

MATRIX SCHUBERT VARIETIES, BINOMIAL IDEALS, AND REDUCED GRÖBNER BASES

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ABSTRACT. We prove a sharp lower bound on the number of terms in an element of the reduced Gröbner basis of a Schubert determinantal ideal I_w under the term order of Knutson–Miller [Ann. of Math. (2) 161 (2005), pp. 1245–1318]. We give three applications. First, we give a pattern-avoidance characterization of the matrix Schubert varieties whose defining ideals are binomial. This complements a result of Escobar–Mészáros [Proc. Amer. Math. Soc. 144 (2016), pp. 5081–5096] on matrix Schubert varieties that are toric with respect to their natural torus action. Second, we give a combinatorial proof that the recent formulas of Rajchgot–Robichaux–Weigandt [J. Algebra 617 (2023), pp. 160–191] and Almousa–Dochtermann–Smith [Preprint, arXiv:2209.09851, 2022] computing the Castelnuovo–Mumford regularity of vexillary I_w and toric edge ideals of bipartite graphs respectively agree for binomial I_w . Third, we demonstrate that the Gröbner basis for I_w given by the minimal generators of Gao–Yong [J. Commut. Algebra 16 (2024), pp. 267–273] is reduced if and only if the defining permutation w is vexillary.

1. INTRODUCTION AND MAIN RESULTS

Let \mathbb{k} be a field and let \mathcal{M}_n be the affine space of $n \times n$ matrices with entries in \mathbb{k} . For a permutation w in the symmetric group S_n , the *permutation matrix* M_w is the element of \mathcal{M}_n with 1s in the positions $(i, w(i))$ for $1 \leq i \leq n$ and 0s elsewhere. Let B_+ and B_- denote the groups of invertible upper- and lower-triangular $n \times n$ matrices over \mathbb{k} . The product $B_- \times B_+$ acts on \mathcal{M}_n by left- and right-multiplication, i.e., $(x, y) \cdot A = xAy^{-1}$. Fulton [6] introduced the following object in his study of degeneracy loci of flagged vector bundles.

Definition 1.1. The *matrix Schubert variety* X_w is the Zariski closure of the $B_- \times B_+$ -orbit of the permutation matrix M_w in $\mathcal{M}_n \cong \mathbb{k}^{n^2}$.

Definition 1.2. The *Schubert determinantal ideal* I_w is the ideal of $R = \mathbb{k}[x_{ij}]_{1 \leq i, j \leq n}$ corresponding to X_w .

For a permutation w , let $r_w(i, j) = r(i, j) = r_{ij}$ be the *rank function* counting the number of 1s weakly northwest of position (i, j) in the permutation matrix M_w .

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Let $M^{[a,b]}$ denote the northwest $a \times b$ submatrix of the generic matrix of variables $[x_{ij}]_{1 \leq i, j \leq n}$. In [6], Fulton described a generating set for I_w :

$$I_w = \langle (r_{ij} + 1) \times (r_{ij} + 1) \text{ minors of } M^{[i,j]} \mid 1 \leq i, j \leq n \rangle.$$

Example 1.3. Consider I_w for $w = 31425$. The rank function is:

$$r_w = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & 2 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline 1 & 2 & 3 & 4 & 4 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

I_w is generated by the two 1×1 minors of $M^{[1,2]}$ and the three 2×2 minors of $M^{[3,2]}$:

$$I_w = \left\langle x_{11}, x_{12}, \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{12} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \right\rangle.$$

In [8, Theorem B], Knutson–Miller showed that Fulton’s generators form a *Gröbner basis* for I_w under “antidiagonal” term orderings. More recently, Gao–Yong refined Fulton’s generators to a minimal generating set that is still a Gröbner basis in [7, Corollary 1.8]. Now, the *reduced* Gröbner basis of an ideal is unique (for a given term order) and contains fewer terms than any other Gröbner basis. Our first theorem establishes a sharp lower bound for the number of terms in the reduced Gröbner basis of I_w under any antidiagonal term order.

Theorem 1.4. *The reduced Gröbner basis \mathcal{G}'_w of I_w under any antidiagonal term order has one generator for each element of the Gao–Yong Gröbner basis \mathcal{G}_w . Each generator of degree d in \mathcal{G}'_w has at least 2^{d-1} terms.*

By minimality, any Gröbner basis using antidiagonal term order must have as many generators as the Gao–Yong Gröbner basis, and Theorem 1.4 says the number of terms in each generator must be exponential in its degree. This suggests that any description of \mathcal{G}'_w (under antidiagonal term order) should be complicated.

We present three applications of Theorem 1.4.

The first two applications are stated in terms of permutation pattern avoidance. A permutation $w \in S_n$ contains a pattern $v \in S_m$ if there exist indices $i_1 < i_2 < \dots < i_m$ such that $w(i_1), \dots, w(i_m)$ appear in the same relative order as $v(1), \dots, v(m)$. If w does not contain the pattern v , then we say it *avoids* v . Permutations avoiding 2143 are called *vexillary*.

Theorem 1.5. *The Gao–Yong Gröbner basis \mathcal{G}_w for I_w is reduced if and only if w is vexillary.*

The number of vexillary permutations in S_n is asymptotic to $c9^n n^{-4}$ for some constant c as shown by Macdonald [11, pg. 22], while the number of permutations in S_n is asymptotic to $\sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ by Stirling’s approximation. Theorem 1.5 therefore shows that the Gao–Yong Gröbner basis is reduced for a super-exponentially small percentage of $w \in S_n$ as $n \rightarrow \infty$.

All matrix Schubert varieties are normal [6]. Therefore, those that are (affine) toric varieties with respect to some algebraic torus correspond exactly to *binomial ideals* I_w (meaning they can be generated by binomials). Theorem 1.6 characterizes when I_w is binomial:

Theorem 1.6. *The Schubert determinantal ideal I_w is binomial if and only if w avoids the patterns 1243 and 2143.*

In [5, Theorem 3.5], Escobar–Mészáros presented a combinatorial characterization of matrix Schubert varieties that are toric with respect to a certain natural torus action. This characterization leaves open the possibility of other matrix Schubert varieties that are toric with respect to a different action. Theorem 1.6 complements their result by showing that the toric matrix Schubert varieties Escobar–Mészáros identified are the only ones that exist.

By Theorem 1.6, toric matrix Schubert varieties are a subclass of vexillary matrix Schubert varieties. Special tools apply in the vexillary case (see [9] and the references therein).

The permutations in S_n avoiding the patterns 1243 and 2143 have been previously studied. In [10, Corollary 9], Kremer proved that they are enumerated by the *large Schröder numbers* s_{n-1} (OEIS sequence A006318). The large Schröder numbers have generating function

$$G(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}$$

and satisfy the recurrence relation

$$s_n = \frac{6n - 3}{n + 1}s_{n-1} - \frac{n - 2}{n + 1}s_{n-2} \quad (n \geq 2).$$

As a third application we consider formulas for the *(Castelnuovo–Mumford) regularity* of binomial I_w . Regularity is a homological invariant that roughly describes the complexity of a module. In [14, Theorem 1.5], Rajchgot–Robichaux–Weigandt gave a formula for the regularity of vexillary I_w^1 , while in [1, Corollary 6.7] Almousa–Dochtermann–Smith gave a formula for the regularity of toric edge ideals of bipartite graphs. By Theorem 1.6 and Portakal’s interpretation of binomial I_w as toric edge ideals [13, p. 1273], these two formulas must agree for binomial I_w . We give a direct proof of this fact.

Organization. Section 2 contains preliminary definitions and results needed for the proof of Theorem 1.4. We introduce standard facts about reduced Gröbner bases from [2] and recall work of Gao–Yong [7] that produces a Gröbner basis for I_w that is also a minimal generating set. Section 3 contains the proof of Theorem 1.4. In Section 4 we prove Theorem 1.5, and Section 5 contains the proof of Theorem 1.6. Section 6 reviews regularity along with the formulas of Rajchgot–Robichaux–Weigandt [14] and Almousa–Dochtermann–Smith [1] before proving their formulas agree for binomial I_w .

2. BACKGROUND

2.1. Reduced Gröbner bases. We review standard facts about Gröbner bases needed in our proofs. We use Cox, Little, and O’Shea’s book [2] as our reference, following their terminology and notation. All ideals in this section belong to the ring $\mathbb{k}[x_1, \dots, x_n]$.

Definition 2.1. The *lead term* of a polynomial f with respect to a term order $<$ on $\mathbb{k}[x_1, \dots, x_n]$ is denoted $LT(f)$. The *initial ideal* of an ideal I is $\text{init}(I) = \langle LT(f) \mid f \in I \rangle$.

¹Pechenik–Speyer–Weigandt gave a formula for the regularity of *all* I_w in [12, Theorem 1.1].

Definition 2.2. A *Gröbner basis* for an ideal I with respect to a term order $<$ is a finite subset $\mathcal{G} = \{g_1, \dots, g_s\} \subseteq I$ such that $\text{init}(I)$ is generated by $\{LT(g_1), \dots, LT(g_s)\}$.

Any ideal $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ has a Gröbner basis, and every Gröbner basis for I is a generating set for I [2, Corollary 2.5.6]. Gröbner bases are non-minimal and non-unique in general, motivating the construction of *minimal* and *reduced* Gröbner bases.

Definition 2.3. A Gröbner basis \mathcal{G} for an ideal I with respect to a term order $<$ is *minimal* if $LT(g_1)$ does not divide $LT(g_2)$ for all $g_1, g_2 \in \mathcal{G}$. Equivalently, \mathcal{G} is a minimal Gröbner basis if the lead terms of its elements form the unique minimal generating set for $\text{init}(I)$.

Remark 2.4. Since any Gröbner basis for I is a generating set, a minimal generating set for I that is a Gröbner basis is a minimal Gröbner basis. The converse does not hold in general.

Definition 2.5. A Gröbner basis \mathcal{G}' for an ideal I with respect to a term order $<$ is *reduced* if for any two generators $g_1, g_2 \in \mathcal{G}'$, $LT(g_1)$ does not divide any term of g_2 .

Let \bar{f}^A be the remainder when f is divided by the (ordered) elements of A .

Theorem 2.6 ([2, Theorem 2.7.5]). *Fix a term order $<$ and ideal I . Then*

- (1) I has a unique reduced Gröbner basis \mathcal{G}' up to scalar multiplication.
- (2) Any Gröbner basis \mathcal{G} for I can be reduced to \mathcal{G}' via the following finite algorithm. First reduce \mathcal{G} to a minimal Gröbner basis \mathcal{G}_0 by removing each $g \in \mathcal{G}$ such that $LT(h)$ divides $LT(g)$ for some $h \in \mathcal{G}$. Say $\mathcal{G}_0 = \{g_1, \dots, g_k\}$. For $1 \leq i \leq k$ let $\mathcal{G}'_i = (\mathcal{G}_{i-1} \setminus \{g_i\})$, let $g'_i = \bar{g}_i^{\mathcal{G}'_{i-1}}$, and set $\mathcal{G}_i = \mathcal{G}'_i \cup \{g'_i\}$. Then $\mathcal{G}_k = \mathcal{G}'$ is the reduced Gröbner basis of I .

The proof of Theorem 1.6 uses an additional fact characterizing the reduced Gröbner bases of binomial ideals. This appears as Proposition 1.1(a) in [4].

Proposition 2.7 ([4, Proposition 1.1(a)]). *If I is a binomial ideal, then for any term order $<$ the reduced Gröbner basis \mathcal{G}' of I with respect to $<$ consists of binomials.*

2.2. Generators for I_w . Our main results build on prior work concerning Gröbner bases of Schubert determinantal ideals I_w . We collect the key facts needed for our proof below.

Definition 2.8. The *Rothe diagram* of a permutation $w \in S_n$ is the set $D(w) = \{(i, j) \in [n] \times [n] : j < w(i), i < w^{-1}(j)\}$.

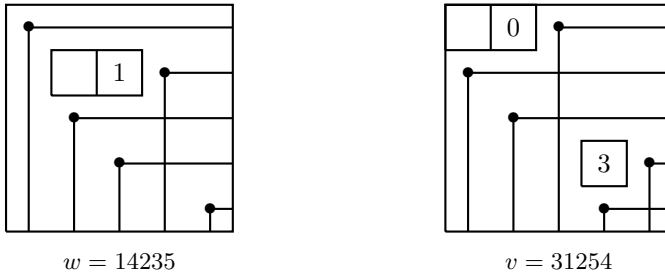
Definition 2.9 ([6]). For a permutation $w \in S_n$, *Fulton’s essential set* is the subset $E(w) \subseteq D(w)$ consisting of pairs $(i, j) \in D(w)$ such that $(i + 1, j)$ and $(i, j + 1)$ are not in $D(w)$. Visually, $E(w)$ consists of all “southeast corners” of connected components of $D(w)$.

The first part of Theorem 2.10 was proved by Fulton as [6, Lemma 3.10], while the second part was established by Knutson and Miller as [8, Theorem B].

Theorem 2.10 ([6, 8]). *Let $w \in S_n$ be a permutation. Then*

- (1) I_w is generated by the $(r_{ij} + 1) \times (r_{ij} + 1)$ minors of $M^{[i,j]}$ as (i, j) varies over the essential set $E(w)$.
- (2) This generating set forms a Gröbner basis for I_w under any antidiagonal term order (i.e., any term order such that the antidiagonal of a generic minor is the lead term).

Example 2.11. Consider the permutations $w = 14235$ and $v = 31254$. Their Rothe diagrams are pictured below, with the values of the rank function r_{ij} displayed only in the elements of the essential sets.



Theorem 2.10 asserts that I_w is generated by the three 2×2 minors of $M^{[2,3]}$. Similarly, I_v is generated by the two 1×1 minors of $M^{[1,2]}$ and the one 4×4 minor of $M^{[4,4]}$.

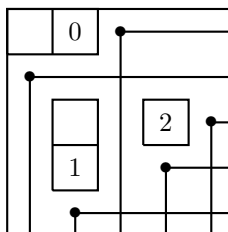
In [7, Theorem 1.6], Gao and Yong refine Fulton’s generators to a *minimal generating set*, that is, one where no generator can be removed without changing the ideal. To state their result, which is presented as Theorem 2.14, we need some additional definitions. For sets $I, J \subseteq [n]$, let $m_{I,J}$ denote the minor of the generic matrix $[x_{ij}]_{1 \leq i,j \leq n}$ using rows I and columns J . We say $m_{I,J}$ belongs to some (i, j) in the essential set $E(w)$ if it is a Fulton generator associated to (i, j) , i.e. if $I \subseteq [i]$, $J \subseteq [j]$, and the rank of $m_{I,J}$ is $r(i, j) + 1$.

Definition 2.12. A minor $m_{I,J}$ attends $M^{[i',j']}$ if $|I \cap [i']| > r(i', j')$ and $|J \cap [j']| = r(i, j) + 1$, or if $|I \cap [i']| = r(i, j) + 1$ and $|J \cap [j']| > r(i', j')$.

Definition 2.13. A minor $m_{I,J}$ belonging to $(i, j) \in E(w)$ is *elusive* if it does not attend $M^{[i',j']}$ for all elements (i', j') in $E(w)$ such that $r(i', j') < r(i, j)$.

Theorem 2.14 ([7, Theorem 1.6 and Corollary 1.8]). *The Schubert determinantal ideal I_w is minimally generated by the set \mathcal{G}_w of elusive minors, which includes at least one minor with southeast corner b for each $b \in D(w)$. \mathcal{G}_w forms a Gröbner basis with respect to any antidiagonal term order.*

Example 2.15. Consider I_w for $w = 31542$, which has the following Rothe diagram:



The Knutson–Miller Gröbner basis \mathcal{G}_1 consists of two 1×1 minors, $\binom{4}{2}$ 2×2 minors, and $\binom{4}{3}$ 3×3 minors. The elusive minors are those which cannot be expressed in terms of the other Fulton generators via cofactor expansion. For example, the Fulton generator $g = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}$ attends $M^{[1,2]}$. This means that $g = z_{22}g_1 - z_{21}g_2$, where $g_1 = z_{11}$ and $g_2 = z_{12}$ are also Fulton generators of I_w . The minimal Gröbner basis of elusive minors for I_w is:

$$\mathcal{G}_w = \left\{ x_{11}, x_{12}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} \\ x_{41} & x_{42} \end{vmatrix}, \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix}, \begin{vmatrix} x_{11} & x_{13} & x_{14} \\ x_{21} & x_{23} & x_{24} \\ x_{31} & x_{33} & x_{34} \end{vmatrix}, \begin{vmatrix} x_{12} & x_{13} & x_{14} \\ x_{22} & x_{23} & x_{24} \\ x_{32} & x_{33} & x_{34} \end{vmatrix} \right\}.$$

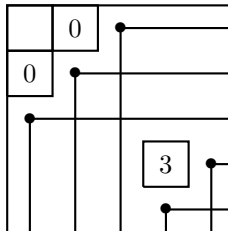
3. THE REDUCED GRÖBNER BASIS FOR I_w

The work of Gao–Yong [7] summarized in Theorem 2.14 describes a minimal Gröbner basis for any Schubert determinantal ideal I_w . Examples 3.1 and 3.2 illustrate that although the minimal Gröbner basis \mathcal{G}_w of elusive minors and the reduced Gröbner basis \mathcal{G}'_w have the same number of generators, they may not be identical.

Example 3.1. Let $w = 31542$ and recall the minimal Gröbner basis \mathcal{G}_w from Example 2.15. Note that some terms of the 3×3 minors are divisible by the generators x_{11} and x_{12} , so this Gröbner basis is not reduced. Applying the algorithm of Theorem 2.6(2), we find that in the reduced Gröbner basis \mathcal{G}'_w the degree-3 generators each have only four terms. Explicitly,

$$\mathcal{G}'_w = \left\{ x_{11}, x_{12}, \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \begin{vmatrix} x_{21} & x_{22} \\ x_{41} & x_{42} \end{vmatrix}, \begin{vmatrix} x_{31} & x_{32} \\ x_{41} & x_{42} \end{vmatrix}, \right. \\ \left. x_{14} \begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} - x_{13} \begin{vmatrix} x_{21} & x_{24} \\ x_{31} & x_{34} \end{vmatrix}, x_{14} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} - x_{13} \begin{vmatrix} x_{22} & x_{24} \\ x_{32} & x_{34} \end{vmatrix} \right\}.$$

Example 3.2. Consider I_w for $w = 32154$. The Rothe diagram for w is pictured below.



The Knutson–Miller Gröbner basis is equal to \mathcal{G}_w in this case, consisting of the 4×4 minor $g = \det M^{[4,4]}$ along with the three 1×1 minors x_{11}, x_{12}, x_{21} . The reduced Gröbner basis \mathcal{G}'_w is obtained by removing all terms of g containing these three variables, leaving behind a degree-4 generator with $8 = 2^{4-1}$ terms.

Proposition 3.3 generalizes Example 3.2 (where $n = 4$), proving that the lower bound on the number of terms in elements of \mathcal{G}'_w given by Theorem 1.4 is sharp. Our proof of Theorem 1.4 relies on reduction to this special case.

Proposition 3.3. *Let $w = (n - 2)(n - 3) \dots (2)(1)(n + 1)(n)$ for some $n \geq 3$. Then the Gao–Yong Gröbner basis for I_w is $\mathcal{G}_w = \{x_{ij}\}_{i+j \leq n-1} \cup \{g\}$, where $g = \det M^{[n,n]}$. The reduced Gröbner basis is $\mathcal{G}'_w = \{x_{ij}\}_{i+j \leq n-1} \cup \{g'\}$, where g' has degree n and 2^{n-1} terms.*

Proof. As in Example 3.2, it follows immediately from definitions that the Gao–Yong Gröbner basis \mathcal{G}_w has the claimed form in this case. Dividing g by $\mathcal{G}_w \setminus \{g\}$ leaves a remainder g' consisting of all terms in g not containing any x_{ij} with $i + j \leq n - 1$. By the reduction algorithm of Theorem 2.6(2) it follows that $\mathcal{G}'_w = (\mathcal{G}_w \setminus \{g\}) \cup \{g'\}$. It remains only to show that g' contains 2^{n-1} terms. Since g is the determinant of an $n \times n$ matrix, terms of g are in bijection with permutations in S_n . Permutations $v \in S_n$ corresponding to terms avoiding $\{x_{ij}\}_{i+j \leq n-1}$ are those satisfying $v(i) \geq n - i$ for all i . Constructing these permutations by iteratively choosing the value of $v(i)$ with two options at each step except the last shows that there are precisely 2^{n-1} terms in g' as claimed. \square

In order to reduce the proof of Theorem 1.4 to Proposition 3.3, we employ three lemmas which establish properties of elusive minors and their positioning relative to each other. For all of these lemmas, let $m_{I,J}$ be an elusive minor of rank d in \mathcal{G}_w with row indices $I = \{i_1, \dots, i_d\}$ and column indices $J = \{j_1, \dots, j_d\}$.

Lemma 3.4. *Let $m_{I,J}$ be as above. If (i_a, j) lies in the Rothe diagram $D(w)$ for some $1 \leq a < d$ and $j \geq j_d$, then $r_w(i_a, j) \geq a$. Similarly, if (i, j_b) lies in $D(w)$ for some $1 \leq b < d$ and $i \geq i_d$, then $r_w(i, j_b) \geq b$.*

Proof. This is immediate from the definition of an elusive minor. If $r(i_a, j) < a$ for some $(i_a, j) \in D(w)$ with $1 \leq a \leq d$ and $j \geq j_d$, then $m_{I,J}$ attends $M^{[i_a, j]}$. Any minor that attends an element of $D(w)$ attends some element of $E(w)$, so $m_{I,J}$ is not elusive. \square

Lemma 3.5. *Let $m_{I,J}$ be as above. Then (i_d, j_d) lies in $D(w)$.*

Proof. Let $m_{I,J}$ belong to some element (i, j) of the essential set $E(w)$, so $i \geq i_d$ and $j \geq j_d$. Since (i, j) lies in $E(w)$, we know that $w(i) > j$, and since $m_{I,J}$ is a $d \times d$ minor belonging to (i, j) we also know that $r(i, j) = d - 1$. Consider the sequence of elements (i, j_a) for $1 \leq a \leq d$. Since $w(i) > j$, each element (i, j_a) lies in $D(w)$ unless $w(k) = j_a$ for some $k < i$. The definition of the rank function implies that there cannot be more than $d - 1$ values of k satisfying $w(k) \leq j_d$, so at least one element (i, j_a) must lie in $D(w)$.

Suppose (i, j_c) lies in $D(w)$. If c is strictly less than d , then by Lemma 3.4 we know that $r(i, j_c) \geq c$. Thus we must have $w(k) \leq j_c$ for at least c values of k strictly less than i . This leaves at most $d - 1 - c$ values of k satisfying $j_c < w(k) \leq j_d$. Thus at least one of the $(c - d)$ elements $(i, j_{c+1}), \dots, (i, j_d)$ lies in $D(w)$. Iterating this argument shows that (i, j_d) lies in $D(w)$, and we see analogously that $(i_d, j) \in D(w)$. Thus (i_d, j_d) lies in $D(w)$ as claimed. \square

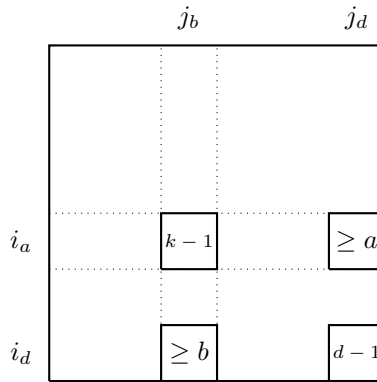


FIGURE 1. The situation of Lemma 3.6. The boxes (i_a, j_b) and (i_d, j_d) are the southeast corners of $m_{I', J'}$ and $m_{I, J}$ respectively. The expression in each box (i, j) indicates the value of $r_w(i, j)$.

Lemma 3.6. *Let $m_{I, J}$ be as above and let $m_{I', J'}$ ($I' \subset I, J' \subset J$) be a $k \times k$ sub-minor ($k < d$) that is also a generator of I_w . Then the main antidiagonal of $m_{I', J'}$ is weakly northwest of the $(d - 2)$ th antidiagonal of $m_{I, J}$, where the first antidiagonal is the northwest corner.*

Proof. Let the southeast corner of $m_{I', J'}$ be (i_a, j_b) for some a and b between k and d . Without loss of generality we may assume that $I' = \{i_{a-k+1}, i_{a-k+2}, \dots, i_a\}$ and $J' = \{j_{b-k+1}, j_{b-k+2}, \dots, j_b\}$, since among all $k \times k$ sub-minors of $m_{I, J}$ with southeast corner (i_a, j_b) this one has the most southeastern main antidiagonal. Writing down coordinates for points on the antidiagonals of $m_{I, J}$ and $m_{I', J'}$ reduces the proof to showing that $a + b \leq d + k - 2$.

Since $m_{I, J}$ has rank d we know that $r(i_d, j_d) = d - 1$. We claim that in addition $r(i_a, j_d) \geq a$. Let $c \in [a]$ be the greatest number such that (i_c, j_d) lies in $D(w)$ (or take $c = 0$ if no such c exists). Then by Lemma 3.4 $r(i_c, j_d) \geq c$. Furthermore, since $(i_{c'}, j_d) \notin D(w)$ for the $(a - c)$ integers c' satisfying $c < c' \leq a$ and $(i_d, j_d) \in D(w)$ by Lemma 3.5, we must have $w(i_{c'}) < j_d$ for all such c' . It follows that $r(i_a, j_d) \geq c + (a - c) = a$, since the rank function counts the number of $x \in [i_a]$ such that $w(x) < j_d$. The same argument shows that $r(i_d, j_b) \geq b$.

We now know that $r(i_a, j_b) = k - 1, r(i_d, j_d) = d - 1, r(i_a, j_d) \geq a$, and $r(i_d, j_b) \geq b$. These rank conditions express the locations of points $(x, w(x))$ in the permutation matrix M_w . It follows that $r(i_a, j_d) + r(i_d, j_b)$ is bounded above by $r(i_a, j_b) + r(i_d, j_d)$ (see Figure 1). Thus

$$a + b \leq r(i_a, j_d) + r(i_d, j_b) \leq r(i_a, j_b) + r(i_d, j_d) = d + k - 2. \quad \square$$

Proof of Theorem 1.4. By Theorem 2.14 and Theorem 2.6(2), the elements of \mathcal{G}'_w are the remainders obtained by dividing elusive minors $m_{I, J}$ in \mathcal{G}_w by each other. Suppose that some term of $m_{I, J}$ is divisible by the lead term (i.e., the antidiagonal term) of another elusive minor $m_{I', J'}$. Then in particular that term is divisible by each variable in the main antidiagonal of $m_{I', J'}$. By Lemma 3.6 any variable x_{ij} in the main antidiagonal of an elusive sub-minor $m_{I', J'}$ lies on or above the $(d - 2)$ th antidiagonal of $m_{I, J}$. We have therefore reduced the problem to showing that the

determinant of a generic $d \times d$ matrix contains 2^{d-1} terms that avoid the variables $x_{i,j}$ with $i + j \leq d - 1$. This is the case of Proposition 3.3. \square

4. VEXILLARY I_w

In this paper we do not attempt to describe the reduced Gröbner basis \mathcal{G}'_w explicitly. However, we can characterize the permutations w such that the minimal Gröbner basis \mathcal{G}_w is already reduced. This happens precisely when w is a *vexillary* (2143-avoiding) permutation. Our proof uses the equivalent characterization of vexillary permutations as those w such that all elements of the essential set $E(w)$ can be ordered into a list $\{e_1, e_2, \dots, e_n\}$ with each e_i weakly southwest of e_{i+1} . This characterization appears as Remark 9.17 in [6].

Proof of Theorem 1.5. Suppose first that w is vexillary and let $g \in \mathcal{G}_w$ be an elusive minor belonging to some $e \in E(w)$. If $LT(g')$ divides some term of g , then g' must belong to an $e' \in E(w)$ strictly northwest of the southeast corner of g . But then e' is strictly northwest of e since the southeast corner of g is (weakly) northwest of e , so w cannot be vexillary. It follows that no generators divide any terms of g , so this minimal Gröbner basis is in fact reduced.

Conversely, suppose that w is not vexillary, so there exist elements $e = (i, j)$ and $e' = (i', j')$ such that e' is strictly southeast of e (i.e., $i < i'$ and $j < j'$). After fixing e , we may choose e' among the elements of $E(w)$ with this property to minimize the difference k between $r' = r(i', j')$ and $r = r(i, j)$. Let $m_{I,J}$ be the maximally southeastern minor belonging to e and let $m_{I',J'}$ be the maximally southeastern minor belonging to e' and containing $m_{I,J}$ as a subminor. Then $m_{I,J}$ is an elusive minor belonging to e by Proposition 2.2 of [7], so to complete the proof it suffices to show that $m_{I',J'}$ is also elusive.

If $m_{I',J'}$ is the maximally southeastern minor belonging to e' then $m_{I',J'}$ is elusive by Proposition 2.2 of [7]. We therefore reduce to the case where

$$I' = I \cup \{i' - k + 1, i' - k + 2, \dots, i'\} \text{ and } J' = J \cup \{j' - k + 1, j' - k + 2, \dots, j'\}.$$

Suppose that $m_{I',J'}$ attends $M^{[a,b]}$ for some $(a, b) \in E(w)$. Then $r(a, b) < r'$, so $r(a, b) - r < r' - r$, which contradicts our choice of e' if (a, b) is strictly southeast of e . But if (a, b) is not strictly southeast of e , then $m_{I',J'}$ attends (a, b) if and only if $m_{I,J}$ does and we also obtain a contradiction. \square

5. BINOMIAL SCHUBERT DETERMINANTAL IDEALS

In order to establish Theorem 1.6, we first characterize, for each positive integer k , the permutations w such that $r_w(i, j) < k$ for all $(i, j) \in E(w)$. We only need the special case of this proposition where $k = 2$. However, the general proof requires no additional effort and may be of independent interest, so we provide it anyway.

Proposition 5.1. *The rank function r_w satisfies $r_w(i, j) < k$ for all $(i, j) \in E(w)$ if and only if w avoids the $k!$ patterns $\{v(k+2)(k+1) \mid v \in S_k\}$ in S_{k+2} . In particular, r_w evaluates to 0 or 1 on each element of $E(w)$ if and only if w avoids the patterns 1243 and 2143.*

Proof. We prove the contrapositive. Suppose w contains a pattern of the form $v(k+2)(k+1)$ for some $v \in S_k$ (written in one-line notation). Let $a_1 < a_2 < \dots < a_{k+2}$ witness this pattern, so $w(a_{k+2}) > w(a_i)$ for all $i \leq k$ and $w(a_{k+1}) > w(a_{k+2})$. Let $b_i = w(a_i)$ for $1 \leq i \leq k+2$. Then $d = (a_{k+1}, b_{k+2})$ lies in $D(w)$, as it is west

of (a_{k+1}, b_{k+1}) and north of (a_{k+2}, b_{k+2}) . Furthermore, since d lies southeast of (a_i, b_i) for each $i \leq k$, we know that $r_w(a_{k+1}, b_{k+2}) \geq k$. Let $e = (i, j) \in E(w)$ be a southeast corner of the connected component of $D(w)$ containing d . Then $r_w(i, j) = r_w(a_{k+1}, b_{k+2}) \geq k$.

Conversely, suppose that for some $e = (i, j)$ in $E(w)$ we have $r_w(i, j) \geq k$. Since $r_w(i, j) \geq k$ there must exist points $(a_i, w(a_i))$ for $1 \leq i \leq k$ such that $a_1 < \dots < a_k < i$ and each $w(a_i)$ is less than j . Also, since e lies in the Rothe diagram of w it follows that $w(i) = b$ for some $b > j$ and $j = w(a)$ for some $a > i$. Thus we have points $(i, w(i))$ and $(a, w(a))$ such that $i < a$ and $w(i) > w(a)$. Putting these pieces together, we see that the sequence (a_1, \dots, a_k, i, a) witnesses a pattern embedding of $v(k+2)(k+1)$ in w for some $v \in S_k$. \square

Proof of Theorem 1.6. By Proposition 5.1, the permutation w avoids the patterns 1243 and 2143 if and only if $r_w(i, j)$ evaluates to 0 or 1 on every element of the essential set $E(w)$. This happens if and only if the Fulton generators of I_w from Theorem 2.10 are all binomials by definition. By Theorem 2.14 it follows that the Fulton generators of I_w are all binomials if and only if the elusive minors of the Gao–Yong minimal Gröbner basis \mathcal{G}_w are all binomials. It therefore suffices to show that I_w is a binomial ideal if and only if the set \mathcal{G}_w of elusive minors consists of binomials. One direction is immediate, so we need only prove that if I_w is a binomial ideal then \mathcal{G}_w consists of binomials.

We know from Proposition 2.7 that I_w is a binomial ideal if and only if its reduced Gröbner basis \mathcal{G}'_w consists of binomials. We therefore need to show that if \mathcal{G}_w contains a $k \times k$ minor for $k \geq 3$, then \mathcal{G}'_w contains a generator with ≥ 3 terms. This follows immediately from Theorem 1.4: $k \times k$ minors are polynomials of degree k , so if \mathcal{G}_w contains such a minor then the corresponding generator in \mathcal{G}'_w has at least $2^{3-1} = 4$ terms. \square

6. REGULARITY OF BINOMIAL I_w

As our final application, we will demonstrate that two recently-proven formulas for the regularity of different classes of ideals coincide for binomial I_w . On one side, Rajchgot–Robichaux–Weigandt provided a formula in [14, Theorem 1.5] for vexillary Schubert determinantal ideals, which includes the binomial I_w as a special case by Theorem 1.6. On the other side, Almousa–Dochtermann–Smith gave a formula for the regularity of *toric edge ideals* of bipartite graphs in [1, Corollary 6.7]. Binomial I_w can be realized as toric edge ideals as shown by Portakal in [13, p. 1273]. We begin by recalling the definition of regularity. A standard reference is [3, Chapter 20].

View $R = \mathbb{k}[x_{ij}]_{1 \leq i, j \leq n}$ as a graded ring in the standard way (so each x_{ij} has degree 1), and let $R(-a)$ be R with all degrees shifted by a (so each x_{ij} has degree $1 + a$). For a homogeneous ideal $I \subseteq R$, a (*graded*) *free resolution* of R/I is an exact sequence of free graded R -modules in the following form:

$$0 \rightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{b_{ij}} \xrightarrow{\partial_i} \dots \xrightarrow{\partial_1} \bigoplus_{j \in \mathbb{Z}} R(-j)^{b_{0j}} \xrightarrow{\partial_0} R/I \rightarrow 0.$$

Exactness means that $\ker(\partial_i) = \text{im}(\partial_{i+1})$ for all i . The maps ∂_i may be written as matrices, and a free resolution is called *minimal* if no units of R appear in these matrices. Equivalently, a minimal free resolution simultaneously minimizes all the numbers b_{ij} . It turns out that R/I has a minimal free resolution, which is unique

up to isomorphism [3, Theorem 20.2]. The constants b_{ij} appearing in the minimal free resolution are denoted by β_{ij} and called the *graded Betti numbers* of R/I . The (Castelnuovo–Mumford) *regularity* of R/I is defined as

$$\text{reg}(I) = \max\{j - i \mid \beta_{ij} \neq 0\}.$$

The following result is a well-known general fact about minimal free resolutions. Once one recalls the standard notions from homological algebra (which we omit here) it follows immediately from the fact that tensoring over a field is an exact functor.

Proposition 6.1. *Let $I \subseteq R = \mathbb{k}[x_1, \dots, x_r]$ be an ideal and suppose a generating set $G = \{g_1, \dots, g_s\}$ for I can be partitioned into subsets G_1 and G_2 such that $G_1 \subset R_1 = \mathbb{k}[x_1, \dots, x_k]$ and $G_2 \subset R_2 = \mathbb{k}[x_{k+1}, \dots, x_r]$ for some k . Let I_1 and I_2 be the ideals generated by G_1 in R_1 and G_2 in R_2 respectively. Let F^\bullet_1 and F^\bullet_2 be minimal free resolutions of R_1/I_1 and R_2/I_2 . Then the minimal free resolution of R/I is given by the tensor product $(F^1 \otimes_{\mathbb{k}} F^2)^\bullet$. In particular, this implies that*

$$\beta_{a,b}(I) = \sum_{i+i'=a} \sum_{j+j'=b} \beta_{i,j}(I_1)\beta_{i',j'}(I_2)$$

and

$$\text{reg}(I) = \text{reg}(I_1) + \text{reg}(I_2).$$

Definition 6.2. A permutation v is *dominant* if r_v evaluates to 0 on all of $E(v)$. When v is dominant, its Rothe diagram consists of a single connected component λ , which we call its *shape*. (This is a shape of a Young diagram.)

Definition 6.3. Let w be a permutation and let D_1, \dots, D_k be the connected components of $D(w)$. For $1 \leq i \leq k$, let v_i be a dominant permutation of shape D_i and let r_i be the value of the rank function on D_i . Let $u_i = 1^{r_i} \times v_i$ be the permutation such that $u_i(j) = j$ for $1 \leq j \leq r_i$ and $u_i(j) = v_i(j) + r_i$ for $r_i + 1 \leq j$. The *parts* of I_w are the ideals $\{I_{u_i}\}_{i=1}^k$. The *dominant part* of I_w is I_{u_i} for the unique u_i that is dominant.

In the case where $I = I_w$ is a binomial Schubert determinantal ideal, every generator of I_w belonging to an element of D_i can be viewed as a generator of the part I_{u_i} . Furthermore, no generators belonging to distinct D_i share variables. This follows from the characterization of toric matrix Schubert varieties given by Escobar and Mészáros in [5, Theorem 3.5]. Combining this observation with Proposition 6.1 and the fact that the dominant part of I_w has regularity 0 yields the following result:

Theorem 6.4. *Let I_w be a binomial Schubert determinantal ideal with dominant part I_{u_0} and non-dominant parts I_{u_1}, \dots, I_{u_k} . Then*

$$\text{reg}(I_w) = \sum_{i=1}^k \text{reg}(I_{u_i}).$$

Remark 6.5. This decomposition can be done for some non-binomial vexillary I_w . More precisely, it can be done whenever w avoids the patterns 2143, 14253, 15243, and their inverses, although we leave the proof for future work.

Theorem 6.4 implies that the Rajchgot–Robichaux–Weigandt and Almoussa–Dochtermann–Smith formulas for the regularity of binomial I_w are equal, provided they are equal for $I_{1 \times v}$ when v is a dominant permutation. We now present these formulas, beginning with that of Rajchgot–Robichaux–Weigandt.

Definition 6.6. The *canonical antidiagonal* of a partition λ is the antidiagonal sequence $C_\lambda = \{e_1, \dots, e_k\}$ of maximum length in λ such that $e_i = \{k - i + 1, i\}$ for each i .

Theorem 6.7 (Special case of [14, Theorem 1.5]). *Let v be a dominant permutation of shape λ . Then $\text{reg}(I_{1 \times v}) = |C_\lambda|$.*

Portakal expressed $I_{1 \times v}$ (v dominant) as the toric edge ideal of a bipartite graph:

Definition 6.8. The *thickening* of a partition λ is the partition $\bar{\lambda}$ with $\bar{\lambda}_1 = \lambda_1 + 1$ and $\bar{\lambda}_i = \lambda_{i-1} + 1$ for all $i > 1$.

Definition 6.9. The *graph* of a partition λ with m rows and n columns is the connected bipartite graph $B_\lambda \subseteq K_{m,n}$ with edges (i, j) whenever (i, j) lies in the diagram of λ .

Theorem 6.10 ([13, p. 1273]). *Let v be dominant of shape λ . Then $I_{1 \times v}$ is the toric edge ideal of the graph $B_{\bar{\lambda}}$ of the thickening $\bar{\lambda}$.*

Using Theorem 6.10, the following formula of Almousa–Dochtermann–Smith also gives the regularity of $I_{1 \times v}$ for v dominant.

Definition 6.11. Let $B \subset K_{m,n}$ be a bipartite graph and S a subgraph. The *recession graph* $R(S; B)$ is the directed bipartite graph built from B by directing the edges in $G \setminus S$ from $[m]$ to $[n]$ and making the edges of S bidirectional.

Definition 6.12. The *recession connectivity* $r(B)$ of a bipartite graph B is the maximum number of components in a subgraph $S \subset B$ such that $R(S; B)$ is strongly connected.

Theorem 6.13 ([1, Corollary 6.7]). *If B is a connected bipartite graph, then the regularity of the toric edge ideal I_B is $r(B) - 1$.*

Since the formulas in Theorem 6.7 and Theorem 6.13 both compute $\text{reg}(I_{1 \times v})$, they must be equal. We conclude this paper with a direct proof that the formulas agree.

Lemma 6.14. *For any partition λ , $r(B_{\bar{\lambda}}) \leq |C_\lambda| + 1$.*

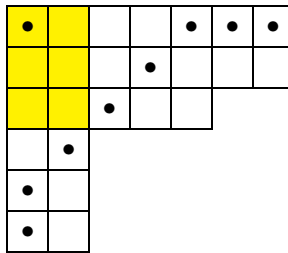
Proof. Note that $r(B_{\bar{\lambda}})$ is bounded above by the matching number $m(B_{\bar{\lambda}})$ of $B_{\bar{\lambda}}$, since taking one edge from each connected component of a subgraph $S \subset B_{\bar{\lambda}}$ yields a matching. If the number of rows or columns in λ is equal to $|C_\lambda|$, then $|C_{\bar{\lambda}}| = |C_\lambda| + 1$. Since $|C_\mu| = m(B_\mu)$ for any partition μ , $r(B_{\bar{\lambda}}) \leq |C_\lambda| + 1$ in this case. Now suppose λ has more than $|C_\lambda|$ rows and columns, so $|C_{\bar{\lambda}}| = |C_\lambda| + 2$. This means there exist $(a + 1, b)$ and $(a, b + 1)$ in $C_{\bar{\lambda}}$ such that $(a + 1, b + 1)$ is not in $\bar{\lambda}$.

Now let $S \subset B_{\bar{\lambda}}$ be such that $R(S; B_{\bar{\lambda}})$ is strongly connected. Note that S must contain an edge $e \in [a] \times [b]$, since otherwise $R(S; B_{\bar{\lambda}})$ cannot have a path from the column vertex 1 to the edge vertex 1. The number of connected components of S is then witnessed by a matching in $B_{\bar{\lambda}}$ containing e . Removing the row and column of $\bar{\lambda}$ intersecting in e yields an auxiliary partition ν . It is clear that $|C_\nu| \geq |C_{\bar{\lambda}}| - 2$, and the reverse inequality must hold because $C_{\bar{\lambda}}$ passes through $(a + 1, b)$ and ν contains neither $(a + 1, b)$ nor (a, b) . This implies that the number of components of S is bounded above by $m(B_\nu) + 1 = |C_{\bar{\lambda}}| - 2 + 1 = |C_\lambda| + 1$, completing the proof. □

Theorem 6.15. *For any partition λ , $r(B_{\bar{\lambda}}) = |C_{\lambda}| + 1$.*

Proof. We explicitly construct a subgraph $S \subset B_{\bar{\lambda}}$ with $|C_{\lambda}| + 1$ connected components such that $R(S; B)$ is strongly connected. Let $S' = C_{\lambda} \cup \{(1, j), (i, 1) : i, j > |C_{\lambda}|\}$. Now let $S = \{(i + 1, j + 1) | (i, j) \in S'\} \cup \{(1, 1)\}$, so S consists of S' (regarded as a subset of $\bar{\lambda}$) along with $(1, 1)$. It is clear that S has $|C_{\lambda}| + 1$ components when viewed as a collection of edges in the graph $B_{\bar{\lambda}}$. It is also straightforward to verify that $R(S; B_{\bar{\lambda}})$ is strongly connected. This proves that $r(B_{\bar{\lambda}}) \geq |C_{\lambda}| + 1$. Equality follows by Lemma 6.14. \square

Example 6.16. With $\lambda = (6, 4, 1, 1, 1)$ and $\bar{\lambda} = (7, 7, 5, 2, 2)$, the subgraph $S \subset B_{\bar{\lambda}}$ constructed in Theorem 6.15 corresponds to the \bullet 's in $\bar{\lambda}$. In this case $|C_{\lambda}| = 3$ while λ has 5 rows and 6 columns, which implies that $|C_{\bar{\lambda}}| = |C_{\lambda}| + 2$. In the notation of the proof of Lemma 6.14 we have $[a] \times [b] = [3] \times [2]$. This box is highlighted in yellow on the diagram.



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