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Braided Thompson groups with and without quasimorphisms

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We study quasimorphisms and bounded cohomology of a variety of braided versions of Thompson groups. Our first main result is that the Brin–Dehornoy braided Thompson group bV has an infinite-dimensional space of quasimorphisms and thus infinite-dimensional second bounded cohomology. This implies that, despite being perfect, bV is not uniformly perfect, in contrast to Thompson's group V. We also prove that relatives of bV like the ribbon braided Thompson group rV and the pure braided Thompson group bF similarly have an infinite-dimensional space of quasimorphisms. Our second main result is that, in stark contrast, the close relative of bV denoted by $\widehat{bV}$, which was introduced concurrently by Brin, has trivial second bounded cohomology. This makes $\widehat{bV}$ the first example of a left-orderable group of type F_∞ that is not locally indicable and has trivial second bounded cohomology. This also makes $\widehat{bV}$ an interesting example of a subgroup of the mapping class group of the plane minus a Cantor set that is nonamenable but has trivial second bounded cohomology, behavior that cannot happen for finite-type mapping class groups.

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1 Introduction

The braided Thompson group bV was introduced independently by Brin [2007] and Dehornoy [2006] as a braided version of the classical Thompson group V. This group and its relatives have proven to be important objects in geometric group theory, in particular thanks to their connections to big mapping class groups. Recall that a surface is said to be *of infinite type* if its fundamental group is not finitely generated, and to such a surface one can associate a mapping class group in the same way as for finite-type surfaces; such mapping class groups are called *big*. As an example of the connection, certain braided Thompson groups are dense in the big mapping class group of a compact surface minus a Cantor set [Skipper and Wu 2021, Corollary 3.20], and hence serve as finitely generated "approximations" of these big mapping class groups. For more on connections between braided Thompson groups and big mapping class groups, see eg [Aramayona et al. 2021; Aramayona and Funar 2021; Funar and Kapoudjian 2004; 2008; 2011; Genevois et al. 2022].

Here we are concerned with the question of which braided Thompson groups have an infinite-dimensional space of quasimorphisms, or second bounded cohomology, and which do not. A function $q\colon \Gamma \to \mathbb{R}$ is called a *quasimorphism* if the quantity $|q(g)+q(h)-q(gh)|$ is uniformly bounded; its supremum is called the *defect* of q and is denoted by $D(q)$. We denote by $Q(\Gamma)$ the space of quasimorphisms of Γ, modulo bounded functions (sometimes this notation is used to denote the space of homogeneous quasimorphisms, which is canonically isomorphic [Calegari 2009b, 2.2.2]). We may sometimes colloquially refer to a group as having "no quasimorphisms" if it only has bounded ones. The objects $Q(\Gamma)$ are of great interest in dynamics, geometric group theory, geometric topology and symplectic geometry. For example, they are intimately connected with bounded cohomology [Frigerio 2017] and stable commutator length [Calegari 2009b]. In this context, Thompson-like groups have played an important role: for instance, they have repeatedly served as the first finitely presented examples achieving certain values of stable commutator length [Ghys and Sergiescu 1987; Zhuang 2008; Fournier-Facio and Lodha 2023].

In addition to bV, we inspect the ribbon braided Thompson group rV, the pure braided Thompson group bF, the kernel bP of the projection $bV \to V$, and most importantly the group $\widehat{bV}$, which was introduced by Brin [2007] along with bV. One can view $\widehat{bV}$ as a braided analogue of a Cantor set point stabilizer in V. See Section 2 for the definitions of all these braided Thompson groups. The group $\widehat{bV}$, despite its strong similarities to bV, has extremely different behavior when it comes to quasimorphisms and bounded cohomology, as our two main results make clear:

Theorem 1.1 *For Γ any of the braided Thompson groups bV, rV, bF or bP, the space $Q(\Gamma)$ is infinite-dimensional, and thus also the second bounded cohomology $\mathrm{H}^2_b(\Gamma)$ is infinite-dimensional.*

Theorem 1.2 *We have $\mathrm{H}^2_b(\widehat{bV}) = 0$.*

Here $\mathrm{H}^2_b(\Gamma)$ denotes the second bounded cohomology of a group Γ, with trivial real coefficients. This invariant was introduced by Johnson [1972] and Trauber in the context of Banach algebras, and has since become a fundamental tool in geometric topology [Gromov 1982], dynamics [Ghys 1987] and rigidity theory [Burger and Monod 2002]. For every group Γ there is a map $Q(\Gamma) \to \mathrm{H}^2_b(\Gamma)$, whose kernel is the space of real-valued homomorphisms (Proposition 3.1). Using this, Theorem 1.2, together with the fact that the abelianization of $\widehat{bV}$ is isomorphic to $\mathbb{Z}$ (Corollary 2.14), implies:

Corollary 1.3 *$Q(\widehat{bV})$ is one-dimensional, spanned by the abelianization of $\widehat{bV}$.*

One consequence of Theorem 1.1 is that, despite being perfect [Zaremsky 2018a], bV is not uniformly perfect (Corollary 4.4). Recall that a group Γ is *uniformly perfect* if there exists $N \in \mathbb{N}$ such that every element in Γ can be written as a product of at most N commutators. This is in contrast to the fact that Thompson's group V is uniformly perfect, and even uniformly simple [Gal and Gismatullin 2017] — in fact, $\mathrm{H}^n_b(V) = 0$ for all $n \geq 1$ [Andritsch 2022]. Since bV is not uniformly perfect, the following natural question emerges:

Question 1.4 Which elements of bV have nonzero stable commutator length?

A characterization of this phenomenon in (finite-type) mapping class groups was given in [Bestvina et al. 2016]; see [Field et al. 2022] for some related results for big mapping class groups.

Theorem 1.2 has interesting consequences for subgroups of big mapping class groups. Pioneering work of Bestvina and Fujiwara [2002] showed that every subgroup of a (finite-type) mapping class group is either virtually abelian or has infinite-dimensional $Q(\Gamma)$; see also [Bestvina et al. 2016]. This can be viewed as a sort of Tits-like alternative, since every quasimorphism on an amenable group is at a bounded distance from a homomorphism [Brooks 1981], whereas groups with hyperbolic features typically have an infinite-dimensional space of quasimorphisms [Brooks 1981; Epstein and Fujiwara 1997; Hull and Osin 2013]. The question of whether something similar happens for the big mapping class group $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ for K a Cantor set was listed in the AIM problem list on big mapping class groups [AIM 2019, Question 4.7]. Namely, it is asked whether every subgroup $\Gamma \leq \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ is either amenable or has infinite-dimensional $Q(\Gamma)$. Theorem 1.2 provides a negative answer to this, since $\widehat{bV}$ is nonamenable (by virtue of containing braid groups), and embeds in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$; see Section 4.

In fact, we should point out that a negative answer to this question was already "almost" available in the literature. Indeed, by a result of Calegari and Chen [2021], every countable circularly orderable group Γ embeds in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$, and there are plenty of countable circularly orderable groups that are nonamenable and have a finite-dimensional space of quasimorphisms, or no quasimorphisms at all [Calegari 2007; Zhuang 2008; Fournier-Facio and Lodha 2023]. The most straightforward example is probably Thompson's group T, which has no quasimorphisms by virtue of being uniformly perfect (and even uniformly simple; see eg [Guelman and Liousse 2023]). In fact, when the groups are even left-orderable, many of them have vanishing second bounded cohomology [Fournier-Facio and Lodha 2023], and sometimes even vanishing bounded cohomology in every positive degree [Monod 2022]. As a remark, since the examples coming from the procedure in [Calegari and Chen 2021] act on the plane by fixing a radial coordinate and acting by rotations, which is really a "one-dimensional" picture, one can view $\widehat{bV}$ as providing the first truly "two-dimensional" example, ie one involving genuine braids.

In order to prove Theorem 1.1, we generally follow the approach used by Bavard [2016] to show that $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ has an infinite-dimensional space of quasimorphisms. Her proof in turn makes use of the approach of Bestvina and Fujiwara [2002] to finite-type mapping class groups, following suggestions of Calegari [2009a] from a blog post. Bavard's result prompted the study of analogues of curve graphs for big mapping class groups, and arguably initiated the recent surge of interest in big mapping class groups; see [Aramayona and Vlamis 2020] for more on the history of big mapping class groups.

In the course of proving Theorem 1.2, we also prove that $\widehat{bV}$ is of type F_∞, meaning it has a classifying space with finitely many cells in each dimension (Corollary 2.15); this is a stronger property than finite generation and finite presentability. It is known that bV and thus $\widehat{bV}$ are left-orderable [Ishida 2018], and that $\widehat{bV}$ contains a copy of bV (see Definition 2.9), which is finitely generated and perfect [Zaremsky 2018a]. Therefore $\widehat{bV}$ serves as the first example of a group with the following properties:

Corollary 1.5 *The group $\widehat{bV}$ is a left-orderable group of type F_∞ that is not locally indicable and has vanishing second bounded cohomology.* □

A finitely generated group is *indicable* if it admits a homomorphism onto $\mathbb{Z}$. A group is *locally indicable* if each of its finitely generated subgroups is indicable. The combination of these properties is interesting because it shows that in the celebrated Witte Morris theorem [2006] the hypothesis of amenability cannot be weakened to the vanishing of second bounded cohomology. The first finitely generated examples were found in [Fournier-Facio and Lodha 2023]; those examples have the additional property of being nonindicable, answering a question of Navas [2018]. Since $\widehat{bV}$ is indicable, the existence of type-F_∞ examples with these stronger properties is still open.

We will always stick to the "$n = 2$ case" to avoid getting bogged down in notation, but the reader should note that all of our results can be adapted to the braided Higman–Thompson groups bV_n (as in [Aroca and Cumplido 2022; Skipper and Wu 2023]) and their analogous subgroups $\widehat{bV_n}$, with appropriate small modifications to the arguments. It would be interesting to try and adapt our arguments to other more complicated Thompson-like groups related to asymptotically rigid mapping class groups, eg for positive genus surfaces [Aramayona and Funar 2021] or for higher-dimensional manifolds [Aramayona et al. 2021].

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2 Braided Thompson groups

The first braided Thompson group, which we denote by bV and which has also been denoted by BV, V_{br} and $\mathrm{br}\,V$ in the literature, was introduced independently by Brin [2007] and Dehornoy [2006], as a braided version of Thompson's group V. Other braided Thompson groups include the "F–like" pure braided Thompson groups bF [Brady et al. 2008], various "T–like" braided Thompson groups [Funar and Kapoudjian 2008; 2011; Witzel 2019], braided Higman–Thompson groups bV_n [Aroca and Cumplido 2022; Skipper and Wu 2023], braided Brin–Thompson groups sV_{br} [Spahn 2021], the "ribbon braided" Thompson group rV [Thumann 2017] and braided Röver–Nekrashevych groups $\mathrm{br}\,V_d(G)$ [Skipper and Zaremsky 2023]. Most relevant to our purposes here is a close relative $\widehat{bV}$ of bV, which was also introduced by Brin [2007] (there denoted by $\widehat{BV}$), and realized up to isomorphism as a concrete subgroup of bV by Brady, Burillo, Cleary and Stein [Brady et al. 2008]; see also [Burillo and Cleary 2009].

Let us recall the definitions of bV and $\widehat{bV}$ using the standard braided tree pair model, as in [Brady et al. 2008; Zaremsky 2018a]. By a *tree* we will always mean a finite rooted planar binary tree. An element of bV is represented by a *representative triple* (T_-, β, T_+), where T_- is a tree, T_+ is a tree with the same number of leaves as T_-, say n, and β is a braid in B_n. Elements of bV are equivalence classes

$[T_-, \beta, T_+]$ of representative triples, where the equivalence relation is given by the notion of expansion, which we now describe.

First, denote by $\rho_n \colon B_n \to S_n$ the usual map from the braid group to the symmetric group, recording how the numbering of the strands at the bottom changes when the strands move to the top. (We may write ρ for ρ_n when we do not need to care about n.) Let (T_-, β, T_+) be a representative triple, say with $T_\pm$ having n leaves and $\beta \in B_n$, and let $1 \le k \le n$. Let T'_+ be the tree obtained from T_+ by adding a caret to the k^{th} leaf, let T'_- be the tree obtained from T_- by adding a caret to the $\rho_n(\beta)(k)^{\text{th}}$ leaf, and let $\beta' \in B_{n+1}$ be the braid obtained from β by bifurcating the k^{th} strand (counting at the bottom) into two parallel strands.

Definition 2.1 (expansion, equivalence) With the above setup, call (T'_-, β', T'_+) the k^{th} *expansion* of (T_-, β, T_+). Declare that two representative triples are equivalent if one is an expansion of the other, and extend this to generate an equivalence relation on the set of representative triples.

The elements of the group bV are the equivalence classes $[T_-, \beta, T_+]$, and the group operation is described as follows. Given two elements $[T_-, \beta, T_+]$ and $[U_-, \gamma, U_+]$, up to expansions we can assume that $T_+ = U_-$. Now we define

$$[T_-, \beta, T_+][T_+, \gamma, U_+] := [T_-, \beta\gamma, U_+].$$

Some immediate subgroups of bV include Thompson's group F, which is the subgroup of elements of the form $[T_-, 1, T_+]$, and the pure braided Thompson group bF, which is the subgroup of elements of the form $[T_-, \beta, T_+]$ for β a pure braid. We will also be especially interested in the following subgroup:

Definition 2.2 (the group $\widehat{bV}$) For each $n \in \mathbb{N}$, let $\widehat{B}_n$ denote the standard copy of B_{n-1} inside B_n which only braids the first $n-1$ strands . Note that if (T'_-, β', T'_+) is an expansion of (T_-, β, T_+), say with $\beta \in B_n$ and $\beta' \in B_{n+1}$, then $\beta \in \widehat{B}_n$ if and only if $\beta' \in \widehat{B}_{n+1}$. Thus the equivalence classes $[T_-, \beta, T_+]$ for $\beta \in \widehat{B}_n$ form a well-defined subgroup of bV, denoted by $\widehat{bV}$.

There is a convenient way to picture elements of bV as (equivalence classes of) so-called *strand diagrams*. For an element $[T_-, \beta, T_+]$, we picture T_+ upside-down and below T_-, with β connecting the leaves of T_+ up to the leaves of T_-. See Figure 1 for an example of an element of bV, and an expansion.

To accurately model the equivalence relation coming from expansion, and the group operation, which amounts to stacking strand diagrams, some equivalences between strand diagrams naturally emerge. The three key equivalences are shown in Figure 2.

See [Brady et al. 2008; Zaremsky 2018a] for more details.

2.1 Using pure braids and ribbon braids

Some more subgroups of bV arise when we restrict to pure braids. As before, consider the standard projection $B_n \to S_n$ from the braid group B_n to the symmetric group S_n. The kernel of this map is the *pure braid group* PB_n. This leads us to the following definition of the pure braided Thompson group bF:

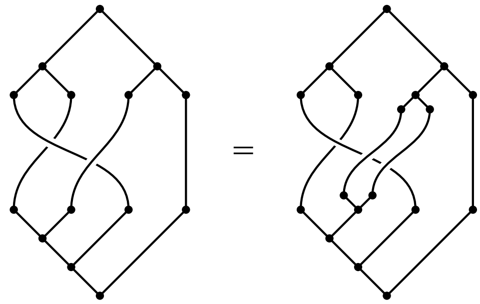

Figure 1: An element $[T_-, \beta, T_+]$ of bV. We draw T_+ upside down, with β as a braid from the leaves of T_+ up to the leaves of T_-. We have $\rho_4(\beta) = (1\ 2\ 3)$, so $\rho_4(\beta)(2) = 3$. Thus to perform the 2nd expansion, we add a caret to the 2nd leaf of T_+, a caret to the 3rd leaf of T_-, and bifurcate the 2nd strand of β (counting from the bottom) into two strands. Note that this element lies in the subgroup $\widehat{bV}$ since the rightmost strand does not braid with any of the others.

Definition 2.3 (the group bF) If (T'_-, β', T'_+) is an expansion of (T_-, β, T_+) then β' is pure if and only if β' is pure, so the equivalence classes $[T_-, \beta, T_+]$ for $\beta \in PB_n$ form a well-defined subgroup of bV, denoted by bF.

Note that bF is not normal in bV, but the following related subgroup is:

Definition 2.4 (the group bP) Let bP denote the subgroup of bV consisting of all $[T, \beta, T]$ such that β is pure.

For reference, the group bP was denoted by PBV in [Brady et al. 2008] and by P_{br} in [Zaremsky 2018a]. Note that the trees in $[T, \beta, T]$ must be the same, so bP is strictly smaller than bF. The quotient bV/bP is isomorphic to Thompson's group V, and the quotient bF/bP is isomorphic to Thompson's group F [Brady et al. 2008]. More precisely, upon passing to a quotient with kernel bP, the elements of bV change from being represented by triples (T_-, β, T_+) for $\beta \in B_n$ to being represented by triples (T_-, σ, T_+) for $\sigma \in S_n$. The notion of expansion has an obvious analogue for permutations, and we get equivalence classes $[T_-, \sigma, T_+]$, which are the elements of V. The image of $\widehat{bV}$ under the projection $bV \to V$ is a group called $\widehat{V}$, which was also considered in [Brin 2007]. The kernel $\widehat{bV} \cap bP$ of the projection $\widehat{bV} \to \widehat{V}$ has the following interesting property, which will be useful later:

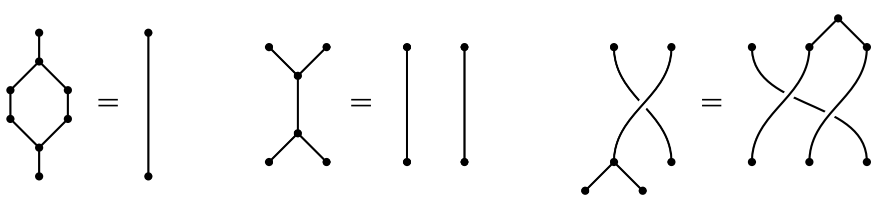

Figure 2: The three key equivalences for strand diagrams, which can occur anywhere inside a strand diagram representing an element of bV. Further equivalences are obtained by combining these, and by rotating and reflecting the one on the right.

Lemma 2.5 *There is an epimorphism $\widehat{bV} \cap bP \to bP$.*

Proof An element of $\widehat{bV} \cap bP$ is of the form $[T, \beta, T]$ for β a pure braid in which the rightmost strand does not braid with any of the others. Let T' be the subtree of T whose root is the left child of the root of T (or if T is trivial, just take T' to also be trivial). Let β' be the (pure) braid obtained from β by deleting any strands corresponding to leaves of T that are closer to the right child of the root than the left (so in particular, in the nontrivial case this includes the rightmost strand). Intuitively, $[T', \beta', T']$ is obtained by taking just the "left part" of $[T, \beta, T]$, from the point of view of the root of T. This operation is well defined up to expansions, and yields a well-defined homomorphism $[T, \beta, T] \to [T', \beta', T']$ from $\widehat{bV} \cap bP$ to bP, which is clearly surjective. □

Finally, let us discuss a "twisted" version of bV, called the *ribbon braided Thompson group rV*. This arises by treating the strands in a strand diagram as ribbons, which are allowed to twist. This first appeared officially in work of Thumann [2017, Section 3.5.3], where he proved that rV (there denoted by RV) is of type F_∞. The idea of using ribbons to represent strands in bV was actually already present in Brin's original paper [2007], but without twisting. We will mostly follow the approach from [Zaremsky 2018b, Example 4.2], which uses the notion of cloning systems from [Witzel and Zaremsky 2018] to provide a framework for elements of rV similar to the one we are using here for bV. An element of rV is represented by a triple $(T_-, \beta(m_1, \dots, m_n), T_+)$ where T_- and T_+ are trees with n leaves and $\beta(m_1, \dots, m_n) \in B_n \wr \mathbb{Z}$. More precisely, $\beta \in B_n$, $m_1, \dots, m_n \in \mathbb{Z}$ and $B_n \wr \mathbb{Z}$ denotes the wreath product $B_n \ltimes \mathbb{Z}^n$ with the action induced by the standard projection $B_n \to S_n$. (We write our wreath products with the acting group on the left, for convenience. Also, we may sometimes write $\beta(0, \dots, 0)$ as β and $1_{B_n}(m_1, \dots, m_n)$ as $(m_1, \dots, m_n)$ for the sake of notational elegance.) An *expansion* of this triple is another triple of the form

$$(T_-', \beta' s_k^{m_k}(m_1, \dots, m_{k-1}, m_k, m_k, m_{k+1}, \dots, m_n), T_+'),$$

where T_+' is T_+ with a caret added to the k^{th} leaf for some $1 \le k \le n$, β' is β with its k^{th} ribbon bifurcated into two parallel ribbons, and T_-' is T_- with a caret added to the $\rho(\beta)(k)^{\text{th}}$ leaf. Here s_k is the k^{th} standard generator of B_n, in the standard presentation

$$B_n = \langle s_1, \dots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i, \text{ and } s_i s_j = s_j s_i \text{ for all } i \text{ and } j \text{ with } |i-j| > 1 \rangle.$$

Let us adopt the convention that s_k crosses the k^{th} ribbon (counting at the bottom) under the $(k+1)^{\text{st}}$ ribbon, and a positive single twist of a ribbon involves the left side of the ribbon (looking at the bottom) twisting under the right side. These conventions make the definition of expansion look somewhat natural; see Figure 3.

By taking the equivalence relation generated by expansion, we get equivalence classes of the form $[T_-, \beta(m_1, \dots, m_n), T_+]$, which comprise the group rV. Just like in bV, the group operation is given, roughly, by first expanding until the right tree of the left element equals the left tree of the right element

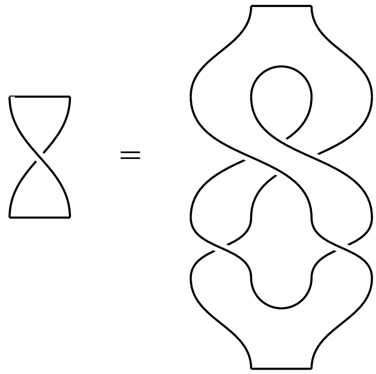

Figure 3: Expansion in rV. Here we see that $[\cdot, 1_{B_1}(1), \cdot] = [\wedge, s_1(1,1), \wedge]$, where $\cdot$ is the trivial tree and $\wedge$ is a single caret.

and then canceling these trees. We could consider various subgroups of rV by restricting to pure braids and/or full twists, but for our purposes we will just stick with all braids and all twists.

At this point we have

$$bP < bF < bV < rV,$$

where we view bV as the subgroup of rV consisting of elements $[T_-, \beta(0,\dots,0), T_+]$, that is, elements with no twisting. As we have said, bP is normal in bF and bV, and in fact it is even normal in rV, as we now show:

Lemma 2.6 *The subgroup bP is normal in rV.*

Proof Let $[U,\gamma,U] \in bP$ and $[T_-, \beta(m_1,\dots,m_n), T_+] \in rV$, expanding so that without loss of generality $U = T_+$. Then

$$\begin{aligned}
&[T_-, \beta(m_1,\dots,m_n), T_+][T_+,\gamma,T_+][T_-, \beta(m_1,\dots,m_n), T_+]^{-1} \\
&\qquad = [T_-, \beta(m_1,\dots,m_n), T_+][T_+,\gamma,T_+][T_+, (-m_1,\dots,-m_n)\beta^{-1}, T_-] \\
&\qquad = [T_-, \beta(m_1,\dots,m_n)\gamma(-m_1,\dots,-m_n)\beta^{-1}, T_-] = [T_-, \beta\gamma\beta^{-1}, T_-] \in bP.
\end{aligned}$$

The last equals sign holds because γ is pure, and hence $(m_1,\dots,m_n)\gamma = \gamma(m_1,\dots,m_n)$. □

As we have said, the quotients bV/bP and bF/bP are isomorphic to V and F, respectively. The quotient rV/bP is isomorphic to a Thompson-like group constructed analogously to rV but using $S_n \wr \mathbb{Z}$ instead of $B_n \wr \mathbb{Z}$; this could be made more precise by putting a cloning system, in the sense of [Witzel and Zaremsky 2018], on the family of groups $S_n \wr \mathbb{Z}$, but we will not need to worry about this here. Indeed, all we will need to use rV/bP for later is to relate quasimorphisms of rV to quasimorphisms of bV, bF and bP, and for this all we need to know about it is the following:

Lemma 2.7 *The quotient rV/bP is uniformly perfect.*

Proof Note that $bV/bP \cong V$ is uniformly perfect [Gal and Gismatullin 2017]. Choose $N \in \mathbb{N}$ such that every element of V is a product of at most N commutators. Set $M = 3N + 2$. We claim that every element of rV/bP is a product of at most M commutators. Let $[T_-, \beta(m_1, \dots, m_n), T_+] \in rV$, and write it as a product of three elements:

$$[T_-, \beta, T_+][T_+, (m_1, 0, \dots, 0), T_+][T_+, (0, m_2, \dots, m_n), T_+].$$

Modulo bP, we know that this first factor is a product of at most N commutators. The second factor is conjugate to $[T_+, (0, m_1, 0, \dots, 0), T_+]$ via the conjugator $[T_+, s_1, T_+]$, and this is of the same form as the third factor. Thus it suffices to focus on the third factor, and show that any element of the form $g = [T, (0, m_2, \dots, m_n), T]$ is, modulo bP, a product of at most $N + 1$ commutators.

Let T' be T with $n-1$ new carets added, one after the other, always attaching each new caret to the leftmost leaf. Thus $g = [T', (0, \dots, 0, m_2, \dots, m_n), T']$, where the number of 0s is n. Let T'' be T with $n-1$ new carets, one on each leaf other than the leftmost. Thus $g = [T'', \gamma(0, m_2, m_2, m_3, m_3, \dots, m_n, m_n), T'']$ for $\gamma \in B_{2n-1}$ the braid that arises from performing this expansion, namely $\gamma = s_2^{m_2} s_4^{m_3} \cdots s_{2n-2}^{m_n}$. Setting $h = [T'', \gamma, T''] \in bV$ we get $h^{-1}g = [T'', (0, m_2, m_2, m_3, m_3, \dots, m_n, m_n), T'']$. Now let $\alpha \in B_{2n-1}$ be any braid satisfying $\alpha(0, \dots, 0, m_2, \dots, m_n)\alpha^{-1} = (0, m_2, 0, m_3, \dots, 0, m_n, 0)$ in $B_{2n-1} \wr \mathbb{Z}$, and set $a = [T'', \alpha, T']$. We get

$$\begin{aligned} aga^{-1} &= [T'', \alpha, T'][T', (0, \dots, 0, m_2, \dots, m_n), T'][T', \alpha^{-1}, T''] \\ &= [T'', \alpha(0, \dots, 0, m_2, \dots, m_n)\alpha^{-1}, T''] = [T'', (0, m_2, 0, m_3, \dots, 0, m_n, 0), T'']. \end{aligned}$$

Hence $h^{-1}gag^{-1}a^{-1} = [T'', (0, 0, m_2, 0, m_3, \dots, m_{n-1}, 0, m_n), T'']$. Now using a similar trick as when we conjugated by a, this is conjugate to $[T', (0, \dots, 0, m_2, \dots, m_n), T']$, which equals g. Thus g is conjugate to $h^{-1}gag^{-1}a^{-1}$, and considered modulo bP this is an element of V times a commutator, so we are done. □

It is worth recording the following consequence:

Corollary 2.8 *The group rV is perfect.*

Proof We already know bV is perfect [Zaremsky 2018a], so the derived subgroup rV' contains bV. In particular it contains bP, and so rV/rV' is a quotient of rV/bP. This is perfect by Lemma 2.7, so we conclude that $rV = rV'$. □

2.2 Algebraic properties of $\widehat{bV}$

Our proof of Theorem 1.2 will rely on some algebraic properties of bV and $\widehat{bV}$, which are the focus of this subsection.

Definition 2.9 (right depth, $\widehat{bV}(1)$) Say that the *right depth* of a tree is the distance from its rightmost leaf to its root. Denote by $\widehat{bV}(1) \le \widehat{bV}$ the subgroup of elements that admit a representative of the form (T_-, β, T_+) such that T_- and T_+ both have right depth 1.

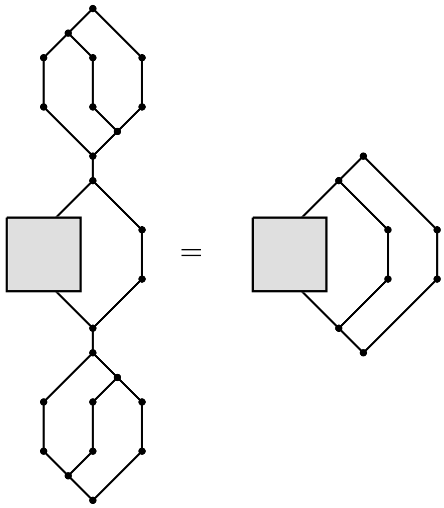

Figure 4: The proof of Lemma 2.11: conjugating an element of $\widehat{bV}(1)$ by x_0 yields another element of $\widehat{bV}(1)$.

Note that $\widehat{bV}(1)$ is naturally isomorphic to bV. Indeed, we have an isomorphism $bV \to \widehat{bV}(1)$ given by $[T_-, \beta, T_+] \to [U_-, \gamma, U_+]$, where U_- is obtained from T_- by adding a new caret whose left leaf is the root of T_-, U_+ is obtained from T_+ by adding a new caret whose left leaf is the root of T_+, and γ is obtained from β by adding one new unbraided strand on the right. This is also discussed in [Brady et al. 2008].

Definition 2.10 (homomorphism χ_1, subgroup $\widehat{D}$) Let $\chi_1 \colon \widehat{bV} \to \mathbb{Z}$ be the homomorphism sending $[T_-, \beta, T_+]$ to the right depth of T_- minus the right depth of T_+. Since expansions preserve this measurement, thanks to the rightmost strand of such a β not braiding, this is well defined, and is clearly a homomorphism. Denote by $\widehat{D}$ the kernel in $\widehat{bV}$ of χ_1.

We call this map χ_1 since its restriction to Thompson's group $F \leq \widehat{bV}$ coincides with a map usually denoted by χ_1. Note that $\widehat{D}$ consists of all $[T_-, \beta, T_+] \in \widehat{bV}$ such that T_- and T_+ have the same right depth. In particular $\widehat{D}$ contains $\widehat{bV}(1)$. We will see in Corollary 2.14 that $\widehat{D}$ equals the derived subgroup of $\widehat{bV}$.

Recall the usual first generator x_0 of Thompson's group F. This is the element $x_0 = [T_2, 1, T_1]$, where T_i is the tree consisting of a caret with a caret attached to its i^{th} leaf, and 1 is the identity in B_3. Note that $\chi_1(x_0) = 1$. Also note that $x_0^{-1} = [T_1, 1, T_2]$.

Lemma 2.11 *We have* $x_0^{-1} \cdot \widehat{bV}(1) \cdot x_0 \leq \widehat{bV}(1)$.

Proof This is clear using strand diagrams; see Figure 4. In the figure, we represent an element of $\widehat{bV}(1)$ by drawing the first carets of each tree and the last (unbraided) strand of the braid, and then drawing a gray box to represent the arbitrary remainder of the picture. Now conjugating by x_0 and applying some of the equivalence moves from Figure 2, we see that in the resulting strand diagram the trees again have right depth 1 and the rightmost strand is unbraided (in fact the two rightmost strands are both unbraided). $\square$

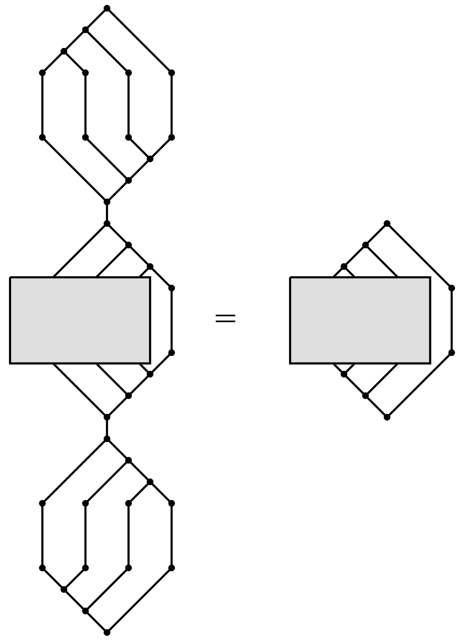

Figure 5: An example of the proof of Lemma 2.12 (for A having just one element): conjugating an element of $\widehat{D}$ in which the trees have right depth 3 by x_0^2 yields an element of $\widehat{bV}(1)$.

Lemma 2.12 *For any finite subset A of $\widehat{D}$, there exists $k \geq 0$ such that $x_0^{-k} \cdot A \cdot x_0^k \leq \widehat{bV}(1)$.*

Proof Since A is finite, we can choose $k \geq 0$ such that every element of A can be represented by a triple (T_-, β, T_+) in which the right depth of T_- (and thus T_+) is at most $k+1$. For any such (T_-, β, T_+), it is clear that $x_0^{-k} \cdot [T_-, \beta, T_+] \cdot x_0^k \in \widehat{bV}(1)$. See Figure 5 for an example. $\square$

Corollary 2.13 *The group $\widehat{bV}$ is isomorphic to an ascending HNN-extension of bV.*

Proof To get our result, we will verify the conditions in [Geoghegan et al. 2001, Lemma 3.1] using $\widehat{bV}(1)$ (which is isomorphic to bV) as the base and x_0 as the stable letter. Clearly no nontrivial power of x_0 lies in $\widehat{bV}(1)$. Lemma 2.11 shows that $x_0^{-1} \cdot \widehat{bV}(1) \cdot x_0 \leq \widehat{bV}(1)$. Finally, we need to show that $\widehat{bV}$ is generated by $\widehat{bV}(1)$ and x_0. Given $[T_-, \beta, T_+] \in \widehat{bV}$, up to right multiplication by a power of x_0 we can assume that $[T_-, \beta, T_+] \in \widehat{D}$, ie T_- and T_+ have the same right depth. Now Lemma 2.12 says we can conjugate by some power of x_0 so that our element lands in $\widehat{bV}(1)$. $\square$

Corollary 2.14 *The derived subgroup $\widehat{bV}'$ equals $\widehat{D}$, so the abelianization of $\widehat{bV}$ is $\mathbb{Z}$, given by the map χ_1.*

Proof Since $\widehat{D}$ is the kernel of a map to $\mathbb{Z}$, it contains $\widehat{bV}'$. Conversely, since $\widehat{bV}(1)$ is isomorphic to bV, and bV is perfect [Zaremsky 2018a], Lemma 2.12 implies that any element of $\widehat{D}$ is conjugate in $\widehat{bV}$ to an element of a perfect subgroup of $\widehat{bV}$, which shows that every element of $\widehat{D}$ lies in $\widehat{bV}'$. This shows $\widehat{bV}' = \widehat{D}$, and the second statement follows since $\widehat{D}$ is the kernel of χ_1 in $\widehat{bV}$. $\square$

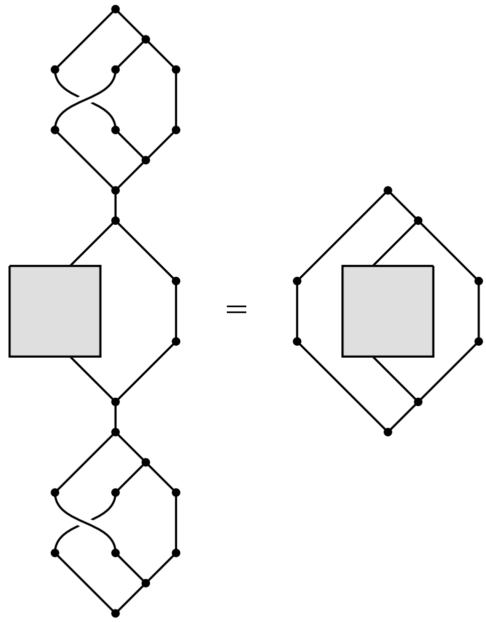

Figure 6: An arbitrary conjugate of an element of $\widehat{bV}(1)$ by g. We see that it will commute with any element of $\widehat{bV}(1)$.

Brin [2006] showed that bV and $\widehat{bV}$ are finitely presented. In fact, bV is even of type F_∞ [Bux et al. 2016]. The techniques in [loc. cit.] could likely be used to show that $\widehat{bV}$ is also of type F_∞, but now, thanks to Corollary 2.13, we can prove this much more quickly:

Corollary 2.15 *The group $\widehat{bV}$ is of type* F_∞.

Proof It is a standard fact that an ascending HNN-extension of a group of type F_n is itself of type F_n; see eg [Baumslag et al. 1980, end of Section 2]. Since bV is of type F_∞ [Bux et al. 2016], Corollary 2.13 implies that $\widehat{bV}$ is as well. □

The key dynamical feature that will make bounded cohomology vanish is contained in the following lemma:

Lemma 2.16 *There exists $g \in \widehat{D}$ such that every element of $\widehat{bV}(1)$ commutes with every element of $g^{-1} \cdot \widehat{bV}(1) \cdot g$.*

Proof We define $g = [T_2, s_1, T_2]$, where as before T_2 is a caret with a second caret hanging on the right, and s_1 is the first standard generator of B_3, ie the element braiding the first two strands with a single twist. Since β does not braid the rightmost strand we have $g \in \widehat{bV}$, and since clearly $\chi_1([T_2, \beta, T_2]) = 0$ we have $g \in \widehat{D}$. We see in Figure 6 that, in any element of $g^{-1} \cdot \widehat{bV}(1) \cdot g$, the trees both have "left depth" 1 and the first strand does not braid with anything. Since elements of $\widehat{bV}(1)$ and $g^{-1} \cdot \widehat{bV}(1) \cdot g$ therefore braid disjoint sets of strands, they commute. □

3 Second bounded cohomology

We will work only with bounded cohomology with trivial real coefficients, and use the definition in terms of the bar resolution. We refer the reader to [Brown 1982; Frigerio 2017] for a general and complete treatment of ordinary and bounded cohomology of discrete groups, respectively. For the more general setting of locally compact groups, we refer the reader to [Monod 2001].

For every $n \geq 0$, denote by $\mathrm{C}^n(\Gamma)$ the set of real-valued functions on Γ^n. By convention, Γ^0 is a single point, so $\mathrm{C}^0(\Gamma) \cong \mathbb{R}$ consists only of constant functions. We define differential operators $\delta^\bullet \colon \mathrm{C}^\bullet(\Gamma) \to \mathrm{C}^{\bullet+1}(\Gamma)$ by $\delta^0 = 0$ and, for $n \geq 1$,

$$\delta^n(f)(g_1, \dots, g_{n+1}) = f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n).$$

One can check that $\delta^{\bullet+1}\delta^\bullet = 0$, so $(\mathrm{C}^\bullet(\Gamma), \delta^\bullet)$ is a cochain complex. We denote by $\mathrm{Z}^\bullet(\Gamma) := \ker(\delta^\bullet)$ the set of *cocycles*, and by $\mathrm{B}^\bullet(\Gamma) := \mathrm{im}(\delta^{\bullet-1})$ the set of *coboundaries*. The quotient $\mathrm{H}^\bullet(\Gamma) := \mathrm{Z}^\bullet(\Gamma)/\mathrm{B}^\bullet(\Gamma)$ is the *cohomology of* Γ *with trivial real coefficients*. We will also call this the *ordinary cohomology* to make a clear distinction from the bounded one, which we proceed to define.

Restricting to functions $f : \Gamma^\bullet \to \mathbb{R}$ that are bounded, meaning that their supremum $\|f\|_\infty$ is finite, leads to a subcomplex $(\mathrm{C}_b^\bullet(\Gamma), \delta^\bullet)$. We denote by $\mathrm{Z}_b^\bullet(\Gamma)$ the *bounded cocycles* and by $\mathrm{B}_b^\bullet(\Gamma)$ the *bounded coboundaries*. The vector space $\mathrm{H}_b^\bullet(\Gamma) := \mathrm{Z}_b^\bullet(\Gamma)/\mathrm{B}_b^\bullet(\Gamma)$ is the *bounded cohomology of* Γ with trivial real coefficients.

The inclusion of the bounded cochain complex into the ordinary one induces a linear map at the level of cohomology, called the *comparison map*:

$$c^\bullet \colon \mathrm{H}_b^\bullet(\Gamma) \to \mathrm{H}^\bullet(\Gamma).$$

This map is in general neither injective nor surjective. In degree 2, the kernel admits a description in terms of quasimorphisms:

Proposition 3.1 [Calegari 2009b, Theorem 2.50] *Let $Q(\Gamma)$ denote the space of quasimorphisms on Γ up to bounded distance, and $\mathrm{Z}^1(\Gamma)$ the space of homomorphisms $\Gamma \to \mathbb{R}$. Then the sequence*

$$0 \to \mathrm{Z}^1(\Gamma) \to Q(\Gamma) \xrightarrow{[\delta^1(-)]} \mathrm{H}_b^2(\Gamma) \xrightarrow{c^2} \mathrm{H}^2(\Gamma)$$

is exact. In particular, c^2 is injective if and only if every quasimorphism on Γ is at a bounded distance from a homomorphism.

While many applications of bounded cohomology in geometric group theory, eg the study of stable commutator length, are only concerned with quasimorphisms, in different settings the full knowledge of H_b^2 is of interest. Notable instances include the classification of circle actions [Ghys 1987], and the construction of manifolds with prescribed simplicial volume [Heuer and Löh 2021; 2023].

In order to prove Theorem 1.2, that $\widehat{bV}$ has vanishing second bounded cohomology, it is enough to prove this for the subgroup $\widehat{D}$ (from Definition 2.10), thanks to the following fact:

Proposition 3.2 ([Monod 2001, 8.6]; see also [Monod and Popa 2003]) *Let $n \geq 0$. Let Γ be a group and N a normal subgroup such that Γ/N is amenable. Then the inclusion $N \to \Gamma$ induces an injection in bounded cohomology $\mathrm{H}_b^n(\Gamma) \to \mathrm{H}_b^n(N)$. In particular, if $\mathrm{H}_b^n(N) = 0$ then $\mathrm{H}_b^n(\Gamma) = 0$.*

To prove vanishing of $\mathrm{H}_b^2(\widehat{D})$, we will use the following notion:

Definition 3.3 Let Γ be a group. We say that Γ has *commuting conjugates* if for every finitely generated subgroup $H \leq \Gamma$ there exists $g \in \Gamma$ such that every element of H commutes with every element of $g^{-1}Hg$.

Theorem 3.4 [Fournier-Facio and Lodha 2023] *If Γ is a group with commuting conjugates, then $\mathrm{H}_b^2(\Gamma) = 0$.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2 Since $\widehat{bV}/\widehat{D} \cong \mathbb{Z}$ by definition, using Proposition 3.2 it suffices to show that $\mathrm{H}_b^2(\widehat{D}) = 0$. By Theorem 3.4 it suffices to show that $\widehat{D}$ has commuting conjugates. Let $H \leq \widehat{D}$ be a finitely generated subgroup. By Lemma 2.12 there exists $k \geq 0$ such that $x_0^{-k} \cdot H \cdot x_0^k \leq \widehat{bV}(1)$. Then by Lemma 2.16 there exists $g \in \widehat{D}$ such that every element of $g^{-1} \cdot x_0^{-k} \cdot H \cdot x_0^k \cdot g$ commutes with every element of $\widehat{bV}(1)$, and so in particular with every element of $x_0^{-k} \cdot H \cdot x_0^k$. Thus, every element of the conjugate of H by $x_0^k \cdot g \cdot x_0^{-k}$ commutes with every element of H. Finally, note that $x_0^k \cdot g \cdot x_0^{-k} \in \widehat{D}$ since $g \in \widehat{D}$, $x_0 \in \widehat{bV}$ and $\widehat{D}$ is normal in $\widehat{bV}$. This shows that $\widehat{D}$ has commuting conjugates and concludes the proof. □

4 Quasimorphisms on rV and bV

In this section we prove Theorem 1.1. We will first work with the ribbon braided Thompson group rV and prove that $Q(rV)$ is infinite-dimensional (Proposition 4.2), and then prove that unbounded quasimorphisms of rV restrict to unbounded quasimorphisms of bV, bF and bP (and indeed, any Γ satisfying $bP \leq \Gamma \leq rV$).

First we need to make the connection between rV and $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. This was done implicitly in [Aramayona and Funar 2021; Funar and Kapoudjian 2004], and more explicitly in [Skipper and Wu 2021, Theorem 3.24]. In short, rV is isomorphic to a certain subgroup of mapping classes of $S^2 \setminus K$, namely those that are "asymptotically quasirigid" with respect to some "rigid structure" involving choices of "admissible subsurfaces" and only act on half of $S^2 \setminus K$ in some sense; see [Skipper and Wu 2021, Definition 3.7] for all the details. We can view this as describing certain mapping classes of $D^2 \setminus K$ that do not require the boundary of D^2 to be fixed, but rather allow it to be half-twisted. For an example providing the intuition for how to view an element of bV as a mapping class, see Figure 7.

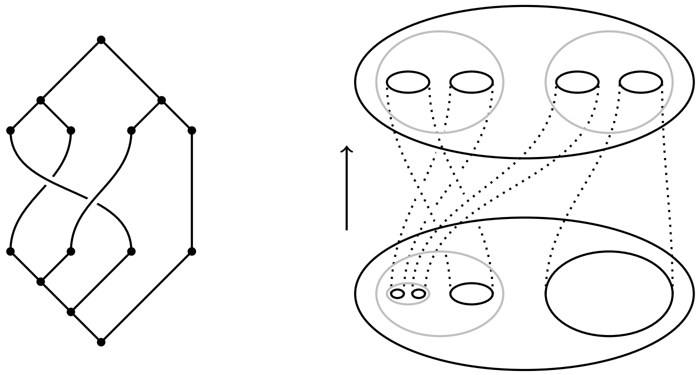

Figure 7: A visualization of the element $[T_-, \beta, T_+]$ of bV from Figure 1 as a mapping class on the disk. The bottom (domain) tree T_+ describes a decomposition of D^2 into the pieces shown, and the top (range) tree T_- describes another such decomposition. The braid β then treats the four smallest subdisks in the domain as "holes", with the range viewed similarly, and gives a homeomorphism from the former to the latter, indicated by the dotted lines. This element does not involve twists, but one could picture the holes twisting as well, yielding an element of rV.

Now we pass from this picture to $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ by viewing $D^2 \setminus K$ inside $\mathbb{R}^2 \setminus K$ at the expense of modding out the cyclic subgroup generated by a full twist around the boundary of D^2. This is represented by the element $[\,\cdot\,, 1_{B_1}(2), \cdot\,]$ of rV, which generates the center $Z(rV)$, so at this point we have embedded $rV/Z(rV)$ inside $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. In particular, we can work in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ to prove that $Q(rV/Z(rV))$ is infinite-dimensional, from which it will immediately follow that $Q(rV)$ is as well. It is also worth mentioning that $bV \cap Z(rV) = \{1\}$, so this provides an explicit embedding of bV into $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$.

4.1 Quasimorphisms on rV

The proof that $Q(rV/Z(rV))$ is infinite-dimensional closely follows Bavard's proof [2016, Théorème 4.8] that $Q(\mathrm{MCG}(\mathbb{R}^2 \setminus K))$ is infinite-dimensional.[1] We will especially use the constructions from [Bavard 2016, Section 4.1].

Bavard [2016] constructs the so-called *ray graph* X_r associated to the surface $\mathbb{R}^2 \setminus K$, and shows that it is hyperbolic. She proceeds to show that the action of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ on X_r satisfies the hypotheses of Bestvina and Fujiwara's main theorem [2002], which implies that $Q(\mathrm{MCG}(\mathbb{R}^2 \setminus K))$ is infinite-dimensional. To prove our Proposition 4.2, we will show that the action of $rV/Z(rV)$ also satisfies these properties, and make reference to [Bavard 2016, Section 4] throughout.

We start by reviewing Bavard's proof for $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. By the main theorem of [Bestvina and Fujiwara 2002], it suffices to exhibit elements $h_1, h_2 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ with the following properties:

(1) h_1 and h_2 are hyperbolic elements for the action of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ on X_r, acting by translation on axes l_1 and l_2, which are equipped with the orientation of the action of the respective elements.

[1] This is Theorem 4.9 in the English translation.

(2) h_1 and h_2 are independent, meaning that their fixed point sets in ∂X_r are disjoint.

(3) There exist constants B and C such that for every segment w of l_2 longer than C, for every $g \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$, if the segment $g \cdot w$ is contained in the B–neighborhood of l_1, then it is oriented in the opposite direction.

Identify K with the set $\{0,1\}^{\mathbb{N}}$ of infinite words κ in the alphabet $\{0,1\}$, and for each finite word $w \in \{0,1\}^*$ let $K(w) := \{w\kappa \mid \kappa \in K\}$ be the *cone* corresponding to w. Then define

$$K_0 = K(00), \quad K_1 = K(010), \quad K_2 = K(0110), \quad \dots, \quad K_\infty = \{0\bar{1}\},$$
$$K_{-1} = K(11), \quad K_{-2} = K(101), \quad K_{-3} = K(1001), \quad \dots, \quad K_{-\infty} = \{1\bar{0}\}.$$

This provides a partition of K into sets K_i for $-\infty \le i \le \infty$, where each K_i for $i \in \mathbb{Z}$ is a clopen set and each $K_{\pm\infty}$ contains one point.

Let us now be more precise about how we would like K to live inside of $\mathbb{R}^2$. Assume that K lies on the horizontal axis $\mathbb{R}$ and is symmetric around $0 \notin K$, and that $K_i \subset \mathbb{R}_{<0}$ for all $0 \le i \le \infty$ and $K_i \subset \mathbb{R}_{>0}$ for all $-\infty \le i \le -1$. Let $I \subset \mathbb{R}$ be a symmetric open neighborhood of 0 that is disjoint from K. Finally, let $\mathcal{C} \subseteq \mathbb{R}^2$ be a homeomorphic copy of a circle, formed as the union of a segment in the horizontal axis $\mathbb{R}$ containing all of K and a semicircle in the upper half-plane. Let ϕ denote the homeomorphism of $\mathbb{R}^2$ that is a half-turn rotation about the origin, so ϕ stabilizes K, and denote by ϕ the mapping class of ϕ in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$.

Theorem 4.1 [Bavard 2016, Théorème 4.8] *Let $\tilde{t}_1$ be any homeomorphism of $\mathbb{R}^2$ that stabilizes $\mathcal{C}$, restricts to the identity on I and sends K_i to K_{i+1} for each $i \in \mathbb{Z}$. Let $t_1 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ be the class of $\tilde{t}_1$, let $t_2 := \phi t_1 \phi^{-1}$, let $h_1 := t_1 t_2 t_1$ and let $h_2 := \phi h_1^{-1} \phi^{-1}$. Then the elements h_1 and h_2 satisfy the three properties above, and hence any subgroup of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ containing h_1 and h_2 has an infinite-dimensional space of quasimorphisms.*

As we have seen, rV maps to $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ with kernel $Z(rV) \cong \mathbb{Z}$. Note that the image in $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ of the element $[\,\cdot\,, 1_{B_1}(1), \cdot\,] \in rV$, which is a single half-twist on one ribbon, is precisely the mapping class ϕ.

Proposition 4.2 *The space $Q(rV)$ is infinite-dimensional.*

Proof We will prove that $Q(rV/Z(rV))$ is infinite-dimensional, which implies our result. By Theorem 4.1 it suffices to show that the elements h_1 and h_2 can be realized inside of $rV/Z(rV)$. Since each of h_1 and h_2 is obtained as a product of conjugates of t_1 by ϕ and since $\phi \in rV/Z(rV)$, it suffices to show that t_1 can be realized inside $rV/Z(rV)$.

Recall that, in Theorem 4.1, the homeomorphism $\tilde{t}_1$ representing t_1 can be any homeomorphism of $\mathbb{R}^2$ satisfying

(1) $\tilde{t}_1$ stabilizes the topological circle $\mathcal{C}$,

(2) $\tilde{t}_1|_I$ is the identity, and

(3) $\tilde{t}_1(K_i) = K_{i+1}$ for each $i \in \mathbb{Z}$.

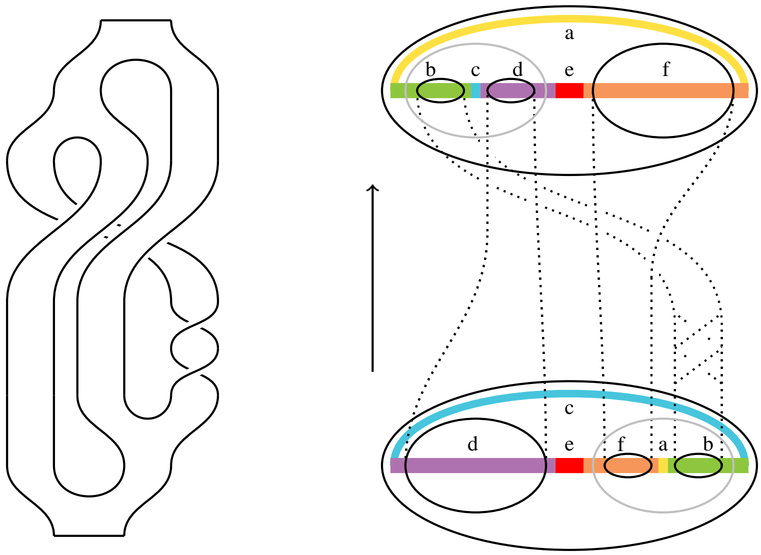

Figure 8: The desired element. Left: a ribbon strand diagram representing this element. Right: the corresponding mapping class, as in Figure 7. We indicate the circle $\mathcal{C}$ with rainbow colors and labels a–f, to make it clear how the mapping class acts on it. The fixed interval I is red (e).

Note that $\tilde{t}_1$ is defined on all of $\mathbb{R}^2$, so the image $\tilde{t}_1(K_i)$ is well defined, even if what we are interested in is a class $t_1 \in \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ — this is the usual equivocation between punctures and marked points.

Consider the element $[T_1, s_1^{-1}s_2^{-1}(0,0,-2), T_2]$ of rV represented as in Figure 8.

With $\mathcal{C}$ and I as indicated in the picture, it is clear that up to isotopy $\mathcal{C}$ is stabilized (thanks to the third strand twisting), and that I is fixed pointwise. One can also check that K_i is sent to K_{i+1} for each $i \in \mathbb{Z}$. We conclude that all the criteria are satisfied, and so we are done. $\square$

4.2 Quasimorphisms on bV

The final step in the proof of Theorem 1.1 is to show that the quasimorphisms on rV constructed in the previous subsection restrict to nontrivial quasimorphisms on bV, bF and bP. This will be a consequence of the following general statement applied to rV and bP, which follows from left exactness of Q; see [Calegari 2009b, Remark 2.90].

Lemma 4.3 *Let Γ be a group and $\Lambda \leq \Gamma$ a normal subgroup, and suppose that $Q(\Gamma/\Lambda) = 0$. Then the restriction $Q(\Gamma) \to Q(\Lambda)$ is injective.*

Proof of Theorem 1.1 Let Γ be any group such that $bP \leq \Gamma \leq rV$, for instance any of the groups in the statement of the theorem. Note that the quotient rV/bP is uniformly perfect by Lemma 2.7, so every quasimorphism on rV/bP is bounded [Calegari 2009b, Lemma 2.2.4], ie $Q(rV/bP) = 0$. Hence Lemma 4.3 applies, and the restriction $Q(rV) \to Q(bP)$ is injective. Since this map factors through $Q(rV) \to Q(\Gamma)$, this restriction is also injective. We conclude by Proposition 4.2. $\square$

Since every quasimorphism on a uniformly perfect group is bounded [Calegari 2009b, Lemma 2.2.4], an immediate corollary of Theorem 1.1 is the following:

Corollary 4.4 *The group bV is not uniformly perfect.* □

Of course we also conclude that rV is not uniformly perfect, despite being perfect (Corollary 2.8). Note that bF is not perfect (it has abelianization $\mathbb{Z}^4$), but it follows from [Zaremsky 2018a] that bF' is perfect. However, we can deduce in the same way:

Corollary 4.5 *The group bF' is not uniformly perfect.* □

We should also mention another group fitting between bP and rV and thus having infinite-dimensional space of quasimorphisms, namely the "braided T" group from [Witzel 2019]. This is the subgroup of bV consisting of elements $[T_-, \beta, T_+]$ such that $\rho(\beta) \in S_n$ is a cyclic permutation.

Let us discuss restricting quasimorphisms of bV to $\widehat{bV}$. If $\Gamma \leq \mathrm{MCG}(\mathbb{R}^2 \setminus K)$ has a bounded orbit in X_r, then the quasimorphisms produced via this action are bounded on Γ. This is analogous to the behavior of finite-type mapping class groups, in particular for the braid group [Feller 2022], and was already noted by Calegari [2009a] for $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$. We see that in fact this happens for $\widehat{bV}$, since it fixes the isotopy class of the ray going from the rightmost point of the Cantor set to infinity on the right. Thanks to Theorem 1.2, we can actually prove a stronger version of this statement. Namely, not only are these quasimorphisms bounded on $\widehat{bV}$, but the same is true for every quasimorphism of bV.

Corollary 4.6 *For every quasimorphism q of bV, the image of $\widehat{bV}$ under q is bounded.*

Proof Let $\psi = [\cdot, 1_{B_1}(1), \cdot]$. Then viewing $rV/Z(rV)$ as a subgroup of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$, we have $\psi Z(rV) = \phi$. Note that the conjugate $\psi^{-1}\widehat{bV}\psi$ equals the subgroup of bV consisting of all elements where the *leftmost* strand does not braid with anything. Let $\chi_0 \colon \psi^{-1}\widehat{bV}\psi \to \mathbb{Z}$ be the map sending $\psi^{-1}g\psi$ to $\chi_1(g)$. Since conjugation by ψ is an isomorphism, Corollary 2.14 implies that the kernel of χ_0 equals the derived subgroup $(\psi^{-1}\widehat{bV}\psi)'$.

Let $g \in \widehat{bV}$, and choose $h \in \widehat{bV} \cap \psi^{-1}\widehat{bV}\psi$ such that $\chi_1(h) = \chi_1(g)$ and $\chi_0(h) = 0$. In particular, h lies in $(\psi^{-1}\widehat{bV}\psi)'$ and gh^{-1} lies in $\widehat{bV}'$. By Corollary 1.3, there exists a scalar $\lambda \in \mathbb{R}$ such that $q|_{\widehat{bV}}$ is at a bounded distance $r(q)$ from $\lambda \cdot \chi_1$, where χ_1 is the abelianization map of $\widehat{bV}$ (Corollary 2.14). It follows that

$$|q(gh^{-1})| \leq |\lambda \cdot \chi_1(gh^{-1})| + r(q) = r(q).$$

We also get $|q(h)| \leq r(q)$ by the same argument applied to $\psi^{-1}\widehat{bV}\psi$ (which is isomorphic to $\widehat{bV}$), up to taking a larger $r(q)$. Thus

$$|q(g)| \leq |q(gh^{-1})| + |q(h)| + D(q) \leq 2r(q) + D(q).$$

This shows that $q|_{\widehat{bV}}$ is bounded, which concludes the proof. □

Together with Theorem 1.1, this implies that, analogously to Corollary 4.4, there is no uniform-length factorization for elements of bV in terms of conjugates of elements in $\widehat{bV}$. Indeed, if such a factorization did exist, we could run a similar argument as in the proof of Corollary 4.6, and obtain that every quasimorphism of bV is bounded.

As one last indication of braided Thompson groups exhibiting unusual bounded cohomological behavior, consider the short exact sequence $1 \to \widehat{bV} \cap bP \to \widehat{bV} \to \widehat{V} \to 1$. By Theorem 1.2 $\mathrm{H}^2_b(\widehat{bV}) = 0$, and in fact the proof works using permutations instead of braids, mutatis mutandis, to show that $\mathrm{H}^2_b(\widehat{V}) = 0$ (also, it is true in general that a quotient of a group with vanishing second bounded cohomology itself has vanishing second bounded cohomology [Bouarich 1995]). However, by Theorem 1.1 $Q(bP)$ is infinite-dimensional, and thus so is $Q(\widehat{bV} \cap bP)$ since $\widehat{bV} \cap bP$ surjects onto bP (Lemma 2.5); in particular, $\mathrm{H}^2_b(\widehat{bV} \cap bP)$ is infinite-dimensional.

This gives a concrete example of the failure of a 2-out-of-3 property for vanishing of second bounded cohomology: if a group Γ has vanishing second bounded cohomology and a quotient Γ/N has the same property, then the kernel N can still have infinite-dimensional second bounded cohomology. For comparison, if $\mathrm{H}^2_b(N) = 0$, then $\mathrm{H}^2_b(\Gamma) = 0$ if and only if $\mathrm{H}^2_b(\Gamma/N) = 0$ [Moraschini and Raptis 2023, Corollary 4.2.2]. The failure of this 2-out-of-3 property was observed in [Fournier-Facio et al. 2023, Theorem 4.5] for every degree, but this is to our knowledge the first "naturally occurring" example in degree 2, as well as the first finitely generated one (and even type F_∞).

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