

Control of a Multiplicative Observation Noise System: MAP-based Achievable Strategy

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Abstract—This paper considers the stabilization of a discrete-time linear system in the presence of multiplicative observation noise, i.e., when the controller observes the state multiplied by a random variable with a continuous density. In the case where this multiplicative observation noise has zero mean, the controller does not even have access to the sign of the state. Given this lack of sign information, a linear memoryless control strategy that optimizes the second moment would do nothing, i.e., would choose the control to be equal to zero. It is known that non-linear strategies can unboundedly outperform linear strategies for this system, yet the optimal strategy for this simple problem remains unknown.

This paper provides a new achievable scheme based on the maximum a-posteriori (MAP) estimation of the state that provably stabilizes the system in *any* moment sense. Additionally, we can compute an explicit convergence rate for the system state. This MAP-controller emerges from a study of the evolution of the conditional density of the state. These densities illustrate the dual nature of the MAP-controller: it extracts information about the system while also driving the state towards zero. Simulations show that the MAP-controller outperforms neural-network-based control strategies as well as the previously best-known non-linear strategies.

I. INTRODUCTION

Multiplicative noise in systems can arise in a variety of settings, in particular, in systems with timing jitter or phase-locked loops [1], voltage-controlled resistors, pulse-width modulation [2], linearization [3], or in physical systems where model parameters are dependent on changing quantities [4]. Multiplicative noise is also thought to be a helpful lens through which to view the performance of stochastic gradient descent and related algorithms [5]–[7]. Multiplicative noise has the interesting property that if the mean of the multiplicative noise is close to zero, it can destroy all sign information in the signal, which can make it challenging to understand using a standard linear perspective. This paper considers the η -moment stabilization and rate of convergence of a simple linear system with multiplicative observation noise as below:

$$\begin{cases} X_{n+1} = aX_n + U_n \\ Y_n = C_n X_n. \end{cases} \quad (1)$$

The initial state X_0 has density $f_X(\cdot)$, and is independent of all other problem parameters. Similarly, the C_n 's are continuous i.i.d. random variables with density $f_C(\cdot)$. We are interested in identifying the largest growth factor $a \in \mathbb{R}$ such that we can maintain $\mathbb{E}[|X_n|^\eta] < \infty$ for all n . We are particularly interested in the case where the C_n have zero mean. In this

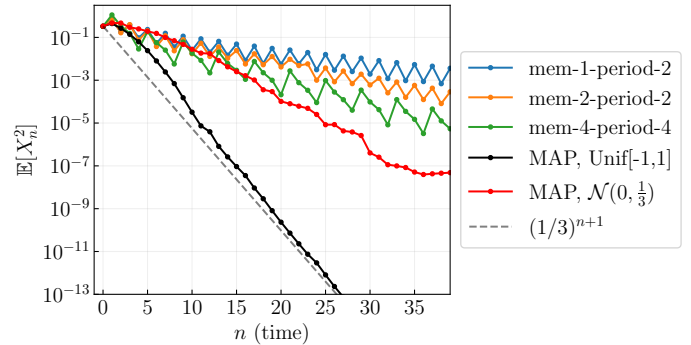


Fig. 1. This figure plots the empirical second moment of the state under MAP-control with $\text{Unif}[-1, 1]$ (black) and $\mathcal{N}(0, \frac{1}{3})$ (red) distributions on each of C_n and X_0 . The neural-network comparison strategies follow those in [9] and are for the uniform noise case. The second moments were computed with 10^4 trajectories for the neural-network and Gaussian MAP strategies, and 10^6 for the uniform MAP strategy.

case, a linear memoryless strategy ($U_n = d \cdot Y_n$ for some $d \in \mathbb{R}$) that optimizes the second moment (or takes a certainty equivalence perspective) would choose $U_n = 0$, and thus could not stabilize the system for any growth $a > 1$ as seen in [8].

[8] showed that a non-linear strategy can unboundedly outperform linear strategies for this problem. However, this was only a proof-of-concept, and an explicit calculation of the largest a that could be stabilized was not provided for this strategy. The non-linear strategy in [8] had a periodic structure — at even times the controller takes a probing action to gain information about the system state, and at odd times it uses that information to control, and explicitly used the dual nature of control [10]. The work in [9] built on this observation and showed that neural networks could be trained to obtain non-linear, periodic strategies that empirically performed better than the strategy from [8]. The periodic strategies in [9] seemed to use certain time-steps for information gain and others for control. However, there were no provable guarantees.

In order to further explore the dual nature of control [10] in this problem, we calculate the conditional probability distributions of the state conditioned on the observations. This allows us to understand the information gain possible from certain control actions. This leads to a MAP-control strategy where we can calculate the maximum stabilizable growth rate a for any moment of stability. We show in Fig. 1 the empirical decay rate of the MAP-based strategy in a second-moment sense for

a system with $a = 1$, compared to that of the neural-network strategies from [9].

A. Related work

Control problems involving multiplicative noise have been long studied by the community. However, they have been challenging to deal with and hence are not yet fully resolved. For instance, early work focused on studying linear strategies for systems with multiplicative observation noise [11]–[13]. More recent work also focuses on LTI strategies for the problem [14], [15]. The past few years have seen a resurgence of interest in control systems with multiplicative noise [16]–[23], where more modern techniques (e.g. policy gradient) have been explored to identify control strategies. However, the optimal strategy for the multiplicative observation noise problem remains open.

The simple system in (1) is particularly interesting from an information-theoretic perspective, as it considers the control of a system over a non-coherent channel [24], with a capacity that grows as $\log \log \text{SNR}$. The traditional approach to control over rate-limited channels leads to the work on data-rate theorems [25]–[37]. However, the data-rate theorems or their style of analysis cannot apply here, since the rate of the channel is potentially unbounded. This is because the magnitude of X_n is not bounded. Finally, it is important to note the contrast of our continuous, multiplicative noise system with the system considered in [38], which considers discrete Bernoulli multiplicative observation noise, and provides a provably optimal strategy in the second-moment sense.

A related system is one with multiplicative actuation noise [39] ($X_{n+1} = X_n + B_n U_n$ and $Y_n = X_n$, where the B_n are i.i.d. random variables). For mean-zero B_n , the largest a that can be stabilized in a second-moment sense is $a = 1$. However, we can stabilize $a > 1$ in the case of multiplicative observation noise in (1).

The control problem considered here is closely related to the problem of state estimation over multiplicative noise [40]–[42]. The early work [41], [43] focused only on linear estimation strategies, and it was shown in [42] that the second-moment of the estimation error must diverge for the system in (1) for $a > 1$ with linear and non-linear estimation strategies in the absence of control. The current MAP-based strategy reframes the problem as the estimation of X_0 . However, it is important to note that the control plays a fundamental role in allowing a successful estimation strategy for X_0 .

B. Main contributions

This paper provides a novel MAP-based strategy that is interpretable through the evolution of the conditional densities of the state given the observation as discussed in Sec. III. We prove that this strategy provably converges in Sec. IV and V. Simulations show that this strategy empirically outperforms learned neural-network strategies in Sec. VI.

II. PROBLEM STATEMENT

We specifically focus on the system in (1) where $a = 1$, and we focus on the maximum decay rate of the system. In

particular, we consider the scalar system with state $X_n \in \mathbb{R}$ at time $n \in \mathbb{Z}_{\geq 0}$, control U_n , observation Y_n , and multiplicative noise C_n . The system is defined as follows in (2).

$$\begin{cases} X_{n+1} = X_n + U_n, & X_0 \sim \text{Unif}[-1, 1] \\ Y_n = C_n X_n, & C_n \sim \text{Unif}[-1, 1]. \end{cases} \quad (2)$$

The control at a time n , i.e. U_n , is chosen causally as a function of the observations $Y_0^n := (Y_0, Y_1, \dots, Y_n)$. The distributions of X_0 and the C_n are known to the controller, and focus on the case where they are both drawn from a uniform distribution on $[-1, 1]$. The realizations of X_0 and the C_n 's are unknown to the controller. Our goal is to identify control strategies that can stabilize the system in an η -moment sense (i.e. strategies that achieve $\lim_{n \rightarrow \infty} \mathbb{E}[|X_n|^\eta] = 0$), and identify the rate of convergence of the η -moment, i.e. $r = \liminf_{n \rightarrow \infty} -\frac{1}{n\eta} \log \mathbb{E}[|X_n|^\eta / |X_0|^\eta]$. The largest growth rate $\log a$ that can be stabilized by a given strategy in an η -moment sense in system (1) is then given by r , as in [39].

Let $\mathcal{H}_n$ be the σ -algebra generated by all observations Y_0^n . Note that this represents information that is always available to the controller, including the controls U_0^n .

III. MAP-CONTROL STRATEGY

Our proposed strategy is simple (in hindsight): at every time step, choose the control U_n to be the negative of the maximum a-posteriori (MAP) estimator of the state given all the observations. The analysis of the strategy is easier if we consider the repeated estimation of the initial state X_0 instead of estimating the current state X_n . Hence, we define

$$\hat{X}_{0|n}^{\text{MAP}} := \arg \max_x f_X(X_0 = x | \mathcal{H}_n). \quad (3)$$

With this, we define the MAP-control strategy as the difference between the previous and current estimate of X_0 .

Definition 3.1 (MAP-control): For the system in (2) define the sequence of MAP-controls as

$$U_0 = -\hat{X}_{0|0}^{\text{MAP}} \quad (4)$$

$$U_n = \hat{X}_{0|n-1}^{\text{MAP}} - \hat{X}_{0|n}^{\text{MAP}}, n \geq 1. \quad (5)$$

Note that under this strategy $\sum_{i=0}^n U_i = -\hat{X}_{0|n}^{\text{MAP}}$. We can also show in a straightforward manner that $U_n = -\hat{X}_{n|n}^{\text{MAP}}$, where $\hat{X}_{n|n}^{\text{MAP}} := \arg \max_x f_X(X_n = x | \mathcal{H}_n)$.

Under this strategy, the system state X_n is always equal to the difference between X_0 and the current MAP estimate.

Lemma 3.2: For the system in (2) with MAP-control, the state for all $n \geq 1$ is:

$$X_n = X_0 - \hat{X}_{0|n-1}^{\text{MAP}}. \quad (6)$$

Lemma 3.3 (Conditional Density of $X_0 | \mathcal{H}_n$): For the system in (2), with controls u_0^n that are functions of the observations y_0^n , the conditional density of X_0 given $\mathcal{H}_n$ for $n \geq 1$ is

$$f_X(X_0 = x | \mathcal{H}_n) = \frac{f_X(x)}{f_{Y_0^n}(y_0^n)} \left(\prod_{j=0}^n \frac{f_C\left(\frac{y_j}{x + \sum_{i=0}^{j-1} u_i}\right)}{|x + \sum_{i=0}^{j-1} u_i|} \right), \quad (7)$$

where $f_Y(Y_0^n = y_0^n \mid X_0 = x) = \prod_{j=0}^n \frac{f_C\left(\frac{y_j}{x + \sum_{i=0}^{j-1} u_i}\right)}{|x + \sum_{i=0}^{j-1} u_i|}$. Additionally, the density of X_0 given $\mathcal{H}_0$ is

$$f_X(X_0 = x \mid \mathcal{H}_0) = \frac{f_X(x)}{f_{Y_0}(y_0)} \left(\frac{f_C\left(\frac{y_0}{x}\right)}{|x|} \right). \quad (8)$$

In our setting $f_X(\cdot)$ and $f_C(\cdot)$ are both given by $f(x) = \frac{1}{2} \mathbb{1}\{x : |x| \leq 1\}$, where $\mathbb{1}\{S\}$ is the indicator function for the set S . Now, the MAP estimator at the first time step follows immediately from Lemma 3.3.

Lemma 3.4: The MAP estimate of X_0 given Y_0 is $\hat{X}_{0|0}^{\text{MAP}} = \pm|Y_0|$.

Even beyond the first time step, the MAP-control strategy does not care about the sign of the observation Y_n and only uses $|Y_n|$. This is because Y_n enters the conditional density of $f_X(X_0 = x \mid \mathcal{H}_n)$ through f_C , which is even (see Lemma 3.3).

When MAP-control is applied we can limit our attention to at most two candidate values for $\hat{X}_{0|n}^{\text{MAP}}$ (i.e. candidate maximizers for the density in Lemma 3.3). One is positive and one is negative, and we denote these by $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ respectively. These candidates update with each new observation.

Definition 3.5 (Positive and Negative $\hat{X}_{0|n}^{\text{MAP}}$ Candidates): We define the positive candidate for the MAP estimator as

$$\hat{X}_{0|n}^+ = \arg \max_{x \geq 0} f_Y(Y_0^n = y_0^n \mid X_0 = x) \quad (9)$$

i.e. the value $x \geq 0$ that maximizes the conditional density of Y_0^n given X_0 . Similarly, we define the negative candidate as

$$\hat{X}_{0|n}^- = \arg \max_{x \leq 0} f_Y(Y_0^n = y_0^n \mid X_0 = x). \quad (10)$$

The next lemma shows that the MAP estimator must be equal to one of $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ and provides update rules.

Lemma 3.6 (Update rule for $\hat{X}_{0|n}^+$, $\hat{X}_{0|n}^-$): At each time n , $\hat{X}_{0|n}^{\text{MAP}}$ must be one of the two values $\hat{X}_{0|n}^+$ or $\hat{X}_{0|n}^-$. Furthermore,

$$\hat{X}_{0|n+1}^+ = \max\left(\hat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|, \hat{X}_{0|n}^+\right). \quad (11)$$

If $\hat{X}_{0|K}^+ > 1$, then $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^-, \forall n \geq K$. Similarly,

$$\hat{X}_{0|n+1}^- = \min\left(\hat{X}_{0|n}^{\text{MAP}} - |Y_{n+1}|, \hat{X}_{0|n}^-\right). \quad (12)$$

If $\hat{X}_{0|K}^- < -1$ then $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^+, \forall n \geq K$.

Example: To clarify the significance of $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ when implementing MAP-control, we consider an example of how $f_X(X_0 = x \mid \mathcal{H}_n)$ and $\hat{X}_{0|n}^{\text{MAP}}$ evolve at times 0 and time 1. This is illustrated in Fig. 2, which shows the densities $f_X(x)$, $f_X(X_0 = x \mid \mathcal{H}_0)$, and $f_X(X_0 = x \mid \mathcal{H}_1)$. The prior density $f_X(x)$ is uniform on $[-1, 1]$. On observing $Y_0 = y_0$ the conditional density $f_X(X_0 = x \mid \mathcal{H}_0)$ is given by (8), which is even and has support $[-1, -|y_0|] \cup [|y_0|, 1]$. This is because we must have that $|Y_0| = |C_0 \cdot X_0| \leq |X_0|$ and hence the density contains a product of $\mathbb{1}\{x : |x| \leq 1\}$ and $\mathbb{1}\{x : |y_0| \leq |x|\}$. This density has peaks at $\pm|y_0|$, and

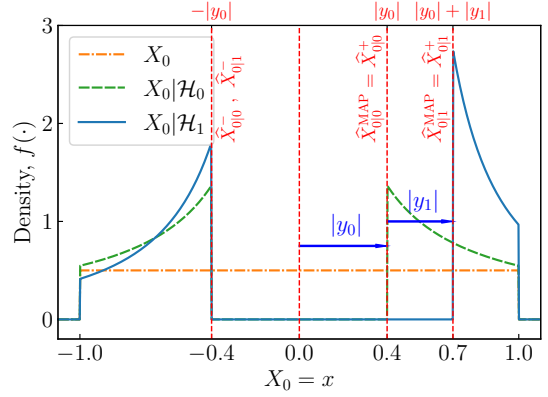


Fig. 2. Example evolution of densities from that of X_0 to $X_0 \mid \mathcal{H}_0$, then to $X_0 \mid \mathcal{H}_1$ with observation realization $|y_0| = 0.4$, MAP-control $U_0 = -\hat{X}_{0|0}^{\text{MAP}} = -0.4$. The following observation realization has $|y_1| = 0.3$.

hence $\hat{X}_{0|0}^+ = |y_0|$, $\hat{X}_{0|0}^- = -|y_0|$. We choose (arbitrarily) $\hat{X}_{0|0}^{\text{MAP}} = \hat{X}_{0|0}^+ = |y_0| = 0.4$. This leads to a control input of $U_0 = -\hat{X}_{0|0}^{\text{MAP}} = -0.4$.

Now, say the next observation is such that $|y_1| = 0.3$. Since $|Y_1| \leq |X_0 - \hat{X}_{0|0}^{\text{MAP}}|$, the support of the density $f(X_0 = x \mid \mathcal{H}_1)$ is changed from that of $f(X_0 = x \mid \mathcal{H}_0)$ by a removal of the interval $(\hat{X}_{0|0}^{\text{MAP}} - |y_1|, \hat{X}_{0|0}^{\text{MAP}} + |y_1|)$. Thus, new observations reduce the possible intervals with non-zero density for X_0 . The structure of the MAP strategy is such that it only maintains at most two disjoint intervals of support, with maxima occurring on the inner bounds of these support intervals. It is these maxima that are $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ as seen and labeled in Fig. 2. In Fig. 2, $\hat{X}_{0|0}^{\text{MAP}} - |y_1| = 0.1$ is greater than $\hat{X}_{0|0}^-$, and thus $\hat{X}_{0|1}^- = \hat{X}_{0|0}^-$ but it is possible for $|y_0|$ and $|y_1|$ to be such that both $\hat{X}_{0|1}^+ > \hat{X}_{0|0}^+$ and $\hat{X}_{0|1}^- < \hat{X}_{0|0}^-$.

This example illustrates the dual nature of control at play. In particular, when the conditional density of $f_X(X_0 = x \mid \mathcal{H}_0)$ is even, as in the first time step, the symmetry prevents us from learning any information about the sign of X_0 . However, as soon as we take the first control action, the next observation y_1 breaks the symmetry and gives us information about the sign of the initial state X_0 .

IV. MAIN RESULTS

We prove the MAP-control strategy stabilizes the system in (2) almost surely (a.s.) and prove a η -moment rate result. The main theorems are stated below.

Theorem 4.1 (Almost-sure convergence): Consider the system in (2) with MAP-control, i.e. $U_0 = -\hat{X}_{0|0}^{\text{MAP}}$, and $U_n = \hat{X}_{0|n-1}^{\text{MAP}} - \hat{X}_{0|n}^{\text{MAP}}$ for $n \geq 1$. Then, $\hat{X}_{0|n}^{\text{MAP}} \rightarrow X_0$ a.s., and $X_n \rightarrow 0$ a.s.

Theorem 4.2 (Convergence rate): For the system in (2) with MAP-control,

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n|^\eta / |X_{n-1}|^\eta] = \frac{1}{\eta + 1}.$$

Therefore, the largest a in (1) that can be stabilized in an η -moment sense is given by $(\eta + 1)^{\frac{1}{\eta}}$.

V. PROOFS

We provide proofs of the main theorems here but proofs of the lemmas in section III are deferred to the full version.

A. Proof of Theorem 4.1

We first consider the case $X_0 > 0$. We will show that in this case $\hat{X}_{0|n}^{\text{MAP}} \rightarrow \hat{X}_{0|n}^+$ and also $\hat{X}_{0|n}^+ \rightarrow X_0$ a.s. as $n \rightarrow \infty$. For this we first show the following lemma.

Lemma 5.1: Under MAP-control, $\hat{X}_{0|n}^+$ is increasing and $\hat{X}_{0|n}^-$ is decreasing in n . Furthermore, if $X_0 > 0$, then $\forall n, \hat{X}_{0|n}^+ \leq X_0$ and $\hat{X}_{0|n}^- \leq X_0$.

Proof: We first note that $\hat{X}_{0|n}^+$ is increasing and $\hat{X}_{0|n}^-$ is decreasing in n . This follows from Lemma 3.6 because:

$$\begin{aligned}\hat{X}_{0|n+1}^+ &= \max\left(\hat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|, \hat{X}_{0|n}^+\right) \geq \hat{X}_{0|n}^+ \\ \hat{X}_{0|n+1}^- &= \min\left(\hat{X}_{0|n}^{\text{MAP}} - |Y_{n+1}|, \hat{X}_{0|n}^-\right) \leq \hat{X}_{0|n}^-.\end{aligned}$$

Now, we show by induction that $\forall n \geq 0, \hat{X}_{0|n}^+ \leq X_0$ and $\hat{X}_{0|n}^- \leq X_0$. The base case holds:

$$\begin{aligned}\hat{X}_{0|0}^+ &= |Y_0| = |C_0 X_0| = |C_0| X_0 \leq X_0 \\ \hat{X}_{0|0}^- &= -|Y_0| = -|C_0 X_0| \leq 0 \leq X_0.\end{aligned}$$

Assume as the inductive hypothesis that $\hat{X}_{0|n-1}^+ \leq X_0$ and $\hat{X}_{0|n-1}^- \leq X_0$. Since $\hat{X}_{0|n-1}^-$ is decreasing, the bound on $\hat{X}_{0|n}^-$ follows immediately as $\hat{X}_{0|n}^- \leq \hat{X}_{0|n-1}^- \leq X_0$.

Now, we consider $\hat{X}_{0|n}^+ = \max\left(\hat{X}_{0|n-1}^{\text{MAP}} + |Y_n|, \hat{X}_{0|n-1}^+\right)$. There are two cases. First if $\hat{X}_{0|n}^+ = \hat{X}_{0|n-1}^+$, we immediately conclude $\hat{X}_{0|n}^+ \leq X_0$ from the induction hypothesis. Else if $\hat{X}_{0|n}^+ = \hat{X}_{0|n-1}^{\text{MAP}} + |Y_n|$, using Lemma 3.2 we have that:

$$\hat{X}_{0|n}^+ = \hat{X}_{0|n-1}^{\text{MAP}} + |C_n| \left(X_0 - \hat{X}_{0|n-1}^{\text{MAP}}\right).$$

Since we know that $\hat{X}_{0|n-1}^+ \leq X_0$ and $\hat{X}_{0|n-1}^- \leq X_0$, we have that $\hat{X}_{0|n-1}^{\text{MAP}} \leq X_0$. Hence,

$$\begin{aligned}\hat{X}_{0|n}^+ &= \hat{X}_{0|n-1}^{\text{MAP}} + |C_n| \left(X_0 - \hat{X}_{0|n-1}^{\text{MAP}}\right) \\ &= (1 - |C_n|) \hat{X}_{0|n-1}^{\text{MAP}} + |C_n| X_0 \leq X_0,\end{aligned}\quad (13)$$

where (13) again follows from the induction hypothesis. $\square$

The rest of the proof is in two parts. We first show that $\hat{X}_{0|n}^{\text{MAP}} \rightarrow \hat{X}_{0|n}^+$ a.s. and then that $\hat{X}_{0|n}^+ \rightarrow X_0$ a.s.

Lemma 5.2: $\hat{X}_{0|n}^{\text{MAP}} \rightarrow \hat{X}_{0|n}^+$ a.s. as $n \rightarrow \infty$.

Proof: To show that $\hat{X}_{0|n}^{\text{MAP}} \rightarrow \hat{X}_{0|n}^+$ a.s., it suffices to show that with probability 1 we have $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^-$ only finitely many times, since $\hat{X}_{0|n}^{\text{MAP}}$ must be equal to either $\hat{X}_{0|n}^+$ or $\hat{X}_{0|n}^-$.

We proceed by proof by contradiction. If possible, let $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^-$ infinitely often. A sample path where this occurs will have a sequence of strictly increasing time indices,

$k_1 < k_2 < \dots$, at which $\hat{X}_{0|k_i}^{\text{MAP}} = \hat{X}_{0|k_i}^-$. Consider the transition from time k_i to $k_i + 1$. At this time, we know that the update rule for $\hat{X}_{0|k_i+1}^-$ must resolve as:

$$\hat{X}_{0|k_i+1}^- = \hat{X}_{0|k_i}^- - |Y_{k_i+1}|,$$

since $\hat{X}_{0|k_i}^{\text{MAP}} = \hat{X}_{0|k_i}^-$. Since $\hat{X}_{0|n}^- \leq X_0$ from Lemma 5.1:

$$\begin{aligned}\hat{X}_{0|k_i+1}^- &= \hat{X}_{0|k_i}^- - |C_{k_i+1}| \left(X_0 - \hat{X}_{0|k_i}^-\right) \\ &= (1 + |C_{k_i+1}|) \hat{X}_{0|k_i}^- - |C_{k_i+1}| X_0.\end{aligned}\quad (14)$$

Since the k_i are strictly increasing ($k_{i+1} \geq k_i + 1$) and $\hat{X}_{0|n}^-$ is decreasing we have $\hat{X}_{0|k_{i+1}}^- \leq \hat{X}_{0|k_i+1}^-$. Thus from (14):

$$\hat{X}_{0|k_{i+1}}^- \leq (1 + |C_{k_i+1}|) \hat{X}_{0|k_i}^- - |C_{k_i+1}| X_0. \quad (15)$$

Recursively applying this bound gives:

$$\begin{aligned}\hat{X}_{0|k_{i+1}}^- &\leq \hat{X}_{0|k_1}^- \prod_{j=1}^i (1 + |C_{k_j+1}|) \\ &\quad - X_0 \sum_{j=1}^i |C_{k_j+1}| \prod_{l=j+1}^i (1 + |C_{k_l+1}|).\end{aligned}\quad (16)$$

Note that this upper bound on $\hat{X}_{0|k_{i+1}}^-$ is negative. The second term on the RHS of (16) is negative with probability one since we assume that $X_0 > 0$. The first term on the RHS of (16) is non-positive because we have $\hat{X}_{0|k_1}^- \leq 0$. Further, we have that

$$\hat{X}_{0|k_1}^- \leq \hat{X}_{0|0}^- = -|Y_0| = -|C_0| X_0 \leq 0, \quad (17)$$

and we conclude that $\hat{X}_{0|k_1}^- < 0$ with probability 1.

Now $(1 + |C_{k_j+1}|) > 1$ with probability 1 for all k_j , and therefore $\prod_{j=1}^i (1 + |C_{k_j+1}|) \rightarrow \infty$ as $i \rightarrow \infty$. Hence, the first term on the RHS of (16) tends to $-\infty$. As the second term is negative, $\hat{X}_{0|k_{i+1}}^- \rightarrow -\infty$ as $i \rightarrow \infty$. Hence, there exists some first k^* such that $\hat{X}_{0|k^*}^- < -1$, which by Lemma 3.6 gives $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^+$ for all $n \geq k^*$, which contradicts our assumption that $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^-$ infinitely often. $\square$

Lemma 5.3: $\hat{X}_{0|n}^+ \rightarrow X_0$ a.s. as $n \rightarrow \infty$.

Proof: We know from Lemma 5.1 that X_0 is an upper bound on $\hat{X}_{0|n}^+$ and that $\hat{X}_{0|n}^+$ is an increasing sequence. Therefore, by the monotone convergence theorem it must converge to its supremum. We will show that it converges to X_0 .

Since $\hat{X}_{0|n}^{\text{MAP}} \rightarrow \hat{X}_{0|n}^+$ a.s., there exists N_+ such that for all $n \geq N_+$ we have $\hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^+$ (since $\hat{X}_{0|n}^{\text{MAP}}$ equals one of $\hat{X}_{0|n}^+$ or $\hat{X}_{0|n}^-$). Using the update rule in (11), since $\hat{X}_{0|n}^+ + |Y_{n+1}| \geq \hat{X}_{0|n}^+$, we have that $n \geq N_+$,

$$\hat{X}_{0|n+1}^+ = \hat{X}_{0|n}^+ + |Y_{n+1}| = \hat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|. \quad (18)$$

We will show that the essential supremum of $\hat{X}_{0|n}^+$ is X_0 . If possible, say that $\text{ess sup}_n \hat{X}_{0|n}^+ = M < X_0$. Consider the observation for $n \geq N_+$, and using Lemma 3.2 gives:

$$\begin{aligned} |Y_{n+1}| &= |C_{n+1}X_{n+1}| \\ &= |C_{n+1}(X_0 - \hat{X}_{0|n}^{\text{MAP}})| = |C_{n+1}(X_0 - \hat{X}_{0|n}^+)|. \end{aligned} \quad (19)$$

Now consider for $n \geq N_+$:

$$\mathbb{P}(|Y_{n+1}| > M - \hat{X}_{0|n}^+) = \mathbb{P}\left(|C_{n+1}| > \frac{M - \hat{X}_{0|n}^+}{X_0 - \hat{X}_{0|n}^+}\right). \quad (20)$$

Note that $0 < X_0 - \hat{X}_{0|n}^+$ since we assume $M < X_0$. Furthermore, $M < X_0$ also implies that $M - \hat{X}_{0|n}^+ < X_0 - \hat{X}_{0|n}^+$. Hence, $(M - \hat{X}_{0|n}^+)/(X_0 - \hat{X}_{0|n}^+) < 1$. Therefore, from (20), we have that $\mathbb{P}(|Y_{n+1}| > M - \hat{X}_{0|n}^+) > 0$. Then, using the update rule from (18):

$$\mathbb{P}(\hat{X}_{0|n+1}^+ > M) = \mathbb{P}(\hat{X}_{0|n}^+ + |Y_{n+1}| > M) > 0,$$

which contradicts our assumption that M is the essential supremum. Hence $\hat{X}_{0|n}^+ \rightarrow X_0$ a.s. as $n \rightarrow \infty$. $\square$

Combining Lemmas 5.2 and 5.3, we obtain the theorem for the case $X_0 > 0$. The case for $X_0 < 0$ can be handled in a nearly identical argument, except with $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ swapped in considerations. $X_0 = 0$ is a zero probability event. $\square$

B. Proof of Theorem 4.2

Let $X_0 > 0$ and thus $\hat{X}_{0|n}^+ \rightarrow X_0$ a.s. from below by the previous proof of Theorem 4.1. Under this condition, $\exists N_+$ such that $\forall n \geq N_+ : \hat{X}_{0|n}^{\text{MAP}} = \hat{X}_{0|n}^+$. Consider such a trajectory of $\hat{X}_{0|n}^{\text{MAP}}$ for $n \geq N_+$. Then, the update equation in (11) can be rewritten in terms of $\hat{X}_{0|n}^{\text{MAP}}$ as:

$$\hat{X}_{0|n+1}^{\text{MAP}} = \max\left(\hat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}|, \hat{X}_{0|n}^{\text{MAP}}\right).$$

Therefore, we can write

$$\hat{X}_{0|n+1}^{\text{MAP}} = \hat{X}_{0|n}^{\text{MAP}} + |Y_{n+1}| = \hat{X}_{0|n}^{\text{MAP}} + |C_{n+1}X_{n+1}|.$$

This implies

$$X_0 - \hat{X}_{0|n+1}^{\text{MAP}} = X_0 - \hat{X}_{0|n}^{\text{MAP}} - |C_{n+1}|X_{n+1} \quad (21)$$

$$X_{n+2} = X_{n+1} - |C_{n+1}|X_{n+1} \quad (22)$$

$$= (1 - |C_{n+1}|)X_{n+1}. \quad (23)$$

Note that $X_{n+1} = X_0 - \hat{X}_{0|n}^{\text{MAP}} = X_0 - \hat{X}_{0|n}^+ \geq 0$ in (21) from Lemma 5.1 and (22) follows from Lemma 3.2.

Now consider the expectation of the ratio of the η -moments for $n \geq N_+$, and use (23):

$$\mathbb{E}\left[\frac{|X_{n+2}|^\eta}{|X_{n+1}|^\eta}\right] = \mathbb{E}[|1 - |C_{n+1}||^\eta] = \frac{1}{\eta + 1}, \quad (24)$$

since $(1 - |C_{n+1}|)$ is distributed as $\text{Unif}[0, 1]$, which has η -moment $\frac{1}{\eta + 1}$. When $\eta = 2$, this is a rate of $\frac{1}{3}$, and as $\eta \rightarrow \infty$ this rate tends to zero. The case $X_0 < 0$ proceeds similarly. $\square$

VI. SIMULATIONS AND DISCUSSION

Fig. 1 compares the performance of the MAP-control strategy (black) to the neural-network control strategies described in [9] for the case of C_n and X_0 drawn from $\text{Unif}[-1, 1]$. Three different neural-network controllers from [9], with different memory and periodicity parameters (e.g. memory-1 period-2 etc.), were trained for the system in (2) as comparisons.

The MAP-control (for the uniform distribution case) was computed using the update rules on $\hat{X}_{0|n}^+$ and $\hat{X}_{0|n}^-$ as in (11) and (12). Using the update rules avoids the challenges of numerically computing the MAP estimator. The MAP strategy empirically outperforms the neural-network strategies, with second-moment decay rate of $1/3$ as expected.

For C_n and X_0 drawn from $\text{Unif}[-1, 1]$ we plot an ensemble of the state trajectories in Fig. 3(a) on a log scale. In fig. 3(b) we see that the MAP strategy outperforms the strategy from [8] in both the “log”-moment (i.e. $\mathbb{E}[\ln |X_0|]$) and first-moment senses. The empirical slope of the “log”-moment for the MAP strategy is about -1 .

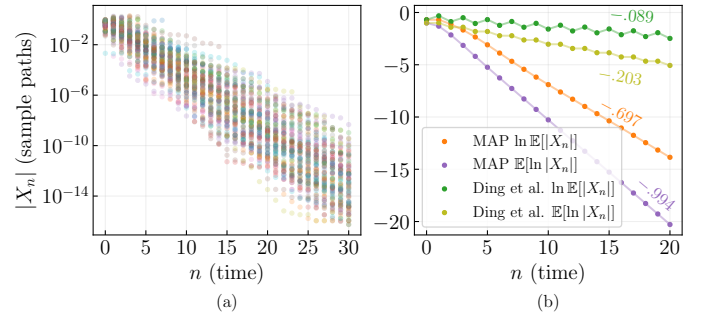


Fig. 3. (a) 100 trajectory realizations of $|X_n|$. (b) “Log” moment and first moment of the state under the MAP strategy as well as the non-linear strategy proposed from [8] with computed slopes. Each expectation is computed over 10^6 trajectories. Note the different y-axis scales.

While this paper focuses on the case of C_n as $\text{Unif}[-1, 1]$, our proof technique for almost-sure convergence can likely be extended to handle more general bounded distributions. For example, we believe the MAP strategy will converge a.s. for C_n distributed on an interval $[a, b]$, such that $a \leq 0 \leq b$, as long as C_n has no probability atoms at 0 and has density strictly decreasing on $[a, 0)$ and strictly increasing on $(0, b]$.

Empirically, the MAP-control strategy seems to also work well for unbounded distributions. Fig. 1 additionally shows the empirical performance of the MAP-control strategy (red) when C_n and X_0 are both drawn from $\mathcal{N}(0, 1/3)$, i.e. a Gaussian with variance matching the uniform distribution on $[-1, 1]$. Here the MAP estimate is computed by a grid search. We find that empirically the MAP-controller in the Gaussian case also outperforms the strategies from [9] trained for Gaussian densities, but do not include these plots for space reasons.

ACKNOWLEDGEMENTS

We thank the reviewers for their helpful suggestions, as well as the NSF for CAREER grant ECCS-2240031.

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