DECONVOLUTIONAL DETERMINATION OF THE NONLINEARITY IN A SEMILINEAR WAVE EQUATION

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Abstract. We demonstrate that in three space dimensions, the scattering behaviour of semilinear wave equations with quintic-type nonlinearities uniquely determines the nonlinearity. The nonlinearity is permitted to depend on both space and time.

1. Introduction

We consider the semilinear wave equation

$$\begin{cases} (\partial_{tt} - \Delta_x)u(t,x) = F(t,x,u(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3; \\ u(0,\,\cdot\,) = u_0; \\ \partial_t u(0,\,\cdot\,) = u_1. \end{cases} \tag{1.1}$$

Under mild assumptions on the nonlinearity $F : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we show that this equation admits a small-data scattering theory and that the scattering operator *determines* the nonlinearity. The specific class of nonlinearities we consider is given in Definition 1.1 and may be regarded as a generalization of the energy-critical case. The main inspiration for the problem we study is the paper [12] of Sá Barreto, Uhlmann, and Wang. Our methods, however, are more strongly influenced by Killip, Murphy, and Vișan [6].

The requirements that we impose on the nonlinearity are as follows:

Definition 1.1 (Admissible nonlinearity)**.** A measurable function $F : \mathbb{R}\times\mathbb{R}^3\times\mathbb{R} \to \mathbb{R}$ will be called *admissible* for equation (1.1) if

(i) $F(t,x,0) = 0$ for all t, x;
(ii) $|F(t,x,u) - F(t,x,v)| \lesssim (|u|^4 + |v|^4)|u-v|$ for all u, v uniformly in t, x; and
(iii) $F(t,x,-u) = -F(t,x,u)$ for all t, x.

If $F(t,x,u) = \pm|u|^4 u$, the resulting equation is the defocusing/focusing (depending on the sign of the nonlinearity) energy-critical wave equation. This name reflects the fact that in this case, the equation enjoys a scaling symmetry

$$u(t,x) \mapsto u^\lambda(t,x) = \lambda^{\frac{1}{2}} u\big(\lambda t, \lambda x\big) \qquad \text{for } \lambda > 0$$

that preserves the energy of solutions

$$E(u) = \int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla u(t,x)|^2 + \tfrac{1}{2}|\partial_t u(t,x)|^2 \pm \tfrac{1}{6}|u(t,x)|^6 \, dx.$$

Accordingly, we will be studying equation (1.1) with initial data (u_0, u_1) in the energy space $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

Definition 1.2 (Solution)**.** A function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is said to be a *strong global solution* of equation (1.1) if $(u, \partial_t u) \in C_t^0 \dot{H}_x^1(\mathbb{R} \times \mathbb{R}^3) \times C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^3)$, $u \in L_t^5 L_x^{10}(K \times \mathbb{R}^3)$ for all compact sets $K \subseteq \mathbb{R}$, and u satisfies the Duhamel formula

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds.$$

Here $\mathcal{U}$ denotes the propagator for the linear wave equation, that is,

$$\mathcal{U}(t) := \begin{bmatrix} \cos(t|\nabla|) & \dfrac{\sin(t|\nabla|)}{|\nabla|} \\ -|\nabla| \sin(t|\nabla|) & \cos(t|\nabla|) \end{bmatrix}.$$

Here and in what follows we abbreviate $u(t, \cdot\,)$ as $u(t)$ and $F(t, \cdot\,, u(t))$ as $F(t)$.

For the definitions of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ and the mixed-norm spaces above, see Section 1.1. The spaces in Definition 1.2 are chosen to capture the dispersive behaviour of solutions while matching the criticality of the problem. The Strichartz estimates in Theorem 2.1 show that solutions to the linear wave equation also belong to these spaces and allow us to construct solutions to the semilinear wave equation (1.1) via contraction mapping. In fact, we will prove that this equation admits a small-data global well-posedness and scattering theory for admissible nonlinearities.

Theorem 1.3 (Small-data scattering)**.** *Let F be an admissible nonlinearity for equation* (1.1)*. Then there exists an $\eta > 0$ such that equation* (1.1) *has a unique global solution u satisfying*

$$\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \tag{1.2}$$

whenever $(u_0, u_1) \in B_\eta$, where

$$B_\eta := \{(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) : \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} < \eta\}.$$

This solution scatters in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ as $t \to \pm\infty$, meaning that there exist (*necessarily unique*) *asymptotic states $(u_0^\pm, u_1^\pm) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for which*

$$\left\| \begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} - \mathcal{U}(t) \begin{bmatrix} u_0^\pm \\ u_1^\pm \end{bmatrix} \right\|_{\dot{H}^1 \times L^2} \to 0 \quad \text{as } t \to \pm\infty. \tag{1.3}$$

In addition, for all $(u_0^-, u_1^-) \in B_\eta$, there exists a unique global solution u to equation (1.1) *and a unique asymptotic state $(u_0^+, u_1^+) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ so that both limits in* (1.3) *hold.*

The map $(u_0, u_1) \mapsto (u_0^+, u_1^+)$ defined implicitly by Theorem 1.3 on the open ball $B_\eta \subseteq \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is the inverse of what is often called the *forward wave operator*; in this paper, we will refer to it simply as the *wave operator* and we will denote it by W_F. The map $(u_0^-, u_1^-) \mapsto (u_0^+, u_1^+)$ is the *scattering operator* and will be denoted S_F. Our principal result is that either operator determines the nonlinearity completely.

Our hypotheses on the nonlinearity F do not demand any continuity in t or x. Avoiding such a restriction is important for us as we wish to allow nonlinearities of the form $1_\Omega(x)u^5$, which model a nonlinear medium (whose shape we wish to determine) surrounded by vacuum.

Without a continuity requirement, complete determination of the nonlinearity means determination at (Lebesgue) almost every spacetime point. We can be very precise about the spacetime points at which we determine the nonlinearity:

Definition 1.4 (Determinable point)**.** Suppose that F is an admissible nonlinearity for equation (1.1). A point $(t,x)\in\mathbb{R}\times\mathbb{R}^3$ will be called *determinable* for F if it is a Lebesgue point of $F(\,\cdot\,,\,\cdot\,,u)$ for every rational u. The set of all such points will be denoted D_F.

For each fixed u, the map $(t,x)\mapsto F(t,x,u)$ is bounded and measurable and so almost every point is a Lebesgue point. The countability of the rational numbers then guarantees that almost every spacetime point is determinable.

Theorem 1.5. *Suppose that F and $\widetilde{F}$ are admissible nonlinearities for equation* (1.1) *and that B_η and $B_{\widetilde{\eta}}$ are corresponding balls given by Theorem 1.3. If W_F and $W_{\widetilde{F}}$, or S_F and $S_{\widetilde{F}}$, agree on $B_\eta\cap B_{\widetilde{\eta}}$ (that is, the smaller of the two balls), then $F(t,x,\,\cdot\,)=\widetilde{F}(t,x,\,\cdot\,)$ for all $(t,x)\in D_F\cap D_{\widetilde{F}}$.*

The question of whether the nonlinearity in a dispersive PDE is determined by its scattering behaviour has been extensively studied [1, 8, 10, 12–16, 18–27]. Usually, rather strong assumptions are imposed on the nonlinearity in order to obtain a positive answer.

In contrast, Killip, Murphy, and Vișan's deconvolution-based approach [6] enabled them to determine power-type nonlinearities in a semilinear Schrödinger equation with only moderate growth restrictions on the nonlinearities. Their approach is flexible and technically simple, as demonstrated by its subsequent application to the determination of coefficients [4, 9] and inhomogeneities [2, 3] of nonlinear Schrödinger equations.

In this paper, we revisit the setting considered by Sá Barreto, Uhlmann, and Wang [12], who determined nonlinearities of the form $F=F(u)$ in equation (1.1) under the following assumptions:

(i) $F(u)=h(u)u$ for some even function h satisfying $|h(u)|\approx|u|^4$ for all u;
(ii) $F'(u)u\sim F(u)$ as $u\to 0$ and as $u\to\pm\infty$;
(iii) $u\mapsto\int_0^u F(v)\,dv$ is convex;
(iv) $|F^{(j)}(u)|\lesssim|u|^{5-j}$ for each $0\le j\le 5$; and
(v) $F^{(4)}(u)=0$ if and only if $u=0$.

By adapting the deconvolution technique of [6] to the setting of the wave equation, we will prove that even more general nonlinearities of the form $F=F(t,x,u)$ can be determined under the weaker conditions of Definition 1.1.

Let us now turn to an overview of the paper, the method of [6], and the principal challenges to be overcome in applying it in the wave equation setting.

Our first task is to establish the existence, uniqueness, and long-time behaviour of solutions to (1.1) for small initial data and for admissible nonlinearities. This is Theorem 1.3, which we prove in Section 2.

Following [6], our approach to identifying the nonlinearity is through the small-data asymptotics of the scattering and wave operators. These are presented in Corollary 2.2, which gives a precise estimate on the difference between the full operators and what is known as their *Born approximation*.

Under the Born approximation, the scattering/wave operators capture the spacetime integral of $u(t,x)F(t,x,u(t,x))$, where $u(t,x)$ is a solution of the *linear* wave equation. This evidently represents a substantial 'blurring' of the nonlinearity across different values of t, x, and u. If the nonlinearity did not depend on t and x, then this blurring would take the form of a convolution (over the multiplication group). By switching to exponential variables, this then would become a convolution in the traditional sense. In this way, the question of identifying the nonlinearity reduces to a deconvolution problem. As we will discuss in Section 4, the uniqueness criterion for such deconvolution problems is the well-known L^1 Tauberian theorem of Wiener; see Theorem 4.1.

To overcome the dependence of the nonlinearity on space and time, we will employ a solution of the linear wave equation that concentrates tightly at a single point in spacetime (while also remaining small in scaling-critical norms). As noted earlier, we do not assume that the nonlinearity is continuous in t or x; consequently, there are some subtleties to be overcome in localizing the nonlinearity to a single spacetime point. This is the role of Lemma 3.3. With this hurdle overcome, the uniqueness question is reduced to the deconvolution problem presented in Proposition 3.2.

We now arrive at the crux of the matter: we need to find solutions to the linear wave equation that lead to a deconvolution problem that can actually be solved. Concretely, we must find a linear solution whose distribution function we can compute sufficiently explicitly that we will be able to verify the hypotheses of Wiener's Tauberian theorem. The distribution function for the solution we choose is computed in Lemma 3.1. Although we are unable to compute the resulting Fourier transform precisely, we are nonetheless able to verify that it is nonvanishing (see Proposition 4.2) and consequently to apply the Tauberian theorem.

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1.1. **Notation.** Throughout this paper, we employ the standard notation $A \lesssim B$ to indicate that $A \leq CB$ for some constant $C > 0$; if $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$. Occasionally, we adjoin subscripts to this notation to indicate dependence of the constant C on other parameters; for instance, we write $A \lesssim_{\alpha,\beta} B$ when $A \leq CB$ for some constant $C > 0$ depending on α, β.

The homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ is defined as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{\dot{H}^1(\mathbb{R}^3)} := \||\xi|\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^3)} \approx \||\nabla|f\|_{L^2(\mathbb{R}^3)},$$

where we use the Fourier transform convention

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}\,dx.$$

We write $L^p_t L^q_x(I \times \mathbb{R}^3)$ for the mixed Lebesgue space on a spacetime region $I \times \mathbb{R}^3$ with norm

$$\|u\|_{L^p_t L^q_x(I\times\mathbb{R}^3)} := \big\|\|u(t,x)\|_{L^q_x(\mathbb{R}^3)}\big\|_{L^p_t(I)}$$

(with the understanding that $I = \mathbb{R}$ if it is unspecified). Similarly, we define the mixed-norm space $L^\infty_t \dot{H}^1_x(I \times \mathbb{R}^3)$ with norm

$$\|u\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^3)} := \big\|\|u(t,x)\|_{\dot{H}^1_x(\mathbb{R}^3)}\big\|_{L^\infty_t(I)}.$$

We write $C_t^0\dot{H}_x^1$ for the subspace of functions that are continuous in time with values in $\dot{H}^1(\mathbb{R}^3)$.

2. Small-data scattering

We begin by establishing the small-data scattering theory described in Theorem 1.3. This relies on a standard contraction mapping argument using Strichartz estimates.

Theorem 2.1 (Strichartz estimates, [5, 11, 17])**.** *If* $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ *is a global solution of equation* (1.1)*, then*

$$\|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|F(t, x, u(t, x))\|_{L_t^1 L_x^2}.$$

The contraction mapping argument constructs the solution from the Duhamel formula

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds. \tag{2.1}$$

Similarly, the solution with prescribed asymptotic state (u_0^-, u_1^-) as $t \to -\infty$ is constructed from the formula

$$\begin{bmatrix} u(t) \\ \partial_t u(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0^- \\ u_1^- \end{bmatrix} + \int_{-\infty}^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds. \tag{2.2}$$

Proof of Theorem 1.3. Let

$$\begin{aligned} X := \big\{ & u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} : \\ & (u, \partial_t u) \in C_t^0 \dot{H}_x^1(\mathbb{R} \times \mathbb{R}^3) \times C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^3),\ u \in L_t^5 L_x^{10}(\mathbb{R} \times \mathbb{R}^3), \\ & \|(u, \partial_t u)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u\|_{L_t^5 L_x^{10}} \le 2C \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \big\}, \end{aligned}$$

where C is the implicit constant in the Strichartz estimates. Equipping X with the metric

$$d(u, v) := \|(u, \partial_t u) - (v, \partial_t v)\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|u - v\|_{L_t^5 L_x^{10}},$$

we obtain a nonempty complete metric space (X, d).

For $u \in X$, we then define

$$(\Phi(u))(t) := \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} u_1 + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} F(s)\, ds$$

so that

$$\begin{bmatrix} (\Phi(u))(t) \\ (\partial_t \Phi(u))(t) \end{bmatrix} = \mathcal{U}(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t \mathcal{U}(t-s) \begin{bmatrix} 0 \\ F(s) \end{bmatrix} ds. \tag{2.3}$$

To construct the solution of equation (1.1), we will show that Φ is a contraction on (X, d) whenever $(u_0, u_1) \in B_\eta$ and η is sufficiently small. The solution sought will then be the fixed point of Φ whose existence and uniqueness are guaranteed by the Banach fixed point theorem.

We first verify that Φ maps X into itself. Let C_F be a constant such that $|F(t, x, u)| \le C_F |u|^5$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. If $u \in X$, then by the Strichartz estimates, we have

$$\begin{aligned} & \|(\Phi(u), \partial_t \Phi(u))\|_{L_t^\infty \dot{H}_x^1 \times L_t^\infty L_x^2} + \|\Phi(u)\|_{L_t^5 L_x^{10}} \\ & \quad \le C(\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|F(t, x, u(t, x))\|_{L_t^1 L_x^2}) \end{aligned}$$

$$\begin{aligned}
&\le C(\|(u_0,u_1)\|_{\dot H^1\times L^2} + C_F\|u\|^5_{L^5_tL^{10}_x})\\
&\le C[1 + C_F(2C\eta)^4(2C)]\|(u_0,u_1)\|_{\dot H^1\times L^2}\\
&\le 2C\|(u_0,u_1)\|_{\dot H^1\times L^2},
\end{aligned}$$

provided that η is sufficiently small.

To show that $(\Phi(u))(t)$ and $(\partial_t\Phi(u))(t)$ are also *continuous* in t, fix a $t_0 \in \mathbb{R}$ and consider, without loss of generality, the case when $t \ge t_0$. The first term on the right-hand side of formula (2.3) converges to $\mathcal{U}(t_0)(u_0,u_1)$ in $\dot H^1 \times L^2$ as $t \to t_0$ since $\mathcal{U}(t)$ is strongly continuous in t. As for the second term, we observe that

$$\begin{aligned}
&\left\| \int_0^t \mathcal{U}(-s)\begin{bmatrix}0\\F(s)\end{bmatrix}\,ds - \int_0^{t_0}\mathcal{U}(-s)\begin{bmatrix}0\\F(s)\end{bmatrix}\,ds\right\|_{\dot H^1\times L^2}\\
&\quad\le \left\|\int_{t_0}^t \frac{\sin(-s|\nabla|)}{|\nabla|}F(s)\,ds\right\|_{\dot H^1} + \left\|\int_{t_0}^t \cos(-s|\nabla|)F(s)\,ds\right\|_{L^2}\\
&\quad\lesssim \int_{t_0}^t \|F(s)\|_{L^2}\,ds\\
&\quad\lesssim \|u\|^5_{L^5_tL^{10}_x([t_0,t]\times\mathbb{R}^3)} \to 0 \quad \text{as } t\to t_0,
\end{aligned}$$

by the dominated convergence theorem. Consequently, the second term converges to $\mathcal{U}(t_0)\int_0^{t_0}\mathcal{U}(-s)(0,F(s))\,ds$ in $\dot H^1\times L^2$ as $t\to t_0$ since $\mathcal{U}(t)$ is strongly continuous and uniformly bounded in t. Altogether, this shows that $\Phi(u)\in X$ as required.

Now if $u, v \in X$, the Strichartz estimates also yield

$$\begin{aligned}
d(\Phi(u),\Phi(v)) &\lesssim \|F(t,x,u(t,x)) - F(t,x,v(t,x))\|_{L^1_tL^2_x}\\
&\lesssim \|(|u|^4+|v|^4)|u-v|\|_{L^1_tL^2_x}\\
&\lesssim \big(\|u\|^4_{L^5_tL^{10}_x} + \|v\|^4_{L^5_tL^{10}_x}\big)\|u-v\|_{L^5_tL^{10}_x}\\
&\lesssim [(2C\eta)^4 + (2C\eta)^4]\,d(u,v),
\end{aligned}$$

which shows that Φ is a contraction for sufficiently small η.

Next, we prove that the solution u scatters in $\dot H^1\times L^2$. As $\mathcal{U}(t)$ is unitary on $\dot H^1\times L^2$, this amounts to showing that the functions $\mathcal{U}^{-1}(t)(u(t),\partial_t u(t))$ converge in $\dot H^1\times L^2$ as $t\to\pm\infty$. By time reversal symmetry, it suffices to consider $t\to+\infty$. For $t_2\ge t_1\ge T$,

$$\begin{aligned}
&\left\|\mathcal{U}^{-1}(t_2)\begin{bmatrix}u(t)\\\partial_t u(t)\end{bmatrix} - \mathcal{U}^{-1}(t_1)\begin{bmatrix}u(t)\\\partial_t u(t)\end{bmatrix}\right\|_{\dot H^1\times L^2}\\
&\quad= \left\|\int_0^{t_2}\mathcal{U}(-s)\begin{bmatrix}0\\F(s)\end{bmatrix}\,ds - \int_0^{t_1}\mathcal{U}(-s)\begin{bmatrix}0\\F(s)\end{bmatrix}\,ds\right\|_{\dot H^1\times L^2}\\
&\quad\lesssim \|u\|^5_{L^5_tL^{10}_x([t_1,t_2]\times\mathbb{R}^3)} \to 0 \quad\text{as } T\to\infty,
\end{aligned}$$

by the dominated convergence theorem. We conclude that $\{\mathcal{U}^{-1}(t)(u(t),\partial_t u(t))\}$ is Cauchy in $\dot H^1\times L^2$ as $t\to\infty$ and therefore convergent.

This completes the construction of the wave operator. The construction of the scattering operator, using (2.2) in place of (2.1), is entirely analogous. □

We note that the foregoing argument shows that the wave operator is given by

$$W_F\left(\begin{bmatrix}u_0\\u_1\end{bmatrix}\right) = \begin{bmatrix}u_0\\u_1\end{bmatrix} + \int_0^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(t)\end{bmatrix}dt, \tag{2.4}$$

where u is the solution of equation (1.1) with initial data (u_0, u_1). Similarly, the scattering operator is given by

$$S_F\left(\begin{bmatrix}u_0^-\\u_1^-\end{bmatrix}\right) = \begin{bmatrix}u_0^-\\u_1^-\end{bmatrix} + \int_{-\infty}^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(t)\end{bmatrix}dt, \tag{2.5}$$

where u is the solution of equation (1.1) that scatters to (u_0^-, u_1^-) as $t \to -\infty$.

Corollary 2.2 (Small-data asymptotics for the wave and scattering operators)**.** *Suppose that F is an admissible nonlinearity for equation* (1.1) *and that B_η is a corresponding ball given by Theorem 1.3. If u_{lin} denotes the solution of the linear wave equation with initial data $(u_0, u_1) \in B_\eta$, then (in $\dot{H}^1 \times L^2$) we have*

$$W_F\left(\begin{bmatrix}u_0\\u_1\end{bmatrix}\right) = \begin{bmatrix}u_0\\u_1\end{bmatrix} + \int_0^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F_{\text{lin}}(t)\end{bmatrix}dt + \mathcal{O}\left(\left\|\begin{bmatrix}u_0\\u_1\end{bmatrix}\right\|_{\dot{H}^1\times L^2}^9\right). \tag{2.6}$$

Similarly, given $(u_0^-, u_1^-) \in B_\eta$, let u be the solution of equation (1.1) *that scatters to (u_0^-, u_1^-) as $t \to -\infty$. If u_{lin} denotes the solution of the linear wave equation with initial data $(u_0, u_1) := (u(0), \partial_t u(0)) \in B_\eta$, then*

$$S_F\left(\begin{bmatrix}u_0^-\\u_1^-\end{bmatrix}\right) = \begin{bmatrix}u_0^-\\u_1^-\end{bmatrix} + \int_{-\infty}^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F_{\text{lin}}(t)\end{bmatrix}dt + \mathcal{O}\left(\left\|\begin{bmatrix}u_0\\u_1\end{bmatrix}\right\|_{\dot{H}^1\times L^2}^9\right). \tag{2.7}$$

Here $F_{\text{lin}}(t)$ is an abbreviation for $F(t, \cdot, u_{\text{lin}}(t))$.

Proof. We will derive the asymptotic expansion (2.7) from formula (2.5) for the scattering operator; the derivation of (2.6) from formula (2.4) for the wave operator is similar.

Comparing (2.5) with (2.7), we see that the latter follows from

$$\left\|\int_{-\infty}^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(t)-F_{\text{lin}}(t)\end{bmatrix}dt\right\|_{\dot{H}^1\times L^2} \lesssim \left\|\begin{bmatrix}u_0\\u_1\end{bmatrix}\right\|_{\dot{H}^1\times L^2}^9,$$

which we will prove by duality. To this end, fix some $(v_0, v_1) \in \dot{H}^1 \times L^2$ and let v_{lin} denote the solution of the linear wave equation with initial data (v_0, v_1). Then

$$\begin{aligned}
&\left\langle \int_{-\infty}^\infty \mathcal{U}(-t)\begin{bmatrix}0\\F(t)-F_{\text{lin}}(t)\end{bmatrix}dt, \begin{bmatrix}v_0\\v_1\end{bmatrix}\right\rangle_{\dot{H}^1\times L^2}\\
&= \int_{-\infty}^\infty \left\langle \begin{bmatrix}0\\F(t)-F_{\text{lin}}(t)\end{bmatrix}, \mathcal{U}(t)\begin{bmatrix}v_0\\v_1\end{bmatrix}\right\rangle_{\dot{H}^1\times L^2} dt\\
&= \int_{-\infty}^\infty \left\langle \begin{bmatrix}0\\F(t)-F_{\text{lin}}(t)\end{bmatrix}, \begin{bmatrix}v_{\text{lin}}(t)\\\partial_t v_{\text{lin}}(t)\end{bmatrix}\right\rangle_{\dot{H}^1\times L^2} dt\\
&= \int_{-\infty}^\infty \langle F(t)-F_{\text{lin}}(t), \partial_t v_{\text{lin}}(t)\rangle_{L^2}\, dt.
\end{aligned}$$

As a result, it will suffice to show that

$$\left|\int_{-\infty}^\infty \langle F(t)-F_{\text{lin}}(t), \partial_t v_{\text{lin}}(t)\rangle_{L^2}\, dt\right| \lesssim \left\|\begin{bmatrix}u_0\\u_1\end{bmatrix}\right\|_{\dot{H}^1\times L^2}^9 \left\|\begin{bmatrix}v_0\\v_1\end{bmatrix}\right\|_{\dot{H}^1\times L^2}. \tag{2.8}$$

To estimate this integral, we first employ Hölder's inequality to deduce that

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty}\big\langle F(t)-F_{\mathrm{lin}}(t),\partial_t v_{\mathrm{lin}}(t)\big\rangle_{L^2}\,dt\Big|\\
&\quad\le \|F(t,x,u(t,x))-F(t,x,u_{\mathrm{lin}}(t,x))\|_{L^1_tL^2_x}\cdot\|\partial_t v_{\mathrm{lin}}\|_{L^\infty_tL^2_x}\\
&\quad\lesssim \big(\|u\|^4_{L^5_tL^{10}_x}+\|u_{\mathrm{lin}}\|^4_{L^5_tL^{10}_x}\big)\|u-u_{\mathrm{lin}}\|_{L^5_tL^{10}_x}\cdot\|\partial_t v_{\mathrm{lin}}\|_{L^\infty_tL^2_x}. \qquad (2.9)
\end{aligned}$$

By (1.2) and the Strichartz estimates, we have

$$\begin{aligned}
\|u\|^4_{L^5_tL^{10}_x}&\lesssim\|(u_0,u_1)\|^4_{\dot H^1\times L^2},\\
\|u_{\mathrm{lin}}\|^4_{L^5_tL^{10}_x}&\lesssim\|(u_0,u_1)\|^4_{\dot H^1\times L^2},\\
\|u-u_{\mathrm{lin}}\|_{L^5_tL^{10}_x}&\lesssim\|F(t,x,u(t,x))\|_{L^1_tL^2_x}\lesssim\|u\|^5_{L^5_tL^{10}_x}\lesssim\|(u_0,u_1)\|^5_{\dot H^1\times L^2},\\
\|\partial_t v_{\mathrm{lin}}\|_{L^\infty_tL^2_x}&\lesssim\|(v_0,v_1)\|_{\dot H^1\times L^2}.
\end{aligned}$$

Inserting these estimates into (2.9) yields (2.8), completing the proof of the corollary. □

3. Reduction to a convolution equation

The next step is the reduction of the proof of Theorem 1.5 to the consideration of a convolution equation. As in [6], the central idea is to exploit the Born approximation for well-chosen solutions of the linear wave equation. Indeed, the principal obstacle to be overcome in implementing that strategy is to find solutions of the linear wave equation with the key properties we need. Most fundamentally, we need solutions for which we are not only able to compute the distribution function (i.e., the measure of spacetime superlevel sets), but can also prove that the Fourier transform of a certain function w connected with it does not vanish.

Our solutions will be built from the radially symmetric solution

$$u_{\mathrm{lin}}(t,r):=\frac{f(r-t)-f(r+t)}{r}$$

of the linear wave equation $(\partial_{tt}-\Delta_x)u(t,x)=0$ on $\mathbb{R}\times\mathbb{R}^3$, where $r:=|x|$ and $f(s):=\max\{1-|s|,0\}$. This solution arises from the initial data

$$\begin{aligned}
u_0(x)&:=u_{\mathrm{lin}}(0,|x|)=0\in\dot H^1(\mathbb{R}^3),\\
u_1(x)&:=\partial_t u_{\mathrm{lin}}(0,|x|)=\begin{cases}\frac{2}{|x|} & \text{if } 0<|x|\le 1,\\ 0 & \text{if } |x|>1\end{cases}\in L^2(\mathbb{R}^3).
\end{aligned} \qquad (3.1)$$

In addition, $u_{\mathrm{lin}}(t,x)\ge 0$ for $t>0$ and $u_{\mathrm{lin}}(t,x)$ is odd in t.

The next lemma gives a formula for the distribution function of u_{lin}. The function w connected with this solution is presented in (3.7). The nonvanishing of the Fourier transform of w will be demonstrated in Proposition 4.2.

Lemma 3.1. *For $\lambda>0$, let*

$$m(\lambda):=\big|\{(t,x)\in(0,\infty)\times\mathbb{R}^3:u_{\mathrm{lin}}(t,x)>\lambda\}\big|.$$

Then

$$m(\lambda)=\frac{4\pi}{3}\left(\frac{1}{2\lambda^3}-\frac{2}{(\lambda+2)^3}\right)1_{(0,2)}(\lambda).$$

Proof. For $t, \lambda > 0$, let

$$m(t;\lambda) := \left|\{x \in \mathbb{R}^3 : u_{\text{lin}}(t,x) > \lambda\}\right|$$

so that

$$m(\lambda) = \int_0^\infty m(t;\lambda)\,dt. \tag{3.2}$$

We will evaluate this integral by analyzing u_{lin} on the spacetime regions $0 < t < \frac{1}{2}$, $\frac{1}{2} < t < 1$, and $t > 1$.

On the region $0 < t < \frac{1}{2}$, we have

$$u_{\text{lin}}(t,r) = \begin{cases} 2 & \text{if } 0 < r < t, \\ \frac{2t}{r} & \text{if } t \le r < 1-t, \\ \frac{1-r+t}{r} & \text{if } 1-t \le r < 1+t, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $0 < t < \frac{1}{2}$,

$$m(t;\lambda) = \frac{4\pi}{3} \cdot \begin{cases} (\frac{1+t}{1+\lambda})^3 & \text{if } 0 < \lambda < \frac{2t}{1-t}, \\ (\frac{2t}{\lambda})^3 & \text{if } \frac{2t}{1-t} \le \lambda < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the contribution of this region to the right-hand side of (3.2) is

$$\begin{aligned} \int_0^{\frac{1}{2}} m(t;\lambda)\,dt &= \int_0^{\frac{1}{2}} \frac{4\pi}{3}\left(\frac{1+t}{1+\lambda}\right)^3 1_{\{0<\lambda<\frac{2t}{1-t}\}}(t)\,dt + \int_0^{\frac{1}{2}} \frac{4\pi}{3}\left(\frac{2t}{\lambda}\right)^3 1_{\{\frac{2t}{1-t}\le\lambda<2\}}(t)\,dt \\ &= \left[\int_{\frac{\lambda}{\lambda+2}}^{\frac{1}{2}} \frac{4\pi}{3}\left(\frac{1+t}{1+\lambda}\right)^3 dt\right] 1_{(0,2)}(\lambda) + \left[\int_0^{\frac{\lambda}{\lambda+2}} \frac{4\pi}{3}\left(\frac{2t}{\lambda}\right)^3 dt\right] 1_{(0,2)}(\lambda) \\ &= \frac{4\pi}{3}\left(\frac{81}{64(\lambda+1)^3} - \frac{4(\lambda+1)}{(\lambda+2)^4}\right) 1_{(0,2)}(\lambda) + \frac{4\pi}{3}\cdot\frac{2\lambda}{(\lambda+2)^4}\,1_{(0,2)}(\lambda) \\ &= \frac{4\pi}{3}\left(\frac{81}{64(\lambda+1)^3} - \frac{2}{(\lambda+2)^3}\right) 1_{(0,2)}(\lambda). \end{aligned} \tag{3.3}$$

On the region $\frac{1}{2} < t < 1$, we have

$$u_{\text{lin}}(t,r) = \begin{cases} 2 & \text{if } 0 < r < 1-t, \\ \frac{1+r-t}{r} & \text{if } 1-t \le r < t, \\ \frac{1-r+t}{r} & \text{if } t \le r < 1+t, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $\frac{1}{2} < t < 1$,

$$m(t;\lambda) = \frac{4\pi}{3} \cdot \begin{cases} (\frac{1+t}{1+\lambda})^3 & \text{if } 0 < \lambda < \frac{1}{t}, \\ (\frac{1-t}{\lambda-1})^3 & \text{if } \frac{1}{t} \le \lambda < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the contribution of this region to the right-hand side of (3.2) is

$$\int_{\frac{1}{2}}^1 m(t;\lambda)\,dt = \int_{\frac{1}{2}}^1 \frac{4\pi}{3}\left(\frac{1+t}{1+\lambda}\right)^3 1_{\{0<\lambda<\frac{1}{t}\}}(t)\,dt + \int_{\frac{1}{2}}^1 \frac{4\pi}{3}\left(\frac{1-t}{\lambda-1}\right)^3 1_{\{\frac{1}{t}\le\lambda<2\}}(t)\,dt$$

$$
\begin{aligned}
&= \left[\int_{\frac12}^{1} \frac{4\pi}{3}\left(\frac{1+t}{1+\lambda}\right)^3 dt\right] 1_{(0,1]}(\lambda) + \left[\int_{\frac12}^{\frac1\lambda} \frac{4\pi}{3}\left(\frac{1+t}{1+\lambda}\right)^3 dt\right] 1_{(1,2)}(\lambda) \\
&\quad + \left[\int_{\frac1\lambda}^{1} \frac{4\pi}{3}\left(\frac{1-t}{\lambda-1}\right)^3 dt\right] 1_{(1,2)}(\lambda) \\
&= \frac{4\pi}{3}\cdot\frac{175}{64(\lambda+1)^3}\, 1_{(0,1]}(\lambda) + \frac{4\pi}{3}\left(\frac{\lambda+1}{4\lambda^4} - \frac{81}{64(\lambda+1)^3}\right) 1_{(1,2)}(\lambda) \\
&\quad + \frac{4\pi}{3}\cdot\frac{\lambda-1}{4\lambda^4}\, 1_{(1,2)}(\lambda) \\
&= \frac{4\pi}{3}\cdot\frac{175}{64(\lambda+1)^3}\, 1_{(0,1]}(\lambda) + \frac{4\pi}{3}\left(\frac{1}{2\lambda^3} - \frac{81}{64(\lambda+1)^3}\right) 1_{(1,2)}(\lambda). \qquad (3.4)
\end{aligned}
$$

On the region $t > 1$, we have

$$
u_{\mathrm{lin}}(t,r) = \begin{cases} \frac{1+r-t}{r} & \text{if } t-1 \le r < t, \\ \frac{1-r+t}{r} & \text{if } t \le r < t+1, \\ 0 & \text{otherwise.} \end{cases}
$$

Hence, for $t > 1$,

$$
m(t;\lambda) = \frac{4\pi}{3}\cdot \begin{cases} (\frac{t+1}{\lambda+1})^3 - (\frac{t-1}{1-\lambda})^3 & \text{if } 0<\lambda<\frac1t, \\ 0 & \text{otherwise.} \end{cases}
$$

Therefore, the contribution of this region to the right-hand side of (3.2) is

$$
\begin{aligned}
\int_1^\infty m(t;\lambda)\,dt &= \int_1^\infty \frac{4\pi}{3}\left[\left(\frac{t+1}{\lambda+1}\right)^3 - \left(\frac{t-1}{1-\lambda}\right)^3\right] 1_{\{0<\lambda<\frac1t\}}(t)\,dt \\
&= \left\{\int_1^{\frac1\lambda} \frac{4\pi}{3}\left[\left(\frac{t+1}{\lambda+1}\right)^3 - \left(\frac{t-1}{1-\lambda}\right)^3\right] dt\right\} 1_{(0,1)}(\lambda) \\
&= \frac{4\pi}{3}\left(\frac{1}{2\lambda^3} - \frac{4}{(\lambda+1)^3}\right) 1_{(0,1)}(\lambda). \qquad (3.5)
\end{aligned}
$$

Finally, combining (3.2) with (3.3), (3.4), and (3.5) completes the proof of the lemma. □

To continue, we generate further solutions of the linear wave equation using the scaling symmetry. Specifically, for positive parameters α and ε, the rescaled function $u_{\mathrm{lin}}^{\alpha,\varepsilon}$ defined as

$$
u_{\mathrm{lin}}^{\alpha,\varepsilon}(t,x) := \alpha u_{\mathrm{lin}}((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x)
$$

solves the linear wave equation with initial data

$$
\begin{aligned}
u_0^{\alpha,\varepsilon}(x) &:= u_{\mathrm{lin}}^{\alpha,\varepsilon}(0,x) = \alpha u_0((\alpha/\varepsilon)^2 x) = 0, \\
u_1^{\alpha,\varepsilon}(x) &:= \partial_t u_{\mathrm{lin}}^{\alpha,\varepsilon}(0,x) = (\alpha/\varepsilon)^2 \alpha u_1((\alpha/\varepsilon)^2 x).
\end{aligned}
$$

Under this rescaling, we have $\|u_0^{\alpha,\varepsilon}\|_{\dot H^1} = \varepsilon\|u_0\|_{\dot H^1}$ and $\|u_1^{\alpha,\varepsilon}\|_{L^2} = \varepsilon\|u_1\|_{L^2}$, so from (3.1) we compute that

$$
\|(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon})\|_{\dot H^1\times L^2}^2 = \varepsilon^2\|(u_0,u_1)\|_{\dot H^1\times L^2}^2 = 16\pi\varepsilon^2.
$$

In particular, if F is an admissible nonlinearity for equation (1.1) and B_η is a corresponding ball given by Theorem 1.3, then $(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon}) \in B_\eta$ for all sufficiently small ε.

We will also rely on the observation that $u_{\text{lin}}(t,x) = \partial_t v_{\text{lin}}(t,x)$, where v_{lin} is itself a radially symmetric solution of the linear wave equation on $\mathbb{R} \times \mathbb{R}^3$ with initial data

$$\begin{aligned} v_0(x) &:= v_{\text{lin}}(0,x) = \begin{cases} |x| - 2 & \text{if } 0 < |x| \le 1, \\ -\frac{1}{|x|} & \text{if } |x| > 1 \end{cases} \in \dot{H}^1(\mathbb{R}^3), \\ v_1(x) &:= \partial_t v_{\text{lin}}(0,x) = 0 \in L^2(\mathbb{R}^3). \end{aligned}$$

Thus, $u_{\text{lin}}^{\alpha,\varepsilon}(t,x) = \partial_t v_{\text{lin}}^{\alpha,\varepsilon}(t,x)$, where the rescaled function $v_{\text{lin}}^{\alpha,\varepsilon}$ defined as

$$v_{\text{lin}}^{\alpha,\varepsilon}(t,x) := (\alpha/\varepsilon)^{-2} \alpha v_{\text{lin}}((\alpha/\varepsilon)^2 t, (\alpha/\varepsilon)^2 x)$$

solves the linear wave equation with initial data

$$\begin{aligned} v_0^{\alpha,\varepsilon}(x) &:= v_{\text{lin}}^{\alpha,\varepsilon}(0,x) = (\alpha/\varepsilon)^{-2} \alpha v_0((\alpha/\varepsilon)^2 x), \\ v_1^{\alpha,\varepsilon}(x) &:= \partial_t v_{\text{lin}}^{\alpha,\varepsilon}(0,x) = \alpha v_1((\alpha/\varepsilon)^2 x) = 0. \end{aligned}$$

Under this rescaling, we have

$$\|(v_0^{\alpha,\varepsilon}, v_1^{\alpha,\varepsilon})\|_{\dot{H}^1 \times L^2}^2 = (\alpha/\varepsilon)^{-6} \alpha^2 \|(v_0, v_1)\|_{\dot{H}^1 \times L^2}^2 = 16\pi\varepsilon^6/3\alpha^4.$$

Proposition 3.2 (Reduction to a convolution equation)**.** *Suppose that F and $\widetilde{F}$ are admissible nonlinearities for equation* (1.1). *For $(t_0, x_0) \in D_F \cap D_{\widetilde{F}}$ and $\tau \in \mathbb{R}$, define*

$$\begin{aligned} H(\tau; t_0, x_0) &:= e^{-4\tau} \frac{\partial F}{\partial u}(t_0, x_0, e^\tau) + e^{-5\tau} F(t_0, x_0, e^\tau), \\ \widetilde{H}(\tau; t_0, x_0) &:= e^{-4\tau} \frac{\partial \widetilde{F}}{\partial u}(t_0, x_0, e^\tau) + e^{-5\tau} \widetilde{F}(t_0, x_0, e^\tau). \end{aligned}$$

Then H and $\widetilde{H}$ are bounded and, under the hypotheses of Theorem 1.5, we have

$$H * w = \widetilde{H} * w, \tag{3.6}$$

where

$$w(\tau) := \left(e^{-3\tau} - \frac{4e^{-6\tau}}{(e^{-\tau}+1)^3} \right) 1_{(0,\infty)}(\tau). \tag{3.7}$$

The proof of this proposition relies on the following result, which shows that in the Born approximation described in Corollary 2.2, we may replace $F(t,x,u)$ by $F(t_0, x_0, u)$ up to acceptable errors.

Lemma 3.3. *Suppose that F is an admissible nonlinearity for equation* (1.1). *Then for all $(t_0, x_0) \in D_F$, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} \left\langle F(t,x,u_{\text{lin}}^{\alpha,\varepsilon}(t-t_0, x-x_0)), u_{\text{lin}}^{\alpha,\varepsilon}(t-t_0, x-x_0) \right\rangle_{L_x^2} dt \\ &\quad = \int_{-\infty}^{\infty} \left\langle F(t_0, x_0, u_{\text{lin}}^{\alpha,\varepsilon}(t-t_0, x-x_0)), u_{\text{lin}}^{\alpha,\varepsilon}(t-t_0, x-x_0) \right\rangle_{L_x^2} dt + o_\alpha(\varepsilon^8) \end{aligned}$$

as $\varepsilon \to 0$.

We postpone the proof of Lemma 3.3 until after we have completed that of Proposition 3.2.

Proof of Proposition 3.2. We only consider the case where the scattering operators agree, as the wave operators can be treated similarly. By time and space translation symmetry, it suffices to treat the case $(t_0, x_0) = (0, 0)$.

Let $G(t, x, u) := F(t, x, u)u$ so that

$$\int_{-\infty}^{\infty} \big\langle F(0, 0, u_{\text{lin}}(t, x)), u_{\text{lin}}(t, x)\big\rangle_{L^2_x} \, dt = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} G(0, 0, u_{\text{lin}}(t, x)) \, dx \, dt.$$

By the fundamental theorem of calculus, Fubini's theorem, and Lemma 3.1,

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} G(0, 0, u_{\text{lin}}(t, x)) \, dx \, dt = 2 \int_0^{\infty} \frac{\partial G}{\partial u}(0, 0, \lambda) m(\lambda) \, d\lambda.$$

Hence,

$$\int_{-\infty}^{\infty} \big\langle F(0, 0, u_{\text{lin}}^{\alpha,\varepsilon}(t, x)), u_{\text{lin}}^{\alpha,\varepsilon}(t, x)\big\rangle_{L^2_x} \, dt = 2(\alpha/\varepsilon)^{-8} \int_0^{\infty} \frac{\partial G}{\partial u}(0, 0, \lambda) m(\lambda/\alpha) \, d\lambda.$$

Performing the change of variables $\lambda =: e^{\tau}$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big\langle F(0, 0, u_{\text{lin}}^{\alpha,\varepsilon}(t, x)), u_{\text{lin}}^{\alpha,\varepsilon}(t, x)\big\rangle_{L^2_x} \, dt \\
&= \frac{2\varepsilon^8}{\alpha^8} \int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(0, 0, e^{\tau}) e^{\tau} m(e^{\tau - \log \alpha}) \, d\tau \\
&= \frac{2\varepsilon^8}{\alpha^8} \int_{-\infty}^{\infty} H(\tau) e^{6\tau} m(e^{\tau - \log \alpha}) \, d\tau \\
&= \frac{2\varepsilon^8}{\alpha^8} \cdot \frac{(2\alpha)^6 \pi}{12} \int_{-\infty}^{\infty} H(\tau) \cdot \frac{12}{\pi} e^{-6(\log 2\alpha - \tau)} m(e^{\tau - \log \alpha}) \, d\tau \\
&= \frac{32\pi\varepsilon^8}{3\alpha^2} \int_{-\infty}^{\infty} H(\tau) w(\log 2\alpha - \tau) \, d\tau \\
&= \frac{32\pi\varepsilon^8}{3\alpha^2} (H * w)(\log 2\alpha), \qquad (3.8)
\end{aligned}$$

where $w(\tau) = \frac{12}{\pi} e^{-6\tau} m(e^{-(\tau - \log 2)})$ is as given by (3.7).

On the other hand, if $F_{\text{lin}}^{\alpha,\varepsilon}(t) := F(t, \cdot, u_{\text{lin}}^{\alpha,\varepsilon}(t))$, then

$$\begin{aligned}
\int_{-\infty}^{\infty} \big\langle F_{\text{lin}}^{\alpha,\varepsilon}(t), u_{\text{lin}}^{\alpha,\varepsilon}(t)\big\rangle_{L^2} \, dt &= \int_{-\infty}^{\infty} \big\langle F_{\text{lin}}^{\alpha,\varepsilon}(t), \partial_t v_{\text{lin}}^{\alpha,\varepsilon}(t)\big\rangle_{L^2} \, dt \\
&= \int_{-\infty}^{\infty} \left\langle \begin{bmatrix} 0 \\ F_{\text{lin}}^{\alpha,\varepsilon}(t) \end{bmatrix}, \mathcal{U}(t) \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2} dt \\
&= \left\langle \int_{-\infty}^{\infty} \mathcal{U}(-t) \begin{bmatrix} 0 \\ F_{\text{lin}}^{\alpha,\varepsilon}(t) \end{bmatrix} dt, \begin{bmatrix} v_0^{\alpha,\varepsilon} \\ v_1^{\alpha,\varepsilon} \end{bmatrix} \right\rangle_{\dot{H}^1 \times L^2}.
\end{aligned}$$

It follows from Corollary 2.2 that agreement of the scattering operators implies that

$$\begin{aligned}
\int_{-\infty}^{\infty} \big\langle F_{\text{lin}}^{\alpha,\varepsilon}(t), u_{\text{lin}}^{\alpha,\varepsilon}(t)\big\rangle_{L^2} \, dt &= \int_{-\infty}^{\infty} \big\langle \widetilde{F}_{\text{lin}}^{\alpha,\varepsilon}(t), u_{\text{lin}}^{\alpha,\varepsilon}(t)\big\rangle_{L^2} \, dt \\
&\quad + \mathcal{O}(\|(u_0^{\alpha,\varepsilon}, u_1^{\alpha,\varepsilon})\|_{\dot{H}^1 \times L^2}^9) \cdot \|(v_0^{\alpha,\varepsilon}, v_1^{\alpha,\varepsilon})\|_{\dot{H}^1 \times L^2} \\
&= \int_{-\infty}^{\infty} \big\langle \widetilde{F}_{\text{lin}}^{\alpha,\varepsilon}(t), u_{\text{lin}}^{\alpha,\varepsilon}(t)\big\rangle_{L^2} \, dt + \mathcal{O}_\alpha(\varepsilon^{12}). \qquad (3.9)
\end{aligned}$$

Now given a $\tau_0 \in \mathbb{R}$, let $\alpha := \frac{1}{2}e^{\tau_0}$ so that $\tau_0 = \log 2\alpha$. Combining Lemma 3.3, (3.8), and (3.9), we deduce that

$$(H * w)(\tau_0) = (\widetilde{H} * w)(\tau_0) + o(1) + \mathcal{O}(\varepsilon^4) \quad \text{as } \varepsilon \to 0.$$

Taking $\varepsilon \to 0$, we arrive at the conclusion. □

Proof of Lemma 3.3. Fix a point $(t_0, x_0) \in D_F$ and let

$$G^{\alpha,\varepsilon}(t, x, u) := F(t_0 + (\alpha/\varepsilon)^{-2}t, x_0 + (\alpha/\varepsilon)^{-2}x, u)u$$

so that

$$\begin{aligned}\int_{-\infty}^{\infty} \big\langle F(t, x, u_{\text{lin}}^{\alpha,\varepsilon}(t - t_0, x - x_0)), u_{\text{lin}}^{\alpha,\varepsilon}(t - t_0, x - x_0)\big\rangle_{L^2_x} dt \\ = (\alpha/\varepsilon)^{-8} \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} G^{\alpha,\varepsilon}(t, x, \alpha u_{\text{lin}}(t, x))\, dx\, dt.\end{aligned}$$

Then the conclusion sought can be written as follows: as $\varepsilon \to 0$,

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} G^{\alpha,\varepsilon}(t, x, \alpha u_{\text{lin}}(t, x)) - G^{\alpha,\varepsilon}(0, 0, \alpha u_{\text{lin}}(t, x))\, dx\, dt = o_\alpha(1). \tag{3.10}$$

To prove this, we first recall from the proof of Lemma 3.1 that

$$u_{\text{lin}}(t, x) \le \begin{cases} 2 \cdot 1_{\{0<|x|<2\}}(t, x) & \text{if } 0 < t < 1, \\ \frac{1}{t} \cdot 1_{\{t-1\le|x|<t+1\}}(t, x) & \text{if } t > 1. \end{cases}$$

Hence

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u_{\text{lin}}(t, x)|^6\, dx\, dt = 2 \int_0^{\infty} \int_{\mathbb{R}^3} |u_{\text{lin}}(t, x)|^6\, dx\, dt \lesssim 1 + \int_1^{\infty} \left(\frac{1}{t}\right)^6 t^2\, dt < \infty.$$

Thus, given any $\eta > 0$, the dominated convergence theorem guarantees that there exists an $R > 0$ (depending on η) so that

$$\begin{aligned}\left| \iint_{|t|+|x|>R} G^{\alpha,\varepsilon}(t, x, \alpha u_{\text{lin}}(t, x)) - G^{\alpha,\varepsilon}(0, 0, \alpha u_{\text{lin}}(t, x))\, dx\, dt \right| \\ \lesssim_\alpha \iint_{|t|+|x|>R} |u_{\text{lin}}(t, x)|^6\, dx\, dt < \eta.\end{aligned} \tag{3.11}$$

To estimate the integral in (3.10) over the complementary region $|t| + |x| \le R$, we partition it into the sets

$$U_n^R := \{(t, x) \in \mathbb{R}\times\mathbb{R}^3 : |t|+|x| \le R \text{ and } \lceil 2\alpha\rceil n/N \le \alpha u_{\text{lin}}(t, x) < \lceil 2\alpha\rceil(n+1)/N\},$$

where N is some large positive integer and $|n| \le N$. For $(t, x) \in U_n^R$, we then have

$$\begin{aligned}|G^{\alpha,\varepsilon}(t, x, \alpha u_{\text{lin}}(t, x)) - G^{\alpha,\varepsilon}(t, x, \lceil 2\alpha\rceil n/N)| \lesssim_\alpha 1/N, \\ |G^{\alpha,\varepsilon}(0, 0, \lceil 2\alpha\rceil n/N) - G^{\alpha,\varepsilon}(0, 0, \alpha u_{\text{lin}}(t, x))| \lesssim_\alpha 1/N,\end{aligned}$$

with implicit constants depending only on α. As $\lceil 2\alpha\rceil n/N \in \mathbb{Q}$, replacing the true values of αu_{lin} with these approximations will allow us to exploit the hypothesis that (t_0, x_0) is a determinable point. To employ these approximations, we first note that

$$\left| \iint_{|t|+|x|\le R} G^{\alpha,\varepsilon}(t, x, \alpha u_{\text{lin}}(t, x)) - G^{\alpha,\varepsilon}(0, 0, \alpha u_{\text{lin}}(t, x))\, dx\, dt \right|$$

$$\lesssim_\alpha \sum_{|n|\leq N} \iint_{U_n^R} \Big| G^{\alpha,\varepsilon}(t,x,\lceil 2\alpha\rceil n/N) - G^{\alpha,\varepsilon}(0,0,\lceil 2\alpha\rceil n/N)\Big|\, dx\, dt + \frac{R^4}{N}\,.$$

For each n, a change of variables gives

$$\begin{aligned}
&\iint_{U_n^R} \Big| G^{\alpha,\varepsilon}(t,x,\lceil 2\alpha\rceil n/N) - G^{\alpha,\varepsilon}(0,0,\lceil 2\alpha\rceil n/N)\Big|\, dx\, dt \\
&\lesssim_{\alpha,R} \varepsilon^{-8} \iint_{|t-t_0|+|x-x_0|\leq(\alpha/\varepsilon)^{-2}R} \Big| F(t,x,\lceil 2\alpha\rceil n/N) - F(t_0,x_0,\lceil 2\alpha\rceil n/N)\Big|\, dx\, dt,
\end{aligned}$$

which tends to zero as $\varepsilon \to 0$ because (t_0, x_0) is a determinable point.

Therefore, choosing N sufficiently large (depending on η) and then ε sufficiently small (depending on η), we obtain

$$\bigg| \iint_{|t|+|x|\leq R} G^{\alpha,\varepsilon}(t,x,\alpha u_{\text{lin}}(t,x)) - G^{\alpha,\varepsilon}(0,0,\alpha u_{\text{lin}}(t,x))\, dx\, dt \bigg| \lesssim_\alpha \eta.$$

Combining this with (3.11) and recalling that η was arbitrary, we deduce (3.10). □

In view of Proposition 3.2, the proof of Theorem 1.5 reduces to showing that the convolution equation (3.6) implies equality of the nonlinearities F and $\widetilde{F}$. We turn our attention to this task in the next section.

4. Deconvolutional determination of the nonlinearity

The final step in the proof of Theorem 1.5 consists of formally "deconvolving" both sides of equation (3.6) with w to arrive at $H = \widetilde{H}$. This in turn implies that $F(t_0, x_0, \cdot\,) = \widetilde{F}(t_0, x_0, \cdot\,)$. The tool that will enable us to do so is a Tauberian theorem of Wiener [28]. For the following formulation of the Tauberian theorem, as well as a very elegant proof, see Korevaar [7].

Theorem 4.1 (Wiener's Tauberian theorem)**.** *Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. If $f * g = 0$ and $\widehat{f}$ has no zeroes, then $g = 0$.*

Proposition 4.2. *Let w be as defined in Proposition 3.2. Then $\widehat{w}$ has no zeroes.*

Assuming that this proposition holds (so that Wiener's Tauberian theorem is applicable to w), Theorem 1.5 follows immediately, as we demonstrate next.

Proof of Theorem 1.5. Fix a point $(t_0, x_0) \in D_F \cap D_{\widetilde{F}}$ and define H and $\widetilde{H}$ as in Proposition 3.2 so that $(H - \widetilde{H}) * w = 0$. It follows from Theorem 4.1 and Proposition 4.2 that $H = \widetilde{H}$. In particular,

$$\tfrac{d}{d\tau}\big[e^\tau F(t_0,x_0,e^\tau) - e^\tau \widetilde{F}(t_0,x_0,e^\tau)\big] = 0,$$

from which it follows that $F(t_0, x_0, \cdot\,) = \widetilde{F}(t_0, x_0, \cdot\,)$. □

Proof of Proposition 4.2. We decompose w as $w = w_0 + w_1$, where

$$\begin{aligned}
w_0(\tau) &:= \left(\frac{e^{-3\tau}}{2}\right) 1_{(0,\infty)}(\tau), \\
w_1(\tau) &:= \left(\frac{e^{-3\tau}}{2} - \frac{4e^{-6\tau}}{(e^{-\tau}+1)^3}\right) 1_{(0,\infty)}(\tau).
\end{aligned}$$

First, we compute that

$$\widehat{w_0}(\xi) = \int_0^\infty \frac{e^{-3\tau}}{2} \cdot e^{-i\xi\tau}\, d\tau = \frac{1}{6+2i\xi}\,. \tag{4.1}$$

As $8e^{-3\tau} \leq (e^{-\tau}+1)^3 \leq 8$ for all $\tau \in (0,\infty)$, we also have $w_1(\tau) \geq 0$ and so

$$|\widehat{w_1}(\xi)| \leq \int_0^\infty \frac{e^{-3\tau} - e^{-6\tau}}{2}\, d\tau = \frac{1}{12}\,.$$

Using the expression (4.1) for $\widehat{w_0}(\xi)$, we find that $|\widehat{w_0}(\xi)| > \frac{1}{12}$ whenever $|\xi|^2 < 27$, which implies that $|\widehat{w}(\xi)| > 0$ for all such ξ.

To handle the remaining ξ, we integrate by parts to obtain

$$\widehat{w_1}(\xi) = \int_0^\infty \frac{d}{d\tau}\left(\frac{e^{-3\tau}}{2} - \frac{4e^{-6\tau}}{(e^{-\tau}+1)^3}\right)\frac{e^{-i\xi\tau}}{i\xi}\, d\tau.$$

It is straightforward to verify that

$$A(\tau) := -\frac{d}{d\tau}\left(\frac{e^{-3\tau}}{2}\right) \quad \text{and} \quad B(\tau) := \frac{d}{d\tau}\left(-\frac{4e^{-6\tau}}{(e^{-\tau}+1)^3}\right)$$

satisfy $0 \leq \frac{2}{3}B(\tau) \leq A(\tau)$ for all $\tau \in (0,\infty)$. Hence

$$|\widehat{w_1}(\xi)| \leq \frac{1}{|\xi|}\int_0^\infty |B(\tau) - A(\tau)|\, d\tau \leq \frac{1}{|\xi|}\int_0^\infty A(\tau) - \frac{1}{3}B(\tau)\, d\tau = \frac{1}{3|\xi|}\,.$$

Using the expression (4.1) for $\widehat{w_0}(\xi)$ again, we find that $|\widehat{w_0}(\xi)| > \frac{1}{3|\xi|}$ whenever $|\xi|^2 > \frac{36}{5}$, which implies that $|\widehat{w}(\xi)| > 0$ for all such ξ.

As $\frac{36}{5} < 27$, we conclude that $\widehat{w}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, as was to be shown. □

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