THE MINKOWSKI EQUALITY OF BIG DIVISORS

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Abstract. We give conditions characterizing equality in the Minkowski inequality for big divisors on a projective variety. Our results draw on the extensive history of research on Minkowski inequalities in algebraic geometry.

1. Introduction

Suppose that X is a projective d-dimensional algebraic variety over a field k and D is an $\mathbb{R}$-Cartier divisor on X. Then the volume of D is

$$\mathrm{vol}(D)=\lim_{n\to\infty}\frac{\dim_k\Gamma(X,\mathcal{O}_X(nD))}{n^d/d!}.$$

If D is nef, then the volume of D is the self intersection number $\mathrm{vol}(D)=(D^d)$. For an arbitrary $\mathbb{R}$-Cartier divisor D,

$$\mathrm{vol}(D)=\begin{cases}\langle D^d\rangle & \text{if } D \text{ is pseudo effective}\\ 0 & \text{otherwise.}\end{cases}$$

Here $\langle D^d\rangle$ is the positive intersection product. The positive intersection product $\langle D^d\rangle$ is the ordinary intersection product (D^d) if D is nef, but these products are different in general. More generally, given pseudo effective $\mathbb{R}$-Cartier divisors $D_1,\ldots,D_p$ on X with $p\le d$, there is a positive intersection product $\langle D_1\cdot\ldots\cdot D_p\rangle$ which is a linear form on $N^1(\mathcal{X})^{d-p}$, where $\mathcal{X}$ is the limit of all birational models of X. We have that

$$\mathrm{vol}(D)=\langle D^p\rangle=\langle D\rangle\cdot\ldots\cdot\langle D\rangle=\langle D\rangle^d.$$

We denote the linear forms on $N^1(\mathcal{X})^{d-p}$ by $L^{d-p}(\mathcal{X})$. The intersection theory and theory of volumes which is required for this paper is reviewed in Section PrelSect 2.

Suppose that D_1 and D_2 are pseudo effective $\mathbb{R}$-Cartier divisors on X. We have the Minkowski inequality

$$\mathrm{vol}(D_1+D_2)^{\frac{1}{d}}\ge\mathrm{vol}(D_1)^{\frac{1}{d}}+\mathrm{vol}(D_2)^{\frac{1}{d}}$$

which follows from Theorem Ineq+ 1.2 below. Further, we have the following characterization of equality in the Minkowski inequality.

Theorem22+ **Theorem 1.1.** *Let X be a d-dimensional projective variety over a field k. For any two big $\mathbb{R}$-Cartier divisors D_1 and D_2 on X,*

Neweq20+ (1)
$$\mathrm{vol}(D_1+D_2)^{\frac{1}{d}}\ge\mathrm{vol}(D_1)^{\frac{1}{d}}+\mathrm{vol}(D_2)^{\frac{1}{d}}$$

with equality if and only if $\langle D_1\rangle$ and $\langle D_2\rangle$ are proportional in $L^{d-1}(\mathcal{X})$.

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In the case that D_1 and D_2 are nef and big, this is proven in [4, Theorem 2.15] (over an algebraically closed field of characteristic zero) and in [9, Theorem 6.13] (over an arbitrary field). In this case of nef divisors, the condition that $\langle L_1\rangle$ and $\langle L_2\rangle$ are proportional in $L^{d-1}(\mathcal{X})$ is just that D_1 and D_2 are proportional in $N^1(X)$.

Theorem 1.1 is obtained in the case that D_1 and D_2 are big and movable and k is an algebraically closed field of characteristic zero in [26, Proposition 3.7]. In this case the condition for equality is that D_1 and D_2 are proportional in $N^1(X)$. Theorem 1.1 is established in the case that D_1 and D_2 are big $\mathbb{R}$-Cartier divisors and X is nonsingular, over an algebraically closed field k of characteristic zero in [26, Theorem 1.6]. In this case, the condition for equality is that the positive parts of the σ decompositions of D_1 and D_2 are proportional; that is, $P_\sigma(D_1)$ and $P_\sigma(D_2)$ are proportional in $N^1(X)$.

In Section 5, we modify the proof sketched in [26] of [26, Proposition 3.7] to be valid over an arbitrary field. Characteristic zero is required in the proof in [26] as the existence of resolution of singularities is assumed and an argument using the theory of multiplier ideals is used, which requires characteristic zero as it relies on both resolution of singularities and Kodaira vanishing.

We will write

$$s_i = \langle D_1^i \cdot D_2^{d-i}\rangle \text{ for } 0 \le i \le d.$$

We have the following generalization of the Khovanskii-Teissier inequalities to positive intersection numbers.

Ineq+ **Theorem 1.2.** *(Minkowski Inequalities) Suppose that X is a projective algebraic variety of dimension d over a field k and D_1 and D_2 are pseudo effective $\mathbb{R}$-Cartier divisors on X. Then*

1) $s_i^2 \ge s_{i+1}s_{i-1}$ *for* $1 \le i \le d-1$.
2) $s_i s_{d-i} \ge s_0 s_d$ *for* $1 \le i \le d-1$.
3) $s_i^d \ge s_0^{d-i}s_d^i$ *for* $0 \le i \le d$.
4) $\mathrm{vol}(D_1+D_2)^{\frac{1}{d}} \ge \mathrm{vol}(D_1)^{\frac{1}{d}} + \mathrm{vol}(D_2)^{\frac{1}{d}}$.

Theorem 1.2 follows from [4, Theorem 2.15] when k has characteristic zero and from [9, Theorem 6.6] in general. When D_1 and D_2 are nef, the inequalities of Theorem 1.2 are proven by Khovanskii and Teissier [33], [34], [23, Example 1.6.4]. In the case that D_1 and D_2 are nef, we have that $s_i = \langle D_1^i \cdot D_2^{d-i}\rangle = (D_1^i \cdot D_2^{d-i})$ are the ordinary intersection products.

We have the following characterization of equality in these inequalities.

Minkeq+ **Theorem 1.3.** *(Minkowski equalities) Suppose that X is a projective algebraic variety of dimension d over a field k of characteristic zero, and D_1 and D_2 are big $\mathbb{R}$-Cartier divisors on X. Then the following are equivalent:*

1) $s_i^2 = s_{i+1}s_{i-1}$ *for all* $1 \le i \le d-1$.
2) $s_i s_{d-i} = s_0 s_d$ *for all* $1 \le i \le d-1$.
3) $s_i^d = s_0^{d-i}s_d^i$ *for all* $0 \le i \le d$.
4) $s_{d-1}^d = s_0 s_d^{d-1}$.
5) $\mathrm{vol}(D_1+D_2)^{\frac{1}{d}} = \mathrm{vol}(D_1)^{\frac{1}{d}} + \mathrm{vol}(D_2)^{\frac{1}{d}}$.
6) $\langle D_1\rangle$ *is proportional to* $\langle D_2\rangle$ *in* $L^{d-1}(\mathcal{X})$.

Theorem 1.3 is valid over any field k when $\dim X \le 3$, since resolution of singularities is true in these dimensions. When D_1 and D_2 are nef and big, then Theorem 1.3 is proven in [4, Theorem 2.15] when k has characteristic zero and in [9, Theorem 6.13] for arbitrary

k. When D_1 and D_2 are nef and big, the condition 6) of Theorem 1.3 is just that D_1 and D_2 are proportional in $N^1(X)$.

Suppose that $s_0, \ldots, s_d$ are nonnegative real numbers, $s_0 > 0$, $s_d > 0$ and the inequalities of 1), 2) and 3) of the statement of Theorem 1.2 hold. These last conditions always hold when $s_i = \langle D_1^i \cdot D_2^{d-i} \rangle$ with D_1, D_2 big $\mathbb{R}$-Cartier divisors on a projective d-dimensional algebraic variety. The assumption that $s_0, s_d > 0$ and inequalities 3) imply $s_i > 0$ for $0 \le i \le d$.

Suppose that the equality 4) also holds, $s_{d-1}^d = s_0 s_d^{d-1}$. By the inequalities 1) we have that

$$\frac{s_{d-1}}{s_0} = \left(\frac{s_{d-1}}{s_{d-2}}\right)\left(\frac{s_{d-2}}{s_{d-3}}\right)\cdots\left(\frac{s_1}{s_0}\right) \ge \left(\frac{s_d}{s_{d-1}}\right)^{d-1}.$$

Thus $s_{d-1}^d = s_0 s_d^{d-1}$ implies the equalities 1), $s_i^2 = s_{i+1}s_{i-1}$, hold for $1 \le i \le d-1$, and so the equalities 2) and 3) also hold.

However, we get weaker conclusions if we only assume that $s_j^d = s_0^{d-j} s_d^j$, for some $j < d-1$. In this case we have the equality

$$\frac{s_j^{d-j}}{s_0^{d-j}} = \left(\frac{s_j}{s_{j-1}}\right)^{d-j}\cdots\left(\frac{s_1}{s_0}\right)^{d-j} = \left(\frac{s_d}{s_{d-1}}\right)^{j}\cdots\left(\frac{s_{j+1}}{s_j}\right)^{j} = \frac{s_d^j}{s_j^j}$$

implying that $s_i^2 = s_{i+1}s_{i-1}$ for $1 \le i \le j$.

The proof of Theorem 1.3 relies on the following Diskant inequality for big divisors.

Suppose that X is a projective variety and D_1 and D_2 are big $\mathbb{R}$-Cartier divisors on X. The slope $s(D_1, D_2)$ is the largest real number s such that $\langle D_1 \rangle \ge s \langle D_2 \rangle$.

Theorem 1.4. *(Diskant inequality for big divisors) Suppose that X is a projective d-dimensional variety over a field k of characteristic zero and D_1, D_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

$$\text{(2)} \quad \langle D_1^{d-1} \cdot D_2 \rangle^{\frac{d}{d-1}} - \mathrm{vol}(D_1)\mathrm{vol}(D_2)^{\frac{1}{d-1}} \ge [\langle D_1^{d-1} \cdot D_2 \rangle^{\frac{1}{d-1}} - s(D_1, D_2)\mathrm{vol}(D_2)^{\frac{1}{d-1}}]^d.$$

The Diskant inequality is proven for nef and big divisors in [4, Theorem G] in characteristic zero and in [9, Theorem 6.9] for nef and big divisors over an arbitrary field. In the case that D_1 and D_2 are nef and big, the condition that $\langle D_1 \rangle - s\langle D_2 \rangle$ is pseudo effective in $L^{d-1}(\mathcal{X})$ is that $D_1 - sD_2$ is pseudo effective in $N^1(X)$. The Diskant inequality is proven when D_1 and D_2 are big and movable divisors and X is a projective variety over an algebraically closed field of characteristic zero in [26, Proposition 3.3, Remark 3.4]. Theorem 1.4 is a consequence of [13, Theorem 3.6].

Let D_1 and D_2 be big $\mathbb{R}$-Cartier divisors on a projective variety X. Generalizing Teissier [33], we define the inradius of D_1 with respect to D_2 as

$$r(D_1, D_2) = s(D_1, D_2)$$

and the outradius of D_1 with respect to D_2 as

$$R(D_1, D_2) = \frac{1}{s(D_2, D_1)}.$$

We deduce the following consequence of the Diskant inequality.

TheoremH+

Theorem 1.5. *Suppose that X is a d-dimensional projective variety over a field k of characteristic zero and D_1, D_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

(3)
$$\frac{s_{d-1}^{\frac{1}{d-1}} - (s_{d-1}^{\frac{d}{d-1}} - s_0^{\frac{1}{d-1}} s_d)^{\frac{1}{d}}}{s_0^{\frac{1}{d-1}}} \le r(D_1, D_2) \le \frac{s_d}{s_{d-1}} \le \frac{s_1}{s_0} \le R(D_1, D_2) \le \frac{s_d^{\frac{1}{d-1}}}{s_1^{\frac{1}{d-1}} - (s_1^{\frac{d}{d-1}} - s_d^{\frac{1}{d-1}} s_0)^{\frac{1}{d}}}.$$

This gives a solution to [33, Problem B] for big $\mathbb{R}$-Cartier divisors. The inequalities of Theorem 1.5 are proven by Teissier in [33, Corollary 3.2.1] for divisors on surfaces satisfying some conditions. In the case that D_1 and D_2 are nef and big on a projective variety over a field of characteristic zero, Theorem 1.5 follows from the Diskant inequality [4, Theorem F]. In the case that D_1 and D_2 are nef and big on a projective variety over an arbitrary field, Theorem 1.5 is proven in [9, Theorem 6.11], as a consequence of the Diskant inequality [9, Theorem 6.9] for nef divisors.

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PrelSect

2. Preliminaries

In this section we review some properties of cycles and intersection theory on projective varieties over an arbitrary field.

2.1. **Codimension 1 cycles.** To establish notation we give a quick review of some material from [21], [18, Chapter 2] and [23, Chapter 1]. Although the ongoing assumption in [23] is that $k = \mathbb{C}$, this assumption is not needed in the material reviewed in this subsection.

Let X be a d-dimensional projective variety over a field k. The group of Cartier divisors on X is denoted by $\mathrm{Div}(X)$. There is a natural homomorphism from $\mathrm{Div}(X)$ to the $(d-1)$-cycles (Weil divisors) $Z_{d-1}(X)$ of X written as $D \mapsto [D]$. Further, there is a natural homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ given by $D \mapsto \mathcal{O}_X(D)$.

Denote numerical equivalence on $\mathrm{Div}(X)$ by $\equiv$. For D a Cartier divisor, $D \equiv 0$ if and only if $(C \cdot D)_X := \deg(\mathcal{O}_X(D) \otimes \mathcal{O}_C) = 0$ for all integral curves C on X.

The group $N^1(X)_{\mathbb{Z}} = \mathrm{Div}(X)/\equiv$ and $N^1(X) = N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$. An element of $\mathrm{Div}(X) \otimes \mathbb{Q}$ will be called a $\mathbb{Q}$-Cartier divisor and an element of $\mathrm{Div}(X) \otimes \mathbb{R}$ will be called an $\mathbb{R}$-Cartier divisor. In an effort to keep notation as simple as possible, the class in $N^1(X)$ of an $\mathbb{R}$-Cartier divisor D will often be denoted by D.

We will also denote the numerical equivalence on $Z_{d-1}(X)$ defined on page 374 [18] by $\equiv$. Let $N_{d-1}(X)_{\mathbb{Z}} = Z_{d-1}(X)/\equiv$ and $N_{d-1}(X) = N_{d-1}(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. There is a natural homomorphism $N^1(X) \to N_{d-1}(X)$ which is induced by associating to the class of a $\mathbb{R}$-Cartier divisor D the class in $N_{d-1}(X)$ of its associated Weil divisor $[D]$ [18, Section 2.1]. If $f : Y \to X$ is a morphism, the cycle map $f_* : Z_{d-1}(Y) \to Z_{d-1}(X)$ of [18, Section 1.4] induces a homomorphism $f_* : N_{d-1}(Y) \to N_{d-1}(X)$ ([18, Example 19.1.6]).

Suppose that $f : Y \to X$ is a dominant morphism where Y is projective variety. Then $f^* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ is defined by taking local equations of D on X as local equations of $f^*(D)$ on Y. There is an induced homomorphism $f^* : N^1(X) \to N^1(Y)$ which is an injection by [21, Lemma 1]. By [18, Proposition 2.3], we have that if D is an $\mathbb{R}$-Cartier divisor on X, then

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(4)
$$f_*[f^*D] = \deg(Y/X) D$$

where $\deg(Y/X)$ is the index of the function field of X in the function field of Y.

In this subsection, we will use the notation for intersection numbers of [18, Definition 2.4.2].

The first statement of the following lemma follows immediately from [28] or [22, Corollary XIII.7.4] if k is algebraically closed. The second statement is [18, Example 19.1.5].

Lemma55 **Lemma 2.1.** *Let X be a d-dimensional projective variety over a field k. Then:*

1) *The homomorphism $N^1(X) \to N_{d-1}(X)$ is an injection.*
2) *If X is nonsingular, then the homomorphism $N^1(X) \to N_{d-1}(X)$ is an isomorphism.*

Proof. Suppose that $N^1(X) \to N_{d-1}(X)$ is not injective. The homomorphism $N^1(X) \to N_{d-1}(X)$ is obtained by tensoring the natural map $N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to N_{d-1}(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathbb{R}$ over $\mathbb{Q}$. Thus $N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to N_{d-1}(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is not injective, and so there exists a Cartier divisor D on X such that the Weil divisor $[D]$ associated to D is numerically equivalent to zero (its class is zero in $N_{d-1}(X)$) but the class of D is not zero in $N^1(X)$. Thus there exists an integral curve C on X such that

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$$(C \cdot D)_X \neq 0. \tag{5}$$

Let $\overline{k}$ be an algebraic closure of k. There exists an integral subscheme $\overline{X}$ of $X \otimes_k \overline{k}$ such that $\overline{X}$ dominates X. Thus $\overline{X}$ is a projective variety over $\overline{k}$. Let $\psi : \overline{X} \to X$ be the induced dominant morphism. Let $U \subset X$ be an affine open subset such that $U \cap C \neq \emptyset$. $\psi^{-1}(U)$ is affine since it is a closed subscheme of the affine scheme $U \otimes_k \overline{k}$. Let $A = \Gamma(U, \mathcal{O}_X)$ and $B = \Gamma(\psi^{-1}(U), \mathcal{O}_{\overline{X}})$. The ring extension $A \to B$ is integral. Let $P = \Gamma(U, \mathcal{I}_C)$, a prime ideal of A such that $\dim A/P = 1$, and let M be a maximal ideal of A containing P. By the going up theorem, there exists a prime ideal Q of B such that $Q \cap A = P$ and prime ideal N of B such that $Q \subset N$ and $N \cap A = M$. Now $A/M \to B/N$ is an integral extension from a field to a domain, so B/N is a field. Thus N is a maximal ideal of B and since there are no prime ideals of B properly between Q and N (by [3, Corollary 5.9]) we have that $\dim B/Q = 1$. Let $\overline{C}$ be the closure of $V(Q) \subset \psi^{-1}(U)$ in $\overline{X}$. Then $\overline{C}$ is an integral curve on X which dominates C. There exists a field of definition k' of $\overline{X}$ and $\overline{C}$ over k which is a subfield of $\overline{k}$ which is finite over k. That is, there exist subvarieties $C' \subset X'$ of $X \otimes_k k'$ such that $X' \otimes_{k'} \overline{k} = \overline{X}$ and $C' \otimes_{k'} \overline{k} = \overline{C}$. We factor $\psi : \overline{X} \to X$ by morphisms

$$\overline{X} \stackrel{\alpha}{\to} X' \stackrel{\varphi}{\to} X$$

where $\alpha = \mathrm{id}_{X'} \otimes_{\mathrm{id}_{k'}} \mathrm{id}_{\overline{k}}$. The morphism φ is finite and surjective and α is flat (although it might not be of finite type). Let H be an ample Cartier divisor on X. Then $\varphi^* H$ is an ample Cartier divisor on X' (by [20, Exercise III.5.7(d)]). Thus for some positive integer m we have that global sections of $\mathcal{O}_{X'}(m\varphi^*(H))$ give a closed embedding of X' in $\mathbb{P}^n_{k'}$ for some n. Thus global sections of $\mathcal{O}_{\overline{X}}(m\psi^*(H))$ give a closed embedding of $\overline{X} = X' \otimes_{k'} \overline{k}$ in $\mathbb{P}^n_{\overline{k}}$. In particular, we have that $\psi^*(H)$ is an ample Cartier divisor on $\overline{X}$. We have natural morphisms

$$N^1(X) \to N^1(X') \to N^1(\overline{X}).$$

Here X is a k-variety and $\overline{X}$ is a $\overline{k}$-variety. X' is both a k-variety and a k'-variety. When we are regarding X' as a k-variety we will write X'_k and when we are regarding X' as a k'-variety we will write $X'_{k'}$.

We may use the formalism of Kleiman [21], using the Snapper polynomials [31] to compute intersection products of Cartier divisors. This is consistent with the intersection products of Fulton [18] by [18, Example 18.3.6]. This intersection theory is also presented in [10, Chapter 19].

Since D is numerically equivalent to zero as a Weil divisor, we have that

$$(D\cdot H^{d-1})_X=(D^2\cdot H^{d-2})_X=0. \tag{6}$$

We have that

$$(\psi^*D\cdot\psi^*H^{d-1})_{\overline{X}}=(\varphi^*D\cdot\varphi^*H^{d-1})_{X'_{k'}}=\frac{1}{[k':k]}(\varphi^*D\cdot\varphi^*H^{d-1})_{X'_k}$$

using [18, Example 18.3.6] and the fact that

$$H^i(\overline{X},\mathcal{O}_{\overline{X}}(\psi^*(mD)+\psi^*(nH)))=H^i(X'_{k'},\mathcal{O}_{X'}(\varphi^*(mD)+\varphi^*(nH)))\otimes_{k'}\overline{k}$$

for all m,n since α is flat. We thus have that

$$(\psi^*D\cdot\psi^*H^{d-1})_{\overline{X}}=\frac{1}{[k':k]}(\varphi^*D\cdot\varphi^*H^{d-1})_{X'_k}=\frac{\deg(X'/X)}{[k':k]}(D\cdot H^{d-1})_X=0 \tag{7}$$

by [18, Proposition 2.3] and (6). Similarly,

$$(\psi^*D^2\cdot\psi^*H^{d-2})_{\overline{X}}=0. \tag{8}$$

Since $\overline{k}$ is algebraically closed and the equations (7) and (8) hold, we have that

$$(\psi^*D\cdot\overline{C})_{\overline{X}}=0$$

by [28] and [22, Corollary XIII.7.4]. Thus by [18, Example 18.3.6 and Proposition 2.3],

$$\begin{aligned}0 &= (\psi^*D\cdot\overline{C})_{\overline{X}}=(\varphi^*D\cdot C')_{X'_{k'}}=\tfrac{1}{[k':k]}(\varphi^*D\cdot C')_{X'_k}\\ &= \tfrac{1}{[k':k]}(D\cdot\varphi_*C')_X=\tfrac{\deg(C'/C)}{[k':k]}(D\cdot C)_X,\end{aligned}$$

giving a contradiction to (5). Thus the map $N^1(X)\to N_{d-1}(X)$ is injective.

This homomorphism is always an isomorphism if X is nonsingular by [18, Example 19.1.5]. □

As defined and developed in [21], [23, Chapter 2], there are important cones $\mathrm{Amp}(X)$ (the ample cone), $\mathrm{Big}(X)$ (the big cone), $\mathrm{Nef}(X)$ (the nef cone) and $\mathrm{Psef}(X):=\overline{\mathrm{Eff}}(X)$ (the pseudo effective cone) in $N^1(X)$.

If D is a Cartier divisor on the projective variety X, then the complete linear system $|D|$ is defined by

$$|D|=\{\mathrm{div}(\sigma)\mid\sigma\in\Gamma(X,\mathcal{O}_X(D))\}. \tag{9}$$

Let $\mathrm{Mov}'(X)$ be the convex cone in $N^1(X)$ generated by the classes of Cartier divisors D such that $|D|$ has no codimension 1 fixed component. Define $\overline{\mathrm{Mov}}(X)$ to be the closure of $\mathrm{Mov}'(X)$ in $N^1(X)$. An $\mathbb{R}$-Cartier divisor D is said to be movable if the class of D is in $\overline{\mathrm{Mov}}(X)$. Define $\mathrm{Mov}(X)$ to be the interior of $\overline{\mathrm{Mov}}(X)$. As explained in [30, page 85], we have inclusions

$$\mathrm{Amp}(X)\subset\mathrm{Mov}(X)\subset\mathrm{Big}(X)$$

and

$$\mathrm{Nef}(X)\subset\overline{\mathrm{Mov}}(X)\subset\mathrm{Psef}(X).$$

The following lemma is also proven over algebraically closed fields k in [17, Corollary 3.17].

Lemma 2.2. *Suppose that X is a d-dimensional variety over a field k, D is a pseudo effective $\mathbb{R}$-Cartier divisor on X, H is an ample $\mathbb{Q}$-Cartier divisor on X and $(H^{n-1}\cdot D)_X = 0$. Then $D \equiv 0$.*

Proof. We will establish the lemma when k is algebraically closed. The lemma will then follow for arbitrary k by the method of the proof of Lemma 2.1.

We consider two operations on varieties. First suppose that Y is a projective variety of dimension $d \geq 2$ over k, $\tilde{H}$ is an ample $\mathbb{Q}$-Cartier divisor and $\tilde{D}$ is a pseudo effective $\mathbb{R}$-Cartier divisor on Y and $\tilde{C}$ is an integral curve on Y. Let $\pi : \overline{Y} \to Y$ be the normalization of Y. Then there exists an integral curve $\overline{C}$ in $\overline{Y}$ such that $\pi(\overline{C}) = \tilde{C}$ (as in the proof of Lemma 2.1). We have that

$$(\pi^*(\tilde{H})^{d-1} \cdot \pi^*(\tilde{D}))_{\overline{Y}} = (\tilde{H}^{d-1} \cdot \tilde{D})_Y$$

and

$$(\overline{C} \cdot \pi^*(\tilde{D}))_{\overline{Y}} = \deg(\overline{C}/\tilde{C})(\tilde{C} \cdot \tilde{D})_Y.$$

We further have that $\pi^*(\tilde{D})$ is pseudo effective.

For the second operation, suppose that Y is a normal projective variety over k. Let $\tilde{H}$ be an ample $\mathbb{Q}$-Cartier divisor on Y and $\tilde{D}$ be a pseudo effective $\mathbb{R}$-Cartier divisor on Y. Let $\tilde{C}$ be an integral curve on Y. Let $\varphi : Z := B(\tilde{C}) \to Y$ be the blow up of $\tilde{C}$. Let E be the effective Cartier divisor on Z such that $\mathcal{O}_Z(-E) = \mathcal{I}_{\tilde{C}}\mathcal{O}_Z$. There exists a positive integer m such that $m\tilde{H}$ is a Cartier divisor and $\varphi^*(m\tilde{H}) - E$ is very ample on Z. Let L be the linear system

$$L = \{F \in |mH| \mid \tilde{C} \subset \mathrm{Supp}(F)\}$$

on Y. The base locus of L is $\tilde{C}$. We have an induced rational map $\Phi_L : X \dashrightarrow \mathbb{P}^n$ where n is the dimension of L. Let Y' be the image of Φ_L. Then $Y' \cong Z$ since $\varphi^*(m\tilde{H}) - E$ is very ample on Z. Thus $\dim Y' = d$ and we have equality of function fields $k(Y') = k(Y)$. By the first theorem of Bertini, [29], [35, Section I.7], [10, Theorem 22.12], a general member W of L is integral, so that it is a variety. By construction, $\tilde{C} \subset W$. Let $\alpha : W \to Y$ be the inclusion. We have that $\alpha^*(\tilde{H})$ is ample on W. A general member of L is not a component of the support of $\tilde{D}$ so $\alpha^*(\tilde{D})$ is pseudo effective. We have that $(\alpha^*(\tilde{H})^{d-2} \cdot \alpha^*(\tilde{D}))_W = (\tilde{H}^{d-1} \cdot \tilde{D})_Y$. Further, $(\tilde{C} \cdot \alpha^*(\tilde{D}))_W = (\tilde{C} \cdot \tilde{D})_Y$.

Suppose that D is not numerically equivalent to zero. We will derive a contradiction. There then exists an integral curve C on X such that $(C \cdot D)_X \neq 0$. By iterating the above two operations, we construct a morphism of k-varieties $\beta : S \to X$ such that S is a two dimensional projective variety, with an integral curve $\tilde{C}$ on S, an ample $\mathbb{Q}$-Cartier divisor $\tilde{H}$ on S and a pseudo effective $\mathbb{R}$-Cartier divisor on S such that $(\tilde{H} \cdot \tilde{D})_S = 0$ but $(\tilde{D} \cdot \tilde{C})_S \neq 0$. Let $\gamma : T \to S$ be a resolution of singularities (which exists by [1], [27] or [5]). There exists an exceptional divisor E on T and a positive integer m such that $m\tilde{H}$ is a Cartier divisor on S and $A := \gamma^*(m\tilde{H}) - E$ is an ample $\mathbb{Q}$-Cartier divisor. There exists an integral curve $\overline{C}$ on T such that $\gamma(\overline{C}) = \tilde{C}$ and $\gamma^*(\tilde{D})$ is a pseudo effective $\mathbb{R}$-Cartier divisor. Since E is exceptional for γ, We have that

$$(A \cdot \gamma^*(\tilde{D}))_T = (\gamma^*(m\tilde{H}) - E) \cdot \gamma^*(\tilde{D}))_T = (\gamma^*(m\tilde{H}) \cdot \gamma^*(\tilde{D}))_T = m(\tilde{H} \cdot \tilde{D})_S = 0$$

and

$$(\gamma^*(\tilde{D}) \cdot \overline{C}) = \deg(\overline{C}/\tilde{C})(\tilde{C} \cdot \tilde{D})_S \neq 0$$

by [21, Chapter I], [10, Proposition 19.8 and Proposition 19.12]. But this is a contradiction to [21, Theorem 1, page 317], [23, Theorem 1.4.29], since $N^1(T) = N_1(T)$ by Lemma 2.1. □

2.2. **Normal varieties.** In this section we review some material from [15]. Suppose that X is a normal projective variety over a field k. The map $D \to [D]$ is an inclusion of $\mathrm{Div}(X)$ into $Z_{d-1}(X)$, and thus induces an inclusion of $\mathrm{Div}(X) \otimes \mathbb{R}$ into $Z_{d-1}(X) \otimes \mathbb{R}$. We may thus identify a Cartier divisor D on X with its associated Weil divisor $[D]$.

Let x be a real number. Define $\lfloor x \rfloor$ to be the round down of x and $\{x\} = x - \lfloor x \rfloor$. Let E be an $\mathbb{R}$-Weil divisor on a normal variety X (an element of $Z_{d-1}(X) \otimes \mathbb{R}$). Expand $E = \sum a_i E_i$ with $a_i \in \mathbb{R}$ and E_i prime divisors on X. Then we have associated divisors

$$\lfloor E \rfloor = \sum \lfloor a_i \rfloor E_i \text{ and } \{E\} = \sum \{a_i\} E_i.$$

There is an associated coherent sheaf $\mathcal{O}_X(E)$ on X defined by

$$\Gamma(U, E) = \{f \in k(X)^* \mid \mathrm{div}(f) + E|_U \geq 0\} \text{ for } U \text{ an open subset of } X.$$

We have that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. If D and D' are $\mathbb{R}$-Weil divisors on X, then define $D' \sim_{\mathbb{Z}} D$ if $D' - D = \mathrm{div}(f)$ for some $f \in k(X)$. Define $D' \sim_{\mathbb{Q}} D$ if there exists $m \in \mathbb{Z}_{>0}$ such that $mD' \sim_{\mathbb{Z}} mD$.

For D an $\mathbb{R}$-Weil divisor, the complete linear system $|D|$ is defined as

$$|D| = \{\mathbb{R}\text{-Weil divisors } D' \mid D' \geq 0 \text{ and } D' \sim_{\mathbb{Z}} D\}.$$

If D is an integral Cartier divisor, then this is in agreement with the definition of (9). For D an $\mathbb{R}$ Weil divisor, we define

$$|D|_{\mathbb{Q}} = \{\mathbb{R}\text{-Weil divisors } D' \mid D' \geq 0 \text{ and } D' \sim_{\mathbb{Q}} D\}.$$

2.3. **σ-decomposition.** In this subsection we assume that X is a nonsingular projective variety over a field k. We will restrict our use of σ-decompositions to this situation. Nakayama defined and developed σ-decompositions for nonsingular complex projective varieties in [30, Chapter III]. The theory and proofs in this chapter extend to arbitrary fields. The σ-decomposition is extended to normal projective varieties in [15].

Since X is nonsingular, the map $D \to [D]$ is an isomorphism from $\mathrm{Div}(X)$ to $Z_{d-1}(X)$, and thus induces an isomorphism $\mathrm{Div}(X) \otimes \mathbb{R} \to Z_{d-1}(X) \otimes \mathbb{R}$. Thus we may identify $\mathbb{R}$-Cartier divisors and $\mathbb{R}$-Weil divisors on X, which we will refer to as $\mathbb{R}$-divisors. Since X is normal, we may use the theory of Subsection 2.2.

Let D be an $\mathbb{R}$-divisor. We define

$$|D|_{\mathrm{num}} = \{\mathbb{R} \text{ divisors } D' \text{ on } X \mid D' \geq 0 \text{ and } D' \equiv D\}.$$

Let D be a big $\mathbb{R}$-divisor and Γ be a prime divisor on X. Then we define

$$\sigma_\Gamma(D)_{\mathbb{Z}} := \begin{cases} \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |D|\} & \text{if } |D| \neq 0 \\ +\infty & \text{if } |D| = \emptyset, \end{cases}$$

$$\sigma_\Gamma(D)_{\mathbb{Q}} := \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |D|_{\mathbb{Q}}\},$$

$$\sigma_\Gamma(D) := \inf\{\mathrm{mult}_\Gamma \Delta \mid \Delta \in |D|_{\mathrm{num}}\}.$$

These three functions $\sigma_\Gamma(D)_*$ satisfy

$$\sigma_\Gamma(D_1 + D_2)_* \leq \sigma_\Gamma(D_1)_* + \sigma_\Gamma(D_2)_*.$$

We have that

$$\sigma_\Gamma(D)_{\mathbb{Q}} = \sigma_\Gamma(D) \tag{10}$$

by [30, Lemma III.1.4]. The function σ_Γ is continuous on $\mathrm{Big}(X)$ by [30, Lemma 1.7]. If D is a pseudo effective $\mathbb{R}$-divisor and Γ is a prime divisor, then

$$\sigma_\Gamma(D) := \lim_{t\to 0^+} \sigma_\Gamma(D+tA)$$

where A is any ample $\mathbb{R}$-divisor on X. The limits of these sequences exist and these sequences converge to the same number by [30, Lemma 1.5]. By [30, Corollary 1.11], there are only finitely many prime divisors Γ on X such that $\sigma_\Gamma(D) > 0$. For a given pseudo effective $\mathbb{R}$-divisor D, the $\mathbb{R}$-divisors

$$N_\sigma(D) = \sum_\Gamma \sigma_\Gamma(D)\Gamma \text{ and } P_\sigma(D) = D - N_\sigma(D)$$

are defined in [30, Definition 1.12]. The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the σ-decomposition of D.

Suppose that D is a pseudo effective $\mathbb{R}$-divisor, A and H are ample $\mathbb{R}$-divisors and $t, \varepsilon > 0$. Then, since $D + tA + \varepsilon H$, $D + \varepsilon H$ and tA are big, we have that for any prime divisor Γ,

$$\sigma_\Gamma(D + tA + \varepsilon H) \leq \sigma_\Gamma(D + \varepsilon H) + \sigma_\Gamma(tA) = \sigma_\Gamma(D + \varepsilon H).$$

Thus

$$\sigma_\Gamma(D + tA) = \lim_{\varepsilon\to 0^+} \sigma_\Gamma(D + tA + \varepsilon H) \leq \lim_{\varepsilon\to 0^+} \sigma_\Gamma(D + \varepsilon H) = \sigma_\Gamma(D).$$

In particular, if $\Gamma_1, \ldots, \Gamma_s$ are the prime divisors such that $N_\sigma(D) = \sum_{i=1}^s a_i\Gamma_i$ where $a_i > 0$ for all i, then for all $t > 0$, there is an expansion $N_\sigma(D + tA) = \sum_{i=1}^s a_i(t)\Gamma_i$ where $a_i(t) \in \mathbb{R}_{\geq 0}$. Thus $\lim_{t\to 0^+} N_\sigma(D + tA) = N_\sigma(D)$ and $\lim_{t\to 0^+} P_\sigma(D + tA) = P_\sigma(D)$.

Lemma31 **Lemma 2.3.** *Suppose that D is a pseudo effective $\mathbb{R}$-divisor on a nonsingular projective variety X. Then*

1) *$P_\sigma(D)$ is pseudo effective.*
2) *$\sigma_\Gamma(P_\sigma(D)) = 0$ for all prime divisors Γ on X, so that the class of $P_\sigma(D)$ is in $\overline{\mathrm{Mov}}(X)$.*
3) *$N_\sigma(D) = 0$ if and only if the class of D is in $\overline{\mathrm{Mov}}(X)$.*

Proof. Let A be an ample $\mathbb{R}$-divisor on X. For all $\varepsilon > 0$, $D + \varepsilon A$ is big. Thus the class of $D + \varepsilon A - \sum \sigma_\Gamma(D + \varepsilon A)\Gamma$ is in $\mathrm{Big}(X)$. Thus $P_\sigma(D) = \lim_{\varepsilon\to 0^+} D + \varepsilon A - \sum \sigma_\Gamma(D + \varepsilon A)\Gamma$ is pseudo effective. Statement 2) follows from [30, Lemma III.1.8] and [30, Proposition III.1.14]. Statement 3) is [30, Proposition III.1.14]. $\square$

2.4. **Positive intersection products.** Let X be a d-dimensional projective variety over a field k. In [9], we generalize the positive intersection product on projective varieties over an algebraically closed field of characteristic zero defined in [4] to projective varieties over an arbitrary field. We give a quick survey of this theory in this section, referring to [9] for details.

Let $I(X)$ be the directed set of projective varieties Y which have a birational morphism to X. If $f : Y' \to Y$ is in $I(X)$ and $\mathcal{L} \in N^1(Y)$, then $f^*\mathcal{L} \in N^1(Y')$. We may thus define $N^1(\mathcal{X}) = \lim_\to N^1(Y)$. If D is a Cartier $\mathbb{R}$-divisor on Y, we will sometimes abuse notation and identify D with is class in $N^1(\mathcal{X})$. In [9], $N^1(Y)$ is denoted by $M^1(Y)$. For $p \in \mathbb{N}$, we define $N^p(Y)$ to be the direct product of $N^1(Y)$ p times and define $N^p(\mathcal{X}) = \lim_\to N^p(Y)$.

We define $\alpha \in N^1(\mathcal{X})$ to be $\mathbb{Q}$-Cartier (respectively nef, big, effective, pseudoeffective) if there exists a representative of α in $N^1(Y)$ which has this property for some $Y \in I(X)$. We define subsets $\mathrm{Nef}^p(\mathcal{X})$, $\mathrm{Big}^p(\mathcal{X})$ and $\mathrm{Psef}^p(\mathcal{X})$ to be the respective subsets of $N^p(\mathcal{X})$ of nef, big and pseudoeffective divisors. They are all convex cones in the vector space $N^p(\mathcal{X})$.

If $p = 1$, we will often write $\mathrm{Nef}(\mathcal{X})$, $\mathrm{Big}(\mathcal{X})$ and $\mathrm{Psef}(\mathcal{X})$. We have that $\{\mathrm{Nef}(Y)^p\}$, $\{\mathrm{Big}(Y)^p\}$ and $\{\mathrm{Psef}(Y)^p\}$ also form directed systems. As sets, we have that

$$\mathrm{Nef}\,^p(\mathcal{X}) = \lim_{\rightarrow}(\mathrm{Nef}(Y)^p),\ \mathrm{Big}\,^p(\mathcal{X}) = \lim_{\rightarrow}(\mathrm{Big}(Y)^p),\ \mathrm{Psef}\,^p(\mathcal{X}) = \lim_{\rightarrow}(\mathrm{Psef}(Y)^p).$$

We give all of these sets their respective strong topologies.

The set $\mathrm{Psef}(\mathcal{X})$ is a strict cone in the vector space $N^1(\mathcal{X})$, since $\mathrm{Psef}(Y)$ are strict cones for $Y \in I(X)$. Thus we have an induced partial order on $N^1(\mathcal{X})$. More generally, if V is a vector space and $C \subset V$ is a pointed (containing the origin) convex cone which is strict ($C \cap (-C) = \{0\}$), then we have a partial order on V defined by $x \le y$ if $y - x \in C$. An element $\alpha \in N^1(\mathcal{X})$ satisfies $\alpha \ge 0$ if there exists a representative $\alpha' \in N^1(Y)$ of α for some $Y \in I(X)$ such that $\alpha' \ge 0$ in $N^1(Y)$ ($\alpha' \in \mathrm{Psef}(Y)$).

For $Y \in I(X)$ and $0 \le p \le d$, we let $L^p(Y)$ be the real vector space of p-multilinear forms on $N^1(Y)$. Giving the finite dimensional real vector space $L^p(Y)$ the Euclidean topology, we define

eq300 (11)
$$L^p(\mathcal{X}) = \lim_{\leftarrow} L^p(Y).$$

$L^p(\mathcal{X})$ is a Hausdorff topological real vector space. We define $L^0(\mathcal{X}) = \mathbb{R}$. The pseudo effective cone $\mathrm{Psef}(L^p(Y))$ in $L^p(Y)$ is the closure of the cone generated by the natural images of the p-dimensional closed subvarieties of Y. The inverse limit of the $\mathrm{Psef}(L^p(Y))$ is then a closed convex and strict cone $\mathrm{Psef}(L^p(\mathcal{X}))$ in $L^p(\mathcal{X})$, defining a partial order $\ge$ in $L^p(\mathcal{X})$. The pseudo effective cone in $L^0(\mathcal{X})$ is the set of nonnegative real numbers. For $Y \in I(X)$, let $\rho_Y : N^1(Y) \to N^1(\mathcal{X})$ and $\pi_Y : L^p(\mathcal{X}) \to L^p(Y)$ be the induced continuous linear maps. In [4] they consider a related but different vector space from $L^p(\mathcal{X})$.

Suppose that $\alpha_1, \ldots, \alpha_r \in N^1(\mathcal{X})$ with $r \le d$. Let $f : Y \to X \in I(X)$ be such that $\alpha_1, \ldots, \alpha_r$ are represented by classes in $N^1(Y)$ of $\mathbb{R}$-Cartier divisors $D_1, \ldots, D_r$ on Y. Then the ordinary intersection product $D_1 \cdot \ldots \cdot D_r$ induces a linear map $D_1 \cdot \ldots \cdot D_r \in L^{d-r}(\mathcal{X})$. If $r = d$, then this linear map is just the intersection number $(D_1 \cdot \ldots \cdot D_d)_Y \in \mathbb{R}$ of [18, Definition 2.4.2].

If $\alpha_1, \ldots, \alpha_p \in N^1(\mathcal{X})$ are big (elements of $\mathrm{Big}(\mathcal{X})$), we define the positive intersection product ([4, Definition 2.5, Proposition 2.13] in characteristic zero, [9, Definition 4.4, Proposition 4.12]) to be

eq33 (12)
$$\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle = \mathrm{lub}\ \begin{array}{l}\{(\alpha_1 - D_1) \cdot \ldots \cdot (\alpha_p - D_p) \in L^{d-p}(\mathcal{X}) \mid D_i \text{ are effective } \mathbb{R}\text{-Cartier} \\ \text{divisors on some } Y_i \in I(X) \text{ and } \alpha - D_i \text{ are big}\}\end{array}$$

where lub denotes the least upper bound of the set. This is well defined by [9, Proposition 4.3].

Prop35 **Proposition 2.4.** *([4, Proposition 2.13], [9, Proposition 4.12]) If $\alpha_1, \ldots, \alpha_p \in N^1(\mathcal{X})$ are big, we have that $\langle \alpha_1 \cdot \ldots \cdot \alpha_p \rangle$ is the least upper bound in $L^{d-p}(\mathcal{X})$ of all intersection products $\beta_1 \cdot \ldots \cdot \beta_p$ where β_i is the class of a nef $\mathbb{R}$-Cartier divisor such that $\beta_i \le \alpha_i$ for all i.*

If $\alpha_1, \ldots, \alpha_p \in N^1(\mathcal{X})$ are pseudo effective, their positive intersection product is defined ([4, Definition 2.10], [9, Definition 4.8, Lemma 4.9]) as

$$\lim_{\varepsilon \to 0^+} \langle (\alpha_1 + \varepsilon H) \cdot \ldots \cdot (\alpha_p + \varepsilon H) \rangle$$

where H is a big $\mathbb{R}$-Cartier divisor on some $Y \in I(X)$.

Lemma36 **Lemma 2.5.** *([4, Proposition 2.9, Remark 2.11], [9, Lemma 4.13], [9, Proposition 4.7]) The positive intersection product $\langle\alpha_1\cdot\ldots\cdot\alpha_p\rangle$ is homogeneous and super additive on each variable in the p-fold product $(\mathrm{Psef}(\mathcal{X}))^p$. Further, it is continuous on the p-fold product of the big cone.*

Remark50 **Remark 2.6.** *Since a positive intersection product is always in the pseudo effective cone, if $\alpha_1,\ldots,\alpha_d\in N^1(\mathcal{X})$ are pseudo effective, then $\langle\alpha_1\cdot\ldots\cdot\alpha_d\rangle\in\mathbb{R}_{\ge 0}$. Since the intersection product of nef and big $\mathbb{R}$-Cartier divisors is positive, it follows from Proposition 2.4 that if $\alpha_1,\ldots,\alpha_d\in N^1(\mathcal{X})$ are big, then $\langle\alpha_1\cdot\ldots\cdot\alpha_d\rangle\in\mathbb{R}_{>0}$.*

Lemma34 **Lemma 2.7.** *Let H be an ample $\mathbb{R}$-Cartier divisor on some $Y\in I(X)$ and let $\alpha\in N^1(\mathcal{X})$ be pseudo effective. Then*

$$\langle H^{d-1}\cdot\alpha\rangle = H^{d-1}\cdot\langle\alpha\rangle.$$

Proof. By Proposition 2.4, for all $\varepsilon>0$,

$$\langle((1+\varepsilon)H))^{d-1}\cdot(\alpha+\varepsilon H)\rangle = (1+\varepsilon)^{d-1}\left(H^{d-1}\cdot\langle\alpha+\varepsilon H\rangle\right).$$

Taking the limit as ε goes to zero, we have the conclusions of the lemma. □

Theorem17 **Theorem 2.8.** *Suppose that X is a d-dimensional projective variety, $\alpha\in N^1(X)$ is big and $\gamma\in N^1(X)$ is arbitrary. Then*

$$\frac{d}{dt}\mathrm{vol}(\alpha+t\gamma) = d\langle(\alpha+t\gamma)^{d-1}\rangle\cdot\gamma$$

whenever $\alpha+t\gamma$ is big.

This is a restatement of [4, Theorem A], [9, Theorem 5.6]. The proof shows that

$$\lim_{\Delta t\to 0}\frac{\mathrm{vol}(\alpha+(t+\Delta t)\gamma)-\mathrm{vol}(\alpha+t\gamma)}{\Delta t} = d\langle(\alpha+t\gamma)^{d-1}\rangle\cdot\gamma.$$

Suppose $\alpha\in N^1(\mathcal{X})$ is pseudo effective. Then we have for varieties over arbitrary fields, the formula of [4, Corollary 3.6],

eq40 (13) $$\langle\alpha^d\rangle = \langle\alpha^{d-1}\rangle\cdot\alpha.$$

To establish this formula, first suppose that α is big. Then taking the derivative at $t=0$ of $\langle(\alpha+t\alpha)^d\rangle=(1+t)^d\langle\alpha^d\rangle$, we obtain formula (13) from Theorem 2.8. If α is pseudo effective, we obtain (13) by regarding α as a limit of the big divisors $\alpha+tH$ where H is an ample $\mathbb{R}$-Cartier divisor.

The natural map $N^1(X)\to L^{d-1}(X)$ is an injection, as follows from the proof of Lemma 2.1. Let $WN^1(X)$ be the image of the homomorphism of $Z_{d-1}(X)\otimes\mathbb{R}$ to $L^{d-1}(X)$ which associates to $D\in Z_{d-1}(X)\otimes\mathbb{R}$ the natural map $(\mathcal{L}_1,\ldots,\mathcal{L}_{d-1})\mapsto(\mathcal{L}_1\cdot\ldots\cdot\mathcal{L}_{d-1}\cdot D)_X$. We have that $WN^1(X)$ is the subspace of $L^{d-1}(X)$ generated by $\mathrm{Psef}(X)$. We always have a factorization $N^1(X)\to N_{d-1}(X)\to WN^1(X)$. In this way we can identify the map $D\cdot$ which is the image of an element of $Z_{d-1}(X)\otimes\mathbb{R}$ in $L^{d-1}(X)$ with its class in $WN^1(X)$. If X is nonsingular, then $WN^1(X)=N_{d-1}(X)=N^1(X)$.

2.5. **Volume of divisors.** Suppose that X is a d-dimensional projective variety over a field k and D is a Cartier divisor on X. The volume of D is ([23, Definition 2.2.31])

$$\mathrm{vol}(D)=\limsup_{n\to\infty}\frac{\dim_k(\Gamma(X,\mathcal{O}_X(nD))}{n^d/d!}.$$

This lim sup is actually a limit. When k is an algebraically closed field of characteristic zero, this is shown in Example 11.4.7 [23], as a consequence of Fujita Approximation [19] (c.f. [23, Theorem 10.35]). The limit is established in [24] and [32] when k is algebraically closed of arbitrary characteristic. A proof over an arbitrary field is given in [7, Theorem 10.7].

Since vol is a homogeneous function, it extends naturally to a function on $\mathbb{Q}$-divisors, and it extends to a continuous function on $N^1(X)$ ([23, Corollary 2.2.45]), giving the volume of an arbitrary $\mathbb{R}$-Cartier divisor.

We have ([4, Theorem 3.1], [9, Theorems 5.2 and 5.3]) that for a pseudo effective $\mathbb{R}$-Cartier divisor D on X,

eq44
$$\operatorname{vol}(D) = \langle D^d \rangle. \tag{14}$$

Further, we have by [15, Theorem 3.5], that for an arbitrary $\mathbb{R}$-Weil divisor D on a normal variety X, that

$$\operatorname{vol}(D) = \lim_{n\to\infty} \frac{\dim_k(\Gamma(X, \mathcal{O}_X(nD))}{n^d/d!}.$$

An $\mathbb{R}$-Weil divisor D is said to be big if $\operatorname{vol}(D) > 0$.

Lemma 2.9. *Suppose that L is an $\mathbb{R}$-Cartier divisor on a d-dimensional projective variety X over a field k, Y is a projective variety and $\varphi : Y \to X$ is a generically finite morphism. Then*

eq43
$$\operatorname{vol}(\varphi^* L) = \deg(Y/X) \operatorname{vol}(L). \tag{15}$$

Proof. First assume that L is a Cartier divisor. The sheaf $\varphi_*\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-modules. Let R be the coordinate ring of X with respect to some closed embedding of X in a projective space. Then $R = \oplus_{i\ge 0} R_i$ is a standard graded domain over R_0, and R_0 a finite extension field of k. There exists a finitely generated graded R-module M such that the sheafication $\tilde{M}$ of M is isomorphic to $\varphi_*\mathcal{O}_Y$ (by [20, Proposition II.5.15 and Exercise II.5.9] or [10, Theorem 11.46]). Let S be the multiplicative set of nonzero homogeneous elements of R and η be the generic point of X. The ring $R_{(0)}$ is the set of homogeneous elements of degree 0 in the localization $S^{-1}R$ and the $R_{(0)}$-module $M_{(0)}$ is the set of homogeneous elements of degree 0 in the localization $S^{-1}M$. The function field of X is $k(X) = \mathcal{O}_{X,\eta} = R_{(0)}$ and $(\varphi_*\mathcal{O}_Y)_\eta = M_{(0)}$ is a $k(X)$-vector space of rank $r = \deg(Y/X)$. Let $f_1, \ldots, f_r \in M_{(0)}$ be a $k(X)$-basis. Write $f_i = \frac{z_i}{s_i}$ where $z_i \in M$ is homogeneous of some degree d_i and $s_i \in R$ is homogeneous of degree d_i. Multiplication by z_i induces a degree 0 graded R-module homomorphism $R(-d_i) \to M$ giving us a degree 0 graded R-module homomorphism $\oplus_{i=1}^r R(-d_i) \to M$. Let K be the kernel of this homomorphism and F be the cokernel. Let $\tilde{K}$ be the sheafification of K and $\tilde{F}$ be the sheafification of F. We have a short exact sequence of coherent $\mathcal{O}_X$-modules $0 \to \tilde{K} \to \oplus_{i=1}^r \mathcal{O}_X(d_i) \to \pi_*\mathcal{O}_Y \to \tilde{F} \to 0$. Localizing at the generic point, we see that $\tilde{K}_\eta = 0$ and $\tilde{F}_\eta = 0$ so that the supports of $\tilde{K}$ and $\tilde{F}$ have dimension less than $\dim X$, and thus $K = 0$ since it is a submodule of a torsion free R-module. Tensoring the short exact sequence $0 \to \oplus_{i=1}^r \mathcal{O}_X(d_i) \to \pi_*\mathcal{O}_Y \to \tilde{F} \to 0$ with L^n, we see that

$$\operatorname{vol}(\varphi^* L) = \lim_{n\to\infty} \frac{\dim_k \Gamma(Y, \varphi^* L^n)}{n^d/d!} = \lim_{n\to\infty} \frac{\dim_k(\oplus_{n=1}^r \Gamma(X, \mathcal{O}_X(d_i) \otimes L^n))}{n^d/d!} = \deg(Y/X) \operatorname{vol}(L).$$

Since volume is homogeneous, (15) is valid for $\mathbb{Q}$-Cartier divisors, and since volume is continuous on $N^1(X)$ and $N^1(Y)$, (15) is valid for $\mathbb{R}$-Cartier divisors. ☐

SubSecMN

2.6. **Big and Movable divisors on a normal variety.** Let X be a normal projective variety over a field, and Γ be a prime divisor on X. Recall that an $\mathbb{R}$-Weil divisor D on X is said to be big if $\mathrm{vol}(D) > 0$.

As explained in [15], the definitions of $\sigma_\Gamma(D)_\mathbb{Z}$ and $\sigma_\Gamma(D)_\mathbb{Q}$ of Subsection 2.3 extend to big $\mathbb{R}$-Weil divisors D on X, leading to the definition of the σ-decomposition $D = P_\sigma(D) + N_\sigma(D)$ as in Subsection 2.3. The inequalities

$$\sigma_\Gamma(D_1+D_2)_\mathbb{Z} \le \sigma_\Gamma(D_1)_\mathbb{Z} + \sigma_\Gamma(D_2)_\mathbb{Z} \text{ and } \sigma_\Gamma(D_1+D_2)_\mathbb{Q} \le \sigma_\Gamma(D_1)_\mathbb{Z} + \sigma_\Gamma(D_2)_\mathbb{Q}$$

continue to hold.

Let D be a big and movable $\mathbb{R}$-Cartier divisor on X and A be an ample $\mathbb{R}$-Cartier divisor on X. Then $D + tA \in \mathrm{Mov}(X)$ for all positive t, so that $\sigma(D+tA)_\mathbb{Q} = 0$ for all $t > 0$. Since D is big, there exists $\delta > 0$ such that $D \sim_\mathbb{Q} \delta A + \Delta$ where Δ is an effective $\mathbb{R}$-Cartier divisor. Then for all $\varepsilon > 0$, $(1+\varepsilon)D \sim_\mathbb{Q} D + \varepsilon\delta A + \varepsilon\Delta$ and so

$$(1+\varepsilon)\sigma_\Gamma(D)_\mathbb{Q} \le \sigma(D+\varepsilon\delta A)_\mathbb{Q} + \varepsilon\mathrm{mult}_\Gamma(\Delta) = \varepsilon\mathrm{mult}_\Gamma(\Delta)$$

for all $\varepsilon > 0$. Thus, with our assumption that D is a big and movable $\mathbb{R}$-Cartier divisor, we have that

eq50 (16) $$\sigma_\Gamma(D)_\mathbb{Q} = 0 \text{ for all prime divisors } \Gamma \text{ on } X.$$

If D is a big $\mathbb{R}$-Cartier divisor on a normal projective variety X, then $\mathrm{vol}(D) = \mathrm{vol}(P_\sigma(D))$ and so if $P_\sigma(D)$ is $\mathbb{R}$-Cartier, then $\mathrm{vol}(D) = \langle P_\sigma(D)^d\rangle$.

Lemma200 **Lemma 2.10.** *Suppose that X is a projective variety and D is a big $\mathbb{R}$-Cartier divisor on X. Let $f : Y \to X \in I(X)$ be such that Y is normal. Then*

eq91 (17) $$\pi_Y(\langle D\rangle) = P_\sigma(f^*(D)).$$

Proof. We may assume that $Y = X$ so that $f^*D = D$. After replacing D with an $\mathbb{R}$-Cartier divisor numerically equivalent to D, we may assume that $D = \sum_{i=1}^r a_iG_i$ is an effective divisor, where G_i are prime divisors and $a_i \in \mathbb{R}_{>0}$. For $m \in \mathbb{Z}_{>0}$, write $mD = N_m + \sum_{i=1}^r \sigma_{G_i}(mD)_\mathbb{Z} G_i$. Then $|mD| = |N_m| + \sum_{i=1}^r \sigma_{G_i}(mD)_\mathbb{Z}G_i$ where $|N_m|$ has no codimension one components in its base locus.

There exists a birational morphism $\varphi_m : X_m \to X$ such that X_m is normal and is a resolution of indeterminancy of the rational map determined by $|N_m|$ on X. Thus $\varphi_m^*(mD) = M_m + \sum_{i=1}^r \sigma_{G_i}(mD)_\mathbb{Z}\overline{G}_i + F_m$ where M_m and F_m are effective, F_m has exceptional support for φ_m, $\overline{G}_i$ is the proper transform of G_i on X_m and $|\varphi_m^*(mD)| = |M_m| + \sum_{i=1}^r \sigma_{G_i}(mD)_\mathbb{Z}\overline{G}_i + F_m$ where $|M_m|$ is base point free. Thus M_m is a nef integral Cartier divisor on X_m.

Set $D_m = \sum_{i=1}^r \frac{\sigma_{G_i}(mL)_\mathbb{Z}}{m}\overline{G}_i + \frac{F_m}{m}$, so that D_m is an effective $\mathbb{R}$-Cartier divisor on X_m. We have that $\frac{1}{m}M_m \le \langle D\rangle$ in $L^{d-1}(\mathcal{X})$ so that $\pi_X(\frac{1}{m}M_m) \le \pi_X\langle D\rangle$ in $L^{d-1}(X)$. Now

$$\begin{aligned}\pi_X(\tfrac{1}{m}M_m) &= (\varphi_m)_*(\tfrac{1}{m}M_m) = \tfrac{1}{m}(\varphi_m)_*((\varphi_m)^*(mD) - \textstyle\sum_{i=1}^r \sigma_{G_i}(mD)_\mathbb{Z}\overline{G}_i - F_m)\\ &= D - \textstyle\sum_{i=1}^r \frac{\sigma_{G_i}(mD)_\mathbb{Z}}{m}G_i.\end{aligned}$$

Thus

$$P_\sigma(D) = D - \sum_{i=1}^r \sigma_{G_i}(mD)G_i = \lim_{m\to\infty}(D - \sum_{i=1}^r \frac{\sigma_{G_i}(mD)_\mathbb{Z}}{m}G_i) \le \pi_X(\langle D\rangle)$$

in $L^{d-1}(X)$.

Let $Z \in I(X)$ be normal, with birational map $g : Z \to X$ and N be a nef and big $\mathbb{R}$-Cartier divisor on Z and E be an effective $\mathbb{R}$-Cartier divisor on Z such that $N + E = g^*(D)$. Let Γ be a prime divisor on Z. Then

$$\sigma_\Gamma(g^*(D)) \le \sigma_\Gamma(N) + \mathrm{ord}_\Gamma(E) = \mathrm{ord}_\Gamma(E).$$

Thus $N_\sigma(g^*(D)) \le E$ and so $N \le P_\sigma(g^*(D))$.

Let $\tilde{\Gamma}$ be a prime divisor on X and let Γ be the proper transform of $\tilde{\Gamma}$ on Z. Then $\sigma_\Gamma(g^*(D)) = \sigma_{\tilde{\Gamma}}(D)$ so that $\pi_X(N) \le P_\sigma(D)$ in $WN^1(X)$. Thus $\pi_X(\langle D\rangle) \le P_\sigma(D)$ in $L^{d-1}(X)$. ☐

Let X be a projective variety and $L_1, \ldots, L_{d-1} \in N^1(X)$. Suppose that D is a big and movable $\mathbb{R}$-Cartier divisor on X. Then the intersection product in $L^0(\mathcal{X}) = \mathbb{R}$ is

$$\begin{aligned} L_1 \cdot \ldots \cdot L_{d-1} \cdot \langle D\rangle &= \rho_X(L_1) \cdot \ldots \cdot \rho_X(L_{d-1}) \cdot \langle D\rangle = L_1 \cdot \ldots \cdot L_{d-1} \cdot \pi_X(\langle D\rangle) \\ &= (L_1 \cdot \ldots \cdot L_{d-1} \cdot P_\sigma(D))_X = (L_1 \cdot \ldots \cdot L_{d-1} \cdot D)_X \end{aligned} \tag{18}$$

3. A theorem on volumes

In this section we generalize [11, Theorem 4.2]. The proof given here is a variation of the one given in [11], using the theory of divisorial Zariski decomposition of $\mathbb{R}$-Weil divisors on normal varieties of [15]. Let X be a d-dimensional normal projective variety over a field k. Suppose that D is a big $\mathbb{R}$-Weil divisor on X; that is, $\mathrm{vol}(D) > 0$. Let E be a codimension one prime divisor on X. In [15, Lemma 4.1] the function σ_E of Subsection 2.3 is generalized to give the following definition ([15, Lemma 4.1])

$$\sigma_E(D) = \lim_{m\to\infty} \min \frac{1}{m}\{\mathrm{mult}_E D' \mid D' \sim_{\mathbb{Z}} mD, D' \ge 0\}.$$

Suppose that D is a big $\mathbb{R}$-Weil divisor and $E_1, \ldots, E_r$ are distinct prime divisors on X. Then by [15, Lemma 4.1], for all $m \in \mathbb{N}$,

$$\Gamma(X, \mathcal{O}_X(mD)) = \Gamma(X, \mathcal{O}_X(mD - \sum_{i=1}^r m\sigma_{E_i}(D)E_i)). \tag{19}$$

We now recall the method of [24] to compute volumes of graded linear series on X, as extended in [11] to arbitrary fields. We restrict to the situation of our immediate interest; that is, D is a big $\mathbb{R}$-Weil divisor and H is an ample Cartier divisor on X such that $D \le H$.

Suppose that $p \in X$ is a nonsingular closed point and

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_d = \{p\} \tag{20}$$

is a flag; that is, the Y_i are subvarieties of X of dimension $d - i$ such that there is a regular system of parameters $b_1, \ldots, b_d$ in $\mathcal{O}_{X,p}$ such that $b_1 = \cdots = b_i = 0$ are local equations of Y_i in X for $1 \le i \le d$.

The flag determines a valuation ν on the function field $k(X)$ of X as follows. We have a sequence of natural surjections of regular local rings

(21)

$$\mathcal{O}_{X,p} = \mathcal{O}_{Y_0,p} \xrightarrow{\sigma_1} \mathcal{O}_{Y_1,p} = \mathcal{O}_{Y_0,p}/(b_1) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_{d-1}} \mathcal{O}_{Y_{d-1},p} = \mathcal{O}_{Y_{d-2},p}/(b_{d-1}) \xrightarrow{\sigma_d} \mathcal{O}_{X,p}/m_p = k(p).$$

Define a rank d discrete valuation ν on $k(X)$ by prescribing for $s \in \mathcal{O}_{X,p}$,

$$\nu(s) = (\mathrm{ord}_{Y_1}(s), \mathrm{ord}_{Y_2}(s_1), \cdots, \mathrm{ord}_{Y_d}(s_{d-1})) \in (\mathbb{Z}^d)_{\mathrm{lex}}$$

where

$$s_1 = \sigma_1\left(\frac{s}{b_1^{\mathrm{ord}_{Y_1}(s)}}\right), s_2 = \sigma_2\left(\frac{s_1}{b_2^{\mathrm{ord}_{Y_2}(s_1)}}\right), \ldots, s_{d-1} = \sigma_{d-1}\left(\frac{s_{d-2}}{b_{d-1}^{\mathrm{ord}_{Y_{d-1}}(s_{d-2})}}\right).$$

Let $g = 0$ be a local equation of H at p. For $m \in \mathbb{N}$, define

$$\Phi_{mD} : \Gamma(X, \mathcal{O}_X(mD)) \to \mathbb{N}^d$$

by $\Phi_{mD}(f) = \nu(fg^m)$. The Okounkov body $\Delta(D)$ of D is the closure of the set

$$\cup_{m\in\mathbb{N}} \frac{1}{m}\Phi_{mD}(\Gamma(X, \mathcal{O}_X(mD)))$$

in $\mathbb{R}^d$. $\Delta(D)$ is a compact and convex set by [24, Lemma 1.10] or the proof of [7, Theorem 8.1].

By the proof of [7, Theorem 8.1] and of [8, Lemma 5.4] we see that

GR4 (22) $$\mathrm{vol}(D) = \lim_{m\to\infty} \frac{\dim_k \Gamma(X, \mathcal{O}_X(mD))}{m^d/d!} = d![\mathcal{O}_{X,p}/m_p : k]\mathrm{vol}(\Delta(D)).$$

The following proposition is proven with the assumption that the ground field k is perfect in i) implies ii) of [15, Theorem B]. The assumption that k is perfect is required in their proof as they use [32], which proves that a Fujita approximation exists to compute the volume of a Cartier divisor when the ground field is perfect. The theorem of [12] is used in [15] to conclude that a separable alteration exists if the ground field k is perfect.

Prop1 **Proposition 3.1.** *Suppose that X is a normal projective variety over a field k and D_1, D_2 are big $\mathbb{R}$-Weil divisors on X such that $D_1 \le D_2$ and $\mathrm{vol}(D_1) = \mathrm{vol}(D_2)$. Then*

$$\Gamma(X, \mathcal{O}_X(nD_1)) = \Gamma(X, \mathcal{O}_X(nD_2))$$

for all $n \in \mathbb{N}$.

Proof. Write $D_2 = D_1 + \sum_{i=1}^r a_iE_i$ where the E_i are prime divisors on X and $a_i \in \mathbb{R}_{>0}$ for all i. By induction on r, we may suppose that $r = 1$. Let H be an ample Cartier divisor on X such that $D_2 \le H$.

Choose a flag (20) with $Y_1 = E_1$ and p a point such that $p \in X$ is a nonsingular closed point of X and E_1. Let $\pi_1 : \mathbb{R}^d \to \mathbb{R}$ be the projection onto the first factor. For $f \in \Gamma(X, \mathcal{O}_X(mD_j))$,

$$\frac{1}{m}\mathrm{ord}_{E_1}(fg^m) = \frac{1}{m}\mathrm{ord}_{E_1}((f) + mD_j) + \mathrm{ord}_{E_1}(H - D_j).$$

Thus

$$\pi_1^{-1}(\sigma_{E_1}(D_j) + \mathrm{ord}_{E_1}(H - D_j)) \cap \Delta(D_j) \ne \emptyset$$

and

$$\pi_1^{-1}(a) \cap \Delta(D_j) = \emptyset \text{ if } a < \sigma_{E_1}(D_j) + \mathrm{ord}_{E_1}(H - D_j).$$

Further, $\Delta(D_1) \subset \Delta(D_2)$ and $\mathrm{vol}(D_1) = \mathrm{vol}(D_2)$, so $\Delta(D_1) = \Delta(D_2)$ by [11, Lemma 3.2]. Thus

$$\sigma_{E_1}(D_1) + \mathrm{ord}_{E_1}(H - D_1) = \sigma_{E_1}(D_2) + \mathrm{ord}_{E_1}(H - D_2).$$

We obtain that

$$D_2 - \sigma_{E_1}(D_2)E_1 = D_1 - \sigma_{E_1}(D_1)E_1.$$

By (19), for all $m \ge 0$,

$$\Gamma(X, \mathcal{O}_X(mD_1)) = \Gamma(X, \mathcal{O}_X(mD_2)).$$

□

Lemma 3.2. *Suppose that X is a nonsingular projective variety and $D_1 \leq D_2$ are big $\mathbb{R}$-divisors on X. Then the following are equivalent*

1) $\mathrm{vol}(D_1) = \mathrm{vol}(D_2)$
2) $\Gamma(X, \mathcal{O}_X(nD_1)) = \Gamma(X, \mathcal{O}_X(nD_2))$ *for all* $n \in \mathbb{N}$
3) $P_\sigma(D_1) = P_\sigma(D_2)$.

Proof. The implication 1) implies 2) is a consequence of Proposition 3.1. We now assume 2) holds and prove 3). Then $|nD_2| = |nD_1| + n(D_2 - D_1)$ for all $n \geq 0$. Thus

$$\sigma_\Gamma(D_2) = \sigma_\Gamma(D_1) + \mathrm{ord}_\Gamma(D_2 - D_1),$$

and so

$$\begin{aligned} P_\sigma(D_2) &= D_2 - N_\sigma(D_2) = D_1 + (D_2 - D_1) - (N_\sigma(D_1) + D_2 - D_1) \\ &= D_1 - N_\sigma(D_1) = P_\sigma(D_1). \end{aligned}$$

Finally, we prove 3) implies 1). Suppose that $P_\sigma(D_1) = P_\sigma(D_2)$. Then

$$\mathrm{vol}(D_1) = \mathrm{vol}(P_\sigma(D_1)) = \mathrm{vol}(P_\sigma(D_2)) = \mathrm{vol}(D_2)$$

by (19). □

4. The Augmented Base Locus

Let X be a normal variety over a field. Let D be a big $\mathbb{R}$-Cartier divisor on X. The augmented base locus $B_+(D)$ is defined in [14, Definition 1.2] and extended to $\mathbb{R}$-Weil divisors in [15, Definition 5.1]. $B_+^{\mathrm{div}}(D)$ is defined to be the divisorial part of $B_+(D)$. It is shown in [14, Proposition 1.4] that if D_1 and D_2 are big $\mathbb{R}$-Cartier divisors and $D_1 \equiv D_2$ then $B_+(D_1) = B_+(D_2)$. In [15, Lemma 5.3], it is shown that if A is an ample $\mathbb{R}$-Cartier divisor on X, then

$$B_+^{\mathrm{div}}(D) = \mathrm{Supp}(N_\sigma(D - \varepsilon A)) \tag{23}$$

for all sufficiently small positive ε.

The following Lemma is $i)$ equivalent to $ii)$ of [15, Theorem B], in the case that X is nonsingular, over an arbitrary field. We use Lemma 3.2 to remove the assumption in [15, Theorem B] that the ground field is perfect.

Lemma 4.1. *Let X be a nonsingular projective variety over a field. Let D be a big $\mathbb{R}$-divisor on X and E be an effective $\mathbb{R}$-divisor. Then* $\mathrm{vol}(D + E) = \mathrm{vol}(D)$ *if and only if Supp*$(E) \subset B_+^{\mathrm{div}}(D)$.

Proof. Suppose that $\mathrm{vol}(D + E) = \mathrm{vol}(D)$.

Let D' be an $\mathbb{R}$-divisor such that $D' \equiv D$. Then $\mathrm{vol}(D' + E) = \mathrm{vol}(D')$. Lemma 3.2 implies $\Gamma(X, \mathcal{O}_X(nD')) = \Gamma(X, \mathcal{O}_X(nD' + sE))$ for all $n > 0$ and $0 \leq s \leq n$. Thus $\Gamma(X, \mathcal{O}_X(nD')) = \Gamma(X, \mathcal{O}_X(nD' + rE))$ for all $n > 0$ and $r \geq 0$ by [30, Lemma III.1.8, Corollary III.1.9] or [15, Lemma 4.1]. Let A be an ample $\mathbb{R}$-divisor on X and suppose that F is an irreducible component of E and $F \not\subset \mathrm{Supp}(N_\sigma(D - \varepsilon A))$ for ε sufficiently small. By [15, Lemma 4.9], there exists $m > 0$ such that

$$mD + F = (\frac{1}{2}m\varepsilon A + F) + (\frac{1}{2}m\varepsilon A + mP_\sigma(D - \varepsilon A)) + mN_\sigma(D - \varepsilon A)$$

is numerically equivalent to an effective divisor G that does not contain F in its support. Let $D' = \frac{1}{m}(G - F) \equiv D$. Then for r sufficiently large,

$$\dim_k \Gamma(X, \mathcal{O}_X(mD' + rE)) \geq \dim_k \Gamma(X, \mathcal{O}_X(mD' + F)) > \dim_k \Gamma(X, \mathcal{O}_X(mD')),$$

giving a contradiction, and so by (23), $\mathrm{Supp}(E) \subset B_+^{\mathrm{div}}(D)$.

Now suppose that $\mathrm{Supp}(E) \subset B_+^{\mathrm{div}}(D)$. Let A be an ample $\mathbb{R}$-divisor on X. By (23), we have that $\mathrm{Supp}(E) \subset \mathrm{Supp}(N_\sigma(D - \varepsilon A))$ for all sufficiently small positive ε. By [15, Lemma 4.13], we have that $\mathrm{vol}(D+E-\varepsilon A) = \mathrm{vol}(D-\varepsilon A)$ for all sufficiently small $\varepsilon > 0$. Thus $\mathrm{vol}(D+E) = \mathrm{vol}(D)$ by continuity of volume of $\mathbb{R}$-divisors. □

5. The Minkowski equality

In this section, we modify the proof sketched in [26] of [26, Proposition 3.7] to be valid over an arbitrary field. Characteristic zero is required in the proof in [26] as the existence of resolution of singularities is assumed and an argument using the theory of multiplier ideals is used, which requires characteristic zero as it relies on both resolution of singularities and Kodaira vanishing. I thank the referee for pointing out, that in the case that k is algebraically closed, Proposition 5.1 also follows from [16, Proposition 5.3].

Proposition 5.1. *Let X be a nonsingular projective d-dimensional variety over a field k. Suppose that L is a big $\mathbb{R}$-divisor on X, and P and N are $\mathbb{R}$-divisors on X such that $L \equiv P + N$ where $\mathrm{vol}(L) = \mathrm{vol}(P)$ and N is pseudo effective. Then $P_\sigma(P) \equiv P_\sigma(L)$.*

Proof. Write $N = P_\sigma(N) + N_\sigma(N)$.

Since L and P are big $\mathbb{R}$-Cartier divisors, by superadditivity and positivity of intersection products,

$$\begin{array}{rcl} \mathrm{vol}(L) & = & \langle L^d\rangle \geq \langle L^{d-1}\cdot P\rangle + \langle L^{d-1}\cdot N\rangle \\ & = & \langle (P+N)^{d-1}\cdot P\rangle + \langle L^{d-1}\cdot N\rangle \\ & \geq & \langle P^d\rangle + \langle L^{d-1}\cdot N\rangle = \mathrm{vol}(P) + \langle L^{d-1}\cdot N\rangle. \end{array}$$

Thus $\langle L^{d-1}\cdot N\rangle = 0$. Let A be an ample Cartier divisor on X. There exists a small real multiple $\overline{A}$ of A such that $B := L - \overline{A}$ is a big $\mathbb{R}$-Cartier divisor.

$$0 = \langle (\overline{A}+B)^{d-1}\cdot N\rangle \geq \langle \overline{A}^{d-1}\cdot N\rangle = \langle \overline{A}^{d-1}\cdot (P_\sigma(N)+N_\sigma(N))\rangle \geq \langle \overline{A}^{d-1}\cdot P_\sigma(N)\rangle = \overline{A}^{d-1}\cdot \langle P_\sigma(N)\rangle$$

by superadditivity and Lemma 2.7.

By Lemma 2.3, $P_\sigma(N) + \varepsilon\overline{A}$ is big and movable, so by (18),

$$\overline{A}^{d-1}\cdot \langle P_\sigma(N) + \varepsilon\overline{A}\rangle = \overline{A}^{d-1}\cdot (P_\sigma(N) + \varepsilon\overline{A}),$$

so

$$\overline{A}^{d-1}\cdot \langle P_\sigma(N)\rangle = \lim_{\varepsilon\to 0}\overline{A}^{d-1}\cdot \langle P_\sigma(N) + \varepsilon\overline{A}\rangle = \overline{A}^{d-1}\cdot P_\sigma(N).$$

Thus

$$(A^{d-1}\cdot P_\sigma(N))_X = 0 \tag{24}$$

and so $P_\sigma(N) \equiv 0$ by Lemma 2.2. Thus $N \equiv N_\sigma(N)$. Thus, replacing P with the numerically equivalent divisor $P + P_\sigma(N)$, we may assume that N is effective. By Lemma 3.2, we have that

$$P_\sigma(P) = P_\sigma(P+N) \equiv P_\sigma(L).$$

□

Lemma 5.2. *Let X be a nonsingular d-dimensional projective variety over a field k. Suppose that L_1 and L_2 are big $\mathbb{R}$-divisors on X. Set s to be the largest real number s such that $L_1 - sL_2$ is pseudo effective. Then*

$$s^d \leq \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)} \tag{25}$$

and if equality holds in (25), then $P_\sigma(L_1) \equiv sP_\sigma(L_2)$.

Proof. The pseudo effective cone is closed, so s is well defined. We have $L_1 \equiv sL_2 + \gamma$ where γ is pseudo effective. Thus $\mathrm{vol}(L_1) \geq \mathrm{vol}(sL_2) = s^d\mathrm{vol}(L_2)$. If this is an equality, then $sP_\sigma(L_2) \equiv P_\sigma(L_1)$ by Proposition 5.1. ☐

Let X be a projective variety over a field k. An alteration $\varphi : Y \to X$ is a proper and dominant morphism such that Y is a nonsingular projective variety and $[k(Y) : k(X)] < \infty$. If X is normal and D is a pseudo effective $\mathbb{R}$-Cartier divisor on X, then by [15, Lemma 4.12],

eqNew20

$$\varphi_* N_\sigma(\varphi^* D) = \deg(Y/X) N_\sigma(D). \tag{26}$$

It is proven in [12] that for such X, an alteration always exists (although it may be that $k(Y)$ is not separable over $k(X)$ if k is not perfect).

Lemma21

Lemma 5.3. *Suppose that X is a projective variety over a field k, $\varphi : Y \to X$ is an alteration and L_1, L_2 are pseudo effective $\mathbb{R}$-Cartier divisors on X. Suppose that $s \in \mathbb{R}_{>0}$. Then $\varphi^*(L_1) - sP_\sigma(\varphi^*(L_2))$ is pseudo effective if and only if $P_\sigma(\varphi^*(L_1)) - sP_\sigma(\varphi^*(L_2))$ is pseudo effective.*

Proof. Certainly if $P_\sigma(\varphi^* L_1) - sP_\sigma(\varphi^* L_2)$ is pseudo effective then $\varphi^*(L_1) - sP_\sigma(\varphi^* L_2)$ is pseudo effective. Suppose $\varphi^*(L_1) - sP_\sigma(\varphi^*(L_2))$ is pseudo effective. Then there exists a pseudo effective $\mathbb{R}$-divisor γ on Y such that

$$P_\sigma(\varphi^* L_1) + N_\sigma(\varphi^* L_1) = \varphi^* L_1 \equiv sP_\sigma(\varphi^* L_2) + \gamma = (sP_\sigma(\varphi^* L_2) + P_\sigma(\gamma)) + N_\sigma(\gamma).$$

The effective $\mathbb{R}$-divisor $N_\sigma(\gamma)$ has the property that $\varphi^*(L_1) - N_\sigma(\gamma)$ is movable by Lemma 2.3, so $N_\sigma(\gamma) \geq N_\sigma(\varphi^* L_1)$ by [30, Proposition III.1.14]. Thus $P_\sigma(\varphi^* L_1) - sP_\sigma(\varphi^* L_2)$ is pseudo effective. ☐

Lemma22

Lemma 5.4. *Let X be a d-dimensional projective variety over a field k. Suppose that L_1 and L_2 are big and movable $\mathbb{R}$-Cartier divisors on X. Let s be the largest real number such that $L_1 - sL_2$ is pseudo effective. Then*

eq23

$$s^d \leq \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)} \tag{27}$$

and if equality holds in (27), then L_1 and L_2 are proportional in $N^1(X)$.

Proof. Let $\varphi : Y \to X$ be an alteration.

Let L be a big and movable $\mathbb{R}$-Cartier divisor on X. Let $\Gamma \subset Y$ be a prime divisor which is not exceptional for φ. Let $\tilde{\Gamma}$ be the codimension one subvariety of X which is the support of $\varphi_*\Gamma$. Since L is movable, there exist effective $\mathbb{R}$-Cartier divisors D_i on X such that $\lim_{i\to\infty} D_i = L$ in $N^1(X)$ and $\tilde{\Gamma} \not\subset \mathrm{Supp}(D_i)$ for all i. We thus have that $\varphi^*(L) = \lim_{i\to\infty} \varphi^*(D_i)$ in $N^1(Y)$ and $\Gamma \not\subset \mathrm{Supp}(\varphi^*(D_i))$ for all i, so that $\sigma_\Gamma(\varphi^*(D_i)) = 0$ for all i. Thus $\sigma_\Gamma(\varphi^*(L)) = 0$ since σ_Γ is continuous on the big cone of Y. Thus $N_\sigma(\varphi^* L)$ has exceptional support for φ and thus $\varphi_*(P_\sigma(\varphi^* L)) = \varphi_*(\varphi^* L) = \deg(Y/X)L$ by (4).

Let s_Y be the largest real number such that $P_\sigma(\varphi^* L_1) - s_Y P_\sigma(\varphi^* L_2)$ is pseudo effective. Then $s_Y \geq s$ since $\varphi^* L_1 - s\varphi^* L_2$ is pseudo effective and by Lemma 5.3, and so

$$s^d \leq s_Y^d \leq \frac{\mathrm{vol}(\varphi^* L_1)}{\mathrm{vol}(\varphi^* L_2)} = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$$

by Lemma 5.2 and (15).

If $s^d = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$, then $P_\sigma(\varphi^*(L_1)) = sP_\sigma(\varphi^*(L_2))$ in $N^1(Y)$ by Lemma 5.2, and so

$$\deg(Y/X)(L_1 - sL_2) = \varphi_*(\varphi^*(L_1) - s\varphi^*(L_2)) = \varphi_*(P_\sigma(\varphi^*(L_1)) - s\varphi_*(P_\sigma(\varphi^*(L_2)) = 0$$

in $N_{d-1}(X)$, so that $0 = L_1 - sL_2$ in $N^1(X)$ by Lemma 2.1. ☐

The following proposition is proven over an algebraically closed field of characteristic zero in [26, Proposition 3.3].

Prop13 **Proposition 5.5.** *Suppose that X is a projective d-dimensional variety over a field k and L_1, L_2 are big and moveable $\mathbb{R}$-Cartier divisors on X. Then*

$$\langle L_1^{d-1}\rangle \cdot L_2 \geq \mathrm{vol}(L_1)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}$$

with equality if and only if L_1 and L_2 are proportional in $N^1(X)$.

Proof. Let $f : \overline{X} \to X$ be the normalization of X. Since $\overline{X}$ has no exceptional divisors for f, f^*L_1 and f^*L_2 are movable. We have that $\langle f^*L_1^{d-1}\rangle \cdot f^*L_2 = \langle L_1^{d-1}\rangle \cdot L_2$ and $\mathrm{vol}(f^*L_i) = \mathrm{vol}(L_i)$ for $i = 1, 2$. Further, $f^* : N^1(X) \to N^1(\overline{X})$ is an injection, so L_1 and L_2 are proportional in $N^1(X)$ if and only if f^*L_1 and f^*L_2 are proportional in $N^1(\overline{X})$. We may thus replace X with its normalization $\overline{X}$, and so we can can assume for the remainder of the proof that X is normal.

We construct birational morphisms $\psi_m : Y_m \to X$ with numerically effective $\mathbb{R}$-Cartier divisors $A_{i,m}$ and effective $\mathbb{R}$-Cartier divisors $E_{i,m}$ on Y_m such that $A_{i,m} = \psi_m^*(L_i) - E_{i,m}$ and $\langle L_i\rangle = \lim_{m\to\infty} A_{i,m}$ in $L^{d-1}(\mathcal{X})$ for $i = 1, 2$. We have that $\pi_X(A_{i,m}) = \psi_{m*}(A_{i,m})$ comes arbitrarily closed to $\pi_X(\langle L_j\rangle) = P_\sigma(L_j) = L_j$ in $L^{d-1}(X)$ by Lemma 2.10.

Let s_L be the largest number such that $L_1 - s_L L_2$ is pseudo effective and let s_m be the largest number such that $A_{1,m} - s_m A_{2,m}$ is pseudo effective.

We will now show that given $\varepsilon > 0$, there exists a positive integer m_0 such that $m > m_0$ implies $s_m < s_L + \varepsilon$. Since $\mathrm{Psef}(X)$ is closed, there exists $\delta > 0$ such that the open ball $B_\delta(L_1 - (s_L + \varepsilon)L_2)$ in $N^1(X)$ of radius δ centered at $L_1 - (s_L + \varepsilon)L_2$ is disjoint from $\mathrm{Psef}(X)$. There exists m_0 such that $m \geq m_0$ implies $\psi_{m*}(A_{1,m}) \in B_{\frac{\delta}{2}}(L_1)$ and $\psi_{m*}(A_{2,m}) \in B_{\frac{\delta}{(s_L+\varepsilon)2}}(L_2)$. Thus $\psi_{m*}(A_{1,m} - (s_L + \varepsilon)A_{2,m}) \notin \mathrm{Psef}(X)$ for $m \geq m_0$ so that $s_m < s_L + \varepsilon$.

By the Khovanski Teissier inequalities for nef and big divisors ([4, Theorem 2.15] in characteristic zero, [9, Corollary 6.3]),

eq14 (28) $$(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{d}{d-1}} \geq \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}$$

for all m. By Proposition 2.4, taking limits as $m \to \infty$, we have

$$\langle L_1^{d-1} \cdot L_2\rangle \geq \mathrm{vol}(L_1)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}.$$

Now for each m, we have

$$A_{1,m}^{d-1} \cdot \psi_m^*(L_2) = A_{1,m}^{d-1} \cdot (A_{2,m} + E_{2,m}) \geq A_{1,m}^{d-1} \cdot A_{2,m}$$

since $E_{2,m}$ is effective and $A_{1,m}$ is nef. Taking limits as $m \to \infty$, we have $\langle L_1^{d-1}\rangle \cdot L_2 \geq \langle L_1^{d-1} \cdot L_2\rangle$. Thus

eq15 (29) $$\langle L_1^{d-1}\rangle \cdot L_2 \geq \langle L_1^{d-1} \cdot L_2\rangle \geq \mathrm{vol}(L_1)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}.$$

The Diskant inequality for big and nef divisors, [9, Theorem 6.9], [4, Theorem F] implies

$$(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{d}{d-1}} - \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}} \geq ((A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{1}{d-1}} - s_m\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}})^d.$$

We have that $(A_{1,m}^{d-1}\cdot A_{2,m})^{\frac{1}{d-1}} - s_m \mathrm{vol}(A_{2,m})^{\frac{1}{d-1}} \ge 0$ since $s_m^d \le \frac{\mathrm{vol}(A_{1,m})}{\mathrm{vol}(A_{2,m})}$ by Lemma 5.4 and by (28).

We have that

$$\begin{aligned}\left[(A_{1,m}^{d-1}\cdot A_{2,m})^{\frac{d}{d-1}} - \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\right]^{\frac{1}{d}} &\ge (A_{1,m}^{d-1}\cdot A_{2,m})^{\frac{1}{d-1}} - s_m\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\\ &\ge (A_{1,m}^{d-1}\cdot A_{2,m})^{\frac{1}{d-1}} - (s_L+\varepsilon)\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\end{aligned}$$

for $m \ge m_0$. Taking the limit as $m\to\infty$, we have

$$\langle L_1^{d-1}\cdot L_2\rangle^{\frac{d}{d-1}} - \mathrm{vol}(L_1)\mathrm{vol}(L_2)^{\frac{1}{d-1}} \ge [\langle L_1^{d-1}\cdot L_2\rangle^{\frac{1}{d-1}} - s_L\mathrm{vol}(L_2)^{\frac{1}{d-1}}]^d. \tag{30}$$

If $(\langle L_1^{d-1}\rangle\cdot L_2)^{\frac{d}{d-1}} = \mathrm{vol}(L_1)\mathrm{vol}(L_2)^{\frac{1}{d-1}}$ then $\langle L_1^{d-1}\rangle\cdot L_2 = \langle L_1^{d-1}\cdot L_2\rangle$ by (29) and $(\langle L_1^{d-1}\rangle\cdot L_2)^{\frac{1}{d-1}} = s_L\mathrm{vol}(L_2)^{\frac{1}{d-1}}$, so that $s_L^d = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$ and thus L_1 and L_2 are proportional in $N^1(X)$ by Lemma 5.4.

Suppose L_1 and L_2 are proportional in $N^1(X)$, so that $L_1 \equiv s_L L_2$ and $s_L^d = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$. Then

$$\langle L_1^{d-1}\rangle\cdot L_2 = s_L^{d-1}\langle L_2^{d-1}\rangle\cdot L_2 = s_L^{d-1}\langle L_2^d\rangle = \frac{\mathrm{vol}(L_1)^{\frac{d-1}{d}}}{\mathrm{vol}(L_2)^{\frac{d-1}{d}}}\mathrm{vol}(L_2) = \mathrm{vol}(L_1)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}$$

where the second equality is by (13). □

The proof of the following theorem is deduced from Proposition 5.5 by extracting an argument from [25, Theorem 4.11]. Over algebraically closed fields of characteristic zero, it is [26, Proposition 3.7].

Theorem 5.6. *Let L_1 and L_2 be big and moveable $\mathbb{R}$-Cartier divisors on a d-dimensional projective variety X over a field k. Then*

$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} \ge \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}} \tag{31}$$

with equality if and only if L_1 and L_2 are proportional in $N^1(X)$.

Proof. By Theorem 2.8, we have that

$$\frac{d}{dt}\mathrm{vol}(L_1+tL_2) = d\langle (L_1+tL_2)^{d-1}\rangle\cdot L_2$$

for t in a neighborhood of $[0,1]$. By Proposition 5.5,

$$\langle (L_1+tL_2)^{d-1}\rangle\cdot L_2 \ge \mathrm{vol}(L_1+tL_2)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}.$$

Thus

$$\begin{aligned}\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} - \mathrm{vol}(L_1)^{\frac{1}{d}} &= \int_0^1 \mathrm{vol}(L_1+tL_2)^{\frac{1-d}{d}}\langle (L_1+tL_2)^{d-1}\rangle\cdot L_2 dt\\ &\ge \int_0^1 \mathrm{vol}(L_1+tL_2)^{\frac{1-d}{d}}\mathrm{vol}(L_1+tL_2)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}dt\\ &= \int_0^1 \mathrm{vol}(L_2)^{\frac{1}{d}}dt = \mathrm{vol}(L_2)^{\frac{1}{d}}.\end{aligned} \tag{32}$$

Since positive intersection products are continuous on big divisors, we have equality in (32) if and only if

$$\langle (L_1+tL_2)^{d-1}\rangle\cdot L_2 = \mathrm{vol}(L_1+tL_2)^{\frac{d-1}{d}}\mathrm{vol}(L_2)^{\frac{1}{d}}$$

for $0\le t\le 1$. Thus if equality holds in (31), then L_1 and L_2 are proportional in $N^1(X)$ by Proposition 5.5.

Since vol is homogeneous, if L_1 and L_2 are proportional in $N^1(X)$, then equality holds in (31). □

The following theorem is proven over algebraically closed fields of characteristic zero in [26, Theorem 1.6].

Theorem20 **Theorem 5.7.** *Let X be a nonsingular d-dimensional projective variety over a field k. For any two big $\mathbb{R}$-divisors L_1 and L_2 on X,*

$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} \geq \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}$$

with equality if and only if $P_\sigma(L_1)$ and $P_\sigma(L_2)$ are proportional in $N^1(X)$.

Proof. We have $\mathrm{vol}(P_\sigma(L_i)) = \mathrm{vol}(L_i)$ for $i=1,2$. Since $L_i = P_\sigma(L_i) + N_\sigma(L_i)$ for $i=1,2$ where $P_\sigma(L_i)$ is pseudo effective and movable and $N_\sigma(L_i)$ is effective, we have by super additivity of positive intersection products of pseudo effective divisors and Theorem 5.6 that

$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} \geq \mathrm{vol}(P_\sigma(L_1)+P_\sigma(L_2))^{\frac{1}{d}} \geq \mathrm{vol}(P_\sigma(L_1))^{\frac{1}{d}} + \mathrm{vol}(P_\sigma(L_2))^{\frac{1}{d}} = \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}.$$

Thus if we have the equality $\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} = \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}$, we have

$$\mathrm{vol}(P_\sigma(L_1)+P_\sigma(L_2))^{\frac{1}{d}} = \mathrm{vol}(P_\sigma(L_1))^{\frac{1}{d}} + \mathrm{vol}(P_\sigma(L_2))^{\frac{1}{d}}.$$

Then $P_\sigma(L_1)$ and $P_\sigma(L_2)$ are proportional in $N^1(X)$ by Theorem 5.6.

Now suppose that $P_\sigma(L_1)$ and $P_\sigma(L_2)$ are proportional in $N^1(X)$. Then there exists $s \in \mathbb{R}_{>0}$ such that $P_\sigma(L_2) \equiv sP_\sigma(L_1)$, so that $B_+^{\mathrm{div}}(P_\sigma(L_1)) = B_+^{\mathrm{div}}(P_\sigma(L_2))$. Since $\mathrm{vol}(L_i) = \mathrm{vol}(P_\sigma(L_i))$ for $i=1,2$, we have that $\mathrm{Supp}(N_\sigma(L_1)), \mathrm{Supp}(N_\sigma(L_2)) \subset B_+^{\mathrm{div}}(P_\sigma(L_1))$ by Lemma 4.1. Thus $\mathrm{Supp}(N_\sigma(L_1)+N_\sigma(L_2)) \subset B_+^{\mathrm{div}}(P_\sigma(L_1))$, so that by Lemma 4.1,

$$\mathrm{vol}(L_1+L_2) = \mathrm{vol}(P_\sigma(L_1)+sP_\sigma(L_1)) = (1+s)^d \mathrm{vol}(P_\sigma(L_1)).$$

Thus

$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} = (1+s)\mathrm{vol}(P_\sigma(L_1))^{\frac{1}{d}} = \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}.$$

□

6. Characterization of equality in the Minkowski inequality

Theorem21 **Theorem 6.1.** *Let X be a normal d-dimensional projective variety. For any two big $\mathbb{R}$-Cartier divisors L_1 and L_2 on X,*

$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} \geq \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}.$$

If equality holds, then $P_\sigma(L_1) = sP_\sigma(L_2)$ in $N_{d-1}(X)$, where $s = \left(\frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}\right)^{\frac{1}{d}}$.

Proof. Here we use the extension of σ-decomposition to $\mathbb{R}$-Weil divisors on a normal projective variety of [15]. Let $\varphi : Y \to X$ be an alteration. We have that $\varphi^* L_1$ and $\varphi^* L_2$ are big $\mathbb{R}$-Cartier divisors. By [15, Lemma 4.12], for $i=1,2$, $\varphi_* N_\sigma(\varphi^* L_i) = \deg(Y/X)\, N_\sigma(L_i)$. Since $\varphi_*\varphi^* L = \deg(Y/X)\, L$ by (4), we have that $\varphi_* P_\sigma(\varphi^* L_i) = \deg(Y/X)\, P_\sigma(L_i)$. Now $\mathrm{vol}(\varphi^* L_i) = \deg(Y/X)\,\mathrm{vol}(L_i)$ for $i=1,2$ and $\mathrm{vol}(\varphi^* L_1 + \varphi^* L_2) = \deg(Y/X)\,\mathrm{vol}(L_1+L_2)$ by (15).

Thus the inequality of the statement of the theorem holds for L_1 and L_2 since it holds for $\varphi^* L_1$ and $\varphi^* L_2$ by Theorem 5.7. Suppose that equality holds in the inequality. Then by Theorem 5.7, we have that there exists $s \in \mathbb{R}_{>0}$ such that $P_\sigma(\varphi^* L_1) = sP_\sigma(\varphi^* L_2)$ in $N^1(Y)$. Thus $\varphi_* P_\sigma(\varphi^* L_1) = s\varphi_* P_\sigma(\varphi^* L_2)$ in $N_{d-1}(X)$, so that $P_\sigma(L_1) = sP_\sigma(L_2)$

in $N_{d-1}(X)$. Since volume is homogeneous and $P_\sigma(\varphi^*L_1)$, $sP_\sigma(\varphi^*L_2)$ are numerically equivalent $\mathbb{R}$-Cartier divisors,

$$\frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)} = \frac{\mathrm{vol}(\varphi^*L_1)}{\mathrm{vol}(\varphi^*L_2)} = \frac{\mathrm{vol}(P_\sigma(\varphi^*L_1))}{\mathrm{vol}(P_\sigma(\varphi^*L_2))} = s^d.$$

□

Theorem22 **Theorem 6.2.** *Let X be a d-dimensional projective variety over a field k. For any two big $\mathbb{R}$-Cartier divisors L_1 and L_2 on X,*

Neweq20 (33)
$$\mathrm{vol}(L_1+L_2)^{\frac{1}{d}} \ge \mathrm{vol}(L_1)^{\frac{1}{d}} + \mathrm{vol}(L_2)^{\frac{1}{d}}$$

with equality if and only if $\langle L_1\rangle$ and $\langle L_2\rangle$ are proportional in $L^{d-1}(\mathcal{X})$. When this occurs, we have that $\langle L_1\rangle = s\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$, where $s = \left(\frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}\right)^{\frac{1}{d}}$.

In the case that D_1 and D_2 are nef and big, this is proven in [4, Theorem 2.15] (over an algebraically closed field of characteristic zero) and in [9, Theorem 6.13] (over an arbitrary field). In this case of nef divisors, the condition that $\langle L_1\rangle$ and $\langle L_2\rangle$ are proportional in $L^{d-1}(\mathcal{X})$ is just that D_1 and D_2 are proportional in $N^1(X)$.

Theorem 6.2 is obtained in the case that D_1 and D_2 are big and movable and k is an algebraically closed field of characteristic zero in [26, Proposition 3.7]. In this case the condition for equality is that D_1 and D_2 are proportional in $N^1(X)$. Theorem 6.2 is established in the case that D_1 and D_2 are big $\mathbb{R}$-Cartier divisors and X is nonsingular, over an algebraically closed field k of characteristic zero in [26, Theorem 1.6]. In this case, the condition for equality is that the positive parts of the σ decompositions of D_1 and D_2 are proportional; that is, $P_\sigma(D_1)$ and $P_\sigma(D_2)$ are proportional in $N^1(X)$.

Proof. Let $f : Y \to X \in I(X)$ with Y normal. Then $\mathrm{vol}(f^*(L_1) + f^*(L_2)) = \mathrm{vol}(L_1 + L_2)$ and $\mathrm{vol}(f^*L_j) = \mathrm{vol}(L_j)$ for $j = 1, 2$ so that the inequality (33) holds by Theorem 6.1.

Suppose that equality holds in (33). Let $s = \left(\frac{\mathrm{vol}(L_2)}{\mathrm{vol}(L_1)}\right)^{\frac{1}{d}}$. Then by Theorem 6.1, $P_\sigma(f^*L_1) = sP_\sigma(f^*L_2)$ in $N_{d-1}(Y)$. Thus $\pi_Y(\langle L_1\rangle) = s\pi_Y(\langle L_2\rangle)$ by (17). Since the normal $Y \in I(X)$ are cofinal in $I(X)$, we have that $\langle L_1\rangle = s\langle L_2\rangle$.

Suppose that $\langle L_1\rangle = s\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$ for some $s \in \mathbb{R}_{>0}$. Then equality holds in (33) by Proposition 2.4 and the fact that the positive intersection product is homogeneous. □

Defslope **Definition 6.3.** *Suppose that X is a projective variety and $\alpha, \beta \in N^1(X)$ are big. The slope $s(\alpha, \beta)$ is the largest real number s such that $\langle\alpha\rangle \ge s\langle\beta\rangle$.*

Let X be a projective variety and $f : Z \to X$ be a resolution of singularities. Suppose that L_1 and L_2 are $\mathbb{R}$-Cartier divisors on X. Let $\overline{L}_1 = f^*(L_1)$ and $\overline{L}_2 = f^*L_2$. Suppose that $\varphi : Y \to Z$ is a birational morphism of nonsingular projective varieties where Y is nonsingular and $t \in \mathbb{R}$. We will show that

eqZ1 (34)
$P_\sigma(\overline{L}_1) - tP_\sigma(\overline{L}_2)$ is pseudo effective if and only if $P_\sigma(\varphi^*\overline{L}_1) - tP_\sigma(\varphi^*\overline{L}_2)$ is pseudo effective.

The fact that $P_\sigma(\overline{L}_1) - tP_\sigma(\overline{L}_2)$ pseudo effective implies $P_\sigma(\varphi^*\overline{L}_1) - tP_\sigma(\varphi^*\overline{L}_2)$ pseudo effective follows from Lemma 5.3. If $P_\sigma(\varphi^*\overline{L}_1) - tP_\sigma(\varphi^*\overline{L}_2)$ is pseudo effective, then $\varphi_*(P_\sigma(\varphi^*\overline{L}_1) - tP_\sigma(\varphi^*\overline{L}_2)) = P_\sigma(\overline{L}_1) - tP_\sigma(\overline{L}_2)$ is pseudo effective.

Let $s = s(L_1, L_2)$. Since the $Y \to Z$ with Y nonsingular are cofinal in $I(X)$, we have that

(35) s is the largest positive number such that $\pi_Z(\langle L_1\rangle - s\langle L_2\rangle) = P_\sigma(\overline{L}_1) - sP_\sigma(\overline{L}_2)$ is pseudo effective.

Proposition 6.4. *Suppose that X is a variety over a field of characteristic zero and L_1, L_2 are big $\mathbb{R}$-Cartier divisors on X. Let $s = s(L_1, L_2)$. Then*

$$s^d \le \frac{\langle L_1^d\rangle}{\langle L_2^d\rangle} \tag{36}$$

and we have equality in this equation if and only if $\langle L_1\rangle$ is proportional to $\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$. If we have equality, then $\langle L_1\rangle = s\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$.

Proof. Let $Y \in I(X)$ be nonsingular, with birational morphism $f : Y \to X$. Then by Lemma 2.10,

$$P_\sigma(f^*L_1) - sP_\sigma(f^*L_2) = \pi_Y(\langle L_1\rangle) - s\langle L_2\rangle) \in \mathrm{Psef}(Y).$$

Thus by Lemma 5.2,

$$s^d \le \frac{\mathrm{vol}(P_\sigma(f^*L_1))}{\mathrm{vol}(P_\sigma(f^*L_2))} = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)} = \frac{\langle L_1^d\rangle}{\langle L_2^d\rangle},$$

and so the inequality (36) holds.

Suppose we have equality in (36). We have that $\pi_Y(\langle L_1\rangle) - s\pi_Y(\langle L_2\rangle) = P_\sigma(f^*L_1) - sP_\sigma(f^*L_2)$ is pseudo effective and $s^d = \frac{\mathrm{vol}(P_\sigma(f^*L_1))}{\mathrm{vol}(P_\sigma(f^*L_2))}$, so we have that $P_\sigma(f^*L_1) = sP_\sigma(f^*L_2)$ in $N^1(Y)$ by (35) and Lemma 5.2. Since the nonsingular Y are cofinal in $I(X)$, we have that $\langle L_1\rangle = s\langle L_2\rangle$ by Lemma 2.10 and (11).

Suppose that $\langle L_1\rangle = t\langle L_2\rangle$ for some $t \in \mathbb{R}_{>0}$. Then $s = t$ and by Proposition 2.4,

$$\langle L_1^d\rangle = \langle L_1\rangle \cdot \ldots \cdot \langle L_1\rangle = \langle sL_2\rangle \cdot \ldots \cdot \langle sL_2\rangle = s^d\langle L_2\rangle \cdot \ldots \cdot \langle L_2\rangle = s^d\langle L_2^d\rangle.$$

$\square$

Theorem 6.5. *(Diskant inequality for big divisors) Suppose that X is a projective d-dimensional variety over a field k of characteristic zero and L_1, L_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

$$\langle L_1^{d-1}\cdot L_2\rangle^{\frac{d}{d-1}} - \mathrm{vol}(L_1)\mathrm{vol}(L_2)^{\frac{1}{d-1}} \ge [\langle L_1^{d-1}\cdot L_2\rangle^{\frac{1}{d-1}} - s(L_1,L_2)\mathrm{vol}(L_2)^{\frac{1}{d-1}}]^d. \tag{37}$$

The Diskant inequality is proven for nef and big divisors in [4, Theorem G] in characteristic zero and in [9, Theorem 6.9] for nef and big divisors over an arbitrary field. In the case that D_1 and D_2 are nef and big, the condition that $\langle D_1\rangle - s\langle D_2\rangle$ is pseudo effective in $L^{d-1}(\mathcal{X})$ is that $D_1 - sD_2$ is pseudo effective in $N^1(X)$. The Diskant inequality is proven when D_1 and D_2 are big and movable divisors and X is a projective variety over an algebraically closed field of characteristic zero in [26, Proposition 3.3, Remark 3.4]. Theorem 6.5 is a consequence of [13, Theorem 3.6].

Proof. Let $s = s(L_1, L_2)$. Let $f : Z \to X$ be a resolution of singularities. After replacing L_i with f^*L_i for $i = 1, 2$, we may assume that X is nonsingular.

We construct birational morphisms $\psi_m : Y_m \to X$ with numerically effective $\mathbb{R}$-Cartier divisors $A_{i,m}$ and effective $\mathbb{R}$-Cartier divisors $E_{i,m}$ on Y_m such that $A_{i,m} = \psi_m^*(L_i) - E_{i,m}$ and $\langle L_i\rangle = \lim_{m\to\infty} A_{i,m}$ in $L^{d-1}(\mathcal{X})$ for $i = 1, 2$. We have that $\pi_X(A_{i,m}) = \psi_{m,*}(A_{i,m})$ comes arbitrarily closed to $\pi_X(\langle L_j\rangle) = P_\sigma(L_j)$ in $L^{d-1}(X)$ by Lemma 2.10.

By (35), s is the largest number such that $P_\sigma(L_1) - sP_\sigma(L_2)$ is pseudo effective (in $N^1(X)$). Let s_m be the largest number such that $A_{1,m} - s_m A_{2,m}$ is pseudo effective (in $N^1(Y_m)$).

We will now show that given $\varepsilon > 0$, there exists a positive integer m_0 such that $m > m_0$ implies $s_m < s + \varepsilon$. Since $\mathrm{Psef}(X)$ is closed, there exists $\delta > 0$ such that the open ball $B_\delta(P_\sigma(L_1) - (s+\varepsilon)P_\sigma(L_2))$ in $N^1(X)$ of radius δ centered at $P_\sigma(L_1) - (s+\varepsilon)P_\sigma(L_2)$ is disjoint from $\mathrm{Psef}(X)$. There exists m_0 such that $m \ge m_0$ implies $\psi_{m*}(A_{1,m}) \in B_{\frac{\delta}{2}}(P_\sigma(L_1))$ and $\psi_{m*}(A_{2,m}) \in B_{\frac{\delta}{(s+\varepsilon)2}}(P_\sigma(L_2))$. Thus $\psi_{m*}(A_{1,m} - (s+\varepsilon)A_{2,m}) \notin \mathrm{Psef}(X)$ for $m \ge m_0$ so that $s_m < s + \varepsilon$.

By the Khovanski Teissier inequalities for nef and big divisors ([4, Theorem 2.15] in characteristic zero, [9, Corollary 6.3]),

$$(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{d}{d-1}} \ge \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}} \tag{38}$$

for all m. By Proposition 2.4, taking limits as $m \to \infty$, we have

$$\langle L_1^{d-1} \cdot L_2\rangle \ge \mathrm{vol}(L_1)^{\frac{d-1}{d}} \mathrm{vol}(L_2)^{\frac{1}{d}}. \tag{39}$$

The Diskant inequality for big and nef divisors, [9, Theorem 6.9], [4, Theorem F] implies

$$(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{d}{d-1}} - \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}} \ge ((A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{1}{d-1}} - s_m \mathrm{vol}(A_{2,m})^{\frac{1}{d-1}})^d.$$

We have that $(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{1}{d-1}} - s_m \mathrm{vol}(A_{2,m})^{\frac{1}{d-1}} \ge 0$ since $s_m^d \le \frac{\mathrm{vol}(A_{1,m})}{\mathrm{vol}(A_{2,m})}$ by Lemma 5.4 and by (38).

We have that

$$\begin{aligned}\left[(A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{d}{d-1}} - \mathrm{vol}(A_{1,m})\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\right]^{\frac{1}{d}} &\ge (A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{1}{d-1}} - s_m \mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\\ &\ge (A_{1,m}^{d-1} \cdot A_{2,m})^{\frac{1}{d-1}} - (s+\varepsilon)\mathrm{vol}(A_{2,m})^{\frac{1}{d-1}}\end{aligned}$$

for $m \ge m_0$. Taking the limit as $m \to \infty$, we have that (37) holds. □

Proposition 6.6. *Suppose that X is a projective d-dimensional variety over a field k of characteristic zero and L_1, L_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

$$\langle L_1^{d-1} \cdot L_2\rangle \ge \mathrm{vol}(L_1)^{\frac{d-1}{d}} \mathrm{vol}(L_2)^{\frac{1}{d}}.$$

If equality holds, then $\langle L_1\rangle = s\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$, where $s = s(L_1, L_2) = \left(\frac{\mathrm{vol}(L_2)}{\mathrm{vol}(L_1)}\right)^{\frac{1}{d}}$.

Proof. The inequality holds by (39). Let $s = s(L_1, L_2)$. By (37), if $\langle L_1^{d-1} \cdot L_2\rangle^{\frac{d}{d-1}} = \mathrm{vol}(L_1)\mathrm{vol}(L_2)^{\frac{1}{d-1}}$ then $\langle L_1^{d-1} \cdot L_2\rangle^{\frac{1}{d-1}} = s\mathrm{vol}(L_2)^{\frac{1}{d-1}}$, so that $s^d = \frac{\mathrm{vol}(L_1)}{\mathrm{vol}(L_2)}$ and thus $\langle L_1\rangle = s\langle L_2\rangle$ in $L^{d-1}(\mathcal{X})$ by Proposition 6.4. □

Suppose that X is a complete d-dimensional algebraic variety over a field k and D_1, D_2 are pseudo effective $\mathbb{R}$-Cartier divisors on X. We will write

$$s_i = \langle D_1^i \cdot D_2^{d-i}\rangle \text{ for } 0 \le i \le d.$$

We have the following generalization of the Khovanskii-Teissier inequalities to positive intersection numbers.

Theorem 6.7. *(Minkowski Inequalities) Suppose that X is a projective algebraic variety of dimension d over a field k and D_1 and D_2 are pseudo effective $\mathbb{R}$-Cartier divisors on X. Then*

1) $s_i^2 \ge s_{i+1}s_{i-1}$ *for* $1 \le i \le d-1$.
2) $s_i s_{d-i} \ge s_0 s_d$ *for* $1 \le i \le d-1$.
3) $s_i^d \ge s_0^{d-i} s_d^i$ *for* $0 \le i \le d$.
4) $\langle (D_1 + D_2)^d\rangle^{\frac{1}{d}} \ge \langle D_1^d\rangle^{\frac{1}{d}} + \langle D_2^d\rangle^{\frac{1}{d}}$.

Proof. Statements 1) - 3) follow from the inequality of [9, Theorem 6.6] ([4, Theorem 2.15] in characteristic zero). Statement 4) follows from 3) and the super additivity of the positive intersection product. □

When D_1 and D_2 are nef, the inequalities of Theorem 6.7 are proven by Khovanskii and Teissier [33], [34], [23, Example 1.6.4]. In the case that D_1 and D_2 are nef, we have that $s_i = \langle D_1^i \cdot D_2^{d-i}\rangle = (D_1^i \cdot D_2^{d-i})$ are the ordinary intersection products.

We have the following characterization of equality in these inequalities.

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Theorem 6.8. *(Minkowski equalities) Suppose that X is a projective algebraic variety of dimension d over a field k of characteristic zero, and D_1 and D_2 are big $\mathbb{R}$-Cartier divisors on X. Then the following are equivalent:*
1) $s_i^2 = s_{i+1}s_{i-1}$ *for all* $1 \le i \le d-1$.
2) $s_i s_{d-i} = s_0 s_d$ *for all* $1 \le i \le d-1$.
3) $s_i^d = s_0^{d-i} s_d^i$ *for all* $0 \le i \le d$.
4) $s_{d-1}^d = s_0 s_d^{d-1}$.
5) $\langle (D_1 + D_2)^d\rangle^{\frac{1}{d}} = \langle D_1^d\rangle^{\frac{1}{d}} + \langle D_2^d\rangle^{\frac{1}{d}}$.
6) $\langle D_1\rangle$ *is proportional to* $\langle D_2\rangle$ *in* $L^{d-1}(\mathcal{X})$.

When D_1 and D_2 are nef and big, then Theorem 6.8 is proven in [4, Theorem 2.15] when k has characteristic zero and in [9, Theorem 6.13] for arbitrary k. When D_1 and D_2 are nef and big, the condition 6) of Theorem 6.8 is just that D_1 and D_2 are proportional in $N^1(X)$.

Proof. All the numbers s_i are positive by Remark 2.6. Proposition 2.4 shows that 6) implies 1), 2), 3), 4) and 5). Theorem 6.2 shows that 5) implies 6). Proposition 6.6 shows that 4) implies 6). Since the condition of 3) is a subcase of the condition 4), we have that 3) implies 6).

Suppose that 2) holds. By the inequality 3) of Theorem 6.7 and the equality 2), we have that

$$s_i^d s_{d-i}^d \ge (s_0^{d-i}s_d^i)(s_0^i s_d^{d-i}) = (s_0 s_d)^d = (s_i s_{d-i})^d.$$

Thus the equalities 3) hold.

Suppose that the inequalities 1) hold. Then

$$\frac{s_{d-1}}{s_0} = \frac{s_{d-1}}{s_{d-2}}\frac{s_{d-2}}{s_{d-3}}\cdots\frac{s_1}{s_0} = \left(\frac{s_d}{s_{d-1}}\right)^{d-1}$$

so that 4) holds. □

Remark 6.9. *The existence of resolutions of singularities is the only place where characteristic zero is used in the proof of Theorem 6.8. Thus the conclusions of Theorem 6.8 are valid over an arbitrary field for varieties of dimension $d \le 3$ by [2], [6].*

Let D_1 and D_2 be big $\mathbb{R}$-Cartier divisors on a projective variety X. Generalizing Teissier [33], we define the inradius of D_1 with respect to D_2 as

$$r(D_1, D_2) = s(D_1, D_2)$$

where $s(D_1, D_2)$ is the slope defined in Definition 6.3 and define the outradius of D_1 with respect to D_2 as

$$R(D_1, D_2) = \frac{1}{s(D_2, D_1)}.$$

Theorem 6.10. *Suppose that X is a d-dimensional projective variety over a field k of characteristic zero and D_1, D_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

$$\frac{s_{d-1}^{\frac{1}{d-1}} - (s_{d-1}^{\frac{d}{d-1}} - s_0^{\frac{1}{d-1}} s_d)^{\frac{1}{d}}}{s_0^{\frac{1}{d-1}}} \le r(D_1, D_2) \le \frac{s_d}{s_{d-1}}. \tag{40}$$

Proof. Let $s = s(D_1, D_2) = r(D_1, D_2)$. Since $\langle D_1 \rangle \ge s\langle D_2 \rangle$, we have that $\langle D_1^d \rangle \ge s\langle D_2 \cdot D_1^{d-1} \rangle$ by Lemma 2.5. This gives us the upper bound. We also have that

$$\langle D_1^{d-1} \cdot D_2 \rangle^{\frac{1}{d-1}} - s\langle D_2^d \rangle^{\frac{1}{d-1}} \ge 0. \tag{41}$$

We obtain the lower bound from Theorem 6.5 (using the inequality $s_{d-1}^d \ge s_0 s_d^{d-1}$ to ensure that the bound is a positive real number). ☐

Theorem 6.11. *Suppose that X is a d-dimensional projective variety over a field k of characteristic zero and D_1, D_2 are big $\mathbb{R}$-Cartier divisors on X. Then*

$$\frac{s_{d-1}^{\frac{1}{d-1}} - (s_{d-1}^{\frac{d}{d-1}} - s_0^{\frac{1}{d-1}} s_d)^{\frac{1}{d}}}{s_0^{\frac{1}{d-1}}} \le r(D_1, D_2) \le \frac{s_d}{s_{d-1}} \le \frac{s_1}{s_0} \le R(D_1, D_2) \le \frac{s_d^{\frac{1}{d-1}}}{s_1^{\frac{1}{d-1}} - (s_1^{\frac{d}{d-1}} - s_d^{\frac{1}{d-1}} s_0)^{\frac{1}{d}}}. \tag{42}$$

Proof. By Theorem 6.10, we have that

$$\frac{s_1^{\frac{1}{d-1}} - (s_1^{\frac{d}{d-1}} - s_d^{\frac{1}{d-1}} s_0)^{\frac{1}{d}}}{s_d^{\frac{1}{d-1}}} \le s(D_2, D_1) \le \frac{s_0}{s_1}.$$

The theorem now follows from the fact that $R(D_1, D_2) = \frac{1}{s(D_2, D_1)}$ and Theorem 6.7. ☐

This gives a solution to [33, Problem B] for big $\mathbb{R}$-Cartier divisors. The inequalities of Theorem 6.11 are proven by Teissier in [33, Corollary 3.2.1] for divisors on surfaces satisfying some conditions. In the case that D_1 and D_2 are nef and big on a projective variety over a field of characteristic zero, Theorem 6.11 follows from the Diskant inequality [4, Theorem F]. In the case that D_1 and D_2 are nef and big on a projective variety over an arbitrary field, Theorem 6.11 is proven in [9, Theorem 6.11], as a consequence of the Diskant inequality [9, Theorem 6.9] for nef divisors.

References

[1] S. Abhyankar, Local uniformization on algebraic surfaces over ground fields of characteristic $p \ne 0$, Annals of Math. 63 (1956), 491 - 526.

[2] S. Abhyankar, Resolution of singularities of embedded algebraic surfaces, second edition, Springer Verlag, New York, Berlin, Heidelberg, 1998.

[3] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra, Addison Wesley, Reading Massachusetts, 1969.

[4] S. Boucksom, C. Favre and M. Jonsson, Differentiability of volumes of divisors and a problem of Teissier, J. Algebraic Geom. 18 (2009), 279 - 308.

[5] V. Cossart, U. Jannsen, S. Saito, Desingularization: Invariants and strategy, Lecture Notes in Mathematics 2270, Springer, 2020.

CP [6] V. Cossart, O. Piltant, Resolution of singularities of arithmetical threefolds, I and II, Journal of algebra 529 (2019), 268 - 535.

C1 [7] S.D. Cutkosky, Asymptotic multiplicities of graded families of ideals and linear series, Advances in Mathematics 264 (2014), 55 - 113.

C3 [8] S.D. Cutkosky, Asymptotic Multiplicities, Journal of Algebra 442 (2015), 260 - 298.

C [9] S.D. Cutkosky, Teissier's problem on inequalities of nef divisors over an arbitrary field, J. Algebra Appl. 14 (2015).

AG [10] S.D. Cutkosky, Introduction to algebraic geometry, Graduate Studies in Mathematics, 188. American Mathematical Society, Providence RI, 2018.

C2 [11] S.D. Cutkosky, Mixed multiplicities of divisorial filtrations, Advances in Math. 358 (2019).

dJ [12] A.J. de Jong, Smoothness, semistability and alterations, Publ. Math. IHES 83 (1996), 51 - 93.

DF [13] Nguyen-Bac Dang and C. Favre, Intersection theory of nef b-divisor classes, arXiv:2007.04549.

ELM [14] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier 56 (2006), 1701 - 1734.

FKL [15] M. Fulger, J. Kollár and B. Lehmann, Volume and Hilbert function of $\mathbb{R}$-divisors, Michigan Math. J. 65 (2016), 371 - 387.

FL1 [16] M. Fulger and B. Lehmann, Zariski decompositions of numerical cycle classes, J. Algebraic Geom. 26 (2017), 43 - 106.

FL2 [17] M. Fulger and B. Lehmann, Positive cones of dual cycle classes, Alg. Geom. 4 (2017), 1-28.

F [18] W. Fulton, Intersection Theory, Springer Verlag, Berlin, Heidelberg, 1984.

F2 [19] T. Fujita, Approximating Zariski decomposition of big line bundles, Kodai Math. J. 17 (1994), 1-3.

H [20] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, 52, Springer-Verlag, New York-Heidelberg, 1077.

Kl [21] S. Kleiman, A numerical criterion for ampleness, Ann. of Math. 84 (1966), 293 - 344.

Kl2 [22] S. Kleiman, Expose XIII, Les Théorèmes de Finitude pour le Foncteur de Picard, in Théorie des Intersections et Théorème de Riemann-Roch, SGA 6, XIII.4.6.

L [23] R. Lazarsfeld, Positivity in Algebraic Geometry, Vols I and II, Springer Verlag, Berlin, Heidelberg, 2004

LM [24] R. Lazarsfeld and M. Mustaţă, Convex bodies associated to linear series, Ann. Sci. Ec. Norm. Super 42 (2009) 783 - 835.

LX1 [25] B. Lehmann and J. Xiao, Convexity and Zariski decomposition structure, Geom. Funct. Anal. 26 (2016), 1135 - 1189.

LX2 [26] B. Lehmann and J. Xiao, Positivity functions for curves on algebraic varieties, Algebra Number Theory 13 (2019), 1243 - 1279.

Li [27] J. Lipman, Desingularization of 2-dimensional schemes, Annals of Math. 107 (1978), 115 - 207.

M [28] T. Matsusaka, The criteria for algebraic equivalence and the torsion group, Amer. J. Math. 79 (1957), 52 - 66.

M1 [29] T. Matsusaka, The theorem of Bertini on linear systems, Mem. Coll. Sci. Univ. Kyoto 26 (1951), 51 - 62.

N [30] N. Nakayama, Zariski-decomposition and abundance, MSJ Memoirs, vol 14. Math. Soc. Japan, Tokyo, 2004.

Sn [31] E. Snapper, Polynomials associated with divisors, J. Math. and Mech. 9 (1960), 123 - 129.

T [32] S. Takagi, Fujita's approximation theorem in positive characteristics, J. Math. Kyoto Univ. 47 (2007), 179 - 202.

T1 [33] B. Teissier, Bonnesen-type inequalities in algebraic geometry, I. Introduction to the problem, in Seminar on Differential Geometry, 85 - 105, Ann. Math. Studies 102, Princeton Univ. Press, 1982.

T2 [34] B. Teissier, Du théorème de l'index de Hodge aux inégalites isopérimétriques, C.R. Acad. Sci. Paris Sér A-B 288 (1979), A287 - A289.

Z [35] O. Zariski, Introduction to the problem of minimal models in the theory of algebraic surfaces, The Mathematical Society of Japan, 1958.

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