Quantized Axial Charge of Staggered Fermions and the Chiral Anomaly

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In the $1+1$D ultralocal lattice Hamiltonian for staggered fermions with a finite-dimensional Hilbert space, there are two conserved, integer-valued charges that flow in the continuum limit to the vector and axial charges of a massless Dirac fermion with a perturbative anomaly. Each of the two lattice charges generates an ordinary U(1) global symmetry that acts locally on operators and can be gauged individually. Interestingly, they do not commute on a finite lattice and generate the Onsager algebra, but their commutator goes to zero in the continuum limit. The chiral anomaly is matched by this non-Abelian algebra, which is consistent with the Nielsen-Ninomiya theorem. We further prove that the presence of these two conserved lattice charges forces the low-energy phase to be gapless, reminiscent of the consequence from perturbative anomalies of continuous global symmetries in continuum field theory. Upon bosonization, these two charges lead to two exact U(1) symmetries in the XX model that flow to the momentum and winding symmetries in the free boson conformal field theory.

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Introduction—The realization of chiral symmetries on the lattice has been a long-standing problem in lattice field theory [1]. One prominent difficulty is that the global symmetries of interest typically have anomalies in the sense that they cannot be consistently coupled to dynamical gauge fields.

Anomalies are traditionally linked to divergences in continuum field theory and are often considered challenging, if not impossible, to realize on a lattice with finite lattice spacing. However, this piece of lore is not true. Numerous examples of anomalies for discrete global symmetries are realized on the lattice. See, for example, Ref. [2] for a recent survey of this topic. A more refined piece of lore is that perturbative anomalies of continuous global symmetries cannot be realized on a lattice. By perturbative or local anomalies, we mean those captured by Feynman diagrams and local operator product expansions. This piece of lore also turns out to be incorrect: perturbative anomalies can be realized in Villain-type lattice models [2–7]. The local Hilbert spaces of these models are infinite-dimensional. See Ref. [7] for discussions of chiral fermion symmetry in $1+1$D QED in this setup, and Ref. [8] for a Euclidean lattice realization of mixed gravitational anomalies with global symmetries.

Perhaps a more interesting question is whether perturbative anomalies be realized in a lattice model whose local Hilbert space is finite-dimensional (e.g., qubits). There is a simple no-go argument against this hope. This can be seen most readily for U(1) symmetries in $1+1$D, where perturbative anomalies are encoded in the equal-time commutator of the conserved current operators $[j^0(t,x), j^x(t,x')] \sim i\partial_x \delta(x-x')$, known as the Schwinger term [9,10]. Taking the trace of this relation immediately shows that the Schwinger term cannot be realized verbatim on a finite-dimensional Hilbert space since the trace of a commutator in a finite-dimensional vector space necessarily vanishes. This is similar to the fact that $[X,P]=i\hbar$ cannot be realized in a finite-dimensional Hilbert space in quantum mechanics. See Ref. [11] for a rigorous proof of this no-go result. The Schwinger term, however, has recently been realized exactly on a lattice model with an infinite-dimensional Hilbert space [5].

These difficulties are rigorously formulated in the Nielsen-Ninomiya theorem [12–14]. The theorem, most precisely formulated in [15], asserts that in any quadratic lattice fermion Hamiltonian with certain locality properties in odd spatial dimensions, there must be an equal number of left- and right-moving fermions within each irreducible representation of the global symmetries. In particular, this theorem prohibits the existence of an axial charge that (i) has quantized, integer eigenvalues and (ii) commutes with the vector U(1) symmetry. If such an axial charge existed, one could restrict to states with fixed vector and axial charge and find a single left-moving fermion.

Given the above, what fingerprints of anomalies can we hope for on a lattice with a finite-dimensional Hilbert space? In this Letter, we focus on the anomaly between the vector and axial symmetries of a $1+1$D massless Dirac fermion, which is the oldest and arguably the simplest

FIG. 1. In an ultralocal Hamiltonian lattice model, we discuss two conserved, quantized charges Q^{V} and Q^{A}, which become the vector and axial charges $\mathcal{Q}^{\mathrm{V}}$ and $\mathcal{Q}^{\mathrm{A}}$ of the massless Dirac fermion field theory in the continuum limit. We use calligraphic and ordinary fonts for operators in the continuum and on the lattice, respectively. The anomaly between $\mathcal{Q}^{\mathrm{V}}$ and $\mathcal{Q}^{\mathrm{A}}$ is matched on the lattice by the non-Abelian Onsager algebra generated by Q^{V} and Q^{A}.

anomaly of all. In a microscopic Hamiltonian lattice realization, we discuss two conserved, quantized charges that become the vector and axial charges in the continuum limit. While each of the two charges generates an ordinary U(1) symmetry satisfying all the locality properties, they do *not* commute on a finite-size lattice. Together, they form a non-Abelian algebra, known as the Onsager algebra [16], which is consistent with the Nielsen-Ninomiya theorem and matches the continuum anomaly (see Fig. 1).

Symmetries and anomalies in the continuum—We start by reviewing the symmetries and anomalies of the massless Dirac fermion field theory in $1+1$D. We denote the left- and right-moving (complex, one-component) Weyl fermions as $\Psi_{\mathrm{L}}(t,x)$ and $\Psi_{\mathrm{R}}(t,x)$, respectively. In Minkowski spacetime with metric $\eta_{\mu\nu} = (1,-1)$, the action for a free, massless Dirac fermion $\Psi = (\Psi_{\mathrm{R}}\, \Psi_{\mathrm{L}})^{\mathsf{T}}$ is

$$\begin{aligned} S &= i\int \mathrm{d}t\, \mathrm{d}x\, \bar{\Psi}\Gamma^{\mu}\partial_{\mu}\Psi \\ &= i\int \mathrm{d}t\, \mathrm{d}x \Big[\Psi_{\mathrm{L}}^{\dagger}(\partial_t - \partial_x)\Psi_{\mathrm{L}} + \Psi_{\mathrm{R}}^{\dagger}(\partial_t + \partial_x)\Psi_{\mathrm{R}}\Big], \end{aligned} \quad (1)$$

where the Dirac matrices $\Gamma^0 = \sigma^x$ and $\Gamma^1 = -i\sigma^y$ and $\bar{\Psi} = \Psi^{\dagger}\Gamma^0$. The internal global symmetry for the right-moving Weyl fermion is $\mathrm{O}(2)^{\mathrm{R}} \cong \mathrm{U}(1)^{\mathrm{R}} \rtimes \mathbb{Z}_2^{\mathcal{C}^{\mathrm{R}}}$, where $\mathrm{U}(1)^{\mathrm{R}}$ and the charge conjugation symmetry $\mathbb{Z}_2^{\mathcal{C}^{\mathrm{R}}}$ act on the fermion as $e^{i\varphi\mathcal{Q}^{\mathrm{R}}}\Psi_{\mathrm{R}}e^{-i\varphi\mathcal{Q}^{\mathrm{R}}} = e^{-i\varphi}\Psi_{\mathrm{R}}$ and $\mathcal{C}^{\mathrm{R}}\Psi_{\mathrm{R}}(\mathcal{C}^{\mathrm{R}})^{-1} = \Psi_{\mathrm{R}}^{\dagger}$, respectively. Here, $\mathcal{Q}^{\mathrm{R}}$ is the quantized charge for the right movers, which obeys $\mathcal{C}^{\mathrm{R}}\mathcal{Q}^{\mathrm{R}}(\mathcal{C}^{\mathrm{R}})^{-1} = -\mathcal{Q}^{\mathrm{R}}$. A similar global $\mathrm{O}(2)^{\mathrm{L}}$ symmetry applies to the left movers, making the total internal global symmetry $\mathrm{O}(2)^{\mathrm{L}} \times \mathrm{O}(2)^{\mathrm{R}}$.

The (quantized) vector and axial charges are defined as $\mathcal{Q}^{\mathrm{V}} = \mathcal{Q}^{\mathrm{L}} + \mathcal{Q}^{\mathrm{R}}$ and $\mathcal{Q}^{\mathrm{A}} = \mathcal{Q}^{\mathrm{L}} - \mathcal{Q}^{\mathrm{R}}$ and act on the fermions as

$$\begin{aligned} [\mathcal{Q}^{\mathrm{V}}, \Psi_{\mathrm{L}}^{\dagger}] &= \Psi_{\mathrm{L}}^{\dagger}, & [\mathcal{Q}^{\mathrm{V}}, \Psi_{\mathrm{R}}^{\dagger}] &= \Psi_{\mathrm{R}}^{\dagger}, \\ [\mathcal{Q}^{\mathrm{A}}, \Psi_{\mathrm{L}}^{\dagger}] &= \Psi_{\mathrm{L}}^{\dagger}, & [\mathcal{Q}^{\mathrm{A}}, \Psi_{\mathrm{R}}^{\dagger}] &= -\Psi_{\mathrm{R}}^{\dagger}. \end{aligned} \quad (2)$$

The $\mathbb{Z}_2$ subgroups of the vector and axial U(1) symmetries act identically on the fermions as a fermion parity, so the global form of these symmetries is $[\mathrm{U}(1)^{\mathcal{V}} \times \mathrm{U}(1)^{\mathcal{A}}]/\mathbb{Z}_2$. The vector and axial charges are related by

$$\mathcal{Q}^{\mathrm{A}} = \mathcal{C}^{\mathrm{R}}\mathcal{Q}^{\mathrm{V}}(\mathcal{C}^{\mathrm{R}})^{-1}. \quad (3)$$

The vector and axial U(1) symmetries are separately anomaly-free in the sense that there is no obstruction to gauging either one of the two. However, there is a mixed anomaly between them, which implies that when the vector symmetry is gauged (i.e., in QED), the axial symmetry is broken, and vice versa [17,18]. This can be seen from the anomalous conservation equation $\partial^{\mu} j_{\mu}^{\mathrm{A}} = -(1/\pi)E$, where $j_{\mu}^{\mathrm{A}} = \bar{\Psi}\Gamma^5\Gamma_{\mu}\Psi$ (with $\Gamma^5 = \Gamma^0\Gamma^1$) is the axial current and E is the electric field.

Lattice model—Consider the $1+1$D staggered fermion lattice Hamiltonian [19–21]

$$H = -i\sum_{j=1}^{L}\left(c_j^{\dagger}c_{j+1} + c_j c_{j+1}^{\dagger}\right), \quad (4)$$

where there is a single-component complex fermion c_j at every site j, satisfying $\{c_j, c_{j'}\} = \{c_j^{\dagger}, c_{j'}^{\dagger}\} = 0$ and $\{c_j, c_{j'}^{\dagger}\} = \delta_{j,j'}$. This Hamiltonian is ultralocal, with only nearest-neighbor couplings. The continuum limit is a single, free, massless Dirac fermion (1). We consider both periodic and antiperiodic boundary conditions on a closed chain with L sites, i.e., $c_{j+L} = (-1)^{\nu}c_j$ with $\nu = 0, 1$. We assume L to be even for simplicity.

It will be important for the following discussion to decompose the complex fermion into two real fermions as $c_j = \frac{1}{2}(a_j + ib_j)$, where $a_j = a_j^{\dagger}$ and $b_j = b_j^{\dagger}$ are decoupled Majorana fermions satisfying $\{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{j,j'}$. In terms of these Majorana fermions, the Hamiltonian (4) becomes

$$H = -\frac{i}{2}\sum_{j=1}^{L}(a_j a_{j+1} + b_j b_{j+1}). \quad (5)$$

The Hamiltonian (4) has a $\mathrm{U}(1)^{\mathrm{V}}$ fermion-number symmetry whose quantized charge is

$$Q^{\mathrm{V}} = \sum_{j=1}^{L}\left(c_j^{\dagger}c_j - \frac{1}{2}\right) = \frac{i}{2}\sum_{j=1}^{L} a_j b_j \equiv \sum_{j=1}^{L} q_j^{\mathrm{V}}. \quad (6)$$

It flows to the vector charge $\mathcal{Q}^{\mathrm{V}}$ of the Dirac fermion field theory in the continuum limit. We will hence refer to it as the vector charge or the fermion number. The $\mathrm{U}(1)^{\mathrm{V}}$ symmetry acts on the lattice fermions as $e^{i\varphi Q^{\mathrm{V}}}c_j e^{-i\varphi Q^{\mathrm{V}}} = e^{-i\varphi}c_j$, or equivalently as

$$\begin{aligned} e^{i\varphi Q^{\mathrm{V}}}a_j e^{-i\varphi Q^{\mathrm{V}}} &= \cos\varphi\, a_j + \sin\varphi\, b_j, \\ e^{i\varphi Q^{\mathrm{V}}}b_j e^{-i\varphi Q^{\mathrm{V}}} &= \cos\varphi\, b_j - \sin\varphi\, a_j. \end{aligned} \quad (7)$$

Quantized axial charge—Is the axial symmetry exact on the lattice? We claim that the lattice operator

$$Q^{\mathrm{A}} = \frac{1}{2}\sum_{j=1}^{L}(c_j + c_j^\dagger)(c_{j+1} - c_{j+1}^\dagger) = \frac{i}{2}\sum_{j=1}^{L} a_j b_{j+1} \equiv \sum_{j=1}^{L} q^{\mathrm{A}}_{j+\frac{1}{2}} \tag{8}$$

obeys the following properties. (a) It is quantized with integer eigenvalues. (b) It commutes with the ultralocal Hamiltonian (4). (c) It is a sum of local charge density operators $q^{\mathrm{A}}_{j+\frac{1}{2}}$. (d) It is bilinear in the fermions. (e) It becomes the continuum axial charge $\mathcal{Q}^{\mathrm{A}}$ in the continuum limit. Property (a) follows from the fact that the local factors $q^{\mathrm{A}}_{j+\frac{1}{2}}$, which are the lattice version of the (time component of the) Noether current, commute with each other and can be simultaneously diagonalized. Properties (b), (c), and (d) are straightforward to verify. Below, we will argue for the last property.

To find a lattice axial charge, we note that since the vector charge is manifest on the lattice, using (3), it suffices to find a lattice realization of the right-moving charge conjugation $\mathcal{C}^{\mathrm{R}}$. The latter is a chiral fermion parity that flips the sign of a single Majorana-Weyl fermion. Specifically, we decompose the Weyl fermion into two Majorana-Weyl fermions as $\Psi_{\mathrm{R}} = \lambda_{\mathrm{R}} + i\chi_{\mathrm{R}}$, then $\mathcal{C}^{\mathrm{R}}$ only flips the sign of χ_{R}. It is known that such a chiral fermion parity is realized as a lattice translation on the lattice, which we review below.

We focus on the Majorana fermion b_j in the lattice Hamiltonian (5), which flows to the left and right Majorana-Weyl fermions χ_{L} and χ_{R} in the continuum. Up to a constant, the Hamiltonian for b_j in momentum space is $\sum_{0<k<L/2}\sin(2\pi k/L)\beta_{-k}\beta_k$, where $\beta_k = (1/\sqrt{L})\sum_{j=1}^{L} e^{-(2\pi i k j/L)} b_j$ with $k \in \mathbb{Z} + \nu/2$ and $k \sim k + L$. The ground state(s) $|\Omega\rangle$ obeys $\beta_k|\Omega\rangle = 0$ for all $0 < k < (L/2)$. The left- and right-moving modes in the continuum field theory created by χ_{L} and χ_{R} arise from the lattice modes created by β_{-k} with k close to $k = 0$ and $k = (L/2)$, respectively.

The key symmetry in the lattice model (5) is the translation operator T_b that acts as $T_b a_j T_b^{-1} = a_j$ and $T_b b_j T_b^{-1} = b_{j+1}$. In momentum space, $T_b \beta_k T_b^{-1} = e^{(2\pi i k/L)}\beta_k$, hence T_b acts with a relative minus sign between the modes around $k = 0$ and $k = (L/2)$ in the limit $L \to \infty$. We conclude that on the low-lying states, T_b acts as the right-moving charge conjugation times the continuum translation operator [22]

$$T_b = \mathcal{C}^{\mathrm{R}} e^{\frac{2\pi i \mathcal{P}}{L}}, \tag{9}$$

where $\mathcal{P}$ is the continuum momentum operator acting only on the Majorana fermion χ (but not λ). This is to be contrasted with the lattice translation $T = T_a T_b$ that shifts both a_j and b_j by one site. In the continuum limit, T flips the signs of both $\lambda_{\mathrm{R}}, \chi_{\mathrm{R}}$ [20,21], while T_b only flips the sign of χ_{R}.

Having identified the lattice origin of the right-moving charge conjugation $\mathcal{C}^{\mathrm{R}}$, we follow (3) to define the lattice axial charge,

$$Q^{\mathrm{A}} = T_b Q^{\mathrm{V}} T_b^{-1}, \tag{10}$$

which gives (8). On the low-lying states in the $L \to \infty$ limit, $e^{2\pi i \mathcal{P}/L}$ becomes 1 and $T_b \sim \mathcal{C}^{\mathrm{R}}$, so from (3) we see that Q^{A} becomes the continuum axial charge $\mathcal{Q}^{\mathrm{A}}$. From this expression, it is clear that Q^{A} has integer eigenvalues since it is unitarily equivalent to Q^{V}. It is also clear that it commutes with the Hamiltonian since Q^{V} and T_b do.

Since Q^{A} is quantized, we can exponentiate it to find an exact $\mathrm{U}(1)^{\mathrm{A}}$ axial symmetry on the lattice, which acts locally on the fermions as

$$\begin{aligned} e^{i\varphi Q^{\mathrm{A}}} a_j e^{-i\varphi Q^{\mathrm{A}}} &= \cos\varphi a_j + \sin\varphi b_{j+1}, \\ e^{i\varphi Q^{\mathrm{A}}} b_j e^{-i\varphi Q^{\mathrm{A}}} &= \cos\varphi b_j - \sin\varphi a_{j-1}. \end{aligned} \tag{11}$$

This quantized axial charge was first identified in [23,24] from the connection to integrability. Here, we provided an alternative derivation using lattice translation.

While Q^{V} and Q^{A} each generates an ordinary U(1) global symmetry, interestingly, these two lattice charges do *not* commute,

$$[Q^{\mathrm{V}}, Q^{\mathrm{A}}] = -\sum_{j=1}^{L}(c_j c_{j+1} + c_j^\dagger c_{j+1}^\dagger). \tag{12}$$

This is to be contrasted with the continuum where $[\mathcal{Q}^{\mathrm{V}}, \mathcal{Q}^{\mathrm{A}}] = 0$. In the Supplemental Material [25] (SM), we show that the nonvanishing lattice commutator goes to zero on the low-lying states in the $L \to \infty$ limit. Note that $(-1)^F \equiv e^{i\pi Q^{\mathrm{V}}} = e^{i\pi Q^{\mathrm{A}}}$ is the (nonanomalous) fermion parity that flips the sign of all the fermions. It commutes with both U(1)'s, i.e., $[(-1)^F, Q^{\mathrm{A}}] = [(-1)^F, Q^{\mathrm{V}}] = 0$.

Unquantized axial charge—Although Q^{A} does not commute with Q^{V}, there is another conserved operator that does [20,21],

$$\tilde{Q}^{\mathrm{A}} = \frac{1}{2}\sum_{j=1}^{L}(c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger) = \frac{i}{4}\sum_{j=1}^{L}(a_j b_{j+1} - b_j a_{j+1}) \equiv \sum_{j=1}^{L} \tilde{q}^{\mathrm{A}}_{j+\frac{1}{2}}. \tag{13}$$

As we show in the SM, both Q^{A} and $\tilde{Q}^{\mathrm{A}}$ flow to the same continuum axial charge $\mathcal{Q}^{\mathrm{A}}$. However, the eigenvalues of $\tilde{Q}^{\mathrm{A}}$ are generally irrational and are not quantized.

Furthermore, while $e^{i\lambda\tilde{Q}^{\mathrm{A}}}$ is a conserved unitary operator with $\lambda \in \mathbb{R}$, it fails to send local operators to local operators when $\lambda \sim \mathcal{O}(L)$ (i.e., it is not a locality-preserving unitary). See the SM for further discussion.

The commutator between the local charge densities q_j^{V} and $\tilde{q}^{\mathrm{A}}_{j'+\frac{1}{2}}$ is [46]

$$[q_j^{\mathrm{V}}, \tilde{q}^{\mathrm{A}}_{j'+\frac{1}{2}}] = \frac{i}{2}(\delta_{j,j'} - \delta_{j-1,j'})h_{j'+\frac{1}{2}}, \tag{14}$$

where $h_{j+\frac{1}{2}} = -(i/2)(a_j a_{j+1} + b_j b_{j+1})$ is the Hamiltonian density. This is the lattice avatar of the Schwinger term, which encodes the mixed anomaly of $\mathrm{U}(1)^{\mathcal{V}}$ and $\mathrm{U}(1)^{\mathcal{A}}$ in the continuum.

The simultaneous realization of both vector and axial U(1) symmetries seems to conflict with the well-known Nielsen-Ninomiya theorem. However, neither of the conserved lattice operators Q^{A} and $\tilde{Q}^{\mathrm{A}}$ satisfies both conditions (i) and (ii) mentioned in the introduction: Q^{A} is quantized but does not commute with Q^{V}, while $\tilde{Q}^{\mathrm{A}}$ commutes with Q^{V} but is not quantized (see Table I). Therefore, we cannot define a notion of lattice chirality using them. The fact that these two conditions cannot be met simultaneously is in harmony with the Nielsen-Ninomiya theorem. Relatedly, we do not have $\mathrm{U}(1)^{\mathrm{L}}$ or $\mathrm{U}(1)^{\mathrm{R}}$ symmetries on the lattice (which would have violated the no-go theorem in Ref. [11]) since $\frac{1}{2}(Q^{\mathrm{V}} \pm Q^{\mathrm{A}})$ do not have quantized eigenvalues. In the SM, we compare our construction with others in the literature.

Chiral anomaly from the Onsager algebra—We now discuss the non-Abelian algebra generated by the charges Q^{V} and Q^{A}. Let us define $G_n = (i/2)\sum_j (a_j a_{j+n} - b_j b_{j+n})$ and $Q_n = (i/2)\sum_j a_j b_{j+n}$ with $n \in \mathbb{Z}$. These operators all commute with the Hamiltonian and obey the following closed algebra:

$$[Q_n, Q_m] = iG_{m-n}, \qquad [G_n, G_m] = 0,$$
$$[Q_n, Q_m] = 2i(Q_{n-m} - Q_{n+m}), \tag{15}$$

which is precisely the Onsager algebra [16]. In particular, $Q^{\mathrm{V}} = Q_0$, $Q^{\mathrm{A}} = Q_1$, $\tilde{Q}^{\mathrm{A}} = \frac{1}{2}(Q_1 + Q_{-1})$. While $Q_n = T_b^n Q_0 T_b^{-n}$ have integer eigenvalues, G_n do not.

For a fixed finite n, these operators are sum of local operators, and using (9), we find

$$\lim_{L\to\infty} Q_n = \begin{cases} \mathcal{Q}^{\mathrm{V}} & \text{for } n \text{ even}, \\ \mathcal{Q}^{\mathrm{A}} & \text{for } n \text{ odd}. \end{cases} \tag{16}$$

Therefore, we find an infinite tower of conserved lattice operators, with Q^{A} and $\tilde{Q}^{\mathrm{A}}$ special cases of them, that flow to the same axial charge in the continuum limit. Furthermore, this implies that $\lim_{L\to\infty} G_n = 0$. Thus, the anomalous $[\mathrm{U}(1)^{\mathcal{V}} \times \mathrm{U}(1)^{\mathcal{A}}]/\mathbb{Z}_2$ symmetry in the massless Dirac fermion field theory arises from the Onsager algebra in the staggered fermion lattice model.

TABLE I. Properties of the two lattice operators, both flowing to the same axial charge of a free Dirac fermion in the continuum limit.

Lattice operators	Q^{A}	$\tilde{Q}^{\mathrm{A}}$
Quantized eigenvalues?	✓	
$[\bullet, Q^{\mathrm{V}}] = 0$?		✓
$[\bullet, H] = 0$?	✓	✓
Sum of local terms?	✓	✓

Since the model (4) is integrable, it is expected to have many conserved quantities. However, the majority of these conserved quantities are nonlocal, meaning they cannot be expressed as a sum of operators supported in a finite region, nor do they map local operators to other local operators. The interesting point, however, is that among these conserved operators in the Onsager algebra, Q^{A} (along with Q_n for small, odd n) is local and flows to the axial charge in the continuum, which has an anomaly with the vector charge.

Gauging the vector symmetry—Let us now discuss the fate of the two axial charges after we gauge $\mathrm{U}(1)^{\mathrm{V}}$. The gauged Hamiltonian is a lattice regularization of $1+1$D QED, i.e., the Schwinger model [17]. We introduce a U(1)-valued gauge field $U_{j+\frac{1}{2}}$ and an integer-valued electric field operator $L_{j+\frac{1}{2}}$ on each link. They satisfy $[L_{j+\frac{1}{2}}, U_{j'+\frac{1}{2}}] = \delta_{j,j'} U_{j'+\frac{1}{2}}$. The gauged Hamiltonian is [20,47]

$$H_{\mathrm{V}} = -\sum_j \left(i c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} + \mathrm{H.c} \right) + K \sum_j L^2_{j+\frac{1}{2}}. \tag{17}$$

Furthermore, we impose the Gauss constraint, $L_{j+\frac{1}{2}} - L_{j-\frac{1}{2}} = q_j^{\mathrm{V}} + [(-1)^j/2]$ [48,49]. The Gauss law restricts the Hilbert space to a subsector of fixed Q^{V} charge. Since the quantized axial charge Q^{A} does not commute with Q^{V}, it does not act within the gauge-invariant subspace. Q^{A} is therefore no longer a symmetry when we couple to the dynamical gauge field for $\mathrm{U}(1)^{\mathrm{V}}$. This is analogous to gauging a U(1) subgroup of SU(2): if the S^z symmetry is gauged, the S^x and S^y charges cannot be made gauge-invariant and are, therefore, explicitly broken.

On the other hand, the unquantized axial charge $\tilde{Q}^{\mathrm{A}}$ can be made gauge-invariant as $\tilde{Q}^{\mathrm{A}}(U) = \frac{1}{2}\sum_j (c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} - c_j U^\dagger_{j+\frac{1}{2}} c^\dagger_{j+1})$. Even though it is a gauge-invariant operator acting in the gauged theory, it fails to commute with the gauged Hamiltonian,

$$-i[\tilde{Q}^{\mathrm{A}}(U), H_{\mathrm{V}}] = -\frac{K}{2}\sum_j \{L_{j+\frac{1}{2}}, h_{j+\frac{1}{2}}\}, \tag{18}$$

where $h_{j+\frac{1}{2}}(U) = -i\left(c_j^\dagger U_{j+\frac{1}{2}} c_{j+1} + c_j U_{j+\frac{1}{2}}^\dagger c_{j+1}^\dagger\right)$ is the Hamiltonian density for the fermions. This is the lattice avatar of the Schwinger anomaly of the continuum axial charge $(d/dt)\mathcal{Q}^{\mathrm{A}} = -i[\mathcal{Q}^{\mathrm{A}}, H] = -(1/\pi)\int \mathrm{d}x\, E$ [50]. We conclude that both the quantized Q^{A} and the unquantized $\tilde{Q}^{\mathrm{A}}$ axial symmetries are broken as we gauge the vector $\mathrm{U}(1)^{\mathrm{V}}$ symmetry.

Gauging the axial symmetry—The main advantage of the quantized axial charge Q^{A}, compared to the unquantized one $\tilde{Q}^{\mathrm{A}}$, is that it can be coupled to dynamical (compact) U(1) gauge fields. The gauged Hamiltonian is obtained by conjugating H_{V} with T_b,

$$H_{\mathrm{A}} = \sum_j \left[-\frac{i}{4}(a_j - ib_{j+1}) U_{j+\frac{1}{2}} (a_{j+1} + ib_{j+2}) + \mathrm{h.c.} \right] + K \sum_j L_{j+\frac{1}{2}}^2. \tag{19}$$

The Gauss law is obtained similarly, and found to be $L_{j+\frac{1}{2}} - L_{j-\frac{1}{2}} = q_{j+\frac{1}{2}}^{\mathrm{A}} + [(-1)^j/2]$. We conclude that Q^{V} and Q^{A} can individually be gauged on the lattice and are free of self-anomalies.

A lattice anomaly as an obstruction to gapped phases—Do the lattice axial and vector symmetries have an anomaly? Conventionally, the anomaly of a global symmetry is defined as the obstruction to gauging the symmetry. However, the lattice charges Q^{A} and Q^{V} generate the non-Abelian Onsager algebra (15), which includes highly nonlocal charges, so it is not clear if there is a sensible prescription for gauging it. On the other hand, it has been advocated in Refs. [51–54] that a global symmetry should be called anomalous if there does not exist a trivially gapped phase (i.e., a gapped phase with a nondegenerate ground state and no long-range entanglement) preserving the symmetry. This definition is inspired by the 't Hooft anomaly matching argument and avoids the need to discuss the gauging of said global symmetry. Below, we will show that the lattice axial and vector symmetries together are anomalous in this sense. See Refs. [55–57] for the relation between these two definitions of anomalies.

Concretely, we prove in the SM that local deformations of the Hamiltonian (4) preserving both Q^{V} and Q^{A} are necessarily quadratic in the fermions. In fact, the only Q^{V} and Q^{A} symmetric local deformations are of the form $\sum_{j=1}^{L} -i(c_j^\dagger c_{j+n} + c_j c_{j+n}^\dagger)$, which flow to irrelevant deformations of the Dirac conformal field theory (CFT) in the IR. For a small deformation strength, such terms only renormalize the velocities of the left- and right-moving fermions in the continuum limit. In particular, the quartic Thirring coupling $c_j^\dagger c_j c_{j+1}^\dagger c_{j+1}$ preserves Q^{V} but breaks Q^{A}. This is to be contrasted with its continuum limit; the continuum Thirring coupling $\frac{1}{4}(\bar{\Psi}\Gamma^\mu\Psi)(\bar{\Psi}\Gamma_\mu\Psi) = \Psi_L^\dagger \Psi_L \Psi_R^\dagger \Psi_R$ preserves both the vector and axial symmetries $\mathcal{Q}^{\mathrm{V}}, \mathcal{Q}^{\mathrm{A}}$, and leads to an exactly marginal deformation of the Dirac fermion CFT.

One interesting corollary is that any Hamiltonian that commutes with both Q^{V} and Q^{A} must be gapless. This is reminiscent of the constraint from perturbative anomalies of continuous global symmetries in continuum field theory, which are encoded in the local operator product expansion and, therefore, cannot be matched by a gapped phase. Even when the symmetry is spontaneously broken, there are gapless Goldstone boson modes. The symmetries Q^{V} and Q^{A} present an example of such a gapless constraint on a lattice with a finite-dimensional local Hilbert space.

This constraint is much stronger than the typical discrete anomalies or the Lieb-Schultz-Mattis constraints [58–60], where the low-energy phase is constrained to be either gapless *or* gapped with some nontrivial features, such as degenerate ground states and/or long-range entanglement described by a topological quantum field theory. It is also stronger than the "symmetry-enforced gaplessness" discussed in Refs. [61–67], where a gapped phase with spontaneous broken discrete symmetries (e.g., time-reversal symmetry) remains a possibility. The charges Q^{V} and Q^{A} entirely exclude any gapped phases, leaving the gapless phase as the only possibility.

While the preservation of both Q^{V} and Q^{A} imposes a nontrivial gapless constraint, each of these two conserved charges can individually be coupled to respective gauge fields and is free of anomalies. Moreover, it is possible to deform the gapless Hamiltonian (4) to a trivially gapped phase with a unique vacuum while preserving either Q^{V} or Q^{A} (but not both). If we choose to preserve Q^{V}, while violating Q^{A}, we can add the deformation $\delta H = \sum_j (-1)^j c_j^\dagger c_j$ to open an energy gap. In the continuum, this corresponds to deforming the Lagrangian density by the mass term $\Psi_{\mathrm{R}}^\dagger \Psi_{\mathrm{L}} + \Psi_{\mathrm{L}}^\dagger \Psi_{\mathrm{R}}$ [20,21,49]. On the other hand, the deformation $T_b \delta H T_b^{-1} = \sum_j (-1)^j q_{j+\frac{1}{2}}^{\mathrm{A}}$ preserves Q^{A} but violates Q^{V} and drives the system to a trivially gapped phase. In the continuum, it corresponds to $\Psi_{\mathrm{R}}\Psi_{\mathrm{L}} + \Psi_{\mathrm{L}}^\dagger \Psi_{\mathrm{R}}^\dagger$.

Finally, we note that the charge Q^{V} is on-site in the sense that the local terms q_j^{V} involve fermions only on site j, and it maps a local operator at site j to another at the same location. On the other hand, Q^{A} is not on-site since it smears a local operator at site j to its nearest neighbors. The unitary transformation by T_b^{-1} renders Q^{A} on-site, but this comes at the cost of Q^{V} no longer being on-site. However, there does not seem to be a unitary transformation that makes both Q^{V} and Q^{A} on-site simultaneously, hinting at a mixed anomaly between them. See Refs. [68,69] for related discussions.

Exact winding symmetry in the XX model—It is well-known that upon bosonization, the fermionic Hamiltonian (4) becomes the XX model, $H = \sum_j X_j X_{j+1} + Y_j Y_{j+1}$ [58]. The vector charge Q^{V} becomes $\frac{1}{2}\sum_j Z_j$, the charge of the manifest U(1) spin rotation symmetry of the XX Hamiltonian. In the continuum, this flows to the momentum U(1) symmetry of the $c = 1$ compact boson CFT at radius $R = \sqrt{2}$ (our convention for the radius R is such that $R = 1$ is the self-dual point under T-duality). On the other hand, the axial charge Q^{A} is mapped to a second U(1) symmetry of the XX model, which flows to the winding U(1) symmetry in the continuum; its explicit form on a chain with even L is

$$\frac{1}{4}\sum_{j=1}^{L/2}(X_{2j-1}Y_{2j} - Y_{2j}X_{2j+1}). \tag{20}$$

These two lattice charges first appeared in [16], and were later discussed in [41,70–72] in the context of symmetries of the XX model.

Conclusion—While a lattice model with a finite-dimensional Hilbert space cannot host the exact chiral anomaly, the symmetry generators for the axial and vector symmetries—which in the continuum limit have a mixed anomaly—exist exactly in the staggered fermion lattice model. The quantized charges $Q^{\mathrm{V}}, Q^{\mathrm{A}}$ resemble their continuum counterparts closely: each one of them generates a (compact) U(1) global symmetry and can be gauged. However, they do not commute with each other on the lattice. By contrast, the unquantized axial charge $\tilde{Q}^{\mathrm{A}}$ generates an $\mathbb{R}$ global symmetry, and it is not clear how to couple it to U(1) gauge fields.

It would be interesting to investigate the fate of these charges for interacting fermions, e.g., in $1+1$D QED with multiple flavors of fermions and in QCD, where gauge interactions make the models nonintegrable. Another generalization is to explore quantized axial charges in $3+1$D staggered fermions, which is more phenomenologically relevant. One qualitative difference compared to the $1+1$D case is that the axial symmetry in $3+1$D not only has a mixed anomaly with the vector symmetry but it is also anomalous by itself.

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