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ON FEASIBILITY OF EXTRAPOLATION OF COMPLETELY MONOTONE FUNCTIONS*

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Abstract. The feasibility of extrapolation of completely monotone functions can be quantified by examining the worst-case scenario, whereby a pair of completely monotone functions agree on a given interval to a given relative precision, but differ as much as it is theoretically possible at a given point. We show that extrapolation is impossible to the left of the interval, while the maximal discrepancy to the right exhibits a power law typical for extrapolation of similar classes of complex analytic functions. The power law exponent is derived explicitly and shows a precipitous drop immediately beyond the right end-point, with a subsequent decay to zero inversely proportional to the distance from the interval. The local extrapolation problem, where the worst discrepancy from a given completely monotone function is sought, is also analyzed. In this case explicit and easily verifiable optimality conditions are derived, enabling us to solve the problem exactly for a single decaying exponential. In the general case, our approach leads to a natural algorithm for computing solutions to the local extrapolation problem numerically. The methods developed in this paper can easily be adapted to other classes of analytic functions represented as integral transforms of positive measures with analytic kernels.

Key words. analytic continuation, completely monotone functions, multiexponential models, inverse problems, optimality conditions and duality, sensitivity, stability, well-posedness

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1. Introduction. The theory of completely monotone functions (CMFs) was developed in the 1920s and 1930s in the works of Bernstein [2], Hausdorff [24], Widder [48, 46], and Feller [14] in connection with the Markov moment problem [29]. This class of functions arises in several areas of mathematics [27, 1, 31, 49] and remains of current research interest (see reviews [34, 32]). Its importance in applications is rapidly becoming more and more appreciated. Multiexponential models, whereby a quantity of interest is a linear combination of decaying exponentials *with positive coefficients,* are abundant in physics [25, 17], engineering [22, 40], medicine [41, 11, 10], and industry [42, 38].

While the problem of central practical importance in applications is the estimation of parameters of a multiexponential model [35, 12, 38, 36], our goal is a theoretical analysis of reliability of such procedures. To quantify the feasibility of recovery of such functions from noisy measurements, we look for a pair of CMFs with relative discrepancy ϵ on $[a,b]\subset[0,\infty)$, as measured by the L^2 norm, that differ as much as possible at a given point $x_0\notin(a,b)$. We show that the discrepancy can be made as large as one wishes for $0\le x_0\le a$, while for $x_0\ge b$ the relative discrepancy scales as $\epsilon^{\gamma(x_0)}$, where

$$\gamma(x_0)=\frac{2}{\pi}\arcsin\left(\frac{b-a}{x_0-a}\right),\quad x_0\ge b. \tag{1.1}$$

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An analogous problem has been considered for the class of Stieltjes functions (see, e.g., [43, 29, 30]) in [20].

Our general methodology, developed in [19, 21, 20] for the Stieltjes class, can be applicable to many different classes of functions that can be represented by integral transforms of positive measures with analytic kernels. For example, CMFs are the Laplace transforms of positive measures, while the Stieltjes functions, for which this approach was first developed, are the Stieltjes transforms of positive measures [47]. The main technical difficulty is to link the problem of the worst discrepancy between a pair of functions in our function class to the much better understood problem of largest deviation from 0 among functions in reproducing kernel Hilbert spaces of analytic functions (e.g., Hardy spaces) that are small on a curve in their domain of analyticity [6, 33, 16, 45, 9, 44]. The latter problem can be reduced to the analysis of the asymptotics of eigenvalues and eigenfunctions of specific integral operators [37, 23, 39, 19, 21]. The former is treated using the same methodology as in [20], where a family of Hilbert space norms was constructed that bridge the gap between the Hardy space norm and the L^2 norm on the given curve.

We also investigate the local problem of finding a CMF $g(x)$, such that $\|f_0 - g\|_{L^2(a,b)} \leq \epsilon$, that maximizes and minimizes $f_0(x) - g(x)$, $x \notin [a,b]$, where $f_0(x)$ is a given CMF, normalized by $\|f_0\|_{L^2(a,b)} = 1$. For this problem, we derive necessary and sufficient conditions for the extremals $g(x)$, using the direct analysis of the variation due to Caprini [3, 4, 5]. Caprini's method has the advantage of suggesting an algorithm for computing the extremals numerically. The implementation of this algorithm suggested the exact solutions for $f_0(x) = e^{-x}$, which are then explicitly exhibited and analyzed. The Caprini analysis-based approach has already been exploited in the context of extrapolation of Stieltjes functions [18]. The details and implementation of an analogous algorithm for CMFs will be addressed elsewhere.

There are three main innovations in this paper. The reduction to an integral equation is now done using a new version of the Kuhn–Tucker theorem, which is valid in all locally convex topological vector spaces, making it applicable to a broader class of problems. In the case under study, the resulting integral operator has already been fully analyzed in [26]. The theory in [19, 21] shows how the asymptotic behavior of eigenfunctions for large eigenvalues leads to explicit formulas for exact exponents in the power laws, like (1.1).

The second innovation is a nontrivial construction of a continuous family of Hilbert space norms that bridge the gap between the Hardy space norm and the $L^2(a, b)$ norm. While the constructed family of norms does not bridge the gap completely, it does so asymptotically. The explicit form of the power law (1.1) and the explicit asymptotics of the solution to the integral equation are essential to establishing the link.

The third is the worst-case analysis of the local problem. There, the necessary and sufficient conditions for extremality are found and used to compute the two CMFs deviating the most from a single decaying exponential, with which they agree up to a relative precision ϵ on a finite interval.

2. Preliminaries and problem formulation.

We say that $f : (0, \infty) \to [0, \infty)$ belongs to the class CM[1] if it can be represented as

[1] The original definition of CMF is a nonnegative C^∞ function on $(0, \infty)$, whose kth derivative is either always positive or always negative, depending on whether k is even or odd. It was shown by Bernstein [2] that the two definitions are equivalent.

$$f(x) = f_\sigma(x) = \int_0^\infty e^{-xt} d\sigma(t), \tag{2.1}$$

where σ is a positive, Borel-regular measure on $[0,\infty)$, such that $f(x) < \infty$ for all $x > 0$. In what follows, we will adopt the notation $f_\sigma(x)$ to denote the function given by (2.1). Formula (2.1) implies that $f \in \mathcal{H}(\mathcal{R})$, where $\mathcal{R} = \{z \in \mathbb{C} : \Re z > 0\}$ is the complex right half-plane, and $\mathcal{H}(\Omega)$ denotes the space of all complex analytic functions on the open set $\Omega \subset \mathbb{C}$. The uniqueness property of analytic functions suggests that the knowledge of a CMF on an interval $[a,b]$ should determine such a function uniquely. In practice, where $f(x)$ is known only approximately, the feasibility of extrapolation becomes a nontrivial question that we address in this paper. Specifically, we assume that we know the values of a CMF $f(x)$ on the interval $[a,b]$ up to a given relative precision ϵ in $L^2(a,b)$. We want to know how accurately we can extrapolate this function outside of $[a,b]$. One immediately observes that for any given CMF $f(x)$ the function $f_K(x) = f(x) + \epsilon\sqrt{2K}e^{-K(x-a)}$ is completely monotone for any $K > 0$, and that $\|f_K(x) - f(x)\|_{L^2(a,b)} \le \epsilon$. However, for any $c \in [0,a]$, we can make $f_K(c) - f(c)$ as large as we wish by choosing K sufficiently large. This shows that if we know that a pair of CMFs has a relative discrepancy ϵ in $L^2(a,b)$, their discrepancy at $x \le a$ can be made as large as one wishes. We therefore conclude that we may assume, without loss of generality, that $a = 0$ and rescale b to 1. For this reason, we restrict our attention to a subclass $\mathfrak{C}_2$ of CMFs defined by

$$\mathfrak{C}_2 = \{f \in \mathrm{CM} : \|f\|_2 < +\infty\}, \tag{2.2}$$

where $\|\cdot\|_2$ denotes the $L^2(0,1)$ norm. We note that $\mathfrak{C}_2$ is not a vector space, but a convex cone. The natural vector space the cone $\mathfrak{C}_2$ lies in is $\mathfrak{X} = \mathfrak{C}_2 - \mathfrak{C}_2$, which is a real vector space, even though its elements are complex-analytic functions on $\mathcal{R}$.

To formulate the problem of the worst-case extrapolation, we denote

$$\Delta[f,g](x) = \frac{f(x) - g(x)}{\|f\|_2 + \|g\|_2}, \tag{2.3}$$

describing the relative discrepancy at the point x between the two functions $\{f,g\} \subset \mathfrak{C}_2$. The worst-case extrapolation problem is

$$\Delta^{x_0}(\epsilon) = \max_{\|\Delta[f,g]\|_2 \le \epsilon} |\Delta[f,g](x_0)|, \tag{2.4}$$

where $x_0 \ge 1$ is a given point. In other words, we seek the largest relative discrepancy between two $\mathfrak{C}_2$ functions, which are at most ϵ apart on $[0,1]$ in the L^2 sense. Our primary goal is to prove formula (1.1), which is equivalent to the following theorem.

THEOREM 2.1. *Let $x_0 \ge 1$; then*

$$\gamma(x_0) \overset{\text{def}}{=} \lim_{\epsilon \to 0^+} \frac{\ln \Delta^{x_0}(\epsilon)}{\ln \epsilon} = \frac{2}{\pi} \arcsin\left(\frac{1}{x_0}\right), \tag{2.5}$$

where $\Delta^{x_0}(\epsilon)$ is given by (2.4), *and the limit in* (2.5) *exists.*

The idea of the proof is to relate (2.4), which we call the (f,g)-problem, to a simpler problem that we know how to solve explicitly:

$$\Delta_*^{x_0}(\epsilon) = \max_{\phi \in \mathcal{A}_\epsilon} \phi(x_0), \quad \mathcal{A}_\epsilon = \{\phi \in H : \|\phi\| \le 1, \|\phi\|_2 \le \epsilon\}, \tag{2.6}$$

where $H=\{\phi\in H^2(\mathcal{R}):\overline{\phi(z)}=\phi(\bar{z})\}$ is a real subspace of the standard Hardy Hilbert space $H^2(\mathcal{R})$, and where $\|\cdot\|$ is a multiple of the standard Hardy space norm

$$\|\phi\|^2=\sup_{x>0}\frac{1}{2\pi}\int_{\mathbb{R}}|\phi(x+iy)|^2dy=\frac{1}{2\pi}\int_{\mathbb{R}}|\phi(iy)|^2dy=\frac{1}{\pi}\int_0^\infty|\phi(iy)|^2dy. \tag{2.7}$$

We call (2.6) the ϕ-problem. We note that the Hardy space $H^2(\mathcal{R})$ is a reproducing kernel Hilbert space (see, e.g., [7]), and problems like (2.6) have been well understood [19, 20]. Our goal is to show both that

$$\gamma_*(x_0)\stackrel{\text{def}}{=}\lim_{\epsilon\to 0}\frac{\ln\Delta_*^{x_0}(\epsilon)}{\ln\epsilon}=\gamma(x_0) \tag{2.8}$$

and that $\gamma_*(x_0)$ is equal to the right-hand side of (2.5). We follow here the same strategy that was used in [20] in an analogous problem about Stieltjes functions. The main difference (and therefore difficulty) is that the Hardy norm $\|\cdot\|$ is not equivalent to $\|\cdot\|_2$ on the convex cone $\mathfrak{C}_2$. This makes the direct comparison between $\gamma(x_0)$ and $\gamma_*(x_0)$ impossible.

Our way of resolving this difficulty is to bridge the gap between the two norms by introducing a continuous family of intermediate Hardy space-like norms of increasing strength on $\mathfrak{X}=\mathfrak{C}_2-\mathfrak{C}_2$, all of which are equivalent to $\|\cdot\|_2$ on $\mathfrak{C}_2$. Each norm in the family gives rise to the corresponding ϕ-problem (2.6), where it replaces the Hardy norm $\|\cdot\|$. What permits us to close the circle of inequalities between the corresponding power law exponents γ is our ability to solve the the original ϕ-problem (2.6) explicitly and thus estimate all of its intermediate norms directly. We remark that it is the absence of the explicit solution of the ϕ-problem in [20] that prevented us from completing the rigorous proof of the analogue of (2.8) in the context of Stieltjes functions.

3. Existence of maximizers.

The goal of this section is to prove the attainment of the maxima both in (2.4) and in (2.6). We start by proving the representation property of functions in H.

LEMMA 3.1. *For any $\phi\in H$, there exists $\sigma\in L^2(0,\infty)$, $\sigma(t)\in\mathbb{R}$, such that*

$$\phi(z)=\int_0^\infty\sigma(t)e^{-zt}dt,\quad \Re z>0. \tag{3.1}$$

Proof. If $\phi\in H$, then $\phi(iy)\in L^2(\mathbb{R})$, and therefore, there exists $\sigma\in L^2(\mathbb{R})$, such that

$$\phi(iy)=\hat{\sigma}(y)=\int_{\mathbb{R}}\sigma(t)e^{-iyt}dt.$$

The symmetry of functions in H, i.e., $\overline{\phi(iy)}=\phi(-iy)$, implies that $\sigma(t)\in\mathbb{R}$. Since H is a subspace of the Hardy space $H^2(\mathcal{R})$, for any $\phi\in H$ there is the Kramers–Kronig relation [8, 28], which says that the real part of $\phi(iy)$ is the Hilbert transform of its imaginary part. Since the Hilbert transform is a Fourier multiplier operator by $i\,\mathrm{sign}(t)$, the Kramers–Kronig relation can be written as $\Re\hat{g}(y)=0$, where $g(t)=\sigma(t)-\sigma(t)\mathrm{sign}(t)$. But then, $g(t)$ has to be an odd function on $\mathbb{R}$. We conclude that $g(t)$ must be identically zero since it is zero on $(0,\infty)$. It follows that $\sigma(t)=0$ for all $t<0$, and

$$\phi(iy)=\hat{\sigma}(y)=\int_0^\infty\sigma(t)e^{-iyt}dt,\quad y\in\mathbb{R},$$

so that the representation (3.1) holds, since Hardy functions possess a unique analytic extension into the complex right half-plane. ∎

We remark that in view of representation (3.1) the Hardy inner product in H can also be computed as

$$(\phi_\sigma,\phi_\mu)=(\sigma,\mu)_{L^2(0,\infty)}. \tag{3.2}$$

To establish attainment in (2.6), we need the following lemma.

LEMMA 3.2. *For any* $\phi\in H$

$$\|\phi\|_2\le\sqrt{\pi}\|\phi\|. \tag{3.3}$$

Proof. Using representation (3.1), we have

$$\|\phi\|_2^2\le\|\phi\|^2_{L^2(0,\infty)}=\int_0^\infty\int_0^\infty\frac{\sigma(s)\sigma(t)}{s+t}dsdt=\pi((H\sigma)(-t),\sigma(t))_{L^2(\mathbb{R})},$$

where $H\sigma$ is the Hilbert transform and $\sigma\in L^2(0,\infty)$ is extended by zero on $(-\infty,0)$ to yield a function in $L^2(\mathbb{R})$. Hence

$$\|\phi\|_2^2\le\pi\|(H\sigma)(-t)\|_{L^2(\mathbb{R})}\|\sigma\|_{L^2(\mathbb{R})}=\pi\|\sigma\|^2_{L^2(\mathbb{R})}=\pi\|\phi\|^2.$$ ∎

The attainment in (2.6) is now obvious since $\mathcal{A}_\epsilon$ is closed, convex, and bounded in H, and the evaluation functional $H\ni\phi\mapsto\phi(x_0)$ is continuous. (It is obvious, for example, from representation (3.1) and the fact that $e^{-x_0t}\in L^2(0,\infty)$.)

To prove the attainment in (2.4), we need the following lemma.

LEMMA 3.3. *For any* $f_\sigma\in\mathfrak{C}_2$

$$\|f_\sigma\|_2\ge\|\sigma\|_*, \tag{3.4}$$

where

$$\|\sigma\|_*=\int_0^\infty\frac{d\sigma(t)}{t+1}. \tag{3.5}$$

Proof. Using representation (2.1), we compute

$$\|f_\sigma\|_2^2=\int_0^\infty\int_0^\infty\frac{1-e^{-(t+s)}}{t+s}d\sigma(t)d\sigma(s).$$

Now observe that since $\min_{x\ge0}x^{-1}(x+1)(1-e^{-x})=1$, for any $s>0$ and $t>0$ we have

$$\frac{1-e^{-(t+s)}}{t+s}\ge\frac{1}{t+s+1}\ge\frac{1}{(t+1)(s+1)}.$$

Inequality (3.4) follows. ∎

We are now ready to prove the attainment in (2.4).

THEOREM 3.4. *The maximum in* (2.4) *is attained.*

Proof. Let (f_n, g_n) be a minimizing sequence for (2.4). Then sequences

$$\widetilde{f}_n = \frac{f_n}{\|f_n\|_2 + \|g_n\|_2}, \quad \widetilde{g}_n = \frac{g_n}{\|f_n\|_2 + \|g_n\|_2}$$

are bounded in $L^2(0,1)$. By Lemma 3.3 the corresponding measures $\widetilde{\sigma}_n$, $\widetilde{\mu}_n$ are bounded in X^*, where

$$X = \left\{\Phi \in C([0,\infty)) : \lim_{t\to\infty}(1+t)\Phi(t) = 0\right\} \tag{3.6}$$

is a Banach space with $\|\Phi\|_X = \sup_{t>0}(t+1)|\Phi(t)|$. Since X is separable, there are weak-* converging subsequences, not relabeled, $\widetilde{\sigma}_n \overset{*}{\rightharpoonup} \sigma_0 \in X^*$, $\widetilde{\mu}_n \overset{*}{\rightharpoonup} \mu_0 \in X^*$. Since $e^{-x_0 t} \in X$, we conclude that $\widetilde{f}_n(x_0) - \widetilde{g}_n(x_0) \to f_0(x_0) - g_0(x_0)$, where

$$f_0(x) = \int_0^\infty e^{-xt} d\sigma_0(t), \quad g_0(x) = \int_0^\infty e^{-xt} d\mu_0(t).$$

In fact, the pointwise convergence of $\widetilde{f}_n$ and $\widetilde{g}_n$ together with their weak precompactness in $L^2(0,1)$ implies that $\widetilde{f}_n \rightharpoonup f_0$, and $\widetilde{g}_n \rightharpoonup g_0$ in $L^2(0,1)$. The weak lower semicontinuity of the norm in $L^2(0,1)$ implies that $\|\Delta[f_0,g_0]\|_2 \le \epsilon$ and therefore the pair (f_0, g_0) attains the maximum in (2.4). ∎

4. The ϕ-problem.

The goal of this section is to solve the ϕ-problem (2.6).

4.1. Reduction to an integral equation.

The ϕ-problem (2.6) asks to maximize a linear continuous functional on the Hilbert space H over a convex and closed subset $\mathcal{A}_\epsilon \subset H$. A new general version of the Kuhn–Tucker theorem, valid in all locally convex topological vector spaces, is formulated and proved in Appendix A. In order to apply it, we need to describe the admissible set of functions $\mathcal{A}_\epsilon$ in the standard form (A.1). To do so, we first observe that

$$\|\phi_\sigma\| = \|\sigma\|_{L^2(0,+\infty)} = \sup_{\|\Psi\|_{L^2(0,+\infty)}\le 1} \int_0^\infty \Psi(t)\sigma(t)dt, \qquad \|\phi\|_2 = \sup_{\|\psi\|_2\le 1}\int_0^1 \phi(x)\psi(x)dx.$$

Let us show that $L^2(0,1)$ acts on H by weakly continuous functionals, where the action of $\psi \in L^2(0,1)$ on H is defined by

$$\phi \mapsto (\phi,\psi)_2 = \int_0^1 \phi(x)\psi(x)dx.$$

Indeed, $|(\phi,\psi)_2| \le \|\psi\|_2\|\phi\|_2 \le \sqrt{\pi}\|\psi\|_2\|\phi\|$, by Lemma 3.2. We also have

$$(\phi_\sigma,\psi)_2 = \int_0^1 \phi_\sigma(x)\psi(x)dx = \int_0^\infty (\Lambda\psi)(t)\sigma(t)dt, \qquad (\Lambda\psi)(t) = \int_0^1 \psi(x)e^{-xt}dx,$$

while the bound

$$|(\phi_\sigma,\psi)_2| \le \sqrt{\pi}\|\psi\|_2\|\phi_\sigma\| = \sqrt{\pi}\|\psi\|_2\|\sigma\|_{L^2(0,\infty)}$$

implies

$$\|\Lambda\psi\|_{L^2(0,\infty)} \le \sqrt{\pi}\|\psi\|_2.$$

Thus, we obtain the desired description of $\mathcal{A}_\epsilon$:

$$\mathcal{A}_\epsilon = \{\phi_\sigma \in H : (\sigma, \Psi)_{L^2(0,\infty)} \le 1 \ \forall \|\Psi\|_{L^2(0,\infty)} \le 1,\ (\sigma, \Lambda\psi)_{L^2(0,\infty)} \le \epsilon \ \forall \|\psi\|_2 \le 1\}.$$

In order to apply the Kuhn–Tucker theorem, we need to compute the smallest closed convex cone $\widehat{\mathcal{F}} \subset H \times \mathbb{R}$ containing the set

$$\mathcal{F} = \{(\Psi, 1) : \|\Psi\|_{L^2(0,\infty)} \le 1\} \cup \{(\Lambda\psi, \epsilon)_{L^2(0,\infty)} : \|\psi\|_2 \le 1\}.$$

We can characterize $\widehat{\mathcal{F}}$ as

$$\widehat{\mathcal{F}} = \{(\Psi + \Lambda\psi, A + \epsilon B) : \|\Psi\|_{L^2(0,\infty)} \le A,\ \|\psi\|_2 \le B,\ A \ge 0,\ B \ge 0\}.$$

Indeed, it is obvious both that $\widehat{\mathcal{F}}$ is a convex cone and that each element of $\widehat{\mathcal{F}}$ is a nonnegative linear combination of two elements from $\mathcal{F}$. To prove that $\widehat{\mathcal{F}}$ is closed suppose that

$$\Psi_n + \Lambda\psi_n \to P \text{ in } L^2(0,\infty), \quad A_n + \epsilon B_n \to \alpha, \quad \|\Psi_n\|_{L^2(0,\infty)} \le A_n, \quad \|\psi_n\|_2 \le B_n.$$

Then $A_n \le A_n + \epsilon B_n$ and $B_n \le (A_n + \epsilon B_n)/\epsilon$. Hence, we can extract convergent subsequences (not relabeled) of $A_n \to A$ and $B_n \to B$. We can also extract the weakly convergent subsequences (not relabeled) $\Psi_n \rightharpoonup \Psi$, $\psi_n \rightharpoonup \psi$. The weak lower semicontinuity of the norms implies that $\|\Psi\|_{L^2(0,\infty)} \le A$, $\|\psi\|_2 \le B$, while $A + \epsilon B = \alpha$ and $\Psi + \Lambda\psi = P$. Thus, $(P, \alpha) \in \widehat{\mathcal{F}}$, and we conclude that $\widehat{\mathcal{F}}$ is weakly closed. Now, according to the Kuhn–Tucker Theorem A.1,

$$\Delta_*^{x_0}(\epsilon) = \max_{\phi \in \mathcal{A}_\epsilon} \phi(x_0) = \min_{\psi \in L^2(0,1)} \left(\epsilon\|\psi\|_2 + \left\|\Lambda\psi - e^{-x_0 t}\right\|_{L^2(0,\infty)} \right). \tag{4.1}$$

The minimizer ψ_ϵ in (4.1) exists for any fixed $\epsilon > 0$, because this is a convex and coercive variational problem. However, this problem is difficult to analyze. Hence, we are going to modify the maximization problem (4.1) to make it more tractable, while using our understanding of the relation between solutions of (2.6) and (4.1) to obtain the maximizer in (2.6). Using that for $1/p + 1/q = 1$,

$$\frac{1}{p}\left(\frac{p}{q}a^2 + b^2\right) = \frac{a^2}{q} + \frac{b^2}{p} \le (a+b)^2 \le pa^2 + qb^2 = q\left(\frac{p}{q}a^2 + b^2\right), \tag{4.2}$$

we conclude that for the sake of understanding the asymptotic behavior of $\Delta_*^{x_0}(\epsilon)$, we can replace the variational problem (4.1) by a quadratic one:

$$Q_{x_0}(\varepsilon) = \min_{\psi \in L^2(0,1)} \varepsilon^2\|\psi\|_2^2 + \left\|\Lambda\psi - e^{-x_0 t}\right\|_{L^2(0,\infty)}^2, \tag{4.3}$$

where $\varepsilon = \epsilon\sqrt{p(\epsilon)/q(\epsilon)}$, and where the parameters $p(\epsilon)$, $q(\epsilon)$, satisfying $1/p(\epsilon) + 1/q(\epsilon) = 1$, will be chosen later to optimize the upper bound that, according to (4.2), reads

$$\Delta_*^{x_0}(\epsilon)^2 \le q(\epsilon) Q_{x_0}(\varepsilon), \qquad \varepsilon = \epsilon\sqrt{\frac{p(\epsilon)}{q(\epsilon)}}. \tag{4.4}$$

The advantage of the quadratic minimization problem (4.3) over (4.1) is that the minimizer ψ_ε of (4.3) solves a linear integral equation

$$\varepsilon^2\psi(x) + (K\psi)(x) = \frac{1}{x_0 + x}, \quad x \in [0,1], \tag{4.5}$$

where $K : L^2(0,1) \to L^2(0,1)$, and

$$(K\psi)(x) = (\Lambda^*\Lambda\psi)(x) = \int_0^1 \frac{\psi(y)dy}{x+y}$$

is a bounded, nonnegative, and self-adjoint operator. Hence, (4.5) has a unique solution $\psi_\varepsilon \in L^2(0,1)$.

Representing the kernel $(x+y)^{-1}$ of the integral operator in the form

$$\frac{1}{x+y} = \int_0^\infty e^{-xt}e^{-yt}dt,$$

we conclude that the solution ψ_ε of (4.5) satisfies

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^2}\int_0^\infty (e^{-x_0 t} - \Lambda\psi_\varepsilon)e^{-xt}dt. \tag{4.6}$$

This shows that $\psi_\varepsilon \in L^2(0,1)$ has the unique extension, also denoted $\psi_\varepsilon \in H$, which has a representation (3.1), with $\sigma = \varepsilon^{-2}(e^{-x_0 t} - \Lambda\psi_\varepsilon) \in L^2(0,\infty)$. Therefore, in view of (3.2), we have

$$\|\psi_\varepsilon\| = \frac{1}{\varepsilon^2}\|\Lambda\psi_\varepsilon - e^{-x_0 t}\|_{L^2(0,\infty)}. \tag{4.7}$$

Setting $x = x_0$ in (4.6), we obtain

$$\psi_\varepsilon(x_0) = \frac{1}{\varepsilon^2}\int_0^\infty (e^{-x_0 t} - \Lambda\psi_\varepsilon)e^{-x_0 t}dt.$$

Multiplying (4.6) by ψ_ε and integrating over $[0,1]$, we get

$$\|\psi_\varepsilon\|_2^2 = \frac{1}{\varepsilon^2}\int_0^\infty (e^{-x_0 t} - \Lambda\psi_\varepsilon)\Lambda\psi_\varepsilon dt.$$

Subtracting the two equations and taking (4.7) into account yields

$$\|\psi_\varepsilon\|_2^2 + \varepsilon^2\|\psi_\varepsilon\|^2 = \psi_\varepsilon(x_0). \tag{4.8}$$

This relation implies that $Q_{x_0}(\varepsilon) = \varepsilon^2\psi_\varepsilon(x_0)$, while the upper bound (4.4) becomes

$$\Delta_*^{x_0}(\epsilon)^2 \le q(\epsilon)\varepsilon^2\psi_\varepsilon(x_0) = \epsilon^2 p(\epsilon)\psi_\varepsilon(x_0). \tag{4.9}$$

The lower bound for $\Delta_*^{x_0}(\epsilon)$ is obtained by using a test function

$$\phi_\epsilon = \frac{\epsilon\psi_\varepsilon}{\|\psi_\varepsilon\|_2} \in H, \tag{4.10}$$

which obviously satisfies $\|\phi_\epsilon\|_2 = \epsilon$, and where $p(\epsilon)$ is chosen so that $\|\phi_\epsilon\| = 1$. Specifically, using (4.8), we have

$$\|\phi_\epsilon\|^2 = \frac{\epsilon^2\|\psi_\varepsilon\|^2}{\|\psi_\varepsilon\|_2^2} = \frac{q(\epsilon)}{p(\epsilon)}\left(\frac{\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|_2^2} - 1\right) = \frac{\frac{\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|_2^2} - 1}{p(\epsilon) - 1}.$$

Setting $\|\phi_\epsilon\|^2 = 1$, we obtain

$$p(\epsilon) = \frac{\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|_2^2} = 1 + \frac{\varepsilon^2\|\psi_\varepsilon\|^2}{\|\psi_\varepsilon\|_2^2} \in (1, +\infty), \tag{4.11}$$

due to (4.8). The choice (4.11) of $p(\epsilon)$ implies that $\phi_\epsilon \in \mathcal{A}_\epsilon$, yielding the lower bound for $\Delta_*^{x_0}(\epsilon)$

$$(\Delta_*^{x_0}(\epsilon))^2 \ge (\phi_\epsilon(x_0))^2 = \frac{\epsilon^2\psi_\varepsilon(x_0)^2}{\|\psi_\varepsilon\|_2^2} = \epsilon^2 p(\epsilon)\psi_\varepsilon(x_0),$$

provided $p(\epsilon)$ is given by (4.11). Hence, the lower bound for $\Delta_*^{x_0}(\epsilon)$ agrees with the upper bound (4.4), and therefore,

$$\Delta_*^{x_0}(\epsilon) = \frac{\epsilon\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|_2}, \tag{4.12}$$

where ψ_ε solves (4.5) and ε and ϵ are related by

$$\epsilon = \frac{\|\psi_\varepsilon\|_2}{\|\psi_\varepsilon\|}, \tag{4.13}$$

which is easy to obtain combining (4.11) and the formula for ε from (4.4). Substituting this into (4.12), we also obtain

$$\Delta_*^{x_0}(\epsilon) = \frac{\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|}. \tag{4.14}$$

We can use formulas (4.13) and (4.14) to establish the explicit leading order asymptotics of $\Delta_*^{x_0}(\epsilon)$ if we can compute the explicit leading order asymptotics of the right-hand sides in (4.13) and (4.14). Specifically, if $E_0(\varepsilon)$ and $E_1(\varepsilon)$ are continuous and monotone increasing functions on $[0,1)$, such that $E_0(0) = E_1(0) = 0$, and

$$\lim_{\varepsilon\to 0^+} \frac{\psi_\varepsilon(x_0)}{E_0(\varepsilon)\|\psi_\varepsilon\|} = 1, \qquad \lim_{\varepsilon\to 0^+} \frac{\|\psi_\varepsilon\|_2}{E_1(\varepsilon)\|\psi_\varepsilon\|} = 1,$$

then we want to conclude that

$$\lim_{\epsilon\to 0^+} \frac{\Delta_*^{x_0}(\epsilon)}{E_0(E_1^{-1}(\epsilon))} = 1. \tag{4.15}$$

Since $\epsilon(\varepsilon) \sim E_1(\varepsilon)$, then the assumed properties of $E_1(\varepsilon)$ imply that $\epsilon \to 0^+$ if and only if $\varepsilon \to 0^+$. Then,

$$\lim_{\epsilon\to 0^+} \frac{\Delta_*^{x_0}(\epsilon)}{E_0(E_1^{-1}(\epsilon))} = \lim_{\varepsilon\to 0^+} \frac{E_0(\varepsilon)\frac{\psi_\varepsilon(x_0)}{E_0(\varepsilon)\|\psi_\varepsilon\|}}{E_0\left(E_1^{-1}\left(E_1(\varepsilon)\frac{\|\psi_\varepsilon\|_2}{E_1(\varepsilon)\|\psi_\varepsilon\|}\right)\right)}.$$

Thus, (4.15) follows if functions E_0 and E_1 have the additional property

$$\lim_{\varepsilon\to 0^+} \frac{E_0(\varepsilon)}{E_0(E_1^{-1}(E_1(\varepsilon)r(\varepsilon)))} = 1, \tag{4.16}$$

whenever $r(\varepsilon) \to 1$, as $\varepsilon \to 0^+$. It is not difficult to give an example of continuous and monotone increasing functions E_0 and E_1, with $E_0(0) = E_1(0) = 0$, that fail to satisfy (4.16).

4.2. Solution of the integral equation. To solve the integral equation (4.5), we diagonalize the bounded self-adjoint operator K. The problem of computing the eigenfunctions of K can be related to a problem about the truncated Hilbert transform

$$(H_1u)(\xi) = P.V.\int_0^1 \frac{u(y)dy}{\xi - y},$$

regarded as a map $H_1 : L^2(0,1) \to L^2(-1,0)$, that has been solved in [26]. The relation between the operators K and H_1 is expressed by the formula $K^2 = H_1^*H_1$, which shows that if u is an eigenfunction of K with eigenvalue $\nu > 0$, then u is also a singular function of H_1 with singular value ν. Conversely, if u is a singular function of H_1 with singular value ν, then $K^2u = \nu^2u$, which implies that $(K+\nu)(K-\nu)u = 0$. Since K is a bounded nonnegative operator, the operator $K+\nu$ is invertible, and we conclude that $Ku = \nu u$. In [26] it was shown that the spectrum of $H_1^*H_1$ is continuous, and its eigenfunctions can be found explicitly by observing that the differential operator

$$(Lu)(x) = -(x^2(1-x^2)u'(x))' + 2x^2u(x)$$

commutes with $H_1^*H_1$. We can easily verify that L also commutes with K. That means that if u is an eigenfunction of L corresponding to the eigenvalue λ, then $\lambda Ku = KLu = LKu$. Hence, Ku is also an eigenfunction of L with the eigenvalue λ. As computed in [26], the eigenspaces of L are all one-dimensional, spanned by

$$u(x;\mu) = x^{-\frac{1}{2}+i\mu} F\left(\left[\frac{1}{4}+\frac{i\mu}{2}, \frac{3}{4}+\frac{i\mu}{2}\right], [1]; 1-x^2\right), \quad \lambda = \mu^2 + \frac{1}{4},\ \mu \ge 0, \tag{4.17}$$

where $F([a,b],[c];z)$ is the Gauss hypergeometric function. We conclude that functions $u(x;\mu)$ are eigenfunctions of K. The corresponding eigenvalues are the singular values of H_1, which, according to [26], are given by

$$\nu(\mu) = \frac{\pi}{\cosh(\pi\mu)}. \tag{4.18}$$

We note that the function $z \mapsto F([a,b],[c];z)$ is analytic in $\mathbb{C} \setminus [1,+\infty)$. Therefore, $u(x;\mu)$, given by (4.17), is analytic in the complex right half-plane. The orthogonality of the eigenfunctions is conveniently expressed in terms of the "u-transform" and its inverse (see [26]):

$$\hat{f}(\mu) = \int_0^1 f(x)u(x;\mu)dx, \qquad f(x) = \int_0^\infty \hat{f}(\mu)u(x;\mu)\mu\tanh(\pi\mu)d\mu. \tag{4.19}$$

Multiplying the second equation by $f(x)$ and integrating gives the generalized Plancherel formula

$$\|f\|_2^2 = \int_0^\infty |\hat{f}(\mu)|^2\mu\tanh(\pi\mu)d\mu. \tag{4.20}$$

The knowledge of the eigenfunctions of K permits us to solve the integral equation (4.5):

$$\psi_\varepsilon(x) = \int_0^\infty \frac{u(x_0;\mu)u(x;\mu)\mu\tanh(\pi\mu)}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}d\mu, \quad \hat{\varepsilon} = \frac{\varepsilon}{\sqrt{2\pi}}. \tag{4.21}$$

Moreover,

$$\|\psi_\varepsilon\|_2^2=\int_0^\infty \frac{u(x_0;\mu)^2\mu\tanh(\pi\mu)}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}d\mu, \tag{4.22}$$

while

$$\psi_\varepsilon(x_0)=\int_0^\infty \frac{u(x_0;\mu)^2\mu\tanh(\pi\mu)}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}d\mu. \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.8) gives

$$\|\psi_\varepsilon\|^2=\frac{1}{\pi}\int_0^\infty \frac{u(x_0;\mu)^2\mu\sinh(\pi\mu)}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}d\mu. \tag{4.24}$$

When $x=x_0>1$ the coefficient $-x^2(1-x^2)$ in the differential operator L becomes positive, and we expect the eigenfunctions $u(x_0;\mu)$ to grow exponentially as $\mu\to\infty$. Thus, if we set $\varepsilon=0$ in (4.22) and (4.23), we obtain exponentially divergent integrals, while they remain convergent for each $\varepsilon>0$. Thus, $\|\psi_\varepsilon\|_2\to\infty$ and $\psi_\varepsilon(x_0)\to\infty$, as $\varepsilon\to 0$, and the precise asymptotics of these quantities, as $\varepsilon\to 0$, would depend on the rate of exponential growth of $u(x_0;\mu)$, as $\mu\to\infty$.

4.3. Asymptotics of $\boldsymbol{\Delta_*^{x_0}(\epsilon)}$. In this section the notation $A(\epsilon)\sim B(\epsilon)$ means $A(\epsilon)/B(\epsilon)\to 1$, as $\epsilon\to 0^+$. Similarly, $A(\mu)\sim B(\mu)$ means $A(\mu)/B(\mu)\to 1$, as $\mu\to+\infty$. The goal of this section is to prove the following explicit asymptotics[2] of $\Delta_*^{x_0}(\epsilon)$.

THEOREM 4.1.

$$\Delta_*^{x_0}(\epsilon)\sim\begin{cases} C_*(x_0)\epsilon^{\frac{2}{\pi}\arcsin\left(\frac{1}{x_0}\right)}, & x_0>1,\\ \frac{\sqrt{2}}{\pi}\epsilon|\ln\epsilon|, & x_0=1,\end{cases} \tag{4.25}$$

where

$$C_*(x_0)=\frac{1}{2}\sqrt{\frac{x_0}{2(x_0^2-1)\arcsin\left(\frac{1}{x_0}\right)}}\left(\frac{2\pi\arcsin\left(\frac{1}{x_0}\right)}{\arccos\left(\frac{1}{x_0}\right)}\right)^{\frac{\arccos\left(\frac{1}{x_0}\right)}{\pi}}. \tag{4.26}$$

Formula (4.15) expresses the asymptotics of $\Delta_*^{x_0}(\epsilon)$ in terms of the asymptotics of $\|\psi_\varepsilon\|_2$, $\psi_\varepsilon(x_0)$, and $\|\psi_\varepsilon\|$, given by (4.22), (4.23), and (4.24), respectively. In turn, these depend on the asymptotics of $u(x_0;\mu)$, as $\mu\to\infty$. The following lemma gives the asymptotics of $u(z;\mu)$, as $\mu\to\infty$ for all z in the complex right half-plane, excluding the interval $[0,1]$. While in this section we will only need the asymptotics of $u(z;\mu)$ for real $z>1$, the asymptotics for other values of z will also be required later on.

LEMMA 4.2. *Let $u(x;\mu)$ be the eigenfunctions of the integral operator K. Then formula* (4.17) *gives the analytic extension of $u(x;\mu)$ from $[0,1]$ to the complex right half-plane. Moreover,*

$$u(z;\mu)\sim R(z)\frac{e^{\mu\alpha(z)}}{\sqrt{2\pi\mu}},\ \ as\ \mu\to\infty, \tag{4.27}$$

[2]For our purposes, we only need the exponent. We derive the explicit formula for $C_*(x_0)$ because we can, and because the technique we use may be of broader interest.

for every $z\in\Omega=\{z\in\mathbb{C}:\Re z>0,\ z\notin[0,1]\}$, *where*

$$R(z)=z^{-1/2}(z^2-1)^{-1/4},\qquad \alpha(z)=\arccos\left(\frac{1}{z}\right)=i\ln\left(\frac{1-i\sqrt{z^2-1}}{z}\right), \tag{4.28}$$

and where the principal branches of the natural logarithm and all fractional powers are used.

The proof is a straightforward application of the asymptotic formulas for the Gauss hypergeometric function from [13]. The required calculations needed to apply these formulas to our specific case are detailed in Appendix B. We also remark that $u(1;\mu)=1$ and that the asymptotics of $u(x;\mu)$ for $x\in[0,1)$ is given in [26, formula (4.34)].

The exponential growth of $u(z;\mu)$ as $\mu\to\infty$, described by Lemma 4.2, permits us to compute the explicit asymptotics of $\psi_\varepsilon(z)$, $\|\psi_\varepsilon\|_2$, and $\|\psi_\varepsilon\|$, given by (4.21), (4.22), and (4.24), respectively. This is made possible by the following lemma.

LEMMA 4.3. *Suppose that* $v\in C([0,\infty))$ *is such that* $v(\mu)\to1$, *as* $\mu\to+\infty$, $k\in\{1,2\}$, *and* $\Re\beta\in(0,k)$. *Then*

$$\int_0^\infty\frac{e^{\pi\beta\mu}v(\mu)d\mu}{(2\hat\varepsilon^2\cosh(\pi\mu)+1)^k}\sim\frac{(1-\beta)^{k-1}\hat\varepsilon^{-2\beta}}{\sin(\pi\beta)}\ \text{ as } \hat\varepsilon\to0^+. \tag{4.29}$$

Proof. Changing the variable of integration $\mu'=\pi\mu+2\ln\hat\varepsilon$, we obtain

$$\int_0^\infty\frac{e^{\pi\beta\mu}v(\mu)d\mu}{(2\hat\varepsilon^2\cosh(\pi\mu)+1)^k}=\frac{\hat\varepsilon^{-2\beta}}{\pi}\int_{2\ln\hat\varepsilon}^\infty\frac{e^{\beta\mu'}v\left(\frac{\mu'}{\pi}-\frac{2}{\pi}\ln\hat\varepsilon\right)}{(e^{\mu'}+e^{-\mu'+4\ln\hat\varepsilon}+1)^k}d\mu'.$$

Since $\Re\beta\in(0,k)$ and $v(\cdot)$ is a bounded function, the Lebesgue dominated convergence theorem applies, and we obtain[3]

$$\lim_{\hat\varepsilon\to0^+}\int_{2\ln\hat\varepsilon}^\infty\frac{e^{\beta\mu'}v\left(\frac{\mu'}{\pi}-\frac{2}{\pi}\ln\hat\varepsilon\right)}{(e^{\mu'}+e^{-\mu'+4\ln\hat\varepsilon}+1)^k}d\mu'=\int_{\mathbb{R}}\frac{e^{\beta\mu'}d\mu'}{(e^{\mu'}+1)^k}=\frac{\pi(1-\beta)^{k-1}}{\sin(\pi\beta)}.\qquad\square$$

As a corollary, we obtain the explicit asymptotics of $\psi_\varepsilon(z)$, $\|\psi_\varepsilon\|_2$, and $\|\psi_\varepsilon\|$.

THEOREM 4.4. *Let* $x_0>1$ *and* ψ_ε *be the solution of the integral equation* (4.5). *Formula* (4.21) *defines an analytic extension of* $\psi_\varepsilon(x)$ *from* $[0,1]$ *to the complex right half-plane. Suppose* $z\in\Omega=\{z\in\mathbb{C}:\Re z>0,\ z\notin[0,1]\}$. *Then,*

(i) $\psi_\varepsilon(z)\sim\frac{R(x_0)R(z)}{2\pi\sin(\pi\beta(z))}\hat\varepsilon^{-2\beta(z)}$, *where* $\beta(z)=\frac{\alpha(x_0)+\alpha(z)}{\pi}$, *and* $\hat\varepsilon=\frac{\varepsilon}{\sqrt{2\pi}}$.

(ii) $\|\psi_\varepsilon\|_2\sim\frac{1}{\pi}\sqrt{\frac{x_0\arcsin(1/x_0)}{2(x_0^2-1)}}\hat\varepsilon^{-\beta(x_0)}$;

(iii) $\varepsilon\|\psi_\varepsilon\|\sim\frac{1}{\pi}\sqrt{\frac{x_0\arccos(1/x_0)}{2(x_0^2-1)}}\hat\varepsilon^{-\beta(x_0)}$.

Proof. We begin by "substituting" our large μ asymptotics (4.27) from Lemma 4.2 into formulas (4.21), (4.22), and (4.24). We obtain

$$\psi_\varepsilon(z)=\frac{R(x_0)R(z)}{2\pi}\int_0^\infty\frac{e^{\pi\beta(z)\mu}v(z;\mu)}{2\hat\varepsilon^2\cosh(\pi\mu)+1}d\mu,$$
$$\|\psi_\varepsilon\|_2^2=\frac{R(x_0)^2}{2\pi}\int_0^\infty\frac{e^{\pi\beta(x_0)\mu}v(x_0;\mu)}{(2\hat\varepsilon^2\cosh(\pi\mu)+1)^2}d\mu,$$

[3]The formula is correct only for $k=1$ or 2. For general $k\in\mathbb{N}$, the correct right-hand side is more complicated: $\frac{1}{(k-\beta)B(k,\beta-k)\sin(\pi(\beta-k))}$, where $B(x,y)$ is the Euler beta function.

and

$$\|\psi_\varepsilon\|^2 = \frac{R(x_0)^2}{4\pi^2}\int_0^\infty \frac{e^{(1+\beta(x_0))\pi\mu}\widetilde{v}(x_0;\mu)}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}d\mu,$$

where

$$v(z;\mu) = \frac{u(x_0;\mu)}{u_0(x_0;\mu)}\cdot\frac{u(z;\mu)}{u_0(z;\mu)}\tanh(\pi\mu), \quad \widetilde{v}(z;\mu) = 2v(z;\mu)e^{-\pi\mu}\cosh(\pi\mu),$$

and

$$u_0(z;\mu) = R(z)\frac{e^{\mu\alpha(z)}}{\sqrt{2\pi\mu}}. \tag{4.30}$$

In order to apply Lemma 4.3, we must verify that the function $\mu \mapsto v(z;\mu)$ and the exponent $\beta(z)$ satisfy the assumptions in Lemma 4.3. The continuity of $\mu \mapsto v(z;\mu)$ follows from the Euler integral representation of the hypergeometric function combined with formula (4.17), which gives

(4.31)

$$u(z;\mu) = z^{-\frac{1}{2}+i\mu}\frac{\sin\left(\frac{3\pi}{4}+i\frac{\pi\mu}{2}\right)}{\pi}\int_0^1 t^{-\frac{1}{4}+\frac{i\mu}{2}}(1-t)^{-\frac{3}{4}-\frac{i\mu}{2}}(1-(1-z^2)t)^{-\frac{1}{4}-\frac{i\mu}{2}}\,dt.$$

The integrand is continuous in μ and bounded by $t^{-1/4}(1-t)^{-3/4}|(1-(1-z^2)t|^{-1/4}e^{\pi\mu} \in L^1(0,1)$. An application of the Lebesgue dominated convergence theorem implies that $\mu \mapsto u(z;\mu)$ is continuous on $[0,\infty)$ for any $z \in \Omega$. Formula (4.30) shows that $u_0(z;\mu)$ is nonvanishing and continuous in $\mu > 0$, proving the continuity of $\mu \mapsto v(z;\mu)$, while Lemma 4.2 implies that $v(z;\mu) \to 1$, as $\mu \to \infty$, for every $z \in \Omega$. Finally, the required constraint $\Re\beta(z) \in (0,1)$ for any $z \in \Omega$ is guaranteed by the following lemma.

Lemma 4.5. *$\Re\alpha(z) \in (0, \frac{\pi}{2})$ for any $z \in \Omega$, where $\alpha(z)$ is defined as in* (4.28).

Proof. We observe that $\alpha : \Omega \to \mathbb{C}$ is injective since $\cos\alpha(z) = 1/z$. Thus, $\partial_\infty\alpha(\Omega) = \alpha(\partial_\infty\Omega)$, where $\partial_\infty\Omega$ refers to the boundary of Ω in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. It is easy to see that $\alpha(z)$ maps the ray $i(0,+\infty)$ to the line $\pi/2 + i\mathbb{R}$ and the ray $i(-\infty,0)$ to the same line $\pi/2 + i\mathbb{R}$. It maps the interval $[0,1]+0i$ to the ray $i[0,+\infty]$ and the interval $[0,1]-0i$ to the ray $i[-\infty,0]$, while $\sqrt{z^2-1} = z\sqrt{1-z^{-2}}$, when $z \to \infty$, $z \in \Omega$. Therefore, $\alpha(z) \to i\ln(-i) = \pi/2$, as $z \to \infty$. We conclude that $\partial_\infty\alpha(\Omega) = i\mathbb{R} \cup (\pi/2 + i\mathbb{R}) \cup \{\infty\}$, and $\alpha(\Omega) = \{w \in \mathbb{C} : 0 < \Re w < \pi/2\}$ since $\alpha(\Omega)$ must be a connected subset of $\mathbb{C}$. ∎

Lemma 4.3 can now be applied, and we obtain

$$\psi_\varepsilon(z) \sim \frac{R(x_0)R(z)\hat{\varepsilon}^{-2\beta(z)}}{2\pi\sin\pi\beta(z)},$$

$$\|\psi_\varepsilon\|_2^2 \sim \frac{R(x_0)^2(1-\beta(x_0))\hat{\varepsilon}^{-2\beta(x_0)}}{2\pi\sin\pi\beta(x_0)},$$

and

$$\|\psi_\varepsilon\|^2 \sim \frac{R(x_0)^2\beta(x_0)\hat{\varepsilon}^{-2\beta(x_0)-2}}{4\pi^2\sin\pi\beta(x_0)}.$$

Substituting the values of $R(x_0)$, $\alpha(x_0)$, and $\beta(x_0)$, we obtain the claimed asymptotic formulas (i)–(iii). ∎

Now, we can compute the explicit asymptotics of $\Delta_*^{x_0}(\epsilon)$, given in (4.15). We compute

$$\frac{\psi_\varepsilon(x_0)}{\|\psi_\varepsilon\|}\sim\hat{\varepsilon}^{1-\beta(x_0)}\sqrt{\frac{x_0}{2(x_0^2-1)\beta(x_0)}}=:E_0(\varepsilon),$$

and

$$\frac{\|\psi_\varepsilon\|_2}{\|\psi_\varepsilon\|}\sim\varepsilon\sqrt{\frac{\arcsin(1/x_0)}{\arccos(1/x_0)}}=:E_1(\varepsilon).$$

It is now evident that functions $E_0(\varepsilon)$ and $E_1(\varepsilon)$ are continuous and monotone increasing on $[0,1)$, such that $E_0(0)=E_1(0)=0$. Since $E_1(\varepsilon)$ is linear and $E_0(\varepsilon)$ is a constant multiple of a power, property (4.16) reads

$$\frac{E_0(\varepsilon)}{E_0\left(E_1^{-1}\left(E_1(\varepsilon)r(\varepsilon)\right)\right)}=\frac{E_0(\varepsilon)}{E_0\left(\varepsilon r(\varepsilon)\right)}=(r(\varepsilon))^{\beta(x_0)-1}\to 1,\ \text{as }\varepsilon\to 0^+,$$

for any function $r(\varepsilon)$ such that $r(\varepsilon)\to 1$ as $\varepsilon\to 0^+$. Thus, formula (4.15) applies, and

$$\Delta_*^{x_0}(\epsilon)\sim E_0(E_1^{-1}(\epsilon))=\sqrt{\frac{x_0}{2(x_0^2-1)\beta(x_0)}}\left(\sqrt{\frac{2\pi\arcsin(1/x_0)}{\arccos(1/x_0)}}\right)^{\beta(x_0)-1}\epsilon^{1-\beta(x_0)}.$$

Substituting the values of $\alpha(x_0)=\arccos(1/x_0)$ and $\beta(x_0)=2\alpha(x_0)/\pi$ into the above formula, we obtain Theorem 4.1 for all $x_0>1$. In particular, we see that for any $x_0>1$

$$\gamma_*(x_0)=\lim_{\epsilon\to 0}\frac{\ln\Delta_*^{x_0}(\epsilon)}{\ln\epsilon}=\frac{2}{\pi}\arcsin\left(\frac{1}{x_0}\right). \tag{4.32}$$

The singular behavior at $x_0=1$ of coefficients in all of our asymptotic formulas indicates that the asymptotic analysis for $x_0=1$ needs to be done separately.

Theorem 4.6. *Let ψ_ε be the solution of the integral equation* (4.5) *with $x_0=1$. Then*

(i) $\|\psi_\varepsilon\|_2^2\sim\psi_\varepsilon(1)\sim\frac{2(\ln\varepsilon)^2}{\pi^2}$;
(ii) $\|\psi_\varepsilon\|^2\sim\frac{-2\ln\varepsilon}{\pi^2\varepsilon^2}$.

Proof. Whenever $x_0=1$, our formulas (4.22), (4.23), and (4.24) simplify because $u(1;\mu)\equiv 1$:

$$\psi_\varepsilon(1)=\int_0^\infty\frac{\mu\tanh(\pi\mu)}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}d\mu,\quad \|\psi_\varepsilon\|_2^2=\int_0^\infty\frac{\mu\tanh(\pi\mu)}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}d\mu, \tag{4.33}$$

$$\|\psi_\varepsilon\|^2=\frac{1}{\pi}\int_0^\infty\frac{\mu\sinh(\pi\mu)}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}d\mu. \tag{4.34}$$

The situation here is similar to the one for $x_0>1$ in that setting $\hat{\varepsilon}=0$ still results in divergent integrals. This indicates that it is the behavior of the integrands at $\mu=\infty$ that determines the asymptotics of the integrals when $\hat{\varepsilon}\to 0^+$. When μ is large $\tanh(\pi\mu)$ will be replaced by 1, and both $2\cosh(\pi\mu)$ and $2\sinh(\pi\mu)$ by $e^{\pi\mu}$. To make this heuristic argument rigorous, we make a simple observation that we formulate as a lemma for easy reference.

LEMMA 4.7. *Let* (G,σ) *be an arbitrary measure space. Suppose that for any* $\varepsilon\in(0,\varepsilon_0)$ $\{W_\varepsilon,\hat{W}_\varepsilon\}\subset L^1(G;d\sigma)$ *and*

(i) $\underline{\lim}_{\varepsilon\to0^+}\left|\int_G W_\varepsilon(\mu)d\sigma(\mu)\right|=\infty$;

(ii) $\overline{\lim}_{\varepsilon\to0^+}\|W_\varepsilon-\hat{W}_\varepsilon\|_{L^1(G;d\sigma)}<\infty$.

Then $\int_G W_\varepsilon(\mu)d\sigma(\mu)\sim\int_G\hat{W}_\varepsilon(\mu)d\sigma(\mu)$, *as* $\varepsilon\to0^+$.

Proof. $\overline{\lim}_{\varepsilon\to0^+}\left|\frac{\int_G\hat{W}_\varepsilon(\mu)d\sigma(\mu)}{\int_G W_\varepsilon(\mu)d\sigma(\mu)}-1\right|\le\frac{\overline{\lim}_{\varepsilon\to0^+}\|\hat{W}_\varepsilon-W_\varepsilon\|_{L^1(G;d\sigma)}}{\underline{\lim}_{\varepsilon\to0^+}\left|\int_G W_\varepsilon(\mu)d\sigma(\mu)\right|}=0.$ ∎

As we have already pointed out, the integrals in (4.33) and (4.34) satisfy condition (i) of the lemma. Then estimates

$$|\tanh(\pi\mu)-1|\le2e^{-2\pi\mu},\qquad|2\sinh(\pi\mu)-e^{\pi\mu}|=e^{-\pi\mu}$$

ensure that condition (ii) of the lemma is satisfied, and we conclude that

$$\psi_\varepsilon(1)\sim\int_0^\infty\frac{\mu d\mu}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1},\quad\|\psi_\varepsilon\|_2^2\sim\int_0^\infty\frac{\mu d\mu}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2},$$

and

$$\|\psi_\varepsilon\|^2\sim\frac{1}{2\pi}\int_0^\infty\frac{\mu e^{\pi\mu}d\mu}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2}.$$

Similarly, the estimate

$$\left|\frac{\mu}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}-\frac{\mu}{\hat{\varepsilon}^2e^{\pi\mu}+1}\right|\le2\hat{\varepsilon}^2\mu e^{-\pi\mu}$$

implies that

$$\psi_\varepsilon(1)\sim\int_0^\infty\frac{\mu d\mu}{\hat{\varepsilon}^2e^{\pi\mu}+1}.$$

To handle the remaining two integrals, we define

$$W_\varepsilon(\mu)=\frac{\mu e^{\pi\mu}}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2},\quad\hat{W}_\varepsilon(\mu)=\frac{\mu e^{\pi\mu}}{(\hat{\varepsilon}^2e^{\pi\mu}+1)^2}.$$

We first compute

$$|W_\varepsilon(\mu)-\hat{W}_\varepsilon(\mu)|=\frac{\mu\hat{\varepsilon}^2(2\hat{\varepsilon}^2e^{\pi\mu}+2+\hat{\varepsilon}^2e^{-\pi\mu})}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2(\hat{\varepsilon}^2e^{\pi\mu}+1)^2}$$

and estimate

$$2\hat{\varepsilon}^2e^{\pi\mu}+2+\hat{\varepsilon}^2e^{-\pi\mu}\le3(\hat{\varepsilon}^2e^{\pi\mu}+1),$$

so that

$$|W_\varepsilon(\mu)-\hat{W}_\varepsilon(\mu)|\le\frac{3\mu\hat{\varepsilon}^2}{(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2(\hat{\varepsilon}^2e^{\pi\mu}+1)}.$$

Next, we estimate

$$\hat{\varepsilon}^2e^{\pi\mu}+1\ge\hat{\varepsilon}^2e^{\pi\mu},\qquad(2\hat{\varepsilon}^2\cosh(\pi\mu)+1)^2\ge1,$$

and obtain

$$|W_\varepsilon(\mu) - \hat{W}_\varepsilon(\mu)| \le 3\mu e^{-\pi\mu} \in L^1(0,\infty).$$

Thus, Lemma 4.7 is applicable and

$$\psi_\varepsilon(1) \sim \int_0^\infty \frac{\mu d\mu}{\hat{\varepsilon}^2 e^{\pi\mu}+1} := I_1(\hat{\varepsilon}), \qquad \|\psi_\varepsilon\|_2^2 \sim \int_0^\infty \frac{\mu d\mu}{(\hat{\varepsilon}^2 e^{\pi\mu}+1)^2} := I_2(\hat{\varepsilon}),$$

$$\|\psi_\varepsilon\|^2 \sim I_0(\hat{\varepsilon}) := \frac{1}{2\pi}\int_0^\infty \frac{\mu e^{\pi\mu} d\mu}{(\hat{\varepsilon}^2 e^{\pi\mu}+1)^2} = \frac{\ln\left(1+\frac{1}{\hat{\varepsilon}^2}\right)}{2\pi^3\hat{\varepsilon}^2} \sim -\frac{\ln\hat{\varepsilon}}{\pi^3\hat{\varepsilon}^2},$$

establishing part (ii) of the theorem. Part (i) is proved by means of the l'Hôpital rule:

$$\lim_{\hat{\varepsilon}\to 0^+}\frac{I_1(\hat{\varepsilon})}{(\ln\hat{\varepsilon})^2} = \lim_{\hat{\varepsilon}\to 0^+}\frac{\hat{\varepsilon}I_1'(\hat{\varepsilon})}{2\ln\hat{\varepsilon}} = -\lim_{\hat{\varepsilon}\to 0^+}\frac{2\pi\hat{\varepsilon}^2 I_0(\hat{\varepsilon})}{\ln\hat{\varepsilon}} = \frac{2}{\pi^2}.$$

To apply the l'Hôpital rule to $I_2(\hat{\varepsilon})$, we compute

$$I_2'(\hat{\varepsilon}) = -4\hat{\varepsilon}\int_0^\infty \frac{\mu e^{\pi\mu} d\mu}{(\hat{\varepsilon}^2 e^{\pi\mu}+1)^3} = -2\frac{(\hat{\varepsilon}^2+1)\ln(1+\hat{\varepsilon}^{-2})-1}{\pi^2\hat{\varepsilon}(\hat{\varepsilon}^2+1)} \sim \frac{4\ln\hat{\varepsilon}}{\pi^2\hat{\varepsilon}}.$$

Thus,

$$\lim_{\hat{\varepsilon}\to 0^+}\frac{I_2(\hat{\varepsilon})}{(\ln\hat{\varepsilon})^2} = \lim_{\hat{\varepsilon}\to 0^+}\frac{\hat{\varepsilon}I_2'(\hat{\varepsilon})}{2\ln\hat{\varepsilon}} = \frac{2}{\pi^2}.$$

The theorem is now proved. □

According to Theorem 4.6,

$$\frac{\psi_\varepsilon(1)}{\|\psi_\varepsilon\|} \sim \frac{\sqrt{2}}{\pi}\varepsilon|\ln\varepsilon|^{3/2} =: E_0(\varepsilon), \qquad \frac{\|\psi_\varepsilon\|_2}{\|\psi_\varepsilon\|} \sim \varepsilon\sqrt{|\ln\varepsilon|} =: E_1(\varepsilon).$$

This shows that both $E_0(\varepsilon)$ and $E_1(\varepsilon)$ are continuous, monotone increasing functions on $[0, e^{-3/2})$, satisfying $E_0(0) = E_1(0) = 0$. In order to use formula (4.15) for the exact asymptotics of $\Delta_*^1(\epsilon)$, we need to verify property (4.16). This is somewhat tedious. Let $r(\varepsilon) \to 1$, as $\varepsilon \to 0^+$, be arbitrary. To make the calculations more compact, we define $\delta = \delta(\varepsilon) = r(\varepsilon)E_1(\varepsilon)$. Then,

$$\rho(\varepsilon) = \frac{E_0(\varepsilon)}{E_0(E_1^{-1}(E_1(\varepsilon)r(\varepsilon)))} = \frac{\varepsilon|\ln\varepsilon|^{3/2}}{E_1^{-1}(\delta)|\ln E_1^{-1}(\delta)|^{3/2}} = \frac{\varepsilon|\ln\varepsilon|^{3/2}}{\delta|\ln E_1^{-1}(\delta)|} = \frac{|\ln\varepsilon|}{r(\varepsilon)|\ln E_1^{-1}(\delta)|},$$

where we have used the relation $E_1^{-1}(\delta)|\ln E_1^{-1}(\delta)|^{1/2} = E_1(E_1^{-1}(\delta)) = \delta$ together with the formula for $\delta(\varepsilon)$. Next, we write

$$|\ln\varepsilon| = |\ln r(\varepsilon)E_1(\varepsilon) - \ln(r(\varepsilon)\sqrt{|\ln\varepsilon|})| = |\ln\delta(\varepsilon)|\tilde{r}(\varepsilon),$$

where

$$\tilde{r}(\varepsilon) = \left|1 - \frac{\ln(r(\varepsilon)\sqrt{|\ln\varepsilon|})}{\ln(r(\varepsilon)\varepsilon\sqrt{|\ln\varepsilon|})}\right| \to 1, \text{ as } \varepsilon \to 0^+.$$

Thus,

$$\rho(\varepsilon) = \frac{\tilde{r}(\varepsilon)|\ln\delta|}{r(\varepsilon)|\ln E_1^{-1}(\delta)|}.$$

It remains to observe that $\delta(\varepsilon)\to 0^+$, as $\varepsilon\to 0^+$ and therefore, $\eta(\varepsilon)=E_1^{-1}(\delta(\varepsilon))\to 0^+$. Hence,

$$\lim_{\varepsilon\to 0^+}\rho(\varepsilon)=\lim_{\varepsilon\to 0^+}\frac{\tilde{r}(\varepsilon)}{r(\varepsilon)}\cdot\lim_{\delta\to 0^+}\frac{|\ln\delta|}{|\ln E_1^{-1}(\delta)|}=\lim_{\eta\to 0^+}\frac{|\ln E_1(\eta)|}{|\ln\eta|}=1.$$

Formula (4.15) is now applicable, and we compute, using $E_1^{-1}(\epsilon)|\ln E_1^{-1}(\epsilon)|^{1/2}=\epsilon$,

$$\Delta_*^1(\epsilon)\sim\frac{\sqrt{2}}{\pi}E_1^{-1}(\epsilon)|\ln E_1^{-1}(\epsilon)|^{3/2}=\frac{\sqrt{2}}{\pi}\epsilon|\ln\epsilon|\frac{|\ln E_1^{-1}(\epsilon)|}{|\ln\epsilon|}\sim\frac{\sqrt{2}}{\pi}\epsilon|\ln\epsilon|$$

since

$$\lim_{\epsilon\to 0^+}\frac{|\ln E_1^{-1}(\epsilon)|}{|\ln\epsilon|}=\lim_{\eta\to 0^+}\frac{|\ln\eta|}{|\ln E_1(\eta)|}=1.$$

In particular, we can conclude that

$$\gamma_*(1)=\lim_{\epsilon\to 0^+}\frac{\ln\Delta_*^1(\epsilon)}{\ln\epsilon}=1=\lim_{x_0\to 1^+}\gamma_*(x_0).$$

This completes the proof of Theorem 4.1 for $x_0=1$.

5. A continuous family of Hilbert space norms.

Our task now is to connect the explicit exponent $\gamma_*(x_0)$, given by (4.32) to the desired exponent $\gamma(x_0)$ coming from the (f,g)-problem (2.4). This is done by introducing a family of norms that help us to bridge the gap between the $L^2(0,1)$ norm and the $H^2(\mathcal{R})$ norm on the convex cone $\mathfrak{C}_2$. In reference to $f\in\mathfrak{C}_2$, we will use the notation

$$(\mathcal{F}_p[f])(z)=\frac{f(z^{1/p})}{z^{\frac{p-1}{2p}}(z^{1/p}+1)},\quad p\ge 1,\tag{5.1}$$

where the principal branch of z^α is always chosen. For all $p\ge 1$ and $f\in\mathfrak{C}_2$ the functions $\mathcal{F}_p[f]$ are analytic on the complex right half-plane $\mathcal{R}$. We then define the family of spaces

$$\mathfrak{H}_p=\{f\in\mathcal{H}(\mathcal{R}):\mathcal{F}_p[f]\in H\},\quad p\ge 1,\tag{5.2}$$

equipped with norms $\|f\|_{\mathfrak{H}_p}=\|\mathcal{F}_p[f]\|$.

THEOREM 5.1. *$\mathfrak{C}_2\subset\mathfrak{H}_p$, for every $p>1$, and there exists a constant $C_p>0$, such that*

$$\|f\|_{\mathfrak{H}_p}\le C_p\|f\|_2\tag{5.3}$$

for every $f\in\mathfrak{C}_2$.

Proof. The fact that the functions $f_{\delta_t}=e^{-xt}$ belong to $\mathfrak{H}_p$ follows from the observations that for each fixed $y>0$ and $q\in[0,1]$ the functions

$$\nu_1(x)=\mathfrak{Re}\,[(x+iy)^q],\quad \nu_2(x)=|(x+iy)^q|,\quad \nu_3(x)=|(x+iy)^q+1|^2,\quad q\in[0,1],$$

are monotone increasing in $x\in(0,+\infty)$. This is evident from the polar representation of $x+iy=r(x)e^{i\theta(x)}$ and the observation that $r(x)$ is an increasing function of x, while $\theta(x)\in(0,\pi/2)$ is a decreasing one. Then

$$\nu_1(x)=r(x)^q\cos(q\theta(x)),\quad \nu_2(x)=r(x)^q,\quad \nu_3(x)=r(x)^{2q}+1+2r(x)^q\cos(q\theta(x))$$

are obviously increasing functions since $q\theta(x) \in [0,\pi/2]$ for all $x \geq 0$ and $q \in [0,1]$. Thus,

$$|(\mathcal{F}_p[f_{\delta_t}])(x+iy)|^2 = \frac{e^{-2t\nu_1(x)}}{\nu_2(x)\nu_3(x)} \leq \frac{e^{-2t\nu_1(0)}}{\nu_2(0)\nu_3(0)} = |(\mathcal{F}_p[f_{\delta_t}])(iy)|^2.$$

It is also easy to see that

$$\int_0^\infty |(\mathcal{F}_p[f_{\delta_t}])(iy)|^2 dy = \int_0^\infty \frac{e^{-2ta_p y^{1/p}}}{y^{\frac{p-1}{p}}(y^{2/p}+1+y^{1/p}a_p)} = p\int_0^\infty \frac{e^{-2ta_p u}}{u^2+1+2a_p u}du < \infty,$$

where $a_p = \cos(\pi/(2p))$. We conclude that

$$f(x) = \sum_{j=1}^N c_j e^{-xt_j} \in \mathfrak{H}_p \tag{5.4}$$

for all $p \geq 1$.

Now, let σ be a positive measure, such that $f_\sigma \in \mathfrak{H}_p \cap \mathfrak{C}_2$. Let us show that (5.3) holds for all such functions f_σ. Indeed, for any $f_\sigma \in \mathfrak{H}_p \cap \mathfrak{C}_2$, we have (see (2.7))

$$\|f_\sigma\|_{\mathfrak{H}_p} = \frac{1}{\sqrt{\pi}}\|(\mathcal{F}_p[f_\sigma])(iy)\|_{L^2(0,\infty)}.$$

Then, in order to establish (5.3), we need to prove the inequality

$$\|(\mathcal{F}_p[f_\sigma])(iy)\|_{L^2(0,\infty)} \leq \sqrt{\pi}C_p\|f_\sigma\|_2. \tag{5.5}$$

To prove (5.5), we estimate

$$\left|f_\sigma\left((iy)^{1/p}\right)\right| \leq \int_0^\infty e^{-a_p y^{1/p}t}d\sigma(t) = f_\sigma\left(a_p y^{1/p}\right). \tag{5.6}$$

We conclude that

$$\|(\mathcal{F}_p[f_\sigma])(iy)\|^2_{L^2(0,\infty)} \leq \int_0^\infty \frac{\left|f_\sigma\left(a_p y^{1/p}\right)\right|^2}{y^{\frac{p-1}{p}}|i^{1/p}y^{1/p}+1|^2}dy.$$

Making a change of variables $u = a_p y^{1/p}$, we obtain

$$\|(\mathcal{F}_p[f_\sigma])(iy)\|^2_{L^2(0,\infty)} \leq \frac{p}{a_p}\int_0^\infty \frac{|f_\sigma(u)|^2}{(u+1)^2+u^2\tan^2\left(\frac{\pi}{2p}\right)}du.$$

Writing

$$\int_0^\infty \frac{|f_\sigma(u)|^2}{(u+1)^2+u^2\tan^2\left(\frac{\pi}{2p}\right)}du = \sum_{n=0}^\infty \int_n^{n+1} \frac{|f_\sigma(u)|^2}{(u+1)^2+u^2\tan^2\left(\frac{\pi}{2p}\right)}du,$$

and estimating

$$\frac{1}{(u+1)^2+u^2\tan^2\left(\frac{\pi}{2p}\right)} \leq \frac{1}{(n+1)^2+n^2\tan^2\left(\frac{\pi}{2p}\right)},$$

when $u\in[n,n+1]$, we obtain the bound

$$\|(\mathcal{F}_p[f_\sigma])(iy)\|^2_{L^2(0,\infty)}\le\frac{p}{a_p}\sum_{n=0}^{\infty}\frac{\int_0^1|f_\sigma(x+n)|^2dx}{(n+1)^2+n^2\tan^2\left(\frac{\pi}{2p}\right)}.$$

Finally, using the fact that $0\le f_\sigma(x+n)\le f_\sigma(x)$ for any CMF f_σ, we conclude that

$$\|(\mathcal{F}_p[f_\sigma])(iy)\|^2_{L^2(0,\infty)}\le\frac{p\|f_\sigma\|_2^2}{a_p}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2+n^2\tan^2\left(\frac{\pi}{2p}\right)}.$$

If we replace $n+1$ by n in the bound above for $n>0$, we obtain a simpler formula for the constant C_p:

$$C_p^2=\frac{p}{\pi a_p}+\frac{\pi p a_p}{6},\qquad a_p=\cos\left(\frac{\pi}{2p}\right).$$

To finish the proof of the theorem, we need the following density lemma.

LEMMA 5.2. *Suppose $f\in\mathfrak{C}_2$. Then there exists a sequence of functions $f_n\in\mathfrak{C}_2$ of the form* (5.4)*, such that $f_n\to f$ in $L^2(0,1)$.*

Proof. Let K be the closure in $L^2(0,1)$ of the set of positive finite linear combinations of functions $f_{\delta_t}(x)=e^{-xt}$. Then, K is a closed, convex subset of $L^2(0,1)$. Suppose there exists $f_0\in\mathfrak{C}_2\setminus K$. Then, by the Hahn–Banach separation theorem there exists $g_0\in L^2(0,1)$, such that for all $t\ge0$

$$\int_0^1 e^{-xt}g_0(x)dx\ge0>\int_0^1 f_0(x)g_0(x)dx.$$

If σ_0 is the spectral measure of $f_0\in\mathfrak{C}_2$, then integrating the left inequality above with respect to σ_0, we obtain

$$\int_0^1 f_0(x)g_0(x)dx\ge0,$$

which contradicts the right inequality. We conclude that $K=\mathfrak{C}_2$. ∎

Now, if $f\in\mathfrak{C}_2$ and $f_n=f_{\sigma_n}$ is as in the lemma, then by Lemma 3.3 $\|\sigma_n\|_*\le\|f_n\|_2$. Thus, we can extract a weak-* convergent subsequence in X^*, not relabeled, so that $\sigma_n\overset{*}{\rightharpoonup}\sigma$, where X is defined as in (3.6). It follows that along this subsequence $f_{\sigma_n}(z)\to f_\sigma(z)$ for all $z\in\mathcal{R}$ since $e^{-zt}\in X$. Thus, since $f_{\sigma_n}\to f$ in $L^2(0,1)$, then $f_\sigma=f$, and consequently $(\mathcal{F}_p[f_n])(z)\to(\mathcal{F}_p[f])(z)$ pointwise on $\mathcal{R}$. In addition, by the already proved inequality (5.3) for functions (5.4), we have $\|\mathcal{F}_p[f_n]\|=\|f_n\|_{\mathfrak{H}_p}\le C_p\|f_n\|_2$. Hence, there exists a further subsequence, not relabeled, along which $\mathcal{F}_p[f_n]\rightharpoonup F$ in $H^2(\mathcal{R})$. But, since $H^2(\mathcal{R})$ is a reproducing kernel Hilbert space, weak convergence implies pointwise convergence, showing that $\mathcal{F}_p[f]=F\in H$. We conclude that $f\in\mathfrak{H}_p$, and the theorem is now proved. ∎

We emphasize that inequality (5.3) is valid only for all $f\in\mathfrak{C}_2$. It does not hold for $f\in\mathfrak{X}=\mathfrak{C}_2-\mathfrak{C}_2$, for which the reverse inequality holds, as shown in our next theorem.

THEOREM 5.3. *For every $p\ge1$*

$$\|f\|_2\le2\sqrt{\frac{2\pi}{p}}\|f\|_{\mathfrak{H}_p}\tag{5.7}$$

for every $f\in\mathfrak{X}$ (every $f\in\mathfrak{X}\cap\mathfrak{H}_1$ if $p=1$).

Proof. To prove this theorem, we use the analyticity of $(\mathcal{F}_p[f])(z)$ in the right half-plane. Let Γ_L be the boundary of the rectangle $[0,1]\times[0,L]$ traversed in the positive direction. We first observe that similarly to (5.6), we can estimate

$$|f_\sigma((x+iL)^{1/p})|\le f_{|\sigma|}\left(L^{1/p}a_p\right)\le f_{|\sigma|}(a_p).$$

We conclude that

$$\lim_{L\to\infty}\int_0^1|(\mathcal{F}_p[f_\sigma])(x+iL)|^2dx=0,$$

and using the Cauchy theorem $\int_{\Gamma_L}(\mathcal{F}_p[f_\sigma])(z)^2dz=0$, we obtain the formula

$$\|\mathcal{F}_p[f_\sigma]\|_2^2=\int_0^1(\mathcal{F}_p[f_\sigma])(x)^2dx=\int_0^\infty(\mathcal{F}_p[f_\sigma])(iy)^2idy-\int_0^\infty(\mathcal{F}_p[f_\sigma])(1+iy)^2idy.$$

By the symmetry of CMFs, we have $\overline{(\mathcal{F}_p[f_\sigma])(z)}=(\mathcal{F}_p[f_\sigma])(\overline{z})$. Therefore, we obtain the inequality

$$\begin{aligned}\|\mathcal{F}_p[f_\sigma]\|_2^2&\le\frac{1}{2}\int_{\mathbb{R}}|(\mathcal{F}_p[f_\sigma])(iy)|^2dy+\frac{1}{2}\int_{\mathbb{R}}|(\mathcal{F}_p[f_\sigma])(1+iy)|^2dy\\&\le\int_{\mathbb{R}}|(\mathcal{F}_p[f_\sigma])(iy)|^2dy=2\pi\|f_\sigma\|^2_{\mathfrak{H}_p},\end{aligned}$$

where we used the property of Hardy functions that $\int_{\mathbb{R}}|F(x+iy)|^2dy$ is a nonincreasing function of x. Finally, changing variable $u=x^{1/p}$, we estimate

$$\|\mathcal{F}_p[f_\sigma]\|_2^2=\int_0^1(\mathcal{F}_p[f_\sigma])(x)^2dx=p\int_0^1\frac{f_\sigma(u)^2}{(u+1)^2}du\ge\frac{p}{4}\|f_\sigma\|_2^2.\qquad\square$$

Now, in reference to the $\|\cdot\|_{\mathfrak{H}_p}$ norm, we can define the ϕ_p-problem by analogy with the ϕ-problem (2.6):

$$\Delta_p^{x_0}(\epsilon)=\sup_{\phi\in\mathcal{A}_\epsilon^p}\phi(x_0),\quad\mathcal{A}_\epsilon^p=\{\phi\in\mathfrak{H}_p:\|\phi\|_{\mathfrak{H}_p}\le1,\|\phi\|_2\le\epsilon\}.\tag{5.8}$$

6. The relations between (f,g)-, ϕ-, and ϕ_p-problems. In this section, we are going to examine the relations between the (f,g)-, ϕ-, and ϕ_p-problems, given by (2.4), (2.6), and (5.8), respectively, with the goal of establishing (2.8), thereby proving Theorem 2.1.

Let $p>1$, and let $\psi_\epsilon^{(n)}\in\mathfrak{H}_p$ be a maximizing sequence in the ϕ_p-problem (5.8). Define $\phi_\epsilon^{(n)}=\mathcal{F}_p[\psi_\epsilon^{(n)}]\in H$. Then $\|\phi_\epsilon^{(n)}\|=\|\psi_\epsilon^{(n)}\|_{\mathfrak{H}_p}\le1$, while

$$\|\phi_\epsilon^{(n)}\|_2^2=\int_0^1\frac{|\psi_\epsilon^{(n)}(x^{1/p})|^2}{x^{(p-1)/p}(1+x^{1/p})^2}dx=p\int_0^1\frac{|\psi_\epsilon^{(n)}(u)|^2}{(1+u)^2}du\le p\|\psi_\epsilon^{(n)}\|_2^2\le p\epsilon^2.$$

Thus, $\phi_\epsilon^{(n)}/\sqrt{p}$ is a valid test function for the ϕ-problem, for every $n\ge1$, where x_0 was replaced by x_0^p. Therefore,

$$\Delta_*^{x_0^p}(\epsilon)\ge\frac{\phi_\epsilon^{(n)}(x_0^p)}{\sqrt{p}}=\frac{\psi_\epsilon^{(n)}(x_0)}{\sqrt{p}x_0^{(p-1)/2}(1+x_0)}\to\frac{\Delta_p^{x_0}(\epsilon)}{\sqrt{p}x_0^{(p-1)/2}(1+x_0)},\text{ as }n\to\infty.\tag{6.1}$$

Now, let (f_ϵ, g_ϵ) be the solution of the (f,g)-problem. Define $\phi_\epsilon(x) = \Delta[f_\epsilon, g_\epsilon](x)$ (see (2.3) for notation). Then, $\|\phi_\epsilon\|_2 \le \epsilon$, and by Theorem 5.1

$$\|\phi_\epsilon\|_{\mathfrak{H}_p} \le \frac{\|f_\epsilon\|_{\mathfrak{H}_p} + \|g_\epsilon\|_{\mathfrak{H}_p}}{\|f_\epsilon\|_2 + \|g_\epsilon\|_2} \le C_p.$$

Thus, $\phi_\epsilon/(C_p+1)$ is a valid test function in the ϕ_p-problem for any $p>1$. Therefore,

$$\Delta_p^{x_0}(\epsilon) \ge \frac{\phi_\epsilon(x_0)}{C_p+1} = \frac{\Delta[f_\epsilon, g_\epsilon](x_0)}{C_p+1} = \frac{\Delta^{x_0}(\epsilon)}{C_p+1}. \tag{6.2}$$

An essential benefit of using the Hardy norm $\|\cdot\|$ is that it permits a controlled split of functions $\phi \in H$ into the difference of two CMFs. Here is the construction. By Lemma 3.1, if $\phi \in H$, then there is a unique $\sigma \in L^2(0,\infty)$, such that

$$\phi(z) = \int_0^\infty e^{-zt}\sigma(t)dt, \quad \Re z > 0.$$

Let $\sigma_+(t) = \max\{0, \sigma(t)\}$, $\sigma_-(t) = \max\{0, -\sigma(t)\}$. Then, we define

$$\phi_\pm(z) = \int_0^\infty e^{-zt}\sigma_\pm(t)dt, \quad \Re z > 0.$$

In this construction

$$\int_0^\infty \sigma_+(t)\sigma_-(t)dt = 0.$$

Therefore, by Plancherel's identity

$$\int_{\mathbb{R}} \phi_+(iy)\overline{\phi_-(iy)}dy = 0.$$

But then

$$\|\phi\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\phi_+(iy) - \phi_-(iy)|^2 dy = \|\phi_+\|^2 + \|\phi_-\|^2 \ge \frac{1}{4}(\|\phi_+\| + \|\phi_-\|)^2,$$

which shows that

$$\|\phi\| \le \|\phi_+\| + \|\phi_-\| \le 2\|\phi\|.$$

In order to complete the circle of inequalities, we take ϕ_ϵ to be the solution of the ϕ-problem and define $f = \phi_\epsilon^+$, $g = \phi_\epsilon^-$. We then have, using Lemma 3.2,

$$|\Delta[f,g](x_0)| = \frac{|\phi_\epsilon(x_0)|}{\|\phi_\epsilon^+\|_2 + \|\phi_\epsilon^-\|_2} \ge \frac{|\phi_\epsilon(x_0)|}{\sqrt{\pi}(\|\phi_\epsilon^+\| + \|\phi_\epsilon^-\|)} \ge \frac{|\phi_\epsilon(x_0)|}{2\sqrt{\pi}\|\phi_\epsilon\|} \ge \frac{|\phi_\epsilon(x_0)|}{2\sqrt{\pi}} = \frac{\Delta_*^{x_0}(\epsilon)}{2\sqrt{\pi}}.$$

We also estimate

$$\|\Delta[f,g](x)\|_2 = \frac{\|\phi_\epsilon\|_2}{\|\phi_\epsilon^+\|_2 + \|\phi_\epsilon^-\|_2} \le \frac{C_p\epsilon}{\|\phi_\epsilon^+\|_{\mathfrak{H}_p} + \|\phi_\epsilon^-\|_{\mathfrak{H}_p}} \le \frac{C_p\epsilon}{\|\phi_\epsilon\|_{\mathfrak{H}_p}}.$$

To complete the circle of inequalities, we need the following theorem.

THEOREM 6.1. *For any* $x_0 \geq 1$ *and* $p > 1$, *there exists* $c_p(x_0) > 0$, *such that*

$$\|\phi_\epsilon\|_{\mathfrak{H}_p} \geq c_p(x_0)\epsilon^{1-\frac{1}{p}} \tag{6.3}$$

for all sufficiently small ϵ.

The proof is in Appendix C. It is based on the fact that the solution ϕ_ϵ of the ϕ-problem, given by (4.10) and (4.21), is expressed in terms of the explicitly known eigenfunctions $u(x;\mu)$, given by (4.17), of the integral operator K.

We can now complete the circle of inequalities and prove (2.8). According to Theorem 6.1, $(\phi_\epsilon^+, \phi_\epsilon^-)$ is an admissible pair for the (f, g)-problem, where ϵ is replaced with $(C_p/c_p(x_0))\epsilon^{\frac{1}{p}}$, permitting us to conclude that $\overline{\gamma}(x_0)/p \leq \gamma_*(x_0)$, where

$$\underline{\gamma}(x_0) = \varliminf_{\epsilon\to 0} \frac{\ln \Delta^{x_0}(\epsilon)}{\ln \epsilon}, \quad \overline{\gamma}(x_0) = \varlimsup_{\epsilon\to 0} \frac{\ln \Delta^{x_0}(\epsilon)}{\ln \epsilon}.$$

Combining this inequality with inequalities (6.1) and (6.2), we get

$$\gamma_*(x_0^p) \leq \underline{\gamma_p}(x_0) \leq \overline{\gamma_p}(x_0) \leq \overline{\gamma}(x_0) \leq p\gamma_*(x_0), \tag{6.4}$$

where

$$\underline{\gamma_p}(x_0) = \varliminf_{\epsilon\to 0} \frac{\ln \Delta_p^{x_0}(\epsilon)}{\ln \epsilon}, \quad \overline{\gamma_p}(x_0) = \varlimsup_{\epsilon\to 0} \frac{\ln \Delta_p^{x_0}(\epsilon)}{\ln \epsilon}.$$

The explicit form of $\gamma_*(x_0)$, given by (4.32) implies that $\gamma_*(x_0)$ is a continuous function of x_0. Then, passing to the limit as $p \to 1^+$ in (6.4), we obtain the existence of the limits

$$\lim_{p\to 1^+} \underline{\gamma_p}(x_0) = \lim_{p\to 1^+} \overline{\gamma_p}(x_0) = \gamma_*(x_0).$$

Inequality (6.2) then implies that

$$\underline{\gamma_p}(x_0) \leq \underline{\gamma}(x_0) \leq \overline{\gamma}(x_0) \leq p\gamma_*(x_0).$$

Passing to the limit in this inequality as $p \to 1^+$ proves the existence of the limit

$$\gamma(x_0) = \lim_{\epsilon\to 0} \frac{\ln \Delta^{x_0}(\epsilon)}{\ln \epsilon},$$

as well as the desired equality (2.8).

7. The local problem. Suppose $f_0 \in \mathfrak{C}_2$ is given, as well as $x_0 \geq 1$. Let

$$\mathcal{K}_\epsilon[f_0] = \{f \in \mathfrak{C}_2 : \|f - f_0\|_2 \leq \epsilon\}.$$

We note that $\mathcal{K}_\epsilon[f_0]$ is a convex set. The goal is to compute

$$M_\epsilon(x_0; f_0) = \max_{f\in\mathcal{K}_\epsilon[f_0]} f(x_0), \qquad m_\epsilon(x_0; f_0) = \min_{f\in\mathcal{K}_\epsilon[f_0]} f(x_0). \tag{7.1}$$

While the Kuhn–Tucker theorem is applicable to the local problem (7.1) and leads to optimality conditions that are easy to check numerically, they are not very useful as a guide for finding the extremals in (7.1).

For this reason, we forgo the details of the Kuhn–Tucker-based analysis and opt instead for the direct variational approach due to Caprini [3, 4, 5], which is narrower in scope than Kuhn–Tucker but leads directly to a natural algorithm for computing the extremals in (7.1) approximately. The method is applicable for the minimization of general positive definite quadratic functionals and necessitates the dual reformulation of the variational problems (7.1). Given $f_0 \in \mathfrak{C}_2$ and $\delta \in (-\delta_-, \delta_+)$, for some small $\delta_\pm > 0$, we seek to solve

$$\min_{\substack{f \in \mathfrak{C}_2 \\ f(x_0)-f_0(x_0)=\delta}} \|f - f_0\|_2^2. \tag{7.2}$$

Suppose that f_{σ_*} satisfies the constraint $f_{\sigma_*}(x_0) - f_0(x_0) = \delta$ and minimizes the functional $J[\sigma] = \|f_\sigma - f_0\|_2^2$. The Caprini method is based on the following representation of the variation $\Delta J = J[\sigma] - J[\sigma_*] \geq 0$:

$$\Delta J = \|f_\sigma - f_0\|_2^2 - \|f_{\sigma_*} - f_0\|_2^2 = 2\int_0^\infty C(t)d\Delta\sigma(t) + \|f_{\Delta\sigma}\|_2^2, \tag{7.3}$$

where $\Delta\sigma = \sigma - \sigma_*$, and

$$C(t) = (\Lambda f_{\sigma_*})(t) - (\Lambda f_0)(t) = \int_0^1 e^{-xt}(f_{\sigma_*}(x) - f_0(x))dx \tag{7.4}$$

is the Caprini function.

THEOREM 7.1. *The minimizer σ_* in* (7.2) *exists and is unique and has either a finite support or a countable support $\{t_n : n \geq 1\}$ with*

$$\sum_{n=1}^\infty \frac{1}{t_n} < \infty. \tag{7.5}$$

In either case

$$C(t) \geq \frac{e^{-x_0 t}}{f_0(x_0) + \delta} \int_0^1 f_{\sigma_*}(x)(f_{\sigma_*}(x) - f_0(x))dx, \quad t \geq 0, \tag{7.6}$$

with equality at all $t = t_n$ in the support of σ_*. Conversely, if σ_* is a positive measure, whose support $\{t_n : n \geq 1\}$ satisfies (7.5)*, and is such that* (7.6) *holds, then it is a minimizer in* (7.2)*, provided $\sigma_* \neq \sigma_0$, where $f_{\sigma_0} = f_0$.*

Proof. To prove existence, we let σ_n be a minimizing sequence. Then the boundedness of $\|f_{\sigma_n}\|_2$ implies the boundedness of $\|\sigma_n\|_*$, according to Lemma 3.3. Hence, we can extract a subsequence, not relabeled, such that $f_{\sigma_n} \rightharpoonup f_*$ in $L^2(0,1)$ and $\sigma_n \overset{*}{\rightharpoonup} \sigma_*$ in X^*, where X is given by (3.6). Then $f_{\sigma_n}(x) \to f_{\sigma_*}(x)$ for all $x > 0$ since $e^{-xt} \in X$ for all $x > 0$. We conclude that $f_* = f_{\sigma_*}$, and that $f_{\sigma_*}(x_0) - f_0(x_0) = \delta$. The weak lower semicontinuity of the $L^2(0,1)$ norm implies that

$$\|f_* - f_0\|_2 \leq \varliminf_{n\to\infty} \|f_{\sigma_n} - f_0\|_2 = \min_{\substack{f \in \mathfrak{C}_2 \\ f(x_0)-f_0(x_0)=\delta}} \|f - f_0\|_2^2.$$

The uniqueness of the minimizer follows from the convexity of the constraint and the strict uniform convexity of the $L^2(0,1)$ norm.

Now, let σ_* be the minimizer in (7.2). Assume first that σ_* has a point mass at t_*. Then, we remove $\epsilon\delta_{t_*}(t)$ from σ_*, while placing the mass $\epsilon e^{x_0(t_0 - t_*)}$ at t_0, preserving the constraint. In that case

$$\Delta J = 2\epsilon(e^{x_0(t_0-t_*)}C(t_0) - C(t_*)) + O(\epsilon^2) \geq 0,$$

and therefore, $e^{x_0t_0}C(t_0) \geq e^{x_0t_*}C(t_*)$. Hence, any point mass t_* in the support of σ_* must be a point of global minimum of $e^{x_0t}C(t)$ on $[0,+\infty)$. If t_* is in the support of σ_*, but is not a point mass, then $m(\epsilon) = \sigma_*((t_* - \epsilon)^+, t_* + \epsilon)) \to 0$, as $\epsilon \to 0$, while $m(\epsilon) > 0$ for any $\epsilon > 0$. In that case, we remove $\sigma_*|((t_* - \epsilon)^+, t_* + \epsilon)$ from σ_* and place the appropriate mass $m(\epsilon)e^{x_0(t_0-t_*)}$ at t_0, so as to maintain the constraint. This time, we obtain

$$\Delta J = m(\epsilon)(e^{x_0(t_0-t_*)}C(t_0) - C(t_*)) + o(m(\epsilon)).$$

Once again, we conclude that t_* must be a point of global minimum of $e^{x_0t}C(t)$. Since $e^{x_0t}C(t)$ is an entire function of t, as is evident from (7.4), the support of σ_* must be discrete. If the support of σ_* is infinite,

$$\sigma_* = \sum_{n=1}^{\infty} a_n \delta_{t_n}(t), \quad a_n > 0, \tag{7.7}$$

and does not satisfy (7.5), then, by the Müntz–Szász theorem [15], the set of functions u^{t_n} is dense in $C_0([0,1])$. But then the functions e^{-xt_n} are dense in $C_0(0,\infty)$. In that case the equation

$$e^{x_0t_n}C(t_n) = m \stackrel{\text{def}}{=} \min_{t\geq 0} e^{x_0t}C(t) \tag{7.8}$$

would imply that

$$\int_0^1 g(x)(f_*(x) - f_0(x))dx = g(x_0)m \quad \forall g \in C_0(0,\infty),$$

where f_* is shorthand for f_{σ_*}. This easily leads to a contradiction if, for example, we take a delta-like sequence $g_n(x)$ converging to $\delta_a(x)$ for an arbitrary $a \in (0,1)$.

Now, (7.8) written as $C(t_n) = e^{-x_0t_n}m$ implies

$$\int_0^1 f_*(x)(f_*(x) - f_0(x))dx = f_*(x_0)m,$$

giving a formula for m,

$$m = \frac{1}{f_*(x_0)} \int_0^1 f_*(x)(f_*(x) - f_0(x))dx. \tag{7.9}$$

The constraint $f_*(x_0) - f_0(x_0) = \delta$ can then be incorporated into the optimality conditions by replacing $f_*(x_0)$ by $f_0(x_0) + \delta$ in (7.9), obtaining (7.6).

To see that (7.6) with the equality provision is sufficient for optimality, we integrate (7.6) with respect to σ_* and obtain $f_0(x_0) + \delta = f_*(x_0)$, taking (7.4) into account, unless

$$\int_0^1 f_*(x)(f_*(x) - f_0(x))dx = 0. \tag{7.10}$$

However, if (7.10) holds, then (7.6) reads $C(t) \geq 0$. Integrating this inequality with respect to σ_0, such that $f_0 = f_{\sigma_0}$, we obtain

$$\int_0^1 f_0(x)(f_*(x) - f_0(x))dx \geq 0. \tag{7.11}$$

Subtracting (7.11) from (7.10), we obtain $\|f_0-f_*\|\le 0$, which implies that $f_*=f_0$ and hence $\sigma_*=\sigma_0$. This shows that (7.6) implies $f_*(x_0)-f_0(x_0)=\delta$, provided $\sigma_*\ne\sigma_0$.

Now, if σ is any competitor measure, satisfying the constraint, then (7.3) becomes

$$\Delta J=2\left(\int_0^\infty C(t)d\sigma(t)-f_*(x_0)m\right)+\|f_{\Delta\sigma}\|_2^2,$$

where

$$m=\frac{1}{f_0(x_0)+\delta}\int_0^1 f_{\sigma_*}(x)(f_{\sigma_*}(x)-f_0(x))dx.$$

Discarding $\|f_{\Delta\sigma}\|_2^2$ and using inequality (7.6), we obtain

$$\Delta J\ge 2\left(m\int_0^\infty e^{-x_0t}d\sigma(t)-f_*(x_0)m\right)=2m(f_\sigma(x_0)-f_*(x_0))=0$$

since, due to the constraint, we must have $f_\sigma(x_0)=f_*(x_0)$ for any competitor measure. $\square$

To illustrate the optimality conditions, let us consider an example with $f_0(x)=e^{-x}$. In this case, the solutions of (7.2) can be computed explicitly. The forms of these solutions were, in fact, suggested by first solving these problems numerically with an algorithm based on formula (7.3). If $\delta>0$, then $f_*(x)=f_*^+(x)=a+be^{-x\tau}$ for appropriately chosen $a>0$, $b>0$, and $\tau>1$. If $\delta\in(-e^{-x_0},0)$, then $f_*(x)=f_*^-(x)=ae^{-x\tau}$ for appropriately chosen $a>0$ and $\tau>1$. If $\delta>0$, the optimality condition (7.6) gives equations $\widehat{C}(0)=\widehat{C}(\tau)=0$ and $\widehat{C}'(\tau)=0$, where $\widehat{C}(t)=C(t)-me^{-x_0t}$. Together with the constraint, $f_*(x_0)=f_0(x_0)+\delta$, this results in 4 equations for the 4 unknowns a,b,τ, and m. Similarly, if $\delta<0$, the optimality condition (7.6) gives equations $\widehat{C}(\tau)=0$ and $\widehat{C}'(\tau)=0$, which together with the constraint results in 3 equations for the 3 unknowns a,τ, and m. The resulting system of equations is linear in (a,b,m) for $\delta>0$ and in (a,m) for $\delta<0$, so that these parameters can be easily eliminated, leading to a single algebraic equation for τ. This equation is too complicated to display here, but it can be easily investigated, either numerically or by means of a computer algebra system, and shown to have a unique solution $\tau(x_0,\delta)$, for all $x_0\ge 1$ and $\delta\in(-e^{-x_0},0)$ if $\delta<0$, and $\delta\in(0,+\infty)$ if $\delta>0$. When ϵ is small, we find

$$M_\epsilon(x_0;e^{-x})=E_+(x_0)\epsilon+O(\epsilon^2),\quad m_\epsilon(x_0;e^{-x})=E_-(x_0)\epsilon+O(\epsilon^2),$$

where $E_+(x_0)$ is an increasing function of x_0 from

$$E_+(1)=\sqrt{-\frac{e^4-8e^3+14e^2+8e-19}{(e^2-2e-1)(3e^2-10e+5)}}\approx 2.67788263$$

to

$$E_+(\infty)=\sqrt{-\frac{e^2+2e-1}{3e^2-10e+5}}\approx 27.488747597.$$

The function $E_-(x_0)$ behaves in a more complicated manner. It increases from

$$E_-(1)=2\sqrt{\frac{e^2-1}{e^4-6e^2+1}}\approx 1.5$$

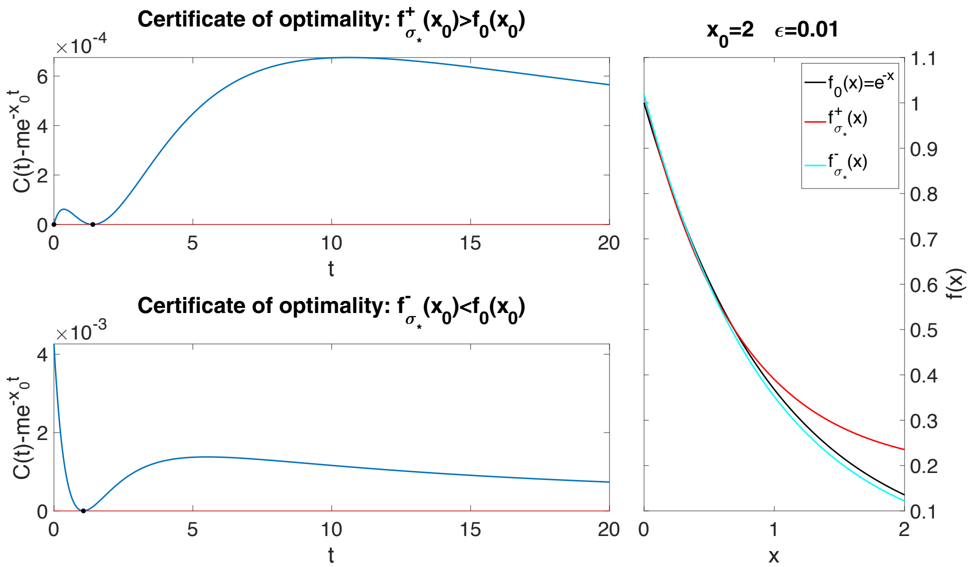

FIG. 1. *Solutions of the local worst case extrapolation problems* (7.2) *with* $f_0(x)=e^{-x}$, $x_0=2$, $\epsilon=0.01$, *and their respective certificates of optimality.*

to its maximal value $E_-((e+2)/(e+1))\approx 1.566$ and then decreases to 0 as x_0 increases from $(e+2)/(e+1)$ to $+\infty$, In fact, $\hat{E}_-(x_0)=x_0^{-1}e^{x_0}E_-(x_0)$ is a monotone increasing function from $eE_-(1)\approx 4.1$ to

$$\hat{E}_-(\infty)=2e\sqrt{\frac{2(e^2-1)}{(e^2-1)^2-4e^2}}\approx 5.8.$$

The plots of $f_*^{\pm}(x)$ and $f_0(x)$ together with their respective certificates of optimality $\widehat{C}(t)=C(t)-me^{-x_0t}$ are shown in Figure 1.

Appendix A. Kuhn–Tucker in topological vector spaces. Let X be a locally convex topological vector space. Let $\mathcal{F}\subset X^*\oplus\mathbb{R}$ be any subset. Define

$$K=\{x\in X: f(x)\le\alpha\ \forall(f,\alpha)\in\mathcal{F}\}. \tag{A.1}$$

Then $K\subset X$ is both closed and convex. Let $h\in X^*$ be a given functional. The maximization problem

$$m=\sup_{x\in K}h(x) \tag{A.2}$$

is called the linear programming problem. If the set K is empty the value of m is set to $-\infty$ by convention.

Let $\widehat{\mathcal{F}}$ denote the smallest closed (in weak-* topology of $X^*\oplus\mathbb{R}$) convex cone containing $\mathcal{F}$. We remark that

$$K=\{x\in X: f(x)\le\alpha\ \forall(f,\alpha)\in\widehat{\mathcal{F}}\}.$$

We also define

$$K^*=\{(f,\alpha)\in X^*\oplus\mathbb{R}: f(x)\le\alpha\ \forall x\in K\}.$$

Obviously, $\widehat{\mathcal{F}} \subset K^*$. It is easy to give an example where $K^* \neq \widehat{\mathcal{F}}$. Let $X = \mathbb{R}$ and $\mathcal{F} = \{(1,0)\}$, so that $K = \{x \in \mathbb{R} : x \le 0\}$ and $\widehat{\mathcal{F}} = \{(f,0) \in \mathbb{R}^2 : f \ge 0\}$. But

$$K^* = \{(f,\alpha) \in \mathbb{R}^2 : fx \le \alpha \ \forall x \le 0\} = \{(f,\alpha) \in \mathbb{R}^2 : f \ge 0,\ \alpha \ge 0\}.$$

Our goal is to obtain a dual formulation of (A.2). We observe that if $m < +\infty$, then $(h,m) \in K^*$, while $(h, m-\epsilon) \notin K^*$ for any $\epsilon > 0$. Thus, m is the smallest of the numbers α, such that $(h,\alpha) \in K^*$. For this reason, we introduce the following notation. For any subset $S \subset X^* \times \mathbb{R}$ and any $f \in X^*$, we define

$$S_f = \{\alpha \in \mathbb{R} : (f,\alpha) \in S\}.$$

Our remark can then be stated as follows: $m < +\infty$ if and only if $K_h^* \neq \emptyset$, in which case $m = \min K_h^*$. The dual set K^* is a maximal set of inequalities defining K, while the set $\widehat{\mathcal{F}} \subset K^*$ describes the weak-* closure of the set of inequalities obtained by positive linear combinations of finite subsets of inequalities in (A.1). The remarkable fact of the Kuhn–Tucker theorem is that even though $\widehat{\mathcal{F}}$ can be a lot smaller than K^*, as our example showed, it still contains all the bottom extremal points of K^*.

Theorem A.1. *Suppose that the set K, given by* (A.1)*, is nonempty. Let $\widehat{\mathcal{F}}$ be the smallest weak-* closed convex cone containing $\mathcal{F}$. Let m be given by* (A.2)*. Then*

$$m = \min \widehat{\mathcal{F}}_h, \tag{A.3}$$

where we have indicated that the minimum is achieved if $\widehat{\mathcal{F}}_h \neq \emptyset$.

We remark that requiring $K \neq \emptyset$ is essential. For example, we can take $X = \mathbb{R}^2$ and $\mathcal{F} = \{(\boldsymbol{e}_1, 0), (-\boldsymbol{e}_1, -1)\}$, corresponding to constraints $x_1 \le 0$ and $-x_1 \le -1$, which are inconsistent, so that $K = \emptyset$. We compute

$$\widehat{\mathcal{F}} = \{((\lambda_1 - \lambda_2)\boldsymbol{e}_1, -\lambda_2) : \lambda_1 \ge 0,\ \lambda_2 \ge 0\}.$$

For $h = \boldsymbol{e}_2$ the set of pairs $(\boldsymbol{e}_2, \alpha) \in \widehat{\mathcal{F}}$ is empty resulting in the minimum in (A.3) to be $+\infty$, while the supremum over the empty set is $-\infty$.

Proof. We have already observed that

$$\sup_{x \in K} h(x) < +\infty \quad \Longleftrightarrow \quad K_h^* \neq \emptyset.$$

Therefore, if $K_h^* = \emptyset$, then $\widehat{\mathcal{F}}_h = \emptyset$ since $\widehat{\mathcal{F}} \subset K^*$. Thus, if $m = +\infty$, then formula (A.3) is valid. It only remains to consider the case $m < +\infty$, whereby $(h,m) \in K^*$. The theorem below asserts that $(h,m) \in \widehat{\mathcal{F}}$, and therefore, that m has to be equal to the right-hand side of (A.3) since $(h, m-\epsilon) \notin K^*$ for every $\epsilon > 0$. ∎

Theorem A.2. *Under the assumptions of Theorem* A.1 *assume additionally that $m < +\infty$. Then $(h,m) \in \widehat{\mathcal{F}}$.*

Proof. If $(h,m) \notin \widehat{\mathcal{F}}$, then, by the Hahn–Banach convex separation theorem, there exist $\xi_0 \in X$, $\mu_0 \in \mathbb{R}$, $\gamma \in \mathbb{R}$, such that

$$h(\xi_0) + \mu_0 m < \gamma \le f(\xi_0) + \mu_0 \alpha \quad \forall (f,\alpha) \in \widehat{\mathcal{F}}. \tag{A.4}$$

Here, we used the fact that the set of all linear continuous functionals on X^*, equipped with its weak-* topology, is parametrized by X; i.e., for any $F \in (X^*, \text{weak-*})^*$ there exists a unique $x \in X$, such that $F(f) = f(x)$ for all $f \in X^*$.

We first observe that if there exists $(f_0,\alpha_0)\in\widehat{\mathcal{F}}$, such that $f_0(\xi_0)+\mu_0\alpha_0<0$, then the second inequality in (A.4) cannot hold since $(\lambda f_0,\lambda\alpha_0)\in\widehat{\mathcal{F}}$ for any $\lambda>0$. However, if $f_0(\xi_0)+\mu_0\alpha_0\geq 0$, then $\lambda f_0(\xi_0)+\mu_0\lambda\alpha_0$ can be made as close to 0 as one wishes. It follows that $\gamma=0$. We thus restate (A.4) in a more convenient form:

$$h(\xi_0)+\mu_0 m<0,\qquad f(\xi_0)+\mu_0\alpha\geq 0\quad\forall(f,\alpha)\in\widehat{\mathcal{F}}. \tag{A.5}$$

We need to consider 3 possibilities for μ_0.

1. $\mu_0>0$. In this case

$$f\left(-\frac{\xi_0}{\mu_0}\right)\leq\alpha\quad\forall(f,\alpha)\in\widehat{\mathcal{F}},$$

which implies that $-\xi_0/\mu_0\in K$. But then, according to the first inequality in (A.5),

$$h\left(-\frac{\xi_0}{\mu_0}\right)>m,$$

which contradicts the definition (A.2) of m.

2. $\mu_0=0$. Since $K\neq\emptyset$, there exists $u\in K$. But then for any $\lambda\geq 0$, we have

$$f(u-\lambda\xi_0)\leq\alpha\quad\forall(f,\alpha)\in\widehat{\mathcal{F}}.$$

This implies that $u-\lambda\xi_0\in K$. But $h(u-\lambda\xi_0)=h(u)-\lambda h(\xi_0)$, which can be made arbitrarily large and positive by a choice of $\lambda>0$ since $h(\xi_0)<0$. This contradicts the assumption that $m<+\infty$.

3. $\mu_0<0$. For convenience of working with positive numbers, we set $\mu_0=-\nu_0$, and $\nu_0>0$. In that case, we have $f(\xi_0)\geq\nu_0\alpha$ for every $(f,\alpha)\in\widehat{\mathcal{F}}$. Then for every $x\in K$, we have for any $t>0$

$$f(x-t\xi_0)\leq(1-t\nu_0)\alpha.$$

Thus, for all $x\in K$ and $t\in(0,1/\nu_0)$, we conclude that

$$y(x,t)=\frac{x-t\xi_0}{1-t\nu_0}\in K.$$

We will get a contradiction by showing that

$$\sup_{\substack{x\in K\\ 0<t<\nu_0^{-1}}} h(y(x,t))>m.$$

We compute

$$h(y(x,t))=m+\frac{h(x)-m-t(h(\xi_0)-\nu_0 m)}{1-t\nu_0}.$$

By definition of the supremum there exist $x_0\in K$, such that

$$h(x_0)>m+\frac{h(\xi_0)-\nu_0 m}{2\nu_0}$$

since $h(\xi_0)-\nu_0 m<0$. But then $h(y(x_0,(2\nu_0)^{-1}))>m$.

The obtained contradictions imply that $(h,m)\in\widehat{\mathcal{F}}$, establishing (A.3). ∎

Appendix B. Asymptotics of $u(z;\mu)$ for large μ. To compute the asymptotics of $u(z;\mu)$, as $\mu\to+\infty$ for $z\in\Omega=\{z\in\mathbb{C}:\Re z>0,\ z\notin[0,1]\}$, we first apply the Pfapf transformation [13, formula (1.8)] and obtain

$$u(z;\mu)=\frac{1}{z}F\left(\left[\frac{1}{4}+\frac{i\mu}{2},\frac{1}{4}-\frac{i\mu}{2}\right],[1];1-\frac{1}{z^2}\right).$$

We note that $g(z)=1-z^{-2}$ maps Ω into $\widehat{\Omega}=\mathbb{C}\setminus\{w\in\mathbb{R}:w(w-1)\ge 0\}$, to which the asymptotic expansion from [13, Theorem 3.2] applies. Substituting our parameters into the expansion [13, (3.8)–(3.11)] and retaining only the leading term ($n=1$ in the expansion), we obtain

$$\pi iF\left(\left[\frac{1}{4}+\frac{i\mu}{2},\frac{1}{4}-\frac{i\mu}{2}\right],[1];1-\frac{1}{z^2}\right)\sim \left(\frac{\xi}{2}\right)^{\frac{1}{2}}\left(e^{\left(\frac{i\mu}{2}-\frac{1}{4}\right)\pi i}K_{-\frac{1}{2}}\left(-\frac{i\xi\mu}{2}\right)-e^{\left(\frac{3}{4}-\frac{i\mu}{2}\right)\pi i}K_{-\frac{1}{2}}\left(\frac{i\xi\mu}{2}\right)\right)c_0+O(\Phi_1(\mu,\xi)),$$

where

$$\xi=\ln\left(1-\frac{2}{z^2}-2i\sqrt{\left(1-\frac{1}{z^2}\right)\frac{1}{z^2}}\right),\qquad c_0=-\frac{\sqrt{z}}{(z^2-1)^{1/4}}.$$

$$\Phi_1(\mu,\xi)=e^{-\frac{\pi\mu}{2}}\frac{\sqrt{|\xi|}}{\mu}\left|K_{-\frac{1}{2}}\left(-\frac{i\mu\xi}{2}\right)\right|+e^{\frac{\pi\mu}{2}}\frac{\sqrt{|\xi|}}{\mu}\left|K_{-\frac{1}{2}}\left(\frac{i\mu\xi}{2}\right)\right| +\frac{e^{-\frac{\pi\mu}{2}}}{\sqrt{|\xi|}\mu}\left|K_{\frac{1}{2}}\left(-\frac{i\mu\xi}{2}\right)\right|+\frac{e^{\frac{\pi\mu}{2}}}{\sqrt{|\xi|}\mu}\left|K_{\frac{1}{2}}\left(\frac{i\mu\xi}{2}\right)\right|.$$

Here the transformation $\zeta=1-2z^{-2}$ maps

$$\Omega=\{z\in\mathbb{C}:\Re z>0,\ z\notin[0,1]\}$$

onto

$$G=\mathbb{C}\setminus\{\zeta\in\mathbb{R}:|\zeta|\ge 1\}.$$

Then,

$$\xi=\ln(\zeta-i\sqrt{1-\zeta^2}).$$

We observe that $\zeta=\cosh\xi$, and therefore, $\xi(\zeta)$ is injective on G. Thus, $\partial_\infty\xi(G)=\xi(\partial_\infty G)$. Computing the images of $(-\infty,-1]\pm 0i$ and $[1,+\infty)\pm 0i$ and noting that $\xi(\infty)=\infty$, we conclude that $\xi(\zeta)$ maps G onto the strip $-\pi<\Im\xi<0$ bijectively. We also note that

$$\cosh\xi=\zeta=1-\frac{1}{z^2},$$

which implies that

$$\frac{1}{z^2}=-\sinh^2\left(\frac{\xi}{2}\right).$$

Since $z\in\Omega$ lies in the right half-plane, while $\Im\xi\in(-\pi,0)$, we conclude that

$$\frac{1}{z}=i\sinh\left(\frac{\xi}{2}\right)=\sin\left(\frac{i\xi}{2}\right).$$

We can write this as

$$\cos\left(\frac{\pi}{2}-\frac{i\xi}{2}\right)=\frac{1}{z},\quad -\pi<\Im\xi<0.$$

Since $\eta=\pi/2-i\xi/2$ maps the strip $-\pi<\Im\xi<0$ onto the strip $\Re\eta\in(0,\pi/2)$, we conclude that $\eta=\alpha(z)$, where $\alpha(z)$ was defined in (4.28) and, thus,

$$\xi=i(2\alpha(z)-\pi). \tag{B.1}$$

Using (B.1) and the formulas

$$K_{\frac{1}{2}}(z)=K_{-\frac{1}{2}}(z)=\sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}},$$

we obtain the error estimate

$$O(\Phi_1(\mu,\xi))=O\left(\frac{e^{\frac{\pi\mu}{2}(1+\Im\xi/\pi)}}{\mu\sqrt{\mu}}\right)=O\left(\frac{|e^{\pi\mu\alpha(z)}|}{\mu\sqrt{\mu}}\right).$$

Since $|\Im\xi|<\pi$, we conclude that the term $e^{\left(\frac{i\mu}{2}-\frac{1}{4}\right)\pi i}K_{-\frac{1}{2}}(-\frac{i\xi\mu}{2})$ is negligible compared to $e^{\left(\frac{3}{4}-\frac{i\mu}{2}\right)\pi i}K_{-\frac{1}{2}}(\frac{i\xi\mu}{2})$. Therefore, we obtain the asymptotics

$$u(z;\mu)\sim\frac{e^{i\pi/4}}{\sqrt{2\pi\mu}}\frac{e^{\frac{\pi\mu}{2}\left(1-\frac{i\xi}{\pi}\right)}}{(z^2-1)^{1/4}\sqrt{z}}\frac{\sqrt{\xi}}{\sqrt{i\xi}}+O\left(\frac{|e^{\pi\mu\alpha(z)}|}{\mu\sqrt{\mu}}\right).$$

Since $-\pi<\Im\xi<0$, we conclude that

$$\frac{\sqrt{\xi}}{\sqrt{i\xi}}=e^{-\frac{i\pi}{4}}.$$

Thus, for all $z\in\Omega$,

$$u(z;\mu)=\frac{1}{\sqrt{2\pi\mu}}\frac{e^{\pi\mu\alpha(z)}}{(z^2-1)^{1/4}\sqrt{z}}+O\left(\frac{|e^{\pi\mu\alpha(z)}|}{\mu\sqrt{\mu}}\right). \tag{B.2}$$

Appendix C. Estimate of $\|\phi_\epsilon\|_{\mathfrak{H}_p}$. The goal of this section is to prove the lower bound (6.3) on $\|\phi_\epsilon\|_{\mathfrak{H}_p}$. When $x_0>1$, part (i) of Theorem 4.4 can be used to estimate $\|\psi_\varepsilon\|_{\mathfrak{H}_p}$ from below. If $x_0=1$,

$$\psi_\varepsilon(z)=\int_0^\infty\frac{u(z;\mu)\mu\tanh(\pi\mu)}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}d\mu. \tag{C.1}$$

Its asymptotics as $\varepsilon\to0^+$ is given by the following theorem.

THEOREM C.1. *Let $z\in\Omega=\{z\in\mathbb{C}:\Re z>0,\ z\notin[0,1]\}$ and ψ_ε be the solution of the integral equation* (4.5) *with $x_0=1$. Then*

$$\psi_\varepsilon(z)\sim\frac{R(z)\sqrt{|\ln\hat{\varepsilon}|}}{\pi\sin(\alpha(z))}\hat{\varepsilon}^{\frac{-2\alpha(z)}{\pi}},\quad \hat{\varepsilon}=\frac{\varepsilon}{\sqrt{2\pi}}, \tag{C.2}$$

where $R(z)$ and $\alpha(z)$ are defined as in (4.28).

Proof. As we have argued before, the asymptotics of $\psi_\varepsilon(z)$ is determined by the asymptotics of the integrand in (C.1), as $\mu\to\infty$. Thus, we would want to replace $u(z;\mu)$ by its asymptotics (4.27), $\tanh(\pi\mu)$ by 1, and $2\cosh(\pi\mu)$ by $e^{\pi\mu}$. We therefore, rewrite (C.1) as

$$\psi_\varepsilon(z)=\frac{R(z)}{\sqrt{2\pi}}\int_0^\infty\frac{\sqrt{\mu}e^{\alpha(z)\mu}v(z;\mu)}{2\hat{\varepsilon}^2\cosh(\pi\mu)+1}d\mu, \tag{C.3}$$

where

$$v(z;\mu)=\frac{u(z;\mu)}{u_0(z;\mu)}\tanh(\pi\mu),$$

and where $u_0(z;\mu)$ is given by (4.30). Then, $v(z;\cdot)\in C([0,\infty))$, due to the representation (4.31), as argued in the proof of Theorem 4.4, and $v(z;\mu)\to 1$, as $\mu\to\infty$, by Lemma 4.2. Thus, there exists $M(z)>0$, such that $|v(z;\mu)|\le M(z)$ for any $z\in\Omega$.

Let $I(\hat{\varepsilon})$ denote the integral in (C.3). Changing variables $\mu'=\pi\mu+2\ln(\hat{\varepsilon})$, we obtain

$$I(\hat{\varepsilon})=\frac{1}{\pi}\sqrt{\frac{-2\ln\hat{\varepsilon}}{\pi}}\,\hat{\varepsilon}^{\frac{-2\alpha(z)}{\pi}}\int_{2\ln\hat{\varepsilon}}^\infty\sqrt{\frac{\mu'-2\ln\hat{\varepsilon}}{-2\ln\hat{\varepsilon}}}\left(\frac{e^{\frac{\alpha(z)\mu'}{\pi}}v(z;\frac{\mu'}{\pi}-\frac{2\ln\hat{\varepsilon}}{\pi})}{e^{\mu'}+e^{-\mu'+4\ln\hat{\varepsilon}}+1}\right)d\mu'.$$

The estimate

$$\left|\sqrt{\frac{\mu'-2\ln\hat{\varepsilon}}{-2\ln\hat{\varepsilon}}}\left(\frac{e^{\frac{\alpha(z)\mu'}{\pi}}v(z;\frac{\mu'}{\pi}-\frac{2\ln\hat{\varepsilon}}{\pi})}{e^{\mu'}+e^{-\mu'+4\ln\hat{\varepsilon}}+1}\right)\right|\chi_{(2\ln\hat{\varepsilon},\infty)}(\mu')\le\Phi(\mu'),$$

where $\Phi(\mu')$ is given by

$$\Phi(\mu')=\begin{cases}M(z)e^{\left(\frac{\Re\alpha(z)}{\pi}-1\right)\mu'}, & \mu'<0,\\ M(z)\sqrt{\mu'+1}e^{\left(\frac{\Re\alpha(z)}{\pi}-1\right)\mu'}, & \mu'>0,\end{cases}$$

shows that the Lebesgue dominated convergence theorem is applicable since $\Re\alpha(z)\in(0,\pi/2)$, by Lemma 4.5. Therefore,

$$\psi_\varepsilon(z)\sim\frac{R(z)}{\pi^2}\sqrt{|\ln\hat{\varepsilon}|}\,\hat{\varepsilon}^{\frac{-2\alpha(z)}{\pi}}\int_{\mathbb{R}}\frac{e^{\frac{\alpha(z)\mu'}{\pi}}}{e^{\mu'}+1}d\mu'=\frac{R(z)\sqrt{|\ln\hat{\varepsilon}|}}{\pi\sin(\alpha(z))}\hat{\varepsilon}^{\frac{-2\alpha(z)}{\pi}}.$$

The theorem is proved. ∎

In order to estimate $\|\psi_\epsilon\|_{\mathfrak{H}_p}$ (for any $x_0\ge 1$), we need a tighter bound on $\Re\alpha((iy)^{1/p})$, when $y>0$ and $p>1$, which becomes optimal as $p\to 1^+$. Formula (4.28) shows that $\Re\alpha(iy+0)=\pi/2$ for any $y>0$. In fact, we have the following estimate.

LEMMA C.2. *Let $y>0$ and $p>1$. Then $\Re\alpha((iy)^{1/p})\in(\frac{\pi}{2p},\frac{\pi}{2})$.*

Proof. We first observe that for any $z\in\Omega$

$$\alpha(z)=-i\ln z+i\ln(1-i\sqrt{z^2-1}).$$

Indeed, it is easy to see that the right-hand side of the above formula is analytic in Ω and agrees with $\arccos(1/z)$ for $z>1$. The same is true for the left-hand side.

Therefore, they must agree everywhere in Ω. If $z=(iy)^{1/p}$, then $z^2=re^{i\theta_p}$, where $\theta_p=\pi/p\in(0,\pi)$, and $r>0$. It is now easy to see that $\arg(z^2-1)$, as a function of r, decreases from π at $r=0$ to θ_p at $r=+\infty$. Hence, $\arg(-i\sqrt{z^2-1})$ decreases from 0 at $r=0$ to $\theta_p/2-\pi/2$ at $r=+\infty$. Therefore, $\arg(1-i\sqrt{z^2-1})$ will also be between 0 and $\theta_p/2-\pi/2$. Thus,

$$\Re\alpha(z)=\frac{\theta_p}{2}-\arg(1-i\sqrt{z^2-1})\in\left(\frac{\theta_p}{2},\frac{\pi}{2}\right). \qquad \square$$

THEOREM C.3. *For $x_0\geq 1$ and $p>1$, there is a constant $s_p(x_0)>0$ such that*

$$\|\psi_\varepsilon\|_{\mathfrak{H}_p}\geq s_p(x_0)\begin{cases}\varepsilon^{-\frac{2\alpha(x_0)}{\pi}-\frac{1}{p}}, & x_0>1,\\ \varepsilon^{-\frac{1}{p}}\sqrt{|\ln\varepsilon|}, & x_0=1,\end{cases}$$

for all sufficiently small $\varepsilon>0$.

Proof. Let

$$\psi_\varepsilon^{x_0}(z)=\begin{cases}\dfrac{R(x_0)R(z)}{2\pi\sin(\pi\beta(z))}\hat{\varepsilon}^{-2\beta(z)}, & x_0>1,\\ \dfrac{R(z)\sqrt{|\ln\hat{\varepsilon}|}}{\pi\sin(\alpha(z))}\hat{\varepsilon}^{\frac{-2\alpha(z)}{\pi}}, & x_0=1,\end{cases}\qquad \beta(z)=\frac{\alpha(x_0)+\alpha(z)}{\pi}.$$

Then, Theorems 4.4(i) and C.1 say that $\psi_\varepsilon(z)\sim\psi_\varepsilon^{x_0}(z)$ for any $z\in\Omega$ and any $x_0\geq 1$. We then write

$$\|\psi_\varepsilon\|^2_{\mathfrak{H}_p}=\frac{1}{\pi}\int_0^\infty\frac{|\psi_\varepsilon((iy)^{1/p})|^2}{N_p(y)|\psi_\varepsilon^{x_0}((iy)^{1/p})|^2}|\psi_\varepsilon^{x_0}((iy)^{1/p})|^2dy,$$

where $N_p(y)=y^{\frac{p-1}{p}}|1+(iy)^{1/p}|^2$. By Lemma C.2, we estimate

$$|\psi_\varepsilon^{x_0}((iy)^{1/p})|\geq A_p(x_0,y)K_p^{x_0}(\varepsilon), \tag{C.4}$$

where

$$A_p(x_0,y)=\begin{cases}\dfrac{R(x_0)|R((iy)^{1/p})|(2\pi)^{\frac{\alpha(x_0)}{\pi}+\frac{1}{2p}}}{2\pi|\sin(\pi\beta((iy)^{1/p}))|}, & x_0>1,\\ \dfrac{|R((iy)^{1/p})|(2\pi)^{\frac{1}{2p}}}{\pi|\sin(\pi\alpha((iy)^{1/p}))|}, & x_0=1,\end{cases}$$

$$K_p^{x_0}(\varepsilon)=\begin{cases}\varepsilon^{-\frac{2\alpha(x_0)}{\pi}-\frac{1}{p}}, & x_0>1,\\ \varepsilon^{-\frac{1}{p}}\sqrt{|\ln\varepsilon|}, & x_0=1.\end{cases}$$

Thus, we obtain the lower bound

$$\|\psi_\varepsilon\|^2_{\mathfrak{H}_p}\geq\frac{K_p^{x_0}(\varepsilon)^2}{\pi}\int_0^\infty\frac{|\psi_\varepsilon((iy)^{1/p})|^2}{N_p(y)|\psi_\varepsilon^{x_0}((iy)^{1/p})|^2}A_p(x_0,y)^2dy.$$

Now, by Fatou's lemma, we have, taking into account $\psi_\varepsilon(z)\sim\psi_\varepsilon^{x_0}(z)$, as $\varepsilon\to 0^+$,

$$\varliminf_{\varepsilon\to 0}\frac{\|\psi_\varepsilon\|^2_{\mathfrak{H}_p}}{K_p^{x_0}(\varepsilon)^2}\geq\frac{1}{\pi}\int_0^\infty\frac{A_p(x_0,y)^2}{N_p(y)}dy=:2s_p(x_0)^2>0.$$

It follows that for all sufficiently small $\varepsilon>0$, we have $\|\psi_\varepsilon\|_{\mathfrak{H}_p}\geq s_p(x_0)K_p^{x_0}(\varepsilon)$. $\square$

We now have everything we need to prove Theorem 6.1.

Proof of Theorem 6.1. We have, using (4.13),

$$\|\phi_\epsilon\|_{\mathfrak{H}_p} = \frac{\epsilon\|\psi_\varepsilon\|_{\mathfrak{H}_p}}{\|\psi_\varepsilon\|_2} = \frac{\|\psi_\varepsilon\|_{\mathfrak{H}_p}}{\|\psi_\varepsilon\|}. \tag{C.5}$$

It only remains to observe that Theorems 4.4(iii) and 4.6(ii) can be written as

$$\|\psi_\varepsilon\| \sim C_0(x_0) K_p^{x_0}(\varepsilon)\varepsilon^{\frac{1}{p}-1},$$

where

$$C_0(x_0) = \begin{cases} \dfrac{(2\pi)^{\beta(x_0)/2}}{\pi}\sqrt{\dfrac{x_0 \arccos(1/x_0)}{2(x_0^2-1)}}, & x_0 > 1, \\ \dfrac{\sqrt{2}}{\pi}, & x_0 = 1. \end{cases}$$

Combining this with Theorem C.3 and applying it to (C.5), we obtain that

$$\|\phi_\epsilon\|_{\mathfrak{H}_p} \geq \frac{s_p(x_0)}{2C(x_0)}\varepsilon^{1-\frac{1}{p}}$$

for all sufficiently small $\varepsilon > 0$. Theorem 6.1 is now proved. ∎

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