MINIMAL EQUATIONS FOR MATRIX SCHUBERT VARIETIES

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Explicit minimal generators for Fulton's Schubert determinantal ideals are determined along with some implications.

1. Introduction and results

Let $\mathrm{Mat}_{n \times n}$ be the space of $n \times n$ matrices over a field $\Bbbk$; the coordinate ring is $R = \Bbbk[x_{ij}]_{1 \le i, j \le n}$. Let GL_n be the general linear group of invertible $n \times n$ matrices with a Borel subgroup B of upper triangular matrices and B_- of lower triangular matrices. Let $B_- \times B$ act on $\mathrm{Mat}_{n \times n}$ by $(b_-, b) \cdot M = b_- M b^{-1}$. Let w be a permutation in the symmetric group $\mathfrak{S}_n$ on $[n] = \{1, 2, \ldots, n\}$, and suppose M_w is its permutation matrix with 1 in position $(i, w(i))$ and 0 elsewhere.

Definition 1.1 [2; 7]. The *matrix Schubert variety* X_w is the $B_- \times B$ orbit closure of M_w.

Definition 1.2 [2; 7]. The *Schubert determinantal ideal* $I_w \subset R$ is the defining ideal of X_w.

Since [2], there has been interest in matrix Schubert varieties and the Schubert determinantal ideals; see, e.g., [1; 3; 4; 5; 6; 7; 8; 9; 10].

Let $r_{i,j} = r_{i,j}(w)$ be the *rank function* of w. It counts the number of 1's weakly northwest of position (i, j) in M_w. Let $M^{[i,j]}$ denote the northwest $i \times j$ submatrix of a generic matrix $M \in \mathrm{Mat}_{n \times n}$. In [2], it is proved that I_w is indeed generated by determinants

$$(1) \qquad I_w = \big\langle (r_{i,j} + 1) \times (r_{i,j} + 1) \text{ size minors of } M^{[i,j]} : (i, j) \in [n]^2 \big\rangle,$$

and that this ideal is prime [2, Corollary 3.13].

In *loc. cit.*, Fulton minimizes the *description* of the generators (1). The *Rothe diagram* of w is

$$D(w) = \{(i, j) \in [n]^2 : j < w(i), \ i < w^{-1}(j)\}.$$

Fulton's essential set is

$$E(w) = \{(i, j) \in D(w) : (i + 1, j), (i, j + 1) \notin D(w)\}.$$

Fulton proved that $I_w = \langle (r_{i,j} + 1) \times (r_{i,j} + 1) \text{ size minors of } M^{[i,j]} : (i, j) \in E(w) \rangle$. This is a minimal list of rank conditions needed to describe I_w, but does not provide a minimal set of generators. The minors in this description are called the *essential minors* of I_w.
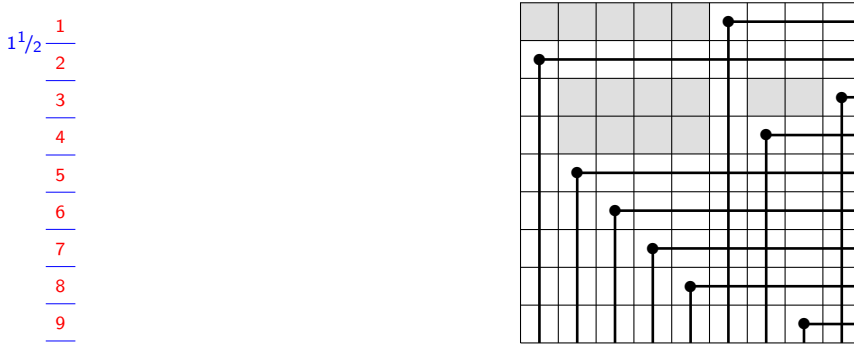
Figure 1. Rothe diagram of $w = 619723458$, the boxes of $D(w)$ are shaded.

Example 1.3 (essential minors do not form a minimal generating set). The reader can check that

$$I_{3142} = \left\langle x_{11}, x_{12}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix} \right\rangle.$$

The latter two essential minors can be dispensed with; they are implied by the first two.

For $I, J \subseteq [n]$ with $|I| = |J|$, define $m_{I,J}$ to be the determinant of the submatrix of M with row and column indices I and J, respectively. An essential generator $m_{I,J}$ *belongs to* $(i, j) \in E(w)$ if $I \subseteq [i]$, $J \subseteq [j]$ and $r = |I| = |J| = r_{i,j} + 1$.

Definition 1.4. A minor $m_{I,J}$ *attends* $M^{[i',j']}$ if $|I \cap [i']| > r_{i',j'}$ and $|J \cap [j']| = r_{i,j} + 1$ or $|I \cap [i']| = r_{i,j} + 1$ and $|J \cap [j']| > r_{i',j'}$.

Definition 1.5. A minor $m_{I,J}$ that belongs to $(i, j) \in E(w)$ is *elusive* if it does not attend $M^{[i',j']}$ for all $(i', j') \in E(w)$ such that $r_{i',j'} < r_{i,j}$.

Theorem 1.6. *The ideal I_w is minimally generated by elusive minors. Moreover, for any $b \in D(w)$ there exists an elusive minor whose southeast corner is b.*

Example 1.7. Let $w = 619723458$ (see Figure 1). An example of an elusive minor is $m_{\{1,2,3\},\{5,7,8\}}$, whereas $m_{\{1,2,3\},\{4,5,8\}}$ is not elusive since it attends $M^{[4,5]}$.

Theorem 1.6 is a handy way to hand compute the size of a minimal generating set. Here, the minimal generating set contains five generators of degree 1, $\binom{3}{2}\binom{5}{2}$ generators of degree 2 and $1 + \binom{5}{1}\binom{3}{2}$ generators of degree 3. All five degree 1 essential minors are elusive, a degree 2 essential minor is elusive if and only if $1 \notin I$, and a degree 3 essential minor is elusive if and only if $|J \cap [5]| \leq 1$.

An exercise is $w = 13865742$ [7, Example 1.3.5]. A minimum generating set is of size 104 and consists of 21 (2×2) minors and 83 (3×3) minors.[1]

Knutson and Miller [7, Theorem B] proved that the essential minors of I_w form a Gröbner basis with respect to any *antidiagonal term order* $\prec$, that is, a monomial order that picks the antidiagonal term of a minor as the initial term (one example is the lexicographic ordering obtained by reading the rows of the generic matrix right-to-left in rows, and from top-to-bottom). Knutson, Miller and Yong [8, Theorem 3.8]

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[1]This example is also considered in the unpublished Section 3 of earlier arXiv preprint versions 1 and 2 of [7]. The notion of *attends* is more general than "*causes*" used in those preprints and the published version.

prove the same result for any *diagonal term order* $\prec'$ (where the order picks the diagonal term of a minor as the initial term), but under the hypothesis that w is *vexillary* (that is 2143-avoiding; see the definition of pattern avoidance below). We refine these statements.

Corollary 1.8. *The set of elusive minors is a Gröbner basis for I_w under an antidiagonal term order $\prec$. If we assume w is vexillary, the same statement is true under diagonal term order $\prec'$. That is, in either case, I_w has a Gröbner basis given by a set of minimal generators.*

The codimension of $X_w \subseteq \Bbbk^{n^2}$ is $\ell(w) = \#D(w)$, that is, the number of inversions of w [2, Corollary 3.13]. Since the size of a minimal generating set is an invariant, X_w is a complete intersection if and only if the size of the set of its elusive minors is $\ell(w)$. Using this, we give a self-contained proof of the result below of Ulfarsson and Woo [11, Corollary 6.3], which is a pattern avoidance characterization of matrix Schubert varieties that are complete intersections. Their result came after an earlier characterization by Hsiao [4, Theorem 5.2], which depends on the Gröbner basis theorem of [7, Theorem B].

Recall $w \in \mathfrak{S}_n$ *pattern includes* $u \in S_m$ if there exist indices $i_1 < i_2 < \cdots < i_m$ such that $w(i_1), \ldots, w(i_m)$ is in the same relative order as $u(1), \ldots, u(m)$. Furthermore, w *avoids* u if no such indices exist.

Corollary 1.9 [4; 11]. *The matrix Schubert variety X_w is a complete intersection if and only if w avoids 1342, 1432 and 1423.*[2]

In fact, the proofs in [4; 11] construct a minimal set of generators for I_w to prove "$\Longleftarrow$". However, their arguments do not do this outside of those cases.

2. Proofs

Proof of Theorem 1.6. Suppose a minor m belonging to $(i, j) \in E(w)$ is not elusive; say it attends $M^{[i', j']}$ for $(i', j') \in E(w)$ satisfying $r_{i', j'}(w) < r_{i, j}(w)$. Then it follows by induction using cofactor expansion that m is in the ideal generated by the $(r_{i', j'} + 1) \times (r_{i', j'} + 1)$ minors belonging to $M^{[i', j']}$. Hence m can be dispensed with.

Conversely, suppose $m = m_{I, J}$ belonging to $e = (i, j) \in E(w)$ is elusive. Let

$$I = \{1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq i\} \quad \text{and} \quad J = \{1 \leq j_1 < j_2 < \cdots < j_{r+1} \leq j\},$$
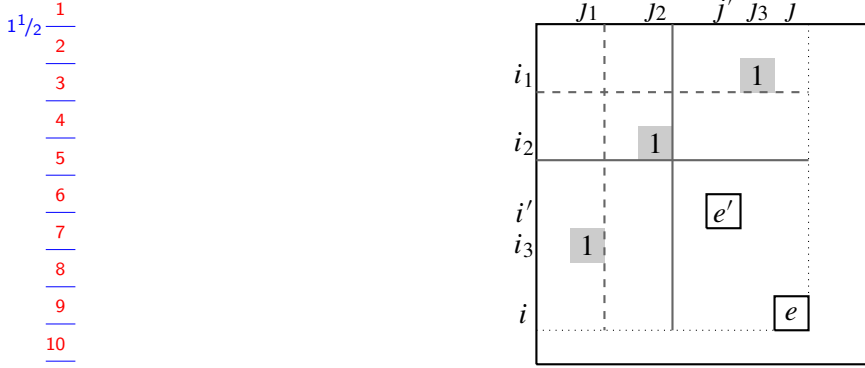
where $r = r_{i, j}(w)$.

Claim 2.1. *For any $1 \leq k < r + 1$, we have $r_{i_k, j} \geq k$ and $r_{i, j_k} \geq k$.*

Proof of Claim 2.1: We will prove $r_{i_k, j} \geq k$; $r_{i, j_k} \geq k$ follows from the same reasoning. We proceed by induction. For $k = 1$, suppose $r_{i_1, j} = 0 < k$, then $(i_1, j) \in D(w)$. There is $(a, b) \in E(w)$ weakly southeast of (i_1, j) and in the same connected component of $D(w)$. Since $r_{a,b} = r_{i_1, j} = 0$, $b \geq j$ and $I \cap [a] \geq 1 > 0$, the minor $m_{I, J}$ attends $M^{[a,b]}$, a contradiction.

Now suppose $r_{i_k, j} \geq k$ for all $1 \leq k < s$ for some $s < r + 1$. If $(i_s, j) \in D(w)$, there is a $(a, b) \in E(w)$ that is in the same connected component as (i_s, j) and weakly southeast of (i_s, j). If $r_{i_s, j} = s - 1$, then $m_{I, J}$ attends $M^{[a,b]}$, a contradiction. So $r_{i_s, j} \geq s$ in this case. Now if $(i_s, j) \notin D(w)$, since $(i, j) \in D(w)$

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[2]In [11, Corollary 6.3], the additional patterns $31524, 24153$ and 351624 are listed. However these follow from the size 4 patterns.

Figure 2. Relative position of e, e' and P

and $i > i_s$, we know that $w(i_s) < j$. Since $r_{i_{s-1},j} \geq s-1$, we see $r_{i_s,j} \geq r_{i_{s-1},j} + 1 = s$, completing the induction step. $\square$

To prove that $m = m_{I,J}$ is necessary as a generator, it suffices to find a point $P \in \mathrm{Mat}_{n \times n}$ such that m does not vanish at P but every other essential generator does vanish. Set $P_{a,b} = 1$ if $a = i_t$ and $b = j_{r-t+2}$ for $1 \leq t \leq r+1$, and let all other entries be 0. In words, P places 1's on the antidiagonal of m. Evidently, m does not vanish at P.

It remains to prove all other essential minors do vanish at P. Suppose, to the contrary that m' is a minor that belongs to $e' \in E(w)$ but does not vanish at P. Since the only minor of size at least $r_e(w) + 1$ that does not vanish at P is m, $r_{e'}(w) < r_e(w)$. Let $e' = (i', j')$. If e' is not in the rectangle with corners $(1,1)$ and $(i_{r+1} - 1, j_{r+1} - 1)$, by definition, m attends $M^{[e']}$ contradicting the assumption that m is elusive. Thus, the only possibility is that $i' < i_{r+1}$ and $j' < j_{r+1}$ as depicted in Figure 2.

Let us assume that $i_p \leq i' < i_{p+1}$ and $j_\ell \leq j' < j_{\ell+1}$ for some $0 \leq p, \ell < r+1$, where $i_0 = j_0 := 1$ (in the figure, $p = \ell = 2$). Since m' does not vanish at P, it is straightforward that $p + \ell > r + 1$ (otherwise, m' only involves 0 entries) and $r_{i', j'} < p + \ell - (r+1)$ (the right-hand side is the number of 1's that appear in the northwest $i_p \times j_\ell$ rectangle of M_w). In particular, this implies that

$$r_{i_p, j_\ell} < p + \ell - (r+1).$$

By Claim 2.1, $r_{i,j_\ell} \geq \ell$. Since $r_{i,j} = r$, we obtain

$$r_{i_p, j} - r_{i_p, j_\ell} \leq r_{i,j} - r_{i,j_\ell} \leq r - \ell.$$

Hence,

$$r_{i_p, j} < p + \ell - (r+1) + r - \ell = p - 1.$$

Yet by Claim 2.1, we have $r_{i_p, j} \geq p$, a contradiction. So all other essential minors vanish on P.

We now turn to the second statement of the theorem, restated here in more exact form.

Proposition 2.2. *For $b = (i, j) \in D(w)$, the minor $m_{[i-r,i],[j-r,j]}$ is elusive with southeast corner b.*

Proof. Let $r = r_b(w)$, $I = [i - r, i]$ and $J = [j - r, j]$. We will show that $m_{I,J}$ is an elusive minor. Since $r_e(w) = r$ for any $e \in E(w)$ that is in the same connected component as b in $D(w)$, the minor $m_{I,J}$

belongs to any such e. Suppose $m_{I,J}$ attends $M^{[e']}$ for some $e' = (i', j') \in E(w)$. We can assume, without loss, that $i' < i$ and $j' > j$.

Since $b, e' \in D(w)$, we know that $(i', j) \in D(w)$. Let $k = i - i'$, we then have

$$r_{i',j}(w) \geq r - k + 1.$$

Since $r_{e'}(w) \geq r_{i',j}(w)$,

$$r_{e'}(w) \geq r - k + 1.$$

Thus any minor that belongs to e' has size at least $r - k + 2$. Since

$$|I \cap [i']| = r - k + 1 < r - k + 2,$$

$m_{I,J}$ does not attend $M^{[e']}$, a contradiction. Therefore $m_{I,J}$ is elusive, as claimed. $\square$

Proof of Corollary 1.8. For $f \in R$, let $\mathrm{init}_{\prec}(f)$ be the initial term of f under $\prec$. By definition, a generating set $\mathcal{S}$ of I_w is a Gröbner basis if

$$\langle \mathrm{init}_{\prec}(f) : f \in \mathcal{S} \rangle = \langle \mathrm{init}_{\prec}(f) : f \in I_w \rangle.$$

By [7, Theorem B], the essential minors are a Gröbner basis under $\prec$. Therefore it suffices to show that if m is a nonelusive minor then $\mathrm{init}_{\prec} m$ is divisible by $\mathrm{init}_{\prec} m'$ where m' is an elusive minor. We proceed by induction, ordering the essential set by rank value. In the base case where the rank is 0, all the associated 1×1 minors are elusive, trivially. Suppose m belongs to $(i, j) \in E(w)$ but is not elusive. Since $m = m_{I,J}$ is not elusive we may suppose, without loss of generality, that there is $(i', j') \in E(w)$ with $r_{i',j'}(w) < r_{i,j}(w)$ such that $|I \cap [i']| > r_{i',j'}$ and $|J \cap [j']| = r_{i,j} + 1$ (the argument in the other case is similar). Thus, there is a minor $m_{I',J'}$ that belongs to (i', j') (and of size $(r_{i',j'} + 1) \times (r_{i',j'} + 1)$) whose antidiagonal term divides that of $m_{I,J}$. More precisely I' consists of the $r_{i',j'} + 1$ smallest indices of I, and J' consists of the $r_{i',j'} + 1$ largest indices of J. If $m_{I',J'}$ is elusive we are done. Otherwise by induction $\mathrm{init}_{\prec} m_{I',J'}$ is divisible by $\mathrm{init}_{\prec} m_{I'',J''}$ for some elusive $m_{I'',J''}$ in which case $\mathrm{init}_{\prec} m_{I'',J''}$ divides $\mathrm{init}_{\prec} m_{I,J}$, as desired.

If w is vexillary and $e, e' \in E(w)$ then e cannot be strictly northwest of e'. Using this and the Gröbner basis result [8, Theorem 3.8], the second sentence follows like the first.

The final sentence of the statement then is immediate from Theorem 1.6.

Proof of Corollary 1.9.

Lemma 2.3. *The variety X_w is a complete intersection if and only if $X_{w^{-1}}$ is a complete intersection.*

Proof. Notice that $D(w)$ is the transpose of $D(w^{-1})$. The lemma then follows from (1). $\square$

Lemma 2.4 (shifting). *If $m_{I,J}$ is an elusive minor of I_w, then $m_{I',J'}$ is elusive whenever $I' = (I - \{i\}) \cup \{t\}$, where $t > i$ and $t \notin I$, or similarly for J'.*

Proof. Since $|I' \cap [i']| \leq |I \cap [i']|$ and $|J' \cap [j']| \leq |J \cap [j']|$ for all $i', j' \in [n]$, the lemma is immediate from the definitions. $\square$

$(\Longrightarrow)$: We prove the contrapositive. Suppose w pattern embeds 1342 or 1432. Let $a_1 < a_2 < a_3 < a_4$ be such that $w(a_1) < w(a_4) < w(a_2), w(a_3)$. Set $b_1 = w(a_1)$, $b_2 = w(a_4)$, $b_3 = w(a_2)$ and $b_4 = w(a_3)$, then $e = (a_3, b_2) \in D(w)$ as shown in Figure 3.
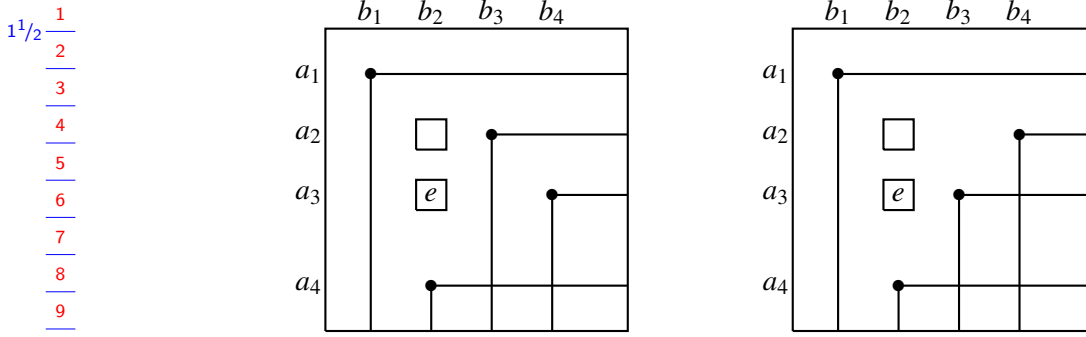
Figure 3. w embeds 1342 (left); w embeds 1432 (right).

Furthermore, we can assume, without loss, that for any $a < a_1$, we have $w(a) > b_2$. That is, we can pick (a_1, b_1) to be the highest nonzero entry in M_w that is strictly northwest of e. Since $w(a_2) = b_3 > b_2$, we have $a_3 > r_e(w) + 1$. Set $r = r_e(w)$, and let

$$I = [a_3 - r, a_3], \quad I' = \{a_3 - r - 1\} \cup [a_3 - r + 1, a_3], \quad J = [b_2 - r, b_2].$$

Since $m_{I,J}$ is elusive (by Proposition 2.2), to show that $m_{I',J}$ is also elusive, it is enough to prove that there does not exist $(i', j') \in E(w)$ such that $J \cap [j'] = J$, $r_{i',j'} = 0$ and $I \cap [i'] = \{a_3 - r - 1\}$. Suppose not. Indeed, since $w(a_2) > b_2$, we know that $a_3 - r - 1 \geq a_1$ and thus $r_{i',j'} \geq 1$, a contradiction. Therefore, there are at least two elusive minors whose southeast corner is e and therefore, by Theorem 1.6, the variety X_w is not a complete intersection.

Since w includes 1342 if and only if w^{-1} includes 1423, we are done by Lemma 2.3.

($\Longleftarrow$): We again argue the contrapositive. Suppose X_w is not a complete intersection. By Theorem 1.6, there is either $e = (i, j) \in D(w)$ having more than one elusive minor m with southeast corner e, or there is an $e = (i, j) \in [n]^2 - D(w)$ that is the southeast corner of an elusive minor m' that belongs to $e' \in E(w)$. In the second case, by using Lemma 2.4 to repeatedly shift the southmost row and/or eastmost column used by m' one obtains another elusive minor m'' with southeast corner e' which is different than the elusive minor from the proof of Proposition 2.2. Hence we assume we are in the first case.

Let $r = r_e(w)$. By using Lemma 2.3 or Lemma 2.4, we may assume that $m_{I',J}$ is an elusive minor, where

$$I' = \{i - r - 1\} \cup [i - r + 1, i] \quad \text{and} \quad J = [j - r, j].$$

Since $m_{I',J}$ is elusive, $r_{i-r-1,j} \geq 1$. Also, since $(i, j) \in D(w)$, either $w(i-r-1) < j$ or $(i-r-1, j) \in D(w)$.

Suppose $w(i - r - 1) < j$, by the pigeonhole principle, there exists a such that $(a, j) \in D(w)$ and $i - r - 1 < a < i$. As a result,

$$i - r - 1 < a < i < w^{-1}(j) \quad \text{and} \quad w(i - r - 1) < j < w(a), w(i).$$

Therefore w embeds 1342 or 1432, and we are done.

Hence $(i - r - 1, j) \in D(w)$. Since $r_{i-r-1,j} \geq 1$, there exists $a < i - r - 1$ such that $w(a) < j$. Since $(i - r - 1, j) \in D(w)$, we get $w(i - r - 1) > j$. We then have

$$a < i - r - 1 < i < w^{-1}(j) \quad \text{and} \quad w(a) < j < w(i - r - 1), w(i).$$

Therefore, w embeds 1342 or 1432. $\square$

Acknowledgements

References

[1] A. Fink, J. Rajchgot, and S. Sullivant, "Matrix Schubert varieties and Gaussian conditional independence models", *J. Algebraic Combin.* **44**:4 (2016), 1009–1046.

[2] W. Fulton, "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas", *Duke Math. J.* **65**:3 (1992), 381–420.

[3] Z. Hamaker, O. Pechenik, and A. Weigandt, "Gröbner geometry of Schubert polynomials through ice", *Adv. Math.* **398** (2022), art. id. 108228.

[4] J.-C. Hsiao, "On the F-rationality and cohomological properties of matrix Schubert varieties", *Illinois J. Math.* **57**:1 (2013), 1–15.

[5] P. Klein, "Diagonal degenerations of matrix Schubert varieties", *Algebr. Comb.* **6**:4 (2023), 1073–1094.

[6] P. Klein and A. Weigandt, "Bumpless pipe dreams encode Gröbner geometry of Schubert polynomials", preprint, 2021. arXiv 2108.08370

[7] A. Knutson and E. Miller, "Gröbner geometry of Schubert polynomials", *Ann. of Math.* (2) **161**:3 (2005), 1245–1318.

[8] A. Knutson, E. Miller, and A. Yong, "Gröbner geometry of vertex decompositions and of flagged tableaux", *J. Reine Angew. Math.* **630** (2009), 1–31.

[9] O. Pechenik, D. E. Speyer, and A. Weigandt, "Castelnuovo–Mumford regularity of matrix Schubert varieties", preprint, 2021. arXiv 2111.10681

[10] J. Rajchgot, Y. Ren, C. Robichaux, A. St. Dizier, and A. Weigandt, "Degrees of symmetric Grothendieck polynomials and Castelnuovo–Mumford regularity", *Proc. Amer. Math. Soc.* **149**:4 (2021), 1405–1416.

[11] H. Ulfarsson and A. Woo, "Which Schubert varieties are local complete intersections?", *Proc. Lond. Math. Soc.* (3) **107**:5 (2013), 1004–1052.

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