A COUNTEREXAMPLE TO STRONG LOCAL MONOMIALIZATION IN A TOWER OF TWO INDEPENDENT DEFECT ARTIN-SCHREIER EXTENSIONS

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ABSTRACT. We give an example of an extension of two dimensional regular local rings in a tower of two independent defect Artin-Schreier extensions for which strong local monomialization does not hold.

RÉSUMÉ. Nous donnons un exemple d'extension d'anneaux locaux réguliers à deux dimensions dans une tour de deux extensions d'Artin-Schreier de défauts indépendants pour lesquelles la monomialisation locale forte ne tient pas.

1. Introduction

In characteristic zero, there is a very nice local form for morphisms, called local monomialization. This result is a little stronger than what comes immediately from the assumption that toroidalization is possible. If $R \rightarrow S$ is an extension of local rings such that the maximal ideal of S contracts to the maximal ideal of R then we say that S dominates R. If S is dominated by the valuation ring $\mathcal{O}_\omega$ of a valuation ω we say that ω dominates S.

Theorem 1.1. *(local monomialization)([2], [3]) Suppose that k is a field of characteristic zero and $R \rightarrow S$ is an extension of regular local rings such that R and S are essentially of finite type over k and ω is a valuation of the quotient field of S which dominates S and S dominates R. Then there is a commutative diagram*

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

such that ω dominates S_1, S_1 dominates R_1 and the vertical arrows are products of monoidal transforms; that is, these arrows are factored by the local rings of blowups of prime ideals whose quotients are regular local rings. In particular, R_1 and S_1 are regular local rings. Further, $R_1 \rightarrow S_1$ has a locally monomial form; that is, there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $x_1, \ldots, x_n$ in S_1, an $m \times n$ matrix $A = (a_{ij})$ with integral coefficients such that rank$(A) = m$ and units $\delta_i \in S_1$ such that

$$u_i = \delta_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

for $1 \leq i \leq m$.

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The difficulty in the proof is to obtain the condition that $\operatorname{rank}(A) = m$. To do this, it is necessary to blow up above both R and S.

In the case when the extension of quotient fields $K \to L$ of the extension $R \to S$ is a finite extension and k has characteristic zero, it is possible to find a local monomialization such that the structure of the matrix of coefficients recovers classical invariants of the extension of valuations in $K \to L$, and this form holds stably along suitable sequences of birational morphisms which generate the respective valuation rings. This form is called strong local uniformization. It is established for rank 1 valuations in [2] and for general valuations in [8]. The case which has the simplest form and will be of interest to us in this paper is when the valuation has rational rank 1. In this case, if $R_1 \to S_1$ is a strong local monomialization, then there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $v_1, \ldots, v_m$ in S_1 , a positive integer a and a unit $\delta \in S_1$ such that

$$u_1 = \delta v_1^a, u_2 = v_2, \ldots, u_m = v_m. \tag{1}$$

The stable forms of mappings in positive characteristic and dimension ≥ 2 are much more complicated. For instance, local monomialization does not always hold. An example is given in [5] where $R \to S$ are local rings of points on nonsingular algebraic surfaces over an algebraically closed field k of positive characteristic p and $k(X) \to k(Y)$ is finite and separable.

The obstruction to local monomialization is the defect. The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [21, Chapter VI, Section 11], [12], [8, Section 7.1]. The defect is discussed in Subsection 2.1. We have the following theorem, showing that the defect is the only obstruction to strong local monomialization for maps of surfaces.

Theorem 1.2. *(*[8, Theorem 7.35]*) Suppose that $K \to L$ is a finite, separable extension of algebraic function fields over an algebraically closed field k of characteristic $p > 0$, $R \to S$ is an extension of local domains such that R and S are essentially of finite type over k and the quotient fields of R and S are K and L respectively such that S dominates R. Suppose that ω is valuation of L which dominates S. Let ν be the restriction of ω to K. Suppose that the extension is defectless ($\delta(\omega/\nu) = 1$). Then the conclusions of Theorem 1.1 hold. In particular, $R \to S$ has a local monomialization (and a strong local monomialization) along ω.*

Suppose that $K \to L$ is a Galois extension of fields of characteristic $p > 0$ and ω is a valuation of L, ν is the restriction of ω to K. Then there is a classical tower of fields ([10, page 171])

$$K \to K^s \to K^i \to K^v \to L.$$

where K^s is the splitting field, K^i is the inertia field, K^v is the ramification field and the extension $K \to K^v$ has no defect. Thus the essential difficulty comes from the extension from K^v to L which could have defect. The extension $K^v \to L$ is a tower of Artin-Schreier extensions, so the Artin-Schreier extension is of fundamental importance in this theory.

Kuhlmann has extensively studied defect in Artin-Schreier extensions in [13]. He separated these extensions into dependent and independent defect Artin-Schreier extensions. This definition is reproduced in Subsection 2.4. Kuhlmann also defined an invariant called the distance to distinguish the natures of Artin-Schreier extensions. This definition is given in Subsections 2.3 and 2.4.

We now specialize to the case of a finite separable extension $K \to L$ of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, and suppose that ω is a valuation of L which is trivial on k and ν is the restriction of ω to K. If L/K has defect then ω must have rational rank 1 and be nondiscrete. We will assume that ω has rational rank 1 and is nondiscrete for the remainder of the introduction.

With these restrictions, the distance δ of an Artin-Schreier extension is $\le 0^-$ when the extension has defect. If it is a defect extension with $\delta = 0^-$ then it is an independent defect extension. If it is a defect extension and the distance is less than 0^- then the extension is a dependent defect extension.

A quadratic transform along a valuation is the center of the valuation at the blow up of a maximal ideal of a regular local ring. There is the sequence of quadratic transforms along ν and ω

$$R \to R_1 \to R_2 \to \cdots \text{ and } S \to S_1 \to S_2 \to \cdots . \tag{2}$$

We have that $\cup_{i=1}^{\infty} R_i = \mathcal{O}_\nu$, the valuation ring of ν, and $\cup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$, the valuation ring of ω. These sequences can be factored by standard quadratic transform sequences (defined in Section 3). It is shown in [8] that given positive integers r_0 and s_0, there exists $r \ge r_0$ and $s \ge s_0$ such that $R_r \to S_s$ has the following form:

$$u = \delta x^a, v = x^b(y^d\gamma + x\Omega) \tag{3}$$

where u, v are regular parameters in R_r, x, y are regular parameters in S_s, γ and τ are units in S_s, $\Omega \in S_s$, a and d are positive integers and b is a non negative integer. If we choose r_0 sufficiently large, then we have that the complexity ad of the extension $R_r \to S_s$ is a constant which depends on the extension of valuations, which we call the stable complexity of (2). When $R_r \to S_s$ has this stable complexity, we call the forms (3) stable forms.

The strongly monomial form is the case when $b = 0$ and $d = 1$; that is, after making a change of variables in y,

$$u = \delta x^a, v = y.$$

As we observed earlier (Theorem 1.2) if the extension $K \to L$ has no defect, then the stable form is the strongly monomial form. If there is defect, then it is possible for the a and d in stable forms along a valuation to vary wildly, even though their product ad is fixed by the extension, as shown in [6, Theorem 5.4].

An example is constructed in [8], showing failure of strong local monomialization. It is a tower of two defect Artin-Schreier extensions, each of the type of [6, Theorem 5.4] referred to above. The first extension is of type 1 for even integers and of type 2 for odd integers. The second extension is of type 2 for even integers and of type 1 for odd integers. The composite gives a sequence of extensions of regular local rings $R_i \to S_i$, where R_i has regular parameters u_i, v_i and S_i has regular parameters x_i, y_i such that the stable form is

$$u_i = \gamma x_i^p, v_i = y_i^p\tau + x_i\Omega \tag{4}$$

for all i. Both of these Artin-Schreier extensions are dependent. This is calculated in [11] and in [6, Section 6]. In keeping with the philosophy that independent Artin-Schreier extensions are better behaved than dependent ones, this leads to the question of if strong monomialization holds in towers of independent Artin-Schreier extensions. However, this is not true as is shown in Theorem 4.1 of this paper. In this theorem, we construct an example in a tower of two independent defect extensions such that strong local monomialization does not hold.

Suppose that $K \to L$ is a finite extension of fields of positive characteristic and ω is a valuation of L with restriction ν to K. It is known that there is no defect in the extension if and only if there is a finite generating sequence in L for the valuation ω over K ([19], [16]). The calculation of generating sequences for extensions of Noetherian local rings which are dominated by a valuation is extremely difficult. This has been accomplished for two dimensional regular local rings in [18] and [9] and for many hypersurface singularities above a regular local ring of arbitrary dimension in [7].

The nature of a generating sequence in an extension of S over R determines the nature of the mappings in the stable forms. It is shown in [4, Theorem 1] that if $R \rightarrow S$ is an extension of two dimensional excellent regular local rings whose quotient fields give a finite extension $K \rightarrow L$ and ω is a valuation of L which dominates S then the extension is without defect if and only if there exist sequences of quadratic transform $R \rightarrow R_1$ and $S \rightarrow S_1$ along ν such that ω has a finite generating sequence in S_1 over R_1. This shows us that we can expect good stable forms (as do hold by Theorem 1.2) if there is no defect, but not otherwise.

2. Preliminaries

2.1. **Some notation.** Let K be a field with a valuation ν. The valuation ring of ν will be donoted by $\mathcal{O}_\nu$, νK will denote the value group of ν and $K\nu$ will denote the residue field of $\mathcal{O}_\nu$.

The maximal ideal of a local ring A will be denoted by m_A. If $A \rightarrow B$ is an extension (inclusion) of local rings such that $m_B \cap A = m_A$ we will say that B dominates A. If a valuation ring $\mathcal{O}_\nu$ dominates A we will say that the valuation ν dominates A.

Suppose that K is an algebraic function field over a field k. An algebraic local ring A of K is a local domain which is a localization of a finite type k-algebra whose quotient field is K. A k-valuation of K is a valuation of K which is trivial on k.

Suppose that $K \rightarrow L$ is a finite algebraic extension of fields, ν is a valuation of K and ω is an extension of ν to L. Then the reduced ramification index of the extension is $e(\omega/\nu) = [\omega L : \nu K]$ and the residue degree of the extension is $f(\omega/\nu) = [L\omega : K\nu]$.

The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [21, Chapter VI, Section 11], [12] and [8, Section 7.1]. In the case that L is Galois over K, we have the formula

$$[L : K] = e(\omega/\nu) f(\omega/\nu) \delta(\omega/\nu) g \tag{5}$$

where g is the number of extensions of ν to L. In fact, we have the equation (c.f. [13] or Section 7.1 [8])

$$|G^s(\omega/\nu)| = e(\omega/\nu) f(\omega/\nu) \delta(\omega/\nu),$$

where $G^s(\omega/\nu)$ is the decomposition group of L/K.

If $K \rightarrow L$ is a finite Galois extension, then we will denote the Galois group of L/K by $\mathrm{Gal}(L/K)$.

2.2. **Initial and final segments and cuts.** We review some basic material about cuts in totally ordered sets from [13]. Let $(S, <)$ be a totally ordered set. An initial segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta < \alpha$ then $\beta \in \Lambda$. A final segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta > \alpha$ then $\beta \in \Lambda$. A cut in S is a pair of sets (Λ^L, Λ^R) such that Λ^L is an initial segment of S and Λ^R is a final segment of S satisfying $\Lambda^L \cup \Lambda^R = S$ and $\Lambda^L \cap \Lambda^R = \emptyset$. If Λ_1 and Λ_2 are two cuts in S, write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$. Suppose that $S \subset T$ is an order preserving inclusion of ordered sets and $\Lambda = (\Lambda^L, \Lambda^R)$ is a cut in S. Then define the cut induced by $\Lambda = (\Lambda^L, \Lambda^R)$ in T to be the cut $\Lambda \uparrow T = (\Lambda^L \uparrow T, T \setminus (\Lambda^L \uparrow T))$ where $\Lambda^L \uparrow T$ is the least initial segment of T in which Λ^L forms a cofinal subset.

We embed S in the set of all cuts of S by sending $s \in S$ to

$$s^+ = (\{t \in S \mid t \le s\}, \{t \in S \mid t > s\}).$$

we may identify s with the cut s^+. Define

$$s^- = (\{t \in S \mid t < s\}, \{t \in S \mid t \ge s\}).$$

Given a cut $\Lambda = (\Lambda^L, \Lambda^R)$, we define $-\Lambda = (-\Lambda^R, -\Lambda^L)$ where $-\Lambda^L = \{-s \mid s \in \Lambda^L\}$ and $-\Lambda^R = \{-s \mid s \in \Lambda^R\}$. We have that if Λ_1 and Λ_2 are cuts, then $\Lambda_1 < \Lambda_2$ if and only if $-\Lambda_2 < -\Lambda_1$.

Observe that for $s \in S$, $-s = -(s^+) = (-s)^-$ and $-(s^-) = (-s)^+ = -s$.

2.3. **Distances.** Let $K \to L$ be an extension of fields and ω be a valuation of L with restriction ν to K. Let $\widetilde{\nu K}$ be the divisible hull of νK. Suppose that $z \in L$. Then the distance of z from K is defined in [13, Section 2.3] to be the cut $\operatorname{dist}(z, K)$ of $\widetilde{\nu K}$ in which the initial segment of $\operatorname{dist}(z, K)$ is the least initial segment of $\widetilde{\nu K}$ in which $\omega(z - K)$ is cofinal. That is,

$$\operatorname{dist}(z, K) = (\Lambda^L(z, K), \Lambda^R(z, K)) \uparrow \widetilde{\nu K}$$

where

$$\Lambda^L(z, K) = \{\omega(z - c) \mid c \in K \text{ and } \omega(z - c) \in \nu K\}.$$

The following notion of equivalence is defined in [13, Section 2.3]. If $y, z \in L$, then $z \sim_K y$ if $\omega(z - y) > \operatorname{dist}(z, K)$.

2.4. **Artin-Schreier extensions.** Let $K \to L$ be an Artin-Schreier extension of fields of characteristic $p > 0$ and ω be a valuation of L with restriction ν to K. The field L is Galois over K with Galois group $G \cong \mathbb{Z}_p$, where p is the characteristic of K.

Let $\Theta \in L$ be an Artin-Schreier generator of K; that is, there is an expression

$$\Theta^p - \Theta = a$$

for some $a \in K$. We have that

$$\operatorname{Gal}(L/K) \cong \mathbb{Z}_p = \{\mathrm{id}, \sigma_1, \ldots, \sigma_{p-1}\},$$

where $\sigma_i(\Theta) = \Theta + i$.

Since L is Galois over K, we have that $ge(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu) = p$ where g is the number of extensions of ν to L. So we either have that $g = 1$ or $g = p$. If $g = 1$, then ω is the unique extension of ν to L and either $e(\omega/\nu) = p$ and $\delta(\omega/\nu) = 1$ or $e(\omega/\nu) = 1$ and $\delta(\omega/\nu) = p$. In particular, the extension is defect if and only if is an immediate extension ($e = f = 1$) and ω is the unique extension of ν to L.

From now on in this subsection, suppose that L is a defect extension of K. By [13, Lemma 4.1], the distance $\delta = \operatorname{dist}(\Theta, K)$ does not depend on the choice of Artin-Schreier generator Θ, so δ can be called the distance of the Artin-Schreier extension. Since L/K is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in νK which has no maximal element by [13, Theorem 2.19].

We have, since the extension is defect, that

$$\delta = \operatorname{dist}(\Theta, K) \le 0^- \tag{6}$$

by [13, Corollary 2.30].

A defect Artin-Schreier extension L is defined in [13, Section 4] to be a dependent defect Artin-Schreier extension if there exists an immediate purely inseparable extension $K(\eta)$ of K of degree p such that $\eta \sim_K \Theta$. Otherwise, L/K is defined to be an independent defect

Artin-Schreier defect extension. We have by [13, Proposition 4.2] that for a defect Artin-Schreier extension,

(7) L/K is independent if and only if the distance $\delta = \text{dist}(\Theta, K)$ satisfies $\delta = p\delta$.

2.5. **Extensions of rank 1 valuations in an Artin-Schreier extension.** In this subsection, we suppose that L is an Artin-Schreier extension of a field K of characteristic p, ω is a rank 1 valuation of L and ν is the restriction of ω to K. We suppose that L is a defect extension of K. To simplify notation, we suppose that we have an embedding of νL in $\mathbb{R}$. Since L has defect over K and L is separable over K, νL is nondiscrete by the corollary on page 287 of [20], so that νL is dense in $\mathbb{R}$.

We define a cut in $\mathbb{R}$ by extending the cut $\text{dist}(\Theta, K)$ in $\widetilde{\nu K}$ to a cut of $\mathbb{R}$ by taking the initial segment of the extended cut to be the least initial segment of $\mathbb{R}$ in which the cut $\text{dist}(\Theta, K)$ is confinal. This cut is then $\text{dist}(\Theta, K) \uparrow \mathbb{R}$. This cut is either s or s^- for some $s \in \mathbb{R}$. If L is a defect extension of K then $\text{dist}(\Theta, K) \uparrow \mathbb{R} = s^-$ where s is a non positive real number by [13, Theorem 2.19] and [13, Corollary 2.30]. We will set $\text{dist}(\omega/\nu)$ to be this real number s, so that

$$\text{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = (\text{dist}(\omega/\nu))^-.$$

The real number $\text{dist}(\omega/\nu)$ is well defined since it is independent of choice of Artin-Schreier generator of L/K by Lemma 4.1 [13].

With the assumptions of this subsection, by (6) and (7), the distance $\delta = \text{dist}(\Theta, K)$ of an Artin-Schreier extension is $\leq 0^-$ when the extension has defect. If it is a defect extension with distance equal to 0^- then it is an independent defect extension. If it is a defect extensions and the distance is less than 0^- then the extension is a dependent defect extension. Thus if L/K is a defect extension, we have that $\text{dist}(\omega/\nu) \leq 0$ and the defect extension L/K is independent if and only if $\text{dist}(\omega/\nu) = 0$.

3. Calculations in two dimensional Artin-Schreier Extensions

Suppose that M is a two dimensional algebraic function field over an algebraically closed field k of characteristic $p > 0$ and μ is a nondiscrete rational rank 1 valuation of M. Suppose that A is an algebraic regular local ring of M such that μ dominates A. A quadratic transform of A is an extension $A \to A_1$ where A_1 is a local ring of the blowup of the maximal ideal of A such that A_1 dominates A and A_1 has dimension two. A quadratic transform $A \to A_1$ is said to be along the valuation μ if μ dominates A_1.

Let

$$A = A_0 \to A_1 \to A_2 \to \cdots$$

be the sequence of quadratic transforms along μ. Then the valuation ring $\mathcal{O}_\mu = \cup A_i$ (by [1, Lemma 12]).

Suppose that $K \to L$ is a finite extension of two dimensional algebraic function fields, R is an algebraic regular local ring of K which is dominated by a regular algebraic local ring S of L such that $\dim R = \dim S = 2$. Let x, y be regular parameters in S and u, v be regular parameters in R. Then we can form the Jacobian ideal

$$J(S/R) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right).$$

This ideal is independent of choice of regular parameters.

The following proposition is proven in [17].

Proposition 3.1. *Suppose that $K \to L$ is an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, ω is a rational rank 1 nondiscrete valuation of L with restriction $\nu = \omega|K$. Further suppose that A is an algebraic local ring of K and B is an algebraic local ring of L which is dominated by ω such that B dominates A. Then there exists a commutative diagram of homomorphisms*

$$\begin{array}{ccc} R & \to & S \\ \uparrow & & \uparrow \\ A & \to & B \end{array}$$

such that R is a regular algebraic local ring of K with regular parameters u, v, S is a regular algebraic local ring of L with regular parameters x, y such that S is dominated by ω, S dominates R, $R \to S$ is quasi finite, $J(S/R) = (x^{\overline{c}})$ for some non negative integer $\overline{c}$ and one of the following three cases holds:

0) *$u = x$, $v = y$ ($R \to S$ is unramified).*
1) *$u = x$, $v = y^p\gamma + x\Sigma$ where γ is a unit in S and $\Sigma \in S$.*
2) *$u = \gamma x^p$, $v = y$ where γ is a unit in S.*

Let $K \to L$ be an Artin-Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$. Let $R \to S$ be an extension from a regular algebraic local ring of K to a regular algebraic local ring of L such that S dominates R.

Let u, v be regular parameters in R and x, y be regular parameters in S. We will say that $R \to S$ is of type 0 with respect to these parameters if

$$\text{Type 0:} \quad u = \gamma x, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$, so that $R \to S$ is unramified. We will say that $R \to S$ is of type 1 with respect to these parameters if

$$\text{Type 1:} \quad u = \gamma x, v = y^p\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$. We will say that $R \to S$ is of type 2 with respect to these parameters if

$$\text{Type 2:} \quad u = \gamma x^p, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$.

These definitions are such that if one these types hold, and $\overline{u}, \overline{v}$ are regular parameters in R, $\overline{x}, \overline{y}$ are regular parameters in S such that $\overline{u}$ is a unit in R times u and $\overline{x}$ is a unit in S times x then $R \to S$ is of the same type for the new parameters $\overline{u}, \overline{v}$ and $\overline{x}, \overline{y}$.

In the construction of our example (Theorem 4.1), we will make use of some results from [6].

Theorem 3.2. ([6, Theorem 4.1]) *Suppose that $R \to S$ is of type 1 with respect to regular parameters x, y in S and u, v in R and that $J(S/R) = (x^{\overline{c}})$. Let $\overline{x} = u$, $\overline{y} = y - g(\overline{x})$ where $g(\overline{x}) \in k[\overline{x}]$ is a polynomial with zero constant term, so that $\overline{x}, \overline{y}$ are regular parameters in S. Computing the Jacobian determinate $J(S/R)$, we see that*

$$u = \overline{x}, v = \overline{y}^p\gamma + \overline{x}^{\overline{c}}\overline{y}\tau + f(\overline{x})$$

where γ, τ are unit series in $\hat{S}$ and $f(\overline{x}) = \sum e_i\overline{x}^i \in k[[\overline{x}]]$. Make the change of variables $\overline{v} = v - \sum e_i u^i$ where the sum is over i such that $i \leq \frac{pq}{m}$ so that $u, \overline{v}$ are regular parameters in R.

Suppose that m, q are positive integers with $m > 1$ and $gcd(m,q) = 1$. Let α be a nonzero element of k. Consider the sequence of quadratic transforms $S \to S_1$ so that S_1 has regular parameters x_1, y_1 defined by

$$\overline{x} = x_1^m (y_1 + \alpha)^{a'}, \overline{y} = x_1^q (y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

We have that $R \to S$ is of type 1 with respect to the regular parameters $\overline{x}, \overline{y}$ and u, v. Let $\sigma = gcd(m, pq)$ which is 1 or p.

There exists a unique sequence of quadratic transforms $R \to R_1$ such that R_1 has regular parameters u_1, v_1 defined by

$$u = u_1^{\overline{m}} (v_1 + \beta)^{c'}, \overline{v} = u_1^{\overline{q}} (v_1 + \beta)^{d'}$$

with $0 \neq \beta \in k$ giving a commutative diagram of homomorphisms

$$\begin{array}{ccc} R_1 & \to & S_1 \\ \uparrow & & \uparrow \\ R & \to & S \end{array}$$

such that $R_1 \to S_1$ is quasi finite. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer c_1 and $R_1 \to S_1$ is quasi finite. Further:

0) *If $\frac{q}{m} \geq \frac{\overline{c}}{p-1}$ then $R_1 \to S_1$ is of type 0.*

1) *If $\frac{q}{m} < \frac{\overline{c}}{p-1}$ and $\sigma = 1$ then $R_1 \to S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1} \right) = \left(\frac{\overline{c}}{p-1} \right) m - q.$$

2) *If $\frac{q}{m} < \frac{\overline{c}}{p-1}$ and $\sigma = p$ then $R_1 \to S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1} \right) = \left(\frac{\overline{c}}{p-1} \right) m - q + 1.$$

In cases 1) and 2), $m = \sigma \overline{m}$, $pq = \sigma \overline{q}$ and $\overline{m} c' - \overline{q} d' = 1$.

Theorem 3.3. *([6, Theorem 4.3]) Suppose that $R \to S$ is of type 2 with respect to regular parameters x, y in S and u, v in R and that $J(S/R) = (x^{\overline{c}})$. Let $g(u) \in k[u]$ be a polynomial with no constant term. Make the change of variables, letting $\overline{v} = v - g(u)$ and $\overline{y} = \overline{v}$, so that $x, \overline{y}$ are regular parameters in S and $u, \overline{v}$ are regular parameters in R.*

Suppose that m, q are positive integers with $gcd(m, q) = 1$. Let α be a nonzero element of k. Consider the sequence of quadratic transforms $S \to S_1$ so that S_1 has regular parameters x_1, y_1 defined by

$$x = x_1^m (y_1 + \alpha)^{a'}, \overline{y} = x_1^q (y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

Let $\sigma = gcd(pm, q)$ which is 1 or p. There exists a unique sequence of quadratic transforms $R \to R_1$ such that R_1 has regular parameters u_1, v_1 defined by

$$u = u_1^{\overline{m}} (v_1 + \beta)^{c'}, \overline{v} = u_1^{\overline{q}} (v_1 + \beta)^{d'}$$

where $pm = \sigma \overline{m}$, $q = \sigma \overline{q}$, $\overline{m} d' - c' \overline{q} = 1$ and $0 \neq \beta \in k$, giving a commutative diagram of homomorphisms

$$\begin{array}{ccc} R_1 & \to & S_1 \\ \uparrow & & \uparrow \\ R & \to & S \end{array}$$

such that $R_1 \to S_1$ is quasi finite. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer c_1. Further:

1) *If $\sigma = 1$ then $R_1 \to S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right) m - m.$$

2) *If $\sigma = p$ then $R_1 \to S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right) m - m + 1.$$

A proof of the following proposition is given in [6, Proposition 7.9]. More general results are proven in [15].

Proposition 3.4. *(Kuhlmann and Piltant, [14]) Suppose that K and L are two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$ and $K \to L$ is an Artin-Schreier extension. Let ω be a rational rank one nondiscrete valuation of L and let ν be the restriction of ω to K. Suppose that L is a defect extension of K.*

Suppose that R is a regular algebraic local ring of K and S is a regular algebraic local ring of L such that ω dominates S, S dominates R and $R \to S$ is of type 1 or 2. Inductively applying Theorems 3.2 and 3.3, we construct a diagram where the horizontal sequences are birational extensions of regular local rings

$$\begin{array}{ccccccc} S = S_0 & \to & S_1 & \to & S_2 & \to & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ R = R_0 & \to & R_1 & \to & R_2 & \to & \cdots \end{array} \tag{8}$$

with $\cup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$. Further assume that for each map $R_i \to S_i$, there are regular parameters u, v in R_i and x, y in S_i such that one of the following forms hold:

$$u = x, v = f \tag{9}$$

where $\dim_k S_i/(x, f) = p$, or

$$u = \delta x^p, v = y \tag{10}$$

where δ is a unit in S_i and in both cases that $x = 0$ is a local equation of the critical locus of $Spec(S_i) \to Spec(R_i)$. Let

$$J_i = J(S_i/R_i) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

be the Jacobian ideal of the map $R_i \to S_i$.

Then the distance $dist(\omega/\nu)$ is computed by the formula

$$-\mathrm{dist}(\omega/\nu) = \frac{1}{p-1} \inf_i \{\omega(J(S_i/R_i))\}$$

where the infimum is over the $R_i \to S_i$ in the sequence (8).

4. An example of a tower of independent defect extensions in which strong local monomialization doesn't hold

Theorem 4.1. *There exists a tower $(K,\nu)\rightarrow(L,\omega)\rightarrow(M,\mu)$ of independent defect Artin-Schreier extensions of valued two dimensional algebraic function fields over an algebraically closed field k of characteristic $p>0$ such that there exist algebraic regular local rings A of K and C of M such that μ dominates C and C dominates A but strong local monomialization along μ does not hold above $A\rightarrow C$.*

Remark 4.2. *Let $\delta\in\mathbb{R}_{\ge 0}$ be a fixed ratio. Suppose that $R\rightarrow S$ is of type 1. By taking m and q sufficiently large in Theorem 3.2 such that $R_1\rightarrow S_1$ is of type 2, we can achieve that $v_1=\lambda y_1+g(x_1)$ where λ is a unit in S_1 and the order of $g(x_1)$ is arbitrarily large. Suppose that $R\rightarrow S$ is of type 2. By taking m and q sufficiently large in Theorem 3.3 such that $R_1\rightarrow S_1$ is of type 1 we can achieve that $v_1=y_1^p\gamma+x_1^{c_1}y_1\tau+f(x_1)$ where γ and τ are unit series in S_1 and the order of $f(x_1)$ is arbitrarily large. In both cases, we can choose m and q so that $\frac{q}{m}$ is arbitrarily close to δ.*

Remark 4.3. *In Theorem 3.3, we have an expression $\overline{v}=\tau y+f(x)$ where τ is a unit in S. Suppose that m and q are positive integers with $gcd(m,q)=1$ and such that ord $f(x)>\frac{q}{m}$. Then the proof of Theorem 3.3 extends to show that the conclusions of Theorem 3.3 hold with $\overline{y}$ replaced with y.*

We now give the proof of Theorem 4.1.

Proof. Let K be a two dimensional algebraic function field over an algebraically closed field, and let R_{-2} be a two dimensional algebraic regular local ring of K. Let u_{-2},v_{-2} be regular parameters in R_{-2}.

Let e be a positive integer. Let $c_{-2}=(p-1)e$. Let Θ be a root of the Artin-Schreier polynomial $X^p-X-v_{-2}u_{-2}^{-pe}$. Let $L=K(\Theta)$. Set $x_{-2}=u_{-2}$, $y_{-2}=u_{-2}^e\Theta$. Let $S_{-2}=R_{-2}[y_{-2}]_{(x_{-2},y_{-2})}$, which is an algebraic regular local ring of L which dominates R_{-2}. The regular parameters x_{-2},y_{-2} in S_{-2} satisfy $u_{-2}=x_{-2},v_{-2}=y_{-2}^p-x_{-2}^{e(p-1)}y_{-2}$, so that the extension $R_{-2}\rightarrow S_{-2}$ is of type 1. We have that $J(S_{-2}/R_{-2})=(x_{-2}^{c_{-2}})$, with $\frac{c_{-2}}{p-1}>0$.

We first construct a commutative diagram

$$\begin{array}{ccc} S_{-2} & \rightarrow & S_{-1}\\ \uparrow & & \uparrow\\ R_{-2} & \rightarrow & R_{-1}\end{array}$$

using Theorem 3.2 so that $R_{-1}\rightarrow S_{-1}$ is of type 2. Let Σ be a root of the Artin-Schreier polynomial $X^p-X-y_{-1}x_{-1}^{-pe}$. Let $M=L(\Sigma)$. Set $z_{-1}=x_{-1}$, $w_{-1}=x_{-1}^e\Sigma$. Let $T_{-1}=S_{-1}[w_{-1}]_{(z_{-1},w_{-1})}$, which is an algebraic regular local ring of M which dominates S_{-1}. The regular parameters z_{-1},w_{-1} in T_{-1} satisfy $x_{-1}=z_{-1},y_{-1}=w_{-1}^p-z_{-1}^{e(p-1)}w_{-1}$, so that the extension $S_{-1}\rightarrow T_{-1}$ is of type 1. We have that $J(T_{-1}/S_{-1})=(z_{-1}^{c'_{-1}})$, with $\frac{c'_{-1}}{p-1}>0$.

From Theorems 3.2 and 3.3, we construct

$$\begin{array}{ccc} T_{-1} & \rightarrow & T_0 \\ \uparrow & & \uparrow \\ S_{-1} & \rightarrow & S_0 \\ \uparrow & & \uparrow \\ R_{-1} & \rightarrow & R_0 \end{array}$$

such that $R_0 \rightarrow S_0$ is of type 1 and $S_0 \rightarrow T_0$ is of type 2. Explicitely, $R_{-1}, R_0, S_{-1}, S_0, T_{-1}, T_0$ have respective regular parameters (u_{-1}, v_{-1}), (u_0, v_0), (x_{-1}, y_{-1}), (x_0, y_0) and (z_{-1}, w_{-1}), (z_0, w_0) which are related by equations

$$\begin{array}{l} u_{-1} = u_0^{pm_0}(v_0+\beta_0)^{d_0'}, v_{-1} = u_0^{q_0}(v_0+\beta_0)^{e_0'} \\ x_{-1} = x_0^{m_0}(y_0+\alpha_0)^{a_0'}, y_{-1} = x_0^{q_0}(w_0+\alpha_0)^{g_0'} \\ z_{-1} = z_0^{pm_0}(v_0+\gamma_0)^{f_0'}, w_{-1} = z_0^{q_0}(w_0+\gamma_0)^{g_0'} \end{array}$$

where $p \not| q_0$ and $\frac{q_0}{pm_0} < \frac{c'_{-1}}{p-1}$ where $J(T_{-1}/S_{-1}) = (z_{-1}^{c'_{-1}})$.

By Remarks 4.2 and 4.3, we can construct $R_0 \rightarrow S_0 \rightarrow T_0$ so that we have expressions $y_0 = \lambda_0 w_0 + g_0(z_0)$ where λ_0 is a unit in T_0 and ord $g_0(z_0)$ is arbitrarily large and $v_0 = \sigma_0 y_0^p + \tau_0 x_0^{c_0} y_0 + f_0(x_0)$ where σ_0, τ_0 are units in S_0 and ord $f_0(x_0)$ is arbitrarily large.

We will inductively construct a commutative diagram within $K \rightarrow L \rightarrow M$ of two dimensional regular algebraic local rings

(11)
$$\begin{array}{ccccccc} T_0 & \rightarrow & T_1 & \rightarrow & T_2 & \rightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ S_0 & \rightarrow & S_1 & \rightarrow & S_2 & \rightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ R_0 & \rightarrow & R_1 & \rightarrow & R_2 & \rightarrow & \cdots \end{array}$$

such that $R_i \rightarrow S_i$ is of type 1 if i is even and is of type 2 if i is odd, $S_i \rightarrow T_i$ is of type 2 if i is even and is of type 1 if i is odd. Further, valuations ν, ω and μ of the respective function fields K, L and M determined by these sequences are such that $K \rightarrow L$ and $L \rightarrow M$ are independent defect extensions. We will have that R_i has regular parameters (u_i, v_i), S_i has regular parameters (x_i, y_i) and T_i has regular parameters (z_i, w_i) such that

$$u_i = u_{i+1}^{\overline{m}_{i+1}}(v_{i+1}+\beta_{i+1})^{d'_{i+1}}, v_i = u_{i+1}^{\overline{q}_{i+1}}(v_{i+1}+\beta_{i+1})^{e'_{i+1}},$$
$$x_i = x_{i+1}^{m_{i+1}}(y_{i+1}+\alpha_{i+1})^{a'_{i+1}}, y_i = x_{i+1}^{q_{i+1}}(y_{i+1}+\alpha_{i+1})^{b'_{i+1}},$$
$$z_i = z_{i+1}^{m'_{i+1}}(w_{i+1}+\gamma_{i+1})^{f'_{i+1}}, w_i = z_{i+1}^{q'_{i+1}}(w_i+\gamma_{i+1})^{g'_{i+1}}$$

with $\overline{m}_i, m_i$ and m'_i larger than 1 for all i.

Let $J(S_i/R_i) = (x_i^{c_i})$ and $J(T_i/S_i) = (z_i^{c'_i})$.

If i is even, then $m_{i+1} = p\overline{m}_{i+1}, m'_{i+1} = \overline{m}_{i+1}, q_{i+1} = \overline{q}_{i+1}, q'_{i+1} = q_{i+1}$ and

$$\frac{q_{i+1}}{m_{i+1}} < \frac{c_i}{p-1}.$$

If i is odd, then $\overline{m}_{i+1} = pm_{i+1}, m'_{i+1} = \overline{m}_{i+1}, q_{i+1} = \overline{q}_{i+1}, q'_{i+1} = q_{i+1}$ and

$$\frac{q'_{i+1}}{m'_{i+1}} < \frac{c'_i}{p-1}.$$

In our construction, if r is even, we will have that

(12)
$$y_r = \lambda_r w_r + g_r(z_r)$$

where λ_r is a unit in T_r and ord $g_r(z_r)$ is arbitrarily large and

$$v_r = \sigma_r y_r^p + \tau_r x_r^{c_r} y_r + f_r(x_r) \tag{13}$$

where σ_r, τ_r are units in S_r and ord $f_r(x_r)$ is arbitrarily large. If r is even, we will have

$$y_r = \sigma_r w_r^p + \tau_r z_r^{c'_r} w_r + f(z_r) \tag{14}$$

where σ_r, τ_r are units in T_r and ord $f(z_r)$ is arbitrarily large and

$$v_r = \lambda_r y_r + g_r(x_r) \tag{15}$$

where λ_r is a unit in S_r and ord $g_r(x_r)$ is arbitrarily large.

Suppose that r is even, and we have constructed $R_r \to S_r \to T_r$. We will construct

$$\begin{array}{ccccc} T_r & \to & T_{r+1} & \to & T_{r+2} \\ \uparrow & & \uparrow & & \uparrow \\ S_r & \to & S_{r+1} & \to & S_{r+2} \\ \uparrow & & \uparrow & & \uparrow \\ R_r & \to & R_{r+1} & \to & R_{r+2}. \end{array}$$

There exists an integer $\lambda(r+1) > 1$ and $q_{r+1} \in \mathbb{Z}_+$ such that $\gcd(q_{r+1}, p) = 1$ and

$$\frac{c_r}{p-1} > \frac{q_{r+1}}{p^{\lambda(r+1)}} > \frac{c_r}{p-1} - \frac{1}{2^{r+1}} m_1 \cdots m_r. \tag{16}$$

In fact, we can find $\lambda(r+1)$ arbitrarily large satisfying the inequality. Set $m_{r+1} = p^{\lambda(r+1)}$. We have that $\frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1}$ with $\gcd(m_{r+1}, pq_{r+1}) = p$. This choice of m_{r+1} and q_{r+1} (along with a choice of $0 \neq \alpha_{r+1} \in k$) determines $S_r \to S_{r+1}$. We have an expression $v_r = \sigma_r y_r^p + \tau_r x_r^{c_r} y_r + f_r(x_r)$ where ord $f_r(x_r)$ is arbitrarily large. In particular, we can assume that ord $f_r(x_r) > \frac{pq_{r+1}}{m_{r+1}}$. Then $R_r \to R_{r+1}$ is defined as desired by Theorem 3.2. By Remark 4.2, since we can take $\lambda(r+1)$ to be arbitrarily large, we can assume that $v_{r+1} = \lambda_{r+1} y_{r+1} + g_{r+1}(x_{r+1})$ where ord $g_{r+1}(x_{r+1})$ is arbitrarily large.

By Remark 4.3 and Theorem 3.3, $T_r \to T_{r+1}$ is defined as desired, with $m'_{r+1} = \frac{m_{r+1}}{p}$, $q'_{r+1} = q_{r+1}$. Since we can take $\lambda(r+1)$ to be arbitrarily large, we can assume that $y_{r+1} = \sigma_{r+1} w_{r+1}^p + \tau_{r+1} z_{r+1}^{c'_{r+1}} w_r + f_{r+1}(z_{r+1})$ where ord $f_{r+1}(z_{r+1})$ is arbitrarily large.

We have defined a commutative diagram

$$\begin{array}{ccc} T_r & \to & T_{r+1} \\ \uparrow & & \uparrow \\ S_r & \to & S_{r+1} \\ \uparrow & & \uparrow \\ R_r & \to & R_{r+1} \end{array} \tag{17}$$

with the desired properties; in particular, $R_{r+1} \to S_{r+1}$ is of type 2 with

$$\frac{c_{r+1}}{p-1} = \left(\frac{c_r}{p-1}\right) m_{r+1} - q_{r+1} + 1$$

and $S_{r+1} \to T_{r+1}$ is of type 1, with

$$\frac{c'_{r+1}}{p-1} = \frac{c'_r}{p-1} m'_{r+1} - m'_{r+1}.$$

Now choose q'_{r+2}, $m'_{r+2} = p^{\lambda(r+2)}$ such that $p \not| q'_{r+2}$ and

$$\frac{c'_{r+1}}{p-1} > \frac{q'_{r+2}}{m'_{r+2}} > \frac{c'_{r+1}}{p-1} - \frac{1}{2^{r+2}} m'_1 \cdots m'_{r+1}. \tag{18}$$

We can take $\lambda(r+2)$ arbitrarily large. Set $m_{r+2} = \frac{m'_{r+2}}{p} = p^{\lambda(r+2)-1}$, $q_{r+2} = q'_{r+2}$. By (18), $\frac{q'_{r+2}}{m'_{r+2}} < \frac{c'_{r+1}}{p-1}$.

Now construct, as in the construction of (17), using Theorems 3.2 and 3.3 and Remark 4.3 and these values of m_{r+2} and q_{r+2},

$$\begin{array}{ccc} T_{r+1} & \rightarrow & T_{r+2} \\ \uparrow & & \uparrow \\ S_{r+1} & \rightarrow & S_{r+2} \\ \uparrow & & \uparrow \\ R_{r+1} & \rightarrow & R_{r+2}, \end{array}$$

so that $R_{r+2} \rightarrow S_{r+2}$ is of type 1 and $S_{r+2} \rightarrow T_{r+2}$ is of type 2. By Remark 4.2, we obtain expressions (12) and (13) for $r+2$.

By induction, we construct the diagram (11).

Let $A = R_0$ and $C = T_0$. We will show that strong local monomialization doesn't hold above $A \rightarrow C$ along μ. Suppose that $R' \rightarrow T'$ has a strongly monomial form above $A \rightarrow C$. Then R' has regular parameters u', v' and T' has regular parameters z', w' such that $u' = \lambda(z')^m$ and $v' = w'$ where $m \in \mathbb{Z}_{>0}$ and λ is a unit in T'. We will show that this cannot occur. There exists a commutative diagram

$$\begin{array}{ccccc} T_s & \rightarrow & T' & \rightarrow & T_{s+1} \\ \uparrow & & \uparrow & & \uparrow \\ R_s & \rightarrow & R' & \rightarrow & R_{s+1} \end{array}$$

for some s. The ring T' has regular parameters $\overline{z}, \overline{w}$ such that

$$z_s = \overline{z}^a \overline{w}^b, w_s = \overline{z}^c \overline{w}^d \tag{19}$$

for some $a, b, c, d \in \mathbb{N}$ with $ad - bc = \pm 1$, and R' has regular parameters $\overline{u}, \overline{v}$ such that $u_s = \overline{u}^{\overline{a}} \overline{v}^{\overline{b}}$, $v_s = \overline{u}^{\overline{c}} \overline{v}^{\overline{d}}$, where $\overline{a}\overline{d} - \overline{b}\overline{c} = \pm 1$. We have an expression

$$u_s = \alpha z_s^p, v_s = \beta w_s^p + \Omega \tag{20}$$

where α, β are units in T_s and where

$$\Omega = \varepsilon z_s^{pc_s} w_s + M \tag{21}$$

or

$$\Omega = \varepsilon z_s^{c'_s} w_s + M \tag{22}$$

where $\varepsilon \in T_s$ is a unit and M is a sum of monomials in z_s, w_s of high order in z_s. Further, $\mu(w_s^p) < \mu(z_s^{pc_s} w_s)$ in (21) and $\mu(w_s^p) < \mu(z_s^{c'_s} w_s)$ in (22).

In particular, $R_s \rightarrow T_s$ is not a strongly monomial form.

Substituting (19) into u_s and v_s in (20), we have

$$u_s = \alpha \overline{z}^{ap} \overline{w}^{bp}, v_s = \beta \overline{z}^{cp} \overline{w}^{dp} + \Omega. \tag{23}$$

We necessarily have that $u_s | v_s$ or $v_s | u_s$ in T'.

First suppose that $c \geq a$ and $d \geq b$. Then we have that

$$u_s = \alpha \overline{z}^{ap} \overline{w}^{bp}, \frac{v_s}{u_s} = \beta \overline{z}^{(c-a)p} \overline{w}^{(d-b)p} + \frac{\Omega}{\alpha \overline{z}^{ap} \overline{w}^{bp}}$$

giving an expression of the form (23). We will show that this is not a strongly monomial form. If it is, then we must have that $a = 0$ or $b = 0$ so that either

$$z_s = \overline{w}, w_s = \overline{z} \overline{w}^d \tag{24}$$

or

$$z_s = \overline{z}, w_s = \overline{z}^c\overline{w} \tag{25}$$

and we must have that $\frac{\Omega}{u_s}$ is part of a regular system of parameters in T'. Substituting into (21) or (22), we see that this cannot occur except possibly in the case that (22) holds and $\frac{z_s^{c'_s}w_s}{u_s}$ is part of a regular system of parameters in T'.

Suppose that (22) and (24) hold with

$$\frac{z_s^{c'_s}w_s}{u_s} = \frac{\overline{w}^{c'_s+d}\overline{z}}{\alpha\overline{w}^p}$$

being part of a regular system of parameters in T'. Now in this case, $\mu(w_s) > \mu(z_s)$ and $\mu(w_s^p) < \mu(z_s^{c'_s}w_s)$ so $p \le c'_s$. Thus $\frac{\overline{w}^{c'_s+d}\overline{z}}{\alpha\overline{w}^p}$ cannot be part of a regular system of parameters in T'. A similar argument shows that we do not obtain a strongly monomial form when (22) and (25) hold.

Suppose that $c < a$ and $d < b$. Then we have expressions

$$v_s = \gamma\overline{z}^{cp}\overline{w}^{dp}, \frac{u_s}{v_s} = \alpha\gamma^{-1}\overline{z}^{(a-c)p}\overline{w}^{(b-d)p}$$

where $\gamma \in T'$ is a unit, giving an expression of the form of (23), which is not strongly monomial. Thus we reduce to the case where $(c-a)(d-b) < 0$. We then have that $u_s \not| v_s$ since $u_s \not| \overline{z}^{cp}\overline{w}^{dp}$. Suppose that $v_s | u_s$. Then $v_s = \lambda\overline{z}^{cp}\overline{w}^{dp}$ where λ is a unit in T'. But this is impossible since $(c-a)(d-b) < 0$. Thus $R' \to T'$ has a form (23) with $a, b, c, d > 0$ and so cannot be a strongly monomial form. We have established that strong local monomialization along μ does not hold above $A \to C$.

From Theorem 3.2, we have that

$$\left(\frac{c_{r+1}}{p-1}\right)\frac{1}{m_1\cdots m_{r+1}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_{r+1}}\left(\frac{1}{m_1\cdots m_r}\right) + \frac{1}{m_1\cdots m_{r+1}}. \tag{26}$$

Then from Theorem 3.3, we have that

$$\frac{c_{r+2}}{p-1} = \left(\frac{c_{r+1}}{p-1}\right)m_{r+2} - m_{r+2},$$

and so

$$\left(\frac{c_{r+2}}{p-1}\right)\frac{1}{m_1\cdots m_{r+2}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_1\cdots m_{r+1}}. \tag{27}$$

By equation (16) we have

$$\frac{1}{2^{r+1}} > \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \left(\frac{q_{r+1}}{m_{r+1}}\right)\frac{1}{m_1\cdots m_r} > 0. \tag{28}$$

By Theorem 3.2,

$$\left(\frac{c'_{r+2}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+2}} = \left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}} - \frac{q'_{r+2}}{m'_1\cdots m'_{r+2}} + \frac{1}{m'_1\cdots m'_{r+2}}$$

and by Theorem 3.3,

$$\frac{c'_{r+3}}{p-1} = \left(\frac{c'_{r+2}}{p-1}\right)m'_{r+3} - m'_{r+3}.$$

We thus have that

$$\left(\frac{c'_{r+3}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+3}}=\left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}}-\frac{q'_{r+2}}{m'_1\cdots m'_{r+2}}. \tag{29}$$

Equation (18) implies

$$\frac{1}{2^{r+2}}>\left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}}-\frac{q'_{r+2}}{m'_1\cdots m'_{r+2}}>0. \tag{30}$$

Now $J(S_i/R_i)=(x_i^{c_i})$ and $x_0=x_i^{m_1\cdots m_i}$ so $\omega(J(S_i/R_i))=\frac{c_i}{m_1\cdots m_i}\omega(x_0)$ and thus by Proposition 3.4, (27) and (28), we have that

$$-\mathrm{dist}(\omega/\nu)=\frac{1}{p-1}\inf_i\{\omega(J(S_i/R_i))\}=0.$$

We have that $J(T_i/S_i)=(z_i^{c'_i})$ and $z_0=z_i^{m'_1\cdots m'_i}$ so $\omega(J(T_i/S_i))=\frac{c'_i}{m'_1\cdots m'_i}\omega(z_0)$ and thus by Proposition 3.4, (29) and (30), we have that

$$-\mathrm{dist}(\mu/\omega)=\frac{1}{p-1}\inf_i\{\omega(J(T_i/S_i))\}=0.$$

□

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