ERRATA TO "A SIMPLER PROOF OF TOROIDALIZATION OF MORPHISMS FROM 3-FOLDS TO SURFACES"

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The page and line numbers refer to the manuscript which is posted on my webpage, www.math.missouri.edu/˜dale. This is the published version (Annales de L'Institut Fourier 63 (2013), 865 - 922), but the page and line numbers are different. A case was missed in Lemma 3.7 (Case (A) and a modification of (15) in the restatement of Definition 3.3 below). The consideration of this new case does not introduce any significant change in the proof. I have written out in detail all of the changes which need to be made in the manuscript to incorporate this new case. Numbers indexing equations, theorems, definitions etc. are as in the earlier manuscript. New equations, theorems etc. are indexed by letters.

Page 10, line 6: "natural numbers $r_2, \ldots, r_{m-2}$ and a positive integer r_{m-1}" should be "natural numbers $r_2, \ldots, r_{m-1}$".

Page 10, insert after line 7: Let $\omega'(m, r_2, \ldots, r_m)$ be a function which associates a positive integer to a positive integer m and natural numbers $r_2, \ldots, r_m$. We will give a precise form of ω' after Theorem 4.2.

Page 10: Definition 3.3 should be modified as follows.

Definition 3.3. *X is 3-prepared (with respect to $f : X \to S$) at a point $p \in D$ if $\sigma_D(p) = 0$ or if $\sigma_D(p) > 0$, f is 2-prepared with respect to D at p and there are permissible parameters x, y, z at p such that x, y, z are uniformizing parameters on an étale cover of an affine neighborhood of p and we have one of the following forms, with $m = \sigma_D(p) + 1$:*

1) *p is a 2-point, and we have an expression (2) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^{r_m} y^{s_m} \tag{13}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero), $r_i + s_i > 0$ whenever $\tau_i \neq 0$ and $(r_m + c)b - (s_m + d)a \neq 0$. Further, $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

2) *p is a 1-point, and we have an expression (1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + \tau_m x^{r_m} \tag{14}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leq i \leq m-1$, $\tau_m \in \hat{\mathcal{O}}_{X,p}$ and $\mathrm{ord}(\tau_m(0, y, 0)) = 1$ (or $\tau_m = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$, and $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

3) *p is a 1-point, and we have an expression (1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + x^t \Omega \tag{15}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \leq i \leq m-1$, $\Omega \in \hat{\mathcal{O}}_{X,p}$, $\tau_i \neq 0$ for some i with $2 \leq i \leq m-1$ and $t > \omega(m, r_2, \ldots, r_{m-1})$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$.

4) *p is a 1-point, and we have an expression (1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} y z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} yz + \tau_m x^{r_m} y + x^t \Omega \tag{A}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ *is a unit,* $\tau_i \in \hat{\mathcal{O}}_{X,p}$ *are units (or zero) for* $2 \le i \le m$, $\Omega \in \hat{\mathcal{O}}_{X,p}$, $\tau_i \ne 0$ *for some* i *with* $2 \le i \le m$ *and* $t > \omega'(m, r_2, \ldots, r_m)$ *(where we set* $r_i = 0$ *if* $\tau_i = 0$*). Further,* $r_i > 0$ *if* $\tau_i \ne 0$.

X is 3-prepared if X is 3-prepared for all $p \in X$.

Pages 12 - 13: Lemma 3.6 and its proof should be changed as follows.

Lemma 3.6. *Suppose that* X *is 2-prepared with respect to* $f : X \to S$. *Suppose that* $p \in D$ *is a 1-point with* $m = \sigma_D(p) + 1 > 1$. *Let* u, v *be permissible parameters for* $f(p)$ *and* x, y, z *be permissible parameters for* D *at* p *such that a form (9) holds at* p. *Let* U *be an étale cover of an affine neighborhood of* p *such that* x, y, z *are uniformizing parameters on* U. *Let* C *be the curve in* U *which has local equations* $x = y = 0$ *at* p.

Let $T_0 = Spec(\mathfrak{k}[x,y])$, $\Lambda_0 : U \to T_0$. *Then there exists a sequence of quadratic transforms* $T_1 \to T_0$ *such that if* $U_1 = U \times_{T_0} T_1$ *and* $\psi_1 : U_1 \to U$ *is the induced sequence of blow ups of sections over* C, $\Lambda_1 : U_1 \to T_1$ *is the projection, then* U_1 *is 2-prepared with respect to* $f \circ \psi_1$ *at all* $p_1 \in \psi_1^{-1}(p)$. *Further, for every point* $p_1 \in \psi_1^{-1}(p)$, *there exist regular parameters* x_1, y_1 *in* $\hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ *such that* x_1, y_1, z *are permissible parameters at* p_1, *and there exist regular parameters* $\tilde{x}_1, \tilde{y}_1$ *in* $\mathcal{O}_{T_1,\Lambda_1(p_1)}$ *such that if* p_1 *is a 1-point,* $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ *where* $\alpha(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ *is a unit series and* $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)$ *with* $\beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$, *and if* p_1 *is a 2-point, then* $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ *and* $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)\tilde{y}_1$, *where* $\alpha(\tilde{x}_1, \tilde{y}_1), \beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ *are unit series. We have one of the following forms:*

1) p_1 *is a 2-point, and we have an expression (2) with*

$$F = \tau z^m + \overline{a}_2(x_1,y_1) x_1^{r_2} y_1^{s_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_1,y_1) x_1^{r_{m-1}} y_1^{s_{m-1}} z + \overline{a}_m x_1^{r_m} y_1^{s_m} \tag{18}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ *is a unit,* $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ *are units (or zero) for* $2 \le i \le m-1$, $\overline{a}_m = 0$ *or 1 and if* $\overline{a}_m = 0$, *then* $\overline{a}_{m-1} \ne 0$. *Further,* $r_i + s_i > 0$ *whenever* $\overline{a}_i \ne 0$ *and* $a(r_m + c)b - (s_m + d)a \ne 0$.

2) p_1 *is a 1-point, and we have an expression (1) with*

$$F = \tau z^m + \overline{a}_2(x_1,y_1) x_1^{r_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_1,y_1) x_1^{r_{m-1}} z + x_1^{r_m} y_1 \tag{19}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ *is a unit,* $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ *are units (or zero) for* $2 \le i \le m-1$. *Further,* $r_i > 0$ *(whenever* $\overline{a}_i \ne 0$*).*

3) p_1 *is a 1-point, and we have an expression (1) with*

$$F = \tau z^m + \overline{a}_2(x_1,y_1) x_1^{r_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_1,y_1) x_1^{r_{m-1}} z + x_1^t y_1 \Omega \tag{20}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ *is a unit,* $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ *are units (or zero) for* $2 \le i \le m-1$ *and* $r_i > 0$ *whenever* $\overline{a}_i \ne 0$. *We also have* $t > \omega(m, r_2, \ldots, r_{m-1})$. *Further,* $\overline{a}_i \ne 0$ *for some* $2 \le i \le m-1$ *and* $\Omega \in \hat{\mathcal{O}}_{U_1,p_1}$.

4) p_1 *is a 1-point, and we have an expression (1) with*

$$F = \tau z^m + \overline{a}_2(x_1,y_1) x_1^{r_2} y_1 z^{m-2} + \cdots + \overline{a}_m(x_1,y_1) x_1^{r_m} y_1 z + x_1^t y_1^2 \Omega \tag{B}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ *is a unit,* $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ *are units (or zero) for* $2 \le i \le m$ *and* $r_i > 0$ *whenever* $\overline{a}_i \ne 0$. *We also have* $t > \omega'(m, r_2, \ldots, r_m)$. *Further,* $\overline{a}_i \ne 0$ *for some* $2 \le i \le m$ *and* $\Omega \in \hat{\mathcal{O}}_{U_1,p_1}$.

Proof. Let $\overline{p} = \Lambda_0(p)$. Let $T = \{i \mid a_i(x,y) \neq 0 \text{ and } 2 \leq i < m\}$. There exists a sequence of blow ups $\varphi_1 : T_1 \to T_0$ of points over $\overline{p}$ such that at all points $q \in \psi_1^{-1}(p)$, we have permissible parameters x_1, y_1, z such that x_1, y_1 are regular parameters in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$ and we have that ug is a monomial in x_1 and y_1 times a unit in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$, where $g = \prod_{i\in T} a_i(x,y)$.

Suppose that $a_m(x,y) \neq 0$. Let $\overline{v} = x^b a_m(x,y)$ if (1) holds and $\overline{v} = x^c y^d a_m(x,y)$ if (2) holds. We have $\overline{v} \notin \mathfrak{k}[[x]]$ (respectively $\overline{v} \notin \mathfrak{k}[[x^a y^b]]$). Then by Lemma 3.5 applied to $u, \overline{v}$, we have that there exists a further sequence of blow ups $\varphi_2 : T_2 \to T_1$ of points over $\overline{p}$ such that at all points $q \in (\psi_1 \circ \psi_2)^{-1}(p)$, we have permissible parameters x_2, y_2, z such that x_2, y_2 are regular parameters in $\hat{\mathcal{O}}_{T_2,\Lambda_2(q)}$ such that $ug = 0$ is a SNC divisor and either

$$u = x_2^{\overline{a}}, \overline{v} = \overline{P}(x_2) + x_2^{\overline{b}} \overline{y}_2^{\overline{c}}$$

with $\overline{c} > 0$ or

$$u = (x_2^{\overline{a}} y_2^{\overline{b}})^t, \overline{v} = \overline{P}(x_2^{\overline{a}} y_2^{\overline{b}}) + x_2^{\overline{c}} \overline{y}_2^{\overline{d}}$$

where $\overline{a}\overline{d} - \overline{b}\overline{c} \neq 0$.

At a 2-point $p_2 \in U_2$ above p we have an expression (2) with

$$F = \tau z^m + \overline{a}_2(x_2,y_2) x_2^{r_2} y_2^{s_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_2,y_2) x_2^{r_{m-1}} y_2^{s_{m-1}} z + \overline{a}_m x_2^{r_m} y_2^{s_m} \tag{21}$$

where τ is a unit series, $\overline{a}_m = 0$ or 1 and $\overline{a}_i$ are unit series (or zero) for $2 \leq i \leq m$.

At a 1-point $p_2 \in U_2$ above p we have by the Weierstrass Preparation Theorem an expression (1) with

(C)

$$F = \tau z^m + \overline{a}_2(x_2,\overline{y}_2) x_2^{r_2} (\overline{y}_2 + \varphi(x_2))^{s_2} z^{m-2} + \cdots + \overline{a}_{m-1}(x_2,\overline{y}_2) x_2^{r_{m-1}} (\overline{y}_2 + \varphi(x_2))^{s_{m-1}} z + \overline{a}_m x_2^{r_m} \overline{y}_2^{s_m}$$

where τ is a unit series, $\overline{a}_m = 0$ or 1, $s_m \geq 1$ and $\overline{a}_i$ are unit series (or zero) for $2 \leq i \leq m$, $\varphi \in k[[x_2]]$ is a series with $0 \leq r := \text{ord } \varphi \leq \infty$, and $x_2, \overline{y}_2, z$ are permissable parameters with $\overline{y}_2 \in \hat{\mathcal{O}}_{T_2}$.

We will show that after a finite number of blow ups of points $T_3 \to_2$, we have an expression (2) with (21) for all 2-points $p_3 \in U_3$ above p and an expression (1) with (21) for all 1-points $p_3 \in U_3$ above p.

Suppose that $p_2 \in U_2$ is a 1-point. If $r = 0$ or ∞, $\overline{a}_m = 0$ or $s_m = 1$, then after a permissible change of variables we have a form (21). Suppose that a form (21) does not hold at p_2. Then $0 < r = \text{ord } \varphi < \infty$, $\overline{a}_m = 1$ and $s_m > 1$. Let $T_3 \to T_2$ be the blow up of $\Lambda_2(p_2) \in T_2$ with induced blow up $\psi_3 : U_3 \to U_2$. Suppose $p_3 \in \psi_3^{-1}(p_2)$. We have permissible parameters x_3, y_3, z in $\hat{\mathcal{O}}_{U_3,p_3}$ such that one of the following cases holds:

a) $x_2 = x_3, \overline{y}_2 = x_3(y_3 + \alpha)$ with $0 \neq \alpha \in \mathfrak{k}$.
b) $x_2 = x_3$, $\overline{y}_2 = x_3 y_3$.
c) $x_2 = x_3 y_3$, $\overline{y}_2 = y_3$.

If a) holds at p_3 then an expression (1) holds at p_3 with a form (C) with $s_m = 1$, so after a change of variables we have a form (21). If b) holds at p_3, then an expression (1) holds at p_3 with a form (C) with ord $\varphi < r$. If ord $\varphi = 0$ we then have an expression (21) (with $s_i = 0$ for $2 \leq i \leq m-1$). If c) holds then

$$\overline{y}_2 + \varphi(x_2) = y_3 + \varphi(x_3 y_3) = y_3 \gamma(x_3, y_3)$$

where γ is a unit series. Thus we have an expression (2) with a form (21).

By induction on r, we must obtain an expression (1) or (2) with a form (21) at all points above p. We may assume that this already holds in U_2.

Suppose that p_2 is a 1-point above p.

Let

$$J = I_2\begin{pmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} & \frac{\partial v}{\partial z}\end{pmatrix} = x^n(\frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial z})$$

for some positive integer n. Since D contains the locus where f is not smooth, we have that the localization $J_{\mathfrak{p}} = (\hat{\mathcal{O}}_{U_2,q})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal (y_2, z_2) in $\hat{\mathcal{O}}_{U_2,q}$.

We compute

$$\frac{\partial F}{\partial z} = \overline{a}_{m-1}x_2^{r_{m-1}}y_2^{s_{m-1}} + \Lambda_1 z$$

and

$$\frac{\partial F}{\partial y_2} = s_m\overline{a}_m y_2^{s_m-1}x_2^{r_m} + \Lambda_2 z$$

for some $\Lambda_1, \Lambda_2 \in \hat{\mathcal{O}}_{U_2,q}$, to see that either

(D) $\qquad \overline{a}_{m-1} \neq 0$ and $s_{m-1} = 0$, or $\overline{a}_m \neq 0$ and $s_m = 1$.

If $s_i \geq 2$ for all i, we have

$$\frac{\partial F}{\partial z} = z\Lambda_1 + y_2^2\Lambda_2, \frac{\partial F}{\partial y_2} = y_2\Lambda_3$$

for some $\Lambda_1, \Lambda_2, \Lambda_3 \in \hat{\mathcal{O}}_{U_2,p_2}$, a contradiction, so

(E) $\qquad$ There exists an i with $2 \leq i \leq m$ such that $\overline{a}_i \neq 0$ and $s_i \leq 1$.

Suppose $p_2 \in U_2$ is a 2-point above p. Deforming p_2 to a 1-point, we see from (D) (and since (21) holds) that (18) holds at p_2.

Let $p_2 \in U_2$ be a 1-point above p where the conclusions of the lemma do not hold. Let $T_3 \rightarrow T_2$ be the blow up of $\Lambda_2(p_2) \in T_2$ with induced blow up $\psi_3 : U_3 \rightarrow U_2$. Suppose $p_3 \in \psi_3^{-1}(p_2)$. We have that the conclusions of the lemma hold in the form (18) if p_3 is the 2-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3y_3$ and $y_2 = y_3$. At a 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$, $y_2 = x_3(y_3+\alpha)$ with $\alpha \neq 0$, we have that a form (19) holds. Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$ and $y_2 = x_3y_3$. We continue to blow up, so that there is at most one point p_3 above p_2 where the conclusions of the lemma do not hold. This point is a 1-point which has permissible parameters x_3, y_3, z with $x_2 = x_3$ and $y_2 = x_3^n y_3$ where we can take n as large as we like. Substituting into (21), we have an expression (1) at p_3 with

$$F = \tau Z^m + \overline{a}_2 x_3^{r_2+s_2n}y_3^{s_2}z^{m-2} + \cdots + \overline{a}_{m-1}x_3^{e_{m-1}+s_{m-1}n}y_3^{s_{m-1}}z + \overline{a}_m x_3^{r_m+s_mn}y_3^{s_m}.$$

Since we must have some $\overline{a}_i \neq 0$ with $s_i \leq 1$ by (E) for $n \gg 0$, we obtain a form (20) or (B). $\square$

Page 13 - 14: The statement of Lemma 3.7 should be changed to:

Lemma 3.7. *Suppose that X is 2-prepared with respect to $f : X \rightarrow S$. Suppose that $p \in D$ is a 1-point with $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Let x, y, z be permissible parameters for D at p such that a form (9) holds at p.*

Let notation be as in Lemma 3.6. For $p_1 \in \psi_1^{-1}(p)$ let $\overline{r}(p_1) = m + 1 + r_m$, if a form (19) holds at p_1, and

$$\overline{r}(p_1) = \begin{cases} max\{m+1+r_m, m+1+s_m\} & if\ \overline{a}_m = 1\\ max\{m+1+r_{m-1}, m+1+s_{m-1}\} & if\ \overline{a}_m = 0\end{cases}$$

if a form (18) holds at p_1*. Let* $\overline{r}(p_1) = \omega(m, r_2, \ldots, r_{m-1}) + m + 1$ *if a form (20) holds at* p_1*,* $\overline{r}(p_1) = \omega'(m, r_2, \ldots, r_m) + m + 1$ *if a form (B) holds at* p_1*. Let*

$$r = \max\{\overline{r}(p_1) \mid p_1 \in \psi_1^{-1}(p)\}. \tag{23}$$

Suppose that $x^* \in \mathcal{O}_{X,p}$ *is such that* $x = \overline{\gamma} x^*$ *for some unit* $\overline{\gamma} \in \hat{\mathcal{O}}_{X,p}$ *with* $\overline{\gamma} \equiv 1 \mod m_p^r \hat{\mathcal{O}}_{X,p}$.

Let V *be an affine neighborhood of* p *such that* $x^*, y \in \Gamma(V, \mathcal{O}_X)$*, and let* C^* *be the curve in* V *which has local equations* $x^* = y = 0$ *at* p.

Let $T_0^* = Spec(\mathfrak{k}[x^*, y])$*. Then there exists a sequence of blow ups of points* $T_1^* \to T_0^*$ *above* (x^*, y) *such that if* $V_1 = V \times_{T_0^*} T_1^*$ *and* $\psi_1^* : V_1 \to V$ *is the induced sequence of blow ups of sections over* C^**,* $\Lambda_1^* : V_1 \to T_1^*$ *is the projection, then* V_1 *is 2-prepared at all* $p_1^* \in (\psi_1^*)^{-1}(p)$*. Further, for every point* $p_1^* \in (\psi_1^*)^{-1}(p)$*, there exist* $\hat{x}_1, \overline{y}_1 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *such that* $\hat{x}_1, \overline{y}_1, z$ *are permissible parameters at* p_1^* *and we have one of the following forms:*

1) p_1^* *is a 2-point, and we have an expression (2) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} \overline{y}_1^{s_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} \overline{y}_1^{s_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \overline{y}_1^{s_m} \tag{24}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero) for* $0 \le i \le m-1$*,* $\overline{\tau}_m$ *is zero or 1,* $\overline{\tau}_{m-1} \ne 0$ *if* $\overline{\tau}_m = 0$*,* $r_i + s_i > 0$ *if* $\overline{\tau}_i \ne 0$*, and*

$$(r_m + c)b - (s_m + d)a \ne 0.$$

2) p_1^* *is a 1-point, and we have an expression (1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \tag{25}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero), and ord* $\overline{\tau}_m(0, \overline{y}_1, 0) = 1$*. Further,* $r_i > 0$ *if* $\overline{\tau}_i \ne 0$.

3) p_1^* *is a 1-point, and we have an expression (1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} z + \hat{x}_1^t \overline{\Omega} \tag{26}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero),* $\overline{\Omega} \in \hat{\mathcal{O}}_{V_1,p_1^*}$*,* $\overline{\tau}_i \ne 0$ *for some* $2 \le i \le m-1$ *and* $t > \omega(m, r_2, \ldots, r_{m-1})$*. Further,* $r_i > 0$ *if* $\overline{\tau}_i \ne 0$.

4) p_1^* *is a 1-point, and we have an expression (1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} \overline{y}_1 z^{m-2} + \cdots + \overline{\tau}_m \hat{x}_1^{r_m} \overline{y}_1 + \hat{x}_1^t \overline{\Omega} \tag{F}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *is a unit,* $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ *are units (or zero) for* $2 \le i \le m$*,* $\overline{\Omega} \in \hat{\mathcal{O}}_{V_1,p_1^*}$*,* $\overline{\tau}_i \ne 0$ *for some* $2 \le i \le m$ *and* $t > \omega'(m, r_2, \ldots, r_m)$*. Further,* $r_i > 0$ *if* $\overline{\tau}_i \ne 0$.

Page 15, line -11: "$\overline{\tau} \in \mathfrak{k}[[x_1, \overline{y}_1]]$ with $\mathrm{ord}(\overline{\tau}_m(0, \overline{y}_1)) = 1$" should be "$\overline{\tau}_m \in \mathfrak{k}[[x_1, \overline{y}_1, z]]$ with $\mathrm{ord}(\overline{\tau}_m(0, \overline{y}_1, 0)) = 1$".

Page 15, line -6: after "form (26)" insert: "and in the case when p_1 has a form (B) a similar argument shows that p_1^* has a form (F)."

Page 22, line -9: At the end of this line, insert "Further, if $I\mathcal{O}_{U',q}$ is principal, then $\sigma_D(q) = 0$".

Page 23, line 7: "$\sigma_{D'}(q)$" should be "$\sigma_D(q)$".

Page 23, line 13: "$\sigma_D(q) < m - h - 1$" should be "$\sigma_D(q) \le m - h - 1$"

Page 24, line 9: "r_m if $\tau_i \ne 0$" should be "r_m if $\tau_m \ne 0$"

Page 24, line 17: "Then x^{r_m} generates $I\hat{\mathcal{O}}_{U',q}$" should be "If $\tau_m \neq 0$, then x^{r_m} generates $I\hat{\mathcal{O}}_{U',q}$".

Page 24, line 18: At end of line, insert: "If $\tau_m = 0$ then $\tau_{m-1} \neq 0$ so $x^{r_{m-1}}z$ generates $I\hat{\mathcal{O}}_{U',q}$. Then $G' = x_1^{r_{m-1}+b_1}\Lambda$ with ord $\Lambda(0,0,z) = 1$ so U' is prepared at q."

Page 24, line 20: Remove "and $r_{m-1} > 0$"

Page 24, lines 22 and 28: "$r_i > 0$" should be $\tau_i \neq 0$".

Page 24, line 27: Should be "$z = x_1^{b_1}z_1$ for some $b_1 \in \mathbb{Z}_+$"

Page 25, lines 2,4,6: "$r_i > 0$" should be $\tau_i \neq 0$".

Page 25, line 3: "For" should be "for".

Page 25, line 19 (at end of proof of Theorem 4.2) add: "The only case that may require a little attention is when $p' \in \psi^{-1}(p)$ is defined by a substitution (41). This is analyzed as follows. Let h be the largest i such that $\tau_i \neq 0$ in (15). Then $x^{r_h}z^{m-h}$ is the local generator of $I\hat{\mathcal{O}}_{U',p'}$. Since $2 \leq h \leq m-1$, we have that

$$\sigma_D(p') = m - h - 1 < m - 1 = \sigma_D(p).''$$

Before the statement of Theorem 4.3 on page 25, add the following:

We construct the function $\omega'(m, r_2, \ldots, r_m)$ in a similar way to the construction of ω. Let I be the ideal in $k[x,z]$ generated by z^m and $x^{r_i}z^{m-i}$ for all i such that $2 \leq i \leq m$ and $\tau_i \neq 0$. We define $\omega'(m, r_2, \ldots, r_m)$ as we define ω, except we allow i to range within $2 \leq i \leq m$.

Theorem G. *Suppose that $p \in Sing_1(X)$ is a 1-point and X is 3-prepared at p. Let x, y, z be permissible parameters at p giving a form (A) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U.*

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \to S$. We have that the curve C_i blown up in $U_{i+1} \to U_i$ is in $Sing_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $Sing_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. The proof is similar to that of Theorem 4.2, using the fact that $t > \omega'(m, r_2, \ldots, r_m)$ as defined above. For instance, in the case when $p' \in \psi^{-1}(p)$ is defined by a substitution (41) and h is the largest i such that $\tau_i \neq 0$, we have

$$\sigma_D(p') = (m - h + 1) - 1 < m - 1 = \sigma_D(p)$$

since $2 \leq h$. □

Page 32, lines 18 and 26: should be "(14), (15) or (A)"

Page 32, line -4: should be "Theorem 4.1, 4.2 or G".

Page 40, lines 13 and -14: should be "Theorem 4.2 and G"

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