EPSILON MULTIPLICITY AND ANALYTIC SPREAD OF FILTRATIONS

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Abstract. We extend the epsilon multiplicity of ideals defined by Ulrich and Validashti to epsilon multiplicity of filtrations, and show that under mild assumptions this multiplicity exists as a limit. We show that in rather general rings, the epsilon multiplicity of a $\mathbb{Q}$-divisorial filtration is positive if and only if the analytic spread of the filtration is maximal (equal to the dimension of the ring). The condition that filtrations $\mathcal{J} \subset \mathcal{I}$ have the same epsilon multiplicity is considered, and we find conditions ensuring that the filtrations have the same integral closure.

1. Introduction

The epsilon multiplicity of an ideal I in a local ring R with maximal ideal m_R is defined in [37] to be

$$\varepsilon(I) = d! \limsup_n \frac{\lambda_R(H^0_{m_R}(R/I^n))}{n^d}.$$

It is shown there, by comparison with the j-multiplicity, that $\varepsilon(I)$ is always a real number. It is shown in [14] that $\varepsilon(I)$ can be an irrational number. In a local ring R, we have that

$$H^0_{m_R}(R/I) = I : m_R^\infty / I.$$

For ideals in analytically unramified local rings, the epsilon multiplicity is a limit.

Theorem 1.1. *([6, Corollary 6.3]) Suppose that I is an ideal in an analytically unramified local ring R. Then $\varepsilon(I)$ is actually a limit,*

$$\varepsilon(I) = d! \lim_{n\to\infty} \frac{\lambda_R(I^n : m_R^\infty / I^n)}{n^d}.$$

In this paper, we extend epsilon multiplicity to filtrations and obtain some results generalizing theorems about epsilon multiplicities for ideals.

We extend the definition of epsilon multiplicity to filtrations by defining the epsilon multiplicity of a filtration $\mathcal{I} = \{I_n\}$ of ideals in R to be

$$\varepsilon(\mathcal{I}) = d! \limsup_n \frac{\lambda_R(H^0_{m_R}(R/I_n))}{n^d}. \tag{1}$$

Let $\mathcal{I} = \{I_n\}$ be a filtration on a local ring R and $c \in \mathbb{Z}_{>0}$. We will say that $\mathcal{I}$ satisfies property $A(c)$ if

$$(I_n : m_R^\infty) \cap m_R^{cn} = I_n \cap m_R^{cn} \text{ for all } n \in \mathbb{N}.$$

As a consequence of [6, Theorem 6.1], we have the following theorem.

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Theorem 1.2. *Let R be an analytically unramified local ring of dimension d, and $\mathcal{I} = \{I_n\}$ be a filtration on R which satisfies the property $A(c)$ for some $c \in \mathbb{Z}_{>0}$. Then*

$$\varepsilon(\mathcal{I}) = d! \lim_{n\to\infty} \frac{\lambda_R(I_n : m_R^{\infty}/I_n)}{n^d} \tag{2}$$

is a real number. That is, the epsilon multiplicity of $\mathcal{I}$ exists as a limit.

We deduce that the epsilon multiplicity exists for many naturally occurring filtrations. Discrete valued filtrations are defined in Section 2.

Theorem 1.3. *Let R be a Noetherian local domain, $\mathcal{J}$ a discrete valued filtration and $\mathfrak{p}$ any prime ideal in R. Then the filtration $\mathcal{J}_{\mathfrak{p}}$ satisfies $A(c_{\mathfrak{p}})$ for some $c_{\mathfrak{p}} \in \mathbb{Z}_{>0}$.*

Moreover, if $R_{\mathfrak{p}}$ is an analytically unramified local domain then $\varepsilon(\mathcal{J}_{\mathfrak{p}})$ exists as a limit. In particular, epsilon multiplicity exists as a limit for discrete valued filtrations of an analytically unramified local domain.

We give examples of filtrations $\mathcal{I}$ for which the epsilon multiplicity is finite, but the epsilon multiplicity does not exist as a limit and filtrations $\mathcal{I}$ for which the epsilon multiplicity is infinite. These filtrations necessarily do not satisfy $A(c)$ for any c.

The following theorem about epsilon multiplicity of ideals follows from results of Ulrich and Validashti.

Theorem 1.4. *Suppose that R is a universally catenary Noetherian local ring of dimension d and I is an ideal of R. Then the analytic spread $\ell(I) = d$ if and only if $\varepsilon(I) > 0$.*

The proof that $\ell(I) < \dim R$ implies $\varepsilon(I) = 0$ follows from the theory of j-multiplicity ([1] and [19]) as explained in the inequality on the second line of page 97 of [37] and Remark 4.2 in [36]. This part of the proof is valid for an arbitrary local ring. By Theorem 4.4 [37], if R is an equidimensional, universally catenary Noetherian local ring and $\ell(I) = d$, then $\varepsilon(I) > 0$.

We obtain the following generalization of Theorem 1.4 to filtrations. Divisorial filtrations and the analytic spread $\ell(\mathcal{I})$ of a filtration are defined in Section 2.

Theorem 1.5. *Let R be a d-dimensional excellent normal local domain of equicharacteristic zero, or an arbitrary excellent local domain of dimension ≤ 3. Let $\mathcal{I}$ be a $\mathbb{Q}$-divisorial filtration of R. Then $\varepsilon(\mathcal{I}) > 0$ if and only if the analytic spread $\ell(\mathcal{I}) = d$.*

Theorem 1.5 is not true for general filtrations. Example 4.2 gives an example of an $\mathbb{R}$-divisorial filtration $\mathcal{I}$ in a regular local ring such that $\varepsilon(\mathcal{I}) > 0$ but $\ell(\mathcal{I}) < \dim R$.

Theorem 1.5 is true for any class of excellent local domains for which resolution of singularities is true. Resolution of singularities is true for reduced finite type schemes over an excellent equicharacteristic zero local ring by [22] and for reduced finite type schemes over an excellent local ring of dimension ≤ 3 by [27] and [4] (in dimension two) and by [5] (in dimension 3). The implication $\varepsilon(\mathcal{I}) > 0$ implies $\ell(\mathcal{I}) = d$ follows from the following theorem.

Theorem 1.6. *([7, Theorem 1.4]) Let R be an excellent local domain of equicharacteristic 0, or of dimension ≤ 3. Let $\mathcal{I} = \{I_n\}$ be a $\mathbb{Q}$-divisorial filtration on R. Then the following are equivalent.*

1) *The analytic spread of $\mathcal{I}$ is $\ell(\mathcal{I}) = \dim R$.*
2) *There exists $n_0 \in \mathbb{Z}_{>0}$ such that $m_R \in Ass(R/I_n)$ if $n \ge n_0$.*
3) *$m_R \in Ass(R/I_{m_0})$ for some $m_0 \in \mathbb{Z}_{>0}$.*

We obtain the following corollary to Theorems 1.5 and 1.6.

Corollary 1.7. *Let R be a d-dimensional excellent normal local domain of equicharacteristic zero, or an arbitrary excellent local domain of dimension ≤ 3. Let $\mathcal{I}$ be a $\mathbb{Q}$-divisorial filtration of R. Then $\varepsilon(\mathcal{I}) > 0$ if and only if for every representation of $\mathcal{I} = \{I_n\}$ as a rational divisorial filtration $I_n = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_r)_{na_r}$ for $n \ge 0$, the maximal ideal m_R is the center $m_{\nu_i} \cap R$ of at least one of the ν_i.*

In the final section we consider epsilon multiplicity under integral closure of filtrations. The following theorem about epsilon multiplicities of ideals is a corollary of Theorem 2.3 [37].

Theorem 1.8. *Suppose that R is a universally catenary local ring and $J \subset I$ are ideals in R. Then $\varepsilon(I_{\mathfrak{p}}) = \varepsilon(J_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ if and only if $\overline{J} = \overline{I}$.*

Example 6.1 shows that there are filtrations $\mathcal{J} \subset \mathcal{I}$ such that $\varepsilon(\mathcal{J}_{\mathfrak{p}}) = \varepsilon(\mathcal{I}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ but $\overline{R[\mathcal{J}]} \ne \overline{R[\mathcal{I}]}$ and Example 6.2 shows that there are filtrations $\mathcal{J} \subset \mathcal{I}$ such that $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$ but $\varepsilon(\mathcal{I}) \ne \varepsilon(\mathcal{J})$. Thus Theorem 1.8 and Theorem 1.9 (stated below), do not extend to arbitrary filtrations.

We show that for some naturally occurring filtrations equality of epsilon multiplicity is equivalent to the integral closures of their Rees algebras being the same. Filtrations which are s-divisorial or s-bounded are defined in Section 6.

Theorem 1.9. *Let (R, m_R) be an excellent local domain and $\mathcal{I}$ be a filtration of ideals in R. Then the following hold.*

(1) *Suppose $\mathcal{J}$ is an s-divisorial filtration such that $\mathcal{J} \subset \mathcal{I}$. Then the following are equivalent.*
   (i) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*
   (ii) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\ell(\mathcal{J}_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$.*
   (iii) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$.*
   (iv) *$\mathcal{I} = \mathcal{J}$.*
   (v) *$\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$.*

(2) *Suppose $\mathcal{J}$ is a bounded s-filtration such that $\mathcal{J} \subset \mathcal{I}$. Then $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$ if and only if $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$.*

2. Notation

Let (R, m_R) be a Noetherian local ring of dimension d. A descending chain

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals in R is called a filtration if $I_m I_n \subset I_{m+n}$ for all $m, n \in \mathbb{N}$. For a filtration $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$, we define the Rees algebra of $\mathcal{I}$ to be the graded R-algebra $R[\mathcal{I}] = \sum_{n \ge 0} I_n t^n$. This generalizes the Rees algebra $R[I] = \sum_{n \ge 0} I^n t^n$ of an ideal I in R. A filtration $\mathcal{I}$ is called a Noetherian filtration if $R[\mathcal{I}]$ is a finitely-generated R-algebra. Otherwise it is called a non-Noetherian filtration. If $I \subset R$ is an ideal, then $V(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subset \mathfrak{p}\}$. For a filtration $\mathcal{I} = \{I_n\}$, we have [16, Lemma 3.1],

$$V(I_1) = V(I_n) \text{ and } \dim R/I_1 = \dim R/I_n \text{ for all } n \ge 1.$$

For a filtration $\mathcal{I} = \{I_n\}$, by $\mathcal{I}_{\mathfrak{p}}$, we denote the filtration $\{I_n R_{\mathfrak{p}}\}$ for any $\mathfrak{p} \in \operatorname{Spec} R$. For any two filtrations $\mathcal{I} = \{I_n\}, \mathcal{J} = \{J_n\}$, by $\mathcal{J} \subset \mathcal{I}$, we mean $J_n \subset I_n$ for all $n \ge 0$ and by $\mathcal{J} = \mathcal{I}$, we mean $J_n = I_n$ for all $n \ge 0$.

The analytic spread of a filtration $\mathcal{I}$ of a local ring is defined to be $\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_R R[\mathcal{I}]$. It is shown in [17, Lemma 3.6] that $\ell(\mathcal{I}) \le \dim R$. Even for symbolic filtrations we can have that $\ell(\mathcal{I}) = 0$ ([17, Theorem 1.11]). A different definition of analytic spread is given in [13], [23] and [18].

Now let R be a Noetherian local domain of dimension d with quotient field K. Let ν be a discrete valuation of K with valuation ring $\mathcal{O}_\nu$ and maximal ideal m_ν. Suppose that $R \subset \mathcal{O}_\nu$. Then for $n \in \mathbb{N}$, define valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \ge n\} = m_\nu^n \cap R.$$

A discrete valued filtration of R is a filtration $\mathcal{I} = \{I_n\}$ such that there exist discrete valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}$$

where $\lceil x \rceil$ denotes the round up of a real number x. A discrete valued filtration is called integral (rational) if $a_i \in \mathbb{Z}_{>0}$ for all i ($a_i \in \mathbb{Q}_{>0}$) for all i.

A divisorial filtration of R is a valuation ν of the quotient field K of R such that ν is positive on R and letting $P = R \cap m_R$, the transcendence degree of the residue field of the valuation ring of ν over the residue field of R/P is $\mathrm{ht}(P) - 1$. If R is excellent, then a valuation ν of K is a divisorial valuation if and only if the valuation ring $\mathcal{O}_\nu$ of ν is essentially of finite type over R. A divisorial valuation is a discrete valuation.

A divisorial filtration of R is a discrete valued filtration

$$\{I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}\}$$

where all the discrete valuations $\nu_1, \ldots, \nu_r$ are divisorial valuations. A divisorial filtration is called integral (rational) if $a_i \in \mathbb{Z}_{>0}$ for all i ($a_i \in \mathbb{Q}_{>0}$) for all i.

It is shown in [16, Lemma 3.6] that for a filtration $\mathcal{I} = \{I_n\}$, the integral closure of $R[\mathcal{I}]$ in $R[t]$ is

$$\overline{R[\mathcal{I}]} = \sum_{m \ge 0} J_m t^m$$

where $\{J_m\}$ is the filtration

$$J_m = \{f \in R \mid f^r \in \overline{I_{rm}} \text{ for some } r > 0\}.$$

The following result follows using the same lines of proof of [10, Lemma 5.7]

Lemma 2.1. *If $\mathcal{I}$ is a discrete valued filtration then $\overline{R[\mathcal{I}]} = R[\mathcal{I}]$.*

If $I \subset R$ is an ideal in R, we define $I^{\mathrm{sat}} = I : m_R^\infty = \cup_{n=1}^\infty I : m_R^n$.

3. Epsilon multiplicity of filtrations

The property $A(c)$ of a filtration is defined in the introduction. We obtain the following theorem, showing that epsilon multiplicity, defined in (1), exits as a limit for discrete valued filtrations.

Theorem 3.1. *Let R be a Noetherian local domain, $\mathcal{J}$ a discrete valued filtration and $\mathfrak{p}$ any prime ideal in R. Then the filtration $\mathcal{J}_\mathfrak{p}$ satisfies $A(c_\mathfrak{p})$ for some $c_\mathfrak{p} \in \mathbb{Z}_{>0}$.*

Moreover, if $R_\mathfrak{p}$ is an analytically unramified local domain then $\varepsilon(\mathcal{J}_\mathfrak{p})$ exists as a limit. In particular, epsilon multiplicity exists as a limit for discrete valued filtrations of an analytically unramified local domain.

Proof. Let $\mathcal{J} = \{J_n = I(\nu_1)_{\lceil na_1\rceil} \cap \cdots \cap I(\nu_r)_{\lceil na_r\rceil}\}$. Reindexing the ν_i as ν_{ij}, let $\mathfrak{p}_i = \mathfrak{m}_{\nu_{ij}} \cap R$ for $i = 1, \ldots, l$ and $j = 1, \ldots, k_i$ be the distinct centers of the discrete valuations $\nu_1, \ldots, \nu_r$ where $k_1 + \cdots + k_l = r$.

Let $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p}_i \not\subseteq \mathfrak{p}$ for all $i = 1, \ldots, l$ then $\mathfrak{p} \notin V(J_1)$ and hence $(J_n R_{\mathfrak{p}})^{sat} = J_n R_{\mathfrak{p}}$. We take $c_{\mathfrak{p}} = 1$ in this case.

Suppose $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some $i \in \{1, \ldots, l\}$. Without loss of generality, we assume $\mathfrak{p}_i \subseteq \mathfrak{p}$ for all $1 \leq i \leq t$ and $\mathfrak{p}_i \not\subseteq \mathfrak{p}$ for all $t + 1 \leq i \leq l$.

Case 1: Suppose $\mathfrak{p}_i \subsetneq \mathfrak{p}$ for all $1 \leq i \leq t$. Then

$$(J_n R_{\mathfrak{p}})^{sat} = \bigcap_{\substack{\mathfrak{m}_{\nu_{ij}} \cap R = \mathfrak{p}_i \\ 1 \leq i \leq t, 1 \leq j \leq k_i}} I(\nu_{ij})_{\lceil na_{ij}\rceil} R_{\mathfrak{p}} = J_n R_{\mathfrak{p}}.$$

Thus we take $c_{\mathfrak{p}} = 1$.

Case 2: Suppose $\mathfrak{p}_i = \mathfrak{p}$ for some $i \in \{1, \ldots, t\}$. Without loss of generality, we assume $i = 1$. Then

$$(J_n R_{\mathfrak{p}})^{sat} = \bigcap_{\substack{\mathfrak{m}_{\nu_{ij}} \cap R = \mathfrak{p}_i \\ 2 \leq i \leq t, 1 \leq j \leq k_i}} I(\nu_{ij})_{\lceil na_{ij}\rceil} R_{\mathfrak{p}}.$$

Now $\mathfrak{p}_1^{\lceil na_{1j}\rceil} \subset I(\nu_{i1})_{\lceil na_{1j}\rceil}$ for all $1 \leq j \leq k_1$. Let $c_{\mathfrak{p}} = \max\{\lceil a_{11}\rceil, \ldots, \lceil a_{1k_1}\rceil\}$. Then

$$\begin{aligned}
(J_n R_{\mathfrak{p}})^{sat} \bigcap \mathfrak{p}^{nc_{\mathfrak{p}}} R_{\mathfrak{p}} &= \bigcap_{\substack{\mathfrak{m}_{\nu_{ij}} \cap R = \mathfrak{p}_i \\ 2 \leq i \leq t, 1 \leq j \leq k_i}} I(\nu_{ij})_{\lceil na_{ij}\rceil} R_{\mathfrak{p}} \cap \mathfrak{p}^{nc_{\mathfrak{p}}} R_{\mathfrak{p}} \\
&\subset \bigcap_{\substack{\mathfrak{m}_{\nu_{ij}} \cap R = \mathfrak{p}_i \\ 2 \leq i \leq t, 1 \leq j \leq k_i}} I(\nu_{ij})_{\lceil na_{ij}\rceil} R_{\mathfrak{p}} \bigcap \Big(\bigcap_{1 \leq j \leq k_1} I(\nu_{1j})_{\lceil na_{1j}\rceil} R_{\mathfrak{p}} \Big) \bigcap \mathfrak{p}^{nc_{\mathfrak{p}}} R_{\mathfrak{p}} \\
&\subset J_n R_{\mathfrak{p}} \bigcap \mathfrak{p}^{nc_{\mathfrak{p}}} R_{\mathfrak{p}}.
\end{aligned}$$

Thus if $R_{\mathfrak{p}}$ is an analytically unramified local domain then by Theorem ThmE1 1.2, $\varepsilon(\mathcal{J}_{\mathfrak{p}})$ exists. $\square$

Example 3.2. *Let $R = k[x,y]_{(x,y)}$ where k is a field and $\mathcal{I} = \{I_n = (x)^{\lceil n\pi\rceil} \cap (x,y)^{\lceil 2n\pi\rceil}\}$ be a non-Noetherian discrete valued filtration in R. Note that $V(I_n) = \{P = (x), m_R = (x,y)\}$. Since $\mathcal{I}_{\mathfrak{p}} = \{I_n R_P = (x)^{\lceil n\pi\rceil} R_P\}$, we have $\varepsilon(\mathcal{I}_P) = \pi$. Now $I_n^{sat} = (x)^{\lceil n\pi\rceil}$. Therefore*

$$\begin{aligned}
\varepsilon(\mathcal{I}) &= 2! \lim_{n\to\infty} \lambda_R\Big((x)^{\lceil n\pi\rceil}/(x)^{\lceil n\pi\rceil} \cap (x,y)^{\lceil 2n\pi\rceil}\Big)/n^2 \\
&= 2! \lim_{n\to\infty} \frac{(\lceil 2n\pi\rceil - \lceil n\pi\rceil)(\lceil 2n\pi\rceil - \lceil n\pi\rceil + 1)}{2n^2} \\
&= \pi^2.
\end{aligned}$$

Now we introduce a family of examples, which will illustrate the essential role of the $A(c)$ condition in the existence of epsilon multiplicity as a limit.

Let $\tau : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be any function such that $\tau(n+1) \geq \tau(n)$ for all n. We will restrict to τ satisfying this condition in this analysis.

Then defining I_n to be the ideal $I_n = (x^2, xy^{\tau(n)})$ in the local ring $R = k[x,y]_{(x,y)}$ over a field k, we have that $\mathcal{I}_\tau = \{I_n\}$ is a filtration. We have that $I_n : m_R^\infty = (x)$ and $(I_n : m_R^\infty)/I_n \cong R/(x, y^{\tau(n)})$ as an R-module. Thus $\lambda_R(I_n : m_R^\infty/I_n) = \tau(n)$.

Note that for any τ and $f \in I_n$ for $n \geq 1$, we have $f^2 \in m_R I_{2n}$. Therefore by [17, Lemma 3.8], $\ell(\mathcal{I}_\tau) = 0$. We conclude that for all τ, the epsilon multiplicity is

$$\varepsilon(\mathcal{I}_\tau) = 2 \limsup \frac{\tau(n)}{n^2}$$

while the analytic spread is

$$\ell(\mathcal{I}_\tau) = 0.$$

The following examples show that we cannot expect the epsilon multiplicity of a filtration $\mathcal{I}$ to exist as a limit if the condition $A(c)$ is not satisfied for any c.

Example 3.3. *Let $\sigma : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be the function defined in (22) of Section 5 of [6]. We have that $\sigma(n+1) \geq \sigma(n)$ for all n, $1 \leq \sigma(n) \leq \frac{n}{2}$ for all n and $\lim_{n\to\infty} \frac{\sigma(n)}{n}$ does not exist, even when n is constrained to lie in any arithmetic sequence. We further have that $\limsup \frac{\sigma(n)}{n} = \frac{1}{2}$. Let $\tau(n) = n\sigma(n)$. Then*

$$\lim_{n\to\infty} \frac{\lambda_R(I_n : m_R^\infty / I_n)}{n^2} = \lim_{n\to\infty} \frac{\sigma(n)}{n}$$

does not exist. In this example, we have that $\varepsilon(\mathcal{I}_\tau) = \frac{1}{2} < \infty$, so that $\varepsilon(\mathcal{I}_\tau)$ is positive but $\ell(\mathcal{I}_\tau) \neq \dim R$.

Example 3.4. *Let $\tau(n)$ be any increasing function such that $\limsup \frac{\tau(n)}{n^2} = \infty$ (such as $\tau(n) = n^3$). We obtain that $\varepsilon(\mathcal{I}_\tau) = \infty$. In particular, $\varepsilon(\mathcal{I}_\tau)$ is not finite.*

Example 3.5. *Suppose that $\mathcal{I}_\tau$ satisfies condition $A(c)$ for some c. Then $(I_n : m_R^\infty) \cap m_R^{cn} = I_n \cap m_R^{cn}$ for all n so that $\tau(n) \leq cn$ for all n. Thus $\frac{\lambda_R(I_n : m_R^\infty / I_n)}{n^2} = \frac{\tau(n)}{n^2} \leq \frac{c}{n}$ for all n. Thus $\varepsilon(\mathcal{I}_\tau) = 0$ and this multiplicity exists as a limit, in agreement with the conclusions of Theorem 1.2. We further have that $\varepsilon(\mathcal{I}_\tau) = 0$ and $\ell(\mathcal{I}_\tau) \neq \dim R$.*

4. Analytic spread and epsilon multiplicity

In this section, we prove Theorem 1.5 from the introduction, which we restate here for the convenience of the reader.

Theorem 4.1. *Let R be a d-dimensional excellent normal local domain of equicharacteristic zero, or an arbitrary excellent local domain of dimension ≤ 3. Let $\mathcal{I}$ be a $\mathbb{Q}$-divisorial filtration of R. Then $\varepsilon(\mathcal{I}) > 0$ if and only if the analytic spread $\ell(\mathcal{I}) = d$.*

The following example shows that the conclusions of Theorem 1.4 and Theorem 1.6 are false for filtrations. We make use of [7, Example 1.5] which shows that the conclusions of Theorem 1.6 do not hold for $\mathbb{R}$-divisorial filtrations.

Example 4.2. *There exists an $\mathbb{R}$-divisorial filtration $\mathcal{I}$ in $R = k[x]_{(x)}$ such that $\varepsilon(\mathcal{I}) > 0$ but $\ell(\mathcal{I}) = 0 < 1 = \dim R$.*

The example satisfies $A(4)$. This is the construction of the example. Let $\mathcal{I} = \{I_n\}$ where $I_n = (x^{\lceil n\pi \rceil})$ in $R = k[x]_{(x)}$. For fixed n, $r\lceil n\pi \rceil > \lceil rn\pi \rceil + 1$ for some $r \in \mathbb{Z}_{>0}$. Thus $f \in I_n$ implies $f^r \in m_R I_{nr}$ and so by [17, Lemma 3.8],

$$\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_R R[\mathcal{I}] = 0 < 1 = \dim R.$$

For all n, $I_n : m_R^\infty = R$ so

$$\varepsilon(\mathcal{I}) = \lim_{n\to\infty} \frac{\lceil n\pi \rceil}{n} = \pi > 0.$$

Let R be a normal excellent local ring. Let $\mathcal{I} = \{I_m\}$ where

$$I_m = I(\nu_1)_{ma_1} \cap \cdots \cap I(\nu_s)_{ma_s}$$

for some divisorial valuations $\nu_1, \ldots, \nu_s$ of R be an $\mathbb{R}$-divisorial filtration of R, with $a_1, \ldots, a_s \in \mathbb{R}_{>0}$. Then there exists a projective birational morphism $\varphi : X \to \mathrm{Spec}(R)$ such that there exist prime divisors $F_1, \ldots, F_s$ on X such that $V_{\nu_i} = \mathcal{O}_{X,F_i}$ for $1 \le i \le s$ ([16, Remark 6.6 to Lemma 6.5]). Let $D = a_1F_1 + \cdots + a_sF_s$, an effective $\mathbb{R}$-divisor on X (an effective $\mathbb{R}$-Weil divisor). Define $\lceil D \rceil = \lceil a_1 \rceil F_1 + \cdots + \lceil a_s \rceil F_s$, an integral divisor. We have coherent sheaves $\mathcal{O}_X(-\lceil nD \rceil)$ on X such that

$$\Gamma(X, \mathcal{O}_X(-\lceil nD \rceil)) = I_n \tag{3}$$

for $n \in \mathbb{N}$. If X is nonsingular then $\mathcal{O}_X(-\lceil nD \rceil)$ is invertible. The formula (3) is independent of choice of X. Further, even on a particular X, there are generally many different choices of effective $\mathbb{R}$-divisors G on X such that $\Gamma(X, \mathcal{O}_X(-\lceil nG \rceil)) = I_n$ for all $n \in \mathbb{N}$. Any choice of a divisor G on such an X for which the formula $\Gamma(X, \mathcal{O}_X(-\lceil nG \rceil)) = I_n$ for all $n \in \mathbb{N}$ holds will be called a representation of the filtration $\mathcal{I}$.

Given an $\mathbb{R}$-divisor $D = a_1F_1 + \cdots + a_sF_s$ on X we have a divisorial filtration $\mathcal{I}(D) = \{I(nD)\}$ where

$$I(nD) = \Gamma(X, \mathcal{O}_X(-\lceil nD \rceil)) = I(\nu_1)_{\lceil na_1 \rceil} \cap \cdots \cap I(\nu_s)_{\lceil na_s \rceil} = I(\nu_1)_{na_1} \cap \cdots \cap I(\nu_s)_{na_s}.$$

We write $R[D] = R[\mathcal{I}(D)]$.

We recall the γ_Γ function defined in [9, Section 3]. We make use of statements and proofs in [7, Section 4] which are based on corresponding statements and proofs for the σ_Γ function on pseudo-effective divisors on a projective nonsingular variety in [29, Chapter III, Section 1].

Let R be a normal excellent local ring and $\pi : X \to \mathrm{Spec}(R)$ be a birational projective morphism such that X is nonsingular. Let $G = \sum a_iE_i$ be an effective $\mathbb{R}$-divisor, and Γ be a prime divisor on X. Let

$$\mathrm{ord}_\Gamma(G) = \begin{cases} a_i & \text{if } \Gamma = E_i \\ 0 & \text{if } \Gamma \not\subset \mathrm{Supp}(D). \end{cases}$$

For D an $\mathbb{R}$-divisor, let

$$\tau_\Gamma(D) = \inf\{\mathrm{ord}_\Gamma(G) \mid G \ge 0 \text{ and } G \sim D\},$$

and define

$$\gamma_\Gamma(D) = \inf\left\{\frac{\tau_\Gamma(mD)}{m} \mid m \in \mathbb{Z}_{>0}\right\}.$$

Since D is linearly equivalent to an effective divisor, there can be only finitely many prime divisors Γ such that $\gamma_\Gamma(D) > 0$.

If R has dimension 2 and D is an integral divisor on X then $\gamma_\Gamma(D)$ is a rational number, but there exist examples where R has dimension 3 and integral divisors D on X such that $\gamma_\Gamma(D)$ is an irrational number ([11, Theorem 4.1]).

We have that

$$\gamma_\Gamma(D) = \lim_{m \to \infty} \frac{\tau_\Gamma(mD)}{m}.$$

Thus if $\alpha \in \mathbb{Q}_{>0}$, we have that

$$\gamma_\Gamma(\alpha D) = \alpha\gamma_\Gamma(D). \tag{4}$$

We also have that for $\mathbb{R}$ divisors D_1 and D_2,

$$\gamma_\Gamma(D_1 + D_2) \le \gamma_\Gamma(D_1) + \gamma_\Gamma(D_2). \tag{5}$$

If $m \in \mathbb{Z}_{>0}$ and $G \sim mD$ is an effective $\mathbb{R}$-divisor, then $\mathrm{ord}_\Gamma(G) \ge m\gamma_\Gamma(D)$.

We now prove Theorem 1.5. Let notation be as in the statement of Theorem 1.5.

Suppose that $\varepsilon(\mathcal{I}) > 0$. Then $H^0_{m_R}(R/I_n) \neq 0$ for some n which implies that $m_R \in \mathrm{Ass}(R/I_n)$. By Theorem 1.6, $\ell(\mathcal{I}) = d$.

We will now show that $\ell(\mathcal{I}) = d$ implies $\varepsilon(\mathcal{I}) > 0$. The first part of the proof is similar to the first part of the proof of [7, Theorem 1.4]. Let $k = R/m_R$. There exists by [7, Lemma 3.1], a projective birational morphism $\pi : X \to \mathrm{Spec}(R)$ such that X is nonsingular, all prime exceptional divisors of π are nonsingular and there exists an effective $\mathbb{Q}$-divisor D on X such that $\mathcal{I} = \mathcal{I}(D)$. Let

$$D = \sum a_i F_i \tag{6}$$

with the F_i distinct prime divisors and $a_i \in \mathbb{Q}_{\ge 0}$.

By Theorem 1.6, $\ell(\mathcal{I}) = d$ implies there exists an m' such that $m_R \in \mathrm{Ass}(R/I_{m'})$. Thus there exists an F_j which contracts to m_R and $m' > 0$ such that if $D_1 = \sum_{i \neq j} a_i F_i$, then

$$I(m'D) \neq I(m'D_1). \tag{7}$$

Without loss of generality, we may assume that $j = 1$ so that $F_j = F_1$. Set $F = F_1$. Since $F \subset \pi^{-1}(m_R)$, F is a projective k-variety. We have that $-D \le -D_1 = -D + a_1F$. Suppose that $\gamma_F(-D) > 0$. Then $\gamma_F(-D_1) = \gamma_F(-D) + a_1$ by [7, Lemma 4.3]. Thus

$$\Gamma(X, \mathcal{O}_X(\lfloor -nD \rfloor)) = \Gamma(X, \mathcal{O}_X(\lfloor -nD_1 \rfloor))$$

for all $n \in \mathbb{N}$ by [7, Lemma 4.1], which is a contradiction to (7). Thus $\gamma_F(-D) = 0$ and $0 \le \gamma_F(-D_1) < a_1$.

Let K be an ideal of R such that $\pi : X \to \mathrm{Spec}(R)$ is the blowup of K. Let A be the Cartier divisor defined by $K\mathcal{O}_X = \mathcal{O}_X(A)$. Expand $A = \sum -b_iF_i$ where we increase the set of F_i if neccessary, so that some a_i and b_j could be zero. A is an anti-effective integral divisor which is ample and all irreducible components of $\pi^{-1}(m_R)$ are in the support of A. After possibly replacing K with a power of K, so that A is replaced with a multiple of A, we may assume that $A - F$ is also ample. We have that

$$F = -\frac{1}{b_1}A - \sum_{i>1} \frac{b_i}{b_1} F_i.$$

We establish the following lemma. For $0 < t \in \mathbb{Q}$, let $D_t = D - tF$.

Lemma 4.3. *Suppose that $t < a_1 - \gamma_F(-D_1)$ is a positive rational number. Then there exists $m_0 > 0$ and an effective integral divisor U on X which does not contain F in it's support such that m_0D_t is an integral divisor and $-m_0D_t \sim U$.*

Proof. Let $\lambda \in \mathbb{Q}_{>0}$ be such that $\lambda + t < a_1 - \gamma_F(-D_1)$. Then $D_t = D_{\lambda+t} + \lambda F$. Suppose that $\gamma_F(-D_t) > 0$. Then by [7, Lemma 4.2] and [7, Lemma 4.3],

$$\begin{aligned} 0 = \gamma_F(-D_1 - \gamma_F(-D_1)F) &= \gamma_F(-D_t + (a_1 - \gamma_F(-D_1) - t)F) \\ &= \gamma_F(-D_t) + (a_1 - \gamma_F(-D_1) - t) > 0, \end{aligned}$$

a contradiction. Thus $\gamma_F(-D_t) = 0$.

We have that

$$-D_t = -D_{\lambda+t} - \lambda F = -D_{\lambda+t} + \frac{\lambda}{b_1}A + \sum_{i>1} \frac{\lambda b_i}{b_1} F_i.$$

Now the lemma follows from Lemma 4.4 [7], since $\gamma_F(-D_{\lambda+t}) = 0$. $\square$

For the rest of the proof, we fix $t \in \mathbb{Q}_{>0}$ with $t < a_1 - \gamma_F(-D_1)$ and allow $s \in \mathbb{Q}_{>0}$ with $0 < s < t$ to vary.

We have that $-D_s = -D_t + (s-t)F$. By Lemma LemmaG1 4.3, there exists $m_0 \in \mathbb{Z}_{>0}$ and an effective integral divisor U whose support does not contain F such that $m_0(-D_t) \sim U$. Thus

$$m_0(-D_s) \sim U + m_0(s-t)F \sim U + m_0(t-s)(\sum_{i>1} \frac{b_i}{b_1} F_i) + \frac{m_0(t-s)}{b_1} A.$$

Since A and $A - F$ are ample, we have that $A - \lambda F$ is ample for $0 \le \lambda \le 1$.

We now impose the further restriction on s that $s \le \frac{t}{2b_1+1}$. This implies that $\frac{sb_1}{t-s} \le \frac{1}{2}$. With this restriction on s, we have that $\frac{t-s}{b_1}A - \lambda F$ is ample for $0 \le \lambda \le s$. In fact,

eqR4
$$A - \frac{b_1\lambda}{t-s}F = \frac{1}{2}A + (\frac{1}{2}A - \frac{b_1\lambda}{t-s}F) = \frac{1}{2}A + \frac{1}{2}(A - \frac{2b_1\lambda}{t-s}F) \tag{8}$$

which is the sum of two ample divisors since $\frac{2b_1\lambda}{t-s} \le 1$. Let $H_s = \frac{t-s}{b_1}A$.

There exists $m_1 \in \mathbb{Z}_{>0}$ such that $m_1\left(\frac{t-s}{b_1}\right) \in \mathbb{Z}_{>0}$, $m_1 s \in \mathbb{Z}_{>0}$ and $m_1 D$ is an integral divisor. Let $m_s = m_0 m_1$.

There exists a section

$$\tau_s \in \Gamma(X, \mathcal{O}_X(-m_s D_s - m_s H_s))$$

such that the divisor (τ_s) of τ_s is $m_1 U + m_s\left(\frac{t-s}{b_1}\right)(\sum_{i>1} b_i F_i)$. Since F is not in the support of (τ_s), τ_s restricts to a nonzero section $\overline{\tau}_s$ of $\Gamma(F, \mathcal{O}_X(-m_s D_s - m_s H_s) \otimes \mathcal{O}_{nm_s sF})$, inducing commutative diagrams with exact columns and rows for $n \in \mathbb{Z}_{\ge 0}$

eqR2
$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \uparrow & & \uparrow \\
0 & \to & \mathcal{O}_X(nm_sH_s) \otimes \mathcal{O}_{nm_ssF} & \overset{\overline{\tau}_s^n}{\to} & \mathcal{O}_X(-nm_sD + nm_ssF) \otimes \mathcal{O}_{nm_ssF} \\
 & & \uparrow & & \uparrow \\
0 & \to & \mathcal{O}_X(nm_sH_s) & \overset{\tau_s^n}{\to} & \mathcal{O}_X(-nm_sD + nm_ssF) \\
 & & \uparrow & & \uparrow \\
0 & \to & \mathcal{O}_X(nm_sH_s - nm_ssF) & \to & \mathcal{O}_X(-nm_sD) \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array} \tag{9}$$

and taking cohomology, we have commutative diagrams with exact columns and rows

$$\begin{array}{ccccc}
0 & \to & H^0(X, \mathcal{O}_X(nm_sH_s) \otimes \mathcal{O}_{nm_ssF}) & \overset{\overline{\tau}_s^n}{\to} & H^0(X, \mathcal{O}_X(-nm_sD + nm_ssF) \otimes \mathcal{O}_{nm_ssF}) \\
 & & \beta_n \uparrow & & \alpha_n \uparrow \\
0 & \to & H^0(X, \mathcal{O}_X(nm_sH_s)) & \overset{\tau_s^n}{\to} & H^0(X, \mathcal{O}_X(-nm_sD + nm_ssF)) \\
 & & \uparrow & & \uparrow \\
0 & \to & H^0(X, \mathcal{O}_X(nm_sH_s - nm_ssF)) & \to & H^0(X, \mathcal{O}_X(-nm_sD)) \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array}$$

We have that

$$\alpha_n(H^0(X, \mathcal{O}_X(-nm_sD + nm_ssF)) \cong H^0(X, \mathcal{O}_X(-nm_sD_s)/H^0(X, \mathcal{O}_X(-nm_sD)).$$

We will show that $\lambda_R(\alpha_n(H^0(X,\mathcal{O}_X(-nm_sD+nm_ssF)))$ grows like n^d. To do this, it suffices to show that $\lambda_R(\beta_n(H^0(X,\mathcal{O}_X(nm_sH_s)))$ grows like n^d, since $\overline{\tau}_s^n$ is an injection and $\overline{\tau}_s^n\beta_n(H^0(X,\mathcal{O}_X(nm_sH_s))\subset\alpha_n(H^0(X,\mathcal{O}_X(-nm_sD+nm_ssF)))$.

Now β_n is surjective for large n since H_s-sF is ample, so that $H^1(X,\mathcal{O}_X(nm_sH_s-nm_ssF))=0$ for $n\gg 0$. Thus it suffices to show that $\lambda_R(H^0(X,O_X(nm_sH_s)\otimes_{\mathcal{O}_{nm_ssF}}))$ grows of order n^d for $n\gg 0$.

We have short exact sequences

$$0\to\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F\to\mathcal{O}_X(nm_sH_s)\otimes\mathcal{O}_{(j+1)F}\to\mathcal{O}_X(nm_sH_s)\otimes\mathcal{O}_{jF}\to 0 \tag{10}$$

for $1\le j\le nm_ss-1$.

For $0\le j\le nm_ss$,

$$nm_sH_s-jF=\frac{nm_s(t-s)}{b_1}\left(\frac{1}{2}A+\frac{1}{2}\left(A-\frac{2b_1}{(t-s)}\frac{j}{nm_s}F\right)\right)$$

is the sum of two ample divisors as shown in (8). This decomposition continues to hold when restricted to F. Thus by Fujita's vanishing theorem ([20] over all fields, also Theorem 1.4.35 page 66 [26] in characteristic zero) we have that there exists n_0 such that $n\ge n_0$ implies that $H^i(F,\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F)=0$ for $i>0$ and $0\le j\le nm_ss$.

Let $h^i(\mathcal{F})=\dim_k H^i(F,\mathcal{F})=\lambda_R(H^i(F,\mathcal{F}))$ and $\chi(\mathcal{F})=h^0(\mathcal{F})-h^1(\mathcal{F})$ if $\mathcal{F}$ is a coherent sheaf on F.

Restricting to $n\ge n_0$ and taking cohomology of the exact sequences (10), we obtain that

$$\begin{array}{rcl}\lambda_R(H^0(X,\mathcal{O}_X(nm_sH_s)\otimes\mathcal{O}_{nm_ssF})) &=& \sum_{j=0}^{nm_ss-1}h^0(\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F)\\ &=& \sum_{j=0}^{nm_ss-1}\chi(\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F).\end{array}$$

Now $\chi(\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F)$ is a polynomial in n and j of total degree $d-1=\dim F$ (the Snapper polynomial [35], [25], [12]). The bi-homogeneous part of $\chi(\mathcal{O}_X(nm_sH_s-jF)\otimes\mathcal{O}_F)$ of total degree $d-1$ is $\frac{((nm_sH_s-jF)^{d-1}\cdot F)}{(d-1)!}$ (by a variation of Theorem 19.16 [12]). We can do these calculations after making a base change by an algebraic closure of k since the intersection theory from the Snapper polynomial is valid over an arbitrary (not necessarily reduced) projective scheme.

Thus for large n,

$$\lambda_R(H^0(X,\mathcal{O}_X(nm_sH_s)\otimes\mathcal{O}_{nm_sF}))$$

is a polynomial $P_s(n)$ in n of degree $\le d$, and

$$P_s(n)=\sum_{j=0}^{nm_ss-1}\frac{((nm_sH_s-jF)^{d-1}\cdot F)}{(d-1)!}+\text{ terms in } n \text{ of degree} < d.$$

Define

$$Q_s(n)=\sum_{j=0}^{ns-1}\frac{\left((nH_s-jF)^{d-1}\cdot F\right)}{(d-1)!}.$$

$Q_s(n)$ is a polynomial in n of degree $\le d$. We have that

$$P_s(n)=Q_s(m_sn)+\text{ terms of degree} < d.$$

eqR7

$$
\begin{aligned}
(d-1)!Q_s(n) &= \textstyle\sum_{j=0}^{ns-1}((nH_s-jF)^{d-1}\cdot F)\\
&= ((nH_s)^{d-1}\cdot F)+\textstyle\sum_{j=1}^{ns-1}\left[\sum_{k=0}^{d-1}\binom{d-1}{k}(-1)^{d-k-1}(H_s^k\cdot F^{d-k})n^kj^{d-1-k}\right]\\
&= ((nH_s)^{d-1}\cdot F)+\textstyle\sum_{k=0}^{d-1}\left[\binom{d-1}{k}(-1)^{d-k-1}(H_s^k\cdot F^{d-k})n^k\left(\sum_{j=1}^{ns-1}j^{d-1-k}\right)\right].
\end{aligned}
\tag{11}
$$

By Faulhaber's formula, $\sum_{k=1}^{n}k^p$ is a polynomial in n with leading term $\frac{n^{p+1}}{p+1}$ (c.f. [2]). Thus $\sum_{j=1}^{ns-1}j^{d-1-k}$ is a polynomial in n of degree $d-k$, whose leading term is $\frac{s^{d-k}}{d-k}n^{d-k}$. Substituting into (11), we see that the coefficient $\sigma(s)$ of the term of degree d of the polynomial $Q_s(n)$ is

$$\sigma(s)=\frac{1}{(d-1)!}\sum_{k=0}^{d-1}(-1)^{d-k-1}\binom{d-1}{k}\frac{s^{d-k}}{d-k}\left(\frac{t-s}{b_1}\right)^k(A^k\cdot F^{d-k})$$

which is a polynomial in s. We have that

$$\sigma(s)=s\frac{1}{(d-1)!}\left(\frac{t}{b_1}\right)^{d-1}(A^{d-1}\cdot F)+\text{ higher degree terms in } s.$$

In particular, $\sigma(s)$ is a nonzero polynomial, and since $\frac{1}{(d-1)!}\left(\frac{t}{b_1}\right)^{d-1}(A^{d-1}\cdot F)>0$ (as A is ample) if s is sufficiently small within the region $0<s\le\frac{t}{2b_1+1}$, we have that $\sigma(s)>0$. We now fix such an s. Since $Q_s(n)$ is a polynomial in n of degree d, $P_s(n)$ is a polynomial in n of degree d.

We have natural inclusions

$$H^0(X,\mathcal{O}(-nD_s))/H^0(X,\mathcal{O}_X(-nD))\subset H^0(X,\mathcal{O}_X(-n(\sum_{\pi(F_i)\ne m_R}a_iF_i)))/H^0(X,\mathcal{O}_X(-nD)).$$

Since $P_s(n)$ is a polynomial in n of degree d (with positive leading coefficient), there exists a constant $c>0$ such that for $n\gg 0$, we have that

$$
\begin{aligned}
0<c &\le \tfrac{P_s(n)}{m_s^dn^d}=\tfrac{\lambda_R(H^0(nm_sF,\mathcal{O}_X(nm_sH_s)\otimes\mathcal{O}_{nm_sF}))}{m_s^dn^d}\\
&\le \tfrac{\lambda_R(\alpha_nH^0(X,\mathcal{O}_X(-nm_sD+nm_sF)))}{m_s^dn^d}\\
&= \tfrac{\lambda_R(H^0(X,\mathcal{O}_X(-nm_sD_s))/H^0(X,\mathcal{O}_X(-nm_sD))}{m_s^dn^d}\\
&\le \tfrac{\lambda_R(H^0(X,\mathcal{O}_X(-nm_s(\sum_{\pi(F_i)\ne m_R}a_iF_i)))/H^0(X,\mathcal{O}_X(-nm_sD))}{m_s^dn^d}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varepsilon(\mathcal{I}) &= d!\lim_{n\to\infty}\tfrac{\lambda_R(I(nD):m_R^\infty/I(nD))}{n^d}\\
&= d!\lim_{n\to\infty}\tfrac{\lambda_R((H^0(X,\mathcal{O}_X(-n(\sum_{\pi(F_i)\ne m_R}a_iF_i)))/H^0(X,\mathcal{O}_X(-nD))}{n^d}\\
&= d!\lim_{n\to\infty}\tfrac{\lambda_R((H^0(X,\mathcal{O}_X(-nm_s(\sum_{\pi(F_i)\ne m_R}a_iF_i)))/H^0(X,\mathcal{O}_X(-nm_sD))}{m_s^dn^d}\\
&\ge d!c>0.
\end{aligned}
$$

5. Epsilon multiplicity under inclusions of filtrations

In this section we consider filtrations $\mathcal{I},\mathcal{J}$ such that $\mathcal{J}\subset\mathcal{I}$ and discuss the existence of their epsilon multiplicities.

finite length **Proposition 5.1.** *Let (R, m_R) be a Noetherian local ring of dimension $d > 0$ and $\mathcal{J} = \{J_n\}, \mathcal{I} = \{I_n\}$ be filtrations of R such that $J_n \subset I_n$ and $\lambda_R(I_n/J_n) < \infty$ for all $n \geq 1$. Suppose $\mathcal{J}$ satisfies $A(c)$ for some $c \in \mathbb{Z}_{>}0$. Then the following hold.*

(i) $\mathcal{I}$ satisfies $A(c)$ and $\varepsilon(\mathcal{I}), \varepsilon(\mathcal{J})$ exist as limits.

(ii) Suppose R is analytically irreducible. If $R[\mathcal{I}]$ is integral over $R[\mathcal{J}]$ then $\varepsilon(\mathcal{I}) = \varepsilon(\mathcal{J})$.

(iii) The converse of (ii) is not true in general.

Proof. Since $\lambda_R(I_n/J_n) < \infty$, we have $J_n \subset I_n \subset J_n^{sat}$ for all $n \geq 1$. Hence

$$J_n \cap {m_R}^{cn} = I_n \cap {m_R}^{cn} = J_n^{sat} \cap {m_R}^{cn}$$

for all $n \geq 1$.

(i) Let $x \in I_n^{sat} \cap {m_R}^{cn}$ for any $n \geq 1$. Then ${m_R}^l x \in I_n \cap {m_R}^{cn} = J_n \cap {m_R}^{cn} \subset J_n$ for some $l \in \mathbb{Z}_{>0}$. Hence $x \in J_n^{sat} \cap {m_R}^{cn} = I_n \cap {m_R}^{cn}$. Therefore $\mathcal{I}$ satisfies $A(c)$ and by Theorem 1.2, $\varepsilon(\mathcal{I})$ and $\varepsilon(\mathcal{J})$ exist as limits and they are real numbers.

(ii) By [6, Theroem 6.1], $\lim\limits_{n\to\infty} \lambda_R(I_n/J_n)/n^d$ exists. Now from the following short exact sequence

$$0 \longrightarrow I_n/J_n \longrightarrow R/J_n \longrightarrow R/I_n \longrightarrow 0$$

of R-modules, we get a short exact sequence

$$0 \longrightarrow H^0_{m_R}(I_n/J_n) = I_n/J_n \longrightarrow H^0_{m_R}(R/J_n) \longrightarrow H^0_{m_R}(R/I_n) \longrightarrow 0$$

of local cohomology modules. Therefore

E (12) $$\varepsilon(\mathcal{J}) = \varepsilon(\mathcal{I}) + d! \lim_{n\to\infty} \frac{\lambda_R(I_n/J_n)}{n^d}.$$

If $R[\mathcal{I}]$ is integral over $R[\mathcal{J}]$ then by [32, Theorem 1.5], we have $\lim\limits_{n\to\infty} \lambda_R(I_n/J_n)/n^d = 0$. Therefore by equation (12), we have $\varepsilon(\mathcal{I}) = \varepsilon(\mathcal{J})$.

(iii) See example 6.1. In this example, the filtrations $\mathcal{I}$ and $\mathcal{J}$ both satisfy $A(2)$, $\varepsilon(\mathcal{I}) = \varepsilon(\mathcal{J})$ but $R[\mathcal{I}]$ is not integral over $R[\mathcal{J}]$. ∎

Using Theorem 1.3 and Proposition 5.1, we get the following.

Corollary 5.2. *Let R be a Noetherian local domain of dimension $d > 0$ and $\mathcal{J} = \{J_n\}$ be a discrete valued filtration. Suppose $\mathcal{I} = \{I_n\}$ is a filtration such that $J_n \subset I_n$ and $\lambda_R(I_n/J_n) < \infty$ for all $n \geq 1$. Then $\mathcal{I}$ satisfies $A(c)$ for some $c \in \mathbb{Z}_{>0}$ and $\varepsilon(\mathcal{I})$ exists as a limit.*

Remark 5.3. *In Proposition 5.1, if we replace "$\mathcal{J}$ satisfies $A(c)$" by "$\mathcal{I}$ satisfies $A(c)$" then $\varepsilon(\mathcal{I})$ exists as a limit but $\varepsilon(\mathcal{J})$ may not exist. For example, let $\mathcal{I} = \{I_n = (x^2, xy^n)\}$ and $\mathcal{J} = \{J_n = (x^2, xy^{n^3})\}$ be filtrations in $k[x,y]_{(x,y)}$. Then $\mathcal{I}$ satisfies $A(2)$, $\varepsilon(\mathcal{I}) = 0$ and $\varepsilon(\mathcal{J}) = \infty$.*

Remark 5.4. *If we drop the condition that $\mathcal{J}$ satisfies $A(c)$ for some $c \in \mathbb{Z}_{>}0$ in Proposition 5.1, then (ii) of Proposition 5.1 is not true in general (see Example 6.2).*

The next example shows that the property of $A(c)$ does not descend in integral extensions of filtrations.

Example4 **Example 5.5.** *There exists a filtration $\mathcal{J}$ which satisfies $A(2)$ with the following properties*

1) *Given $c \in \mathbb{Z}_{\geq 2}$, there exists a subfiltration $\mathcal{K}$ of $\mathcal{J}$ such that $\overline{R[\mathcal{K}]} = \overline{R[\mathcal{J}]}$ and c is the smallest positive integer such that $\mathcal{K}$ satisfies $A(c)$.*

2 *There exists a subfiltration $\mathcal{H}$ of $\mathcal{J}$ such that $\overline{R[\mathcal{H}]} = \overline{R[\mathcal{J}]}$ but $\mathcal{H}$ does not satisfy $A(c)$ for any $c \in \mathbb{Z}_{>0}$.*

We have that $\varepsilon(\mathcal{J}) = \varepsilon(\mathcal{K}) = 0$ but $\varepsilon(\mathcal{H}) = \frac{1}{2}$. Recall that we have defined the epsilon multiplicity of a filtration as a limsup in (1). We now construct the example. Let $\mathcal{J} = \{J_n\}$ where J_n is the ideal (x^2, xy^n) in $R = k[x,y]_{(x,y)}$. Let τ be any increasing function such that $\tau(n) \ge n$ for all n and let $\mathcal{I} = \mathcal{I}_\tau = \{I_n\}$, where $I_n = (x^2, xy^{\tau(n)})$ as defined in Section 3. Then $\mathcal{I}_\tau \subset \mathcal{J}$.

We will show that $\overline{R[\mathcal{I}_\tau]} = \overline{R[\mathcal{J}]}$. To establish this, we need only show that xy^nt^n is integral over $R[I_\tau] = \sum_{n \ge 0} I_n t^n$. This follows since

$$(xy^nt^n)^2 = x^2y^{2n}t^{2n} \in I_{2n}t^{2n}.$$

Let $\mathcal{H}$ be the filtration of Example 3.3. In Example 3.3 we showed that $\mathcal{H}$ does not satisfy $A(c)$ for any c, so that the second statement of Example 5.5 holds.

Let $a \in \mathbb{Z}_{>0}$ and let $\mathcal{K} = \{K_n\}$ where $K_n = (x^2, xy^{an})$.

We will establish that $\mathcal{K}$ satisfies $A(c)$ if and only if $c > a$.

We have that $K_n : m_R^\infty = (x)$ for all a and n. Thus for $c \in \mathbb{Z}_{>0}$,

$$(K_n : m_R^\infty) \cap m_R^{cn} = x m_R^{cn-1}$$

and

$$K_n \cap m_R^{cn} = \begin{cases} m_R^{cn-2}x^2 + xy^{an}m_R^{cn-an-1} = xm_R^{cn-1} & \text{if } cn \ge an+1 \\ m_R^{cn-2}x^2 & \text{if } cn < an+1. \end{cases}$$

Thus $\mathcal{K}$ satisfies $A(c)$ if and only if $c > a$. Taking $a = c-1$, we obtain the conclusions of the first statement of Example 5.5.

We also have that the property of $A(c)$ does not ascend under inclusions of filtrations.

Example5 **Example 5.6.** *There exists an inclusion of filtrations $\mathcal{I} \subset \mathcal{J}$ such that $\mathcal{I}$ satisfies $A(1)$ but $\mathcal{J}$ does not satisfy $A(1)$.*

The example is constructed as follows. Let $R = k[x,y]_{(x,y)}$. Let $\mathcal{I} = \{I_n\}$ where $I_n = (x^{3n})$ and $\mathcal{J} = \{J_n\}$ where $J_n = (x^n m_R^{2n})$. We have that $\mathcal{I}$ satisfies $A(1)$. For all n, $J_n : m_R^\infty = (x^n)$ so that $(J_n : m_R^\infty) \cap m_R^n = (x^n)$ and $J_n \cap m_R^n = J_n$ so $\mathcal{J}$ does not satisfy $A(1)$.

The above examples shows that it is not easy to approximate filtrations which satisfy $A(c)$ by filtrations which also satisfy $A(c)$. In the case that we can, we may compute the epsilon multiplicity of a filtration as a limit of the epsilon multiplicities of the approximating filtrations. The following proposition is proven by an extension of the proof of [6, Theorem 6.1].

PropE4 **Proposition 5.7.** *Suppose that R is an analytically unramified local ring and $\mathcal{I}$ is a filtration of R which satisfies $A(c)$ for some $c \in \mathbb{Z}_{>0}$. Let $\mathcal{I}_i = \{I[i]_n\}$ be subfiltrations of $\mathcal{I}$ for $i \in \mathbb{Z}_{>0}$ such that $\mathcal{I}_j \subset \mathcal{I}_i$ if $j \ge i$, $\cup_{i=1}^\infty = \mathcal{I}$ and $\mathcal{I}_i$ satisfy $A(c)$ for all i. Then*

$$\lim_{i \to \infty} \varepsilon(\mathcal{I}[i]) = \varepsilon(\mathcal{I}).$$

closure

6. Integral closure, multiplicity and epsilon multiplicity of filtrations

In this section we study the relationship between epsilon multiplicities of two filtrations and their integral closures. The following examples show that Theorem 1.8 does not extend to general filtrations.

Example6

Example 6.1. *We have filtrations $\mathcal{J} \subset \mathcal{I}$ in a Noetherian local ring R such that $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ but $\overline{R[\mathcal{I}]} \neq \overline{R[\mathcal{J}]}$.*

In the example, $R = k[x,y]_{(x,y)}$. Further, $\mathcal{I}$ and $\mathcal{J}$ satisfy $A(2)$.

We now construct the example. Consider the filtrations $\mathcal{I} = \{I_n = (x^n)\}$ and $\mathcal{J} = \{J_n = (x^{n+1}, x^n y)\}$ in $R = \mathbb{C}[x,y]_{(x,y)}$. Then $V(I_1) = V(J_1) = \{P = (x), Q = (x,y)\}$ and $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus V(I_1)$. Note that $I_n R_P = J_n R_P = (x^n) R_P$ for all $n \in \mathbb{N}$. Hence $\varepsilon(\mathcal{I}_P) = \varepsilon(\mathcal{J}_P)$. Since I_n does not have Q as an associated prime for all $n \in \mathbb{N}$, we have $\varepsilon(\mathcal{I}) = 0$. For all $n \geq 1$, we have

$$(J_n : m_R^{\infty})/J_n = (x^n)/(x^{n+1}, x^n y) \cong R/(x,y)$$

so $\lambda_R((J_n : m_R^{\infty})/J_n) = 1$. Thus $\varepsilon(\mathcal{J}) = \lim_{n\to\infty} 2/n^2 = 0$.

Now if $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$, since $R \subset R[\mathcal{J}] \subset \overline{R[\mathcal{J}]}$ and $\overline{R[\mathcal{I}]}$ is a finitely generated R-algebra, by the Artin-Tate lemma, we have $R[\mathcal{J}]$ is a finitely generated R-algebra which is a contradiction. Hence $\overline{R[\mathcal{I}]} \neq \overline{R[\mathcal{J}]}$.

Example 0

Example 6.2. *We have filtrations $\mathcal{J} \subset \mathcal{I}$ in a Noetherian local ring R such that $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$ but $\varepsilon(\mathcal{I}) \neq \varepsilon(\mathcal{J})$.*

Consider the filtrations $\mathcal{I} = \{I_n = (x^2, xy^n)\}$ and $\mathcal{J} = \{J_n = (x^2, xy^{n^2})\}$ in $R = \mathbb{C}[x,y]_{(x,y)}$. Since $(xy^n t^n)^2 \in J_{2n} t^{2n}$, we have $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$. Note that

$$\varepsilon(\mathcal{I}) = \lim_{n\to\infty} 2!n/n^2 = 0 \neq 2 = \lim_{n\to\infty} 2!n^2/n^2 = \varepsilon(\mathcal{J}).$$

Let $\mathfrak{a}$ be an m_R-primary ideal of a local ring (R, m_R) and N be a finitely generated R-module with $\dim N = r$. Define

$$e_{\mathfrak{a}}(N) = \lim_{k\to\infty} \frac{l_R(N/\mathfrak{a}^k N)}{k^r/r!}.$$

If $s \geq r = \dim N$, define ([33, V.2], [3, 4.7])

$$e_s(\mathfrak{a}, N) = \begin{cases} e_{\mathfrak{a}}(N) & \text{if } \dim N = s \\ 0 & \text{if } \dim N < s. \end{cases}$$

Let (R, m_R) be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration of ideals of R. In [16, Definition 3.2], we defined the dimension of the filtration $\mathcal{I}$ to be $s(\mathcal{I}) = \dim R/I_n$ (for any $n \geq 1$). If R is an analytically unramified local ring, N is a finitely generated R-module, $\mathfrak{a}$ is an m_R-primary ideal and $\mathcal{I}$ is a filtration on R then by [16, Proposition 4.2], for $s \in \mathbb{N}$ with $s(\mathcal{I}) \leq s \leq d$, we have

$$e_s(\mathfrak{a}, \mathcal{I}; N) := \lim_{m\to\infty} \frac{e_s(\mathfrak{a}, N/I_m N)}{m^{d-s}/(d-s)!}$$

exists and

two

$$e_s(\mathfrak{a}, \mathcal{I}; N) = \sum_{\mathfrak{p}} e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}, N_{\mathfrak{p}}) e_{\mathfrak{a}}(R/\mathfrak{p}) \tag{13}$$

where the sum is over all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = d - s$. We write $e_s(\mathcal{I}) = e_s(m_R, \mathcal{I}; R)$.

Remark 6.3. *Note that for any filtration $\mathcal{I}$ in a local ring R with* $\dim N(\hat{R}) < \dim R$, *by* [6, Theorem 1.1], *we have $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}})$ exists for all $\mathfrak{p} \in \operatorname{MinAss}(R/I_n)$ and $n \geq 1$.*

epeq **Proposition 6.4.** *Let (R, m_R) be an analytically unramified local ring and $\mathcal{I}, \mathcal{J}$ be filtrations in R with $\mathcal{J} \subset \mathcal{I}$ and $\dim R/\mathcal{J} = s$. Then $e_s(\mathcal{I}) = e_s(\mathcal{J})$ if and only if $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = \dim R - s$.*

Proof. Let $V = \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = s \text{ and } \dim R_{\mathfrak{p}} = \dim R - s\}$ and for all $n \geq 1$, $V \cap \operatorname{MinAss}(R/J_1) = V \cap \operatorname{MinAss}(R/J_n) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Let $\mathfrak{p} \in V \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Then $J_n R_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $n \geq 1$. Since $\dim R/\mathcal{I} \leq \dim R/\mathcal{J} = s$, we also have $I_n R_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $n \geq 1$. Hence $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}}) = 0$.

Let $\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Then $\varepsilon(\mathcal{J}_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}})$. Since $\dim R/\mathcal{I} \leq \dim R/\mathcal{J} = s$, we have $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}})$.

Thus by taking $N = R$ and $\mathfrak{a} = m_R$ in equation (13), we have

$$\begin{aligned}
e_s(\mathcal{I}) = e_s(\mathcal{J}) \quad &\Leftrightarrow \quad \sum_{\mathfrak{p} \in V} e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}) e_{m_R}(R/\mathfrak{p}) = \sum_{\mathfrak{p} \in V} e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}}) e_{m_R}(R/\mathfrak{p}) \\
&\Leftrightarrow \quad \sum_{i=1}^{r} [e_{R_{\mathfrak{p}_i}}(\mathcal{J}_{\mathfrak{p}_i}) - e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})] e_{m_R}(R/\mathfrak{p}_i) = 0 \\
&\Leftrightarrow \quad e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i}) = e_{R_{\mathfrak{p}_i}}(\mathcal{J}_{\mathfrak{p}_i}) \text{ for all } i = 1, \ldots, r. \\
&\Leftrightarrow \quad \varepsilon(\mathcal{I}_{\mathfrak{p}_i}) = \varepsilon(\mathcal{J}_{\mathfrak{p}_i}) \text{ for all } i = 1, \ldots, r.
\end{aligned}$$

$\square$

nceq **Remark 6.5.** *Let (R, m_R) be a Noetherian local ring and $\mathcal{I}, \mathcal{J}$ be filtrations of R such that $\mathcal{J} \subset \mathcal{I}$. Suppose $R[\mathcal{J}] = \overline{R[\mathcal{J}]} = \overline{R[\mathcal{I}]}$. Then $\mathcal{I} = \mathcal{J}$.*

We recall a few definitions from [16].

An s-divisorial filtration is a divisorial filtration $\mathcal{J} = \{J_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}\}$ with $\dim R/m_{\nu_i} \cap R = s$ for all $i = 1, \ldots, r$.

A filtration $\mathcal{J}$ is called a bounded filtration if there exists a divisorial filtration

$$\mathcal{C} = \{C_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}\}$$

such that $\overline{R[\mathcal{J}]} = R[\mathcal{C}]$.

A filtration $\mathcal{I}$ is called a bounded s-filtration if there exists an s-divisorial filtration $\mathcal{C}$ such that $\overline{R[\mathcal{J}]} = R[\mathcal{C}]$.

We now prove Theorem 1.9 from the introduction, which we restate here for the convenience of the reader.

Theorem 6.6. *Let (R, m_R) be an excellent local domain and $\mathcal{I}$ be a filtration of ideals in R. Then the following hold.*

(1) *Suppose $\mathcal{J}$ is an s-divisorial filtration such that $\mathcal{J} \subset \mathcal{I}$. Then the following are equivalent.*
   (i) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*
   (ii) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\ell(\mathcal{J}_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$.*
   (iii) *$\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$.*
   (iv) *$\mathcal{I} = \mathcal{J}$.*
   (v) *$\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$.*

(2) *Suppose $\mathcal{J}$ is a bounded s-filtration such that $\mathcal{J} \subset \mathcal{I}$. Then $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$ if and only if $\varepsilon(\mathcal{I}_{\mathfrak{p}}) = \varepsilon(\mathcal{J}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$.*

Proof. Note that $\dim R/\mathcal{I} \leq \dim R/\mathcal{J} = s$. Suppose that $s = 0$. Then $e_R(\mathcal{I}) = \varepsilon(\mathcal{I}), e_R(\mathcal{J}) = \varepsilon(\mathcal{J})$. Thus (1) and (2) follow from [10, Theorem 1.4] , [10, Theorem 13.1] and Remark 6.5. Therefore we assume $s > 0$.

(1) It is clear that $(i) \Rightarrow (ii)$, $(iv) \Rightarrow (i)$ and $(iv) \Rightarrow (v)$. By Lemma 2.1 and Remark 6.5, we have $(v) \Rightarrow (iv)$.

Next we show that (ii) implies (iii). For any $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$, we have either $\mathfrak{p} \notin V(J_1)$ or $\mathcal{J}_\mathfrak{p}$ is a filtration of $\mathfrak{p}R_\mathfrak{p}$-primary ideals. If $\mathfrak{p} \notin V(J_1)$ then $\mathfrak{p} \notin V(I_1)$. Hence $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p}) = 0$. If $\mathcal{J}_\mathfrak{p}$ is a filtration of $\mathfrak{p}R_\mathfrak{p}$-primary ideals then $\ell(\mathcal{J}_\mathfrak{p}) = \dim R_\mathfrak{p}$ and by hypothesis $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p})$.

Now we prove that (iii) implies (iv). By Proposition 6.4, we have $e_s(\mathcal{I}) = e_s(\mathcal{J})$. Therefore by [16, Theorem 6.7], we get $\mathcal{I} = \mathcal{J}$.

(2) Suppose $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$. Then by [16, Theorem 5.1], we have $e_s(\mathcal{I}) = e_s(\mathcal{J})$ and hence by Proposition 6.4, we have $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$.

Suppose $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$. Then by Proposition 6.4, we have $e_s(\mathcal{I}) = e_s(\mathcal{J})$. Using [16, Theorem 7.4], we get the required result. $\square$

Corollary 6.7. *Let (R, m_R) be an excellent local domain and $\mathcal{I}$ be a filtration of ideals in R. Let $\mathcal{J} = \{J^n\}$ where J is an equimultiple ideal such that $J^n \subset I_n$ for all $n \geq 1$. Then the following are equivalent.*

(i) $\overline{R[\mathcal{I}]} = \overline{R[\mathcal{J}]}$.
(ii) $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\dim R/\mathfrak{p} = \dim R - \ell(J)$.
(iii) $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$ *such that* $\ell(\mathcal{J}_\mathfrak{p}) = \dim R_\mathfrak{p}$.

Proof. If height $J = \dim R$ then the result follows from [10, Theorem 1.2]. Suppose height $J < \dim R$. Since J is an equimultiple ideal, by [16, Corollary 8.3], $\mathcal{J}$ is a bounded s-filtration where $s = \dim R - \ell(J)$. Thus the equivalence of (i) and (ii) follows from Theorem 1.9 (2).
Since J is an equimultiple ideal, by [30, Corollary 9, 9.3], all prime divisors of $\overline{J^n}$ are of height $\ell(J)$. Therefore for all $n \geq 1$ and any $\mathfrak{p} \in \operatorname{Ass}(R/\overline{J^n}) = \operatorname{MinAss}(R/J)$, we have $\dim R/\mathfrak{p} = s$. Now using [30, Corollary 4], $\ell(\mathcal{J}_\mathfrak{p}) = \dim R_\mathfrak{p}$ if and only if $\mathfrak{p} \in \operatorname{Ass}(R/\overline{J^n}) = \operatorname{MinAss}(R/J)$ and $\dim R/\mathfrak{p} = s$. Thus (ii) implies (iii).
Now suppose $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = \dim R - \ell(J) = \dim R - \operatorname{height} J$. If $\mathfrak{p} \notin V(J)$ then $J_\mathfrak{p} = R_\mathfrak{p} = I_n R_\mathfrak{p}$ and hence $\varepsilon(\mathcal{I}_\mathfrak{p}) = \varepsilon(\mathcal{J}_\mathfrak{p}) = 0$. Suppose $\mathfrak{p} \in V(J)$. Then $\mathfrak{p} =\in \operatorname{Ass}(R/\overline{J^n}) = \operatorname{MinAss}(R/J)$ and hence $\ell(\mathcal{J}_\mathfrak{p}) = \dim R_\mathfrak{p}$. Thus (iii) implies (ii). $\square$

Example 6.8. *Theorem 1.9 does not extend to be true for general divisorial filtrations. Let $\mathcal{J} = \{\overline{\mathfrak{p}^n}\}$ and $\mathcal{I} = \{\mathfrak{p}^{(n)}\}$ where $\mathfrak{p}$ is a height 2 prime ideal in a regular local ring R of dimension* 3 *such that $\mathcal{I}$ is not finitely generated R-algebra. Hence $\overline{R[\mathcal{I}]} \neq \overline{R[\mathcal{J}]}$. We have* $\dim R/\mathcal{J} = 1$. *Note that $\varepsilon(\mathcal{I}_\mathfrak{p}) = e_1(\mathcal{I}) = e_1(\mathcal{J}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ and $0 = \varepsilon(\mathcal{I}) \neq \varepsilon(\mathcal{J}) = \varepsilon(\mathfrak{p}) > 0$ by Proposition 2.1 (b) [24] , which shows that $\varepsilon(I) = \varepsilon(\{\overline{I^n}\})$ and Theorem 1.4 as $\ell(\mathfrak{p}) = \dim R$. Statement (iii) of Theorem 1.9 is true but statements (i), (ii) (iv) and (v) are not.*

Example 6.9. *Theorem 1.9 does not extend for bounded filtrations. Let $\mathcal{J} = \{\mathfrak{p}^n\}$ and $\mathcal{I} = \{\mathfrak{p}^{(n)}\}$ where $\mathfrak{p}$ is a height 2 prime ideal in a regular local ring R of dimension* 3 *such that $\mathcal{I}$ is not finitely generated R-algebra. Hence $\overline{R[\mathcal{I}]} \neq \overline{R[\mathcal{J}]}$. We have* $\dim R/\mathcal{J} = 1$. *Note that $\varepsilon(\mathcal{I}_\mathfrak{p}) = e_1(\mathcal{I}) = e_1(\mathcal{J}) = \varepsilon(\mathcal{J}_\mathfrak{p})$ and $0 = \varepsilon(\mathcal{I}) \neq \varepsilon(\mathcal{J}) = \varepsilon(\mathfrak{p}) > 0$ as $\ell(\mathcal{J}) =$* $\dim R$.

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