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SEQUENTIAL OPTIMAL CONTRACTING IN CONTINUOUS TIME

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Abstract. This paper studies a principal-agent problem in continuous time with multiple lump-sum payments (contracts) paid at different deterministic times. We reduce the non-zero-sum Stackelberg game between the principal and agent to a standard stochastic optimal control problem. We apply our result to a benchmark model to investigate how different inputs (payment frequencies, payment distribution, discounting factors, agent's reservation utility) affect the principal's value and agent's optimal compensations.

1. **Introduction.** Principal-agent problems describe strategic interactions between two parts: the principal (i.e. the manager), and the agent (i.e. the employee). The principal aims to incentivize good performance from the agent by designing an optimal compensation, i.e. the contract. Such contracts are contingent on an agent's controlled random state, referred to as the output process (i.e. the firm's value), and must satisfy the agent's participation constraint. In most real-life applications, the contracts are structured as multiple lump-sum payments scheduled periodically (e.g. insurance, brokerage, managerial compensations). The contract schedule can be deterministic (e.g. salaries) or random (e.g. spot bonuses). This paper investigates the problem of optimal contract schedule design (multiple lump-sum payments) in continuous time. To our knowledge so far, this is the first paper addressing this problem, as prior studies focused on either (i) continuous-time lump-sum single payment and a possible continuous payment, or (ii) discrete time multiple lump-sum payments. A continuous time approach facilitates tractability and qualitative analysis in the study of optimal contract schedule design. The latter has numerous applications in economics, finance, and management science.

A substantial amount of research has been developed in the continuous time principal-agent literature. The first publication on this matter is authored by Holmström and Milgrom [21]. The authors demonstrated that in a finite-horizon setting the optimal contract is linear with respect to the output process. Since then, multiple extensions have been developed. We highlight the following papers that extend

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the original model: [29], [30],[31], [22], [23], [18], [17]. Thereafter, the following articles [24], [32], [33], [9], [10], [11] employed the stochastic maximum principle and forward-backward stochastic differential equations to characterize optimal compensations in a more general setting. Finally, we mention the latest contributions: [13], [12], [20], [19].

In discrete time, the problem of optimal contracting schedule design has been extensively explored. Rubinstein and Yaari [27], and Rogerson [26] developed discrete-time sequential contracting models providing qualitative properties of the optimal compensations. Subsequently, Fudenberg et al. [15], [16] introduced the concept of short-term compensation corresponding to the case where each contract is renegotiated immediately after the preceding contract is delivered. The authors showed that, when the principal and agent have common knowledge, there is no need to commit to a long-term negotiation.

Our paper has the following contributions. Firstly, in Sections 2 and 3 we consider a general sequential contracting problem with contract schedules consisting of multiple lump-sum payments paid at deterministic pre-established times. Following a similar approach to Cvitanić et al. [8], we represent the agent's value using a recursive system of quadratic BSDEs (backward stochastic differential equations) allowing us to reduce the principal's bi-level optimization problem to a single stochastic control problem with mixed static and continuous controls. The latter problem can be approached using dynamic programming techniques (see, for example, [6]). The assumptions made are fairly general and can be applied to various sequential contracting problems.

Secondly, in Section 4 we introduce a benchmark model to investigate how the contracting environment affects the principal's value and agent's optimal compensations. Inspired by Sannikov [28], our model assumes a risk-neutral principal and a risk-averse (via power utility), time-sensitive agent. In addition to the reservation utility, we enforce a limited liability constraint imposing that every admissible contract schedule provides the agent a non-negative continuation utility. The principal´s problem is to find an optimal contract schedule satisfying the agent's reservation utility and the limited liability constraints.

Using the results from Sections 2, and 3 we reduce the principal's sequential contracting problem to a stochastic control problem with state constraints and mixed continuous-discrete controls which can be approached by carefully applying the results from Bouchard and Nutz [4]. In their paper, Bouchard and Nutz derive a dynamic programming principle for the state-constrained stochastic control problem and write a comparison principle for viscosity solutions of its associated HJB equation. To apply the previous results, we find an appropriate upper and lower bound of the principal´s value function using probabilistic arguments. Finally, in Theorem 6, we show that the principal´s value function is continuous and solves a recursive system of HJB equations. We refer the reader to [7] for a comprehensive review of the theory of viscosity solutions of second-order partial differential equations.

In Section 5 we approximate numerically the solution of the recursive system of HJB equations characterizing the principal's value function allowing us to investigate how the contracting environment affects the principal's value and optimal compensations. Firstly, from our numerical simulations, we observe that the optimal intermediate payments ξ_i^* delivered to the agent are an increasing function of the agent's continuation utility. Furthermore, as the agent's discounting factor increases, the principal's maximum achievable profit decreases. This is connected to the concept of employment interval described by Sannikov [28], indicating that the principal is less inclined to provide a large utility to an impatient agent. Moreover, independently of the agent's discounting, the principal always benefits from increasing the frequency of payments. Regarding the distribution of payments, we conclude that the principal's choice of payment distribution highly depends on the agent's participation constraint. The previous effect is noticeable when the agent is impatient.

Finally, we compare the principal´s value in our model (initial negotiation) with an analogous contracting model when the contracts are renegotiated at every transaction time. In our benchmark model, the negotiation is finalized at the initial time implying that the agent precommits to a long-term contract guaranteeing to remain in the firm until the last payment ξ_N is transacted. In contrast, the renegotiation problem represents the scenario where the principal and agent sign a new contract at the beginning of each contracting period with a possibly different agent's reservation utility level. Our numerical simulations show that the principal's optimal negotiation setting (initial negotiation, renegotiation) depends on the model's inputs.

2. **Set up.**

We start by describing the dynamics of the output process, i.e. the value of the firm. We consider $\Omega := C([0,T],\mathbb{R})$, the space of $\mathbb{R}$-valued, continuous functions endowed with the supremum norm $\|\cdot\|_{C([0,T],\mathbb{R})}$, and the Wiener measure $\mathbb{W} \in \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ denotes the space of probability measures on Ω. We denote by W the canonical random element in Ω, and $\mathbb{F}$ its augmented filtration (with respect to $\mathbb{W}$). The principal compensates the agent with N lump sum payments $\bar{\xi}_N := (\xi_1, \dots, \xi_N)$, i.e. the contract schedule, transacted at $\mathcal{T}_N := \{T_1, \dots, T_N\}$, where $0 < T_1 < \dots < T_{N-1} < T_N = T < \infty$. For each $i \in \{1, \dots, N\}$, we define the set of i-th contracts as follows

$$\mathcal{C}_i^0 := \left\{\xi : (\Omega, \mathcal{F}_{T_i}) \mapsto (E, \mathcal{B}(E)) \text{ measurable, } \mathbb{E}^{\mathbb{W}}\left[\exp\left(p|\xi|\right)\right] < \infty, \text{ for all } p \geq 0\right\}, \tag{1}$$

where $E \subset \mathbb{R}$ is a non-empty convex set. Additionally, for all $i \in \{1, \dots, N\}$, we introduce $\Sigma_i := \prod_{j=1}^{i} \mathcal{C}_j^0$ corresponding to the set of contract schedules $\bar{\xi}_i := (\xi_1, \dots, \xi_i)$ transacted at $\mathcal{T}_i := \{T_1, \dots, T_i\} \subset \mathcal{T}_N$. Let $\bar{\xi}_N \in \Sigma_N$ be a contract schedule. We denote by $\bar{\xi}_{N-1} \in \Sigma_{N-1}$ the first $N-1$ payments of $\bar{\xi}_N$. The output process $X^{\bar{\xi}_{N-1}}$ is defined as the unique strong solution of the following iterative system of SDEs (stochastic differential equations)

$$X_t^{\bar{\xi}_{N-1}} = x_0 + \sum_{i=1}^{N-1} G_i\left(X_{T_i^-}^{\bar{\xi}_{N-1}}, \xi_i\right) \mathbb{1}_{t \geq T_i} + \int_0^t \sigma_s(X_s^{\bar{\xi}_{N-1}}) dW_s, \quad \mathbb{W}-a.s. \tag{2}$$

where $x_0 \in \mathbb{R}$, and $\sigma : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, $G_i : \mathbb{R} \times E \mapsto \mathbb{R}$, $1 \leq i \leq N$, are measurable mappings satisfying the following assumption.

Assumption 1. *The mappings $(G_i)_{i=1}^N$, and σ satisfy the following properties:*

(i) *There exist constants $K_G > 0$, and $p \geq 1$, for which*

$$|G_i(x,y)| \leq K_G\left(1 + |x|^p + |y|^p\right),$$

for all $(i,x,y) \in \{1, \dots, N\} \times \mathbb{R} \times E$.

(ii) *There exists a constant $K_\sigma > 0$, such that*

$$|\sigma(t,x)| \leq K_\sigma\left(1 + |x|\right), \quad |\sigma(t,x) - \sigma(t,x')| \leq K_\sigma |x - x'|,$$

for all $x, x' \in \mathbb{R}$.

(iii) σ *is bounded, and* $\sigma(t,x)$ *is invertible with bounded inverse, for all* $0 \le t \le T$, *and* $x \in \mathbb{R}$.

We notice that under Assumption 1, the SDE (2) admits a unique strong solution. We write $X_{t^-} := \lim_{s \to t^-} X_s$, for all $0 \le t \le T$. Next, we describe the impact of the agent's actions on the output process. We assume the agent's actions consist of $\mathbb{F}$-adapted, $\mathbb{R}$-valued processes α. Furthermore, we introduce the measurable function $\lambda : [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, satisfying the following condition.

Assumption 2. $\lambda(t,x,\cdot) \in C^1(\mathbb{R})$ *for all* $(t,x) \in [0,T] \times \mathbb{R}$. *Moreover, for all* $(t,a,x) \in [0,T] \times \mathbb{R}^2$, *there exists* $K_\lambda > 0$, *such that*

$$|\lambda(t,x,a)| \le K_\lambda \left(1 + |a|\right), \quad \left|\frac{\partial}{\partial a}\lambda(t,x,a)\right| \le K_\lambda.$$

Given a contract schedule $\bar{\xi}_N \in \Sigma_N$, we introduce the set of admissible agent's actions $\mathcal{A}(\bar{\xi}_{N-1})$, consisting of the $\mathbb{F}$-adapted processes α for which there exists $\epsilon > 0$, such that

$$\mathbb{E}\left[\mathcal{E}\left(\int_0^T \sigma_s(X_s^{\bar{\xi}_{N-1}})^{-1}\lambda_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)dW_s\right)^{1+\epsilon}\right] < \infty.$$

Using Girsanov's theorem, for any $(\bar{\xi}_{N-1}, \alpha) \in \Sigma_{N-1} \times \mathcal{A}(\bar{\xi}_{N-1})$ we define the probability measure $\mathbb{P}^\alpha \in \mathcal{P}(\Omega)$, and the $\mathbb{P}^\alpha$-Brownian motion B^α described as follows

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{W}} := \left(\int_0^T \sigma_s(X_s^{\bar{\xi}_{N-1}})^{-1}\lambda_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)dW_s\right),$$

$$B_t^\alpha := W_t - \int_0^t \sigma_s(X_s^{\bar{\xi}_{N-1}})^{-1}\lambda_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)ds.$$

Therefore, under $\mathbb{P}^\alpha$, the output process satisfies the following SDE

$$X_t^{\bar{\xi}_{N-1}} = G_i(X_{T_i^-}^{\bar{\xi}_{N-1}}, \xi_i) + \int_{T_i}^t \lambda_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)ds + \int_{T_i}^t \sigma_s(X_s^{\bar{\xi}_{N-1}})dB_s^\alpha,$$
$$T_i \le t < T_{i+1}, \quad \mathbb{P}^\alpha - a.s,$$

$$X_t^{\bar{\xi}_{N-1}} = x_0 + \int_0^t \lambda_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)ds + \int_0^t \sigma_s(X_s^{\bar{\xi}_{N-1}})dB_s^\alpha, \quad 0 \le t < T_1, \quad \mathbb{P}^\alpha - a.s., \tag{3}$$

for all $i \in \{1, \dots, N-1\}$.

Remark 1. It is worth commenting on the economic interpretation of the state equation (3). For all $(x,y) \in \mathbb{R} \times E$, $G_i(x,y)$ represents the output value at T_i (immediately after the i-th payment is transacted), incorporating potential transaction costs. Finally, given $(t,a,x) \in [0,T] \times \mathbb{R}^2$, the value $\lambda_t(x,a)$ accounts for the effect of the agent's effort on the output level, and $\sigma_t(x)$ denotes the risk level of the output.

3. **The principal-agent problem.** In this section, we introduce a class of sequential optimal contracting problems that can be solved by applying dynamic programming techniques. We start by introducing the agent's objective. Firstly, we define the agent's utility function U_a, running cost c, and discount factor k defined by the measurable functions: $U_a : E \mapsto \mathbb{R}$, $c : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$, and $k : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfying the following condition.

Assumption 3. *The mappings U_a, c, and k satisfy the following properties:*

(i) *There exists a constant $K_u > 0$, for which*

$$|U_a(y)| \leq K_u (1 + |y|), \quad \forall y \in E. \tag{4}$$

(ii) *For all $(t,x) \in [0,T] \times \mathbb{R}$, $c(t,x,\cdot) \in C^1(\mathbb{R})$, strictly convex and non negative. Moreover, there exist constants $K_c \geq 0$, and $p \geq 1$, for which*

$$|c(t,x,a)| \leq K_c \left(1 + |a|^2 + |x|\right), \quad \left|\frac{\partial}{\partial a} c(t,x,a)\right| \geq K_c |a|^p, \quad \lim_{|a|\to\infty} \frac{|c(t,x,a)|}{|a|} = \infty.$$

(iii) *k is bounded. Moreover, for all $(t,x) \in [0,T] \times \mathbb{R}$, $k(t,x,\cdot) \in C^1(\mathbb{R})$, and $\frac{\partial}{\partial a} k(t,x,\cdot)$ is bounded.*

Given a contract schedule $\bar{\xi}_N := (\xi_1, \ldots, \xi_N) \in \Sigma_N$, with transaction times $\mathcal{T}_N := \{T_1, \ldots, T_N\}$, the agent maximizes the following objective

$$J_a\left(t, \alpha, \bar{\xi}_N\right) := \mathbb{E}^{\mathbb{P}^\alpha}\left[\mathcal{K}^\alpha_{t,T} U_a(\xi_N) + \sum_{i=1}^{N-1} \mathcal{K}^\alpha_{t,T_i} U_a(\xi_i) \mathbb{1}_{t<T_i} - \int_t^T \mathcal{K}^\alpha_{t,s} c_s(X^{\bar{\xi}_{N-1}}_s, \alpha_s) ds \middle| \mathcal{F}_t\right], \tag{5}$$

where $(\alpha, X^{\bar{\xi}_{N-1}})$ solves (3), and

$$K^\alpha_{t,s} := \exp\left(-\int_t^s k_r(X^{\bar{\xi}_{N-1}}_r, \alpha_r) dr\right), \quad s \geq t.$$

Additionally, we define the agent's continuation value

$$V^a_t(\bar{\xi}_N) := \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}(\bar{\xi}_{N-1})} J_a\left(t, \alpha, \bar{\xi}_N\right), \quad 0 \leq t \leq T.$$

Next, we introduce the principal's problem. Let $\mathcal{A}^*(\bar{\xi}_N)$ be the set of agent's optimal actions given the contract schedule $\bar{\xi}_N \in \Sigma_N$. The principal maximizes the following objective

$$J_p\left(\bar{\xi}_N\right) = \sup_{\alpha \in \mathcal{A}^*(\bar{\xi}_N)} \mathbb{E}^{\mathbb{P}^\alpha}\left[U_p\left(l(X^{\bar{\xi}_{N-1}}) - \xi_N\right)\right], \tag{6}$$

where U_p is a real utility function, and $l : \Omega \mapsto \mathbb{R}$ is a Borel measurable liquidation function.

Given an agent's reservation utility $R_a \in \mathbb{R}$, we denote by Σ^a_N the set of incentive compatible admissible contracts

$$\Sigma^a_N := \left\{\bar{\xi}_N \in \Sigma_N : V^a_0(\bar{\xi}_N) \geq R_a\right\}.$$

Hence, we define the principal's value as follows

$$V_p = \sup_{\bar{\xi}_N \in \Sigma^a_N} J_p\left(\bar{\xi}_N\right).$$

In conclusion, the principal's problem is to find an incentive-compatible contract schedule $\bar{\xi}_N \in \Sigma_N^a$, for which $J_p(\bar{\xi}_N) = V_p$. We introduce the following class of processes to represent the principal's problem effectively.

For all $0 \leq t < s \leq T$, we define the following sets of processes

$$\mathbb{H}([t,s]) := \left\{ Z, \mathbb{F} - \text{predictable} \;\middle|\; \mathbb{E}^{\mathbb{W}}\left[\left(\int_t^s |Z_r|^2\, dr\right)^{p/2}\right] < \infty, \forall p \geq 0 \right\},$$
$$\mathbb{D}_{\exp}([t,s]) := \left\{ Y, \mathbb{F} - \text{predictable, càdlag} \;\middle|\; \mathbb{E}^{\mathbb{W}}\left[\exp\left(p \sup_{t\leq r\leq s} |Y_r|\right)\right] < \infty, \forall p \geq 0 \right\}.$$

In Proposition 1 we represent any admissible contract schedule in terms of a unique solution of a recursive system of BSDEs. We start by introducing the following functionals

$$\begin{aligned} h_t(a,x,y,z) &:= z\lambda_t(x,a) - k_t(x,a)y - c_t(x,a), \quad (t,a,x,y,z) \in [0,T]\times\mathbb{R}^4, \\ H_t(x,y,z) &:= \sup_{a\in\mathbb{R}} h_t(a,x,y,z). \end{aligned} \tag{7}$$

Remark 2. Note that under Assumptions 1, 2, and 3, the mapping $a \mapsto -h_t(x,y,z,a)$ is coercive implying that $H_t(x,y,z)$ is well defined for all $(t,x,y) \in [0,T]\times\mathbb{R}^2$. Moreover, $z \mapsto H_t(x,y,z)$ is convex as it is the pointwise supremum of affine functions.

Proposition 1. *Let $\bar{\xi}_N := (\xi_1, \dots, \xi_N) \in \Sigma_N$ be a contract schedule. Then, there exists a unique pair of processes $(Y,Z) \in \mathbb{D}_{\exp}([0,T]) \times \mathbb{H}([0,T])$, solving the following recursive system of BSDEs*

$$\begin{aligned} Y_t =& U_a(\xi_N) + \int_t^T H_s\left(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s\right) ds - \int_t^T Z_s\sigma_s(X_s^{\bar{\xi}_{N-1}})dW_s, \\ & \mathbb{W} - a.s. \quad T_{N-1} < t \leq T \\ Y_t =& Y_{T_i} + U_a(\xi_i) + \int_t^{T_i} H_s\left(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s\right) ds - \int_t^{T_i} Z_s\sigma_s(X_s^{\bar{\xi}_{N-1}})dW_s, \\ & \mathbb{W} - a.s. \quad T_{i-1} < t \leq T_i, \end{aligned} \tag{8}$$

for all $i \in \{1, \dots, N-1\}$.

Using the previous recursive system of BSDEs, we characterize the agent's optimal response for any contract schedule $\bar{\xi}_N \in \Sigma_N$ satisfying an integrability constraint.

Proposition 2. *Let $\bar{\xi}_N \in \Sigma_N$, and $(Y,Z) \in \mathbb{D}_{exp}([0,T]) \times \mathbb{H}([0,T])$ be the unique pair of processes defined in Proposition 1. Assume $\mathbb{E}\left[\mathcal{E}\left(\int_0^T Z_s dW_s\right)^{1+\epsilon}\right] < \infty$, for some $\epsilon > 0$. Then,*

$$Y_t = V_t^a(\bar{\xi}_N) \;\; \mathbb{W} - a.s., \quad 0 \leq t \leq T.$$

Moreover, $\alpha \in \mathcal{A}^*(\bar{\xi}_N)$ if and only if

$$\begin{aligned} \alpha_t &= \hat{\alpha}_t(X_t^{\bar{\xi}_{N-1}}, Y_t, Z_t), \quad dt \otimes \mathbb{W} - a.e., \\ \hat{\alpha}_t(x,y,z) &\in \arg\max h_t(\cdot, x, y, z), \quad \forall (t,x,y,z) \in [0,T]\times\mathbb{R}^3. \end{aligned} \tag{9}$$

Remark 3. The previous proposition motivates us to restrict the set of admissible contract schedules by excluding contracts that are not integrable enough. The latter does not suppose a significant limitation and allows us to fully characterize the set of agent's optimal responses.

We introduce the following class of contract schedules:

$$\hat{\Sigma}_N^a := \left\{ \bar{\xi}_N \in \Sigma_N^a : (Y, Z) \text{ solving (8)}, \quad \mathbb{E}\left[\mathcal{E}\left(\int_0^T Z_s dW_s \right)^{1+\epsilon} \right] < \infty, \quad \text{for some } \epsilon > 0 \right\}.$$

In addition, for any $\bar{\xi}_N \in \hat{\Sigma}_N^a$, we denote by $\hat{\mathcal{A}}(\bar{\xi}_N)$ the set of processes $\alpha \in \mathcal{A}(\bar{\xi}_{N-1})$ satisfying

$$\begin{aligned} \alpha_t &= \hat{\alpha}_t(X_t^{\bar{\xi}_{N-1}}, Y_t, Z_t), \quad dt \otimes \mathbb{W} - a.e., \\ \hat{\alpha}_t(x, y, z) &\in \arg\max h_t(\cdot, x, y, z), \quad \forall (t, x, y, z) \in [0, T] \times \mathbb{R}^3. \end{aligned} \tag{10}$$

Finally, we reduce the principal's problem to a weak formulation stochastic control problem.

Theorem 4. *The principal's value satisfies*

$$\begin{aligned} &\sup_{\bar{\xi}_N \in \hat{\Sigma}_N^a} J_p(\bar{\xi}_N) \\ &= \sup_{Y_0 \geq R_a} \sup_{(Z, \bar{\xi}_{N-1}) \in \mathbb{H}([0,T]) \times \Sigma_{N-1}} \sup_{\alpha \in \hat{\mathcal{A}}(\bar{\xi}_{N-1})} \mathbb{E}^{\mathbb{P}^\alpha} \left[U_p \left(l \left(X^{\bar{\xi}_{N-1}} \right) - U_a^{-1} \left(Y_T^{\bar{\xi}_{N-1}, Z} \right) \right) \right], \end{aligned}$$

where $(X^{\bar{\xi}_{N-1}}, Y^{Y_0, Z, \bar{\xi}_{N-1}})$ *solves the following iterative system of SDEs*

$$Y_t = Y_{T_i^-} - U_a(\xi_i) - \int_{T_i}^t H_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s) ds + \int_{T_i}^t Z_s dX_s^{\bar{\xi}_{N-1}}, \quad T_i \leq t < T_{i+1},$$

$$Y_t = Y_0 - \int_0^t H_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s) ds + \int_0^t Z_s dX_s^{\bar{\xi}_{N-1}}, \quad 0 \leq t < T_1,$$

$$\begin{aligned} &X_t^{\bar{\xi}_{N-1}} \\ &= G_i \left(X_{T_i^-}^{\bar{\xi}_{N-1}}, \xi_i \right) + \int_{T_i}^t \left(\lambda_s \left(X_s^{\bar{\xi}_{N-1}}, \hat{\alpha}_s(X_s^{\bar{\xi}_{N-1}}, Z_s, Y_s) \right) ds + \sigma_s(X_s^{\bar{\xi}_{N-1}}) dB_s^\alpha \right), \\ &\quad T_i \leq t < T_{i+1}, \end{aligned}$$

$$\begin{aligned} X_t^{\bar{\xi}_{N-1}} &= x_0 + \int_0^t \left(\lambda_s \left(X_s^{\bar{\xi}_{N-1}}, \hat{\alpha}_s(X_s^{\bar{\xi}_{N-1}}, Z_s, Y_s) \right) ds + \sigma_s(X_s^{\bar{\xi}_{N-1}}) dB_s^\alpha \right), \\ &\quad 0 \leq t < T_1, \quad \mathbb{P}^\alpha - a.s., \quad i \in \{1, \ldots, N-1\}. \end{aligned}$$

4. **The benchmark model.** In this section, we apply the results obtained in the previous sections to a benchmark model with a risk-neutral principal and risk-averse, time-sensitive agent. We assume that the principal compensates the agent with N non-negative payments $\bar{\xi}_N := (\xi_1, \ldots, \xi_N)$ transacted at $\mathcal{T}_N := \{T_1, \ldots, T_N\}$, where $(T_i)_{i=0}^N$ is a strictly increasing sequence satisfying $T_0 = 0, T_N = T$. Given a contract schedule $\bar{\xi}_N \in \Sigma_N$, and an initial output level $x_0 \in \mathbb{R}$, we introduce the output process

$$X_t^{\bar{\xi}_{N-1}} := x_0 + W_t - \sum_{j=1}^{N-1} \xi_j \mathbb{1}_{T_j \leq t}, \quad \mathbb{W} - a.s.$$

We denote by $\mathcal{A}$ the set of admissible agent's actions corresponding to the set of $\mathbb{F}$-adapted processes α satisfying

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^T \alpha_s^2 ds + \int_0^T \alpha_s dW_s\right)^{1+\epsilon}\right] < \infty,$$

for some positive constant $\epsilon > 0$.

Hence, given an agent's action $\alpha \in \mathcal{A}$, and a contract schedule $\bar{\xi}_N \in \Sigma_N$, the output process satisfies the following controlled dynamics

$$X_t^{\bar{\xi}_{N-1}} = x_0 - \sum_{j=1}^{N-1} \xi_j \mathbb{1}_{T_j \le t} + \int_0^t \alpha_s ds + B_t^\alpha, \quad \mathbb{P}^\alpha - a.s., \quad 0 \le t \le T,$$

where $\mathbb{P}^\alpha \in \mathcal{P}(\Omega)$, and B^α is a $\mathbb{P}^\alpha$-Brownian motion defined via Girsanov's theorem as follows

$$B_t^\alpha := W_t - \int_0^t \alpha_r dr, \quad \frac{d\mathbb{P}^\alpha}{d\mathbb{W}} := \exp\left(-\frac{1}{2}\int_0^T \alpha_s^2 ds + \int_0^T \alpha_s dW_s\right).$$

In this scenario, we introduce the agent's objective and continuation value respectively

$$J_a(t,\alpha,\bar{\xi}_N)$$
$$:= \mathbb{E}^{\mathbb{P}^\alpha}\left[e^{-k_a(T-t)}U_a(\xi_N) + \sum_{i=1}^{N-1} e^{-k_a(T_i-t)}U_a(\xi_i)\mathbb{1}_{T_i>t} - \frac{1}{2}\int_t^T e^{-k_a(s-t)}\alpha_s^2 ds \middle| \mathcal{F}_t\right],$$

$$V_t^a(\bar{\xi}_N)$$
$$:= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\left[e^{-k_a(T-t)}U_a(\xi_N) + \sum_{i=1}^{N-1} e^{-k_a(T_i-t)}U_a(\xi_i)\mathbb{1}_{T_i>t} - \frac{1}{2}\int_t^T e^{-k_a(s-t)}\alpha_s^2 ds \middle| \mathcal{F}_t\right],$$

where $U_a(y) := y^{1/\gamma}\mathbb{1}_{y\ge 0} - \infty\mathbb{1}_{y<0}$, for some $\gamma > 1$, and $k_a \ge 0$ represents the agent's discount factor.

On the other hand, we introduce the principal's objective and continuation value

$$J_p\left(\bar{\xi}_N\right) := \mathbb{E}^{\mathbb{P}^\alpha}\left[X_T^{\bar{\xi}_{N-1}} - \xi_N\right], \quad V_p := \sup_{\bar{\xi}_N \in \Sigma_N^a} \sup_{\alpha \in \mathcal{A}^*(\bar{\xi}_N)} J_p\left(\bar{\xi}_N\right), \tag{11}$$

where

$$\Sigma_N^a := \left\{\bar{\xi}_N \in \Sigma_N : V_0^a(\bar{\xi}_N) \ge R_a, V_t^a(\bar{\xi}_N) \ge 0, \mathbb{W} - a.s., 0 \le t \le T\right\},$$

and $R_a \ge 0$, represents the agent's reservation utility level. Additionally, we assume that the agent has limited liability implying that the agent only accepts contracts that generate a non-negative continuation utility at any time.

Applying Proposition 2, we obtain that for any contract schedule $\bar{\xi}_N \in \Sigma_N^a$, there exists a unique pair of processes $(Y,Z) \in \mathbb{D}_{\exp}([0,T]) \times \mathbb{H}([0,T])$ satisfying

$$Y_t = Y_{T_i^-} - U_a(\xi_i) + \int_{T_i}^t \left(-\frac{1}{2}Z_s^2 + k_a Y_s\right) ds + \int_{T_i}^t Z_s dW_s,$$
$$T_i \le t < T_{i+1}, \quad i \in \{1,\dots,N-1\},$$

$$Y_t = Y_0 + \int_0^t \left(-\frac{1}{2}Z_s^2 + k_a Y_s\right) ds + \int_0^t Z_s dW_s, \quad 0 \le t < T_1,$$

where $Y_0 \in \mathbb{R}$.

Next, we invoke ([14], Theorem 4.3) to justify that the previous weak formulation stochastic control problem is equivalent to a stochastic control problem in the strong formulation where the stochastic basis is fixed. Using Theorem 4, and the previous observation, we reduce the principal's problem (11) to the following strong formulation stochastic control problem

$$V(t,x,y) := \sup_{(Z,\bar{\xi}_{N-1})\in\mathcal{U}(t,x,y)} \mathbb{E}\left[X_T^{t,x,Z,\bar{\xi}_{N-1}} - \left(Y_T^{t,y,Z,\bar{\xi}_{N-1}}\right)^{\gamma}\right], \tag{12}$$

for all $(t,x,y) \in [0,T] \times \mathbb{R} \times [0,\infty)$, and, for $s > t$:

$$X_s^{t,x,Z,\bar{\xi}_{N-1}} = x - \sum_{j=1}^{N-1} \xi_j \mathbb{1}_{t<T_j\le s} + \int_t^s Z_r dr + B_s - B_t,$$

$$Y_s^{t,y,Z,\bar{\xi}_{N-1}} = y - \sum_{j=1}^{N-1} U_a(\xi_j)\mathbb{1}_{t<T_j\le s} + \int_t^s \left(\frac{1}{2}Z_r^2 + k_a Y_r^{t,y,Z,\bar{\xi}_{N-1}}\right) dr + \int_t^s Z_r dB_r.$$

In the previous equation, B is a Brownian-motion defined on fixed probability space $(\Omega,\mathbb{F},\mathbb{P})$, and $\mathbb{F}$ is the augmented filtration generated by B. The control Z is an $\mathbb{F}$-predictable process, and $\bar{\xi}_{N-1} := (\xi_1,\dots,\xi_{N-1}) \in \Sigma_{N-1}$, indicating that for every $i \in \{1,\dots,N-1\}$, ξ_i is a $\mathcal{F}_{T_i}$-measurable random variable satisfying the integrability condition stated in (1). Moreover, for all $(t,x,y) \in [0,T] \times \mathbb{R} \times [0,\infty)$, the set $\mathcal{U}(t,x,y)$ denotes the set of controls $(Z,\bar{\xi}_{N-1}) \in \mathbb{H}([0,T]) \times \Sigma_{N-1}$, satisfying the limited liability condition: $Y_t^{t,y,Z,\bar{\xi}_{N-1}} \ge 0\, \mathbb{P}-a.s.$, and the uniform boundedness condition: $|Z_t| \le K, dt \otimes \mathbb{P} - a.e$, for some constant $K > 0$.

Remark 5. From a modeling perspective, the restriction to uniformly bounded continuous controls $|Z_t| \le K dt \otimes \mathbb{P} - a.e.$, establishes a maximum sensitivity level of the agent's value with respect to changes in the output process. Jointly with the limited liability constraint and the reservation utility R_a, comprise the agent's participation constraint.

The following theorem allows us to write the principal's value function as the unique viscosity solution of a recursive system of HJB equations.

Theorem 6. *The principal's value function satisfies the following properties:*

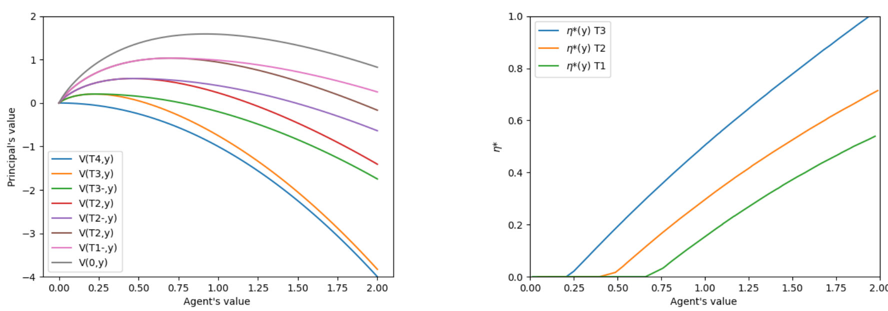
1. *The value function satisfies $V(t,x,y) = x + v(t,y)$, where v is the unique viscosity solution of the following constrained HJB equation:*

$$\begin{aligned} &v_t + G(t,y,v,v_y,v_{yy}) = 0, \quad (t,y) \in [T_{i-1},T_i) \times (0,\infty), \\ &v(T_i^-,y) = f_i(y), \quad y \ge 0, \quad i \in \{1,\cdots,N-1\}, \\ &v(T_N,y) = -y^{\gamma}, \quad y \in [0,\infty], \\ &v(t,0) = 0, \quad t \in [T_{i-1},T_i), \end{aligned} \tag{13}$$

where $G(t, y, v_y, v_{yy}) := k_a y v_y + \sup_{|z| \leq K} \left\{ z + \frac{1}{2} (v_y + v_{yy}) z^2 \right\}$, *and* $f_i(y) := \max_{0 \leq \eta \leq y} -\eta^\gamma + v(T_i, y - \eta)$. *Moreover,* v *is continuous in each region* $\mathcal{R}_i := [T_{i-1}, T_i) \times [0, \infty)$, $i \in \{1 \ldots N\}$.

2. *The function* f_i *is continuous and has polynomial growth for all* $i \in \{1, \ldots, N-1\}$. *Moreover, there exists a minimal function* $\eta_i^* : [0, \infty) \mapsto \mathbb{R}$ *satisfying*
   - $\eta_i^*(y) \in \arg\max_{0 \leq \eta \leq y} v(T_i, y - \eta) - \eta^\gamma$, *for all* $y \geq 0$, $i \in \{1, \ldots, N-1\}$.
   - η_i^* *is lower semicontinuous, for all* $y \geq 0$, $i \in \{1, \ldots, N-1\}$.
3. $V(T_i^-, x, y) = x + f_i(y)$, *for all* $(t, x, y) \in [0, T] \times \mathbb{R} \times [0, \infty)$, $i \in \{1, \ldots N-1\}$.

5. **Numerical results.** In this section, we employ numerical methods to discuss the model presented in the previous section. In the analysis, we fix $U_a(y) = y^{1/2}$, $x_0 = 0$, and $K = 10^6$. The recursive HJB equation (13) is approximated numerically following a standard finite-difference scheme. We refer to [2] to justify the convergence of our numerical scheme to the unique solution of the HJB equation (13), which, by Theorem 6, corresponds to the principal's value function.



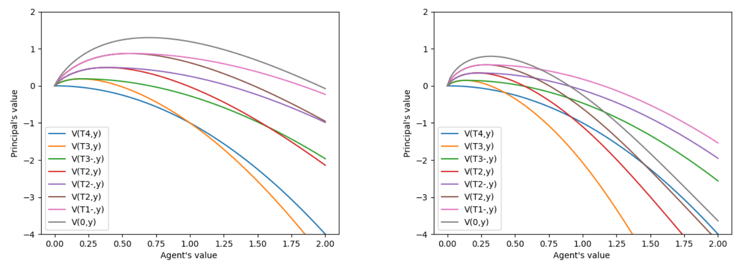
(A) Principal's value with $k_a = 0$.

(B) Static optimizers with $k_a = 0$.

FIGURE 1. Principal's value and optimal utilities $\eta_i^* = U_a(\xi_i^*)$, when $k_a = 0$.

5.1. **The case with no discounting** $k_a = 0$**.** Figure (1a) illustrates the principal's value function at each transaction time when $k_a = 0$. The functions $v(T_i, \cdot)$ represent the principal's value function at time T_i, and $v(T_i^-, y)$ is the principal's value function immediately before T_i. On the other hand, figure (1b) depicts the utilities that the principal provides to the agent at each intermediate transaction time. The function $\eta_i^*(\cdot) := U_a(\xi_i^*(\cdot))$ denotes the optimal utility the principal aims to pay to the agent at T_i, $i \in \{1, \ldots, N-1\}$. We see that the principal's value function in figure (1a) exhibits concavity and it is ultimately decreasing with respect to the agent's continuation utility in a fixed time. The latter is consistent with Sannikov's model ([25], Lemma 8.1). Moreover, our numerical results indicate that an agent with a low reservation utility level has informational rent, meaning that the principal optimally offers a contract providing to the agent a utility strictly greater than the agent's reservation utility.

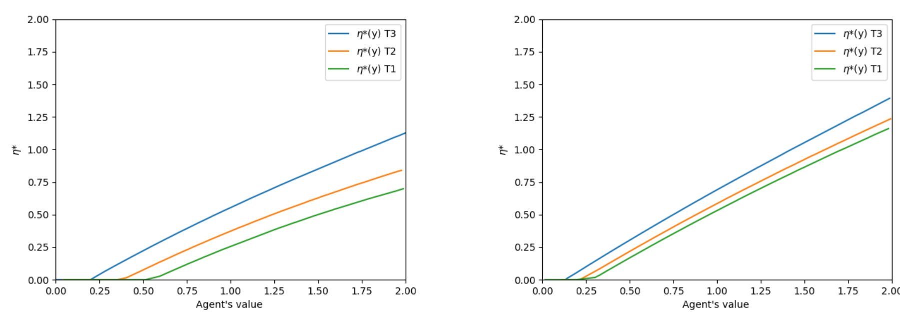
Secondly, figure (1b) shows that the optimal intermediate payments are an increasing function of the agent's utility at a given time. Moreover, for the same agent's utility level, the principal offers more utility to the agent in a posterior payment. Finally, (1b) reflects the fact that the region where the agent has informational rent shrinks as time elapses. The principal only compensates the agent when his utility does not belong to the informational rent region.



(A) Principal's value with k_a= 0.05. (B) Principal's value with $k_a = 0.2$.

FIGURE 2. Principal's value, when $k_a > 0$.

5.2. **The effect of the discounting factor $k_a > 0$.** The figure above represents the principal's value function at each transaction time when the agent is impatient: $k_a > 0$. The above simulations align with the concept of employment interval discussed in Sannikov [28]. In our case, it directly shows that it is harmful for the principal to provide a large utility to the agent when the agent's discounting factor is greater than zero. We define the i-th employment interval to the set $\mathcal{E}_i := \{y \in [0, \infty) : V(T_{i-1}, y) \geq V(T_i, y)\}$. A careful observation of the above figures shows that the employment interval shrinks when the time elapses. Moreover, a comparison between figures (2a) and (2b) shows that an increase of the discounting factor causes a shrinkage of the employment intervals in every contracting period.



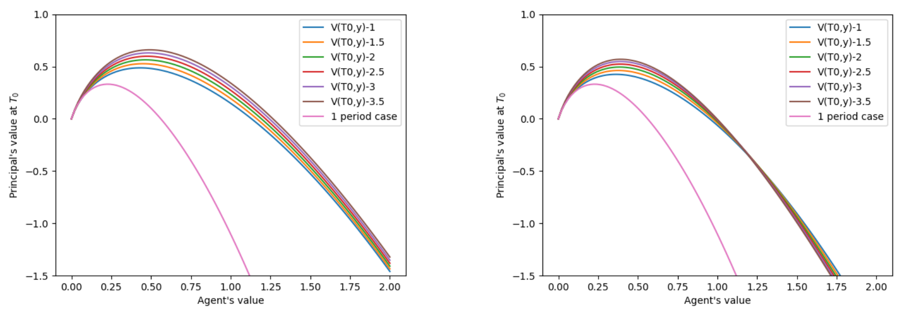
(A) Optimal utilities delivered to the agent. $k_a = 0.05$.

(B) Optimal utilities delivered to the agent. $k_a = 0.2$.

FIGURE 3. Optimal utilities $\eta_i^* = U_a(\xi_i^*)$, when $k_a > 0$.

Figure (3) above describes the optimal utilities delivered to the agent associated with the principal's value functions represented in figure (2). We see that, with an increase in the discounting factor, the truncation region $\{y : \eta_i^*(y) = 0\}$ shrinks.

This means the principal prefers to provide more utilities to an agent with a high discounting factor. Finally, a careful comparison among figures (3a) and (3b) shows that it is not optimal for the principal to fully compensate the agent before the terminal time.



(A) Principal's value $V(T_0, y)$, $k_a = 0$. (B) Principal's value $V(T_0, y)$, $k_a = 0.05$.

FIGURE 4. Principal's value with $N = 2$ payments for different T_1.

5.3. **Analysis of the payments' distribution.** Figure (4) illustrates the problem with $N = 2$ payments with a terminal contract transacted at a fixed time $T = 4$, and a flexible initial payment transacted at one of the following times: $T_1 = 1 + \frac{i}{2}$, $i \in \{0, 1, 2, 3, 4, 5\}$. For example, '$V(T_0, y) - 1.5$' represents the principal's value function at $T_0 = 0$ when the first payment occurs at $T_1 = 1.5$ and the second payment occurs at $T_2 = 4$. Additionally, the '1-period case' refers to the classic principal-agent model with a single payment transacted at the end of the contracting period $T = 4$.

Firstly, we observe that the principal benefits from making multiple payments compared to a single terminal compensation. Secondly, when $k_a = 0$, the principal always benefits from delaying the first payment. Finally, subfigure (4b) depicts the principal's value functions when the agent is impatient ($k_a = 0.05$). It shows that if the agent is impatient with a relatively high reservation utility, the principal benefits from setting the first payment earlier. Conversely, if the agent is impatient with a low reservation utility, the principal benefits from delaying the first payment.

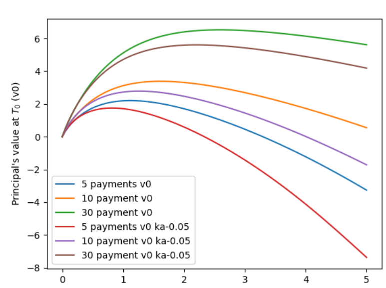


FIGURE 5. Principal's value for different payment frequencies.

5.4. **Analysis of the payment frequency.** Figure (5) illustrates the principal´s value $V(0,\cdot)$ for different payment frequencies. For $N = 5, 10, 30$, we consider a sequential optimal contracting problem with N contracts transacted at $T_i = i\frac{T}{N}$, $i \in \{1, \dots, N\}$, and $T = 10$. As the number of payments in the contract schedule increases, the principal achieves higher profits for any given agent's reservation utility.

5.5. **Initial negotiation vs. renegotiation.** In this subsection, we compare the principal's value in our benchmark model (initial negotiation) with an analogous model assuming that each contract is renegotiated right after the previous payment is transacted. In the renegotiation setting the agent solves a different optimization problem in each contracting region. Moreover, the agent's continuation utility resets when the i-th payment is delivered. Mathematically, the renegotiation problem corresponds to a sequential Stackelberg game with N different reservation utility constraints:

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[e^{-k_a T_i} U_a(\xi_i) - \frac{1}{2}\int_{T_{i-1}}^{T_i} e^{-k_a s}\alpha_s^2 ds \right] \geq e^{-k_a T_{i-1}} R_a^i, \quad i \in \{1, \dots, N\}, \tag{14}$$

where R_a^i is the reservation utility level in the i-th contracting period.

On the other hand, recall the definition of the reservation constraint in the initial negotiation problem:

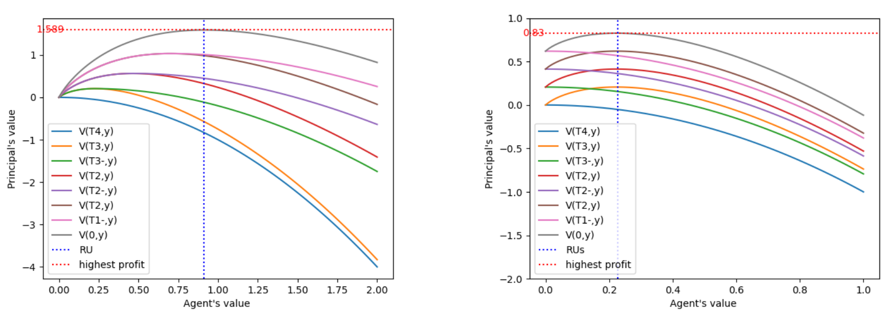
$$\sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\sum_{i=1}^{N} e^{-k_a T_i} U_a(\xi_i) - \frac{1}{2}\int_{0}^{T} e^{-k_a s}\alpha_s^2 ds \right] \geq R_a,$$

where R_a is the agent's reservation utility in the initial negotiation problem.

Assuming that the agent allocates his utility uniformly throughout the duration of the contract, we obtain normalizing the units in (14):

$$R_a^i = e^{k_a T_{i-1}} \frac{(T_i - T_{i-1})}{T} R_a, \quad i \in \{1, \dots, N\}. \tag{15}$$

The following figures represent the principal's value function in both settings assuming $N = 4$, $T = 8$, $T_i = 2i$, $i \in \{0, \dots, 4\}$.

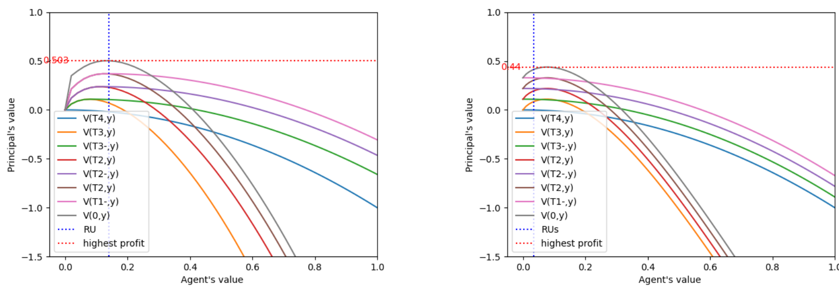


(A) Principal's value. Initial negotiation. $k_a = 0$.

(B) Principal's value. Renegotiation. $k_a = 0$.

FIGURE 6. Initial Negotiation vs. Renegotiation, when $k_a = 0$.

Figure (6) describes the principal's value in the initial negotiation and renegotiation settings. In both cases, we fix $k_a = 0$, and $R_a = 0.909$. The utilities of each contracting period in the renegotiation problem are computed using (15). We observe that the principal benefits from an initial negotiation compared to a renegotiation setting.

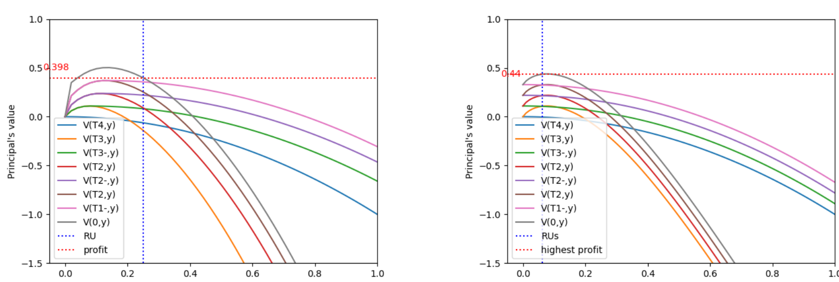


(A) Principal's value function. Initial negotiation, $k_a = 0.4$.

(B) Principal's value function. Renegotiation, $k_a = 0.4$.

FIGURE 7. Initial Negotiation vs. Renegotiation, when $k_a = 0.4$.

Figure (7) describes the principal's value in the initial negotiation and renegotiation settings. In both cases, we fix $k_a = 0.4$, and $R_a = 0.131$. The reservation utilities in the renegotiation problem are computed using (15). We observe that the principal benefits again from an initial negotiation compared to a renegotiation setting.



(A) Principal's value function. Initial negotiation, $k_a = 0.4$.

(B) Principal's value function. Renegotiation, $k_a = 0.4$.

FIGURE 8. Initial Negotiation vs. Renegotiation, when $k_a = 0.4$.

Figure (8) describes the principal's value in the initial negotiation and renegotiation settings. In both cases, we fix $k_a = 0.4$, and $R_a = 0.25$. The reservation utilities in the renegotiation problem are computed using (15). We observe that the principal benefits from renegotiation. The latter shows that the principal's optimal negotiation setting (initial or renegotiation) depends on the contracting environment.

Appendix A. Technical proofs.

A.1. Proof of Proposition 1. Let $\bar{\xi}_N \in \Sigma_N$. Using Assumptions 1, 2, 3, and the uniform boundedness of k, there exists $B_1 > 0$, for which

$$
\begin{aligned}
&H_t(x, y, \cdot) \text{ is convex}, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^2, \\
&|H_t(x, y_1, z) - H_t(x, y_2, z)| \leq B_1 |y_1 - y_2|, \quad \forall (t, x, z, y_1, y_2) \in [0, T] \times \mathbb{R}^4. \qquad (16)
\end{aligned}
$$

Moreover, there exists $B_2 > 0$, for which

$$
\begin{aligned}
|H_t(x, y, z, a)| &= \left| \sup_{a \in \mathbb{R}} \{\lambda(t, x, a) z - c(t, x, a) + y k(t, x, a)\} \right| \\
&\leq \left| \sup_{a \in \mathbb{R}} \{\lambda(t, x, a) z - c(t, x, a)\} \right| + B_2 |y|. \qquad (17)
\end{aligned}
$$

Next, we observe that for all $(t, x) \in [0, T] \times \mathbb{R}$, the mapping $a \mapsto \lambda(t, x, a) z - c(t, x, a)$, admits a global maximizer $a^*(t, x)$ satisfying the first order condition

$$
\frac{\partial}{\partial a} \lambda(t, x, a^*(t, x)) z = \frac{\partial}{\partial a} c(t, x, a^*(t, x)).
$$

Hence, using Assumption 3, there exists $C > 0$ such that

$$
|a^*(t, x)| \leq C\left(1 + |z|\right).
$$

Plugging in the last expression into (17), we obtain

$$
|H_t(x, y, z)| \leq B_2 |y| + C |z|^2.
$$

Applying ([5], Corollary 2), there exists a unique pair of processes $(Y^N, Z^N) \in \mathbb{D}_{\exp}([0, T]) \times \mathbb{H}([0, T])$ solving the BSDE

$$
\begin{aligned}
dY_t^N &= H_t\left(X_t^{\bar{\xi}_{N-1}}, Y_t^N, Z_t^N\right) dt - Z_t^N \sigma_t(X_t^{\bar{\xi}_{N-1}}) dW_t, \\
Y_T^N &= U_a\left(\xi_N\right).
\end{aligned}
$$

Therefore, for all $p \geq 0$, there exists $C_p, p' > 0$, such that

$$
\mathbb{E}\left[\exp\left(p \left| U_a(\xi_{N-1}) + Y_{T_{N-1}}^N \right|\right)\right] < C_p \mathbb{E}\left[\exp\left(p' \left|\xi_{N-1}\right|\right)\right]^{1/2} \mathbb{E}\left[\exp\left(2p \left| Y_{T_{N-1}}^N \right|\right)\right]^{1/2} < \infty,
$$

where we used the Cauchy-Schwarz inequality, and $\left(\bar{\xi}_N, Y\right) \in \Sigma_N \times \mathbb{D}_{\exp}\left([0, T]\right)$.

Repeating the same argument we show by induction that for all $i \in \{1, \dots, N-1\}$, there is a unique pair of processes $(Y^i, Z^i) \in \mathbb{D}_{\exp}([0, T_i]) \times \mathbb{H}([0, T_i])$ satisfying

$$
\begin{aligned}
dY_t^i &= H_t\left(X_t^{\bar{\xi}_{N-1}}, Y_t^i, Z_t^i\right) dt - Z_t^i \sigma_t(X_t^{\bar{\xi}_{N-1}}) dW_t, \\
Y_{T_i}^i &= Y_{T_i}^{i+1} + U_a\left(\xi_i\right).
\end{aligned}
$$

Finally, setting $Y_t := \sum_{i=1}^{N-1} Y_t^i \mathbb{1}_{[T_{i-1}, T_i)} + Y_t^N \mathbb{1}_{[T_{N-1}, T_N]}$, and $Z_t := \sum_{i=1}^{N-1} Z_t^i \mathbb{1}_{[T_{i-1}, T_i)} + Z_t^N \mathbb{1}_{[T_{N-1}, T_N]}$, we obtain that (Y, Z) belongs to $\mathbb{D}_{\exp}([0, T]) \times \mathbb{H}([0, T])$, and solves (8). Moreover, we conclude that $Y_0 \in \mathbb{R}$ by Blumenthal's zero–one law. $\square$

A.2. **Proof of Proposition 2.** Firstly, consider a payment scheme $\bar{\xi}_N \in \Sigma_N$. By Proposition 1, there exists a unique pair of process $(Y, Z) \in \mathbb{D}_{\exp}([0,T]) \times \mathbb{H}([0,T])$ satisfying

$$dY_t = H_t\left(X_t^{\bar{\xi}_{N-1}}, Y_t, Z_t\right)dt - Z_t\sigma_t(X_t^{\bar{\xi}_{N-1}})dW_t, \mathbb{W} - a.s. \quad T_{i-1} < t \le T_i,$$

$$Y_{T_i^-} = Y_{T_i} + U_a(\xi_i), \quad \mathbb{W} - a.s., \quad i \in \{1, \dots, N-1\},$$

$$Y_T = U_a(\xi_N).$$

Without loss of generality, we assume that $0 \le t < T_{N-1}$. Let $\alpha \in \mathcal{A}(\bar{\xi}_{N-1})$. Applying Itô's Lemma, and using Assumptions 1, 2, 3, we obtain

$$\begin{aligned}
&J_a(t, \alpha, \bar{\xi}_N) \\
&= \mathbb{E}^{\mathbb{P}^\alpha}\left[\mathcal{K}^\alpha_{t,T} Y_T + \sum_{i=1}^{N-1} \mathcal{K}^\alpha_{t,T_i} U_a(\xi_i)\, \mathbb{1}_{t<T_i} - \int_t^T \mathcal{K}^\alpha_{t,s} c_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)ds \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}^\alpha}\left[\mathcal{K}^\alpha_{t,T_{N-1}} Y_{T^-_{N-1}} + \sum_{i=1}^{N-2} \mathcal{K}^\alpha_{t,T_i} U_a(\xi_i)\, \mathbb{1}_{t<T_i} - \int_t^{T_{N-1}} \mathcal{K}^\alpha_{t,s} c_s(X_s^{\bar{\xi}_{N-1}}, \alpha_s)ds \middle| \mathcal{F}_t\right] \\
&\quad - \mathbb{E}^{\mathbb{P}^\alpha}\left[\int_{T_{N-1}}^T \mathcal{K}^\alpha_{t,s}\left(H_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s) - h_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s, \alpha_s)\right)ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

Repeating the same argument recursively, we have

$$J_a(t, \alpha, \bar{\xi}_N) = Y_t - \mathbb{E}^{\mathbb{P}^\alpha}\left[\int_t^T \mathcal{K}^\alpha_{t,s}\left(H_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s) - h_s(X_s^{\bar{\xi}_{N-1}}, Y_s, Z_s, \alpha_s)\right)ds \middle| \mathcal{F}_t\right].$$

Therefore,

$$V_t^a(\bar{\xi}_N) \le Y_t, \quad \mathbb{W} - a.s.,$$

and the previous upper bound is attained if and only if $\alpha^*_t = \hat{\alpha}(t, X_t^{\bar{\xi}_{N-1}}, Y_t, Z_t)$ $dt \otimes \mathbb{W} - a.e.$, where $\hat{\alpha}(t,x,y,z) \in \arg\max h_t(x,y,z,\cdot)$. Note that α^* is an $\mathbb{F}$-adapted process. Indeed, using Assumptions 1, 2, 3, we obtain that $\hat{\alpha}$ satisfies

$$\frac{\partial}{\partial a}\lambda(t,x,\hat{\alpha}(t,x,y,z))z - \frac{\partial}{\partial a}k_a(x,\hat{\alpha}(t,x,y,z))y = \frac{\partial}{\partial a}c(t,x,\hat{\alpha}(t,x,y,z)).$$

Hence, applying the measurable selection theorem ([3], Theorem 7.49), we obtain that the process α^* is $\mathbb{F}$-adapted. Moreover, using Assumptions 1, 2 3, there exists $C > 0$ such that $|\hat{\alpha}(t,x,y,z)| \le C(1 + |z| + |y|)$. From the latter estimate and the hypothesis on Z, we obtain that $\alpha^* \in \mathcal{A}(\bar{\xi}_{N-1})$. □

A.3. **Proof of Theorem 6.** We introduce the following auxiliary result before proving Theorem 6. For $n \ge 1$, we denote $\mathbb{R}^{+,n}$ the set of n-tuples with positive entries.

Definition 7. Let $\delta := (b, c, M) \in \mathbb{R}^{+,3}$, $a \in \mathbb{R}^{+,1}$, $\gamma > 1$. We introduce the function $\phi^{\gamma,a,\delta} : [0,T] \times [0,\infty) \mapsto \mathbb{R}$ defined by

$$\phi^{\gamma,a,\delta}(t,y) := -ay^\gamma + be^{\frac{(T-t)}{T}} y^{\frac{1}{M}} + e^{c(T-t)}(1 - e^{-y}).$$

Lemma 1. *For all $\gamma > 1$, there exists $\delta_0 \in \mathbb{R}^{+,3}$ such that*

$$-\phi_t^{\gamma,a,\delta_0}(t,y) - \sup_{|z|\le K}\left\{z + \frac{1}{2}\left(\phi_{yy}^{\gamma,a,\delta_0}(t,y) + \phi_y^{\gamma,a,\delta_0}(t,y)\right)z^2\right\} - k_a y \phi_y^{\gamma,a,\delta_0}(t,y) \ge 0,$$

for all $a \in \left\{1, \frac{1}{2^{\gamma-1}}, \dots, \frac{1}{N^{\gamma-1}}\right\}$, and $(t,y) \in [0,T] \times (0,\infty)$.

Proof. Let $M > \max\{2, \gamma, \frac{1}{\gamma-1}\}$. We obtain

$$\begin{aligned}
&\max_{a\in\left\{1,\dots,\frac{1}{N^{\gamma-1}}\right\}} \sup_{(t,y)\in[0,T]\times(0,\infty)} \{\phi_{yy}+\phi_y\}\\
=&be^{\frac{(T-t)}{T}}\frac{1}{M}\left(y^{\frac{1}{M}-1}-\frac{M-1}{M}y^{\frac{1}{M}-2}\right)-a\gamma y^{\gamma-1}-a\gamma(\gamma-1)y^{\gamma-2}\\
=&y^{\gamma-2}\left(be^{\frac{(T-t)}{T}}\frac{1}{M}\left(y^{\frac{1}{M}+1-\gamma}-\frac{M-1}{M}y^{\frac{1}{M}-\gamma}\right)-a\gamma y-a\gamma(\gamma-1)\right)\\
\le&y^{\gamma-2}\left(be^{\frac{(T-t)}{T}}\frac{1}{M}\left(y^{\frac{1}{M}+1-\gamma}-\frac{M-1}{M}y^{\frac{1}{M}-\gamma}\right)-\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right)\\
<&y^{\gamma-2}\left(be\frac{1}{M}\left(\frac{\gamma M-1}{M}\right)^{\frac{1}{M}-\gamma}\left(\frac{M-1}{M(\gamma-1)-1}\right)^{\frac{1}{M}-\gamma+1}-\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right)\\
<&y^{\gamma-2}\left(be\frac{1}{M}-\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right).
\end{aligned}$$

The fourth inequality comes from applying the first order condition to the function $g_1(y) := y^{\frac{1}{M}+1-\gamma} - \frac{M-1}{M}y^{\frac{1}{M}-\gamma}$, which is satisfied at $y^* := \frac{(M-1)(\gamma M-1)}{M(M(\gamma-1)-1)} \ge \frac{M-1}{M}$. The fifth inequality comes from the monotonicity of the function

$$g_2(M) := \left(\frac{\gamma M-1}{M}\right)^{\frac{1}{M}-\gamma}\left(\frac{M-1}{M(\gamma-1)-1}\right)^{\frac{1}{M}-\gamma+1},$$

restricted to $\left(\max\{2,\gamma,\frac{1}{\gamma-1}\},\infty\right)$. Then, for any $0 < b < \frac{\gamma}{eN^{\gamma-1}} < \frac{M\gamma(\gamma-1)}{eN^{\gamma-1}}$, the following inequality holds

$$\max_{a\in\left\{1,\dots,\frac{1}{N^{\gamma-1}}\right\}} \sup_{(t,y)\in[0,T]\times(0,\infty)} \{\phi_{yy}+\phi_y\} < 0. \tag{18}$$

Using (18), we have that for all $a \in \left\{1,\dots,\frac{1}{N^{\gamma-1}}\right\}$, and $\delta = (b,c,M) \in \mathbb{R}^{+,3}$ such that $M > \max\{2,\gamma,\frac{1}{\gamma-1}\}$, and $b < \frac{\gamma}{eN^{\gamma-1}}$, the following inequality is satisfied

$$-\sup_{|z|\le K}\left\{z+\left(\phi^{a,\delta}_{yy}(t,y)+\phi^{a,\delta}_{y}(t,y)\right)\frac{z^2}{2}\right\} \ge -\sup_{z\in\mathbb{R}}\left\{z+\left(\phi^{a,\delta}_{yy}(t,y)+\phi^{a,\delta}_{y}(t,y)\right)\frac{z^2}{2}\right\}$$

$$=\frac{1}{2\left(be^{(T-t)p}\left(\frac{1}{M}y^{\frac{1}{M}-1}-\frac{M-1}{M^2}y^{\frac{1}{M}-2}\right)-\gamma ay^{\gamma-1}-\gamma(\gamma-1)ay^{\gamma-2}\right)}.$$

Hence, for any $\delta = (b,c,M) \in \mathbb{R}^{+,3}$ such that $M > \max\{2,\gamma,\frac{1}{\gamma-1}\}, b < \frac{\gamma}{eN^{\gamma-1}}$, the following inequality holds for all $a \in \left\{1,\dots,\frac{1}{N^{\gamma-1}}\right\}$:

$$\begin{aligned}
&-\phi^{a,\delta_0}_t(t,y)-\sup_{|z|\le K}\left\{z+\left(\phi^{a,\delta_0}_{yy}(t,y)+\phi^{a,\delta_0}_{y}(t,y)\right)\frac{z^2}{2}\right\}-k_a y\phi^{a,\delta_0}_y(t,y)\\
\ge&\frac{1}{2\left(be^{\frac{(T-t)}{T}}(\frac{1}{M}y^{\frac{1}{M}-1}-\frac{M-1}{M^2}y^{\frac{1}{M}-2})-\gamma ay^{\gamma-1}-\gamma(\gamma-1)ay^{\gamma-2}\right)}\\
&+b(\frac{1}{T}-\frac{k_a}{M})e^{\frac{(T-t)}{T}}y^{\frac{1}{M}}+\gamma(k_a)ay^{\gamma}+e^{(T-t)c}(c-ce^{-y}-k_aye^{-y}).
\end{aligned}$$

We will show that there exists $\delta_0 := (b, c, M) \in \mathbb{R}^{+,3}$, for which

$$1 \leq 2c\left(1 - e^{-y} - \frac{k_a}{c} y e^{-y}\right)\left(-be^{\frac{(T-t)}{T}}\left(\frac{1}{M} y^{\frac{1}{M}-1} - \frac{M-1}{M^2} y^{\frac{1}{M}-2}\right) + \gamma a y^{\gamma-1} + \gamma(\gamma-1) a y^{\gamma-2}\right), \tag{19}$$

holds for all $a \in \{1, \dots, \frac{1}{N^{\gamma-1}}\}$, and $(t, y) \in [0, T] \times (0, \infty)$. The previous implies that

$$\min_{a \in \left\{1,\dots,\frac{1}{N^{\gamma-1}}\right\}} \left\{-\phi_t^{a,\delta_0}(t,y) - \sup_{|z|\leq K}\left\{z + \frac{1}{2}\left(\phi_{yy}^{a,\delta_0}(t,y) + \phi_y^{a,\delta_0}(t,y)\right) z^2\right\} - k_a y \phi_y^{a,\delta_0}(t,y)\right\} \geq 0.$$

To show (19), we fix $c > 3k_a$, and use the following estimate

$$1 - e^{-y} - \frac{k_a}{c} y e^{-y} \geq \frac{1}{4} y \mathbb{1}_{\{0 \leq y < \frac{1}{2}\}} + \frac{1}{4} \mathbb{1}_{\{y \geq \frac{1}{2}\}}. \tag{20}$$

Firstly, we consider the case $y \geq \frac{1}{2}$. Using (20), we obtain that for all $a \in \{1, \dots, \frac{1}{N^{\gamma-1}}\}$:

$$\begin{aligned}
&2c\left(1 - e^{-y} - \frac{k_a}{c} y e^{-y}\right)\left(-be^{\frac{(T-t)}{T}}\left(\frac{1}{M} y^{\frac{1}{M}-1} - \frac{M-1}{M^2} y^{\frac{1}{M}-2}\right) + \gamma a y^{\gamma-1} + \gamma(\gamma-1) a y^{\gamma-2}\right) \\
&\geq \frac{c}{2}\left(-be^{\frac{(T-t)}{T}}\left(\frac{1}{M} y^{\frac{1}{M}-1} - \frac{M-1}{M^2} y^{\frac{1}{M}-2}\right) + \gamma a y^{\gamma-1} + \gamma(\gamma-1) a y^{\gamma-2}\right) \\
&\geq \frac{c y^{\gamma-1}}{2}\left(-be^{\frac{(T-t)}{T}}\left(\frac{1}{M} y^{\frac{1}{M}-\gamma} - \frac{M-1}{M^2} y^{\frac{1}{M}-1-\gamma}\right) + \gamma a\right) \\
&\geq \frac{ca}{2^\gamma}\left(-\frac{be}{a}\left(\frac{M-1}{\gamma M-1}\right)^{\frac{1}{M}-\gamma}\left(\frac{(\gamma+1)M-1}{M}\right)^{\frac{1}{M}-\gamma-1} + \gamma\right) \\
&\geq \frac{ca}{2^\gamma}\left(-\frac{be}{a}\left(\frac{M^*-1}{\gamma M^*-1}\right)^{\frac{1}{M^*}-\gamma} + \gamma\right),
\end{aligned}$$

where $M^* := \max\{2, \frac{1}{\gamma-1}, \gamma\}$.

The first inequality comes from (20), and the third inequality is obtained from applying the first order condition to $(\frac{1}{M} y^{\frac{1}{M}-1} - \frac{M-1}{M^2} y^{\frac{1}{M}-2})$. The second last inequality is a consequence of the nonnegative and nonincreasing property of the function

$$g_3(M) := \left(\frac{M-1}{\gamma M-1}\right)^{\frac{1}{M}-\gamma}\left(\frac{(\gamma+1)M-1}{M}\right)^{\frac{1}{M}-\gamma-1},$$

in the domain $M \geq M^*$. Thus, for all $\delta_0^1 := (b, c, M) \in \mathbb{R}^{+,3}$, satisfying

$$b < \min\left\{\frac{\gamma}{eN^{\gamma-1}}\left(\frac{M^*-1}{\gamma M^*-1}\right)^{\gamma-\frac{1}{M^*}}, \frac{\gamma}{eN^{\gamma-1}}\right\}, \quad M > \max\left\{2, \frac{1}{\gamma-1}, \gamma\right\}, \tag{21}$$

$$c > \max\left\{\frac{N^{\gamma-1}2^\gamma}{\gamma - bN^{\gamma-1}e(\frac{M^*-1}{\gamma M^*-1})^{\frac{1}{M^*}-\gamma}}, 3k_a\right\},$$

we obtain that (19) holds for all $y \geq \frac{1}{2}$.

Next, we consider the case $0 < y < \frac{1}{2}$. Let $M > 2$, $(b, c) \in \mathbb{R}^{+,2}$. Applying the estimate (20), we obtain that for all $a \in \{1, \dots, \frac{1}{N^{\gamma-1}}\}$:

$$\begin{aligned}
&2c\left(1 - e^{-y} - \frac{k_a}{c} y e^{-y}\right)\left(-be^{\frac{(T-t)}{T}}\left(\frac{1}{M} y^{\frac{1}{M}-1} - \frac{M-1}{M^2} y^{\frac{1}{M}-2}\right) + \gamma a y^{\gamma-1} + \gamma(\gamma-1) a y^{\gamma-2}\right) \\
&\geq \frac{c}{2} y\left(be^{\frac{(T-t)}{T}}\left(-\frac{1}{M} y^{\frac{1}{M}-1} + \frac{M-1}{M^2} y^{\frac{1}{M}-2}\right) + \gamma a y^{\gamma-1} + \gamma(\gamma-1) a y^{\gamma-2}\right)
\end{aligned}$$

$$\geq \frac{c}{2} y^{\gamma-1}\left(be^{\frac{(T-t)}{T}}\left(-\frac{1}{M} y^{\frac{1}{M}+1-\gamma}+\frac{M-1}{M^2} y^{\frac{1}{M}-\gamma}\right)+\gamma(\gamma-1)a\right)$$

$$\geq \frac{c}{2} y^{\gamma-1}\left(be^{\frac{(T-t)}{T}}\left(-\frac{1}{M} y^{\frac{1}{M}+1-\gamma}+\frac{M-1}{M^2} y^{\frac{1}{M}-\gamma}\right)+\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right)$$

$$\geq \frac{c}{2} y^{\gamma-1}\left(b\left(-\frac{1}{M} y^{\frac{1}{M}+1-\gamma}+\frac{M-1}{M^2} y^{\frac{1}{M}-\gamma}\right)+\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right),$$

where we used that $g_1(y) = -\frac{1}{M} y^{\frac{1}{M}+1-\gamma} + \frac{M-1}{M^2} y^{\frac{1}{M}-\gamma} > 0$, for all $0 < y < \frac{M-1}{M}$, where $\frac{M-1}{M} > \frac{1}{2}$.

Next, we consider the function

$$\varphi(y) := y^{\gamma-1}\left(b\left(-\frac{1}{M} y^{\frac{1}{M}+1-\gamma}+\frac{M-1}{M^2} y^{\frac{1}{M}-\gamma}\right)+\frac{\gamma(\gamma-1)}{N^{\gamma-1}}\right).$$

Notice that $\lim_{y\to 0^+} \varphi(y) = \infty$, implying that for all $d > 0$, there exists $y_1 := y_1(d, b, M) > 0$, for which $\varphi(y) > d$, for all $0 < y < y_1$. Using that g_2 is non-negative in $(0, \frac{1}{2})$ when $M > 2$, we obtain

$$\varphi(y) \geq d\mathbb{1}_{0<y<y_1} + \frac{\gamma(\gamma-1)}{N^{\gamma-1}} y_1^{\gamma-1} \mathbb{1}_{y_1 \leq y < \frac{1}{2}}.$$

Therefore, taking $d = 1$ (or any arbitrary positive constant), we obtain that for all $\delta_0^2 := (b, c, M) \in \mathbb{R}^{+,3}$ satisfying

$$M > 2, \quad c > \max\left\{2, \frac{2N^{\gamma-1}}{\gamma(\gamma-1)y_1(1,b,M)^{\gamma-1}}, 3k_a\right\}, \tag{22}$$

the inequality (19) is satisfied for all $y \in \left(0, \frac{1}{2}\right)$. Finally, combining (21), and (22), we conclude that for any $\delta_0 := (b, c, M) \in \mathbb{R}^{3,+}$ satisfying

$$b < \min\left\{\frac{\gamma}{eN^{\gamma-1}}\left(\frac{M^*-1}{\gamma M^*-1}\right)^{\gamma-\frac{1}{M^*}}, \frac{\gamma}{eN^{\gamma-1}}\right\}, \quad M > \max\left\{2, \frac{1}{\gamma-1}, \gamma\right\},$$

$$c > \max\left\{2, \frac{2N^{\gamma-1}}{\gamma(\gamma-1)y_1(1,b,M)^{\gamma-1}}, \frac{N^{\gamma-1}2^\gamma}{\left(-bN^{\gamma-1}e\left(\frac{M^*-1}{\gamma M^*-1}\right)^{\frac{1}{M^*}-\gamma}+\gamma\right)}, 3k_a\right\},$$

the inequality in Lemma 1 holds. □

Definition 8. Let $\gamma > 1$ and $\delta_0 \in \mathbb{R}^{+,3}$ be the 3-tuple determined in Lemma 1. We introduce the function $\phi^{\gamma,\delta_0} : [0, T] \times [0, \infty) \mapsto \mathbb{R}$ defined as

$$\phi^{\gamma,\delta_0}(t, y) := \sum_{i=1}^{N} \phi^{\gamma, \frac{1}{(1+N-i)^\gamma}, \delta_0}(t, y)\mathbb{1}_{(T_{i-1},T_i]}.$$

Remark 9. The function ϕ^{γ,δ_0} plays an important role in our approach. It is carefully designed to obtain an appropriate upper bound of the principal´s value function. The latter is crucial to successfully employ the results from [4].

Next, we introduce the open-state constraint value function

$$\hat{V}(t, x, y) := \sup_{Y_0 \geq R_a} \sup_{(Z,\bar{\xi}_{N-1}) \in \hat{\mathcal{U}}(t,x,y)} \mathbb{E}\left[X_T^{t,x,Z,\bar{\xi}_{N-1}} - \left(Y_T^{t,y,Z,\bar{\xi}_{N-1}}\right)^\gamma\right],$$

where

$$X_s^{t,x,Z,\bar{\xi}_{N-1}} = x - \sum_{j=1}^{N-1} \xi_j \mathbb{1}_{t<T_j\leq s} + \int_t^s Z_r dr + B_t - B_s,$$

$$Y_s^{t,y,Z,\bar{\xi}_{N-1}} = y - \sum_{j=1}^{N-1} U_a(\xi_j) \mathbb{1}_{t<T_j\le s} + \int_t^s \left(\frac{1}{2}Z_r^2 + k_a Y_r^{t,y,Z,\bar{\xi}_{N-1}}\right) dr + \int_t^s Z_r dB_r,$$

and $\hat{\mathcal{U}}(t,x,y) := \left\{(Z,\bar{\xi}_{N-1}) \in \mathcal{U}(t,x,y) : \; Y_s^{t,y,Z,\bar{\xi}_N} > 0, \; \mathbb{P}-a.s, \; t \le s \le T\right\}$, for all $(t,x,y) \in [T_{N-1},T] \times \mathbb{R} \times (0,\infty)$.

We introduce the lower and upper semicontinuous envelopes of $\hat{V}$, defined for all $(t,x,y) \in [0,T] \times \mathbb{R} \times (0,\infty)$ as follows

$$\hat{V}_*(t,x,y) := \liminf_{(t',x',y')\to(t,x,y),y'>0} \hat{V}(t',x',y'),$$

$$\hat{V}^*(t,x,y) := \limsup_{(t',x',y')\to(t,x,y),y'>0} \hat{V}(t',x',y').$$

Firstly, we show Theorem 6.1, 6.2, 6.3, simultaneously by backward induction on the regions $\mathcal{R}_i := [T_{i-1},T_i) \times [0,\infty)$, $i \in \{1,\dots,N\}$.

A.3.1. *Proof of Theorem 6.1 on $\mathcal{R}_N$.* Firstly, consider $(t,x,y) \in (T_{N-1},T] \times \mathbb{R} \times (0,\infty)$, a control process $Z \in \hat{\mathcal{U}}(t,x,y)$, and an $\mathbb{F}$-stopping time τ. Applying Itô's Lemma, we obtain

$$\begin{aligned} x + \phi^{\gamma,\delta_0}(t,y) &= X_\tau^{t,x,Z} + \phi^{\gamma,\delta_0}(\tau, Y_\tau^{t,y,Z}) \\ &\quad - \int_t^\tau \left(\phi_t^{\gamma,\delta_0}(s,Y_s^{t,y,Z}) + \mathcal{L}^{Z_s}\phi^{\gamma,\delta_0}(s,Y_s^{t,y,Z})\right) ds + M_\tau^{t,Z}, \end{aligned} \tag{23}$$

where

$$M_\tau^{t,Z} := B_\tau - B_t + \int_t^\tau \phi_y(s,Y_s^{t,y,Z}) Z_s dB_s,$$

and, for any $z \in \mathbb{R}$, the operator $\mathcal{L}^z$ is defined as

$$\mathcal{L}^z f(y) := z + \frac{1}{2}\left(f_{yy}(y) + f_y(y)\right) z^2 + k_a f_y(y) y,$$

for all $f \in C^2((0,\infty))$.

Note that $M_\cdot^{t,Z}$ is a local martingale. Next, we introduce a localizing sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\left\{s \ge t : \left|\phi_y\left(s, Y_s^{t,y,Z}\right) Z_s\right| \ge n\right\} \wedge T.$$

Fixing $\tau := \tau_n$, and taking expectations in (23), we obtain by Lemma 1:

$$x + \phi^{\gamma,\delta_0}(t,y) \ge \mathbb{E}\left[X_{\tau_n}^{t,x,Z} + \phi^{\gamma,\delta_0}(\tau_n, Y_{\tau_n}^{t,y,Z})\right].$$

Next, by the uniform boundedness of the set of admissible controls $\hat{\mathcal{U}}(t,x,y)$, and standard SDE estimates, we have

$$\mathbb{E}\left[\sup_{T_{N-1}\le s\le T} \left|X_s^{t,x,Z} + \phi^{\gamma,\delta_0}(s,Y_s^{t,y,Z})\right|^2\right] < \infty.$$

Thus, $\left(X_{\tau_n}^{t,x,Z} + \phi^{\gamma,\delta_0}(\tau_n, Y_{\tau_n}^{t,y,Z})\right)_{n\ge1}$ is uniformly integrable. Hence, using that $\lim_{n\to\infty} \tau_n = T, \mathbb{P}-a.s.$, and the continuity of ϕ^{γ,δ_0}:

$$x + \phi^{\gamma,\delta_0}(t,y) \ge \mathbb{E}\left[X_T^{t,x,Z} + \phi^{\gamma,\delta_0}(T, Y_T^{t,y,Z})\right]. \tag{24}$$

Taking the supremum over $Z \in \hat{\mathcal{U}}(t,x,y)$ in (24), and using that $\phi^{\gamma,\delta_0}(T,y) \ge -y^\gamma$ for all $y \ge 0$, we obtain

$$x + \phi^{\gamma,\delta_0}(t,y) \ge \hat{V}(t,x,y). \tag{25}$$

Moreover, using the control $\hat{Z} := 0$, we get the following inequalities

$$x - e^{\gamma k_a (T-t)} y^\gamma \le \hat{V}(t,x,y) \le x + \phi^{\gamma,\delta_0}(t,y).$$

The previous implies

$$x = \hat{V}_*(t,x,0) = \lim_{(t',x',y')\to(t,x,0),y'>0} \hat{V}_*(t,x,y). \tag{26}$$

Next, we apply ([4], Proposition 4.11). Firstly, we check that all the assumptions in Proposition 4.11 are satisfied. Indeed, equation (26) implies that $\hat{V}_*$ is continuous on $\{y = 0\}$. Due to the uniform boundedness of the controls, the drift of the state processes $(X^{t,x,Z}_\cdot, Y^{t,y,Z}_\cdot)$ grows linearly in $(x,y) \in \mathbb{R} \times (0,\infty)$. Additionally, for all $y > 0$, the control $\hat{Z} := 0$ satisfies $Y^{t,y,\hat{Z}}_s > 0, \mathbb{P} - a.s.$ Hence, $\hat{V}$ is the unique viscosity solution to the following state-constrained HJB equation

$$-\varphi_t - \sup_{|z|\le K} \left\{ \frac{1}{2}(\varphi_y + \varphi_{yy}) z^2 + \varphi_x z + \frac{1}{2}\varphi_{xx} z^2 + 2\varphi_{xy} z \right\} - k_a y \phi_y = 0, \tag{27}$$
$$\varphi(T,x,y) = x - y^\gamma,$$

in the class of functions with polynomial growth and lower semi-continuous envelope continuous on $[T_{N-1}, T] \times \{y = 0\}$. Moreover, using ([4], Corollary 4.13), we have that $V(t,x,y) = \hat{V}(t,x,y)$, for all $(t,x,y) \in [T_{N-1},T] \times \mathbb{R} \times (0,\infty)$. Using that V is continuous on $[T_{N-1},T] \times \mathbb{R} \times [0,\infty)$, and the fact that the value is separable in (x,y)[1], there exists a continuous function $v : [T_{N-1},T] \times [0,\infty) \mapsto \mathbb{R}$, such that

$$V(t,x,y) = x + v(t,y).$$

Plugging in the last expression into (27), we obtain that v is the unique viscosity solution to the following state-constrained HJB equation

$$-\varphi_t - \sup_{|z|\le K} \left\{ \frac{1}{2}(\varphi_y + \varphi_{yy}) z^2 + z \right\} - k_a y v_y = 0, \quad (t,y) \in [T_{N-1},T) \times (0,\infty),$$
$$\varphi(T,x,y) = -y^\gamma, \quad y > 0,$$

in the class of functions with polynomial growth and lower semi-continuous envelope continuous at $\{y = 0\}$. □

A.3.2. *Proof of Theorem 6.2 on $\mathcal{R}_N$.* From the proof of Theorem 6.1 on $\mathcal{R}_N$, we know that $V(T_{N-1},x,y) = x + v(T_{N-1},y)$ is continuous on $\mathbb{R} \times [0,\infty)$. We write

$$f_{N-1}(x,y) = \max_{\eta\in\beta(x,y)} h(x,y,\eta),$$

where $h(x,y,\eta) = V(T_{N-1}, x - \eta^\gamma, y - \eta)$, and $\beta(x,y) = \{\eta \in [0,\infty) : \eta \le y\}$. Clearly, $h : \mathbb{R} \times [0,\infty) \times [0,y] \to \mathbb{R}$ is a continuous function due to the continuity of $V(T_{N-1},\cdot,\cdot)$ shown in the previous section. Moreover, $\beta(x,y)$ is a continuous set-valued map with non-empty compact values by Theorems 17.20, and 17.21 in [1]. Using Berge's maximum theorem and ([1], Lemma 17.30), we obtain that f_{N-1} is continuous and the lowest maximizer η^*_{N-1} is lower semicontinuous. □

[1] We show this property trivially from the definition of the principal's value in (12).

A.3.3. *Proof of Theorem 6.3 on $\mathcal{R}_N$.* Let $t \in [T_{N-2}, T_{N-1})$. We will show that for all $\epsilon > 0$, and $(t,x,y) \in [T_{N-2}, T_{N-1}) \times \mathbb{R} \times [0,\infty)$, the following dynamic programming equation holds

$$V(t,x,y) = \sup_{(z,\xi_{N-1}) \in \mathcal{U}(t,x,y)} \mathbb{E}\left[V\left(T_{N-1}+\varepsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}\right)\right]. \tag{28}$$

Note that the right-hand side of (28) is well defined as we showed that V is continuous in $[T_{N-1}, T_N] \times \mathbb{R} \times [0,\infty)$, and therefore, measurable. Next, we consider the principal's objective:

$$J(t,x,y,Z,\xi_{N-1}) = \mathbb{E}\left[X^{t,x,Z,\xi_{N-1}}_T - \left(Y^{t,y,Z,\xi_{N-1}}_T\right)^\gamma\right].$$

Evidently, for all $(t,x,y) \in [T_{N-2}, T_{N-1}) \times \mathbb{R} \times [0,\infty)$, and $(Z, \bar{\xi}_{N-1}) \in \mathcal{U}(t,x,y)$, the following inequality is satisfied

$$J(t,x,y,Z,\xi_{N-1}) \leq \sup_{(Z,\bar{\xi}_{N-1}) \in \mathcal{U}(t,x,y)} \mathbb{E}\left[V\left(T_{N-1}+\varepsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}\right)\right]. \tag{29}$$

Hence,

$$V(t,x,y) \leq \sup_{(Z,\xi_{N-1}) \in \mathcal{U}(t,x,y)} \mathbb{E}\left[V\left(T_{N-1}+\varepsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}\right)\right].$$

Repeating the same argument using $\hat{\mathcal{U}}$ as the space of admissible controls, we obtain

$$\hat{V}(t,x,y) \leq \sup_{(Z,\xi_{N-1}) \in \hat{\mathcal{U}}(t,x,y)} \mathbb{E}\left[V\left(T_{N-1}+\varepsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\varepsilon}\right)\right].$$

Next, we show the reverse inequality in (28). Firstly, we observe that

$$V(t,x,y) \geq \hat{V}(t,x,y) \geq \hat{V}(T_{N-1}+\epsilon, x, e^{k_a \max\{T_{N-1}+\epsilon-t,0\}} y), \tag{30}$$

where the second inequality holds by considering the controls $\hat{Z}_s := Z_s \mathbb{1}_{s \geq T_{N-1}+\epsilon}$, where $Z \in \mathcal{U}(T_{N-1}+\epsilon, x, e^{k_a(T_{N-1}+\epsilon-t)} y)$.

Furthermore, we observe that the mapping $(t,x,y) \mapsto \hat{V}(T_{N-1}+\epsilon, x, e^{k_a \max\{T_{N-1}+\epsilon-t,0\}} y)$ is continuous on $[T_{N-2}, T_N] \times \mathbb{R} \times [0,\infty)$, as $\hat{V}$ is continuous on $[T_{N-1}, T_N] \times \mathbb{R} \times [0,\infty)$. Hence, for some $\delta > 0$ small enough, we consider the set $D := (t-\delta, T_{N-1}+\epsilon) \times \mathbb{R} \times (0,\infty)$, and realize that $T_{N-1}+\epsilon$ is the first exit time of $\left(s, X^{t,x,Z,\xi_{N-1}}_s, Y^{t,y,Z,\xi_{N-1}}_s\right)_{s \geq t}$ from D. Using ([4], Lemma 4.9 (ii)), we have

$$V(t,x,y) \geq \hat{V}(t,x,y) \geq \mathbb{E}\left[\hat{V}(T_{N-1}+\epsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\epsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\epsilon})\right], \tag{31}$$

for all $(Z,\xi_{N-1}) \in \mathcal{U}(t,x,y)$. Finally, noticing that $\hat{V} = V$ on $[T_{N-1}, T_N] \times \mathbb{R} \times [0,\infty)$, and taking the supremum in (31), we obtain

$$\hat{V}(t,x,y) \geq \sup_{(Z,\xi_{N-1}) \in \hat{\mathcal{U}}(t,x,y)} \mathbb{E}\left[V(T_{N-1}+\epsilon, X^{t,x,Z,\xi_{N-1}}_{T_{N-1}+\epsilon}, Y^{t,y,Z,\xi_{N-1}}_{T_{N-1}+\epsilon})\right]. \tag{32}$$

Due to the continuity and local boundedness of V on $[T_{N-1}, T_N] \times \mathbb{R} \times [0,\infty)$, taking $\epsilon \to 0$:

$$V(t,x,y) \leq \sup_{Z \in \mathcal{U}(t,x,y)} \mathbb{E}\left[X^{t,x,Z}_{T^-_{N-1}} + f_{N-1}\left(Y^{t,y,Z}_{T^-_{N-1}}\right)\right], \quad (t,x,y) \in [T_{N-2}, T_{N-1}) \times \mathbb{R} \times [0,\infty), \tag{33}$$

$$\hat{V}(t,x,y) = \sup_{Z\in\hat{\mathcal{U}}(t,x,y)} \mathbb{E}\left[X^{t,x,Z}_{T^-_{N-1}} + f_{N-1}\left(Y^{t,y,Z}_{T^-_{N-1}}\right)\right], \quad (t,x,y)\in[T_{N-2},T_{N-1})\times\mathbb{R}\times(0,\infty). \tag{34}$$

Let $(t,x,y)\in[T_{N-2},T_{N-1})\times\mathbb{R}\times[0,\infty)$. We consider the sequence $(t_n,x_n,y_n)\to(T_{N-1},x,y)$, $t_n<T_{N-1}$, $y_n>0$. Based on the claim (34), we get the following inequalities

$$\mathbb{E}\left[X^{t_n,x_n,Z^n}_{T^-_{N-1}} + f_{N-1}\left(Y^{t_n,y_n,Z^n}_{T^-_{N-1}}\right)\right] \le \hat{V}\left(t_n,x_n,y_n\right)$$
$$\le \mathbb{E}\left[X^{t_n,x_n,Z^n}_{T^-_{N-1}} + f_{N-1}\left(Y^{t_n,y_n,Z^n}_{T^-_{N-1}}\right)\right] + \frac{1}{n},$$

where Z^n is a $\frac{1}{n}$-optimal control. Moreover, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(X^{t_n,x_n,Z^n}_{T^-_{N-1}}, Y^{t_n,y_n,Z^n}_{T^-_{N-1}}\right)-(x,y)\right|^2\right]\\
\le&2\mathbb{E}\left[\left|\left(X^{t_n,x_n,Z^n}_{T^-_{N-1}}, Y^{t_n,y_n,Z^n}_{T^-_{N-1}}\right)-(x_n,y_n)\right|^2\right]+2\left(x_n-x\right)^2+2\left(y_n-y\right)^2\\
\le&2Ce^{C(T_{N-1}-t_n)}\left(|x_n|^2+|y_n|^2+C\right)\left(T_{N-1}-t_n\right)+2\left(x_n-x\right)^2+2\left(y_n-y\right)^2,
\end{aligned}$$

for some positive constant $C>0$. In the previous inequalities, we used standard SDE estimates (see, for example, ([24], Theorem 1.3.16)), and the uniform boundedness of Z. Combining the latter bound with the continuity of f_{N-1}, we have

$$\hat{V}(T^-_{N-1},x,y) = x + f_{N-1}(y). \tag{35}$$

Applying the same argument to (33), taking a suitable sequence, we obtain

$$V^*(T^-_{N-1},x,y)\le x+f_{N-1}(y),$$

where for all $(t,x,y)\in[0,T]\times\mathbb{R}\times[0,\infty)$, we define

$$\begin{aligned}
V^*(t^-,x,y) &= \limsup_{(t',x',y')\to(t,x,y),\ t'<t,\ y'>0} V(t',x',y'),\\
V_*(t^-,x,y) &= \liminf_{(t',x',y')\to(t,x,y),\ t'<t,\ y'>0} V(t',x',y').
\end{aligned}$$

Furthermore, we consider a sequence $(\tilde{t}_n,\tilde{x}_n,\tilde{y}_n)\mapsto(T_{N-1},x,y)$, such that $\lim_{n\to\infty}V(\tilde{t}_n,\tilde{x}_n,\tilde{y}_n)=V_*(T^-_{N-1},x,y)$. Then,

$$\hat{V}(T^-_{N-1},x,y)=\lim_{n\to\infty}\hat{V}(\tilde{t}_n,\tilde{x}_n,\tilde{y}_n)\le V_*(T^-_{N-1},x,y)\le V^*(T^-_{N-1},x,y)\le x+f_{N-1}(y).$$

Combining the latter inequality with (35), we obtain $V(T^-_{N-1},x,y)=x+f_{N-1}(y)$.□

Finally, we show the induction step. Assume that the results from Theorem 6 hold on $\mathcal{R}_{i+1}$, for some $i\in\{1,\dots,N-1\}$. We will show that it holds on $\mathcal{R}_i$. As the proofs for 6.2, and 6.3 are identical to the proof in the terminal region $\mathcal{R}_N$, we only write the induction step for Theorem 6.1.

A.3.4. *Proof of Theorem 6.1 on $\mathcal{R}_i$.* Let $(t,x,y) \in [T_{i-1}, T_i) \times \mathbb{R} \times (0,\infty)$. Our hypothesis of induction claims that Theorem 6 holds on $\mathcal{R}_{i+1}$. Hence, $V(T_i^-, x, y) = x + f_i(y) = x + \sup_\eta \{v(T_i, y-\eta) - \eta^\gamma : \eta \in [0,y]\}$, for all $(x,y) \in \mathbb{R} \times [0,\infty)$. By induction, we obtain that for $0 \le \eta \le y$:

$$v(T_i, y-\eta) + x + \eta^\gamma \le \phi^{\gamma,\delta_0}(T_i, y-\eta) + x + \eta^\gamma = \phi^{\gamma, \frac{1}{(N-i+1)^{\gamma-1}}, \delta_0}(T_i, y-\eta) + x + \eta^\gamma.$$

Hence, for all $(x,y) \in \mathbb{R} \times (0,\infty)$, using that $\phi^{\gamma,\delta_0}(T_i, y) \ge v(T_i, y)$ by induction, we obtain

$$\begin{aligned}
&V(T_i^-, x, y) \\
&= f_i(y) + x \\
&= \sup_\eta \Big\{ v(T_i, y-\eta) + x - \eta^\gamma : \eta \in [0,y] \Big\} \\
&\le \sup_\eta \Big\{ \phi^{\gamma,\delta_0}(T_i, y-\eta) - \eta^\gamma : \eta \in [0,y] \Big\} + x \\
&\le \sup_\eta \Big\{ -\frac{1}{(N-i)^{\gamma-1}} (y-\eta)^\gamma - \eta^\gamma : \eta \ge 0 \Big\} + b e^{\frac{T-t}{T}} y^{\frac{1}{M}} + e^{c(T-t)} \left(1 - e^{-y}\right) + x \\
&= -\frac{1}{(N-i+1)^{\gamma-1}} y^\gamma + b e^{\frac{T-t}{T}} y^{\frac{1}{M}} + e^{c(T-t)} \left(1 - e^{-y}\right) + x \\
&= \phi^{\gamma,\delta_0}(T_i^-, y) + x,
\end{aligned} \tag{36}$$

where $\delta_0 := (b,c,M) \in \mathbb{R}^{+,3}$ is the constant found in Lemma 1. Using Lemma 1, and (36), we follow the same probabilistic argument we did for $i = N$, obtaining

$$x - e^{\gamma k_a (T-t)} y^\gamma \le \hat{V}(t,x,y) \le V(t,x,y) \le x + \phi^{\gamma,\delta_0}(t,y),$$

for all $(t,x,y) \in [T_{i-1}, T_i) \times \mathbb{R} \times (0,\infty)$.

Hence, $\hat{V}_*$ is continuous on $[T_{i-1}, T_i) \times \{y = 0\}$. Invoking ([4], Proposition 4.11), and following the same arguments done in the proof of this property in the terminal region $\mathcal{R}_N$, we obtain that $V(t,x,y) = x + v(t,y)$, where v is a continuous function on $\mathcal{R}_i$. Moreover, v is the unique viscosity solution of the following state-constrained HJB equation

$$\begin{aligned}
-\varphi_t - \sup_{|z| \le K} \left\{ \frac{1}{2} \left(\varphi_y + \varphi_{yy}\right) z^2 + z \right\} - k_a y v_y = 0, \quad &(t,y) \in [T_{i-1}, T_i) \times (0,\infty), \\
\varphi(T_i, y) = f_i(y), \quad &y > 0,
\end{aligned}$$

in the class of functions with polynomial growth and lower semi-continuous envelope continuous at $[T_{i-1}, T_i] \times \{y = 0\}$. Hence, we show the induction step. □

The previous completes the proof of Theorem 6.

REFERENCES

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis, a Hitchhiker's Guide*, Springer, Berlin, 2006.

[2] G. Barles and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Analysis*, **4** (1991), 271-283.

[3] D. P. Berstekas and S. E. Shreeve, *Stochastic Control. A Discrete-Time Case,* Athena Scientific, 1996.

[4] B. Bouchard and M. Nutz, Weak dynamic programming for generalized state constraints, *SIAM Journal on Control and Optimization*, **50** (2012), 3344-3373.

[5] P. Briand and Y. Hu, Quadratic BSDEs with convex generators and unbounded terminal conditions, *Probability Theory and Related Fields*, **141** (2008), 543-567.

[6] A. Capponi and Y. Zhang, A continuous time framework for sequential goal-based wealth management, *Management Science*, **70** (2024), 7345-8215.

[7] M. G. Crandall, H. Ishii and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bulletin of the American mathematical society*, **27** (1992), 1-67.

[8] J. Cvitanić, D. Possamaï and N. Touzi, Dynamic programming approach to principal–agent problems, *Finance and Stochastics*, **22** (2018), 1-37.

[9] J. Cvitanić, X. Wan and J. Zhang, Optimal contracts in continuous-time models, *International Journal of Stochastic Analysis*, **2006** (2006), Art. ID 95203, 27 pp.

[10] J. Cvitanić, X. Wan and J. Zhang, Principal-agent problems with exit options, *The BE Journal of Theoretical Economics*, **8** (2008), Art. 23, 43 pp.

[11] J. Cvitanić, X. Wan and J. Zhang, Optimal compensation with hidden action and lump-sum payment in a continuous-time model, *Applied Mathematics and Optimization*, **59** (2009), 99-146.

[12] R. Élie, T. Mastrolia and D. Possamaï, A tale of a principal and many, many agents, *Mathematics of Operations Research*, **44** (2019), 440-467.

[13] R. Élie and D. Possamaï, Contracting theory with multiple competitive agents, *SIAM Journal of Control and Optimization*, **57** (2019), 1157–1188.

[14] N. El Karoui and X. Tan, Capacities, measurable selection and dynamic programming part II: Application in stochastic control problems, École Polytechnique and Université Paris-Dauphiné, 2013.

[15] D. Fudenberg, B. Holmstrom and P. Milgrom, Short-term contracts and long-term agency relationships, *Journal of Economic Theory*, **51** (1990), 1-31.

[16] D. Fudenberg and J. Tirole, Moral hazard and renegotiation in agency contracts, *Econometrica: Journal of the Econometric Society*, **58** (1990), 1279-1319.

[17] M. F. Hellwig, The role of boundary solutions in principal–agent problems of the Holmström–Milgrom type, *Journal of Economic Theory*, **136** (2007), 446-475.

[18] M. F. Hellwig and K. M. Schmidt, Discrete–time approximations of the Holmström–Milgrom brownian–motion model of intertemporal incentive provision, *Econometrica*, **70** (2002), 2225-2264.

[19] C. Hernández and D. Possamaï, Time-inconsistent contract theory, *Mathematical Finance*, **34** (2024), 1022-1085.

[20] N. Hernández-Santibanez and T. Mastrolia, Contract theory in a vuca world, *SIAM Journal on Control and Optimization*, **57** (2019), 3072-3100.

[21] B. Holmstrom and P. Milgrom, Aggregation and linearity in the provision of intertemporal incentives, *Econometrica: Journal of the Econometric Society*, **55** (1987), 303-328.

[22] H. M. Müller, The first-best sharing rule in the continuous-time principal-agent problem with exponential utility, *Journal of Economic Theory*, **79** (1998), 276-280.

[23] H. M. Müller, Asymptotic efficiency in dynamic principal-agent problems, *Journal of Economic Theory*, **91** (2000), 292-301.

[24] H. Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Stoch. Model. Appl. Probab., 61, Springer-Verlag, Berlin, 2009.

[25] D. Possamaï and N. Touzi, Is there a golden parachute in Sannikov's principal–agent problem?, *Mathematics of Operations Research*, 2024.

[26] W. P. Rogerson, Repeated moral hazard, *Econometrica: Journal of the Econometric Society*, **53** (1985), 69-76.

[27] A. Rubinstein and M. E. Yaari, Repeated insurance contracts and moral hazard, *Journal of Economic Theory*, **30** (1983), 74-97.

[28] Y. Sannikov, A continuous-time version of the principal-agent problem, *The Review of Economic Studies*, **75** (2008), 957-984.

[29] H. Schättler and J. Sung, The first-order approach to the continuous-time principal–agent problem with exponential utility, *Journal of Economic Theory*, **61** (1993), 331-371.

[30] J. Sung, Linearity with project selection and controllable diffusion rate in continuous-time principal-agent problems, *The RAND Journal of Economics*, (1995), 720-743.

[31] J. Sung, Corporate insurance and managerial incentives, *Journal of Economic Theory*, **74** (1997), 297-332.

[32] N. Williams, Persistent private information, *Econometrica*, **79** (2011), 1233-1275.

[33] N. Williams, A solvable continuous time dynamic principal–agent model, *Journal of Economic Theory*, **159** (2015), 989-1015.

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