to \mathbb{R}^d$ be a Lipschitz continuous vector field, and $f : \mathbb{R}^d \to [0,\infty)$ be a non-negative Hölder continuous function. We consider $u = u(x,t) \geq 0$ solving the Hele-Shaw type problem:

$$\begin{cases} -\Delta u = f & \text{in } \{u>0\}, \\ u_t = |\nabla u|^2 + \vec{b}\cdot\nabla u & \text{on } \partial\{u>0\}. \end{cases} \tag{1.1}$$

We refer to $\partial\{u>0\}$ as the *free boundary* of u. The second equation is the level set formulation of the velocity law

$$V = (-\nabla u - \vec{b})\cdot\nu = |\nabla u| - \vec{b}\cdot\nu \text{ on } \partial\{u>0\},$$

where V denotes the velocity of the set $\{u>0\}$ along the outward spatial normal $\nu = \frac{-\nabla u}{|\nabla u|}$ at the given free boundary point $(x,t)\in\partial\{u>0\}$.

When f and $\vec{b}$ are both zero, (1.1) corresponds to the classical Hele-Shaw flow describing the motion of incompressible viscous fluid, which occupies part of the space between two parallel, nearby plates, [Sha98, Ric72, ES97]. The general equation (1.1) can be also written as the continuity equation $\rho_t - \nabla\cdot((\nabla u+\vec{b})\rho) = \tilde{f}\rho$, with the density variable $\rho = \chi_{\{u>0\}}$ and growth term $\tilde{f} := f - \nabla\cdot\vec{b}$. In other words, ρ is transported by the velocity field $-(\nabla u+\vec{b})$ and with the growth term $\tilde{f}$. In this context, u can be understood as the pressure variable, and is generated by the incompressibility constraint

1991 *Mathematics Subject Classification.* 35R35, 35B65, 76D27.

I. Kim was partially supported by NSF grant DMS-1900804.

$\rho \leq 1$ to transport density that intends to move with drift $-\vec{b}$ and growth rate $\tilde{f}$. Due to this interpretation of the model, (1.1) has been actively studied in the recent literature, for instance in the context of tumor growth where cells evolve with contact inhibition, [PQV14, DS21, JKT22] and in the context of congested population dynamics [MRCS10, CKY18]. somewhere we need to cite Muskat problem literature, such as Sijue Wu's and Hongji's.

We are interested in the free boundary regularity for viscosity solutions of (1.1). When f and $\vec{b}$ are zero, the regularity property of the flow is by now well-understood in both global and local setting. In the global setting, posed with the presence of a fixed boundary with constant source, it is known that initially Lipschitz free boundary with a small Lipschitz constant immediately regularizes and become smooth for small positive times [CJK07], and for all a.e. times if $d \leq 4$ [FROS20]. In the local setting it is also known that free boundaries that are uniformly close to a Lipschitz graph is smooth, if the Lipschitz constant is small [CJK09].

For our inhomogeneous problem, zooming in at a single point $(x_0, t_0) \in \partial\{u > 0\}$ with the hyperbolic scale $\tilde{u}_r(x,t) := r^{-1}u(r(x-x_0), r(t-t_0))$, one formally sees that the source term tends to zero and the drift becomes a constant vector field as r tends to zero. Thus it seems plausible that similar regularity theory as for the classical Hele-Shaw flow holds. This heuristics however is difficult to quantify. Indeed there are examples of log-Lipschitz continuous function f with $\vec{b} = 0$ that describes tumor growth with nutrients, for which numerical experiments reveal immediate dendrite-like growth on the free boundary [Kit97, PTV14, MRCS14]. The dynamics behind the generation of such irregularities remain mysterious. We will show in this paper that such irregularities must originate from large-scale influx of oscillations. Roughly speaking, we show that "flat boundaries that looks Lipschitz in large scale are indeed Lipschitz and non-degenerate", as long as the Lipschitz constant is small:

Main theorem: *When the solution is close to a cone-monotone profile at each time in a local space-time neighborhood, then the solution is fully cone-monotone with Lipschitz free boundary in a smaller neighborhood, given that the angle of the cone is large. In addition, if $\vec{b}$ is zero, the free boundary is $C^{1,\gamma}$ for some $0 < \gamma < 1$. Lastly, if f and $\vec{b}$ are at least C^3, the solution is also non-degenerate, namely it features faster-than-linear growth near the free boundary.*

Some remarks on the assumption is in order. Our assumption considers solutions which look like cone-monotone solutions up to small scale, which is more general than plane-like profiles. For instance our assumption is satisfied by those who starts from an initially Lipschitz graph with small Lipschitz constant: see Corollary 4.3. [CJK07]. Our proof relies on the local spatially Lipschitz solutions that were constructed in [CJK07], [CJK09] and also in [?], as well as the properties of superharmonic functions given in section 4. For non-homogeneous problem the same example applies, see Corollary **??**. This assumption is also motivated from the well-known *waiting time* phenomena, where the initial free boundary does not move for a finite amount of time. For the classical Hele-Shaw problem with $f = \vec{b} = 0$, it is well-known that there is a waiting time phenomena with a cone-monotone initial data (King-Lacey-Vazquez), where the angle of the cone is small. The same remains true in the presence of the source term $f \in L^\infty$: see Example **??** where the vertex of the cone does not move for a unit amount of time and the profile of the solution stays close to a cone-monotone profile in a unit neighborhood of the vertex. The presence of the drift of course does not change this phenomena either. Hence our requirement on the size of angle is necessary for regularization of the free boundary.

See the next section for the full statements. Our result extends the celebrated free boundary regularity theory introduced by Caffarelli [Caf89, Caf87, ACS98, ACS96, Sav09, DSFS21] as well as the corresponding version for the classical Hele-Shaw flow [CJK09]. In particular our work serves as the first attempt to understand the effect of source and drift on the regularization mechanism of the free boundary evolution. As we will see below, the presence of a nonzero f alone necessitates some significant changes in the standard arguments.

In general, the Lipschitz regularity of the free boundary and the non-degeneracy of the solutions are the two ingredients of further regularity analysis in aforementioned references. We thus suspect that the free boundary in our statement is in fact $C^{1,\gamma}$ in space and time, when f and $\vec{b}$ are smooth. Given the technical nature of these arguments, we do not pursue this next step, to lay out the main arguments to achieve the basic regularity results as clearly as possible.

Let us briefly discuss the optimality of assumptions on f and $\vec{b}$. It is not hard to see that the condition is optimal for the drift term: when $\vec{b}$ is not Lipschitz continuous, one can construct an example where the solution starting with a cone as its positive set maintains the cone shape as its positive set, even developing a cusp at the vertex of the cone (see Example 3.8). On the other hand it is less clear whether the regularity of f is sharp for the theorem. The Hölder regularity of f appears to be close to the optimal condition for the "flat implies Lipschitz" result. We will show an example (see Example 3.10) where this result is false with merely bounded f. We also refer to a counterexample in [Bla01] for the obstacle problem, the time integrated version of our problem, with a continuous f that is not Dini-continuous. For the non-degeneracy, it remains unclear whether smoothness is required for f and $\vec{b}$: see more discussions on this in Section 7.

$\circ$ *Regularization mechanism and new ingredients:* In (1.1) the support of the pressure variable u moves along the velocity field $-(\nabla u + \vec{b})$. Due to the elliptic equation u solves in its support, ∇u acts as the regularizing force in the flow. We largely follow the outline of [Caf89] and [CJK07] for our analysis, which quantitatively and iteratively estimates the regularization effect of the pressure gradient, in the form of its directional monotonicity. We will show that the small-scale monotonicity property improves in the interior of the positive set, and then propagates its improvement from the interior of the positive set to the near boundary region over time. The comparison principle of the flow (Lemma 2.5), viewed as the "ellipticity" of the problem, is a key ingredient of this approach.

There are significant differences in our analysis from the existing literature, necessitated due to the presence of the source and the drift terms that competes with the propagation of directional monotonicity driven by the pressure gradient. Let us briefly discuss some of the highlights. First we point out Proposition 4.1, which compares superharmonic functions in a long strip domain with Lipschitz boundary. This boundary Harnack-type result enables us to compare our solutions to a localized harmonic function, ignoring the effect coming from the far-away regions. Its role in our analysis is indispensable to rule out the effect of external factors in the local regularization process. This result can be viewed as a generalized version of Dahlberg's lemma for harmonic functions, which was crucial for instance in showing that the interior improvement of the monotonicity.

Another important element of our analysis is the estimate on the growth rate of solutions near the free boundary (Lemmas 3.3 and 3.4). Heuristically speaking, such growth rate translates into a strong elliptic effect, competing against the oscillations caused by the source and drift terms. At more technical level, it is used to modify the standard perturbation argument used to show the propagation of the monotonicity (Lemma 6.1 and Proposition 6.2). It is also used to show that the positive set of the solution expands relatively to streamlines. In particular we are able to quantify the expansion rate (Proposition 6.6), which is important to show the non-degeneracy result.

1.1. **Statement of results and Outline of the paper.** For $r > 0$, we denote $\mathcal{Q}_r := B_r \times (-r, r)$. Let us state first the "flat to Lipschitz" result.

Theorem A. Let $\vec{b}$ be a Lipschitz continuous vector field, and f be a non-negative $\bar{\gamma}$-Hölder continuous function with $\bar{\gamma} \in (0,1)$, and for some $\varepsilon \in (0,1)$, let $a_\varepsilon \equiv 0$ if f is constant and $a_\varepsilon := \varepsilon^\alpha$ for some small $\alpha > 0$ otherwise. Suppose that u is a continuous viscosity solution to (1.1) in $\mathcal{Q}_2$ satisfying

- u is $(\varepsilon, a_\varepsilon)$-monotone with respect to $W_{\theta,\mu}$ for some $\theta \in (0, \frac{\pi}{2})$ and $\mu \in \mathbb{S}^{d-1}$,
- $m := \inf_{t\in(-2,2)} u(-\mu, t) > 0$.

If $\frac{\pi}{2} - \theta$ and ε are small enough, then u is non-decreasing along all directions of $W_{\theta',\mu}$ for some $\theta' \in (0, \theta)$ in $\mathcal{Q}_1$. In particular, the free boundary $\Gamma_u(t) \cap B_1$

for each $t \in (-1,1)$ is a Lipschitz continuous graph. Here α_0 only depends on $\bar{\gamma}$, and θ and θ' only depend on $\bar{\gamma}$ and the dimension, and ε also on m, $\|u\|_{L^\infty(\mathcal{Q}_2)}$, $\|\vec{b}\|_{C^1}$ and $\|f\|_{C^{\bar{\gamma}}}$. In addition, when $\vec{b}$ is zero, and when u solves (1.1) in $\mathbb{R}^d \times (-2,2)$, then the free boundary is $C^{1,\gamma}$ in $\mathcal{Q}_1$ for some $\gamma \in (0,1)$.

We refer to Corollary 6.3 and Remark thereafter for further discussion on the case of $\vec{b} = 0$.

The definition of the (ε, a)-monotonicity will be given in Definition 2.7. The $(\varepsilon, 0)$-monotonicity corresponds to the usual ε-monotonicity, which quantifies the scale at which the solution is monotone along a direction. The additional parameter a adds a growth condition at the same scale ε. This is to ensure that away from the boundary the solution is directionally monotone even with smaller scales. While ε-monotonicity is sufficient to guanrantee such "interior improvement" for harmonic functions, it is not the case for the general f: see Remark 2.8 for further discussions. Our condition is also natural. In Lemma 4.6 we show that if the free boundary is known to be Lipschitz continuous, then the solution is monotone and satisfies (ε, a)-monotonicity near the free boundary for any small ε and a.

For general setting, we state our non-degeneracy result.

Theorem B. Under the assumption of Theorem A and further assuming that f is Lipschitz continuous, and

- $u_t \geq \vec{b} \cdot \nabla u - Cu$ in $\mathcal{Q}_2$ in the viscosity sense,
- $C^{-1} \leq \frac{u(-e_d,t)}{u(-e_d,0)} \leq C$ for all $t \in (-2,2)$ for some $C > 0$,

then if $\frac{\pi}{2} - \theta$ and ε are small enough, u is non-degenerate in its positive set $\mathcal{Q}_1$. In other words, $|\nabla u|$ is uniformly positive up to the free boundary. Here θ only depends on $\bar{\gamma}$ and the dimension, and ε and the lower bound of $|\nabla u|$ also on C, m, $\|u\|_{L^\infty(\mathcal{Q}_2)}$, $\|\vec{b}\|_{C^1}$ and $\|f\|_{C^1}$.

Our assumption ensures that u does not decrease too fast in the direction of the streamline generated by $\vec{b}$. This assumption holds for solutions of (1.1) posed in $\mathbb{R}^d \times (0,\infty)$ when f and $\vec{b}$ are smooth, see Corollary 6.6 and Theorem 2.1 in [Chu22].

Remark 1.1. Our results apply to time-dependent f and $\vec{b}$ as well, even though we have only considered stationary ones for simplicity. With $f = f(x,t)$ and $\vec{b} = \vec{b}(x,t)$, Theorem A continues to hold with straightforward modifications in the proof if f and $\vec{b}$ are continuous in time. The same is true for Theorem B if f and $\vec{b}$ are Lipschitz continuous in time.

Here is a brief outline of the paper. In Section 2, we introduce notations and preliminary properties. In Section 3, we prove several tools that will be used, including interior monotonicity and polynomial growth of superharmonic functions near the free boundary, and demonstrate some examples discussing the optimality of our conditions and the formation of cusps on a

Lipschitz free boundary. Section 4 is about superharmonic functions in Lipschitz domains. Section 5 introduces the sup-convolution and its properties. Finally, we give the proof of Theorem A and Theorem B, respectively, in Section 6 and Section 7.

2. Preliminaries

For a space-time function $u : \mathbb{R}^d \times [0, \infty) \to [0, \infty)$, we write

$$\Omega_u := \{u(\cdot, \cdot) > 0\}, \quad \Omega_u(t) := \{u(\cdot, t) > 0\},$$

and

$$\Gamma_u(t) := \partial\Omega_u(t), \quad \Gamma_u := \bigcup_t \Gamma_u(t) \times \{t\}.$$

Similarly, for a function $\omega : \mathbb{R}^d \to [0, \infty)$, we define

$$\Omega_\omega := \{\omega(\cdot) > 0\} \quad \text{and} \quad \Gamma_\omega := \partial\Omega_\omega.$$

Let us recall the notions of viscosity sub- and supersolutions to (1.1) from [Kim03], with trivial modifications due to the drift and source terms and reduced to continuous functions. Consider the domain $\Sigma := D \times (0, T)$ with $T > 0$ and $D \subseteq \mathbb{R}^d$ open and bounded.

Definition 2.1. A non-negative continuous function u defined in Σ is a viscosity subsolution of (1.1) if for every $\phi \in C^{2,1}_{x,t}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega_u} \cap \{t \le t_0\} \cap \Sigma$ at (x_0, t_0), then

$$-(\Delta\phi + f)(x_0, t_0) \le 0 \quad \text{if } u(x_0, t_0) > 0$$
$$(\phi_t - |\nabla\phi|^2 - \vec{b} \cdot \nabla\phi)(x_0, t_0) \le 0 \quad \text{if } (x_0, t_0) \in \Gamma_u \text{ and } -(\Delta\phi + f)(x_0, t_0) > 0.$$

The reason for the intersection of the set $\overline{\Omega_u}$ in the definition is for the simple fact that there are no globally smooth function that crosses the solution from above at a free boundary point.

Definition 2.2. A non-negative continuous function u defined in Σ is a *viscosity supersolution* of (1.1) if for every $\phi \in C^{2,1}_{x,t}(\Sigma)$ such that $u - \phi$ has a local minimum in $\{t \le t_0\} \cap \Sigma$ at (x_0, t_0), then

$$-(\Delta\phi + f)(x_0, t_0) \ge 0 \quad \text{if } u(x_0, t_0) > 0$$
$$(\phi_t - |\nabla\phi|^2 - \vec{b} \cdot \nabla\phi)(x_0, t_0) \ge 0 \quad \text{if } (x_0, t_0) \in \Gamma_u,\ |\nabla\phi(x_0, t_0)| \ne 0 \text{ and } -(\Delta\phi + f)(x_0, t_0) < 0.$$

Definition 2.3. We say that a continuous non-negative function u is a *viscosity solution* of (1.1) if u is both a viscosity subsolution and a viscosity supersolution of (1.1).

To state the comparison principle, we need the following definition:

Definition 2.4. We say that a pair of functions $u_0, v_0 : \overline{D} \to [0, \infty)$ are strictly separated (denoted by $u_0 \prec v_0$) in D if $u_0(x) < v_0(x)$ in $\overline{\Omega_{u_0}} \cap \overline{D}$. This says that the supports of the two functions are separated and in the support of the smaller function, the two functions are strictly ordered.

Below we recall the comparison principle [Kim06, CJK07].

Lemma 2.5. *Let u, v be respectively viscosity sub- and supersolutions in $\Sigma = D \times (0,T)$ with initial data $u_0 \prec v_0$ in D. In addition suppose that $\limsup_{t\to 0^+} \Omega_u(t) = \Omega_{u_0}$. If $u \le v$ on $\partial D \times (0,T)$ and $u < v$ on $(\partial D \times (0,T)) \cap \overline{\Omega_u}$, then $u(\cdot,t) \prec v(\cdot,t)$ in D for all $t \in [0,T)$.*

Parallel argument as in Lemma 2.5 [Kim06] yields that the requirement at the free boundary in Definition 2.1 can be simplified for testing against functions with nonzero gradient.

Lemma 2.6. *Let u be a continuous viscosity subsolution of (1.1) in Σ, and $(x_0,t_0) \in \Gamma_u \cap \Sigma$. Let $\phi \in C^{2,1}_{x,t}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega_u} \cap \{t \le t_0\} \cap \Sigma$ at (x_0,t_0) and $|D\phi(x_0,t_0)| \neq 0$. Then*

$$(\phi_t - |\nabla \phi|^2 - \vec{b}\cdot\nabla\phi)(x_0,t_0) \le 0.$$

2.1. **Monotonicity assumption.** For two vectors $\nu, \mu \in \mathbb{R}^d \setminus \{0\}$, the angle between them is denoted as

$$\langle \nu, \mu \rangle := \arccos\left(\frac{\nu\cdot\mu}{|\nu||\mu|}\right) \in [0,\pi]. \tag{2.1}$$

We denote a spacial cone to direction $\mu \in \mathbb{S}^{d-1}$ with opening 2θ for $\theta \in [0, \frac{\pi}{2}]$ as

$$W_{\theta,\mu} := \left\{ p \in \mathbb{R}^d : \langle p, \mu\rangle \le \theta \right\}. \tag{2.2}$$

Our basic hypothesis will be a monotonicity with respect to the cone $W_{\theta,\mu}$.

For a space-time function $u : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$, we write

$$\Omega_u := \{u(\cdot,\cdot) > 0\}, \quad \Omega_u(t) := \{u(\cdot,t) > 0\},$$

and

$$\Gamma_u(t) := \partial\Omega_u(t), \quad \Gamma_u := \bigcup_t \Gamma_u(t) \times \{t\}.$$

Similarly, for a function $\omega : \mathbb{R}^d \to [0,\infty)$, we define

$$\Omega_\omega := \{\omega(\cdot) > 0\} \quad \text{and} \quad \Gamma_\omega := \partial\Omega_\omega.$$

Definition 2.7. Let $\Omega \subseteq \mathbb{R}^d$, $\theta \in [0,\frac{\pi}{2}]$, $\mu \in \mathbb{S}^{d-1}$, $\varepsilon \in [0,1)$ and $a \ge 0$. We say that a continuous function $\omega : \Omega \to \mathbb{R}$ is (ε, a)-*monotone* with respect to a cone $W_{\theta,\mu}$ in $D \subseteq \Omega$ if for every $\varepsilon' \ge \varepsilon$ and $x \in D$ we have

$$(1 + a\varepsilon)\,\omega(x) \le \inf_{y \in B_{\varepsilon' \sin\theta}(x)\cap\Omega} \omega(y + \varepsilon'\mu)$$

Here we need to assume that the solution also grows slightly in the monotone direction, which amounts to $(\varepsilon, \varepsilon^\alpha)$-monotonicity, to reach the same conclusion (which is proved in Lemma 3.1 and its remark): see Example 3.9, where the interior monotonicity fails with just ε-monotonicity. In Lemma 4.6, we show that if the free boundary is known to be Lipschitz continuous, then the solution is monotone and satisfying the (ε, a)-monotonicity for some $a > 0$ and for any small $\varepsilon > 0$ near the free boundary.

Remark 2.8. 1. It is by now a well-known fact that the $(\varepsilon, 0)$-monotonicity of a positive harmonic function leads to full monotonicity in a smaller neighborhood, see for instance [CS05, Corollary 11.16]. This fact is essential in the regularity analysis for solutions of (1.1) with $f = 0$, since the stronger monotonicity in the positive set propagates to the free boundary so that its small-scale oscillation diminishes in unit time scale. However when f is present and f is not a constant, this is not true. Indeed, in such cases, $\nabla_\mu \omega$ does not necessarily have a sign even if ε is small compared to the C^n norm of f for any $n \geq 1$, see Example 3.9. Thus the assumption of $a \neq 0$ is sharp when f is not a constant. With (ε, a) monotonicity, the interior full monotonicity is shown in Lemma 3.1 and its remark.

2. If f is a non-negative constant, our results hold even if $\alpha = \infty$. We refer readers to the remarks after Lemma 3.1 for the detailed discussion.

Below for any $\bar\gamma$-Hölder continuous function (with $\bar\gamma \in (0,1)$) $g : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^d$ an open set, we denote its $\bar\gamma$-Hölder seminorm and $\bar\gamma$-Hölder norm, respectively, as

$$\|g\|_{C^{0,\bar\gamma}(\Omega)} := \sup_{x,y\in\Omega, x\neq y} \frac{|g(x) - g(y)|}{|x-y|^{\bar\gamma}} \quad \text{and} \quad \|g\|_{C^{\bar\gamma}(\Omega)} := \|g\|_{L^\infty(\Omega)} + \|g\|_{C^{0,\bar\gamma}(\Omega)}.$$

When there is no ambiguity regarding the domain, we will drop Ω from the notations of $C^{0,\bar\gamma}(\Omega)$ and $C^{\bar\gamma}(\Omega)$, and we will simply write $\|g\|_\infty := \|g\|_{L^\infty(\Omega)}$, and the Lipschitz constant $\|g\|_{\mathrm{Lip}} := \|g\|_{C^{0,1}}$.

2.2. **Properties of harmonic and superharmonic functions.** First we recall the well-known Dahlberg lemma.

Lemma 2.9. ([Dah79]) *Let ω_1, ω_2 be two non-negative harmonic functions in a domain $D \subseteq \mathbb{R}^d$ of the form*

$$\Big\{ (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, |x'| < 2,\, |x_d| < 2\bar M,\, x_d < g(x') \Big\}$$

with $g\colon \mathbb{R}^{d-1} \to \mathbb{R}$ a Lipschitz function with Lipschitz constant less than $\bar M$ and $g(0) = 0$. Assume further that $\omega_1 = \omega_2 = 0$ along the graph of g. Then, there exists $C > 1$ depending only on $d, \bar M$ such that

$$\frac{1}{C} \leq \frac{\omega_1(x', x_d)}{\omega_2(x', x_d)} \cdot \frac{\omega_2(0, \bar M)}{\omega_1(0, \bar M)} \leq C$$

in $\big\{(x', x_d) \,:\, |x'| < 1,\, |x_d| < \bar M,\, x_d < g(x')\big\}$.

The following lemma follows from Dahlberg's Lemma and the explicit form of harmonic functions in a cone domain. While the proof is basic, we present it here given the importance of the constant θ_β in our analysis (θ'_β will only be used in Lemma 4.6).

Lemma 2.10. *For given $\theta \in (0, \pi)$, $\mu \in \mathbb{S}^{d-1}$, consider a harmonic function ω in $W_{\theta,\mu} \cap B_2$ such that $\sup_{W_{\theta,\mu}\cap B_1} \omega = 1$ and $\omega = 0$ on $\partial W_{\theta,\mu} \cap B_2$. Then there exists $c \in (0,1)$ such that for any $\beta \in (1,2)$, there are $\theta_\beta, \theta'_\beta \in (0, \frac{\pi}{2})$*

(which are continuous and monotonely decreasing in $\beta \in (1,2)$, and converge to $\frac{\pi}{2}$ as $\beta \to 1$) such that we have

$$\omega(s\mu) \geq c\, s^{\beta} \quad \text{for all } s \in (0,1) \text{ if } \theta \geq \theta_\beta \tag{2.3}$$

and

$$\omega(s\mu) \leq s^{2-\beta}/c \quad \text{for all } s \in (0,1) \text{ if } \theta \leq \pi - \theta'_\beta. \tag{2.4}$$

Proof. This result is a direct consequence of [Anc12, Theorem 1.1]. The theorem proves the existence of a harmonic function in $W_{\theta,\mu}$ such that it vanishes on the boundary of $W_{\theta,\mu}$. Moreover, the harmonic function is of the following form

$$h(r\vartheta) = c\, r^{\beta_\theta} \varphi(\vartheta)$$

where $c, r > 0$, $\vartheta \in \Sigma_\theta$ with $\Sigma_\theta := \mathbb{S}^{d-1} \cap W_{\theta,\mu}$, and φ is a positive function in Σ_θ vanishing on $\partial\Sigma_\theta$. The constant $\beta_\theta > 0$ is given by

$$\beta_\theta := \frac{-d+2+\sqrt{(d-2)^2 + 4\lambda_1(\Sigma_\theta)}}{2}$$

where $\lambda_1(\Sigma_\theta)$ denotes the first eigenvalue of the opposite of the Dirichlet Laplacian in Σ_θ, i.e.

$$\lambda_1(\Sigma_\theta) = \inf\left\{ \int_{\mathbb{S}^{d-1}} |\nabla u|^2 d\sigma \,:\, u \in C^1_c(\Sigma_\theta),\ \int_{\mathbb{S}^{d-1}} |u|^2 d\sigma \geq 1 \right\},$$

with σ the standard Riemannian spherical measure in $\mathbb{S}^{d-1}$. It is not hard to see that β_θ is non-increasing in θ, and $\beta_{\frac{\pi}{2}} = 1$ (since $h = x \cdot \mu$ is a positive harmonic function in $W_{\frac{\pi}{2},\mu}$). We refer readers to [BCG83] for several bounds of $\lambda_1(\Sigma_\theta)$. Since $\lambda_1(\Sigma_\theta)$ and β_θ depend continuously on θ, β_θ can be arbitrarily close to 1 if θ is large (close to $\frac{\pi}{2}$). The conclusions follow immediately from Harnack's inequality and Dahlberg's lemma. □

Remark 2.11. When $d = 2$ the formula can be written as

$$\theta_\beta = \frac{\pi}{2\beta} \quad \text{and} \quad \theta'_\beta = \max\left[\pi - \frac{\pi}{2(2-\beta)}, 0\right].$$

In particular one can deduce that $\theta_\beta \geq \theta_2 \geq \frac{\pi}{4}$ when $d \geq 2$, by comparison principle for harmonic functions.

Next we show some properties of superharmonic functions.

Lemma 2.12. *Let $f : \mathbb{R}^d \to [0,\infty)$ be continuous, let $r > 0$ and let $\omega : \overline{B_{2r}} \to [0,\infty)$, $\omega \in C^2(\overline{B_{2r}})$ be a classical solution to*

$$-\Delta\omega = f, \quad \text{in } B_{2r}.$$

Then there exists a constant $C > 0$, depending only on the dimension, such that for all $x \in B_r$,

$$\omega(x) \leq C\omega(0) + Cr^2\|f\|_{L^\infty(B_{2r})}, \qquad |\nabla\omega(x)| \leq Cr^{-1}\omega(0) + Cr\|f\|_{L^\infty(B_{2r})}.$$

Moreover if $\nabla_\mu\omega \geq 0$ for some $\mu \in \mathbb{S}^{d-1}$ in B_{2r}, then for all $x \in B_r$,

$$\nabla_\mu\omega(x) \leq C\nabla_\mu\omega(0) + Cr\|f\|_{L^\infty(B_{2r})}.$$

Proof. Set $\tilde{\omega}(x) := \omega(rx)$, and so $-\Delta\tilde{\omega} = \tilde{f}$ in B_2 with $\tilde{f}(x) := r^2 f(rx)$. Let G be the Green's function of Laplacian in B_2. Then, we have the representation formula (see e.g., [Eva10])

$$\tilde{\omega}(x) = -\int_{\partial B_2} \tilde{\omega}(y)\partial_n G(x,y)d\sigma(y) + \int_{B_2} \tilde{f}(y)G(x,y)dy, \tag{2.5}$$

where n denotes the outward pointing unit normal to ∂B_1. Notice that there exists $C = C(d) > 0$ such that

$$\sup_{x\in B_1} \left(\int_{B_2} G(x,z)dz + \int_{B_2} |\nabla_x G(x,z)|dz \right) \le C, \tag{2.6}$$

and $0 < -\partial_n G(x,y) \le -C\partial_n G(0,y)$ for $(x,y) \in B_1 \times \partial B_2$. Therefore, also using that $\omega \ge 0$ and (2.5) with $x = 0$, we get for $x \in B_1$ that

$$\begin{aligned}\tilde{\omega}(x) &\le -C\int_{\partial B_2} \tilde{\omega}(y)\partial_n G(0,y)d\sigma(y) + C\int_{B_2} \tilde{f}(y)G(0,y)dy + (C+1)\sup_{B_1} \|\tilde{f}\|_{L^\infty(B_1)} \int_{B_2} G(\cdot,y)dy \\ &\le C\tilde{\omega}(0) + C(C+1)\|\tilde{f}\|_{L^\infty(B_2)}\end{aligned}$$

By rewriting this estimate for ω and f, this yields the first inequality of the conclusion after enlarging C.

Next since $|\nabla\partial_n G(x,y)| \le -C\partial_n G(0,y)$ for any $(x,y) \in B_1 \times \partial B_2$, taking derivatives on both sides of (2.5) yields

$$|\nabla\tilde{\omega}(x)| \le -C\int_{\partial B_2} \tilde{\omega}(y)\partial_n G(0,y)d\sigma(y) + \left|\int_{B_2} \tilde{f}(y)\nabla_x G(x,y)dy\right| \le C\tilde{\omega}(0) + C\|\tilde{f}\|_{L^\infty(B_2)}$$

where in the last inequality we used (2.5) with $x = 0$ and (2.6). This then implies the second inequality, again, by using the definition of $\tilde{\omega}$ and $\tilde{f}$.

For the last claim, without loss of generality, we assume that ω is C^2 in a neighbourhood of B_{2r}. Taking derivatives on both sides of (2.5) and using $\nabla_\mu\omega \ge 0$ yield

$$\begin{aligned}\nabla_\mu\tilde{\omega}(x) &\le -\int_{\partial B_2} \nabla_\mu\tilde{\omega}(y)\partial_n G(x,y)d\sigma(y) + \int_{B_2} \tilde{f}(y)|\nabla G(x,y)|dy \\ &\le -C\int_{\partial B_2} \nabla_\mu\tilde{\omega}(y)\partial_n G(0,y)d\sigma(y) + C\|\tilde{f}\|_{L^\infty(B_2)} \le C\nabla_\mu\tilde{\omega}(0) + C\|\tilde{f}\|_{L^\infty(B_2)}\end{aligned}$$

which implies the last inequality. □

3. Monotonicity properties, Streamlines, and Examples

In this section, we prove several tools that will be used to prove the main theorems, and we discuss by examples the optimality of our monotonicity assumptions and the formation of cusps on the free boundary.

3.1. **Interior monotonicity.** The goal of this section is to show that if a superharmonic function is $(\varepsilon, \varepsilon^\alpha)$-monotone, then under some assumptions it is fully monotone in the interior.

The corresponding result with $f \equiv 0$, $\vec{b} \equiv 0$, and $(\varepsilon, 0)$-monotonicity is proved in the book of Caffarelli and Salsa [CS05, Corollary 11.16]. Here we need ε^α to be positive to compensate the possible loss of monotonicity caused the source function f. One important ingredient of the proof in [CS05, Corollary 11.16] is the Harnack inequality, which is applied to $h := \omega(x) - \omega(x - \varepsilon\mu)$. However when $f \neq 0$, h solves a Poisson equation with the source term $f(x - \varepsilon\mu) - f(x)$ which can be negative at some points, and in such cases the Harnack inequality might fail (because for example, $h := x^2$ solves $-\Delta h = -2$ and $h(x) \geq 0$ with equality holds if and only if $x = 0$). To overcome the problem, we estimate carefully the "error" from the source term in the lemma below. We will later combine this lemma with Lemma 3.3, which provides a lower bound for ω, to conclude the interior monotonicity. Below we use the convention that $\varepsilon^\infty = 0$ for $\varepsilon \in (0, 1)$.

Lemma 3.1. *Let $f \geq 0$ be $\bar{\gamma}$-Hölder continuous on $\overline{B_1}$ for some $\bar{\gamma} \in (0, 1)$, and $\alpha \in [0, \infty]$ and $\varepsilon, \kappa_1 \in (0, 1)$. There exists $C = C(d) > 0$ such that the following holds for all ε small enough (depending only on d, α, κ_1). If ω is a non-negative solution to $-\Delta\omega = f$ in $B_{\varepsilon^{1-\kappa_1}}$, and ω is $(\varepsilon, \varepsilon^\alpha)$-monotone with respect to $W_{0,\mu}$ for $\mu \in \mathbb{S}^{d-1}$, then*

$$\nabla_\mu \omega(x) \geq \varepsilon^\alpha (1 - C\varepsilon^{\kappa_1})\omega(x) - C\varepsilon^{1+\bar{\gamma}-\kappa_1} \|f\|_{C^{0,\bar{\gamma}}(B_1)} \quad \text{for all } x \in B_\varepsilon.$$

Proof. Let us denote $\delta := \varepsilon^{\alpha+1} < 1$. We will only show the conclusion for $x = 0$, and the general case of $x \in B_\varepsilon$ follows the same. For $s \in [\varepsilon, 2\varepsilon]$, define

$$h_s(x) := \omega(x + s\mu) - (1 + \delta)\omega(x), \tag{3.1}$$

and it follows from the $(\varepsilon, \varepsilon^\alpha)$-monotonicity assumption that $h_s \geq 0$. Using the monotonicity again yields for $s \in [\varepsilon, 2\varepsilon]$,

$$\begin{aligned} \sum_{i=0}^{2} (1+\delta)^{-i} h_\varepsilon(x + i\varepsilon\mu) &= (1+\delta)^{-2}\omega(x + 3\varepsilon\mu) - (1+\delta)\omega(x) \\ &\geq (1+\delta)^{-1}\omega(x + s\mu) - (1+\delta)\omega(x) \geq (1+\delta)^{-1} h_s(x) - \delta\omega(x). \end{aligned} \tag{3.2}$$

Note that $-\Delta h_s = (1+\delta)f(\cdot) - f(\cdot + s\mu)$ and

$$|(1+\delta)f(\cdot) - f(\cdot + s\mu)| \leq \delta\|f\|_\infty + s^{\bar{\gamma}}\|f\|_{C^{0,\bar{\gamma}}}.$$

Hence $h_s \geq 0$ and Lemma 2.12 (after shifting 0 to any $y \in B_{3\varepsilon}$) yield for some $C > 0$ (if ε is small) and any $s \in [\varepsilon, 2\varepsilon]$ that

$$h_s(x) \leq C h_s(y) + C\varepsilon^2\delta\|f\|_\infty + C\varepsilon^{2+\bar{\gamma}}\|f\|_{C^{0,\bar{\gamma}}} \quad \text{for all } x, y \in B_{3\varepsilon}. \tag{3.3}$$

This and (3.2) with $x = 0$ yield

$$h_s(0) \leq C \sum_{i=0}^{2} h_\varepsilon(i\varepsilon\mu) + C\delta\omega(0) + C\varepsilon^2\delta\|f\|_\infty + C\varepsilon^{2+\bar{\gamma}}\|f\|_{C^{0,\bar{\gamma}}}$$

$$\leq Ch_\varepsilon(0) + C\delta\omega(0) + C\varepsilon^2\delta\|f\|_\infty + C\varepsilon^{2+\bar\gamma}\|f\|_{C^{0,\bar\gamma}}.$$

Next, by Lemma 2.12 again, for $s \in [\varepsilon, 2\varepsilon]$ and $r := \frac{1}{2}\varepsilon^{1-\kappa_1}$ with $\kappa_1 \in (0,1)$, we get

$$\begin{aligned} |\nabla h_s(0)| &\leq Cr^{-1}h_s(0) + Cr\delta\|f\|_\infty + Cr\varepsilon^{\bar\gamma}\|f\|_{C^{0,\bar\gamma}} \\ &\leq Cr^{-1}h_\varepsilon(0) + Cr^{-1}\delta\omega(0) + Cr\delta\|f\|_\infty + Cr\varepsilon^{\bar\gamma}\|f\|_{C^{0,\bar\gamma}}. \end{aligned} \tag{3.4}$$

Now we estimate $h_\varepsilon(0)$. We obtain from (3.1) and (3.3) with $s = \varepsilon$ that

$$h_\varepsilon(0) \leq C(\omega(2\varepsilon\mu) - (1+\delta)\omega(\varepsilon\mu)) + c_{\varepsilon,f},$$

where $c_{\varepsilon,f} := C\varepsilon^2\delta\|f\|_\infty + C\varepsilon^{2+\bar\gamma}\|f\|_{C^{0,\bar\gamma}}$. Since $\nabla_\mu h_s(0) = \nabla_\mu\omega(s\mu) - (1+\delta)\nabla_\mu\omega(0)$, this implies

$$\begin{aligned} h_\varepsilon(0) &\leq C\left(\int_\varepsilon^{2\varepsilon} \nabla_\mu\omega(s\mu)ds - \delta\omega(\varepsilon\mu)\right) + c_{\varepsilon,f} \\ &\leq C\left(\int_\varepsilon^{2\varepsilon} |\nabla h_s(0)|ds + \varepsilon(1+\delta)\nabla_\mu\omega(0) - (1+\delta)\delta\omega(0)\right) + c_{\varepsilon,f}, \end{aligned} \tag{3.5}$$

where we also used $\omega(\varepsilon\mu) \geq (1+\delta)\omega(0)$. Then by (3.4) with $r = \frac{1}{2}\varepsilon^{1-\kappa_1}$ and the definitions of $c_{\varepsilon,f}$ and δ, we obtain for some $C = C(d) > 0$,

$$h_\varepsilon(0) \leq C\varepsilon^{\kappa_1}h_\varepsilon(0) + C\varepsilon^{1+\kappa_1+\alpha}\omega(0) + C(1+\delta)\left(\varepsilon\nabla_\mu\omega(0) - \delta\omega(0)\right) + C\varepsilon^{2+\bar\gamma-\kappa_1}(\varepsilon^\alpha\|f\|_\infty + \|f\|_{C^{0,\bar\gamma}}).$$

Using $h_\varepsilon \geq 0$, the above estimate yields for all $\varepsilon > 0$ small enough,

$$\nabla_\mu\omega(0) \geq \varepsilon^\alpha(1 - C\varepsilon^{\kappa_1})\omega(0) - C(1+\varepsilon^\alpha)\varepsilon^{1+\bar\gamma-\kappa_1}\|f\|_{C^{0,\bar\gamma}}. \tag{3.6}$$

This yields the conclusion for $x = 0$. □

Remark 3.2. 1. It is clear that $(\varepsilon, \varepsilon^\alpha)$-monotonicity with respect to $W_{\theta,\mu}$ for some $\theta \geq 0$ and $\mu \in \mathbb{S}^{d-1}$ implies $(\varepsilon, \varepsilon^{\alpha'})$-monotonicity with respect to $W_{0,\mu}$ for all $0 \leq \alpha \leq \alpha'$.

2. Let ω be from Lemma 3.1. If either $\alpha \neq \infty$ or f is constant, and $\varepsilon > 0$ is small enough such that $\varepsilon^\alpha\omega(\cdot) \geq 2C\varepsilon^{1+\bar\gamma-\kappa_1}\|f\|_{C^{0,\bar\gamma}}$ and $C\varepsilon^{\kappa_1} < \frac{1}{2}$, then $\omega(s\mu)$ is non-decreasing in s for all $s \in (-\varepsilon, \varepsilon)$.

3. Furthermore, if $\varepsilon^\alpha\omega(\cdot) \geq C\varepsilon^{1+\bar\gamma-2\kappa_1}\|f\|_{C^{0,\bar\gamma}}$ in $B_{2\varepsilon^{1-\kappa_1}}$, then for some larger $C > 0$ and any $j \in (0,1)$, ω is $(j\varepsilon, \varepsilon^\alpha(1 - C\varepsilon^{\kappa_1}))$-monotone with respect to $W_{0,\mu}$ in $B_{\varepsilon^{1-\kappa_1}}$.

3.2. **Polynomial growth near the free boundary.** The goal of this section is to show that a superharmonic function which has cone monotonicity up to ε-scale has a polynomial growth bound up to the same scale. The growth rate lower bound will be used in competition to the irregularity of the source term, to show that the regularity propagates to the boundary over time (see Lemma 6.1, Proposition 6.2 and Theorem 7.3). This bound can be improved to a linear rate once we obtain full monotonicity, later in Section 7.

Next lemma provides a lower bound for the growth rate of $(\varepsilon, 0)$-monotone superharmonic functions.

Lemma 3.3. *Let $\mu \in \mathbb{S}^{d-1}$, and let $\omega \geq 0$ be a continuous function in B_2 such that*

$$-\Delta\omega \geq 0 \text{ in } \Omega_\omega \cap B_2, \quad 0 \in \Gamma_\omega = \partial\Omega_\omega, \quad \omega(\mu) \geq 1,$$

and ω is $(\varepsilon, 0)$-monotone with respect to $W_{\theta,\mu}$ in B_2 for some ε small enough. Then for some dimensional constant $c > 0$ and for any $\beta \in (1, 2)$, if $\theta \geq \theta_\beta$ (with θ_β given in Lemma 2.10) we have

$$\omega(x) \geq c\, d(x, \Gamma_\omega)^\beta$$

for all $x \in B_1 \cap \Omega_\omega$ satisfying $d(x, \Gamma_\omega) \geq 2\varepsilon$.

Proof. For each $x \in B_1 \cap \Omega_\omega$ satisfying $d(x, \Gamma_\omega) \geq 2\varepsilon$, there is $x_0 \in \Gamma_\omega \cap B_2$ such that $x = x_0 + s\mu$ with $s \geq d(x, \Gamma_\omega) \geq 2\varepsilon$. Note that it follows from the monotonicity assumption and $\{0, x_0\} \subset \Gamma_\omega$ that $\omega > 0$ in $((x_0 + \varepsilon\mu + W_{\theta,\mu}) \cup (\varepsilon\mu + W_{\theta,\mu})) \cap B_2$. Thus Harnack's inequality and $\omega(\mu) \geq 1$ yield $\omega(x + \frac{1}{2}\mu) \geq c$ for some dimensional constant $c > 0$. Then by comparing ω with a non-negative harmonic function whose support is $x_0 + \varepsilon\mu + W_{\theta,\mu}$, Lemma 2.9 and Lemma 2.10 yield for some dimensional $c' > 0$ we have

$$\omega(x) \geq c'(s - \varepsilon)^\beta \geq 4^{-1} c' d(x, \Gamma_\omega)^\beta \quad \text{whenever } \theta \geq \theta_\beta.$$

□

For the next lemma the growth rate bound is obtained excluding only a small portion of the original domain B_1, with the expanse of restricting to the near boundary region.

Lemma 3.4. *Under the assumptions of Lemma 3.3 except that ω is only assumed to be $(\varepsilon, 0)$-monotone with respect to $W_{\theta,\mu}$ in B_1 (instead of B_2), then for some $c = c(d) > 0$ and for any $\beta \in (1, 2)$, if $\theta \geq \theta_\beta$ we have*

$$\omega(x) \geq c\, d(x, \Gamma_\omega)^\beta$$

for all $x \in B_{1-\varepsilon^{1/2}} \cap \Omega_\omega$ satisfying $d(x, \Gamma_\omega) \in [2\varepsilon, \varepsilon^{\frac{1}{2}}]$.

Proof. For any $x \in B_{1-\varepsilon^{1/2}} \cap \Omega_\omega$ satisfying $d(x, \Gamma_\omega) \in [2\varepsilon, \varepsilon^{\frac{1}{2}}]$, there exists $x_0 \in \Gamma_\omega \cap B_1$ such that $x = x_0 + s\mu$ with $s \geq 2\varepsilon$. Note that this is not true if $d(x, \Gamma_\omega) >> \varepsilon^{1/2}$. With this $x_0 \in \Gamma_\omega \cap B_1$, we can conclude the proof the same as in Lemma 3.3. □

3.3. **Streamlines.** Here we introduce *streamlines* associated with the drift term, which yields an important monotonicity property for our flow. They are defined as the unique solution $X(t; x_0)$ of the ODE

$$\begin{cases} \partial_t X(t; x_0) = -\vec{b}(X(t; x_0)), & t \in \mathbb{R}, \\ X(0; x_0) = x_0. \end{cases} \tag{3.7}$$

We write $X(t) := X(t; 0)$. In order to analyze the solution along one streamline that passes through $(0, 0)$, we define

$$\bar{u}(x, t) := u(x + X(t), t). \tag{3.8}$$

Then $\bar{u}$ satisfies

$$\begin{cases} -\Delta \bar{u} = f_0(x,t) & \text{in } \{\bar{u} > 0\}, \\ \bar{u}_t = |\nabla \bar{u}|^2 + \vec{b}_0(x,t) \cdot \nabla \bar{u} & \text{on } \partial\{\bar{u} > 0\}, \end{cases} \tag{3.9}$$

where

$$f_0(x,t) := f(x + X(t)), \quad \vec{b}_0(x,t) := \vec{b}(x + X(t)) - \vec{b}(X(t)). \tag{3.10}$$

It was shown in [KPW19, Lemma 3.5] for the drift porous medium equation that $\{u > 0\} =: \Omega_u$ is non-decreasing along the streamlines. The same holds in our case.

Lemma 3.5. *If $(x_0, t_0) \in \Omega_u$, then $(X(t; x_0), t + t_0) \in \Omega_u$ for all $t > 0$.*

Proof. Let us assume $(x_0, t_0) = (0, 0)$. By continuity of the solution, suppose that for some $z \in B_1$ we have $u(t, z) \geq c > 0$ for all $t \in [0, \tau)$ with some small $\tau > 0$. Let D_0 be any strict open subset of $\Omega_u(0) \cap B_1$, and then for $t \in (0, \tau)$ define

$$D_t := \{X(t; x) \,:\, x \in D_0\} \cap B_1.$$

We can assume that $z \in D_t$ for $t \in [0, \tau]$. Let $v(\cdot, t)$ be the largest subharmonic function in $D_t \backslash \{z\}$ such that $v(\cdot, t) = 0$ on ∂D_t and $v(t, z) = c$. It is clear that $v \prec u$ at $t = 0$ and $v < u$ on $(\overline{\Omega_u(t)} \cap \partial B_1) \cup \{z\}$ for $t \in (0, \tau)$.

We claim that that v is a viscosity subsolution to (1.1) in $(B_1 \backslash \{z\}) \times (0, \tau)$. Let us only verify the free boundary condition. Suppose for a smooth function $\phi \in C^{2,1}_{x,t}$ such that $v - \phi$ has a local maximum in $\overline{\Omega_v} \cap \{t \leq t_0\}$ that equals to 0 at $(x_0, t_0) \in \Gamma_v$ and $x_0 \notin \partial B_1$. Note that by the definition of D_t, $\phi(x_0, t_0) \leq \phi(X(-\varepsilon; x_0), t_0 - \varepsilon)$ for all ε sufficiently small. Therefore $\phi_t \leq \vec{b} \cdot \nabla \phi$ at (x_0, t_0), and thus we can conclude with the claim.

Then the comparison principle (Lemma 2.5) yields $v \leq u$. Note that $\{(x, t) \,:\, x \in D_t, t \in [0, \tau)\}$ is non-decreasing along streamlines and D_0 can be arbitrarily close to $\Omega_u(0) \cap B_1$. So Ω_u is non-decreasing along streamlines for $t \in (0, \tau)$, and then the same holds for all positive time. □

3.4. **Lipschitz space-time neighborhood of the free boundary.** In this subsection we show that if the solution u to (1.1) is $(\varepsilon, 0)$-monotone in space, then there exists a Lipschitz space-time neighborhood of the free boundary of u. The interesting feature lies in the time variable component of the Lemma: for the space variable it can be derived from a geometric argument, for instance see Proposition 11.14 in [CS05]. This Lipschitz set will be used as the region where we do comparison later. For simplicity of discussions, we take $\mu := -e_d$ below.

Lemma 3.6. *Suppose $u, f, \vec{b}$ satisfy (1.1), and they are uniformly bounded by L in $\mathcal{Q}_2 = B_2 \times (-2, 2)$ for some $L \geq 1$. If u is $(\varepsilon, 0)$-monotone with respect to $W_{-e_d, \theta}$ for some $\theta \in (0, \frac{\pi}{2})$ in $\mathcal{Q}_2$, then for any $r \in [4\varepsilon, \frac{1}{4}]$ there exists a Lipschitz continuous function $\Phi_r : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\Gamma_u(t) \cap B_{3/2} \subseteq \{(x', x_d) \in B_{3/2} \,:\, |\Phi_r(x', t) - x_d| < r\}$$

for all $t \in (-2,2)$. Moreover, Φ_r is $\cot\theta$-Lipschitz continuous in space and C/r-Lipschitz continuous in time for some $C = C(L,\theta) > 0$.

Proof. From the $(\varepsilon, 0)$-monotonicity assumption, it follows from Proposition 11.14 [CS05] that for each $t \in (-2,2)$, $\Gamma_u(t)$ is contained in a $(1-\sin\theta)\varepsilon$-neighborhood of the graph of a Lipschitz function, with Lipschitz constant $\cot\theta$. Therefore we can find a Lipschitz function $\phi^t : \mathbb{R}^{d-1} \to \mathbb{R}$ with the same Lipschitz constant such that

$$\Gamma_u(t) \cap B_2 \subseteq \{(x', x_d) \in B_2 \,:\, |\phi_t(x') - x_d| < \varepsilon\}. \tag{3.11}$$

Claim. If $r \in (0, \frac{1}{4}]$ and $u(\cdot, t_0) = 0$ in $B_r(x_0)$ for some $(x_0, t_0) \in B_{3/2} \times (-2,2)$, then $u(x_0, t_0 + t) = 0$ for all $t \le cr^2$ for some $c = c(L) > 0$.

Proof of claim. We use a barrier argument to prove the claim for $d \ge 3$ (the proof for $d = 2$ is similar). Also suppose, without loss of generality, that $t_0 = 0$ and $x_0 = 0$. For some $A \ge 1$ to be determined, let

$$w(x,t) := a_t - 2^{-1}L|x|^2 - b_t|x|^{2-d} \quad \text{in } \Sigma := \{(x,t) \,:\, x \in B_{1/2}\backslash B_{r_t},\, t \in [0, r^2/(2A)]\}$$

where

$$a_t := 1 + 8^{-1}L + b_t 2^{d-2}, \quad b_t := \frac{8 + L - 4Lr_t^2}{8r_t^{2-d} - 2^{d+1}}, \quad r_t := r - Ar^{-1}t.$$

Then it is straightforward to verify that for $t \in [0, (2A)^{-1}r^2]$, $-\Delta w(\cdot, t) = dL$, $w(\cdot, t) = 1$ on $\partial B_{1/2}$ and $w(\cdot,t) = 0$ on ∂B_{r_t}. Moreover for these t,

$$|\nabla w(x,t)| \le L|x| + (d-2)b_t|x|^{1-d} \le C/r \quad \text{for } x \in B_{1/2}\backslash B_{r_t},$$

as $1/2 < 1/r$, with $C > 0$ only depending on d, L. Therefore, using that $\frac{d}{dt} r_t = -Ar^{-1}$ and by picking $A := C + L$, we get that w is a supersolution to (1.1). So the assumptions and the comparison principle yield $u \le w$ in Σ. Since $w(\cdot, t) = 0$ on ∂B_{r_t} for all $t \in [0, (2A)^{-1}r^2]$, we proved the claim with $c := (2A)^{-1}$.

Now for each $x' \in \mathbb{R}^{d-1}$ satisfying $|x'| \le \frac{3}{2}$, since $u((x', \phi_t(x') + \varepsilon), t) = 0$, the $(\varepsilon, 0)$-monotonicity yields $u(\cdot, t) = 0$ in $B_{r\sin\theta}((x', \phi_t(x') + r + \varepsilon))$ for all $r \ge \varepsilon$. Hence the above claim implies

$$u\left((x', \phi_t(x') + r + \varepsilon), t + s\right) = 0 \quad \text{for all } s \in [0, c_\theta r^2]$$

where $c_\theta := c\sin\theta$. This yields

$$\phi_{t+s}(x') \le \phi_t(x') + r + 2\varepsilon \quad \text{for all } s \in [0, c_\theta r^2]. \tag{3.12}$$

On the other hand, since Ω_u is non-decreasing along streamlines and $|\vec{b}| \le L$, we obtain

$$\phi_{t+s}(x') \ge \phi_t(x') - c_\theta L r^2 - 2\varepsilon \quad \text{for all } s \in [0, c_\theta r^2]. \tag{3.13}$$

Let $r \in [\varepsilon, \frac{1}{4}]$ and we use ϕ_t to construct a Lipschitz space-time function Φ_r. Let $t_0 := -2$, and define iteratively for $k \in \mathbb{N}$ that $t_k := t_0 + kc_\theta r^2$, and $\Phi_r(x', t_k) := \phi_{t_k}(x')$. Then we extend $\Phi_r(x', \cdot)$ to all $t \in (-2,2)$ by linear interpolation. We see that Φ_r is $\cot\theta$-Lipstchiz continuous in space and $2(c_\theta r)^{-1}$-Lipschitz continuous in time. Finally, (3.11), (3.12) and (3.13) yield that

$$\Gamma_u(t) \cap B_{3/2} \subseteq \{(x', x_d) \in B_{3/2} \,:\, |\Phi_r(x', t) - x_d| < r + 3\varepsilon\}$$

which finishes the proof with $r + 3\varepsilon$ in place of r. □

3.5. **Examples: Waiting time, Formation of cusps and discussion of optimality for the monotonicity assumption.** First let us show that the theorem is false when the angle is small.

Example 3.7. We only consider space dimension 2 and we use the polar coordinates r, θ such that $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. Let us consider $f = 1$, $\vec{b} \equiv 0$ and consider the initial data $u_0(r, \theta) =$u(

In the following first example, we show that the free boundary of solutions starting with a cone as its positive set develops a cusp at the vertex of the cone if the vector field is only Hölder continuous.

Example 3.8. We only consider space dimension 2 and we use the polar coordinates r, θ such that $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. In the example we take $f \equiv 0$, and $\vec{b}$ to be of the form $\vec{b} = (C_0|x_2|^{\gamma_0 - 1}, 0)$ with $C_0 > 1$ and $\gamma_0 \in (1, 2)$.

First we show that the support of the solution is contained in a shrinking cone when C_0 is large. For $t \in [0, 1]$, let

$$\Gamma'_t := \{|\theta| = \theta_t\} \quad \text{ where } \theta_t := (1-t)\frac{\pi}{2\gamma_0} + t\frac{\pi}{2\gamma_1} \in (0, \frac{\pi}{2}) \text{ and } \gamma_1 > \gamma_0.$$

The opening of the cones $\{|\theta| < \theta_t\}$ shrinks from θ_0 to θ_1 for $t \in [0, 1]$. For each t, let $\varphi^t = r^{\gamma_t}(\cos(\gamma_t\theta))_+$ with $\gamma_t := \frac{\pi}{2\theta_t} > 1$. It is easy to see that $\Delta\varphi^t = 0$ in $\{\varphi^t > 0\} = \{|\theta| < \theta_t\}$, and

$$|\nabla\varphi^t| = r^{\gamma_t - 1}\sqrt{\cos^2(\gamma_t\theta) + {\gamma_t}^2\sin^2(\gamma_t\theta)}\Big|_{|\theta| = \theta_t} = \gamma_t r^{\gamma_t - 1} \quad \text{ on } |\theta| = \theta_t.$$

By direct computations, the outer normal direction of Γ'_t is $\nu'_t = (-\sin\theta_t, \pm\cos\theta_t)$, and the normal velocity of Γ'_t at $(r, \pm\theta_t)$ equals to $V'(r, \pm\theta_t) = -(\frac{\pi}{2\gamma_0} - \frac{\pi}{2\gamma_1})r$. We obtain on $\Gamma'_t \cap B_1$,

$$V' - |\nabla\varphi^t| - \vec{b}\cdot\nu'_t = -(\frac{\pi}{2\gamma_0} - \frac{\pi}{2\gamma_1})r - \gamma_t r^{\gamma_t - 1} + C_0|r\cos\theta_t|^{\gamma_0 - 1}\sin\theta_t$$

which is non-negative if C_0 is large enough, due to $\gamma_0 \le \gamma_t$. So φ^t is a supersolution to (1.1) in $B_1 \times (0, 1)$.

Next let u be a solution with initial data $\le \varphi^0$ and with boundary value $\le \varphi^t$ on $\partial B_1 \times \{t > 0\}$, then the origin is on Γ_u by Lemma 3.5 and $\vec{b}(0) = 0$. We claim that the comparison principle (Lemma 2.5) yields $u \le \varphi^t$ and so $\Omega_u(t)$ is contained in $\{\varphi^t > 0\} = \{|\theta| < \theta_t\}$. To justify the use of the comparison principle, by the choice of $\vec{b}$, we first compare u with $\varphi^t(x_1 + \delta, x_2)$ for $\delta > 0$ (the two functions are strictly separated) and then passing $\delta \to 0$ yields the desired inequality $u \le \varphi^t$.

Now we start with $t = 1$ and a solution u such that $(\Omega_u(1) \cap B_1) \subseteq \{|\theta| < \theta_1\}$, and show the formation of cusps. Assume $\gamma_1 > 2$ and $\sigma := \frac{2}{\gamma_1} \in (\gamma_0 - 1, 1)$. For $t \in (1,2)$, define

$$\Gamma_t := \{x_1 = g(x_2,t)\} \quad \text{where } g(x_2,t) := |x_2|\cot\theta_1 + (t-1)|x_2|^\sigma.$$

So a cusp develops at the vertex of the set $\{x_1 > g(x_2,t)\}$ when $t > 1$. For each $t \in (1,2)$, let ϕ^t be a harmonic function in $\{x_1 > g(x_2,t)\}$ with 0 boundary condition and $\phi^t(\frac{1}{2},0) = 1$. If we can show that $\phi^t(x_1,x_2)$ is a supersolution for $t \in (1,2)$, then after further assuming u to be smaller on ∂B_1 and by the comparison principle (which can be justified similarly as before), the support of u is contained in cusps for $t \in (1,2)$, which shows the formation of cusps.

To show that ϕ^t is a supersolution, it suffices to verify the free boundary condition on $\Gamma_t \cap B_1$. Note that the curvature of Γ_t at point $(g(x_2,t),x_2)$ satisfies

$$\frac{|\partial^2_{x_2} g(x_2,t)|}{(1+|\partial_{x_2} g(x_2,t)|^2)^{3/2}} \lesssim \frac{1}{|x_2|} \quad \text{uniformly for all } |x_2| < 1 \text{ and } t \in (1,2).$$

For any fixed $(y_1,y_2) \in \Gamma_t$, let us consider $\tilde{\phi}^t(x_1,x_2) := \phi^t(|y_2|x_1+y_1, |y_2|x_2+y_2)$. Then the free boundary of $\tilde{\phi}_t$ is a graph of finite curvature in a unit neighbourhood of the origin. Thus it follows from Lemma 2.9 (by comparing with radially symmetric harmonic functions, see also [JK05]) that for some $c > 0$,

$$|\nabla\tilde{\phi}^t(0,0)| \le c\,\tilde{\phi}^t(0,-y_2/|y_2|).$$

After scaling back, we get

$$|\nabla\phi^t(x_1,x_2)| \le c|\phi^t(x_1,0)|/|x_2| \le c x_1^{\gamma_1}/|x_2| \quad \text{on } \Gamma_t \cap B_1.$$

In the last inequality we used $\phi^t(x_1,0) \lesssim \varphi^1(x_1,0) \lesssim x_1^{\gamma_1}$, which is due to the support of ϕ^t is contained in $\{|\theta| < \theta_1\}$ and Lemma 2.10. Moreover, by direct computation,

$$\vec{b}\cdot\nu_t = (C_0|x_2|^{\gamma_0-1},0)\cdot\frac{(-1,\pm(\cot\theta_1+\sigma(t-1)|x_2|^{\sigma-1}))}{\sqrt{1+(\cot\theta_1+\sigma(t-1)|x_2|^{\sigma-1})^2}} \approx \frac{-C_0|x_2|^{\gamma_0-1}}{\sqrt{1+(t-1)^2|x_2|^{2\sigma-2}}}$$

where ν_t denotes the unit normal direction on Γ_t. The normal velocity of $\Gamma_t \cap B_1$ is

$$V := (|x_2|^\sigma,0)\cdot\frac{(-1,\pm(\cot\theta_1+\sigma(t-1)|x_2|^{\sigma-1}))}{\sqrt{1+(\cot\theta_1+\sigma(t-1)|x_2|^{\sigma-1})^2}} \approx \frac{-|x_2|^\sigma}{\sqrt{1+(t-1)^2|x_2|^{2\sigma-2}}} \gtrsim \frac{-|x_2|^{\gamma_0-1}}{\sqrt{1+(t-1)^2|x_2|^{2\sigma-2}}}$$

where the last inequality is due to $\sigma > \gamma_0 - 1$.

It remains to show that, if C_0 is large enough, then

$$V \ge \vec{b}\cdot\nu' + |\nabla\phi^t| \quad \text{on } \Gamma_t \cap B_1. \tag{3.14}$$

If $(t-1)|x_2|^{\sigma-1} \ge 1$, then $(t-1)|x_2|^\sigma \approx x_1$ on Γ_t. Due to $\gamma_0 - \sigma < 1$ and $\gamma_1\sigma = 2$, we have for $t \in (1,2)$,

$$V - \vec{b}\cdot\nu' \approx C_0|x_2|^{\gamma_0-\sigma}/(t-1) \gtrsim C_0|x_2| \quad \text{and} \quad |\nabla\phi^t| \lesssim (t-1)^{\gamma_1}|x_2|^{\gamma_1\sigma-1} \lesssim |x_2|.$$

While if $(t-1)|x_2|^{\sigma-1} \le 1$, then $|x_2| \approx x_1$ on Γ_t, and so, by $1 < \gamma_0 < \gamma_1$,

$$V - \vec{b}\cdot\nu' \approx C_0|x_2|^{\gamma_0-1} \quad \text{and} \quad |\nabla\phi_t| \lesssim |x_2|^{\gamma_1-1} \lesssim |x_2|^{\gamma_0-1}.$$

These imply (3.14), and we conclude with the formation of cusps.

Finally, we show that $(\varepsilon, 0)$-monotonicity with respect to $W_{\theta,\mu}$ in a large neighborhood does not imply that the solution is monotone along the direction μ in smaller neighborhood. Here $\theta > 0$ can be large and the source term f is smooth.

Example 3.9. Fix a small $\delta > 0$, and any $\theta \in (0, \frac{\pi}{2})$ and $n \in \mathbb{N}$. Take a smooth function $f \ge 0$ such that f is radially decreasing, f is supported in $B_{2\delta}$, and $f \equiv \delta^n$ in B_δ. Then we can assume that f is uniformly bounded in C^n norm regardless of the choice δ. Now let $\phi_1 : \mathbb{R}^2 \supseteq \overline{B_1} \to \mathbb{R}$ solve

$$-\Delta\phi_1 = f \text{ in } B_1 \quad \text{and} \quad \phi_1 = 0 \text{ on } \partial B_1.$$

Note that $\phi_1(x) = \int_{B_{2\delta}} \frac{1}{2\pi} \ln|x-y| f(y)dy$. Hence by direct computations,

$$\sup_{B_{2\delta}} |\nabla\phi_1| \ge \delta^{n+1}/C \quad \text{and} \quad \phi_1 \in (0, C\delta^{n+2}|\ln\delta|) \text{ in } B_1 \tag{3.15}$$

for some $C > 0$ independent of δ. Moreover, take

$$\phi(x) := \phi_1(x) + 2 + \delta^{n+1}x_1/(2C), \tag{3.16}$$

which is strictly positive, and satisfies $\phi_{x_1} < 0$ at some points in $B_{2\delta}$ by (3.15) and the fact that ϕ_1 is radial. We claim that, with $\varepsilon := \delta^{\frac{1}{2}}$ and δ sufficiently small, ϕ is $(\varepsilon, 0)$-monotone with respect to $W_{\theta,\mu}$ with μ being the positive x_1-direction. Indeed, for any $x, y \in B_1$ and $y \in B_{(\sin\theta)\varepsilon}(x + \varepsilon\mu)$, (3.16) and the second inequality in (3.15) yield

$$\phi(y) - \phi(x) \ge \delta^{n+1}(y_1 - x_1)/(2C) - \phi_1(x) \ge (1-\sin\theta)\varepsilon\delta^{n+1}/(2C) - C\delta^{n+2}|\ln\delta| \ge 0,$$

after taking $\delta = \varepsilon^2$ to be small enough. Thus this yields the claim.

Finally (still in dimension 2), we show that $(\varepsilon, \varepsilon^\alpha)$-monotonicity with respect to $W_{\theta,\mu}$ in a large neighborhood does not imply that the solution is monotone along the direction μ in smaller neighborhood. The source term f is bounded, the constant $\alpha \in (0, 1)$, and the solution can be $>> \varepsilon$.

Example 3.10. Let $\alpha > 0$ be fixed, and let $\min\{0, 1-\alpha\} < \kappa < 1$ and $\delta := \varepsilon^{(\alpha+\kappa+1)/2}$. Then take C, θ, f and ϕ_1 from the previous example with $n = 0$. We define

$$\phi(x) := \phi_1(x) + \delta(x_1 + 1)/2C + \varepsilon^\kappa.$$

By (3.15) with $n = 0$, we have for sufficiently small ε that

$$\varepsilon^\kappa \le \phi \le C\delta^2|\ln\delta| + \delta/C + \varepsilon^\kappa \le 2\varepsilon^\kappa \quad \text{in } B_1. \tag{3.17}$$

From the previous example, ϕ_{x_1} does not have a sign in B_1. We now show that ϕ is $(\varepsilon, \varepsilon^\alpha)$-monotone with respect to $W_{\theta,\mu}$ with μ denoting the positive

x_1-direction. Indeed, for $x, y \in B_1$ and $y \in B_{(\sin\theta)\varepsilon}(x+\varepsilon\mu)$, we get from (3.15) and (3.17) that

$$\begin{aligned}\phi(y) - (1+\varepsilon^{\alpha+1})\phi(x) &\geq \delta(y_1 - x_1)/(2C) - \phi_1(x) - \varepsilon^{\alpha+1}\phi(x) \\ &\geq (1-\sin\theta)\varepsilon\delta/(2C) - C\delta^2|\ln\delta| - 2\varepsilon^{\alpha+\kappa+1}.\end{aligned}$$

This is non-negative when ε is sufficiently small, due to $\varepsilon^{\alpha+\kappa} << \delta << \varepsilon$ by the choice of the parameters.

Note that, later in Proposition 6.2, we will apply the improved interior monotonicity of the solution u in the region that is ε^{γ_1}-away (with $\gamma_1 < 1$ but close to 1) from the free boundary and it is possible that $u \in (\varepsilon^{1/\sigma}, \varepsilon^{\sigma})$ for some $\sigma < 1$ in the region. Thus the above example indeed indicates that merely bounded source function is not sufficient for the purpose.

4. Superharmonic Functions in Lipschitz domains

In this section, motivated by Lemma 3.6 we begin with studying superharmonic functions in Lipschitz domains, starting with an important localization result (Proposition 4.1). Building on this we achieve an important growth estimate for $(\varepsilon, \varepsilon^{\alpha})$ superharmonic functions, up to a small distance away from the free boundary (Lemma 4.5). The challenge lies in the potential oscillation of the source term f, which could affect the distribution of ∇w in small scale.

Throughout the section we denote $g : \mathbb{R}^{d-1} \to \mathbb{R}$ to be a Lipschitz continuous function with Lipschitz constant $c_g > 0$ such that $g(0) = 0$. For any $L \geq 2$, define a strip with width 1 below the graph of g in B_L as

$$\Sigma'_L := B_L \cap \left\{x = (x', x_d) : g(x') - 1 < x_d < g(x')\right\},$$

and denote the bottom part of the boundary as

$$\partial_b \Sigma'_L := B_L \cap \{x = (x', x_d) : x_d = g(x') - 1\}.$$

We consider two non-negative functions $w_{1,L}$ and $w_{2,L}$ such that

$$\begin{cases} -\Delta w_{1,L} = 0, \quad -\Delta w_{2,L} = 1 \text{ in } \Sigma'_L; \\ \\ w_{1,L} = 1, \quad w_{2,L} = 0 \text{ on } \partial_b\Sigma'_L; \quad w_{1,L} = w_{2,L} = 0 \text{ on the rest of } \partial\Sigma'_L. \end{cases}$$

Below we will show that the two functions are comparable, uniformly with respect to the width parameter L. Such result allows us to study our solutions using the well-known properties of harmonic functions in Lipschitz domains. While such result appears to be of classical nature, we were unable to find a relevant version in the literature. It does not appear to be directly verifiable from the Green's function presentation for each functions.

Proposition 4.1. *For $w_{1,L}, w_{2,L}$ and g given as above, let $L \geq 2$ and $c_g < \cot\theta_2$, where θ_2 is from (2.3). Then*

$$w_{2,L} \leq C w_{1,L} \quad \textit{in } \Sigma'_{L-1} \textit{ for some } C = C(d, c_g).$$

Let us remark that if $c_g > \cot\theta_2$ the proposition is false. This is because near the vertex (at which $w_{1,L} = 0$) of a cone with small opening, $w_{1,L}$ grows much slower than quadratic, while $w_{2,L}$ has a quadratic growth.

Proof. 1. First we claim that $w_{2,L}$ is bounded in Σ'_L, with the bound depending only on d and c_g. If this is not true, then we have a sequence of Lipschitz functions g_n and the corresponding w^n_{2,L_n} such that $w^n_{2,L_n}(x_n) = \max_{x\in\Sigma'_L} w^n_{2,L_n}(x) \to \infty$ as $n \to \infty$. Due to the classical regularity results for harmonic functions in Lipschitz domains (see e.g., [JK82]), $w^n_{2,L_n}(\cdot + x_n)/w^n_{2,L_n}(x_n)$ are uniformly continuous with bounds that only depends on d and c_g. By taking a locally uniform convergent subsequence of $w^n_{2,L_n}(\cdot + x_n)/w^n_{2,L_n}(x_n)$, we can easily obtain a contradiction because the limiting function is harmonic in some Lipschitz domain whose dirichlet boundary is a unit distance away from the origin, and it assumes its maximum value 1 at the origin, which is not possible. So we can conclude.

2. We now simplify what we need to prove. First, by Dahlberg's lemma (Lemma 2.9), there is no loss of generality to assume that $w_{1,L} = 1$ on $\partial_b\Sigma'_L \cup (\partial B_L \cap \overline{\Sigma'_L})$. Next we claim that if we can prove the conclusion for $L = 2$, then the general case follows. Indeed since $w_{2,L}$ for all $L > 2$ are uniformly bounded (denote the bound as C_*), $w_{2,L} \le C_* w_{1,2}$ on the boundary of $\Sigma'_L \cap B_2$. This implies that $w_{2,L} \le C_* w_{1,2} + w_{2,2}$ on $\Sigma'_L \cap B_2$. Then by the assumption that the conclusion of the lemma holds for $L = 2$ and by Dahlberg's lemma, we obtain

$$w_{2,L} \le C_* w_{1,2} + w_{2,2} \le C' w_{1,2} \le C'' w_{1,L} \quad \text{ on } \Sigma'_L \cap B_1.$$

The same holds on Σ'_{L-1} by shifting the functions.

3. Now we set $L = 2$ and change the variable

$$y' := x', \quad y_d := x_d - g(x') \quad (\text{write } y := (y', y_d)).$$

Under the transformation, the Lipschitz boundary $x_d = g(x')$ becomes a flat hyperplane $y_d = 0$. The operator $-\Delta$ changes to

$$\mathcal{L} := \mathcal{L}_g = -\nabla\cdot((Dy)^T Dy\nabla) \tag{4.1}$$

where Dy denotes the Jacobian matrix of the transformation. The operator remains uniformly elliptic since $(Dy)^T Dy$ is bounded, measurable and uniformly positive definite.

Working with the new coordinates, let us consider the following two non-negative functions

(4.2)
$$\begin{cases} -\mathcal{L}w'_1 = 0, & -\mathcal{L}w'_2 = 1 \qquad \text{on } \{|x'| < 2,\, x_d \in (-1,0)\} =: \mathcal{T}', \\ \quad w'_1 = 1, & \quad w'_2 = 0 \qquad \text{on } \{|x'| \le 2,\, x_d = -1\}, \\ \quad w'_1 = 0, & \quad w'_2 = 0, \qquad \text{on } \{(|x'| = 2,\, x_d \in (-1,0)) \text{ or } (|x'| \le 2,\, x_d = 0)\}, \end{cases}$$

It suffices to show that

$$w_2' \le C w_1' \text{ on } \mathcal{T}'. \tag{4.3}$$

3. We would like to further reduce the problem to periodic domains. Let us denote $\mathbb{T}^{d-1}$ as the $(d-1)$ dimensional torus, and consider

$$\begin{cases} -\mathcal{L}w_1'' = 0, & -\mathcal{L}w_2'' = 1 & \text{on } \mathcal{T} = \{x' \in \mathbb{T}^{d-1},\, x_d \in (-1,0)\}, \\ w_1'' = 1, & w_2'' = 0 & \text{on } \{x' \in \mathbb{T}^{d-1},\, x_d = -1\}, \\ w_1'' = 0, & w_2'' = 0, & \text{on } \{x' \in \mathbb{T}^{d-1},\, x_d = 0\}. \end{cases} \tag{4.4}$$

We claim that to show (4.3) it suffices to show $w_2'' \le C w_1''$ on $\mathcal{T}$. To prove the claim, we can construct a Lipschitz function $\tilde{g} : 4\mathbb{T}^{d-1} \to \mathbb{R}$ with Lipschitz constant c_g such that $\tilde{g} \equiv g$ on $\{x' \in \mathbb{R}^{d-1} : |x'| < 2\}$. Then the corresponding operator $\mathcal{L}_{\tilde{g}}$ agrees with $\mathcal{L}$ on the same region. Let us still call solutions from (4.4) with $\mathcal{L}_{\tilde{g}}$ and $4\mathbb{T}^{d-1}$ in place of $\mathcal{L}$ and $\mathbb{T}^{d-1}$ as w_1'' and w_2''. Then Lemma 2.9 in pre-transformation coordinates and uniform continuity of w_1', w_1'' yield $w_1'' \le C w_1'$ on $\mathcal{T}'$, and the comparison principle yields $w_2' \le w_2''$. Hence $w_2'' \le C w_1''$ implies (4.3), which shows the claim after rescaling.

4. Now we proceed to show $w_2'' \le C w_1''$ in the periodic domain $\mathcal{T}$ for w_1'', w_2'' from (4.4). We will proceed with induction, to approach the boundary of $x_d = 0$. Let us denote

$$\mathcal{T}_k := \{x \in \mathcal{T} \,:\, x_d \in (-2^{-k}, 0)\}.$$

Since $w_1'' > 0$ is uniformly bounded away from 0 when $x_d \in (-1, -\frac{1}{2}]$ and w_2'' is uniformly bounded, there exists $C_1 > 0$ such that $w_2'' \le C_1 w_1''$ in $\mathcal{T} \backslash \mathcal{T}_1$. Suppose $w_2'' \le C_k w_1''$ in $\mathcal{T} \backslash \mathcal{T}_k$ for some $k \ge 1$ and $C_k > 0$. Let ϕ_k be the unique solution to

$$\begin{cases} -\mathcal{L}\phi_k = 1 & \text{in } \mathcal{T}_k, \\ \phi_k = 0 & \text{on } \partial\mathcal{T}_k. \end{cases}$$

Then by considering $4^k \phi_k(2^{-k}x)$ in $2^k \mathcal{T}_k$, the bound in Step 1. in pre-transformation coordinates yields that $\phi_k \le C_* 4^{-k}$ for some $C_* > 0$ independent of k. Since $w_2'' \le C_k w_1''$ on $\partial\mathcal{T}_k$, we obtain

$$w_2'' \le C_k w_1'' + \phi_k \le C_k w_1'' + C_* 4^{-k} \quad \text{in } \mathcal{T}.$$

Since $c_g \le \cot\theta_\beta < \cot\theta_2$ for some $\beta < 2$, Lemma 3.3 yields that $w_1'' \ge C|x_d|^\beta$. Thus, using that $w_1'' \ge C 2^{-(k+1)\beta}$ in $\mathcal{T} \backslash \mathcal{T}_{k+1}$, there exists $C' = C'(C_*) > 0$ such that

$$C' w_1'' \ge C' 2^{-(k+1)\beta} \ge 2^{(2-\beta)k}(4^{-k} C_*) \quad \text{in } \mathcal{T} \backslash \mathcal{T}_{k+1}.$$

We then obtain

$$w_2'' \le C_{k+1} w_1'' \quad \text{in } \mathcal{T} \backslash \mathcal{T}_{k+1} = \{x \in \mathcal{T} \,:\, x_d \in (-1, -2^{-k-1}]\}.$$

with $C_{k+1} := C_k + C'2^{(\beta-2)k}$. Because C' is independent of k and $\beta < 2$, we have

$$\lim_{k\geq 0} C_k < \infty,$$

and therefore $w_2'' \leq C w_1''$ in $\mathcal{T}$ for some $C > 0$ which finishes the proof. □

Later, instead of applying Proposition 4.1 directly, we are going to use the following corollary. In it, we use Σ_δ which recales the previous Σ_L' to unit length but with δ width (so $\delta \sim 1/L$). For any $\delta \in (0, \frac{1}{2})$, consider the domain

$$\Sigma_\delta := B_1 \cap \{x_d \in (g(x') - \delta, g(x'))\}, \quad \partial_b \Sigma_\delta := B_1 \cap \{x_d = g(x') - \delta\}. \tag{4.5}$$

Corollary 4.2. *Let g be as in Proposition 4.1 with $c_g \leq \theta_\beta$ for some $\beta \in (1,2)$. Let $f : \mathbb{R}^d \to [0,\infty)$ be continuous, and let ω be a non-negative function solving*

$$-\Delta\omega = f \text{ in } B_2 \cap \{x_d < g(x')\}, \quad \omega = 0 \text{ on } B_2 \cap \{x_d = g(x')\}, \omega(-e_d) > 0.$$

Consider w_1 and w_2 each solving

$$\begin{cases} -\Delta w_1 = 0, & -\Delta w_2 = f & \text{in } \Sigma_\delta, \\ w_1 = \omega, & w_2 = 0 & \text{on } \partial_b\Sigma_\delta, \\ w_1 = w_2 = 0 & & \text{on the rest of } \partial\Sigma_\delta. \end{cases}$$

Then there exists $C = C(d,\beta)$ such that

$$\delta^{\beta-2} w_2 \leq C\frac{\|f\|_\infty}{\omega(-e_d)} w_1 \quad \text{in } B_{1-\delta} \cap \Sigma_\delta.$$

Moreover, in the same domain we have

$$w_1 \leq \omega \leq C(1 + \delta^{2-\beta})\frac{\|f\|_\infty}{\omega(-e_d)} w_1.$$

Proof. First, for $m := \omega(-e_d)$, Lemma 2.10 yields that $\omega \geq cm\delta^\beta$ on $\partial_b\Sigma_\delta$ for some $c > 0$. So that $\bar{w}_1(x) := m^{-1}\delta^{-\beta} w_1(\delta x)$ is harmonic in Σ_δ/δ and $\bar{w}_1 \geq c$ on $(\partial_b\Sigma_\delta)/\delta$. Note that $\bar{w}_2(x) := \delta^{-2} w_2(\delta x)$ satisfies

$$-\Delta \bar{w}_2 \leq \|f\|_\infty \text{ in } \Sigma_\delta/\delta \quad \text{and} \quad \bar{w}_2 = 0 \text{ on } (\partial_b\Sigma_\delta)/\delta.$$

Thus, applying the comparison principle and Proposition 4.1 with $\|f\|_\infty^{-1}\bar{w}_2$ (when $\|f\|_\infty > 0$) and $\bar{w}_1$ in place of w_2 and w_1 yield for some $C > 0$,

$$m\delta^{\beta-2} w_2 \leq C\|f\|_\infty w_1 \quad \text{in } \Sigma_\delta \cap B_{1-\delta}.$$

For the second claim, note that $v := \omega - w_2 \geq 0$ is a harmonic function in Σ_δ, $w_1 = \omega = v$ on $\partial_b\Sigma_\delta$ and $w_1 \leq v$ on $\partial\Sigma_\delta$. Hence the comparison principle yields $w_1 \leq v$ in Σ_δ. And by Dahlberg's lemma (Lemma 2.9), we have

$$(w_1 \leq)\, v \leq C w_1 \quad \text{in } \Sigma_{\delta/2} \cap B_{1-\delta/2} \tag{4.6}$$

for some $C = C(d) > 1$. Since w_1 and v are harmonic, $v - w_1 = 0$ on $B_1 \cap \partial_b\Sigma_\delta$, and $0 \leq v - w_1 \leq (C-1)w_1$ on $B_{1-\delta/2} \cap \partial_b\Sigma_{\delta/2}$ by (4.6), we apply Lemma 2.9 again to have $v - w_1 \le C' w_1$ in $B_{1-\delta} \cap (\Sigma_\delta \setminus \Sigma_{\delta/2})$ for some $C' = C'(d)$. Hence we have (4.6) holds (with possibly a different $C = C(d)$) in $B_{1-\delta} \cap \Sigma_\delta$, which finishes the proof by the established first claim. □

Corollary 4.3. *Let u solve (1.1) with $\vec{b} = 0$ and $f \in L^\infty(\mathbb{R}^d)$ in $B_1(0) \times [0,1]$, with a locally Lipschitz domain Ω_0. In particular we have $\{u(x,0) > 0\} = \{x_n < g(x')\}$ in $B_1(0)$, where g is as given in Proposition 4.1. If the Lipschitz constant of Γ_0 is small, Then for any $\varepsilon > 0$ there exists $h > 0$ depending on ε such that $u(\cdot, t)$ is $h\varepsilon$-monotone for the cone W_{θ, e_1} in $B_h(0)$ for $t \in [0, t_h := \frac{h}{u(-he_n, 0)}]$.*

this is only a sketch of the proof, we will see if it makes sense first.

Proof. If we use the same initial data and solve the homogeneous problem (HS) , Theorem 5.7 in [CJK07] in particular states that the corresponding solution v has spatially Lipchitz free boundary which is monotone in the cone $W(\theta, \nu)$ for $t \in [0, t_0]$, where t_0 only depends on Ω_0. Now we construct the barriers for our problem as follows: we solve $-\Delta w(\cdot, t) = f$ on Γ_t, and solve for the harmonic function w_1 in the δ-strip of Γ_t, with inner boundary data the same as w_1. Then by Corollary 4.2, $w_1 \le w \le (1 + C\delta)w$, and so $|Dw| \sim |Dw_1|$ up to $O(\delta)$ error. In particular we know that w is a subsolution and $w(\cdot, (1 + C\delta)t)$ is a supersolution for our original solution u. So we can use the information on w_1 from [CJK07] to bound the free boundary of u. though we have to be careful here since I am assuming here that I have the fixed boundary data from w satisfies $w \le (1 + C\delta)w(\cdot, (1 + C\delta)t)$. But I suppose that this is true as long as f has bounded time derivative. We also need to choose δ so that the resulting gap between the barriers, which is δt_h, is about $h\varepsilon$. The $(h\varepsilon, 0)$-monotonicity I believe follows from comparing u with w and from using Lemma 4.6. □

The next two lemmas connect ω and $\nabla\omega$ in terms of its distance from the free boundary. These were proved in [CS05, Lemma 11.11] for the case $f = 0$. A crucial element in the proof is Harnack inequality for the directional derivative of harmonic functions. In our case Proposition 4.1 is applied to avoid differentiating the source function.

Lemma 4.4. *Let ω and g be as in Corollary 4.2, and in addition suppose that $\omega_{x_d} \le 0$. Then there exists $C > 0$ such that for all sufficiently small δ we have*

$$Cd(x, \{y_d = g(y')\})\omega_{-x_d}(x) \ge \omega(x)$$

holds for all $x \in \Sigma_\delta \cap B_{1-\delta}$, where Σ_δ is given in (4.5).

Proof. Let us fix a point $(y', g(y')) \in B_{1-\delta}$. For simplicity, we may assume that $y' = 0$ and $g(y') = 0$. For $r > 0$ define $z_r := g(y') - re_d$. Then for w_1 given in Corollary 4.2 the following is true due to the boundary Harnack principle [CS05, Theorem 11.5] and the remarks (a)(b) in its proof (see also the proof of [CS05, Lemma 11.11]): There exist $C, \sigma > 0$ such that we have

$$w_1(\tau z_r) \le C\tau^\sigma w_1(z_r) \text{ for any } r \in (0, \delta/2), \tau \in (0,1). \tag{4.7}$$

Next if $\delta^{2-\beta}\|f\|_\infty \le \omega(-e_d)$, then $w_1 \le \omega \le Cw_1$ for some $C > 1$ in $\Sigma_\delta \cap B_{1-\delta}$ by Corollary 4.2. Thus by taking $\tau > 0$ to be small enough (independent of r and y'), we obtain from (4.7) that $\omega(\tau z_r) \le \frac{1}{2}\omega(z_r)$. This implies

$$\frac{1}{2}\omega(z_r) \le \int_{\tau r}^{r} \omega_{-x_d}(-se_d)ds \ (\le \omega(z_r)). \tag{4.8}$$

Now since $\omega_{x_d} \le 0$, by applying the last claim of Lemma 2.12 for possibly multiple times, we get for all $s \in (\tau r, r)$ that

$$\omega_{-x_d}(-se_d) \le C\omega_{-x_d}(z_r) + Cr\|f\|_{L^\infty(B_1)}$$

with $C > 0$ depending on τ and an upper bound of c_g. So (4.8) yields

$$Cr\omega_{-x_d}(z_r) \ge \omega(z_r) - Cr^2\|f\|_\infty. \tag{4.9}$$

For $\theta := \operatorname{arccot} c_g$ ($\ge \theta_\beta$ with $\beta \in (1,2)$), we get

$$r\sin\theta \le d(z_r, \{y_d = g(y')\}) \le r.$$

Also using $\omega(z_r) \ge Cr^\beta\omega(-e_d)$ by Lemma 2.10 and the above, we get from (4.9) that for some positive constants C, C' depending on $\theta, \omega(-e_d)$ and $\|f\|_\infty$,

$$\begin{aligned} Cd(z_r, \{y_d = g(y')\})\omega_{-x_d}(z_r) &\ge \omega(z_r) - Cr^2\|f\|_\infty \\ &\ge \frac{1}{2}\omega(z_r) + Cr^\beta\omega(-e_d) - C'r^2\omega(-e_d) \ge \frac{1}{2}\omega(z_r) \end{aligned}$$

if $r < \delta$ is small enough. □

We now relax the previous assumption and consider $(\varepsilon, \varepsilon^\alpha)$-monotone funcitons. Note that the cone of monotonicity needs to be wider as the regularity of f decreases.

Lemma 4.5. *Let $f \in C^{\bar\gamma}(\mathbb{R}^d)$ for some $\bar\gamma \in (0,1]$ be non-negative, and let $\omega \ge 0$ solve $-\Delta\omega = f$ in $B_2 \cap \Omega_\omega$ with $0 \in \Gamma_\omega$ and $\omega(-e_d) > 0$. Suppose that $\alpha \in (0, \frac{\bar\gamma}{2})$ and $\kappa_2 \in (\frac{\alpha}{\bar\gamma}, \frac{1}{2})$ if f is not a constant. Otherwise take $\alpha = \infty$ and any $\kappa_2 \in (0, \frac{1}{2})$ if f is a constant.*

In addition suppose that ω is $(\varepsilon, \varepsilon^\alpha)$-monotone with respect to $W_{-e_d,\theta}$ in B_1 with $\theta > \theta_{1+\bar\gamma}$. Then there exists $C = C(\theta, \omega(-e_d), \|f\|_\infty) > 0$ such that: for all ε sufficiently small,

(4.10)

$$C|\nabla\omega(x)|d(x, \Gamma_\omega) \ge \omega(x) \quad \text{in } B_{1-\varepsilon^{1/2}} \cap \{x : C\varepsilon^{1-\kappa_2} \le d(x,\Gamma_\omega) \le \varepsilon^{1/2}\} \cap \Omega_\omega.$$

Proof. 1. Let $x_0 \in B_{1-\varepsilon^{1/2}} \cap \Omega_\omega$ satisfying $\delta_0 := d(x_0, \Gamma_\omega) \in [2\varepsilon^{1-\kappa_2}, \varepsilon^{1/2}]$. Below we write $a_0 := \omega(x_0)$. Denoting $x = (x', x_d)$, we consider the domain

$$D_0 := \left\{x : |x' - (x_0)'| < 8^{-1}\delta_0, \, |x_d - (x_0)_d| < 2\delta_0\right\}.$$

Let us also define

$$N_\varepsilon := \{x \in B_{1-\varepsilon^{1/2}} \cap \Omega_\omega : d(x, \Gamma_\omega) > \varepsilon^{1-\kappa_2}\}.$$

By Lemma 3.4 with some $\beta \in (1, 1+\bar{\gamma})$ and the assumption, for some $c > 0$ we have

$$\omega(x) \geq c\,\omega(-e_d)\varepsilon^{(1-\kappa_2)\beta} \quad \text{in } N_\varepsilon. \tag{4.11}$$

Note that when f is not a constant, we have $\varepsilon^{(1-\kappa_2)\beta+\alpha} >> \varepsilon^{1+\bar{\gamma}-\kappa_2}$ and $\varepsilon^{1/2} \geq \varepsilon^{1-\kappa_2}$ due to $\beta \in (1, 1+\bar{\gamma})$ and $\kappa_2 \in (\frac{\alpha}{\bar{\gamma}}, \frac{1}{2})$. So it follows from the second remark of Remark 3.2 that

(4.12)

$\omega(\cdot)$ is fully monotone non-decreasing along all directions in $W_{-e_d,\theta}$ in N_ε.

By our assumption of $(\varepsilon, \varepsilon^\alpha)$-monotonicity and the fact that $\theta \geq \frac{\pi}{4}$, it follows that the set $\{\omega(\cdot) = a_0\} \cap D_0$ is at least $\varepsilon^{1-\kappa_2}$-away from $\Gamma_\omega \cap D_0$, and therefore $\{\omega(\cdot) - a_0 = s\}$ for any $s > 0$ are Lipschitz hypersurfaces in D_0.

2. Now let $w_1(\cdot)$ and $w_1'(\cdot)$ be, respectively, the harmonic functions in $D_0 \cap \Omega_\omega$ with $w_1 = \omega$ on $\partial(D_0 \cap \Omega_\omega)$ and in D_0 with $w_1' = \omega$ on ∂D_0. From (4.11) and classical regularity results of elliptic operators, we get for some $c = c(d) > 0$,

$$w_1'(x_0) \geq w_1(x_0) \geq c\,\omega(-e_d)\delta_0^\beta.$$

Since $w_2 := \omega - w_1$ satisfies $-\Delta w_2 = f$ and $w_2 = 0$ on $\partial(D_0 \cap \Omega_\omega)$, we get $w_2 \leq C\delta_0^2 \|f\|_\infty$ for some $C > 0$ in $D_0 \cap \Omega_\omega$. Therefore, using the fact that $w_2(x_0) \leq C'\delta_0^{2-\beta} w_1(x_0)$ with $C' := C\|f\|_\infty/(c\omega(-e_d))$, we have

$$a_0 \leq (1 + C'\delta_0^{2-\beta})w_1'(x_0). \tag{4.13}$$

Next, similarly as done in the proof of Lemma 5.6 [CS05], let h^x (with $x \in D_0$) be the harmonic measure in D_0. By the $(\varepsilon, \varepsilon^\alpha)$-monotonicity assumption and $0 \in \Gamma_\omega$, we have

$$|\partial D_0 \cap \{w_1' = 0\}| = |\partial D_0 \cap \{\omega = 0\}| \geq c|\partial D_0| \quad \text{for some } c = c(d,\theta) \in (0,1).$$

Hence Lemma 11.9 [CS05] implies that

$$w_1'(x_0) = \int_{\partial D_0} \omega(\sigma) dh^{x_0}(\sigma) \leq (1 - c') \max_{\partial D_0} \omega$$

for some $c' = c'(d,\theta) \in (0,1)$. Thus by taking δ_0 (and so ε) to be small enough and applying (4.13), we obtain

$$a_0 \leq (1 - c'/2) \max_{\partial D_0} w_1.$$

Therefore there exists $x_1 \in \partial D_0$ such that for $C_0 := \frac{1}{1-c'/2} > 1$,

$$w_1(x_1) = \omega(x_1) \geq C_0 a_0 > a_0. \tag{4.14}$$

3. Let us consider the domain

$$D_1 := \left\{x \,:\, |x' - (x_0)'| < 8^{-1}\delta_0,\ -3\delta_0 < x_d - (x_0)_d,\ \omega(x) > a_0\right\},$$

From the full monotonicity (4.12) that the level sets $\{\omega - a_0 = s\} \cap D_1$ for $s > 0$ are Lipschitz graphs. Since $x_1 \in D_1$, the set $\{\omega > C_0 a_0\} \cap D_1$ is at most $C\delta_0$-away from $\Gamma_\omega \cap D_0$. Since $-\Delta(\omega - a_0)^+ = f$ in D_1 and $\omega_{e_d} \le 0$, we can apply Lemma 4.4 to $(\omega - a_0)^+$ to obtain

$$(\omega(x) - a_0)^+ \le C|\nabla\omega(x)|\, d(x_1, \{\omega > a_0\}) \le C'|\nabla\omega(x)|\, d(x_1, \Gamma_\omega)$$

for all $x \in D_1$ when ε is sufficiently small. While we also know from (4.14) that

$$\omega - a_0 \ge (1 - C_0^{-1})\omega \quad \text{ in } \{\omega > C_0 a_0\} \cap D_1.$$

Thus the inequality (4.10) holds for $x \in \{\omega > C_0 a_0\} \cap D_1$. Since x_0 is an arbitrary point that is $\varepsilon^{1-\kappa_2}$-away from Γ_ω, by shifting x_0, $\{\omega > C_0 a_0\} \cap D_1$ contains all points $x \in B_{1-\varepsilon^{1/2}}$ such that $d(x, \Gamma_\omega) \in [C\varepsilon^{1-\kappa_2}, \varepsilon^{\frac{1}{2}}]$. We finished the proof. □

4.1. **Lipschitz free boundary implies cone monotonicity.** Here we show that if the boundary is Lipschitz continuous, then the solutions to $-\Delta\omega = f$ with 0 boundary data are cone-monotone when sufficiently close to the boundary.

Let g be a Lipschitz function as given in the beginning of Section 4.

Lemma 4.6. *Let $D_r := B_r \cap \{x_d < g(x')\}$ for $r > 0$ and let $\omega \ge 0$ be a solution to $-\Delta\omega = f$ in D_1 such that $\omega = 0$ on $g(x') = 0$ and $\omega(-\frac{1}{2}e_d) = 1$. Then if $c_g \le \min\{\cot\theta_\beta, \cot\theta'_\beta\}$ for some $\beta \in (1, 2)$ (where $\theta_\beta, \theta'_\beta$ are from Lemma 2.10), then there are $c, r > 0$ such that $\omega_{-x_d} \ge c\,\omega$ in D_r.*

Proof. For some $\delta \in (0, 1)$ to be determined, let

$$\omega_\delta(x) := a\,\omega(\delta x) \quad \text{ with } a := 1/\omega(-\delta e_d/2).$$

Then ω_δ satisfies $\omega_\delta(-e_d/2) = 1$ and $-\Delta\omega_\delta = f_\delta$ with $f_\delta(x) := a\delta^2 f(\delta x)$. By the assumption on c_g, Lemma 2.10 and Corollary 4.2 yield that

$$C^{-1}\delta^\beta \le a \le C\delta^{2-\beta}, \quad \omega_\delta(-\delta e_d) \le C\delta^{2-\beta}. \tag{4.15}$$

Now let h_1, h_2 be two harmonic functions in $D_1^\delta := B_1 \cap \{\delta x_d < g(\delta x')\}$ such that

$$\begin{aligned} &h_1 = \omega_\delta,\ h_2 = 1 && \text{on } \partial B_1 \cap \overline{D_1^\delta} \\ &h_1 = h_2 = 0 && \text{on } B_1 \cap \{\delta x_d = g(\delta x')\}. \end{aligned}$$

For $y := -\delta e_d$, Corollary 4.2 and (4.15) yield

$$\omega_\delta(y) - h_1(y) \le C\|f_\delta\|_\infty h_1(y) \le C\|f_\delta\|_\infty \omega_\delta(y) \le C\delta^{6-2\beta}.$$

Next, it follows from the last two lines of the proof of Lemma 11.12 [CS05] that if δ is sufficiently small depending only on c_g and d,

$$\nabla_{-x_d} h_1(y) \ge c\,\nabla_{-x_d} h_2(y) \ge c\,h_2(y)/\delta.$$

where c is a dimensional constant. By Lemma 2.10 again,

$$\nabla_{-x_d} h_1(y) \ge c\,\delta^{\beta-1}. \tag{4.16}$$

In view of Lemma 2.12 and $|f_\delta| \le C\delta^{4-\beta}$,

$$|\nabla_{-x_d}(\omega_\delta - h_1)(y)| \le C\delta^{-1}(\omega_\delta(y) - h_1(y)) + C\delta\|f_\delta\|_\infty \le C\delta^{5-2\beta}.$$

Thus (4.16) and $\beta < 2$ yield for all δ sufficiently small that $\nabla_{-x_d}\omega_\delta(y) \geq 0$. This implies that $\nabla_{-x_d}\omega(-\delta^2 e_d) \geq 0$ for all δ sufficiently small. Finally the proof is finished after applying Lemma 4.4. □

5. Sup-convolution

In this section we prove several properties of sup-convolutions, first introduced by Caffarelli (see e.g. [Caf87]). They will be used in the constructions of barriers in the next section.

For non-negative functions u in $C(B_1 \times (0,T))$ and $\varphi \in C^{2,1}_{x,t}(B_1 \times (0,T))$ with $0 < \varphi \leq 1/2$, define

$$v(x,t) := \sup_{B_{\varphi(x,t)}(x)} u(y,t) \quad \text{in } B_{1/2} \times (0,T). \tag{5.1}$$

The following lemma says that if u is $(\varepsilon, 0)$-monotone, then the level surfaces of v are Lipschitz graphs whenever ε/φ and $\nabla\varphi$ are not too big.

Lemma 5.1. (Lemma 5.4 [CS05]) *Let v be as given in (5.1). Suppose that $u(\cdot,t)$ is $(\varepsilon,0)$-monotone with respect to $W_{\theta,\mu}$ for some $\theta \in (0, \frac{\pi}{2}]$ in B_1, and for some $x \in B_{1/2}$ and $\bar\theta \in (0, \frac{\pi}{2})$ we have*

$$\sin\bar\theta \leq \frac{1}{1 + |\nabla_x\varphi(x,t)|}\left(\sin\theta - \frac{\varepsilon\cos^2\theta}{2\varphi(x,t)} - |\nabla_x\varphi(x,t)|\right). \tag{5.2}$$

Then $v(\cdot,t)$ is non-decreasing along all directions in $W_{\bar\theta,\mu}$ at x.

The following lemma estimates Δv. The proof is similar to those in [Caf87, KZ21, CJK07].

Lemma 5.2. *Suppose $-\Delta u = f \geq 0$ in Ω_u with continuous $f : \mathbb{R}^d \to \mathbb{R}$. Let v be given by (5.1), then $v(x,t) = u(y(x,t),t)$ for some $y(x,t) \in B_{\varphi(x,t)}(x)$. Then there are dimensional constants $A_0, A_1 > 1$ such that if φ satisfies*

$$\Delta\varphi \geq \frac{A_0|\nabla\varphi|^2}{|\varphi|} \quad \text{in } B_1 \times (0,T), \tag{5.3}$$

then v satisfies (in the viscosity sense)

$$-\Delta v \leq (1 + A_1\|\nabla\varphi\|_\infty) f \circ y \quad \text{in } \Omega_v \cap [B_{1/2} \times (0,T)].$$

Proof. Since t stays fixed in the proof, we will omit its dependence from the notations of u, v, φ and y. We follow the idea of Lemma 9 in [Caf87] and compute

$$\Delta v(0) = \underline{\lim}_{r\to 0}\left(\fint_{B_r} v(x) - v(0)dx\right), \quad \text{where } \fint_{B_r} v(x)dx := \frac{1}{|B_r|}\int_{B_r} v(x)dx.$$

Let $x_0 \in B_{1/2} \cap \{v(\cdot,t) > 0\}$, which may be set to be the origin. If $y(0)$ is a local supremum of u, then there is nothing to prove since $\Delta v(0) = 0$. Otherwise $y(0) \in \partial B_{\varphi(0)}(0)$, by choosing an appropriate system of coordinates, we can assume for some $\gamma_1, \gamma_2 \in \mathbb{R}$ that

$$v(0) = u(\varphi(0)e_d) \quad \text{and} \quad \nabla\varphi(0) = \gamma_1 e_1 + \gamma_2 e_d. \tag{5.4}$$

Recall that

$$v(x)=\sup_{|\nu|\le 1} u\left(x+\varphi(x)\nu\right)\ge 0.$$

Let us estimate $v(x)$ from below by taking $\nu(x):=\frac{\nu_*(x)}{|\nu_*(x)|}$ where

$$\nu_*(x):=e_d+\frac{\gamma_2 x_1-\gamma_1 x_n}{\varphi(0)}e_1+\frac{\gamma_3}{\varphi(0)}\Big(\sum_{i=2}^{d-1}x_i\,e_i\Big) \tag{5.5}$$

and $\gamma_3\in\mathbb{R}$ satisfies

$$(1+\gamma_3)^2=(1+\gamma_2)^2+\gamma_1^2. \tag{5.6}$$

With this choice of ν, we define $y(x):=x+\varphi(x)\nu(x)$ and so $y(0)=\varphi(0)e_d$. Then direct computations yield (also see [Caf87])

$$y(x)=Y_*(x)+\varphi(0)e_d+o(|x|^2) \tag{5.7}$$

where $Y_*(x)$ denotes the first-order term that is

$$Y_*(x):=x+(\gamma_1x_1+\gamma_2x_n)e_d+(\gamma_2x_1-\gamma_1x_n)e_1+\gamma_3\sum_{i=2}^{d-1}x_ie_i. \tag{5.8}$$

Hence $Y_*(x)$ is a rigid rotation plus a dilation, and (5.4) and (5.6) imply

$$\left|\frac{D(Y_*(x)-x)}{Dx}\right|\le C\|\nabla\varphi\|_\infty. \tag{5.9}$$

Then we have

$$\begin{aligned}\fint_{B_r} v(x)-v(0)dx &\ge \fint_{B_r} u(y(x))-u(y(0))dx\\ &\ge \fint_{B_r} u(y(x))-u(Y_*(x)+y(0))dx+\fint_{B_r} u(Y_*(x)+y(0))-u(y(0))dx.\end{aligned}$$

Using (5.3) and following the computations done in Lemma 9 [Caf87], we find that the first integration in the above ≥ 0. Since u is C^2 near $y(0)$ by the assumption,

$$\lim_{r\to 0}\frac{1}{r^2}\fint_{B_r} u(Y_*(x)+y(0))-u(y(0))dx=\left(\left|\frac{DY_*(x)}{Dx}\right|_{x=0}\right)^2 f(y(0)).$$

Using (5.9) and $\Delta u(y(0))\le 0$, we get

$$\begin{aligned}\liminf_{r\to 0}\frac{1}{r^2}\fint_{B_r} v(x)-v(0)dx &\ge \lim_{r\to 0}\frac{1}{r^2}\fint_{B_r} u(Y_*(x)+y(0))-u(y(0))dx\\ &\ge -(1+C\|\nabla\varphi\|_\infty)f(y(0)).\end{aligned} \tag{5.10}$$

Finally to show the conclusion, suppose ϕ is a smooth function such that ϕ touches v from above at 0. If $r>0$ is small enough, we have $\phi(x)\ge v(x)$ for $x\in B_r$, and thus

$$\Delta\phi(0)=\lim_{r\to 0}\frac{1}{r^2}\fint_{B_r}\phi(x)-\phi(0)dx\ge\limsup_{r\to 0}\frac{1}{r^2}\fint_{B_r} v(x)-v(0)dx.$$

This and (5.10) shows

$$\Delta\phi(0) \geq -(1 + C\|\nabla\varphi\|_\infty)f(y(0))$$

which finishes the proof. □

Below we show that if u satisfies the free boundary condition in (1.1), then v satisfies some appropriate free boundary condition as well.

Lemma 5.3. *Let u, v be as given in Lemma 5.2, where $\varphi \in C^{2,1}_{x,t}$ satisfies*

$$\varphi \leq \varepsilon_1, \quad |\nabla\varphi| \leq \varepsilon_2, \quad -1/2 \leq \varphi_t \leq \varepsilon_3.$$

In addition, if u is a viscosity subsolution of (1.1) in $B_1 \times (0, T)$, and if $\varepsilon_1, \varepsilon_2, |\varepsilon_3|$ are small enough, then v is a viscosity subsolution of

$$\begin{cases} -\Delta v \leq (1 + A_1\|\nabla\varphi\|_\infty)f \circ y & \text{in } \Omega_v \cap (B_{1/2} \times (0,T)),\\ v_t \leq (1+2\varepsilon_2)^2|\nabla v|^2 + \vec{b}\cdot\nabla v + \Big(\varepsilon_1\|\nabla\vec{b}\|_\infty + 2(\varepsilon_1+\varepsilon_2)\|\vec{b}\|_\infty + \varepsilon_3 + |\varepsilon_3|/2\Big)|\nabla v| & \\ & \text{on } \Gamma_v \cap (B_{1/2} \times (0,T)). \end{cases}$$

Proof. Suppose that for a smooth test function ϕ, $v - \phi$ has a local maximum at $(x_0, t_0) \in \Gamma_v$ in $\overline{\Omega_v} \cap B_{1/2} \times \{0 \leq t \leq t_0\}$. We would like to verify the subsolution property for ϕ.

As done before, suppose $x_0 = 0$ and (5.4) holds, and let $\nu_*(x)$ be from (5.5), and $\nu(x) := \frac{\nu_*(x)}{|\nu_*(x)|}$. Then $v(0, t_0) = u(y_0, t_0) = 0$ with $y_0 := \varphi(0, t_0)e_d$, $\nu(0) = e_d$, and $|\nabla\nu| \leq 1$. We now define

$$h(x,t) := x + \varphi(x,t)\nu(x) \quad (\text{ then } h(0,t_0) = y_0).$$

If φ has sufficiently small C^1 norm, h is invertible and h^{-1} is $C^{2,1}_{x,t}$. In particular $\psi(y,t) := \phi(h^{-1}(y,t),t)$ is $C^{2,1}_{x,t}$ in a neighborhood of (y_0, t_0). Since $v(x,t) \geq u(h(x,t),t)$ and $v(0,t_0) = u(y_0,t_0) = 0$, $u - \psi$ has a local maximum in $\overline{\Omega_u} \cap \{t \leq t_0\}$ at (y_0, t_0).

First, suppose that $-(\Delta\psi + f)(0, t_0) > 0$. Since u is a subsolution, ψ satisfies

$$\psi_t(y_0,t_0) \leq |\nabla\psi(y_0,t_0)|^2 + \vec{b}(y_0)\cdot\nabla\psi(y_0,t_0) \tag{5.11}$$

in the classical sense. By taking $\varepsilon_1, \varepsilon_2$ to be small enough, we get

$$\begin{aligned} |\nabla\psi(y_0,t_0) - \nabla\phi(0,t_0)| &\leq \sup_{x\in B_{1/2}} \|D(h^{-1}(x,t_0)) - I_d\||\nabla\phi(0,t_0)| \\ &\leq 2(\varepsilon_1+\varepsilon_2)|\nabla\phi(0,t_0)| \end{aligned} \tag{5.12}$$

Next we estimate $\phi_t(0, x_0)$. To do this, we first show that $\nabla\psi(y_0, t_0)$ is to the direction of e_d. Let us consider the set

$$D := \{x : |x - y_0| < \varphi(x, t_0)\},$$

and then we have $0 \in \partial D$ by $y_0 \in \Gamma_u(t_0)$. Since $\varphi(\cdot, t_0)$ is C^2 and $\nabla\varphi(0, t_0) = \gamma_1 e_1 + \gamma_2 e_d$ by (5.4), ∂D is C^1 and the inner normal direction at 0 equals to $\gamma_1 e_1 + (1 + \gamma_2)e_d$. Note that $v > 0$ in D by the definition of v and $0 \in \Gamma_v$. Thus we get $\phi(\cdot, t_0) > \phi(0, t_0)$ in $D \cap B_r(0)$ for some $r > 0$, which implies that $\nabla\phi(0,t_0)$ is pointing to the direction of $\gamma_1 e_1 + (1+\gamma_2)e_d$. In view of (5.7) and (5.8), we have

$$Dh(0,t_0) = \begin{bmatrix} 1+\gamma_2 & & & & -\gamma_1 \\ & 1+\gamma_3 & & & \\ & & \cdots & & \\ & & & 1+\gamma_3 & \\ \gamma_1 & & & & 1+\gamma_2 \end{bmatrix}.$$

This and

$$\frac{\gamma_1 e_1 + (1+\gamma_2)e_d}{\gamma_1^2 + (1+\gamma_2)^2}|\nabla\phi(0,t_0)| = \nabla\phi(0,t_0) = \nabla\psi(y_0,t_0)\cdot Dh(0,t_0)$$

yield that $\nabla\psi(y_0,t_0) = |\nabla\psi(y_0,t_0)|e_d$.

Now, since $\varphi_t \le \varepsilon_3$, (5.12) shows that if ε_2 is sufficiently small,

(5.13)

$$\begin{aligned}\phi_t(0,t_0) &= \psi_t(y_0,t_0) + \nabla\psi(y_0,t_0)\cdot h_t(0,t_0) \le \psi_t(y_0,t_0) + [\nabla\psi(y_0,t_0)\cdot\nu(0)]\,\varphi_t(0,t_0) \\ &\le \psi_t(y_0,t_0) + (\varepsilon_3 + 2^{-1}|\varepsilon_3|)|\nabla\phi(0,t_0)|.\end{aligned}$$

Then (5.11), (5.12) and (5.13) yield at $(0,t_0)$,

$$\phi_t \le (1+2\varepsilon_2)^2|\nabla\phi|^2 + \vec{b}(y_0)\cdot\nabla\phi + \left(\varepsilon_3 + 2^{-1}|\varepsilon_3| + 2\varepsilon_2\|\vec{b}\|_\infty\right)|\nabla\phi|.$$

Also using $|\vec{b}(y_0) - \vec{b}(0)| \le \varepsilon_1\|\nabla\vec{b}\|$ in the above inequality and rearranging the terms, we obtain

(5.14)

$$\phi_t \le (1+2\varepsilon_2)^2|\nabla\phi|^2 + \vec{b}(0)\cdot\nabla\phi + \left(\varepsilon_3 + 2^{-1}|\varepsilon_3| + \varepsilon_1\|\nabla\vec{b}\|_\infty + 2\varepsilon_2\|\vec{b}\|_\infty\right)|\nabla\phi|.$$

Finally, if $-(\Delta\psi + f)(0,t_0) \le 0$, it follows from the proof of Lemma 5.2 with ψ,ϕ in place of u,v that (note that $\phi(x,t_0) = \psi(x+\varphi(x,t_0)\nu(x),t_0)$ and we only used $v(x) \ge u(x+\varphi(x)\nu(x))$ in the proof before)

$$-\Delta\phi(0) \le (1+C\|\nabla\varphi\|_\infty)f(y_0)$$

which finishes the proof. □

Remark 5.4. The conclusion of Lemma 5.2 holds the same if $f = f(x,t)$ (then we replace $f(y(x,t))$ by $f(y(x,t),t)$). Similarly Lemma 5.3 holds the same if $\vec{b} = \vec{b}(x,t)$ (in this case we don't need any regularity of $\vec{b}$ in t). The proofs are identical.

Lastly, we describe a family of smooth functions φ_η, which will be used in the next section. The proof is parallel to that of Lemma 10.10 in [CS05] with $\alpha := 4\kappa$. In the referenced lemma, the domain is assumed to be uniformly Lipschitz continuous in time. However if we replace L_1 (the graph's Lipschitz constant in time) by $C/r = C\varepsilon^{-\gamma_1}$, the same arguments apply to yield the desired estimates for our setting.

Lemma 5.5. *Let $A_1 > 1$ be the dimensional constant from Lemma 5.2, and let $\gamma_1, \gamma_2, \kappa \in (0,1)$ such that $\gamma_1 - \gamma_2 > 4\kappa$. Take $r := \varepsilon^{\gamma_1}$ and $T \ge 4\varepsilon^{4\kappa}$ with*

$\varepsilon > 0$ sufficiently small. Suppose that $\Phi(x,t)$ is Lipschitz continuous with constant C in space and with constant C/r in time. Then for

$$\Sigma_{r,T} := \{(x,t) \in B_1 \times (-T,T) \,:\, |\Phi(x',t) - x_d| < 2r\},$$

there is $A_2 = A_2(A_1, C) \geq 1$ such that for any $\eta \in [0,1]$, there exists a C^2 function $\varphi_\eta(x,t)$ in $\Sigma_{r,T}$ such that

(1) $0 < \bar{c} \leq \varphi_\eta \leq 1+\eta$ in $\Sigma_{r,T}$ for some universal constant $\bar{c}$,
(2) $\varphi_\eta \Delta \varphi_\eta \geq A_1 |\nabla \varphi_\eta|^2$ holds in $\Sigma_{r,T}$,
(3) $\varphi_\eta \leq 1$ outside $\{(x,t) \in \Sigma_{r,T} \,:\, t > -T + \varepsilon^{4\kappa},\, d(x, \partial B_1) > \frac{1}{2}\varepsilon^\kappa\}$,
(4) $\varphi_\eta \geq 1 + \eta(1 - A_2 \varepsilon^{\gamma_2})$ in $\{(x,t) \in \Sigma_{r,T} \,:\, t > -T + 2\varepsilon^{4\kappa},\, d(x, \partial B_1) > \varepsilon^\kappa\}$,
(5) $|\nabla \varphi_\eta| \leq A_2 \varepsilon^{\gamma_2 - \gamma_1}$ and $0 \leq \partial_t \varphi_\eta \leq A_2 \varepsilon^{\gamma_2 - \gamma_1}$ in $\Sigma_{r,T}$.

In the next section, we will choose Φ_r to be the Lipschitz function from Lemma 3.6, whose graph approximates the free boundary of u up to order r.

6. Flat free boundaries are Lipschitz

In this section we will show by iteration that flat free boundaries are Lipschitz. Similar to [CS05] and [CJK07], the proof is based on the construction of a family of subsolution, building on Section 5. These are constructed as a small perturbation of u of (1.1) in a local domain $B_2(0) \times (-1,1)$.

The family of subsolutions will represent the regularization mechanism of the flow, by the varying size of regularization given as a radius of the sub-convolution we apply to the solution. Due to the presence of the source and drift term with minimal regularity, and their competition with the regularization mechanism, there are additional terms to the perturbation: this makes the construction of barrier function rather versatile and technical. In an effort to make the construction more accessible for interested readers, we list the family of parameters in the next subsection. Readers may also choose to skip to our iterative statement, Proposition 6.2, and the proof of our main theorem thereafter.

6.1. **Parameters and Assumptions.** Our barrier construction involves many parameters, which we put together here for the reader's convenience. First we choose the minimal angle for the cone of monotonicity, from which we will apply our iteration arguments. In light of Lemma 4.5 we will assume that

$$f \in C^{\bar{\gamma}}(\mathbb{R}^d) \quad \text{and} \quad \beta \in (1, 1+\bar{\gamma}).$$

Let θ_β be as in (2.3), and let $\Theta_0 \in (\theta_\beta, \frac{\pi}{2})$ be such that

$$\sin\theta_\beta < \sin\Theta_0 - \cos^2\Theta_0/\bar{c} \tag{6.1}$$

with $\bar{c}$ from Lemma 5.5. We will work with the cone angle $\theta \in (\Theta_0, \pi/2)$ in this section.

Throughout this section we assume that u satisfies the following for some $T \in (0,1]$:

(H-a) $u(\cdot,t)$ is $(\varepsilon,\varepsilon^\alpha)$-monotone with respect to $W_{\theta,-e_d}$ in B_1 for some $\theta \in (\Theta_0,\pi/2)$ for all $t \in (-T,T)$, and with α satisfying

$$0<\alpha<\min\left\{\frac{\bar\gamma^2}{2},\ \frac{\bar\gamma(1-\bar\gamma)}{8},\ \frac{1-\bar\gamma^2}{16}\right\} \tag{6.2}$$

when f is not a constant.

(H-b) $(0,0) \in \Gamma_u$ and $m := \inf_{t\in(-T,T)} u(-e_d,t) > 0$.

Note that if $T<1$, a simple rescaling argument can reduce the problem to the case of $T=1$. So, for simplicity of notations, let us assume $T=1$ from now on.

Let us proceed with the next set of parameters, to be used in the next subsection for the construction of barrier functions. For $\kappa := \frac{2-\beta}{8}$, we choose γ_1 and ι such that γ_1 and ι are respectively close to 1 and 4κ, and $\beta+\iota<2\gamma_1$. More specifically we choose

(6.3)
$$\gamma_1 := \max\left\{\frac{3}{4}+\frac{\beta}{8}, 1-\frac{\bar\gamma}{2}\right\}<1, \quad \iota := 5\kappa = \frac{5}{4}-\frac{5\beta}{8} \quad \text{and} \quad \gamma_2 := \gamma_1-\iota,$$

and so $\gamma_1-\gamma_2>4\kappa$. With this choice of κ,γ_1 and γ_2, let φ_η be from Lemma 5.5, with some $\eta\in(0,1)$.

We also define $0<\alpha_1<\alpha_2<1$ so that

$$1-\gamma_1<\alpha_1<1-(\beta+\iota)/2, \quad \alpha_2<\min\{1-\iota,\bar\gamma\}. \tag{6.4}$$

Note that this is possible since $\beta+\iota<2\gamma_1$ and $\max\{\iota,1-\bar\gamma\}<\gamma_1$.

Lastly we define universal constants: in this section C or c denotes constants that only depend on $d,\alpha,\beta,m,\bar\gamma$, $\|u\|_{L^\infty(B_2(0)\times(-1,1))}$, $\|f\|_\infty$, $\|f\|_{C^{0,\bar\gamma}(B_2)}$, and $\|\vec{b}\|_{C^1(B_2)}$.

6.2. **Construction of the base barrier $\bar v$.** Let us define $r:=\varepsilon^{\gamma_1}$. We will construct our subsolution in the domain

$$\Sigma_{r,1} = \{(x,t)\in\mathcal{Q}_1 \,:\, |\Phi_r(x',t)-x_d|<2r\},$$

where Φ_r is as given in Lemma 3.6 that approximates Γ_u in r-scale.

For a given $\sigma\in[\frac{\varepsilon}{2},\varepsilon]$, define

$$v(x,t) := \sup_{B_{\sigma\varphi_\eta(x,t)}(x)} u(y,t). \tag{6.5}$$

It then follows from (6.1), Lemma 5.1 and Lemma 5.5 that $v(\cdot,t)$ is non-decreasing along all directions of $W_{\theta_\beta,-e_d}$ when ε is sufficiently small. With this choice of v, we define the domains

$$\Sigma^+ := \Sigma_{r,1}\cap\Omega_v, \quad \Sigma^+(t) := \{x \,:\, (x,t)\in\Sigma^+\}, \quad \Sigma_{r,1}(t) := \{x \,:\, (x,t)\in\Sigma_{r,1}\}.$$

and the bottom boundary of $\Sigma^+(t)$ as

$$\partial_b\Sigma^+(t) := (\partial\Sigma_{r,1}(t)\cap\Omega_v)\setminus\partial B_1.$$

Due to the presence of f, we need to adjust v and the adjustments are superharmonic functions.

For each $t\in(-1,1)$, let us define w_1^t and w_2^t by:

(1) $-\Delta w_1^t = 0$ in $\Sigma^+(t)$ with $w_1^t = v(\cdot,t)$ on $\partial_b\Sigma^+(t)$ and zero elsewhere on the boundary.

(2) $-\Delta w_2^t = 1+\|f\|_\infty$ in $\Sigma^+(t)$ with zero boundary condition.

Consider a non-negative harmonic function ϕ in the annulus $(B_1\setminus B_{1-2\varepsilon^\kappa})\cap\Sigma^+(t)$. Since $\varepsilon^\kappa << r$, if $\phi\geq v$ on $\partial_b\Sigma^+(t)\cap(B_{1-\varepsilon^\kappa/2}\setminus B_{1-2\varepsilon^\kappa})$, then Dahlberg's lemma yields a dimensional constant c_* such that

$$c_* w_1^t \leq \phi \quad \text{on } \Sigma^+(t)\cap\partial B_{1-\varepsilon^\kappa} \tag{6.6}$$

With this choice of c_*, we finally define our barrier function by

$$\bar v(\cdot,t) := (1+\varepsilon^{\alpha+1})v(\cdot,t) - \varepsilon^{\alpha_2}w_2^t + c_*\varepsilon^{\alpha_1}w_1^t, \tag{6.7}$$

where α_1,α_2 are given in (6.4).

Lemma 6.1. *For sufficiently small $\varepsilon>0$, $\bar v$ given by (6.7) satisfies the following in the viscosity sense: For any $e\in B_1$,*

$$\begin{cases} -\Delta\bar v \leq f(x-\varepsilon e) & \text{in } \Sigma^+\cap(B_{1-\varepsilon^\kappa}\times(-1,1)),\\ \quad \bar v_t \leq |\nabla\bar v|^2 + \vec b(x-\varepsilon e)\cdot\nabla\bar v & \text{on } \Gamma_{\bar v}\cap(B_{1-\varepsilon^\kappa}\times(-1,1)). \end{cases}$$

Proof. Since $\sigma|\nabla\varphi_\eta|\leq A_2\varepsilon^{1-\iota}$ by Lemma 5.5, $\kappa<1$ and $\sigma\leq\varepsilon$, the proof of Lemma 5.2 yields for small ε,

$$-\Delta v(x,t) \leq (1+A_1A_2\varepsilon^{1-\iota}) \sup_{B_{\sigma\varphi_\eta(x,t)}(x)} f(y). \tag{6.8}$$

Using $\|f\|_{C^{0,\bar\gamma}}\leq C$ and $\varphi_\eta\leq 2$, the right-hand side of the above

$$\begin{aligned} &\leq (1+A_1A_2\varepsilon^{1-\iota})(f(x-\varepsilon e)+C\varepsilon^{\bar\gamma})\\ &\leq (1+A_1A_2\varepsilon^{1-\iota})f(x-\varepsilon e)+C\varepsilon^{\bar\gamma}. \end{aligned}$$

From (6.7) and the fact that $\alpha_2<\min\{1-\iota,\bar\gamma\}$, we obtain for all $\varepsilon>0$ sufficiently small that

$$\begin{aligned} -\Delta\bar v - f(x-\varepsilon e) &\leq -(1+\varepsilon^{\alpha+1})\Delta v - \varepsilon^{\alpha_2} - f(x-\varepsilon e)\\ &\leq C\varepsilon^{1-\iota}f(x-\varepsilon e) + C\varepsilon^{\bar\gamma} - \varepsilon^{\alpha_2} \leq 0 \qquad \text{in } \Sigma^+(t)\cap B_1. \end{aligned}$$

It remains to show the appropriate free boundary condition. By Lemma 5.1 and the choice of Θ_0, for each $t\in(-1,1)$, $\Gamma_v(t)$ is a Lipschitz graph with Lipschitz constant less than $\cot\theta_\beta$ when ε is small. Then, using $u(-e_d,t)\geq m$, Corollary 4.2 with $\delta:=\varepsilon^{\gamma_1}<\varepsilon^\kappa$ yields for some $C>0$,

$$C\varepsilon^{\gamma_1(2-\beta)}w_1^t \geq w_2^t \quad \text{in } \Sigma^+(t)\cap B_{1-\varepsilon^\kappa}.$$

Thus we have, for ε sufficiently small,

$$c_*\varepsilon^{\alpha_1} w_1^t >> \varepsilon^{\alpha_1} w_2^t >> \varepsilon^{\alpha_2} w_2^t \quad \text{in } \Sigma^+(t) \cap B_{1-\varepsilon^\kappa}. \tag{6.9}$$

Next, $-\Delta v \leq 1 + \|f\|_\infty$ by (6.8) for small ε, the construction of w_1^t and w_2^t, and Dahlberg's Lemma imply for some $C > 1$,

$$Cw_1^t + w_2^t \geq v(\cdot, t) \quad \text{in } \Sigma^+(t) \cap B_{1-\varepsilon^\kappa}. \tag{6.10}$$

So for all ε small enough, $\alpha_2 \geq \alpha_1$, (6.7), (6.9) and (6.10) show (for $c_1 := c_*/(4C) > 0$)

$$\begin{aligned} \bar{v} &\geq (1 + \varepsilon^{\alpha+1})v - (\varepsilon^{\alpha_2} + c_*\varepsilon^{\alpha_1}/(2C))w_2^t + c_*\varepsilon^{\alpha_1} w_1^t/2 + c_*\varepsilon^{\alpha_1} v/(2C) \\ &\geq (1 + 2c_1\varepsilon^{\alpha_1})v - C'\varepsilon^{\alpha_1} w_2^t + c_*\varepsilon^{\alpha_1} w_1^t/2 \\ &\geq (1 + 2c_1\varepsilon^{\alpha_1})v \qquad \text{in } \Sigma^+(t) \cap B_{1-\varepsilon^\kappa}. \end{aligned} \tag{6.11}$$

We then show that $\bar{v}$ has a linear growth near the free boundary. For $x_0 \in \Gamma_{\bar{v}}(t_0) \cap B_{1-\varepsilon^\kappa}$ and $t_0 \leq 1$, since $x_0 \in \Gamma_{\bar{v}}(t_0) = \Gamma_v(t_0)$, there exists $y_0 \in \Gamma_u(t_0) \cap \overline{B_{\sigma\varphi_\eta(x_0,t_0)}(x_0)}$. By the definition of sup-convolution, $B_{\sigma\varphi_\eta(x_0,t_0)}(y_0) \subseteq \Omega_v(t_0)$. This means that $\Gamma_v(t_0)$ satisfies the interior ball property at x_0:

$$B_{\bar{c}\varepsilon/2}(y') \subseteq \Omega_v \quad \text{and} \quad x_0 \in \partial B_{\bar{c}\varepsilon/2}(y') \cap \Gamma_v(t_0)$$

for some y'. Thus $\bar{v} \geq \frac{c_*\varepsilon^{\alpha_1}}{2} w_1^t$ (which easily holds for sufficiently small ε by Corollary 4.2) implies that $\bar{v}$ grows at least linearly at (x_0, t_0). Moreover, we use $u(-e_d, t) > 0$ and Lemma 3.4 to obtain that

$$C \max_{B_{3\varepsilon}(x_0)} w_1^{t_0} \geq \max_{B_{3\varepsilon}(x_0)} v(\cdot, t_0) \geq c\,\varepsilon^\beta$$

for some universal $c > 0$. It then follows from the interior ball property, and Dahlberg's Lemma that

$$|\nabla \bar{v}(x_0, t_0)| \geq |\nabla w_1^{t_0}(x_0)| \geq c\,\varepsilon^{\beta-1} \quad \text{with possibly different universal } c > 0. \tag{6.12}$$

Now we check the viscosity subsolution property for $\bar{v}$ at the free boundary. Suppose that a test function ϕ crosses $\bar{v}$ from above at (x_0, t_0). The linear growth of $\bar{v}$ yields $|\nabla\phi(x_0, t_0)| \neq 0$. Due to (6.11) and the fact that $\bar{v}(x_0, t_0) = v(x_0, t_0) = 0$, $(1 + 2c_1\varepsilon^{\alpha_1})v - \phi$ has a local maximum at (x_0, t_0) as well. Using Lemma 2.6, Lemma 5.3 and Lemma 5.5 yields that ϕ satisfies

$$\phi_t \leq (1+C\varepsilon^{1-\iota})^2(1+2c_1\varepsilon^{\alpha_1})^{-1}|\nabla\phi|^2 + \vec{b}(\cdot - \varepsilon e)\cdot\nabla\phi + C\varepsilon^{1-\iota}|\nabla\phi| \quad \text{at } (x_0, t_0)$$

for some $C = C(A_2, \|\vec{b}\|_{C^1})$. Since $1 - \iota > \alpha_1$, we get for small $\varepsilon > 0$,

$$\phi_t \leq (1 - c_1\varepsilon^{\alpha_1})|\nabla\phi|^2 + \vec{b}(\cdot - \varepsilon e)\cdot\nabla\phi + C\varepsilon^{1-\iota}|\nabla\phi| \quad \text{at } (x_0, t_0). \tag{6.13}$$

Next, since $\varepsilon^{\alpha_2} w_2^t \leq v$ and $\bar{v} - \phi$ obtains a local maximum in $\overline{\Omega_v} \cap \{t \leq t_0\}$ at (x_0, t_0), then $c_*\varepsilon^{\alpha_1} w_1^t - \phi$ has a local maximum in the same domain at (x_0, t_0) as well. Combining this with (6.12) shows

$$|\nabla\phi(x_0, t_0)| \geq cc_*\varepsilon^{\beta-1+\alpha_1}.$$

So $\varepsilon^{\alpha_1}|\nabla\phi(x_0,t_0)| >> \varepsilon^{1-\iota}$ since $\beta+2\alpha_1+\iota<2$ by (6.4). From this and (6.13), we obtain

$$\phi_t \le |\nabla\phi|^2 + \vec{b}(\cdot-\varepsilon e)\cdot\nabla\phi \quad \text{ at } (x_0,t_0),$$

and thus the subsolution property is verified. □

6.3. **Flat free boundary is Lipschitz.** Now we can prove the following main inductive proposition.

Proposition 6.2. *Under the assumptions* (H-a)(H-b) *and for any fixed* $\kappa \in (0,\frac{2-\beta}{4})$*, there exist* $C>0$ *and* $j,\delta,\gamma_3\in(0,1)$ *such that if* $\varepsilon>0$ *is sufficiently small,* u *is* $(j\varepsilon,\varepsilon^\alpha(1-C\varepsilon^\delta))$*-monotone with respect to* $W_{\theta-C\varepsilon^{\gamma_3},-e_d}$ *in* $B_{1-\varepsilon^\kappa}\times(-1+2\varepsilon^{4\kappa},1)$.

Proof. First we choose σ and η in the definition of the sup-convolution in (6.5). Since $\theta>\Theta_0\ge\pi/4$, we can take $j\in(0,1)$ so that

$$\sigma := \varepsilon(\sin\theta-(1-j))\in(\varepsilon/2,\varepsilon).$$

Define

$$\gamma_3 := \min\left\{\frac{\alpha_1+\gamma_1-1}{2},\gamma_2\right\}. \tag{6.14}$$

Observe that $\gamma_3\in(0,1)$ by (6.4) and $\gamma_3+1-\gamma_1\in(0,\alpha_1)$. Choose $\eta=\eta_\varepsilon>0$ such that

$$(1+\eta)\left(\sin\theta-(1-j)\right) = j\sin\theta-\varepsilon^{\gamma_3}. \tag{6.15}$$

By taking $\varepsilon>0$ to be small enough, we have

$$\eta\in\left(\frac{j\sin\theta-(\sin\theta-(1-j))}{2(\sin\theta-(1-j))},\frac{j\sin\theta-(\sin\theta-(1-j))}{\sin\theta-(1-j)}\right).$$

It follows from Lemma 5.5 (4), (6.14) and (6.15) that φ_η in (6.5) then satisfies

(6.16)
$$\sigma\varphi_\eta \ge \sigma(1+\eta(1-A_2\varepsilon^{\gamma_2}))\ge\varepsilon(j\sin\theta-C\varepsilon^{\gamma_3}) \text{ in } \Sigma_{r,1}\cap\{t>-1+2\varepsilon^{4\kappa},\, d(x,\partial B_1)>\varepsilon^\kappa\}$$

for some $C=C(A_2)\ge1$, and by Lemma 5.5 (3),

$$\varphi_\eta\le 1 \quad \text{ in } \Sigma_{r,1}\cap\{t<-1+\varepsilon^{4\kappa},\, d(x,\partial B_1)<\varepsilon^\kappa/2\}. \tag{6.17}$$

With above choice of σ and η, we claim that

$$\bar{v}\le\bar{u}:=u(x-j\varepsilon e_d,t) \quad \text{ in } (B_{1-\varepsilon^\kappa}\times(-1,1))\cap\Sigma^+. \tag{6.18}$$

Before showing this claim, we first discuss its consequence. It follows from (6.9) and (6.16) that for $(x,t)\in(B_{1-\varepsilon^\kappa}\times(-1+2\varepsilon^{4\kappa},1))\cap\Sigma^+$,

$$(1+\varepsilon^{\alpha+1})\sup_{B_{\varepsilon(j\sin\theta-C\varepsilon^{\gamma_3})}(x)}u(y,t)\le\bar{v}(x,t)\le u(x-j\varepsilon e_d,t)$$

which yields the conclusion for those (x,t). Next for $(x,t)\in(B_{1-\varepsilon^\kappa}\times(-1+2\varepsilon^{4\kappa},1))\backslash\Sigma^+$, we have $d(x,\Gamma_u(t))\ge\varepsilon^{\gamma_1}$, and thus Lemma 3.4 and the $(\varepsilon,\varepsilon^\alpha)$-monotonicity yield that $u(x,t)\ge c\,\varepsilon^{\gamma_1\beta}$.

Recall the choice of the parameters: $\beta \in (1, 1+\bar\gamma)$, $\kappa = \frac{2-\beta}{8}$ and (6.3). Therefore by taking $\kappa_1 := \frac{1+\bar\gamma-\gamma_1\beta}{4} \in (0,\frac{1}{2})$ and using (6.2), we have $1-\kappa_1 > \kappa$ and $\gamma_1\beta+\alpha < 1+\bar\gamma-2\kappa_1$. Then for small ε,

$$\varepsilon^{\alpha} u(x,t) \geq c\,\varepsilon^{\gamma_1\beta+\alpha} >> \varepsilon^{1+\bar\gamma-2\kappa_1} \quad \text{in } (B_{1-\varepsilon^\kappa}\times(-1+2\varepsilon^{4\kappa},1))\cap\Sigma^+,$$

and so the third remark of Lemma 3.1 concludes the proof of Proposition 6.2 with $\delta := \kappa_1$.

It remains to prove (6.18). To do this, we claim that it suffices to show that for each $t \in (-1,1)$,

$$\bar v(\cdot,t) \leq \bar u(\cdot,t) \text{ on } (\partial B_{1-\varepsilon^\kappa}\cap\Sigma^+(t)) \cup (\partial_b\Sigma^+(t)\cap B_{1-\varepsilon^\kappa}). \tag{6.19}$$

Indeed, by (6.17) and (H-a), when $t < -1+\varepsilon^{4\kappa}$, we have $\Sigma^+(t) = (\Omega_{\bar v}(t) \cap \Sigma_{r,1}(t)) \subseteq \{\bar u(\cdot,t) > 0\}$. Then (6.19) and the comparison principle for Laplacian yield $\bar v \leq \bar u$ in $\Sigma_{r,1}\cap B_{1-\varepsilon^\kappa}\times\{t < -T+\varepsilon^{4\kappa}\}$. This and (6.19) show that $\bar v$ and $\bar u$ are ordered on the parabolic boundary of $\Sigma_{r,1}$. In view of Lemma 6.1 with $e := je_d$, and the equation that $\bar u$ satisfies, we want to apply the comparison principle to conclude with (6.18). To do this rigorously, we replace θ by a slightly smaller θ' at the beginning of the proof, the supports of $\bar v$ and $\bar u$ are then separated (because if $z \in \Gamma_{\bar v}$, then the definitions of v and $\bar v$, and (H-a) yield $z \in \Omega_{\bar v}$). The strict order of $\bar v$ and $\bar u$ in one of their support follows easily from the proof below. Then the comparison principle Lemma 2.5 can now yield (6.18) after passing $\theta' \to \theta$.

Now we show (6.19). For any $t \in (-1,1)$ and $x \in \partial_b\Sigma^+(t)\cap B_{1-\varepsilon^\kappa/2}$, since x is at least ε^{γ_1}-away from $\Gamma_v(t)$, Lemma 2.12 and Lemma 3.4 yield that $\inf_{y\in B_\varepsilon(x)} u(y,t) \approx u(x,t) \geq c\,\varepsilon^{\gamma_1\beta}$. Also note that by (6.2) and (6.3) (when f is not a constant), there exists κ_2 such that $\frac{\alpha}{\bar\gamma} < \kappa_2 < \min\{\frac{1}{2}, 1-\gamma_1\}$. Thus by Lemma 4.5 and (6.15), we have

$$\begin{aligned} v(x,t) &\leq \sup_{B_{(1+\eta)\sigma}(x)} u(y,t) \leq \sup_{B_{j\varepsilon\sin\theta}(x)} u(y,t) - (j\varepsilon\sin\theta - (1+\eta)\sigma) \inf_{y\in B_{j\varepsilon\sin\theta}(x)} |\nabla u(y,t)| \\ &\leq \sup_{B_{j\varepsilon\sin\theta}(x)} u(y,t) - C\varepsilon^{\gamma_3+1-\gamma_1} \inf_{y\in B_{j\varepsilon\sin\theta}(x)} u(y,t) \leq (1 - C\varepsilon^{\gamma_3+1-\gamma_1})\bar u(x,t). \end{aligned}$$

The last inequality is due to the full monotonicity of u in the interior (the second remark after Lemma 3.1) since $\gamma_1\beta+\alpha < 1+\bar\gamma-\kappa_1$. Next using $w_t(x) \leq v(x,t)$, it follows that

$$\begin{aligned} \bar v(x,t) &\leq (1+\varepsilon^{\alpha+1}+c_*\varepsilon^{\alpha_1})v(x,t) \\ &\leq (1+C\varepsilon^{\alpha_1})(1-C\varepsilon^{\gamma_3+1-\gamma_1})\bar u(x,t) \leq \bar u(x,t) \end{aligned} \tag{6.20}$$

when ε is sufficiently small.

Now we consider $(x,t) \in (\partial B_{1-\varepsilon^\kappa}\times(-1,1))\cap\Sigma^+$. We define the following region that contains x:

$$\tilde\partial_l\Sigma^+(t) := (B_1\backslash B_{1-2\varepsilon^\kappa})\cap\Sigma^+(t).$$

The construction of φ_η yields that $\varphi_\eta(\cdot,t) \le 1$ in $\tilde\partial_l \Sigma^+(t)$. Since $B_\sigma(x + j\varepsilon e_d) \subseteq B_{\varepsilon \sin\theta}(x + \varepsilon e_d)$ by the definition of σ, the $(\varepsilon, \varepsilon^\alpha)$-monotonicity assumption yields that

$$\bar u(\cdot,t) \ge (1+\varepsilon^{\alpha+1}) \sup_{B_\sigma(\cdot)} u(y,t) \ge (1+\varepsilon^{\alpha+1}) v(\cdot,t) \quad \text{on } \tilde\partial_l \Sigma^+(t). \tag{6.21}$$

Due to Lemma 6.1 and $\Delta w_1^t = 0$, $\bar u - (1+\varepsilon^{\alpha+1})v + \varepsilon^{\alpha_2} w_2^t \ge 0$ is superharmonic. Note that (6.20) implies

$$\varepsilon^{\alpha_1} w_1^t = \varepsilon^{\alpha_1} v \le \bar u - (1+\varepsilon^{\alpha+1})v + \varepsilon^{\alpha_2} w_2^t$$

on $\partial_b \Sigma^+(t) \cap (B_{1-\varepsilon^\kappa/2} \backslash B_{1-2\varepsilon^\kappa})$. Therefore, the choice of c_* and (6.6) yield

$$c_* \varepsilon^{\alpha_1} w_1^t(x) \le \bar u(x,t) - (1+\varepsilon^{\alpha+1}) v(x,t) + \varepsilon^{\alpha_2} w_2^t(x).$$

We obtain $\bar v(x,t) \le \bar u(x,t)$ again. Overall, we showed (6.19) which implies (6.18) and finishes the proof.

□

Proof of of Theorem A: Let us fix $(x_0,t_0) \in \Gamma_u$, which we may assume to be the origin. Applying Lemma 3.6 with some $r > 0$, the free boundary at any time $t \in (-T,T)$ is contained in a $(r + CT/r)$-neighborhood of a $\cot\theta$-Lipschitz graph. Thus it can not move too far away from $t = 0$ when r and then T are sufficiently small. Then by the assumption, after rescaling and rotating, we can assume that the conditions of Proposition 6.2 hold.

Iterating Proposition 6.2, we generate a sequence of domains

$$\mathcal{Q}^k := B_{R_k} \times (-T_k, 1)$$

where $T_k = 1 - 2\Sigma_{n=1}^k (j^n\varepsilon)^{4\kappa}$, $R_k := 1 - \Sigma_{n=1}^k (j^n \varepsilon)^\kappa$, in which u is $(j^k\varepsilon, \alpha_k)$-monotone with respect to the cone $W_{\theta_k, -e_d}$ where

$$\theta_k := \theta - C\Sigma_{n=1}^k (j^n\varepsilon)^{\gamma_3}, \quad \alpha_k = \varepsilon^\alpha (1 - C\Sigma_{n=1}^k (j^n\varepsilon)^\delta).$$

We claim that for each iteration, the constants C, j, γ_3, δ can be chosen the same. Indeed, by taking ε to be further small enough and $\theta > \Theta_0$, we have for all $k \ge 1$,

$$T_k \ge 1/2, \quad R_k \ge 1/2, \quad \theta_k \ge \Theta_0, \quad \alpha_k \ge \varepsilon^\alpha/2.$$

The claim follows from the proof of Proposition 6.2.

Finally we obtain that u is monotone non-decreasing in all directions of $W_{\Theta_0,-e_d}$ in $B_{1/2} \times (-\frac12, 1)$. The last statement of the theorem follows from Corollary 6.3 below. The proof is then completed.

□

6.4. **$C^{1,\gamma}$ free boundary when $\vec b \equiv 0$.** In the zero-drift case, the support of u increase over time. Using this fact, it is well-known that we can obtain an obstacle problem by integrating u over time (for instance see [EJ81] for the classical setting). We will utilize this fact to derive further regularity result.

Corollary 6.3. *For $f \geq 0$ and $\vec{b} \equiv 0$, let u be a viscosity solution to (1.1) in $\mathbb{R}^d \times (-2,2)$ with bounded support. Suppose the assumptions of Theorem A hold in $\mathcal{Q}_2$. Then there exists $0 < \gamma < 1$ such that $\Gamma_u(t)$ is $C^{1,\gamma}$ in B_1 for each $t \in (-1,1)$.*

Proof. Since $\vec{b} \equiv 0$, Ω_u is non-decreasing in time. and define $w(x) := \int_{-2}^t u(x,s)ds$. Since the positive set of u expands over time, we have $\Omega_w(t) = \Omega_u(t)$ for each $t > -2$, so it suffices to show that $\Omega_w(t)$ is $C^{1,\gamma}$ in B_1 and for each $t \in (-1,1)$.

Since our solution is coming from a globally defined solution, it follows from [KPW19, Theorem 1.1] and [DS21] that the viscosity solution coincides with the weak solution of the divergence form equation

$$(\chi_{\{u>0\}})_t - \Delta u = f\chi_{\{u>0\}} \text{ in } \mathbb{R}^d \times (-2,2).$$

From this weak formulation one can then check that $w(\cdot,t)$ solves the obstacle problem:

$$[1 - F(x,t)]\chi_{\{w>0\}} - \Delta w = 0 \text{ in } \mathbb{R}^d.$$

for each $t > -2$, where $F(x,t) := (t - T(x))f(x)$ and the *hitting time* $T : \mathbb{R}^d \to [0,\infty]$ is given by

$$T(x) := \inf\{t \geq -2 \,:\, u(x,t) > 0\}.$$

Theorem A and Proposition 6.6 yields that $T(x)$ is Hölder continuous in B_1 near $\Gamma_w(t)$, and thus so is $F(x,t)$. Since we already know that the free boundary of w has no cusp singularity, we can conclude from [Bla01, Theorem 7.1] that $\Gamma_w(t) \cap B_1$ is $C^{1,\gamma}$ for each $t \in (-1,1)$ for some γ. □

Remark 6.4. We expect the corollary to hold for local solutions u in $\mathcal{Q}_2$ in general, but the corresponding proof requires coincidence of the notions used in weak and viscosity sense in bounded domains with fixed boundary data. We do not pursue it here.

6.5. **Strict expansion along the streamline.** We finish the section by establishing a uniform, yet sublinear, rate of expansion for the positive set Ω_u along the streamline (so for general Lipschitz $\vec{b}$).

Definition 6.5. We say that the set Ω_u is *strictly expanding relatively to the streamlines* in $\mathcal{Q}_r$, if for all small $t > 0$ there exists $r_t > 0$ such that for any $(x_0,t_0) \in \Gamma_u \cap \mathcal{Q}_r$ we have

$$B_{r_t}(X(t;x_0)) \cap \mathcal{Q}_r \subseteq \Omega_u(t_0 + t).$$

Note that this property is stronger than the conclusion of Lemma 3.5, but is weaker than non-degeneracy.

If the free boundary is Lipschitz continuous, we can quantify the amount of expansion of the free boundary relatively to streamlines based on Lemma 3.3.

Proposition 6.6. *Suppose that in $\mathcal{Q}_1$, $u(\cdot,t)$ is non-decreasing with respect to $W_{\theta,-e_d}$ for some $\theta\in(\theta_\beta,\frac{\pi}{2})$ and $\beta\in(1,2)$, and*

$$(0,0)\in\Gamma_u \quad \textit{and} \quad \inf_{t\in(-1,1)} u(-e_d,t)>0. \tag{6.22}$$

Then Ω_u expands strictly relatively to the streamlines in $\mathcal{Q}_{1/2}$ with $r_t=ct^{1/(2-\beta)}$ for some $c>0$.

Proof. Let us only prove the lemma for $d\geq 3$ and the proof for $d=1,2$ is similar. Let $(x_0,t_0)\in\Gamma_u$, and after shifting, we assume $(x_0,t_0)=(0,0)$. Next we define $\bar u$ from (3.8) which solves (3.9). Lemma 3.5 yields that $0\in\overline{\{\bar u(\cdot,t)>0\}}$ for all $t>0$. Thus, the monotonicity assumption yields

$$W_{\theta,-e_d}\cap B_1\subseteq\{\bar u(\cdot,t)>0\}\text{ for }t>0.$$

Since $\theta\geq\theta_\beta$, this, (6.22) and Lemma 3.3 imply that there exists $c>0$ such that

$$\bar u(\cdot,t)\geq cr^\beta \quad \text{in } B_{2r}(-3re_d)\text{ for all } r\in(0,1). \tag{6.23}$$

Now take $P:=-3re_d$ for some $r>0$, and for c from (6.23) define

$$\varphi(x,t):=cr^{d-2+\beta}(|x-P|^{2-d}-R(t)^{2-d}) \quad \text{with } R(t):=(c_1r^{d-2+\beta}t+(2r)^d)^{\frac1d}$$

with $c_1:=2cd(d-2)$. Then for each $t>0$, $\varphi(\cdot,t)$ is a non-negative harmonic function in $B_{R(t)}(P)\backslash B_r(P)$ such that $\varphi(\cdot,t)=0$ on $\partial B_{R(t)}(P)$ and $\varphi(\cdot,t)\leq cr^\beta$ in $B_{R(t)}(P)\backslash B_r(P)$. Thus, in view of (6.23), we obtain

$$\varphi(x,t)\leq\bar u(x,t) \quad \text{in } (B_{2r}(P)\backslash B_r(P))\times\{0\}\cup\{(x,t):\,t\in(0,1),x\in\partial B_r(P)\}. \tag{6.24}$$

Note that $R(t_*)=4r$ with $t_*:=(4^d-2^d)r^{2-\beta}/c_1\leq 1$ when r is small. So by the definition of φ, if we can show $\varphi\leq\bar u$ for all $t\in[0,t_*]$ and $x\in B_{R(t)}(P)\backslash B_r(P)$, then $B_r(0)\subseteq\Omega_{\bar u}(t_*)$ which concludes the proof.

To do this, in view of (6.24), $f_0\geq 0$ and the comparison principle, it remains to show that φ satisfies the appropriate boundary condition on $|x|=R(t)$. Indeed, direct computation yields

$$R'(t)=c_1r^{d-2+\beta}R(t)^{1-d}/d, \quad |\nabla\varphi(x,t)|=c(d-2)r^{d-2+\beta}|x-P|^{1-d}.$$

Also, by $|\vec b_0(x,t)|\leq\|\nabla\vec b\|_\infty|x|$ and the choice of c_1, we obtain for $x\in\partial B_{R(t)}(P)$ and $t\in[0,t_*]$ that

$$R'(t)-|\nabla\varphi(x,t)|-|\vec b_0(x,t)|\leq c(d-2)r^{d-2+\beta}R(t)^{1-d}-Cr\leq C'r^{\beta-1}-Cr$$

which is non-positive if $r>0$ is sufficiently small. This shows that φ is a subsolution to (3.9), with Dirichlet boundary condition on $\partial B_r(P)\times[0,t_*]$, in $(\mathbb{R}^d\backslash B_r(P))\times[0,t_*]$. Now we can conclude. □

7. Non-degeneracy

The goal of the section is to show non-degeneracy result under additional assumptions. Let us illustrate the outline of the proof, in the setting where there is no drift, namely when $\vec{b} = 0$. Due to the cone-monotonicity proven in the previous section, the free boundary of u is a Lipschitz graph with respect to e_d direction, and u is non-decreasing with respect to a cone $W_{-e_d,\theta}$:

(7.1)
$$\sup_{|y-x|<r\varepsilon} u(y+\varepsilon e_d, t) \leq u(x, t+C\varepsilon) \text{ with } r = \sin\theta \text{ and a uniform constant } C.$$

Our claim is that, if the above inequality is true in a unit space-time neighborhood of a free boundary point x_0 , then by the time $t = t(e_d)$ the free boundary reaches the point $x_0 + e_d$, the constant r in (7.1) increases to a constant strictly larger than 1 near the free boundary. In heuristic terms the claim states that the monotonicity of the solution propagates and improves over time in both space and time variable, as the positive set expands out toward e_d direction. Observe that the claim implies that for some $r' > 0$ we have

$$\sup_{|y-x|<r'\varepsilon} u(y,t) \leq u(x, t+C\varepsilon) \text{ in a small neighborhood of } (x_0 + e_d, t(e_d)),$$

providing uniform linear rate of expansion of the positive set of u, which then yields the non-degeneracy of u due to the velocity law $V = |\nabla u|$.

Our claim above is proved in [CJK07] for the case $f = \vec{b} = 0$. For the proof u was compared with a subsolution of the form $\sup_{|y-x|\leq\varphi(x)\varepsilon} u(y,t)$, where $\varphi(x)$ is a chosen radius function first introduced by Caffrelli [Caf89]. The radius function φ will be small on the boundary of the unit neighborhood but is larger near the point x_0+e_d, which yields the desired result. Of course to elaborate this idea the precise subsolution is more involved than stated, to accomodate a sizable perturbation by the radius function. For our problem we employ this idea but with significant modifications due to the presence of both f ad $\vec{b}$, as we will see below (the barrier construction is given in the proof of Theorem 7.3).

Let us now proceed with the assumptions for this section. For any $\beta \in (1, \frac{3}{2})$, let θ_β be given in Lemma 2.10 so that (2.3) holds. We will assume that u is a solution to (1.1) in $B_2 \times (-1, 1)$ with the following properties in $\mathcal{Q}_1 := B_1 \times (-1, 1)$:

- (H-a') $u(\cdot, t)$ is non-decreasing with respect to $W_{\theta,-e_d}$ for some $\theta \in (\theta_\beta, \frac{\pi}{2})$ and $\beta \in (1, \frac{3}{2})$;
- (H-b) $(0, 0) \in \Gamma_u$ and $m := \inf_{t\in(-1,1)} u(-e_d, t) > 0$;
- (H-c) $u_t \geq \vec{b} \cdot \nabla u - C_0 u$ for some $C_0 > 0$ (in the viscosity sense).

Note that (H-a') is obtained from the previous sections, in particular from Theorem A . (H-b) defines m as a parameter, since it is proportional to the rate the positivity set of u expands over time. The last condition (H-c) states that u almost increases along the streamline. While we showed the monotonicity along the streamline for the positive set in Lemma 3.5, it remains open whether this property holds for the solution u: the difficulty lies in the fact that, if we were to compare $u(x,t)$ with $u(X(t;x),t)$, the corresponding elliptic operator involves higher order derivatives of the drift $\vec{b}$, and thus one cannot directly compare the two functions based on the order of their support, unless $\vec{b}$ is identically zero, or a constant vector field. In the global setting, (H-c) can be derived for the initial value problem with smooth $\vec{b}$ and smooth positive f ([Chu22]).

7.1. **Some properties of the expansion of positive sets.** Recall Definition 6.5 about the expansion of Ω_u. We observe that such property propagates backward in time.

Lemma 7.1. *Let $c := e^{-\|\nabla \vec{b}\|_\infty}$. If for some $t, r_t \in (0,1)$ sufficiently small,*

$$B_{r_t}(X(t;x_0)) \subseteq \Omega_u(t_0+t) \quad \text{for all } (x_0,t_0) \in \Gamma_u \cap \mathcal{Q}_1,$$

then

$$B_{cr_t}(X(-t;x_0)) \subseteq \Omega_u(t_0-t)^c \quad \text{for all } (x_0,t_0) \in \Gamma_u \cap \mathcal{Q}_{1/2}.$$

Proof. Denoting $x_1 := X(-t;x_0)$ with $(x_0,t_0) \in \Gamma_u \cap \mathcal{Q}_{1/2}$, Lemma 3.5 yields that $x_1 \in \Omega_u(t_0-t)^c \cap B_1$. Suppose for contraction that there exists $x_2 \in B_{cr_t}(x_1)$ such that $x_2 \in \Gamma_u(t_0-t)$. If $t, r_t \in (0,1)$ are sufficiently small, then $(x_2, t_0-t) \in \mathcal{Q}_1$. By the assumption, we have

$$B_{r_t}(X(t;x_2)) \subseteq \Omega_u(t_0). \tag{7.2}$$

Next since, for all $s \in (0,t)$,

$$\frac{d}{ds}|X(s;x_1) - X(s;x_2)| \leq \|\nabla \vec{b}\|_\infty |X(s;x_1) - X(s;x_2)|,$$

Gronwall's inequality yields

$$|x_0 - X(t;x_2)| = |X(t;x_1) - X(t;x_2)| \leq e^{\|\nabla \vec{b}\|_\infty t}|x_1 - x_2| < e^{\|\nabla \vec{b}\|_\infty} cr_t = r_t.$$

However this contradicts with (7.2) and $x_0 \in \Gamma_u(t_0)$. ☐

Next we introduce a lemma that says characterizing the movement of the free boundary backward in time is the same as characterizing the growth of solutions forward in time.

Lemma 7.2. *Let $r_1, r_2 \in (0,1)$. Then the following is true for sufficiently small $\varepsilon > 0$: Suppose that there is $\tau > 0$ such that*

$$B_{r_1\varepsilon}(X(-\tau\varepsilon;x) - r_2\varepsilon e_d) \subseteq \Omega_u(t-\tau\varepsilon)^c \text{ for all } (x,t) \in \Gamma_u \cap \mathcal{Q}_1. \tag{7.3}$$

Then, for some universal $C > 0$,

$$u(X(\tau\varepsilon;x) + r_2\varepsilon e_d, t+\tau\varepsilon) > 0 \text{ in } \mathcal{Q}_{1-C\varepsilon}.$$

Proof. Let us fix $(x_0,t_0)\in\Gamma_u\cap\mathcal{Q}_{1-C\varepsilon}$. Suppose for contradiction that

$$u(X(\tau\varepsilon;x_0)+r_2\varepsilon e_d,t_0+\tau\varepsilon)=0.$$

Then, using (H-a') and $(X(\tau\varepsilon;x_0),t_0+\tau\varepsilon)\in\Omega_u$ by Proposition 6.6, there exists $h\in(0,r_2)$ such that $(X(\tau\varepsilon;x_0)+h\varepsilon e_d,t_0+\tau\varepsilon)\in\Gamma_u\cap\mathcal{Q}_1$ if C is large enough. So (7.3) with $t=t_0+\tau\varepsilon$ and $x=X(\tau\varepsilon;x_0)+h\varepsilon e_d$ yields that

$$B_{r_1\varepsilon}(X(-\tau\varepsilon;X(\tau\varepsilon;x_0)+h\varepsilon e_d)-r_2\varepsilon e_d)\subseteq\Omega_u(t_0)^c. \tag{7.4}$$

For $s\in[-\tau\varepsilon,0]$, set

$$Y(s):=X(s;X(\tau\varepsilon;x_0)+h\varepsilon e_d)-X(s;X(\tau\varepsilon;x_0)).$$

It is clear that $Y(0)=h\varepsilon e_d$. Using (3.7) yields for all $s\in[-\tau\varepsilon,0]$,

$$|Y(s)|\leq h\varepsilon+\int_s^0\|\nabla\vec{b}\|_\infty|Y(\tau)|d\tau.$$

Thus, Gronwall's inequality yields

$$|Y(s)|\leq h\varepsilon e^{\|\nabla\vec{b}\|_\infty\tau\varepsilon}\leq 2h\varepsilon.$$

if ε is sufficiently small. Since $X(-\tau\varepsilon;X(\tau\varepsilon;x_0))=x_0$ and $Y(0)=h\varepsilon e_d$, we get

$$\begin{aligned}|X(-\tau\varepsilon;X(\tau\varepsilon;x_0)+h\varepsilon e_d)-x_0-h\varepsilon e_d|&=|Y(-\tau\varepsilon)-Y(0)|\\ &\leq\int_{-\tau\varepsilon}^0\|\nabla\vec{b}\|_\infty|Y(s)|ds\leq 2h\tau\varepsilon^2\|\nabla\vec{b}\|_\infty\end{aligned}$$

which is less than $r_1\varepsilon/2$ if ε is sufficiently small. This and (7.4) imply that

$$B_{r_1\varepsilon/2}(x_0-(r_2-h)\varepsilon e_d)\subseteq\Omega_u(t_0)^c.$$

However since u is non-decreasing along $-e_d$ direction and $h\leq r_2$, this contradicts with $(x_0,t_0)\in\Gamma_u$, which leads to the conclusion. ☐

7.2. **Uniform rate of expansion and non-degeneracy.** Now we are ready to show that the support of our solution strictly expands with respect to streamlines. To show this we apply sup-convolutions as in Section 5 to construct perturbed subsolutions, but our domain is no longer a thin strip near the free boundary. The construction of the barrier function in a thin strip domain was enough in Section 6, since there we showed the propagation of cone monotonicity over time, which came from the interior of the support, ε^κ-away from the boundary. Here we will show propagation of the interior non-degeneracy, which only holds unit distance away from the boundary. This necessitates our construction of the barrier different from the previous section.

Theorem 7.3. *Assume* (H-a')(H-b)(H-c). *If f is Lipschitz continuous, then there exists $C_1>0$ such that*

$$u(X(C_1\varepsilon;x)+\varepsilon e_d,t+C_1\varepsilon)>0\quad\textit{for } (x,t)\in\Gamma_u\cap\mathcal{Q}_{1/2} \textit{ for sufficiently small } \varepsilon>0.$$

Proof. As mentioned earlier in this section, the proof relies on the comparison between u and v, a sup-convolution of u with a varying radius function. More precisely we will compare a perturbed version of these functions, U and V. For the construction of U and V, below we will work with $\theta > \theta_\beta$ that is slightly smaller than the one given in the assumption.

We first choose parameters (t_*, r_* and σ_i with $i = 1, 2, 3$) to be used in the proof. By Lemma 7.1 and Proposition 6.6, there exists $c > 0$ such that for $0 < t_* < 1/3$ we have

$$u(\cdot, t_0 - t_*) = 0 \quad \text{ in } B_{2r_*}(X(-t_*; x_0)) \text{ where } r_* := c\, t_*^{1/(2-\beta)} < \frac{1}{3}, \tag{7.5}$$

for any free boundary point $(x_0, t_0) \in \Gamma_u$ in $\mathcal{Q}_{2/3}$.

Let $A_0, A_1 \geq 1$ be from Lemma 5.2, C_0 from the assumption, and let $M_0 \geq 1$ satisfying (7.15) below which only depends on d, A_0, θ. We call $L := (1 + \|f\|_{C^1} + \|\vec{b}\|_{C^1})^2$, and define

(7.6)

$$\sigma_1 := A_1 M_0, \quad \sigma_2 := L(20M_0^2 + 2M_0((A_1+2)M_0+2)t_*/r_*), \quad \sigma_3 := (A_1+2)M_0+2.$$

Note that $t_*^2 << r_*$ due to $\beta < \frac{3}{2}$, so we can choose $t_* > 0$ to be small enough that

$$t_* \leq \min\left\{\frac{1}{5\sigma_2}, \frac{\sigma_1}{C_0\sigma_3}, \frac{1}{\sigma_3}\right\}. \tag{7.7}$$

Let us fix the reference point $(x_0, t_0) \in \Gamma_u \cap \mathcal{Q}_{2/3}$. After translations, we may assume that $t_0 = t_*$ and $X(-t_*; x_0) = 0$. Then $X(t) := X(t; 0)$ satisfies

$$(X(t_*), t_*) = (X(t_*; X(-t_*; x_0)), t_*) = (x_0, t_0) \in \Gamma_u. \tag{7.8}$$

Define $\bar{u}(x, t) := u(x + X(t), t)$ which solves (3.9) with $f_0, \vec{b}_0$ satisfying

$$\|f_0\|_{C^1},\ \|\nabla\vec{b}_0\|_\infty,\ \|\partial_t\vec{b}_0\|_\infty \leq L, \quad |\vec{b}_0(x, t)| \leq L|x|. \tag{7.9}$$

We will work in the cylindrical domain

$$\Sigma := (B_{r_*}(x_1)\backslash B_{r_{\delta,\theta}}(x_1))\times[0, t_*] \quad \text{ where } x_1 := r_* e_d/5 \text{ and } r_{\delta,\theta} := r_* \sin\theta/10.$$

∘ *Construction of U and V:*

First we perturb $\bar{u}$ to define U. Suppose w^t satisfies $-\Delta w^t = 1$ in $B_{r_*}(x_1) \cap \Omega_{\bar{u}}(t)$ and $w^t = 0$ on $\overline{(B_{r_*}(x_1) \cap \Omega_{\bar{u}}(t))^c}$. Note that, from the cone-monotonicity assumption on u, it follows that $\Gamma_{\bar{u}}(t)$ is a Lipschitz graph with Lipschitz constant smaller than $\cot\theta_\beta$. Corollary 4.2 and (H-b) then yield that

$$w^t \leq C r_*^{2-\beta}\bar{u}(\cdot, t) \quad \text{ in } B_{r_*}(x_1) \text{ for some } C = C(m).$$

Since $\beta < 2$, after further taking t_* to be sufficiently small (then $r_* = c\, t_*^{1/(2-\beta)}$ is small) depending only on c, C, L and M_0, we have for all $t \in [0, t_*]$,

$$L(M_0 + 2)w^t \leq \bar{u}(\cdot, t) \quad \text{ in } B_{r_*}(x_1). \tag{7.10}$$

We define

$$U(x,t) := \bar{u}(x,t) + L(M_0+2)\varepsilon w^t(x). \tag{7.11}$$

We claim that U is a supersolution to

$$(P_\varepsilon) \qquad \begin{cases} -\Delta U = f_0(x,t) + L(M_0+2)\varepsilon & \text{in } \Sigma\cap\Omega_U, \\ U_t = (1-\varepsilon)|\nabla U|^2 + \vec{b}_0\cdot\nabla U & \text{on } \Sigma\cap\Gamma_U. \end{cases}$$

By the construction of w^t, it is direct to see the inequality in $\Sigma\cap\Omega_U$.

Let us check the supersolution property on the free boundary. Suppose $U-\phi$ for some $\phi\in C^{2,1}_{x,t}$ has a local minimum in $\{t\le s_0\}$ at some $(y_0,s_0)\in\Gamma_U\cap\Sigma$ and $|\nabla\phi(y_0,s_0)|\neq 0$ and

$$-(\Delta\phi + f_0 + L(M_0+2)\varepsilon)(y_0,s_0) < 0. \tag{7.12}$$

Because (7.10) yields $U\le(1-\varepsilon)^{-1}\bar{u}$ in Σ, we have that $\bar{u}-(1-\varepsilon)\phi$ obtains a local minimum at $(y_0,s_0)\in\Gamma_{\bar{u}}$. Note that (7.12) and (7.9) yield

$$-(\Delta(1-\varepsilon)\phi + f_0)(y_0,s_0) < 0.$$

So using that $\bar{u}$ is a viscosity solution to (3.9), we get

$$\phi_t \ge (1-\varepsilon)|\nabla\phi|^2 + \vec{b}_0\cdot\nabla\phi \quad \text{at } (y_0,s_0),$$

which proves that U is a supersolution to (P_ε).

We will use the following Φ to construct the radius function for V. Let Φ be the unique solution to

$$\begin{cases} \Delta(\Phi^{-A_0+1}) = 0 & \text{in } B_1\backslash B_{\sin\theta/10} \\ \Phi = A_\theta & \text{on } \partial B_{\sin\theta/10} \\ \Phi = (\sin\theta)/2 & \text{on } \partial B_1 \end{cases} \tag{7.13}$$

where A_θ is chosen sufficiently large so that

$$\Phi(-e_d/5) \ge 3. \tag{7.14}$$

We have $\Delta\Phi = \frac{A_0|\nabla\Phi|^2}{\Phi}$ in $B_1\backslash B_{\sin\theta/10}$, and there exists $M_0 = M_0(d,A_0,\theta)\ge 1$ such that

$$M_0^{-1} \le \Phi \le M_0, \quad \|\nabla\Phi\|_\infty \le M_0 \quad \text{in } B_1\backslash B_{\sin\theta/10}. \tag{7.15}$$

Let $\varphi(x) := r_*\Phi(\frac{x-x_1}{r_*})$ where $x_1 = r_*e_d/5$, and define

$$V(x,t) := (1-\sigma_1\varepsilon)\sup_{y\in B_{\varepsilon(1-\sigma_2 t)\varphi(x)}(x)} \bar{u}(y + r_*\varepsilon e_d, (1-\sigma_3\varepsilon)t). \tag{7.16}$$

We now prove that V is a viscosity subsolution of (P_ε) in Σ. Recall that $\bar{u}$ satisfies (3.9), and $f_0,\vec{b}_0$ given in (3.10) satisfy (7.9). Thus Lemma 5.2 (with $f_0,\vec{b}_0$ in place of $f,\vec{b}$) yields

$$-\Delta V(x,t) \le (1-\sigma_1\varepsilon)(1+A_1M_0\varepsilon)f_0(y(x,t)+r_*\varepsilon e_d,(1-\sigma_3\varepsilon)t)$$

where $y(\cdot,\cdot)$ satisfies $|y(x,t)-x| \le \varepsilon(1-\sigma_2 t)\varphi(x) \le r_* M_0 \varepsilon$. Using this, (7.6) and (7.9) yields

$$\begin{aligned}(1-\sigma_1\varepsilon)(1+A_1M_0\varepsilon)&f_0(y(x,t)+r_*\varepsilon e_d,(1-\sigma_3\varepsilon)t)\\ &\le f_0(x,t)+L\varepsilon((1+M_0)r_*+\sigma_3 t_*).\end{aligned}$$

Due to (7.11), $-\Delta w^t = 1$, $r_* \le 1$ and $t_* \le 1/\sigma_3$, we obtain

$$-\Delta V(x,t) \le f_0(x,t) + L\varepsilon(M_0+2) \quad \text{in } \Sigma. \tag{7.17}$$

Next to prove that V satisfies the free boundary condition on $(y_0,s_0) \in \Gamma_{\bar v} \cap \Sigma$, suppose that for a test function $\phi \in C^{2,1}_{x,t}$, $V-\phi$ has a local maximum in $\overline{\Omega_V} \cap \{t \le s_0\}$ at (y_0,s_0). So

$$\sup_{y\in B_{\varepsilon(1-\sigma_2 t)\varphi(x)}(x)} \bar u(y+r_*\varepsilon e_d,t) - \frac{1}{(1-\sigma_1\varepsilon)}\phi(x,(1-\sigma_3\varepsilon)^{-1}t)$$

has a local maximum at $(y_0,(1-\sigma_3\varepsilon)s_0)$ in $\overline{\Omega_V} \cap \{t \le (1-\sigma_3\varepsilon)s_0\}$.

Recall that

$$M_0^{-1}r_* \le \varphi \le M_0 r_*, \quad |\nabla\varphi| \le M_0.$$

It follows from Lemma 5.3 and its remark (with $f_0, \vec b_0$ in place of $f, \vec b$, and $\varepsilon_1 := M_0 r_*\varepsilon, \varepsilon_2 := M_0\varepsilon, \varepsilon_3 := -\sigma_2 r_*\varepsilon/M_0$) that at $(y_0,(1-\sigma_3\varepsilon)s_0)$,

$$\begin{aligned}(1-\sigma_3\varepsilon)^{-1}\phi_t &\le (1-\sigma_1\varepsilon)^{-1}(1+2\varepsilon_2)^2|\nabla\phi|^2 + \vec b_0\cdot\nabla\phi\\ &\quad + \Big(\varepsilon_1\|\nabla\vec b_0\|_{L^\infty(\Sigma')} + 2(\varepsilon_1+\varepsilon_2)\|\vec b_0\|_{L^\infty(\Sigma')} - \varepsilon_3/2\Big)|\nabla\phi|.\end{aligned}$$

Using (7.6), (7.9) and $(y_0,s_0) \in \Sigma \subseteq B_{2r_*} \times [0,t_*]$ yields for ε sufficiently small,

$$\begin{aligned}\phi_t &\le (1-\varepsilon)|\nabla\phi|^2 + (1-\sigma_3\varepsilon)\vec b_0(y_0,(1-\sigma_3\varepsilon)s_0)\cdot\nabla\phi + (1-\sigma_3\varepsilon)\left(9M_0L - \sigma_2/(2M_0)\right)r_*\varepsilon|\nabla\phi|\\ &\le (1-\varepsilon)|\nabla\phi|^2 + \vec b_0(y_0,s_0)\cdot\nabla\phi + (1-\sigma_3\varepsilon)\left(\sigma_3Lt_* + Lr_* + 9M_0Lr_* - \sigma_2 r_*/(2M_0)\right)\varepsilon|\nabla\phi|\\ &\le (1-\varepsilon)|\nabla\phi|^2 + \vec b_0(y_0,s_0)\cdot\nabla\phi.\end{aligned}$$

$\circ$ *Comparison of V and U:* We are going to show next that

$$V \le U \quad \text{in } \Sigma. \tag{7.18}$$

By the comparison principle applied to $(P)_\varepsilon$, it is enough to show that $V \prec U$ on the parabolic boundary of the domain. Below we always consider $(x,t) \in B_{r_*}(x_1) \times [0,t_*] =: \Sigma'$ unless otherwise stated.

We claim that $V \prec \bar u$ on the parabolic boundary of Σ, which will suffice due to the fact that $\bar u \le U$ by definition. From (7.5) that $\bar u(\cdot,0) = u(\cdot,0) = 0$ in $B_{2r_*}(0) \supseteq B_{9r_*/5}(x_1)$. Because

$$(1-\sigma_2 t)\varphi(x)\varepsilon + r_*\varepsilon \le (1+M_0)r_*\varepsilon \le 4r_*/5$$

in Σ' when ε is small, we obtain

$$V(x,0) = 0 = \bar u(x,0) \quad \text{in } B_{r_*}(x_1).$$

Moreover, the same holds for small $t > 0$, and so U and V cannot cross on the initial boundary of Σ.

Next we consider the inner lateral boundary of Σ. Due to $\bar{u}(0, t_*) = u(X(t_*), t_*) = 0$ and the monotonicity of support along streamlines (Lemma 3.5),

$$\bar{u}(0, t) = u(X(t), t) = 0 \quad \text{ for } t \in [0, t_*].$$

Then since $\bar{u}$ is non-decreasing along all directions of $W_{\theta,-e_d}$, we get $\bar{u} = 0$ in $B_{r_* \sin\theta/5}(x_1 + r_* e_d \varepsilon) \times [0, t_*]$. Thus by taking $\varepsilon > 0$ to be small enough such that

$$(1 - \sigma_2 t)\varphi(x)\varepsilon \leq M_0 r_* \varepsilon \leq r_* \sin\theta/10 = r_{\delta,\theta},$$

we get

$$V(\cdot, \cdot) \leq \sup_{B_{r_{\delta,\theta}}} \bar{u}(\cdot + r_* \varepsilon e_d, (1 - \sigma_3 \varepsilon)\cdot) = 0 \,(= \bar{u}) \quad \text{ in } B_{r_{\delta,\theta}}(x_1) \times [0, t_*].$$

Now it remains to show that $V \prec \bar{u} \,(\leq U)$ on the outer lateral boundary $\partial B_{r_\delta}(x_1) \times [0, t_\delta]$. To do this, we use both the assumptions (H-a') and (H-c). Indeed, it is not hard to derive from the latter that $e^{C_0 t} u(X(t; x), t)$ is non-creasing in t. In particular, writing $x_t := x + X(t)$ for $(x, t) \in \Sigma'$, we get

$$\bar{u}(x, t) = u(x_t, t) \geq e^{-C_0 \sigma_3 \varepsilon t} u(X(-\sigma_3 \varepsilon t; x_t), t - \sigma_3 \varepsilon t).$$

This and the cone-monotonicity then yield

$$\bar{u}(x, t) \geq e^{-C_0 \sigma_3 \varepsilon t} \sup_{y \in B_{r_* \varepsilon \sin\theta}} u(y + r_* \varepsilon e_d + X(-\sigma_3 \varepsilon t; x_t), t - \sigma_3 \varepsilon t). \tag{7.19}$$

Note that $X(0; x_t) = x_t = X(0; X(t)) + x$. Therefore

$$\begin{aligned} &|X(-\sigma_3 \varepsilon t; x_t) - X(-\sigma_3 \varepsilon t; X(t)) - x| \\ &\qquad = |(X(-\sigma_3 \varepsilon t; x_t) - X(0; x_t) - (X(-\sigma_3 \varepsilon t; X(t)) - X(0; X(t)))| \\ &\qquad \leq \int_{-\sigma_3 \varepsilon t}^{0} |\vec{b}(X(-s; x_t)) - \vec{b}(X(-s; X(t)))| ds. \end{aligned} \tag{7.20}$$

By direct computations, for $s \in [-\sigma_3 \varepsilon t, 0]$,

$$\begin{aligned} |\vec{b}(X(-s; x_t)) - \vec{b}(X(-s; X(t)))| &\leq \|\nabla \vec{b}\|_\infty (|X(-s; x_t) - X(-s; X(t))|) \\ &\leq \|\nabla \vec{b}\|_\infty \|\vec{b}\|_\infty |2s| \leq 2\sigma_3 \|\nabla \vec{b}\|_\infty \|\vec{b}\|_\infty t_* \varepsilon. \end{aligned}$$

Then, if ε is small enough, (7.20) yields

$$|X(-\varepsilon t; x_t) - X(-\varepsilon t; X(t)) - x| \leq 2\|\nabla \vec{b}\|_\infty \|\vec{b}\|_\infty t_*^2 \varepsilon^2 \leq r_* \varepsilon \sin\theta/2.$$

Combining this with (7.19) implies

$$\bar{u}(x, t) \geq (1 - \sigma_1 \varepsilon) \sup_{y \in B_{r_* \varepsilon \sin\theta/2}(x)} u(y + r_* \varepsilon e_d + X(t - \sigma_3 \varepsilon t), t - \sigma_3 \varepsilon t). \tag{7.21}$$

Here we also used $e^{-C_0 \sigma_3 \varepsilon t} \geq 1 - \sigma_1 \varepsilon$ for $t \in [0, t_*]$ by (7.7), and $X(-\sigma_3 \varepsilon t; X(t)) = X(t - \sigma_3 \varepsilon t)$.

Take (x,t) on the outer lateral boundary of Σ (then $(x,t) \in \partial B_{r_*}(x_1) \times [0,t_*]$). Since $\varphi(x) = r_*(\sin\theta)/2$, (7.7) yields

$$\varepsilon(1-\sigma_2 t)\varphi(x) = \varepsilon(1-\sigma_2 t)r_* \sin\theta/2 \le r_*\varepsilon \sin\theta/2.$$

Thus (7.21) yields that $V \le \bar{u}$ on $\partial B_{r_*}(x_1) \times [0,t_*]$. If $V(x,t) > 0$ for some $(x,t) \in \partial B_{r_*}(x_1) \times [0,t_*]$, it is easy to get $V < \bar{u}$ at (x,t) from the above proof. In addition, the separation of supports follows from the fact that u is monotone with respect to W_{-e_d,θ_0} for $\theta < \theta_0$. In summary, we conclude that

$$V \prec \bar{u} \quad \text{on } \partial B_{r_*}(x_1) \times [0,t_*].$$

Now we will use (7.18) to conclude the theorem.

∘ *Proof of the Theorem:* Note that (7.14) yields

$$\varphi(0) = r_*\Phi\left(-e_d/5\right) \ge 3r_*.$$

Hence we have

$$B_{r_*\varepsilon/5}(-r_*\varepsilon e_d) \subseteq B_{12r_*\varepsilon/5}(0) + r_*\varepsilon e_d \subseteq B_{\varepsilon\varphi(0)(1-\sigma_2 t_*)} + r_*\varepsilon e_d.$$

With this, by (7.7) and (7.18), we get

(7.22)
$$\begin{aligned}\bar{u}(0,t_*) + L(M_0+2)\varepsilon w^{t_*}(0) = U(0,t_*) &\ge V(0,t_*)\\ &\ge \sup_{|z|\le r_*\varepsilon/5}(1-\sigma_1\varepsilon)\bar{u}(z - r_*\varepsilon e_d, t_* - \sigma_3 t_*\varepsilon).\end{aligned}$$

Due to $(X(t_*),t_*) \in \Gamma_u$ by (7.8), and the definition of w^{t_*}, we get $\bar{u}(0,t_*) = w^{t_*}(0) = 0$. Thus (7.22) yields

$$u(z + X(-\sigma_3 t_*\varepsilon; X(t_*)) - r_*\varepsilon e_d, t_* - \sigma_3 t_*\varepsilon) = 0 \quad \text{for all } z \in B_{r_*\varepsilon/5}.$$

In summary, after translations, we proved for all $(x,t) \in \Gamma_u \cap \mathcal{Q}_{2/3}$,

$$B_{r_*\varepsilon/5}(X(-\sigma_3 t_*\varepsilon; x) - r_*\varepsilon e_d/2) \subseteq \Omega_u(t - \sigma_3 t_*\varepsilon)^c.$$

The proof is now completed by invoking Lemma 7.2. □

Proof of Theorem B. We now show the non-degeneracy result, Theorem B. The proof is a consequence of Theorem 7.3, closely following the arguments given in [CJK07]. We will prove that u grows at least linearly near the free boundary (Theorem 7.5), which readily delivers the desired result.

Heuristically speaking, the strict expansion of the positive set Ω_u along the streamline, along with the velocity law $V = |\nabla u| - \vec{b}\cdot\nu$, should provide a lower bound for $|\nabla u|$ on the free boundary. One needs to ensure however that u does not change too much over time, to be able to relate the rate of expansion of the positive set with the size of the pressure variable. This is where we need a Carleson-type estimate (see also Lemma 2.5-2.6 in [CJK07]), and its proof is parallel to that of Corollary 2.2 in [CJK07].

Let us denote (HS) by the particular case of (1.1) with $f, \vec{b} \equiv 0$.

Lemma 7.4. *Suppose that v is a subsolution to (HS) in $\mathcal{Q}_2$ satisfying $C^{-1} \leq \frac{v(-e_d,t)}{v(-e_d,0)} \leq C$ for some $C \geq 1$. Also suppose that $\Gamma_v(t) \cap B_1(0)$ can be represented by $x_n = g_t(x')$ where $g_t : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz with $\|g\|_{\text{Lip}} \leq c_d$ for some dimensional constant $c_d > 0$. Then there is $\delta_d > 0$ such that the following holds for $0 < \delta < \delta_d$: for any $x_0 \in \Gamma_v(0) \cap B_1$, $x_1 \in \Omega_v(0)$ and $x_2 \in \Omega_v(0)^c$ such that*

$$\frac{\delta}{2} \leq |x_1 - x_0|, \quad |x_2 - x_0|, \quad d(x_1, \Gamma_v(0)), \quad d(x_2, \Gamma_v(0)) \leq \delta,$$

we have for some M depending on C that

$$\frac{\delta^2}{v(x_1,0)} \leq M\, T(x_2), \qquad \textit{where } T(x) := \sup\{t \geq 0 \,:\, v(x,t) = 0\}. \tag{7.23}$$

Now we are ready to prove the non-degeneracy result. Note that Theorem B follows directly from Theorem 7.5 and Lemma 4.4, since Theorem A yields (H-a') in $\mathcal{Q}_1$.

Theorem 7.5. *Assume the conditions of Theorem 7.3. Moreover, suppose that $\theta \geq \operatorname{arccot} c_d$ (with c_d from Lemma 7.4) and $C^{-1} \leq \frac{u(-e_d,t)}{u(-e_d,0)} \leq C$ for all $t \in (-1,1)$ and for some $C \geq 1$. Then there exist $\delta_0, c_0 > 0$ such that for all $\delta \in (0, \delta_0)$,*

$$u(x - \delta e_d, t) \geq c_0 \delta \qquad \textit{for all } (x,t) \in \Gamma_u \cap \mathcal{Q}_{1/2}.$$

Proof. We will only show the conclusion for $(x,t) = (0,0)$, which is on Γ_u by our setting. Let C_1 from Theorem 7.3, and choose $c_1 := (2C_1)^{-1}$. Then $\bar{u}(x,t) := u(x + X(t) + c_1 t e_d, t)$ satisfies

$$\begin{cases} -\Delta \bar{u} = f'(x,t) & \text{in } \{\bar{u} > 0\}, \\ \quad \bar{u}_t = |\nabla \bar{u}|^2 + \vec{b}'(x,t) \cdot \nabla \bar{u} & \text{on } \partial\{\bar{u} > 0\}, \end{cases} \tag{7.24}$$

where

$$f'(x,t) := f(x+X(t)+c_1 t e_d) \text{ and } \vec{b}'(x,t) := \vec{b}(x+X(t)+c_1 t e_d) - \vec{b}(X(t)) + c_1 e_d. \tag{7.25}$$

For each $t \in (-1/2, 1/2)$, let $w_1(\cdot,t)$ be the unique non-negative harmonic function in $\Omega_{\bar{u}}(t) \cap B_1$ such that $w_1(\cdot,t) = 0$ on $\bar{\Gamma}(t)$, and $w_1(\cdot,t) = \bar{u}(\cdot,t)$ on $\Gamma_{\bar{u}}(t) \cap \partial B_1$. It follows from Lemma 11.12 [CS05] that any harmonic function is monotone along the monotonicity direction of its Lipschitz domain, if sufficiently close to its domain boundary where it assumes zero boundary data. In particular, we have $\nabla_{-x_d} w_1(\cdot,t) \geq 0$ in B_r for some $r \in (0,1)$. Let us fix one such r that also satisfies

$$r < \min\left\{1, \frac{c_1}{(1+c_1)} (\|\nabla \vec{b}\|_\infty)^{-1}\right\}.$$

Next, for $w_2 := \bar{u} - w_1$, it follows from Corollary 4.2 that there exists $C_2 > 1$ such that $w_2 \leq (C_2 - 1) w_1$. So we get

$$w_1 \leq \bar{u} \leq C_2 w_1. \tag{7.26}$$

We claim that $C_2 w_1$ is a subsolution to (HS) in $\mathcal{Q}_r$. Since w_1 is harmonic in its support, it suffices to verify the free boundary condition. Suppose there is a smooth function $\phi \in C^{2,1}_{x,t}$ such that $C_2 w_1 - \phi$ has a local maximum zero in $\overline{\Omega_{w_1}} \cap \{t \leq t_0\}$ at $(x_0, t_0) \in \Gamma_{w_1}$. By (7.26), $\bar{u} - \phi$ also obtains a local maximum in $\overline{\Omega_{\bar{u}}} \cap \{t \leq t_0\}$ at (x_0, t_0), and therefore (7.24) and Lemma 2.6 yield

$$\phi_t(x_0, t_0) \leq |\nabla \phi(x_0, t_0)|^2 + \vec{b}'(x_0, t_0) \cdot \nabla \phi(x_0, t_0) \tag{7.27}$$

when $|\nabla \phi(x_0, t_0)| \neq 0$. While when $\nabla \phi(x_0, t_0) = 0$, Lemma 3.5 yields (7.27) again. Hence, to conclude, it is enough to show that

$$\vec{b}'(x_0, t_0) \cdot \nabla \phi(x_0, t_0) \leq 0. \tag{7.28}$$

By the assumption on r, we have for all $(x, t) \in \mathcal{Q}_r$,

$$|\vec{b}(x + X(t) + c_1 t e_d) - \vec{b}(X(t))| \leq r(1 + c_1)\|\nabla \vec{b}\|_\infty \leq c_1.$$

So $\langle \vec{b}'(x, t), e_d \rangle \leq \frac{\pi}{4}$, where the notation (2.1) is used. By the $W_{\theta, -e_d}$-monotonicity of $\bar{u}$, we get $\phi(\cdot, t_0) \geq \phi(x_0, t_0)$ in $x_0 + W_{\theta, -e_d}$ which implies $\langle \nabla \phi(x_0, t_0), -e_d \rangle \leq \pi/2 - \theta$. Consequently, also using $\theta \geq \frac{\pi}{4}$ and $\langle \vec{b}'(x, t), e_d \rangle \leq \frac{\pi}{4}$, we verified (7.28). This concludes that w_1 is a subsolution to (HS) in $\mathcal{Q}_r$.

Lemma 7.4 now yields that for all $\delta > 0$ sufficiently small,

$$\delta^2 / w_1(-\delta e_d, 0) \leq M \sup\{t \geq 0 \,:\, w_1(\delta e_d, t) = 0\}.$$

Thus, (7.26) along with the definition of $\bar{u}$ yields

$$\delta^2 / u(-\delta e_d, 0) \leq M \sup\{t \geq 0 \,:\, u(\delta e_d + X(t) + c_1 t e_d, t) = 0\}. \tag{7.29}$$

Lastly we apply Theorem 7.3 with $\varepsilon := 2\delta$. It follows that if δ is sufficiently small,

$$u(\delta e_d + X(2C_1\delta) + 2C_1\delta c_1 e_d, 2C_1\delta) = u(X(C_1\varepsilon) + \varepsilon e_d, C_1\varepsilon) > 0.$$

Therefore

$$\sup\{t \geq 0 \,:\, u(\delta e_d + X(t) + c_1 t e_d, t) = 0\} \leq 2C_1\delta.$$

From this and (7.29), we obtain $u(-\delta e_d, 0) \geq \delta/(2C_1 M)$, which finishes the proof. □

References

[ACS96] I Athanasopoulos, L Caffarelli, and Sandro Salsa. Regularity of the free boundary in parabolic phase-transition problems. *Acta Mathematica*, 176(2):245–282, 1996.

[ACS98] I Athanasopoulos, LA Caffarelli, and S Salsa. Phase transition problems of parabolic type: Flat free boundaries are smooth. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 51(1):77–112, 1998.

[Anc12] Alano Ancona. On positive harmonic functions in cones and cylinders. *Revista Matemática Iberoamericana*, 28(1):201–230, 2012.

[BCG83] C Betz, GA Cámera, and H Gzyl. Bounds for the first eigenvalue of a spherical cap. *Applied Mathematics and Optimization*, 10(1):193–202, 1983.

[Bla01] Ivan Blank. Sharp results for the regularity and stability of the free boundary in the obstacle problem. *Indiana University Mathematics Journal*, pages 1077–1112, 2001.

[Caf87] Luis A Caffarelli. A harnack inequality approach to the regularity of free boundaries. part i: Lipschitz free boundaries are $c^{1,\alpha}$. *Revista Matemática Iberoamericana*, 3(2):139–162, 1987.

[Caf89] Luis A Caffarelli. A harnack inequality approach to the regularity of free boundaries part ii: Flat free boundaries are lipschitz. *Communications on pure and applied mathematics*, 42(1):55–78, 1989.

[Chu22] Raymond Chu. A hele-shaw limit with a variable upper bound and drift. *arXiv preprint arXiv:2203.02644*, 2022.

[CJK07] Sunhi Choi, David Jerison, and Inwon Kim. Regularity for the one-phase hele-shaw problem from a lipschitz initial surface. *American journal of mathematics*, 129(2):527–582, 2007.

[CJK09] Sunhi Choi, David Jerison, and Inwon Kim. Local regularization of the one-phase hele-shaw flow. *Indiana University mathematics journal*, pages 2765–2804, 2009.

[CKY18] Katy Craig, Inwon Kim, and Yao Yao. Congested aggregation via newtonian interaction. *Archive for Rational Mechanics and Analysis*, 227(1):1–67, 2018.

[CS05] Luis A Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68. American Mathematical Soc., 2005.

[Dah79] B Dahlberg. Harmonic functions in lipschitz domains. *Harmonic analysis in Euclidean spaces*, pages 313–322, 1979.

[DS21] Noemi David and Markus Schmidtchen. On the incompressible limit for a tumor growth model incorporating convective effects. *arXiv preprint arXiv:2103.02564*, 2021.

[DSFS21] Daniela De Silva, Nicolo Forcillo, and Ovidiu Savin. Perturbative estimates for the one-phase stefan problem. *Calculus of Variations and Partial Differential Equations*, 60(6):1–38, 2021.

[EJ81] Charles M Elliott and Vladimır Janovskỳ. A variational inequality approach to hele-shaw flow with a moving boundary. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 88(1-2):93–107, 1981.

[ES97] Joachim Escher and Gieri Simonett. Classical solutions of multidimensional hele–shaw models. *SIAM Journal on Mathematical Analysis*, 28(5):1028–1047, 1997.

[Eva10] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.

[FROS20] Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra. Generic regularity of free boundaries for the obstacle problem. *Publications mathématiques de l'IHÉS*, 132(1):181–292, 2020.

[JK82] David Jerison and Carlos E Kenig. Boundary behavior of harmonic functions in non-tangentially accessible domains. *Advances in Mathematics*, 46(1):80–147, 1982.

[JK05] David Jerison and Inwon Kim. The one-phase hele-shaw problem with singularities. *The Journal of Geometric Analysis*, 15(4):641–667, 2005.

[JKT22] Matt Jacobs, Inwon Kim, and Jiajun Tong. Tumor growth with nutrients: Regularity and stability. *arXiv preprint arXiv:2204.07572*, 2022.

[Kim03] Inwon Kim. Uniqueness and existence results on the hele-shaw and the stefan problems. *Archive for Rational Mechanics & Analysis*, 168(4), 2003.

[Kim06] Inwon Kim. Regularity of the free boundary for the one phase hele–shaw problem. *Journal of Differential Equations*, 223(1):161–184, 2006.

[Kit97] So Kitsunezaki. Interface dynamics for bacterial colony formation. *Journal of the Physical Society of Japan*, 66(5):1544–1550, 1997.

[KPW19] Inwon Kim, Norbert Požár, and Brent Woodhouse. Singular limit of the porous medium equation with a drift. *Advances in Mathematics*, 349:682–732, 2019.

[KZ21] Inwon Kim and Yuming Paul Zhang. Porous medium equation with a drift: Free boundary regularity. *Archive for Rational Mechanics and Analysis*, 242(2):1177–1228, 2021.

[MRCS10] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. A macroscopic crowd motion model of gradient flow type. *Mathematical Models and Methods in Applied Sciences*, 20(10):1787–1821, 2010.

[MRCS14] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. Congestion-driven dendritic growth. *Discrete & Continuous Dynamical Systems*, 34(4):1575, 2014.

[PQV14] Benoît Perthame, Fernando Quirós, and Juan Luis Vázquez. The hele–shaw asymptotics for mechanical models of tumor growth. *Archive for Rational Mechanics and Analysis*, 212(1):93–127, 2014.

[PTV14] Benoît Perthame, Min Tang, and Nicolas Vauchelet. Traveling wave solution of the hele–shaw model of tumor growth with nutrient. *Mathematical Models and Methods in Applied Sciences*, 24(13):2601–2626, 2014.

[Ric72] Stanley Richardson. Hele shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *Journal of Fluid Mechanics*, 56(4):609–618, 1972.

[Sav09] Ovidiu Savin. Regularity of flat level sets in phase transitions. *Annals of Mathematics*, pages 41–78, 2009.

[Sha98] Henry Selby Hele Shaw. Investigation of the nature of surface resistance of water and of stream-line motion under certain experimental conditions. In *Inst. NA.*, 1898.

Department of Mathematics, UCLA, LA CA
Email: `ikim@math.ucla.edu`

Department of Mathematics & Statistics, Auburn University, Auburn AL
Email: `yzhangpaul@auburn.edu`