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Imprimitivity theorems and self-similar actions on Fell bundles ☆

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ABSTRACT

We introduce the notion of self-similar actions of groupoids on other groupoids and Fell bundles. This leads to a new imprimitivity theorem arising from such dynamics, generalizing many earlier imprimitivity theorems involving group and groupoid actions.

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1. Introduction

The dynamics between groups and operator algebras encompass a vast literature in the study of operator algebras. They trace back to the pioneering work of Murray and von Neumann [26] where they encode group dynamics as operators on Hilbert spaces. In its simplest form, a C*-dynamical system arises from a group acting by $*$-automorphisms on a C*-algebra. This system is then encoded by the C*-crossed product, where both the group and the C*-algebra are represented as operators on a Hilbert space. One may refer to Williams's book [36] for a thorough discussion of the subject.

The C*-crossed product construction bears a strong resemblance to the semi-direct product of groups, in which one group H acts on another group G by automorphisms. Their semi-direct product $G \rtimes H$ is a group that encodes both groups and their interaction. But what happens if the group G also acts on H? This leads to a more general construction called the *Zappa–Szép product* of groups (also known as *bicrossed product* or *knit product*), which encodes a two-way action between two groups. Such a two-way action may arise when a group K contains two subgroups H, G such that every element $k \in K$ decomposes uniquely as a product $k = gh$ where $g \in G, h \in H$ (equivalently, $K = G \cdot H$ and $G \cap H = \{e\}$). In this case, for each $g \in G$ and $h \in H$, there exists unique $g' \in G$ and $h' \in H$ such that $hg = g'h'$. This leads to an H-action on G via $(h, g) \mapsto g'$ and a G-action on H via $(h, g) \mapsto h'$. These two actions need to satisfy certain compatibility conditions, and one may recover the enveloping group K as the Zappa–Szép product group $G \bowtie H$ from these compatible actions.

In the realm of operator algebras, the analogous study of Zappa–Szép products is scarce. Representations of Zappa–Szép products of matched pairs of groupoids were studied in [1]. The Zappa–Szép product of étale groupoids and their C*-algebras were first studied in [2]. Recently, we defined and studied an operator algebraic analogue of such products [6]. Just like the C*-crossed product $A \rtimes H$ is an operator algebraic analogue of the semi-direct product of two groups $G \rtimes H$, so is our construction an analogue of the Zappa–Szép product $\mathcal{G} \bowtie \mathcal{H}$ of two groupoids. To achieve this, the operator algebraic data has to 'act' on the groupoid $\mathcal{H}$; this is achieved by replacing the C*-algebra A by a Fell bundle $\mathscr{B} \to \mathcal{G}$ on which the groupoid $\mathcal{H}$ acts in an appropriate sense to form the Fell bundle $\mathscr{B} \bowtie \mathcal{H} \to \mathcal{G} \bowtie \mathcal{H}$. The resulting Fell bundle C*-algebra of these Zappa–Szép dynamics is a generalization of the classical C*-crossed product, and we proved that several properties of the C*-crossed product hold similarly in the Zappa–Szép construction.

Given the vast literature on C*-dynamical systems, our study unlocks a trove of intriguing questions on what properties of C*-crossed products can be generalized to the Zappa–Szép product context. In this paper, we prove a Zappa–Szép analogue of the imprimitivity theorems arising from groupoid actions. Imprimitivity theorems originated from Mackey's study on inducing representations of a locally compact group G from its closed subgroups and giving criteria to identify such representations, known as *Mackey's machine* [21]. Along with the rapid development of the C*-algebra theory, Mackey's imprimitivity theorems were soon recast in terms of C*-algebras in the early 1970s by Rieffel [30,31], where he introduced the notion of Morita equivalence for C*-algebras [32]. One may refer to Rosenberg's survey paper [33] on the rich history of this subject. Since then, the theory of imprimitivity theorems and Morita equivalence among C*-algebras has been further developed. For imprimitivity theorems arising from group dynamics, notable works include Green's [12] and Raeburn's [28] symmetric imprimitivity theorems. One may refer to [36, Chapter 4] for various versions and applications of these results. In [22], Muhly, Renault, and Williams introduced the notion of *equivalent groupoids* which implies the existence of a Morita equivalence between their C*-algebras. This was generalized to Fell bundles by Muhly and Williams in [23] (see also [39]). Applying the technique developed by Muhly and Williams, Kaliszewski et al. [17] recovered and extended *"all known imprimitivity theorems involving groups"* by using a semi-direct product construction of Fell bundles by locally compact groups.

The main theorem of this paper (Theorem 6.1) further generalizes the imprimitivity theorem of Kaliszewski et al. beyond the realm of semi-direct products and to the realm of Zappa–Szép products. This opens a new world of study on the Zappa–Szép-type two-way interactions between groupoids and Fell bundles.

We briefly outline the key ideas and constructions of this paper. We first introduce the notion of *self-similar actions* of a groupoid $\mathcal{H}$ on another groupoid $\mathcal{X}$ in Section 2 and construct their self-similar product groupoid $\mathcal{X} \bowtie \mathcal{H}$. We adopted this terminology in order to differentiate our new construction from earlier, more restrictive Zappa–Szép product constructions [1,2]: we no longer require the groupoids to have the same unit space. Rather, the groupoids are connected using a momentum map, similar to the idea of a semi-direct product of groupoids in [15]. This allows us to study many interesting examples such as group actions on groupoids. We also removed the requirement imposed in our earlier paper [6] that the groupoids be étale: unless stated otherwise, all groupoids are merely assumed to be *locally compact Hausdorff.* Consequently, our new construction is an honest generalization of that in [17], and our notion of a self-similar action is a generalization of self-similar group actions whose close relationship to Zappa–Szép products has already been studied [9,20,27]. At the end of Section 2, we induce Haar systems from $\mathcal{X}$ and $\mathcal{H}$ to a Haar system on $\mathcal{X} \bowtie \mathcal{H}$ under mild assumptions.

In Section 3, we start by studying the orbit space $\mathcal{H}\backslash\mathcal{X}$ of a self-similar left action of $\mathcal{H}$ on $\mathcal{X}$, which is also a groupoid as long as the action is free and proper. In the setup of most symmetric imprimitivity theorems, it is standard to assume that the left $\mathcal{H}$-action on $\mathcal{X}$ commutes with a right action of another groupoid, $\mathcal{G}$, yielding two groupoids of the form $(\mathcal{X}/\mathcal{G}) \rtimes \mathcal{H}$ and $\mathcal{G} \ltimes (\mathcal{H}\backslash\mathcal{X})$ that are equivalent. This assumption is not quite enough in the self-similar product setting. We therefore introduce the notion of *in tune* actions (Definition 3.5), and we call $\mathcal{X}$ a $(\mathcal{H}, \mathcal{G})$*-self-similar para-equivalence* if the $\mathcal{H}$- and $\mathcal{G}$-actions are free, proper, and in tune, and if $\mathcal{X}$ has open source map. Under such assumptions, the $\mathcal{H}$- and $\mathcal{G}$-actions on $\mathcal{X}$ factor through the respective opposite quotient: $\mathcal{H}$ naturally has a self-similar left action on $\mathcal{X}/\mathcal{G}$ and $\mathcal{G}$ a self-similar right action on $\mathcal{H}\backslash\mathcal{X}$, allowing us to build their self-similar product groupoids $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$.

We prove (Theorem 3.10) that these two groupoids are equivalent in the sense of [22]. Moreover, the existence of a Haar system on $\mathcal{X}$ that is equivariant in an appropriate sense allows us to build Haar systems for these equivalent groupoids, so that their groupoid C*-algebras are Morita equivalent.

In Sections 4 and 5, we bootstrap our construction to the more operator algebraic setting of *self-similar actions on Fell bundles* $\mathscr{B} \to \mathcal{X}$ for $\mathcal{X}$ a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence. We define the notions of self-similar left and right actions on $\mathscr{B}$ following similar ideas as in [6]. This allows two constructions: that of their self-similar products $\mathscr{B} \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie \mathscr{B}$, where the color of the symbol distinguishes between left- and right-actions, and that of the orbit spaces $\mathcal{H} \backslash \mathscr{B}$ and $\mathscr{B}/\mathcal{G}$. Assuming the actions are free, proper, and in tune, the orbit spaces become Fell bundles themselves. By iterating these constructions, we obtain two Fell bundles, $(\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H} \backslash \mathscr{B})$.

Our main theorem (Theorem 6.1) in Section 6 states that these two Fell bundles are equivalent in the sense of [23]. Again, under suitable additional assumptions regarding Haar systems, their Fell bundle C*-algebras are therefore Morita equivalent. We note that the imprimitivity theorem of Kaliszewski et al. can be recovered by requiring that half of our two two-way actions be trivial (namely, that $\mathcal{X}$ does not act on $\mathcal{H}$ or $\mathcal{G}$). There are other examples where the $\mathcal{X}$-actions on $\mathcal{G}$ and $\mathcal{H}$ are non-trivial, some of which are briefly discussed (Examples 2.8, 2.13, 3.17, and 3.19). Finally, we apply our result to a certain class of Deaconu–Renault groupoids generated by $*$-commuting endomorphisms in Section 7.

Due to the sheer number of actions involved, we try our best to assign each action a unique symbol to best avoid confusion. By convention, the arrow of each action symbol will point to the element of the space that is acted upon.

2. Self-similar actions

Self-similar groups originated from Grigorchuk's construction of finitely generated groups of intermediate growth [13,14]. Its application in operator algebra was first explored by Nekrashevych [27] where he studied a self-similar group acting on a set. The distinctive feature that set it apart from other group actions is that the set also acts back on the group; this action is often called the *restriction map*. Such a two-way interaction has since been generalized to various contexts; for example to self-similar actions on directed graphs [9], k-graphs [20], and semigroups [3,35]. In this section, we define self-similar groupoid actions on groupoids. Again, the key feature that sets our definition apart from classical groupoid actions is the two-way interactions recorded in these self-similar dynamics.

2.1. Self-similar left actions on groupoids

Notation 2.1. Given continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$ between topological spaces, we write

$$X\,{}_{f}{*}_{g}\,Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

and equip this space with the subspace topology.

Definition 2.2. Let $\mathcal{H}$ and $\mathcal{X}$ be two locally compact Hausdorff groupoids. We say $\mathcal{H}$ has a *self-similar left action* on $\mathcal{X}$ if there exist a continuous surjection $\rho_{\mathcal{X}}^{(0)} : \mathcal{X}^{(0)} \to \mathcal{H}^{(0)}$ and, using the momentum map $\rho_{\mathcal{X}} := \rho_{\mathcal{X}}^{(0)} \circ r_{\mathcal{X}}$, two continuous maps

$$\begin{aligned} \mathcal{H} \curvearrowright \mathcal{X}\colon \qquad & \mathcal{H}\,{}_{s_{\mathcal{H}}}{*}_{\rho_{\mathcal{X}}}\,\mathcal{X} \ni (h,x) \mapsto h \triangleright x \in \mathcal{X} \\ \mathcal{H} \circlearrowleft \mathcal{X}\colon \qquad & \mathcal{H}\,{}_{s_{\mathcal{H}}}{*}_{\rho_{\mathcal{X}}}\,\mathcal{X} \ni (h,x) \mapsto h \triangleleft x \in \mathcal{H} \end{aligned}$$

such that the following hold.

- For any $h \in \mathcal{H}$ and $x \in \mathcal{X}$ such that $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$, we have:

$$r_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(h \triangleright x) \qquad s_{\mathcal{H}}(h \triangleleft x) = \rho_{\mathcal{X}}(x^{-1}) \qquad r_{\mathcal{H}}(h \triangleleft x) = \rho_{\mathcal{X}}((h \triangleright x)^{-1}) \tag{L1}$$

- For all $h \in \mathcal{H}$ and $v \in \mathcal{X}^{(0)}$ such that $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(v)$, and for all $x \in \mathcal{X}$, we have:

$$h \triangleleft v = h \quad \text{and} \quad \rho_{\mathcal{X}}(x) \triangleright x = x \tag{L2}$$

- For all $h \in \mathcal{H}$ and $(x,y) \in \mathcal{X}^{(2)}$ such that $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$, we have $s_{\mathcal{X}}(h \triangleright x) = r_{\mathcal{X}}((h \triangleleft x) \triangleright y)$ and

$$h \triangleleft (xy) = (h \triangleleft x) \triangleleft y \tag{L3}$$

$$h \triangleright (xy) = (h \triangleright x)[(h \triangleleft x) \triangleright y] \tag{L4}$$

- For all $(h,k) \in \mathcal{H}^{(2)}$ and $x \in \mathcal{X}$ such that $s_{\mathcal{H}}(k) = \rho_{\mathcal{X}}(x)$, we have:

$$(hk) \triangleright x = h \triangleright (k \triangleright x) \tag{L5}$$

$$(hk) \triangleleft x = [h \triangleleft (k \triangleright x)](k \triangleleft x) \tag{L6}$$

We will often write $\mathcal{H}\,{}_{s}{*}_{\rho}\,\mathcal{X}$ instead of $\mathcal{H}\,{}_{s_{\mathcal{H}}}{*}_{\rho_{\mathcal{X}}}\,\mathcal{X}$ when the subscripts are clear from context.

Example 2.3. Suppose $\mathcal{X}$ and $\mathcal{H}$ are groupoids with $\mathcal{X}^{(0)} = \mathcal{H}^{(0)}$. Then $(\mathcal{X}, \mathcal{H})$ is a matched pair of groupoids in the sense of [1, Definition 1.1] if and only if $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$ with $\rho_{\mathcal{X}}^{(0)} = \mathrm{id}_{\mathcal{X}^{(0)}}$, meaning that $\rho_{\mathcal{X}} = r_{\mathcal{X}}$. We point out that this is the reason that inverse elements appear in Condition (L1): Here, the condition $s_{\mathcal{H}}(h \triangleleft x) = \rho_{\mathcal{X}}(x^{-1})$ becomes $s_{\mathcal{H}}(h \triangleleft x) = s_{\mathcal{X}}(x)$, which might feel a bit more natural.

Remark 2.4. If $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$, then $(h, x) \mapsto h \triangleright x$ is a left action of the groupoid $\mathcal{H}$ on the space $\mathcal{X}$ with momentum map $\rho_{\mathcal{X}}$ in the sense of [38, Def. 2.1]. Indeed, the algebraic properties needed for an action are

$$\rho_{\mathcal{X}}(h \triangleright x) = r_{\mathcal{H}}(h), \quad \rho_{\mathcal{X}}(x) \triangleright x = x, \quad \text{and} \quad (kh) \triangleright x = k \triangleright (h \triangleright x), \tag{2.1}$$

which are all assumed in (L1), (L2), and (L5), respectively.

Moreover, if $\mathcal{X}^{(0)} = \mathcal{H}^{(0)}$ and $\rho_{\mathcal{X}}^{(0)} = \mathrm{id}_{\mathcal{X}^{(0)}}$, then $(h, x) \mapsto h \triangleleft x$ is a right action of the groupoid $\mathcal{X}$ on the space $\mathcal{H}$ with momentum map $s_{\mathcal{H}}$.

Example 2.5. Suppose $\mathcal{H}$ acts on a groupoid $\mathcal{X}$ by automorphisms, meaning $\mathcal{X}$ has a continuous, surjective momentum map $\rho_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{H}^{(0)}$ and there is a continuous map $\mathcal{H}\,{}_{s}{*}_{\rho}\,\mathcal{X} \to \mathcal{X}$ satisfying not only the conditions in (2.1) but also $h \triangleright (xy) = (h \triangleright x)(h \triangleright y)$ where it makes sense. Then $\triangleright$ is a self-similar left action of $\mathcal{H}$ on $\mathcal{X}$ if and only if we let $\mathcal{X}$ act trivially on $\mathcal{H}$ (meaning $h \triangleleft x = h$). Note that there is no other choice for $\triangleleft$ because of Condition (L4) in combination with the assumption that $\triangleright$ is an action by homomorphisms.

Example 2.6 *(see [1, Example 1.6.]).* Suppose we are given a groupoid $\mathcal{X}$. If we let $\mathcal{H} = \mathcal{X}^{(0)}$ be the trivial groupoid and let $\rho_{\mathcal{X}}^{(0)} = \mathrm{id}_{\mathcal{X}^{(0)}}$, so that $\rho_{\mathcal{X}} = r_{\mathcal{X}}$, then we can define for a tuple $(u, x) = (r_{\mathcal{X}}(x), x) \in \mathcal{X}^{(0)}\,{}_{s}{*}_{r}\,\mathcal{X}$,

$$\begin{aligned} \mathcal{X}^{(0)} \curvearrowright \mathcal{X}\colon \qquad & r_{\mathcal{X}}(x) \triangleright x = x, \\ \mathcal{X}^{(0)} \curvearrowleft \mathcal{X}\colon \qquad & r_{\mathcal{X}}(x) \triangleleft x = s_{\mathcal{X}}(x). \end{aligned}$$

One swiftly verifies that these constitute a self-similar left action of $\mathcal{X}^{(0)}$ on $\mathcal{X}$. (In fact, these groupoids form a matched pair.)

We point out that, in order for the condition $s_{\mathcal{H}}(h \triangleleft x) = \rho_{\mathcal{X}}(x^{-1})$ in (L1) to be satisfied by the pair in Example 2.6, we must define $\mathcal{X}^{(0)} \curvearrowleft \mathcal{X}$ in the above way and cannot let $\mathcal{X}$ act trivially on $\mathcal{X}^{(0)}$. For Example 3.8 later, it will therefore be convenient to know that we can also replace the trivial groupoid $\mathcal{X}^{(0)}$ with the trivial group $\{e\}$ as follows. This also highlights the advantage of not having forced $\mathcal{X}$ and $\mathcal{H}$ to have the same unit space, as was the case in, for example, [1,2].

Example 2.7. Suppose we are given a groupoid $\mathcal{X}$. If we let $\mathcal{H} = \{e\}$ be the trivial group, so that $\rho_{\mathcal{X}}^{(0)} \colon \mathcal{X}^{(0)} \to \mathcal{H}^{(0)} = \{e\}$ is constant and so that $\triangleright$ and $\triangleleft$ must be defined to be trivial, then these constitute a self-similar left action of $\{e\}$ on $\mathcal{X}$.

Example 2.8. Suppose a locally compact Hausdorff group K acts on the left on a locally compact Hausdorff space X; denote the action by $*$. Suppose further that K can be written as an (internal) Zappa–Szép product of two (necessarily closed) subgroups, i.e.,

$K = G \bowtie H$ with the product topology. This means that, for any $h \in H$ and $t \in G$, there exist unique elements $h|_t \in H$ and $h \cdot t \in G$ such that $(e, h)(t, e) = (h \cdot t, h|_t)$, where the product on the left-hand side is the group multiplication of K and where e denotes the identity element of each group.

Consider the transformation groupoid $\mathcal{X} = G \ltimes X = \{(t, x) : x \in X, t \in G\}$; we choose the convention that its range and source maps are $r(t, x) = t * x$ and $s(t, x) = x$, respectively. Then

$$\begin{aligned} H \curvearrowright \mathcal{X}\colon &\quad h \triangleright (t, x) = (h \cdot t, h|_t * x) \\ H \circlearrowleft \mathcal{X}\colon &\quad h \triangleleft (t, x) = h|_t \end{aligned} \tag{2.2}$$

is a self-similar left action of H on $\mathcal{X}$.

Note that units are not necessarily fixed by self-similar actions. Instead, we have the following formulas:

Lemma 2.9. *For any $x \in \mathcal{X}$ and for any $(h, v) \in \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathcal{X}^{(0)}$, we have*

$$\rho_{\mathcal{X}}(x) \triangleleft x = \rho_{\mathcal{X}}(x^{-1}) \tag{L7}$$

$$h \triangleright v \in \mathcal{X}^{(0)} \tag{L8}$$

Moreover, if $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$, then

$$(h \triangleright x)^{-1} = (h \triangleleft x) \triangleright x^{-1} \quad \text{and} \quad (h \triangleleft x)^{-1} = h^{-1} \triangleleft (h \triangleright x) \tag{L9}$$

$$r_{\mathcal{X}}(h \triangleright x) = h \triangleright r_{\mathcal{X}}(x) \quad \text{and} \quad s_{\mathcal{X}}(h \triangleright x) = (h \triangleleft x) \triangleright s_{\mathcal{X}}(x) \tag{L10}$$

Proof. Let $e = \rho_{\mathcal{X}}(x) \in \mathcal{H}^{(0)}$. For (L7),

$$e \triangleleft x = (e^2) \triangleleft x \overset{\text{(L6)}}{=} (e \triangleleft (e \triangleright x))(e \triangleleft x) \overset{\text{(L2)}}{=} (e \triangleleft x)^2.$$

Hence, $e \triangleleft x \in \mathcal{H}^{(0)}$. Therefore,

$$e \triangleleft x = s_{\mathcal{H}}(e \triangleleft x) \overset{\text{(L1)}}{=} \rho_{\mathcal{X}}(x^{-1}).$$

Condition (L8) follows from

$$h \triangleright v = h \triangleright (v^2) \overset{\text{(L4)}}{=} (h \triangleright v)((h \triangleleft v) \triangleright v) \overset{\text{(L2)}}{=} (h \triangleright v)^2.$$

For (L9), note that we have just shown that $h \triangleright (xx^{-1}) \in \mathcal{X}^{(0)}$. By Condition (L4),

$$\mathcal{X}^{(0)} \ni h \triangleright (xx^{-1}) = (h \triangleright x)[(h \triangleleft x) \triangleright x^{-1}].$$

Therefore, $(h \triangleright x)^{-1} = (h \triangleleft x) \triangleright x^{-1}$. Similarly, by what we have proved above, $(h^{-1}h) \triangleleft x \in \mathcal{H}^{(0)}$. Therefore, by Condition (L6),

$$\mathcal{H}^{(0)} \ni (h^{-1}h) \triangleleft x = [h^{-1} \triangleleft (h \triangleright x)](h \triangleleft x).$$

This proves that $(h \triangleleft x)^{-1} = h^{-1} \triangleleft (h \triangleright x)$.

Lastly, for (L10), we compute

$$h \triangleright x = h \triangleright (r_{\mathcal{X}}(x)x) \overset{(L4)}{=} (h \triangleright r_{\mathcal{X}}(x))[(h \triangleleft r_{\mathcal{X}}(x)) \triangleright x] \overset{(L2)}{=} (h \triangleright r_{\mathcal{X}}(x))(h \triangleright x)$$

and

$$h \triangleright x = h \triangleright (x s_{\mathcal{X}}(x)) \overset{(L4)}{=} (h \triangleright x)[(h \triangleleft x) \triangleright s_{\mathcal{X}}(x)]. \quad \square$$

Corollary 2.10. *If $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$ and if $h \triangleright x$ is a unit in $\mathcal{X}$, then x is a unit.*

Proof. By Lemma 2.9, $\mathcal{H}$ maps units to units. In particular, $x \overset{(L5)}{=} h^{-1} \triangleright (h \triangleright x)$ is a unit. $\square$

Since $\triangleright$ is a left groupoid action of $\mathcal{H}$ on the space $\mathcal{X}$ (Remark 2.4), we make the following definitions, which are standard in the literature.

Definition 2.11. If $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$, we call it *free* if $\triangleright$ is free, meaning that the equality $h \triangleright x = x$ implies $h \in \mathcal{H}^{(0)}$. Likewise, we call it *proper* if $\triangleright$ is proper, meaning that the map $\mathcal{H}\,{}_s{*}_\rho \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ defined by $(h, x) \mapsto (h \triangleright x, x)$ is a proper map.

We note that these conditions on $\triangleright$ do not impose conditions on $\triangleleft$.

Example 2.12. Given a groupoid $\mathcal{X}$, the (trivial) self-similar left actions of the trivial groupoid $\mathcal{X}^{(0)}$ and of the trivial group $\{e\}$ on $\mathcal{X}$ (Examples 2.6 and 2.7) are both free and proper.

Example 2.13 *(continuation of Example 2.8).* Suppose again that a locally compact Hausdorff group $K = G \bowtie H$ acts on the left on a locally compact Hausdorff space X, denoted by $*$. We define the self-similar left action $\triangleright$ and $\triangleleft$ of H on the transformation groupoid $\mathcal{X} = G \ltimes X$ as in (2.2).

Note that, if $*$ is free, then so is $\triangleright$: suppose $h \triangleright (t, x) = (t, x)$, i.e., $h \cdot t = t$ and $h|_t * x = x$. By the freeness of the K-action on X, this forces $h|_t = e$. Recall that the Zappa–Szép-structure of K implies that $(e, h)(t, e) = (h \cdot t, h|_t)$. But the right-hand side equals (t, e), which forces $h = e$.

Likewise, if $*$ is proper, then so is $\triangleright$: suppose that we have convergent nets $(t_i, x_i) \to (t, x)$ and $h_i \triangleright (t_i, x_i) \to (s, y)$ in $\mathcal{X}$; we must check that h_i has a convergent subnet.

By definition of $\triangleright$, we know in particular that $h_i|_{t_i} * x_i \to y$ in X. As $x_i \to x$ and as $*$ is proper, it follows that $h_i|_{t_i}$ (has a subnet that) converges to, say, k in K. Since H is closed in K, k is an element of H, and so by continuity of the restriction and inversion map, we conclude that $h_i = (h_i|_{t_i})|_{t_i^{-1}} \to k|_{t^{-1}}$.

Lemma 2.14. *If $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$, then $\triangleright$ restricts to a continuous left action of $\mathcal{H}$ on the unit space, $\mathcal{X}^{(0)}$. The action on $\mathcal{X}$ is free (respectively proper) if and only if the action on $\mathcal{X}^{(0)}$ is free (respectively proper).*

Proof. Notice first that, if $v \in \mathcal{X}^{(0)}$ and $h \in \mathcal{H}$ are such that $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(v)$, then $h \triangleright v \in \mathcal{X}^{(0)}$ by Lemma 2.9 (L8), so the map restricts to a continuous action $\mathcal{H} {}_s*_\rho \mathcal{X}^{(0)} \to \mathcal{X}^{(0)}$ with momentum map $\rho_{\mathcal{X}}^{(0)} : \mathcal{X}^{(0)} \to \mathcal{H}^{(0)}$.

Now suppose the action on $\mathcal{X}^{(0)}$ is free, and assume that $h \triangleright x = x$ for some $x \in \mathcal{X}$. Then

$$x^{-1} = (h \triangleright x)^{-1} \overset{\text{(L9)}}{=} (h \triangleleft x) \triangleright x^{-1},$$

so that

$$h \triangleright (xx^{-1}) \overset{\text{(L4)}}{=} (h \triangleright x)\big[(h \triangleleft x) \triangleright x^{-1}\big] = xx^{-1}.$$

As $xx^{-1} \in \mathcal{X}^{(0)}$, our assumption now implies that h is a unit, proving that $\triangleright$ is free. The other direction of the equivalence is trivial.

Lastly suppose that the action on $\mathcal{X}^{(0)}$ is proper, and assume that the net $\{(h_\lambda \triangleright x_\lambda, x_\lambda)\}_\Lambda$ converges to (y, x) in $\mathcal{X} \times \mathcal{X}$. By (L10) and continuity of $r_{\mathcal{X}}$, this implies that $(h_\lambda \triangleright r_{\mathcal{X}}(x_\lambda), r_{\mathcal{X}}(x_\lambda)) \to (r_{\mathcal{X}}(y), r_{\mathcal{X}}(x))$. By properness on $\mathcal{X}^{(0)}$, it follows from [38, Proposition 2.17] that $\{h_\lambda\}_\Lambda$ has a convergent subnet. By the same proposition, this implies that $\mathcal{H}$ acts properly on $\mathcal{X}$. □

The above implies that a non-trivial groupoid $\mathcal{H}$ cannot admit a free self-similar left action on a group $\mathcal{X}$, because its action on the unit space $\{e\}$ of $\mathcal{X}$ is never free.

Lemma 2.15. *Let $\mathcal{H}$ act on $\mathcal{X}$ by a* free *self-similar left action. If $x, x' \in \mathcal{X}$ satisfy $\mathcal{H} \triangleright x = \mathcal{H} \triangleright x'$ and if $r_{\mathcal{X}}(x) = r_{\mathcal{X}}(x')$, then $x = x'$.*

Proof. Since $\mathcal{H} \triangleright x = \mathcal{H} \triangleright x'$, there exists $h \in \mathcal{H}$ such that $x' = h \triangleright x$. By (L10) (Lemma 2.9), $h \triangleright r_{\mathcal{X}}(x) = r_{\mathcal{X}}(h \triangleright x) = r_{\mathcal{X}}(x')$, which coincides with $r_{\mathcal{X}}(x)$ by assumption. Since the self-similar $\mathcal{H}$-action is free, h must be in $\mathcal{H}^{(0)}$ and thus $x' = x$. □

2.2. The self-similar product groupoid: a generalized Zappa–Szép product

Following [2] and [6, Example 2.4], we can define a Zappa–Szép-type product of $\mathcal{H}$ with $\mathcal{X}$; the main difference is that we do not require the unit spaces of the two groupoids to coincide.

Definition 2.16. Let $\mathcal{H}$ be a groupoid that has a (not necessarily free or proper) self-similar left action on $\mathcal{X}$ (Definition 2.2). The *self-similar product* of $\mathcal{X}$ and $\mathcal{H}$ is the set

$$\mathcal{X} \bowtie \mathcal{H} = \{(x,h) \in \mathcal{X} \times \mathcal{H} : \rho_{\mathcal{X}}(x^{-1}) = r_{\mathcal{H}}(h)\}$$

with the following structure of a groupoid: the unit space is

$$(\mathcal{X} \bowtie \mathcal{H})^{(0)} = (\mathcal{X}^{(0)} \times \mathcal{H}^{(0)}) \cap (\mathcal{X} \bowtie \mathcal{H})$$

and its range and source maps are given by

$$r_{\mathcal{X}\bowtie\mathcal{H}}(x,h) = \left(r_{\mathcal{X}}(x), r_{\mathcal{H}}(h) \triangleleft x^{-1}\right) \quad \text{and respectively,}$$
$$s_{\mathcal{X}\bowtie\mathcal{H}}(x,h) = \left(h^{-1} \triangleright s_{\mathcal{X}}(x), s_{\mathcal{H}}(h)\right).$$

Two elements (x,h) and (y,k) are composable if and only if $s_{\mathcal{X}}(x) = h \triangleright r_{\mathcal{X}}(y)$, in which case their composition is defined by

$$(x,h)(y,k) := (x(h \triangleright y), (h \triangleleft y)k).$$

Lastly, the inverse is

$$(x,h)^{-1} := (h^{-1} \triangleright x^{-1}, h^{-1} \triangleleft x^{-1}).$$

Remark 2.17. Let us do some sanity checks.

The range map lands in the alleged unit space. We trivially have that $v := r_{\mathcal{X}}(x)$ is in $\mathcal{X}^{(0)}$. Since $r_{\mathcal{H}}(h) \triangleleft x^{-1} = \rho_{\mathcal{X}}(v)$ by Lemma 2.9, it is an element of $\mathcal{H}^{(0)}$, and

$$\rho_{\mathcal{X}}(v^{-1}) = \rho_{\mathcal{X}}(v) = r_{\mathcal{H}}(h) \triangleleft x^{-1} = r_{\mathcal{H}}\left(r_{\mathcal{H}}(h) \triangleleft x^{-1}\right),$$

which shows that $r_{\mathcal{X}\bowtie\mathcal{H}}(x,h)$ is in $(\mathcal{X} \bowtie \mathcal{H})^{(0)}$.

Composability condition. The elements (x,h) and (y,k) are composable in $\mathcal{X} \bowtie \mathcal{H}$ if and only if $s_{\mathcal{X}\bowtie\mathcal{H}}(x,h) = r_{\mathcal{X}\bowtie\mathcal{H}}(y,k)$; by our definition of the source and range map, that means

$$h^{-1} \triangleright s_{\mathcal{X}}(x) = r_{\mathcal{X}}(y) \quad \text{and} \quad s_{\mathcal{H}}(h) = r_{\mathcal{H}}(k) \triangleleft y^{-1}.$$

But now notice that the first condition implies the second:

$$\begin{aligned} r_{\mathcal{H}}(k) \triangleleft y^{-1} &= \rho_{\mathcal{X}}(y) && \text{(by (L7) in Lemma 2.9)} \\ &= \rho_{\mathcal{X}}^{(0)}(h^{-1} \triangleright s_{\mathcal{X}}(x)) && \text{(by the first condition)} \\ &= r_{\mathcal{H}}(h^{-1}) = s_{\mathcal{H}}(h) && \text{(by (L1)),} \end{aligned}$$

so (x,h) and (y,k) are composable if and only if $h^{-1} \triangleright s_{\mathcal{X}}(x) = r_{\mathcal{X}}(y)$, as claimed.

The composition makes sense. By assumption, we have $s_{\mathcal{H}}(h) = r_{\mathcal{H}}(k) \triangleleft y^{-1}$. By Lemma 2.9 (L7), the right-hand side is exactly $\rho_{\mathcal{X}}(y)$, so that $h \triangleright y$ and $h \triangleleft y$ are indeed defined. We have $r_{\mathcal{X}}(h \triangleright y) = h \triangleright r_{\mathcal{X}}(y)$ by (L10) (Lemma 2.9); the right-hand side is, by assumption, equal to $h \triangleright [h^{-1} \triangleright s_{\mathcal{X}}(x)]$. By (L5), that is exactly $s_{\mathcal{X}}(x)$, so that $x(h \triangleright y)$ is defined. We have $s_{\mathcal{H}}(h \triangleleft y) = \rho_{\mathcal{X}}(y^{-1})$ by (L1). Since $(y,k) \in \mathcal{X} \bowtie \mathcal{H}$, the right-hand side equals $r_{\mathcal{H}}(k)$, so that $(h \triangleleft y)k$ makes sense. We have $\rho_{\mathcal{X}}((x[h \triangleright y])^{-1}) = \rho_{\mathcal{X}}((h \triangleright y)^{-1})$ which equals $r_{\mathcal{H}}(h \triangleleft y) = r_{\mathcal{H}}([h \triangleleft y]k)$ by (L1), so the product is an element of $\mathcal{X} \bowtie \mathcal{H}$.

Remark 2.18. With the algebraic structure from Definition 2.16 and the subspace topology, $\mathcal{X} \bowtie \mathcal{H}$ is a locally compact Hausdorff groupoid. Indeed, since $\mathcal{X}$ and $\mathcal{H}$ are both locally compact Hausdorff, and since $\mathcal{X} \bowtie \mathcal{H}$ is a closed subspace of $\mathcal{X} \times \mathcal{H}$, it is clear that $\mathcal{X} \bowtie \mathcal{H}$ is itself locally compact Hausdorff. Continuity of multiplication and inversion follow immediately from continuity of $\triangleright$, $\triangleleft$, and of multiplication and inversion in $\mathcal{X}$ and $\mathcal{H}$.

Remark 2.19. Notice that the unit space of the self-similar product,

$$(\mathcal{X} \bowtie \mathcal{H})^{(0)} = \{(u,v) : u \in \mathcal{X}^{(0)}, v \in \mathcal{H}^{(0)}, \rho_{\mathcal{X}}(u) = v\},$$

is homeomorphic to $\mathcal{X}^{(0)}$, since the map $(u,v) \mapsto u$ and its inverse $u \mapsto (u, \rho_{\mathcal{X}}(u))$ are continuous. Under this identification, we can simply write $r_{\mathcal{X}\bowtie\mathcal{H}}(x,h) = r_{\mathcal{X}}(x)$ and $s_{\mathcal{X}\bowtie\mathcal{H}}(x,h) = h^{-1} \triangleright s_{\mathcal{X}}(x)$.

Example 2.20 *(continuation of Example 2.5).* Suppose $\mathcal{H}$ acts on a groupoid $\mathcal{X}$ by automorphisms. Then the self-similar product $\mathcal{X} \bowtie \mathcal{H}$ (where $\mathcal{X}$ acts trivially on $\mathcal{H}$) is identical to the transformation groupoid $\mathcal{X} \rtimes \mathcal{H}$, if we use the convention that $r_{\mathcal{X}\rtimes\mathcal{H}}(x,h) = x$ and $s_{\mathcal{X}\rtimes\mathcal{H}}(x,h) = h^{-1} \triangleright x$.

Example 2.21. Given a groupoid $\mathcal{X}$, it is easy to check that the self-similar product $\mathcal{X} \bowtie \mathcal{X}^{(0)}$ of $\mathcal{X}$ with the trivial groupoid $\mathcal{X}^{(0)}$ (as in Example 2.6) is isomorphic to $\mathcal{X}$ via $(x, s_{\mathcal{X}}(x)) \mapsto x$. Likewise, the self-similar product $\mathcal{X} \bowtie \{e\}$ of $\mathcal{X}$ with the trivial group (as in Example 2.7) is isomorphic to the groupoid $\mathcal{X}$ via $(x,e) \mapsto x$.

In [2, Section 3], the Zappa–Szép product was defined for groupoids that are *matched*: In addition to the left and right actions, groupoids in a matched pair are assumed to have the same unit space, $\mathcal{X}^{(0)} = \mathcal{H}^{(0)}$, and that $\rho_{\mathcal{X}}^{(0)} = \mathrm{id}_{\mathcal{X}^{(0)}}$. Our above definition of the self-similar product $\mathcal{X} \bowtie \mathcal{H}$ does not require $\mathcal{X}$ and $\mathcal{H}$ to be matched; they may have different unit spaces. However, as pointed out in [6, Example 2.4], we can construct a new transformation groupoid $\widetilde{\mathcal{H}}$ such that $\widetilde{\mathcal{H}}$ and $\mathcal{X}$ are matched, and such that their Zappa–Szép product $\mathcal{X} \bowtie \widetilde{\mathcal{H}}$ is isomorphic to the self-similar product $\mathcal{X} \bowtie \mathcal{H}$. We will now make this more precise.

Lemma 2.22. *Suppose a groupoid $\mathcal{H}$ has a self-similar left action on a groupoid $\mathcal{X}$, denoted $\triangleright$ and $\triangleleft$. By Lemma 2.14, we get a left action of $\mathcal{H}$ on $\mathcal{X}^{(0)}$ which gives rise to a transformation groupoid $\widetilde{\mathcal{H}} = \mathcal{H} \ltimes \mathcal{X}^{(0)}$ with unit space $\mathcal{X}^{(0)}$. If we define for $((h,u),x) \in \widetilde{\mathcal{H}}\,{}_s{*}_r\,\mathcal{X}$,*

$$\begin{aligned} \widetilde{\mathcal{H}} \curvearrowright \mathcal{X}: &\qquad (h,u)\cdot x := h \triangleright x, \\ \widetilde{\mathcal{H}} \circlearrowleft \mathcal{X}: &\qquad (h,u)|_x := (h \triangleleft x, s_{\mathcal{X}}(x)), \end{aligned}$$

then $(\mathcal{X}, \widetilde{\mathcal{H}})$ is a matched pair.

Note that the momentum map of $\mathcal{X}$ for these newly defined actions is not $\rho_{\mathcal{X}}$ but $r_{\mathcal{X}}$, as necessary for a matched pair.

Proof. Recall that $\widetilde{\mathcal{H}}$ is the set $\mathcal{H}\,{}_s{*}_\rho\,\mathcal{X}^{(0)}$ with multiplication and inversion defined by

$$(k, h \triangleright u)(h,u) = (hk, u) \qquad \text{and respectively,} \qquad (h,u)^{-1} = (h^{-1}, h \triangleright u).$$

Its unit space is further identified with $\mathcal{X}^{(0)}$; to be precise, the source of (h,u) is $(h^{-1}h, u) = (u,u)$, or simply u.

Let us check that the new actions are well-defined. The actions are only defined for $((h,u),x)$ for which $s_{\widetilde{\mathcal{H}}}(h,u) = u$ equals $r_{\mathcal{X}}(x)$. Since $(h,u) \in \widetilde{\mathcal{H}}$, we have $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}^{(0)}(u)$, and so $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$. This means that $h \triangleright x$ and $h \triangleleft x$ are both defined. Lastly, notice that $s_{\mathcal{H}}(h \triangleleft x) = \rho_{\mathcal{X}}(x^{-1}) = \rho_{\mathcal{X}}(s_{\mathcal{X}}(x))$ by (L1), so that $(h,u)|_x$ is indeed another element of $\widetilde{\mathcal{H}}$.

The ambitious reader can now verify easily that $(\mathcal{X}, \widetilde{\mathcal{H}})$ is a matched pair. □

Proposition 2.23. *With the assumptions and definitions in Lemma 2.22, the Zappa–Szép product $\mathcal{X} \bowtie \widetilde{\mathcal{H}}$ of the matched pair is isomorphic to the self-similar product $\mathcal{X} \bowtie \mathcal{H}$ in the sense of Definition 2.16.*

Proof. By definition of $\mathcal{X} \bowtie \widetilde{\mathcal{H}}$, any of its elements $(x,(h,u))$ satisfies $s_{\mathcal{X}}(x) = r_{\widetilde{\mathcal{H}}}(h,u)$, which is exactly $h \triangleright u$ by definition of the range map of $\widetilde{\mathcal{H}}$. Thus, $u = h^{-1} \triangleright s_{\mathcal{X}}(x)$. Moreover, $\rho_{\mathcal{X}}(x^{-1}) = \rho_{\mathcal{X}}(h \triangleright u) = r_{\mathcal{H}}(h)$ by (L1), which shows that (x,h) is an element of $\mathcal{X} \bowtie \mathcal{H}$. All in all, the maps

$$\varphi\colon \mathcal{X} \bowtie \widetilde{\mathcal{H}} \to \mathcal{X} \bowtie \mathcal{H}, \quad (x,(h,u)) \mapsto (x,h),$$

and

$$\mathcal{X} \bowtie \mathcal{H} \to \mathcal{X} \bowtie \widetilde{\mathcal{H}}, \quad (x,h) \mapsto \big(x, (h, h^{-1} \triangleright s_{\mathcal{X}}(x))\big),$$

are well-defined and mutually inverse. Since they are constructed out of continuous maps, they are themselves continuous. Lastly, notice that φ is a groupoid homomorphism:

$$\begin{aligned}
\varphi\big((x,(h,u))\,(y,(k,v))\big) &= \varphi\big(x[(h,u)\cdot y],(h,u)|_y(k,v)\big) && (\text{def'n of } \mathcal{X}\bowtie\widetilde{\mathcal{H}})\\
&= \varphi\big(x[h\triangleright y],(h\triangleleft y,s_{\mathcal{X}}(y))(k,v)\big) && (\text{def'n of } \cdot \text{ and } |)\\
&= \varphi\big(x[h\triangleright y],([h\triangleleft y]k,v)\big) && (\text{def'n of } \widetilde{\mathcal{H}})\\
&= \big(x[h\triangleright y],[h\triangleleft y]k\big) && (\text{def'n of } \varphi)\\
&= (x,h)\,(y,k) && (\text{def'n of } \mathcal{X}\bowtie\mathcal{H})\\
&= \varphi(x,(h,u))\,\varphi(y,(k,v)).
\end{aligned}$$

This proves that $\mathcal{X}\bowtie\widetilde{\mathcal{H}}$ is isomorphic to $\mathcal{X}\bowtie\mathcal{H}$. □

Example 2.24 *(cf. [2, Section 5.3], [5, Definition 3.6]).* Suppose $\mathcal{G}$ is a locally compact Hausdorff groupoid and H is a group (neither are assumed to be étale), and $\mathbf{c}\colon\mathcal{G}\to H$ is a continuous homomorphism. The *skew-product groupoid* $\mathcal{G}(\mathbf{c})$ is the set $\mathcal{G}\times H$ with the operations given for $(g,g')\in\mathcal{G}^{(2)}$ and $h\in H$ by

$$(g,h)(g',h\mathbf{c}(g))=(gg',h)\quad\text{and}\quad(g,h)^{-1}=(g^{-1},h\mathbf{c}(g)).$$

Note that $\mathcal{G}(\mathbf{c})^{(0)}=\mathcal{G}^{(0)}\times H$. The formula $\varphi_h(g,h'):=(g,h'h^{-1})$ defines a continuous, free action of H on $\mathcal{G}(\mathbf{c})$ by automorphisms. See [18, Section 4] for more details, but note that their convention for $\mathcal{G}(\mathbf{c})$ is slightly different from ours.

In the case where $\mathcal{G}$ and H are étale, [2, Proposition 22] states that the above action induces a left H-action on $\mathcal{G}^{(0)}\times H$ and that the corresponding transformation groupoid

$$\widetilde{H}:=H{}_{\varphi}\!\ltimes\mathcal{G}(\mathbf{c})^{(0)}$$

allows a Zappa–Szép product with $\mathcal{G}(\mathbf{c})$. It was pointed out further that this product $\mathcal{G}(\mathbf{c})\bowtie\widetilde{H}$ *"should be considered as the Zappa–Szép product of the groupoid* $\mathcal{G}(\mathbf{c})$ *with the group* H*"*, since the space $\mathcal{G}(\mathbf{c})\times H$ is homeomorphic to $\mathcal{G}(\mathbf{c})\bowtie\widetilde{H}$ via $\big((g,h),h'\big)\mapsto\big((g,h),(h',s(g),h\mathbf{c}(g)h')\big)$.

Using our machinery above, this comment can be made concrete without the need to go via the transformation groupoid $\widetilde{H}$ (and without assuming étale): Since $H^{(0)}=\{e\}$, the balanced fiber product ${}_{s}*_{\rho}$ just becomes the Cartesian product, and we can define

$$\begin{aligned}
H\curvearrowright\mathcal{G}(\mathbf{c})\colon &\qquad h\triangleright(g,h'):=(g,h'h^{-1})\\
H\curvearrowleft\mathcal{G}(\mathbf{c})\colon &\qquad h\triangleleft(g,h'):=\mathbf{c}(g)^{-1}h\mathbf{c}(g)
\end{aligned}$$

One verifies that these give a self-similar left action of H on $\mathcal{G}(\mathbf{c})$, and so we may construct the self-similar product $\mathcal{G}(\mathbf{c})\bowtie H$ as in Definition 2.16. By Proposition 2.23, $\mathcal{G}(\mathbf{c})\bowtie H$ is isomorphic to the Zappa–Szép product groupoid $\mathcal{G}(\mathbf{c})\bowtie\widetilde{H}$ from [2, Proposition 22].

Remark 2.25. As the last example highlights, the main distinction between the (old) Zappa–Szép product and our (new) self-similar product is that the latter does not require the groupoids with two-way actions to have matching unit spaces. For Zappa–Szép products, there is no inherent distinction between the roles of the two groupoids $\mathcal{H}$ and $\mathcal{X}$ (everything is entirely symmetric), while the self-similar-variant makes a clear distinction between them: Besides its range and source maps, the groupoid $\mathcal{X}$ must also carry a separate momentum map $\rho_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{H}^{(0)}$ with respect to which the $\mathcal{H}$-action is defined. After Proposition 4.3, it is natural to ask whether this added layer of difficulty in Definition 2.2 is worth the effort. But while the self-similar product $\mathcal{X} \bowtie \mathcal{H}$ and the Zappa–Szép product $\mathcal{X} \bowtie \widetilde{\mathcal{H}}$ are isomorphic, there are fundamental differences between the pair $(\mathcal{X}, \mathcal{H})$ and the pair $(\mathcal{X}, \widetilde{\mathcal{H}})$, as we will see in Example 3.8 and its subsequent remark.

Example 2.26 *(reconciliation).* Suppose $\mathcal{H} = \{e\}$ has the trivial self-similar left action on a groupoid $\mathcal{X}$ (Example 2.7). The induced action $\cdot$ of the transformation groupoid $\widetilde{\{e\}} = \{e\} \ltimes \mathcal{X}^{(0)}$ on $\mathcal{X}$ as defined in Lemma 2.22 is then likewise trivial, and the induced action $|$ of $\mathcal{X}$ on $\widetilde{\{e\}}$ is given for $x \in \mathcal{X}$ and $(e, u) \in \widetilde{\{e\}}$ by

$$(e, u)|_x := (e, s_{\mathcal{X}}(x)) \quad \text{where} \quad u = s_{\widetilde{\mathcal{H}}}(e, u) = r_{\mathcal{X}}(x).$$

In other words: If we identify an element (e, u) of $\widetilde{\{e\}}$ with u in $\mathcal{X}^{(0)}$, then the self-similar left action of $\widetilde{\{e\}}$ on $\mathcal{X}$ that we described in Lemma 2.22 is identical to the one of $\mathcal{X}^{(0)}$ on $\mathcal{X}$ that we described in Example 2.6. Under this identification, the concatenation of the isomorphisms $\mathcal{X} \bowtie \{e\} \cong \mathcal{X}$ and $\mathcal{X} \cong \mathcal{X} \bowtie \mathcal{X}^{(0)}$ in Example 2.21 yields exactly the isomorphism $\mathcal{X} \bowtie \{e\} \cong \mathcal{X} \bowtie \widetilde{\{e\}}$ in Proposition 2.23.

One can define an analogous notion of a self-similar action on the right. For the convenience of the reader and to establish notation, we will repeat the main properties in Subsection 2.4.

2.3. Haar systems for self-similar left actions

Definition 2.27. Suppose $\mathcal{H}$ and $\mathcal{X}$ are groupoids and that $\triangleright$ is a left $\mathcal{H}$-action on $\mathcal{X}$ with momentum map $\rho_{\mathcal{X}} = \rho_{\mathcal{X}}^{(0)} \circ r_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{H}^{(0)}$. We say that a left Haar system $\{\lambda^u\}_{u \in \mathcal{X}^{(0)}}$ on $\mathcal{X}$ is $\triangleright$-*invariant* if for all $h \in \mathcal{H}$ and all $u \in \mathcal{X}^{(0)}$ with $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(u)$, we have

$$h \triangleright \lambda^u = \lambda^{h \triangleright u},$$

where $(h \triangleright \lambda^u)(E) = \lambda^u(h^{-1} \triangleright E)$. Equivalently, for all $f \in C_c(\mathcal{X})$,

$$\int f(h \triangleright x)\, \mathrm{d}\lambda^u(x) = \int f(y)\, \mathrm{d}\lambda^{h \triangleright u}(y). \tag{2.3}$$

Proposition 2.28 *(cf. [16, Proposition 6.4]). Suppose $\mathcal{H}$ and $\mathcal{X}$ are locally compact Hausdorff groupoids, that $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$, and that $\mathcal{X}$ has a $\triangleright$-invariant left Haar system λ. If ε is any left Haar system for $\mathcal{H}$, then we get a left Haar system $\lambda \bowtie \varepsilon$ for $\mathcal{X} \bowtie \mathcal{H}$ defined for $u \in \mathcal{X}^{(0)}$ by*

$$\mathrm{d}(\lambda \bowtie \varepsilon)^u(y,k) = \mathrm{d}\varepsilon^{\rho(y^{-1})}(k)\,\mathrm{d}\lambda^u(y).$$

Equivalently, for any $f \in C_c(\mathcal{X} \bowtie \mathcal{H})$,

$$\int f(y,k)\,\mathrm{d}(\lambda \bowtie \varepsilon)^u(y,k) = \int_{\mathcal{X}}\int_{\mathcal{H}} f(y,k)\,\mathrm{d}\varepsilon^{\rho(y^{-1})}(k)\,\mathrm{d}\lambda^u(y).$$

In the above, we have used the fact that $(\mathcal{X} \bowtie \mathcal{H})^{(0)} \approx \mathcal{X}^{(0)}$ by Remark 2.19. To prove the above proposition, we need the following:

Lemma 2.29. *Suppose $u, v \in \mathcal{X}^{(0)}$ and $(x,h) \in \mathcal{X}^u_v \times \mathcal{H}^{\rho(v)} \subseteq \mathcal{X} \bowtie \mathcal{H}$ are fixed. If we let $h_2 = (h^{-1} \triangleleft x^{-1})^{-1}$ and $x_2 = h_2^{-1} \triangleright x$, then $h \triangleright s_{\mathcal{X}}(x_2) = v$, and for all $y \in \mathcal{X}^{h^{-1}\triangleright v}$, we have $x(h \triangleright y) = h_2 \triangleright (x_2 y)$.*

Proof. We compute

$$x_2 = (h^{-1} \triangleleft x^{-1}) \triangleright x \overset{\text{(L9)}}{=} (h^{-1} \triangleright x^{-1})^{-1}, \tag{2.4}$$

so that

$$s_{\mathcal{X}}(x_2) = r_{\mathcal{X}}(h^{-1} \triangleright x^{-1}) \overset{\text{(L10)}}{=} h^{-1} \triangleright r_{\mathcal{X}}(x^{-1}) = h^{-1} \triangleright s_{\mathcal{X}}(x) = h^{-1} \triangleright v,$$

as claimed. By Equation (2.4),

$$h_2 \overset{\text{(L9)}}{=} h \triangleleft (h^{-1} \triangleright x^{-1}) = h \triangleleft x_2^{-1}, \qquad \text{so that} \qquad h_2 \triangleleft x_2 \overset{\text{(L3)}}{=} h.$$

Now, if y is such that $r_{\mathcal{X}}(y) = h^{-1} \triangleright v$, meaning that $x_2 y$ makes sense by our above computation, then

$$\rho_{\mathcal{X}}(x_2 y) = \rho_{\mathcal{X}}(x_2) \overset{\text{(L1)}}{=} s_{\mathcal{H}}(h \triangleleft x_2^{-1}) = s_{\mathcal{H}}(h_2).$$

Therefore, $h_2 \triangleright (x_2 y)$ is likewise defined, and we have:

$$\begin{aligned} h_2 \triangleright (x_2 y) &= (h_2 \triangleright x_2)[(h_2 \triangleleft x_2) \triangleright y] && \text{(by (L4))} \\ &= x[h \triangleright y] && \text{(def'n of } h_2 \text{ and by the above).} \end{aligned}$$ □

Corollary 2.30. *Suppose $u, v \in \mathcal{X}^{(0)}$, $(x, h) \in \mathcal{X}^u_v \times \mathcal{H}^{\rho(v)} \subseteq \mathcal{X} \bowtie \mathcal{H}$, and λ is a $\triangleright$-invariant left Haar system for $\mathcal{X}$ in the sense of Definition 2.27. If $G \in C_c(\mathcal{X})$, then*

$$\int_{\mathcal{X}} G\big(x[h \triangleright y]\big)\, \mathrm{d}\lambda^{h^{-1}\triangleright v}(y) = \int_{\mathcal{X}} G(y)\, \mathrm{d}\lambda^u(y).$$

Proof. Let x_2, h_2 be as in Lemma 2.29. Then

$$\int_{\mathcal{X}} G\big(x[h \triangleright y]\big)\, \mathrm{d}\lambda^{h^{-1}\triangleright v}(y) = \int_{\mathcal{X}} G\big(h_2 \triangleright [x_2 y]\big)\, \mathrm{d}\lambda^{s(x_2)}(y).$$

By left invariance of λ, we have

$$\int_{\mathcal{X}} G\big(h_2 \triangleright [x_2 y]\big)\, \mathrm{d}\lambda^{s(x_2)}(y) = \int_{\mathcal{X}} G(h_2 \triangleright z)\, \mathrm{d}\lambda^{r(x_2)}(z).$$

Since $r_{\mathcal{X}}(x_2) = s_{\mathcal{H}}(h_2)$, we can invoke $\triangleright$-invariance of λ in the form of Equation (2.3) to conclude

$$\int_{\mathcal{X}} G(h_2 \triangleright z)\, \mathrm{d}\lambda^{r(x_2)}(z) = \int_{\mathcal{X}} G(y)\, \mathrm{d}\lambda^{h_2 \triangleright r(x_2)}(y).$$

Since $x_2 = h_2^{-1} \triangleright x$, it follows from (L10) (Lemma 2.9) that $h_2 \triangleright r_{\mathcal{X}}(x_2) = r_{\mathcal{X}}(x) = u$, so that the above right-hand side is as claimed in the statement. $\square$

Proof of Proposition 2.28. For this proof, let $\rho := \rho_{\mathcal{X}} = \rho^{(0)}_{\mathcal{X}} \circ r_{\mathcal{X}}$ and $\rho' := \rho^{(0)}_{\mathcal{X}} \circ s_{\mathcal{X}}$. Fix an arbitrary $u \in \mathcal{X}^{(0)}$ and note that $(\lambda \bowtie \varepsilon)^u$ is a Radon measure on $\mathcal{X} \bowtie \mathcal{H}$, since

$$(\lambda \bowtie \varepsilon)^u\colon \quad C_c(\mathcal{X} \bowtie \mathcal{H}) \to \mathbb{C}, \quad F \mapsto \int_{\mathcal{X}} \int_{\mathcal{H}} F(y, k)\, \mathrm{d}\varepsilon^{\rho'(y)}(k)\, \mathrm{d}\lambda^u(y),$$

is clearly a positive linear functional on $C_c(\mathcal{X} \bowtie \mathcal{H})$. First, we show that $\operatorname{supp}(\lambda \bowtie \varepsilon)^u = (\mathcal{X} \bowtie \mathcal{H})^u$. To see $\supseteq$, fix any

$$\eta = (y, k) \in (\mathcal{X} \bowtie \mathcal{H})^u = \mathcal{X}^u \bowtie \mathcal{H} = \sqcup_{v \in \mathcal{X}^{(0)}} \mathcal{X}^u_v \times \mathcal{H}^{\rho(v)}.$$

For any open neighborhood N_η around η, we must show that $(\lambda \bowtie \varepsilon)^u(N_\eta) > 0$. By monotonicity, it suffices to show this for a *basic* open neighborhood, so we may assume that $N_\eta = (N_y \times N_k) \cap \mathcal{X} \bowtie \mathcal{H}$ for some neighborhoods N_y of y and N_k of k. Thus,

$$(\lambda \bowtie \varepsilon)^u(N_\eta) = \int_{\mathcal{X} \bowtie \mathcal{H}} 1_{N_\eta}(\xi)\, \mathrm{d}(\lambda \bowtie \varepsilon)^u(\xi) = \int_{\mathcal{X}} \int_{\mathcal{H}} 1_{N_\eta}(x, h)\, \mathrm{d}\varepsilon^{\rho'(x)}(h)\, \mathrm{d}\lambda^u(x)$$

$$= \int_{\mathcal{X}} \int_{\mathcal{H}} 1_{N_y}(x)\, 1_{N_k}(h)\, 1_{\mathcal{X}\bowtie\mathcal{H}}(x,h)\, \mathrm{d}\varepsilon^{\rho'(x)}(h)\, \mathrm{d}\lambda^u(x)$$

$$= \int_{\mathcal{X}} 1_{N_y}(x) \left[\int_{\mathcal{H}} 1_{N_k}(h)\, \mathrm{d}\varepsilon^{\rho'(x)}(h)\right] \mathrm{d}\lambda^u(x). \tag{2.5}$$

Since $\mathcal{H}$ is locally compact, we may find a precompact neighborhood M_k of k for which $\overline{M_k} \subseteq N_k$. Since $k \in \mathcal{H}^{\rho(v)} = \operatorname{supp} \varepsilon^{\rho(v)}$, we have $\delta := \varepsilon^{\rho(v)}(M_k) > 0$. Let $f \in C_c(\mathcal{H}, [0,1])$ be a function that is constant 1 on M_k and vanishes outside of N_k, so that for all $w \in \mathcal{H}^{(0)}$,

$$\int_{\mathcal{H}} 1_{N_k}(h)\, \mathrm{d}\varepsilon^w(h) \geq \int_{\mathcal{H}} f(h)\, \mathrm{d}\varepsilon^w(h) \geq \int_{\mathcal{H}} 1_{M_k}(h)\, \mathrm{d}\varepsilon^w(h). \tag{2.6}$$

Note that the middle term is exactly $\varepsilon^w(f)$. As ε is a Haar system for $\mathcal{H}$, the function

$$\varepsilon(f)\colon \mathcal{H}^{(0)} \to \mathbb{C}, w \mapsto \varepsilon^w(f),$$

is continuous, where we followed the notation used in [38, Remark 1.20]. As the right-most side of (2.6) equals δ for $w = \rho(v)$, continuity of $\varepsilon(f)$ implies that $\varepsilon(f)$ is greater than $\frac{\delta}{2}$ in a neighborhood U of $\rho(v)$; let $V := (\rho')^{-1}(U) \subseteq \mathcal{X}$.

Using our computation in (2.5), we see that

$$(\lambda \bowtie \varepsilon)^u(N_\eta) \geq \int_{\mathcal{X}} 1_{N_y}(x)\, \varepsilon(f)\big(\rho'(x)\big)\, \mathrm{d}\lambda^u(x)$$

$$\geq \int_{\mathcal{X}} 1_{N_y \cap V}(x)\, \varepsilon(f)\big(\rho'(x)\big)\, \mathrm{d}\lambda^u(x)$$

$$\geq \delta \int_{\mathcal{X}} 1_{N_y \cap V}(x)\, \mathrm{d}\lambda^u(x) = \delta\, \lambda^u(N_y \cap V).$$

Note that by choice of y, $\rho'(y) = \rho_{\mathcal{X}}^{(0)}(s_{\mathcal{X}}(y)) = \rho_{\mathcal{X}}^{(0)}(v)$ is an element of U, so $N_y \cap V$ is a neighborhood of y. Since $y \in \mathcal{X}^u = \operatorname{supp} \lambda^u$, we must have $\lambda^u(N_y \cap V) > 0$, and hence $(\lambda \bowtie \varepsilon)^u(N_\eta) > 0$. Since y, k, u, v were arbitrary, this proves that $\operatorname{supp}(\lambda \bowtie \varepsilon)^u \supseteq \bigsqcup_{v \in \mathcal{X}^{(0)}} \mathcal{X}_v^u \times \mathcal{H}^{\rho(v)}$.

Conversely, assume that $\eta \notin (\mathcal{X} \bowtie \mathcal{H})^u$, i.e., if we write $\eta = (y,k)$, then $r_{\mathcal{X}}(y) \neq u$. Consider $r_{\mathcal{X}}^{-1}(\mathcal{X}^{(0)} \setminus \{u\}) = \mathcal{X} \setminus \mathcal{X}^u$. Since $\mathcal{X}^{(0)}$ is Hausdorff, this is an open neighborhood around y. Since $\operatorname{supp} \lambda^u = \mathcal{X}^u$, we have $\lambda^u(\mathcal{X} \setminus \mathcal{X}^u) = 0$. In particular, if we let $N_\eta := (\mathcal{X} \setminus \mathcal{X}^u) \bowtie \mathcal{H}$, then we have found a neighborhood of η for which $(\lambda \bowtie \varepsilon)^u(N_\eta) = 0$. Indeed, using our computation in (2.5), we see that

$$(\lambda \bowtie \varepsilon)^u(N_\eta) = \int_{\mathcal{X}} 1_{\mathcal{X}\setminus\mathcal{X}^u}(x) \left[\int_{\mathcal{H}} 1_{\mathcal{H}}(h)\, \mathrm{d}\varepsilon^{\rho'(x)}(h)\right] \mathrm{d}\lambda^u(x) = 0.$$

This means that $\eta \notin \operatorname{supp}(\lambda \bowtie \varepsilon)^u$, as claimed.

Next, for $F \in C_c(\mathcal{X} \bowtie \mathcal{H})$, we need to show that the map $u \mapsto \int F\, \mathrm{d}(\lambda \bowtie \varepsilon)^u$ is continuous. We will first prove the claim for $F = (f \times g)|_{\mathcal{X}\bowtie\mathcal{H}}$, where $f \times g\colon (x,h) \mapsto f(x)g(h)$ for some $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{H})$, so that

$$\int_{\mathcal{X}\bowtie\mathcal{H}} F(\eta)\, \mathrm{d}(\lambda \bowtie \varepsilon)^u(\eta) = \int_{\mathcal{X}} f(y) \int_{\mathcal{H}} g(k)\, \mathrm{d}\varepsilon^{\rho'(y)}(k)\, \mathrm{d}\lambda^u(y).$$

Since ε is a Haar system on $\mathcal{H}$ and since $g \in C_c(\mathcal{H})$, we know that the function

$$\mathcal{H}^{(0)} \to \mathbb{C}, \quad u' \mapsto \int_{\mathcal{H}} g(k)\, \mathrm{d}\varepsilon^{u'}(k),$$

is continuous. Since $f \in C_c(\mathcal{X})$ and since $\rho' = \rho_{\mathcal{X}}^{(0)} \circ s_{\mathcal{X}}$ is continuous, it follows that

$$G\colon \mathcal{X} \to \mathbb{C}, \quad y \mapsto f(y) \left(\int_{\mathcal{H}} g(k)\, \mathrm{d}\varepsilon^{\rho'(y)}(k)\right),$$

is continuous and compactly supported. Since λ is a Haar system on $\mathcal{X}$, we thus know that

$$(\mathcal{X} \bowtie \mathcal{H})^{(0)} \cong \mathcal{X}^{(0)} \to \mathbb{C}, \quad u \mapsto \int_{\mathcal{X}} G(y)\, \mathrm{d}\lambda^u(y) = \int_{\mathcal{X}\bowtie\mathcal{H}} F(\eta)\, \mathrm{d}(\lambda \bowtie \varepsilon)^u(\eta),$$

is continuous, as needed.

For general $F \in C_c(\mathcal{X} \bowtie \mathcal{H})$, let $K_{\mathcal{X}}$ and $K_{\mathcal{H}}$ be the $\mathcal{X}$- and the $\mathcal{H}$-part of $\operatorname{supp}(F)$, respectively, both of which are compact. Pick $f \in C_c(\mathcal{X})$ and $g \in C_c(\mathcal{H})$ which are constant 1 on $K_{\mathcal{X}}$ and $K_{\mathcal{H}}$, respectively, so that for any $v \in \mathcal{X}^{(0)}$ and for $K_{\mathcal{X}} \bowtie K_{\mathcal{H}} := (K_{\mathcal{X}} \times K_{\mathcal{H}}) \cap \mathcal{X} \bowtie \mathcal{H}$,

$$(\lambda \bowtie \varepsilon)^v(\operatorname{supp}(F)) \leq (\lambda \bowtie \varepsilon)^v(K_{\mathcal{X}} \bowtie K_{\mathcal{H}}) \leq \int_{\mathcal{X}\bowtie\mathcal{H}} (f \times g)\, \mathrm{d}(\lambda \bowtie \varepsilon)^v.$$

By our earlier argument, the right-hand side is a continuous function in v. Therefore, if K is some compact set, then for any $v \in K$,

$$(\lambda \bowtie \varepsilon)^v(K_{\mathcal{X}} \bowtie K_{\mathcal{H}}) \leq \max_{v' \in K}\Big[\int_{\mathcal{X}\bowtie\mathcal{H}} (f \times g)\, \mathrm{d}(\lambda \bowtie \varepsilon)^{v'}\Big] =: c_K < \infty. \tag{2.7}$$

Now, assume we are given a convergent net $u_i \to u$ in $\mathcal{X}^{(0)}$ and fix an arbitrary $\epsilon > 0$. By local compactness of $\mathcal{X}$, we may without loss of generality assume that each u_i is contained in a compact neighborhood K of U, so that (2.7) holds for $v = u_i$. By Stone–Weierstrass, we can choose finitely many $f_j \in C_c(\mathcal{X}), g_j \in C_c(\mathcal{H})$ such that

$$\left\| F - \sum_{j=1}^{k} (f_j \times g_j)|_{\mathcal{X} \bowtie \mathcal{H}} \right\|_\infty < \epsilon / \big(3c_K + 1\big).$$

Without loss of generality, the support of each f_j is in $K_{\mathcal{X}}$ and of each g_j is in $K_{\mathcal{H}}$, so that for all $v \in K$,

$$\begin{aligned} &\int \big| F - \sum_j f_j \times g_j \big| \, \mathrm{d}(\lambda \bowtie \varepsilon)^v \\ \leq &(\lambda \bowtie \varepsilon)^v \big(K_{\mathcal{X}} \bowtie K_{\mathcal{H}}\big) \left\| F - \sum_{j=1}^{k} (f_j \times g_j)|_{\mathcal{X} \bowtie \mathcal{H}} \right\|_\infty \overset{(2.7)}{<} \epsilon/3. \end{aligned} \tag{2.8}$$

By our earlier result, we may choose i_0 large enough such that for all $i \geq i_0$ and all $1 \leq j \leq k$, we have

$$\left| \int f_j \times g_j \, \mathrm{d}(\lambda \bowtie \varepsilon)^{u_i} - \int f_j \times g_j \, \mathrm{d}(\lambda \bowtie \varepsilon)^u \right| < \epsilon/3k.$$

Combining this with (2.8), we get for all $i \geq i_0$ that

$$\begin{aligned} &\left| \int F \, \mathrm{d}(\lambda \bowtie \varepsilon)^{u_i} - \int F \, \mathrm{d}(\lambda \bowtie \varepsilon)^u \right| \\ &\quad \leq \int \big| F - \sum_j f_j \times g_j \big| \, \mathrm{d}(\lambda \bowtie \varepsilon)^{u_i} \\ &\qquad + \sum_j \left| \int f_j \times g_j \, \mathrm{d}(\lambda \bowtie \varepsilon)^{u_i} - \int f_j \times g_j \, \mathrm{d}(\lambda \bowtie \varepsilon)^u \right| \\ &\qquad + \int \big| \big(\sum_j f_j \times g_j\big) - F \big| \, \mathrm{d}(\lambda \bowtie \varepsilon)^u < \epsilon, \end{aligned}$$

as needed.

Lastly, we have to show that for any $\xi \in \mathcal{X} \bowtie \mathcal{H}$ and any $F \in C_c(\mathcal{X} \bowtie \mathcal{H})$, we have $\int F(\xi\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{s(\xi)}\eta = \int F(\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{r(\xi)}\eta$. Write $\xi = (x, h) \in \mathcal{X}_v^u \times \mathcal{H}^{\rho(v)}$, so that $s(\xi) = h^{-1} \triangleright v$; as above, it suffices to consider the case where F can be written as $F(\xi) = f(x)g(h)$ for some $f \in C_c(\mathcal{X})$ and some $g \in C_c(\mathcal{H})$. Then

$$\begin{aligned} \int_{\mathcal{X} \bowtie \mathcal{H}} F(\xi\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{s(\xi)}(\eta) &= \int_{\mathcal{X} \bowtie \mathcal{H}} F\big(x[h \triangleright y], [h \triangleleft y]k\big) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{h^{-1} \triangleright v}(y, k) \\ &= \int_{\mathcal{X}} \int_{\mathcal{H}} f\big(x[h \triangleright y]\big) \, g\big([h \triangleleft y]k\big) \, \mathrm{d}\varepsilon^{\rho'(y)}(k) \, \mathrm{d}\lambda^{h^{-1} \triangleright v}(y) \end{aligned}$$

$$= \int_{\mathcal{X}} f\big(x[h \triangleright y]\big) \int_{\mathcal{H}} g\big([h \triangleleft y]k\big) \, \mathrm{d}\varepsilon^{\rho'(y)}(k) \, \mathrm{d}\lambda^{h^{-1} \triangleright v}(y),$$

where the last equation follows from (L1), which guarantees that $\rho'(y) = s_{\mathcal{H}}(h \triangleleft y)$. Since $r_{\mathcal{H}}(h \triangleleft y) = \rho'(h \triangleright y)$, left-invariance of ε implies

$$\int_{\mathcal{X} \bowtie \mathcal{H}} F(\xi\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{s(\xi)}(\eta) = \int_{\mathcal{X}} f\big(x[h \triangleright y]\big) \int_{\mathcal{H}} g(k) \, \mathrm{d}\varepsilon^{\rho'(h \triangleright y)}(k) \, \mathrm{d}\lambda^{h^{-1} \triangleright v}(y).$$

For $z \in \mathcal{X}$, define

$$G(z) := f(z) \int_{\mathcal{H}} g(k) \, \mathrm{d}\varepsilon^{\rho'(z)}(k).$$

Since ε is a Haar system and since $g \in C_c(\mathcal{H})$, we know that

$$\mathcal{H}^{(0)} \to \mathbb{C}, \quad u' \mapsto \int_{\mathcal{H}} g \, \mathrm{d}\varepsilon^{u'},$$

is continuous. Since $\rho' = \rho_{\mathcal{X}}^{(0)} \circ s_{\mathcal{X}}$ is continuous and since $f \in C_c(\mathcal{X})$, we conclude that G is a continuous and compactly supported function on $\mathcal{X}$. Since $s_{\mathcal{X}}(h \triangleright y) = s_{\mathcal{X}}(x[h \triangleright y])$, we conclude that

$$\begin{aligned}
\int_{\mathcal{X} \bowtie \mathcal{H}} F(\xi\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{s(\xi)}(\eta) &= \int_{\mathcal{X}} G\big(x[h \triangleright y]\big) \, \mathrm{d}\lambda^{h^{-1} \triangleright v}(y) \\
&= \int_{\mathcal{X}} G(y) \, \mathrm{d}\lambda^{r(x)}(y) && \text{(Corollary 2.30)} \\
&= \int_{\mathcal{X}} f(y) \int_{\mathcal{H}} g(k) \, \mathrm{d}\varepsilon^{\rho'(y)}(k) \, \mathrm{d}\lambda^{r(\xi)}(y) && \text{(def'n of } G) \\
&= \int_{\mathcal{X} \bowtie \mathcal{H}} F(\eta) \, \mathrm{d}(\lambda \bowtie \varepsilon)^{r(\xi)}(\eta). \quad \square
\end{aligned}$$

Corollary 2.31. *Suppose $\mathcal{H}$ and $\mathcal{X}$ are locally compact Hausdorff groupoids and that $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$.*

(1) If $\mathcal{X}$ is étale, then counting measure on $\mathcal{X}$ is $\triangleright$-invariant in the sense of Definition 2.27.
(2) If $\mathcal{H}$ and $\mathcal{X}$ are both r-discrete, then so is $\mathcal{H} \bowtie \mathcal{X}$.
(3) If $\mathcal{H}$ and $\mathcal{X}$ are both étale, then so is $\mathcal{H} \bowtie \mathcal{X}$.

Proof. If $\mathcal{X}$ is étale, [38, Prop. 1.29] says that counting measures form a Haar system on $\mathcal{X}$. Now, for any fixed $(h,u) \in \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathcal{X}^{(0)}$, the map $\mathcal{X}^{h \triangleright u} \to \mathcal{X}^{u}$, $y \mapsto h^{-1} \triangleright y$, is a bijection (in fact, a homeomorphism), and thus

$$\sum_{x \in \mathcal{X}^{u}} f(h \triangleright x) = \sum_{y \in \mathcal{X}^{h \triangleright u}} f(y)$$

for all $f \in C_c(\mathcal{X})$. In other words, counting measure on $\mathcal{X}$ is $\triangleright$-invariant.

Now suppose the groupoids are r-discrete. Since $\mathcal{X}^{(0)} \times \mathcal{H}^{(0)}$ is open in $\mathcal{X} \times \mathcal{H}$ and since $\mathcal{X} \bowtie \mathcal{H}$ has the subspace topology, we have that $(\mathcal{X}^{(0)} \times \mathcal{H}^{(0)}) \cap (\mathcal{X} \bowtie \mathcal{H}) = (\mathcal{X} \bowtie \mathcal{H})^{(0)}$ is open in $\mathcal{X} \bowtie \mathcal{H}$. Thus, $\mathcal{X} \bowtie \mathcal{H}$ is also r-discrete.

Now, if both $\mathcal{X}$ and $\mathcal{H}$ are étale, then it follows from Proposition 2.28 that $\mathcal{H} \bowtie \mathcal{X}$ admits a Haar system. According to [38, Prop. 1.23 and 1.29], any locally compact and r-discrete groupoid that admits a Haar system is necessarily étale, so our claim follows. □

2.4. Rehash (from left to right)

The definitions we made so far can similarly be made on the right; we have added them here for easy reference.

Definition 2.32 *(cf. Definition 2.2).* Let $\mathcal{G}$ and $\mathcal{X}$ be two locally compact Hausdorff groupoids. We say $\mathcal{G}$ has a *self-similar right action* on $\mathcal{X}$ if there exists a continuous surjection $\sigma_{\mathcal{X}}^{(0)} \colon \mathcal{X}^{(0)} \to \mathcal{G}^{(0)}$ and, using the anchor map $\sigma_{\mathcal{X}} \coloneqq \sigma_{\mathcal{X}}^{(0)} \circ s_{\mathcal{X}}$, two continuous maps

$$\mathcal{X} \curvearrowleft \mathcal{G}\colon \qquad \mathcal{X}\,{}_{\sigma_{\mathcal{X}}}{*}_{r_{\mathcal{G}}}\,\mathcal{G} \ni (x,s) \mapsto x \blacktriangleleft s \in \mathcal{X}$$
$$\mathcal{X} \circlearrowleft \mathcal{G}\colon \qquad \mathcal{X}\,{}_{\sigma_{\mathcal{X}}}{*}_{r_{\mathcal{G}}}\,\mathcal{G} \ni (x,s) \mapsto x \blacktriangleright s \in \mathcal{G}$$

such that the following hold.

- For any $x \in \mathcal{X}$ and $t \in \mathcal{G}$ such that $\sigma_{\mathcal{X}}(x) = r_{\mathcal{H}}(t)$, we have

$$\sigma_{\mathcal{X}}(x \blacktriangleleft t) = s_{\mathcal{G}}(t) \qquad \sigma_{\mathcal{X}}(x^{-1}) = r_{\mathcal{G}}(x \blacktriangleright t) \qquad \sigma_{\mathcal{X}}((x \blacktriangleleft t)^{-1}) = s_{\mathcal{G}}(x \blacktriangleright t) \tag{R1}$$

- For all $v \in \mathcal{X}^{(0)}$ and $s \in \mathcal{G}$ such that $\sigma_{\mathcal{X}}(v) = r_{\mathcal{G}}(s)$ and for all $x \in \mathcal{X}$, we have:

$$v \blacktriangleright s = s \text{ and } x \blacktriangleleft \sigma_{\mathcal{X}}(x) = x \tag{R2}$$

- For all $(x,y) \in \mathcal{X}^{(2)}$ and $s \in \mathcal{G}$ such that $\sigma_{\mathcal{X}}(y) = r_{\mathcal{G}}(s)$, we have $s_{\mathcal{X}}(x \blacktriangleleft (y \blacktriangleright s)) = r_{\mathcal{X}}(y \blacktriangleleft s)$ and

$$(xy) \blacktriangleright s = x \blacktriangleright (y \blacktriangleright s) \tag{R3}$$
$$(xy) \blacktriangleleft s = [x \blacktriangleleft (y \blacktriangleright s)](y \blacktriangleleft s) \tag{R4}$$

- For all $x \in \mathcal{X}$ and $(s,t) \in \mathcal{G}^{(2)}$ such that $\sigma_{\mathcal{X}}(x) = r_{\mathcal{G}}(s)$, we have:

$$x \blacktriangleleft (st) = (x \blacktriangleleft s) \blacktriangleleft t \tag{R5}$$

$$x \blacktriangleright (st) = (x \blacktriangleright s)[(x \blacktriangleleft s) \blacktriangleright t] \tag{R6}$$

We call the self-similar right action *free* (and respectively, *proper*) if $\blacktriangleleft$ is free (and respectively, proper).

Remark 2.33. Similar to our previous computation for the self-similar left actions, for every $t \in \mathcal{G}$, $x \in \mathcal{X}$, and $v \in \mathcal{X}^{(0)}$ with $r_{\mathcal{G}}(t) = \sigma_{\mathcal{X}}(x) = \sigma_{\mathcal{X}}(v)$, we have

$$x \blacktriangleright \sigma_{\mathcal{X}}(x) = \sigma_{\mathcal{X}}(x^{-1}) \tag{R7}$$

$$v \blacktriangleleft t \in \mathcal{X}^{(0)} \tag{R8}$$

$$(x \blacktriangleleft t)^{-1} = x^{-1} \blacktriangleleft (x \blacktriangleright t) \quad \text{and} \quad (x \blacktriangleright t)^{-1} = (x \blacktriangleleft t) \blacktriangleright t^{-1} \tag{R9}$$

$$s_{\mathcal{X}}(x \blacktriangleleft t) = s_{\mathcal{X}}(x) \blacktriangleleft t \quad \text{and} \quad r_{\mathcal{X}}(x \blacktriangleleft t) = r_{\mathcal{X}}(x) \blacktriangleleft (x \blacktriangleright t) \tag{R10}$$

In a very similar fashion, we can define the self-similar product for a right action:

Definition 2.34. Let $\mathcal{G}$ be a groupoid that has a self-similar right action on $\mathcal{X}$. Define their self-similar product as the set

$$\mathcal{G} \bowtie \mathcal{X} = \{(t,x) : s_{\mathcal{G}}(t) = \sigma_{\mathcal{X}}(r_{\mathcal{X}}(x))\}$$

with multiplication

$$(s,x)(t,y) := (s(x \blacktriangleright t), (x \blacktriangleleft t)y), \quad \text{whenever } s_{\mathcal{X}}(x) = r_{\mathcal{X}}(y) \blacktriangleleft t^{-1},$$

and inverse

$$(t,x)^{-1} := (x^{-1} \blacktriangleright t^{-1}, x^{-1} \blacktriangleleft t^{-1}).$$

For a right action, we mimic the construction in Definition 2.27 verbatim, only replacing the left Haar system by a right Haar system:

Definition 2.35. Suppose $\mathcal{G}$ and $\mathcal{X}$ are locally compact Hausdorff groupoids and that $\blacktriangleleft$ is a right $\mathcal{G}$-action on $\mathcal{X}$ with momentum map $\sigma_{\mathcal{G}} \colon \mathcal{X} \to \mathcal{G}^{(0)}$. We say that a right Haar system $\{\lambda_u\}_{u \in \mathcal{X}^{(0)}}$ on $\mathcal{X}$ is $\blacktriangleleft$-*invariant* if for all $t \in \mathcal{G}$ and all $u \in \mathcal{X}^{(0)}$ with $\sigma_{\mathcal{X}}(u) = r_{\mathcal{G}}(t)$, we have

$$\lambda_u \blacktriangleleft t = \lambda_{u \blacktriangleleft t},$$

where $\lambda_u \blacktriangleleft t(E) = \lambda_u(E \blacktriangleleft t^{-1})$.

Given a self-similar right action of $\mathcal{G}$ on $\mathcal{X}$, a right Haar system of $\mathcal{G}$ and a $\blacktriangleleft$-invariant right Haar system on $\mathcal{X}$ yields a right Haar systems on the self-similar product groupoid $\mathcal{G} \bowtie \mathcal{X}$ similarly to the result in Proposition 2.28. The details are omitted here.

3. The orbit space

If $\mathcal{H}$ has a self-similar left action on the groupoid $\mathcal{X}$, then $(h, x) \mapsto h \triangleright x$ is an $\mathcal{H}$-action on the space $\mathcal{X}$ according to Lemma 2.4. We can therefore construct the quotient space, $\mathcal{H}\backslash\mathcal{X}$, whose elements we will denote by $\mathcal{H} \triangleright x$. We will now show that we can equip this space with its own groupoid structure as long as the action is free and proper.

Recall from Lemma 2.14 that $\triangleright$ restricts to an $\mathcal{H}$-action on $\mathcal{X}^{(0)}$, so we may consider $\mathcal{H}\backslash\mathcal{X}^{(0)}$. We define $s_{\mathcal{H}\backslash\mathcal{X}}, r_{\mathcal{H}\backslash\mathcal{X}} \colon \mathcal{H}\backslash\mathcal{X} \to \mathcal{H}\backslash\mathcal{X}^{(0)}$ by

$$s_{\mathcal{H}\backslash\mathcal{X}}(\mathcal{H} \triangleright x) = \mathcal{H} \triangleright s_{\mathcal{X}}(x) \quad \text{and} \quad r_{\mathcal{H}\backslash\mathcal{X}}(\mathcal{H} \triangleright x) = \mathcal{H} \triangleright r_{\mathcal{X}}(x). \tag{3.1}$$

These are well-defined by (L10).

Lemma 3.1. *If $s_{\mathcal{H}}$ and $s_{\mathcal{X}}$ are open, then the map $s_{\mathcal{H}\backslash\mathcal{X}}$ is also open.*

Proof. Since $s_{\mathcal{H}}$ is open, the quotient map $q|_{\mathcal{X}^{(0)}} \colon \mathcal{X}^{(0)} \to \mathcal{H}\backslash\mathcal{X}^{(0)}$ is open by [38, Proposition 2.12]. The claim now follows from continuity of q and commutativity of the diagram below.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad q \quad} & \mathcal{H}\backslash\mathcal{X} \\ \downarrow{\scriptstyle s_{\mathcal{X}}} & & \downarrow{\scriptstyle s_{\mathcal{H}\backslash\mathcal{X}}} \\ \mathcal{X}^{(0)} & \xrightarrow{q|_{\mathcal{X}^{(0)}}} & \mathcal{H}\backslash\mathcal{X}^{(0)} \end{array} \qquad \square$$

Lemma 3.2. *Suppose $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$, and fix two elements ξ, η of $\mathcal{H}\backslash\mathcal{X}$ for which $s_{\mathcal{H}\backslash\mathcal{X}}(\xi) = r_{\mathcal{H}\backslash\mathcal{X}}(\eta)$. Then we can find $x_1 \in \xi$ and $y_1 \in \eta$ such that $s_{\mathcal{X}}(x_1) = r_{\mathcal{X}}(y_1)$.*

Moreover, if the action of $\mathcal{H}$ on $\mathcal{X}$ is free, then any two more such elements x_2, y_2 satisfy $\mathcal{H} \triangleright (x_2 y_2) = \mathcal{H} \triangleright (x_1 y_1)$.

Proof. For existence, start with two arbitrary elements $x \in \xi$ and $y \in \eta$. By construction of $s_{\mathcal{H}\backslash\mathcal{X}}$ and $r_{\mathcal{H}\backslash\mathcal{X}}$, we have $s_{\mathcal{X}}(x) \in s_{\mathcal{H}\backslash\mathcal{X}}(\xi)$ and $r_{\mathcal{X}}(y) \in r_{\mathcal{H}\backslash\mathcal{X}}(\eta)$. As the two equivalence classes coincide by assumption, there exists $h \in \mathcal{H}$ such that $s_{\mathcal{X}}(x) = h \triangleright r_{\mathcal{X}}(y)$. Since the right-hand side equals $r_{\mathcal{X}}(h \triangleright y)$ by (L10) (Lemma 2.9), we see that we can pick $x_1 := x$ and $y_1 := h \triangleright y \in \mathcal{H} \triangleright y = \eta$.

To see the claim about the product, let $k, l \in \mathcal{H}$ be such that $x_2 = k \triangleright x_1$ and $y_2 = l \triangleright y_1$. By (L10),

$$s_{\mathcal{X}}(x_2) = s_{\mathcal{X}}(k \triangleright x_1) = (k \triangleleft x_1) \triangleright s_{\mathcal{X}}(x_1), \qquad \text{and}$$

$$r_{\mathcal{X}}(y_2) = r_{\mathcal{X}}(l \triangleright y_1) = l \triangleright r_{\mathcal{X}}(y_1) = l \triangleright s_{\mathcal{X}}(x_1).$$

Since the left-hand sides of these equations are assumed to be equal and since the $\mathcal{H}$-action is free, we conclude that $l = k \triangleleft x_1$. Therefore, by (L6),

$$x_2 y_2 = (k \triangleright x_1)(l \triangleright y_1) = (k \triangleright x_1)([k \triangleleft x_1] \triangleright y_1) = k \triangleright (x_1 y_1),$$

so $\mathcal{H} \triangleright (x_2 y_2) = \mathcal{H} \triangleright (x_1 y_1)$, as claimed. □

The lemma allows us to turn $\mathcal{H}\backslash\mathcal{X}$ into a groupoid.

Proposition 3.3. *Suppose $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$ for which $\triangleright$ is* free *on $\mathcal{X}$. For two elements ξ, η of the orbit space $\mathcal{H}\backslash\mathcal{X}$ with $s_{\mathcal{H}\backslash\mathcal{X}}(\xi) = r_{\mathcal{H}\backslash\mathcal{X}}(\eta)$, define*

$$\xi\eta = \mathcal{H} \triangleright (xy) \quad \text{where } x \in \xi, y \in \eta \text{ are such that } s_{\mathcal{X}}(x) = r_{\mathcal{X}}(y).$$

Further, define

$$(\mathcal{H} \triangleright x)^{-1} = \mathcal{H} \triangleright x^{-1}.$$

With this structure, $\mathcal{H}\backslash\mathcal{X}$ is a (non-topological) groupoid.

If we further assume that $\triangleright$ is proper and $s_{\mathcal{H}}$ is open, then $\mathcal{H}\backslash\mathcal{X}$ is a locally compact Hausdorff groupoid with the quotient topology, and if $\mathcal{X}$ is étale, then so is $\mathcal{H}\backslash\mathcal{X}$.

Proof. We have seen in Lemma 3.2 that, since the $\mathcal{H}$-action is free, the multiplication is well-defined and independent of the choice of x, y. To see that the inversion is well-defined, suppose that $x_1 \in \mathcal{H} \triangleright x = \xi$, i.e., $x_1 = h \triangleright x$ for some h. Then by (L9), we have $x_1^{-1} = (h \triangleright x)^{-1} = (h \triangleleft x) \triangleright x^{-1}$, so $x_1^{-1} \in \mathcal{H} \triangleright x^{-1}$, and hence the definition of $(\mathcal{H} \triangleright x)^{-1}$ does not depend on the chosen representative.

The algebraic properties of a groupoid are now easy to verify and follow from the algebraic properties that $\mathcal{X}$ satisfies.

Now suppose $\triangleright$ is proper and $s_{\mathcal{H}}$ is open. Since we assume our groupoids $\mathcal{H}$ and $\mathcal{X}$ to be locally compact Hausdorff, it follows from [38, Proposition 2.18] that the quotient is locally compact Hausdorff.

To show that the multiplication map $(\mathcal{H}\backslash\mathcal{X})^{(2)} \to \mathcal{H}\backslash\mathcal{X}$ is continuous, suppose we are given a net $\{(\xi_i, \eta_i)\}_{i \in I}$ in $(\mathcal{H}\backslash\mathcal{X})^{(2)}$ that converges to some composable pair (ξ, η). Because of Lemma A.2, it suffices to show that a subnet of $\{\xi_i \eta_i\}_{i \in I}$ converges to $\xi\eta$.

As $(\mathcal{H}\backslash\mathcal{X})^{(2)}$ has the subspace topology of the product topology on $(\mathcal{H}\backslash\mathcal{X}) \times (\mathcal{H}\backslash\mathcal{X})$, convergence implies that $\xi_i \to \xi$ and $\eta_i \to \eta$ in $\mathcal{H}\backslash\mathcal{X}$. Since $s_{\mathcal{H}}$ is open, the quotient map q is open by [38, Proposition 2.12]. Thus, if we fix $x \in \xi$, then by Proposition A.1 we can find a subnet of $\{\xi_i\}_{i \in I}$ that is the image under q of a net in $\mathcal{X}$ that converges to x; without loss of generality, the subnet is the net itself, meaning there exist $x_i \in \mathcal{X}$ such that $x_i \to x$ and $\mathcal{H} \triangleright x_i = \xi_i$. Once again by passing to a subnet, we can without loss of generality assume that $\{\eta_i\}_{i\in I}$ is the image under q of a convergent net, say of $y_i \to y \in \eta$. In other words, by passing to a subnet of a subnet, we can without loss of generality assume that $\{(\xi_i, \eta_i)\}_{i\in I}$ itself can be lifted to a net $\{(x_i, y_i)\}_{i\in I}$ that converges to $(x, y) \in \xi \times \eta$ in $\mathcal{X} \times \mathcal{X}$. Since $(\xi_i, \eta_i) \in (\mathcal{H}\backslash\mathcal{X})^{(2)}$, we have

$$\mathcal{H} \triangleright s_{\mathcal{X}}(x_i) = s_{\mathcal{H}\backslash\mathcal{X}}(\xi_i) = r_{\mathcal{H}\backslash\mathcal{X}}(\eta_i) = \mathcal{H} \triangleright r_{\mathcal{X}}(y_i),$$

so we can find $h_i \in \mathcal{H}$ such that $s_{\mathcal{X}}(x_i) = h_i \triangleright r_{\mathcal{X}}(y_i)$; note that h_i is unique by freeness. Similarly, there exists a unique h with $s_{\mathcal{X}}(x) = h \triangleright r_{\mathcal{X}}(y)$. Continuity of $s_{\mathcal{X}}$ and $r_{\mathcal{X}}$ implies that

$$\big(h_i \triangleright r_{\mathcal{X}}(y_i), r_{\mathcal{X}}(y_i)\big) = \big(s_{\mathcal{X}}(x_i), r_{\mathcal{X}}(y_i)\big) \xrightarrow{i} \big(s_{\mathcal{X}}(x), r_{\mathcal{X}}(y)\big) = \big(h \triangleright r_{\mathcal{X}}(y), r_{\mathcal{X}}(y)\big) \quad (3.2)$$

Since $\triangleright$ is proper, this convergence implies that (a subnet of) $\{h_i\}_{i\in I}$ converges. Since $\mathcal{X}^{(0)}$ is Hausdorff and $\triangleright$ is free, it must converge to h. In particular, continuity if $\triangleright$ implies that $\{(x_i, h_i \triangleright y_i)\}_{i\in I}$ is a net in $\mathcal{X}^{(2)}$ that converges to the composable pair $(x, h \triangleright y)$. Continuity of the multiplication on $\mathcal{X}$ implies that $\{x_i[h_i \triangleright y_i]\}_{i\in I}$ converges to $x[h \triangleright y]$. Since

$$q(x_i[h_i \triangleright y_i]) = \mathcal{H} \triangleright (x_i[h_i \triangleright y_i]) = (\mathcal{H} \triangleright x_i)(\mathcal{H} \triangleright y_i) = \xi_i\,\eta_i$$

and $q(x[h \triangleright y]) = \xi\,\eta$, continuity of q implies that $\{\xi_i\eta_i\}_{i\in I}$ converges to $\xi\eta$. This proves that the multiplication on $\mathcal{H}\backslash\mathcal{X}$ is continuous.

For the inversion map, the argument is similar: if $\xi_i \to \xi$ in $\mathcal{H}\backslash\mathcal{X}$, then openness of q allows a lift $\{x_j\}_{j\in J}$ of a subnet $\{\xi_j\}_{j\in J}$ which converges to a fixed preimage x of ξ. Continuity of the inversion in $\mathcal{X}$ implies that $x_j^{-1} \to x^{-1}$, and continuity of q implies $\xi_j^{-1} = \mathcal{H} \triangleright (x_j^{-1}) \to \mathcal{H} \triangleright (x^{-1}) = \xi^{-1}$. By Lemma A.2, this suffices to show that the inversion on $\mathcal{H}\backslash\mathcal{X}$ is continuous.

Lastly, assume that $\mathcal{X}$ is étale, so its source map is an open map and its unit space is open. As argued above, the quotient map $q\colon \mathcal{X} \to \mathcal{H}\backslash\mathcal{X}$ is open, and so $(\mathcal{H}\backslash\mathcal{X})^{(0)} = \mathcal{H}\backslash\mathcal{X}^{(0)} = q(\mathcal{X}^{(0)})$ is open, i.e., $\mathcal{H}\backslash\mathcal{X}$ is r-discrete. Since $s_{\mathcal{H}}$ and $s_{\mathcal{X}}$ are open, Lemma 3.1 implies that the source map of $\mathcal{H}\backslash\mathcal{X}$ is open, and so [38, Proposition 1.29] implies that $\mathcal{H}\backslash\mathcal{X}$ is étale. □

Example 3.4. If we consider the self-similar left action of the trivial groupoid $\mathcal{X}^{(0)}$ on $\mathcal{X}$ as defined in Example 2.6, then $\mathcal{X}^{(0)}\backslash\mathcal{X} \cong \mathcal{X}$ via $\mathcal{X}^{(0)} \triangleright x \mapsto x$, since $\triangleright$ is trivial.

Likewise, the trivial group $\{e\}$ with its (trivial) self-similar left action on a groupoid $\mathcal{X}$ as defined in Example 2.7 is (trivially) free and proper. The quotient groupoid $\{e\}\backslash\mathcal{X}$ is exactly the groupoid $\mathcal{X}$ if we identify $\{e\} \triangleright x$ with x.

3.1. Self-similar para-equivalences

We are now in a position where we can define a generalized notion of compatible actions.

Definition 3.5. Suppose the two groupoids $\mathcal{H}, \mathcal{G}$ act on the left and right of a groupoid $\mathcal{X}$ by self-similar actions, respectively. We say the actions are *in tune* if for any $h \in \mathcal{H}$, $x \in \mathcal{X}$, and $s \in \mathcal{G}$ with $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$ and $\sigma_{\mathcal{X}}(x) = r_{\mathcal{G}}(s)$, we have

(C0) $\sigma_{\mathcal{X}}(h \triangleright x) = \sigma_{\mathcal{X}}(x)$ in $\mathcal{G}^{(0)}$ and $\rho_{\mathcal{X}}(x) = \rho_{\mathcal{X}}(x \blacktriangleleft s)$ in $\mathcal{H}^{(0)}$,
(C1) $h \triangleright (x \blacktriangleleft s) = (h \triangleright x) \blacktriangleleft s$ in $\mathcal{X}$,
(C2) $(h \triangleright x) \blacktriangleright s = x \blacktriangleright s$ in $\mathcal{G}$, and
(C3) $h \triangleleft (x \blacktriangleleft s) = h \triangleleft x$ in $\mathcal{H}$.

Note that Condition (C0) ensures that the elements in the other conditions make sense.

Definition 3.6. Suppose the two groupoids $\mathcal{H}, \mathcal{G}$ act on the left and right of a groupoid $\mathcal{X}$ by self-similar actions, respectively. If the self-similar actions are in tune and both free and proper, and if $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{X}$ have open source maps, then we call $\mathcal{X}$ an $(\mathcal{H}, \mathcal{G})$-*self-similar para-equivalence.*

Remark 3.7. In case of the semidirect product construction in [17, Appendix A.2], we have $x \blacktriangleright s = s$ and $h \triangleleft x = h$ for all $h \in \mathcal{H}$, $x \in \mathcal{X}$, and $s \in \mathcal{G}$. Therefore, Conditions (C2) and (C3) are trivially satisfied since both sides of the first equation are s and both sides of the second are h. Thus, in this case, the in-tune conditions simply reduce to the commuting conditions (C0) and (C1).

Example 3.8. Let $\mathcal{G}$ and $\mathcal{X}$ be groupoids whose source maps are open, and suppose that $\mathcal{G}$ has a self-similar right action on $\mathcal{X}$ that is free and proper (Definition 2.32). Then $\mathcal{X}$ is a $(\{e\}, \mathcal{G})$-self-similar para-equivalence. Indeed, the trivial actions $\triangleright, \triangleleft$ constitute a free and proper self-similar left action of $\{e\}$ on $\mathcal{X}$ (see Examples 2.7 and 2.12), and the following computations show that the actions of $\mathcal{X}^{(0)}$ and of $\mathcal{G}$ are in tune, where $x \in \mathcal{X}$ and $s \in \mathcal{G}$ are such that $\sigma_{\mathcal{X}}(x) = r_{\mathcal{G}}(s)$.

Re (C0): Since $\triangleright$ is trivial, we have $\sigma_{\mathcal{X}}(h \triangleright x) = \sigma_{\mathcal{X}}(x)$, and since $\rho_{\mathcal{X}} \colon \mathcal{X} \to \{e\}$ is constant, we have $\rho_{\mathcal{X}}(x) = \rho_{\mathcal{X}}(x \blacktriangleleft s)$.

Re (C1), (C2): Since $\triangleright$ is trivial, we have $e \triangleright (x \blacktriangleleft s) = x \blacktriangleleft s = (e \triangleright x) \blacktriangleleft s$ and $(e \triangleright x) \blacktriangleright s = x \blacktriangleright s$.

Re (C3): Since $\triangleleft$ is trivial, we have $e \triangleleft (x \blacktriangleleft s) = e = e \triangleleft x$.

Note, however, that $\mathcal{X}$ is *not* a $(\mathcal{X}^{(0)}, \mathcal{G})$-self-similar para-equivalence (even though $\mathcal{X}$ has a free and proper self-similar left action of $\mathcal{X}^{(0)}$ by Example 2.6): If there exists one $(x, s) \in \mathcal{X}\,{}_\sigma{*}_s\,\mathcal{G}$ with $s \notin \mathcal{G}^{(0)}$, then freeness of the $\mathcal{G}$-action $\blacktriangleleft$ on $\mathcal{X}^{(0)}$ implies that

$$r_\mathcal{X}(x) \neq r_\mathcal{X}(x) \blacktriangleleft (x \blacktriangleright s) \overset{\text{(R10)}}{=} r_\mathcal{X}(x \blacktriangleleft s).$$

As the momentum map $\rho_\mathcal{X}$ for the $\mathcal{X}^{(0)}$-action on $\mathcal{X}$ is $r_\mathcal{X}$ in this setting, the above inequality conflicts with Condition (C0).

Remark 3.9. In Example 2.26, we showed that the (trivial) self-similar left action of $\{e\}$ on a groupoid $\mathcal{X}$ gives rise to the 'standard' self-similar left action of $\mathcal{X}^{(0)}$ on $\mathcal{X}$ as defined in Example 2.6, and it then also followed that $\mathcal{X} \bowtie \{e\} \cong \mathcal{X} \bowtie \mathcal{X}^{(0)}$. This seemed to indicate that the pairs $(\mathcal{X}, \{e\})$ and $(\mathcal{X}, \mathcal{X}^{(0)})$ are 'the same' in some sense.

However, the above example shows that this point of view is ill-advised, since a self-similar para-equivalence $\mathcal{X}$ between $\mathcal{H}$ and $\mathcal{G}$ need not be one between $\widetilde{\mathcal{H}} = \mathcal{H} \ltimes \mathcal{X}^{(0)}$ and $\mathcal{G}$. The reason is that, Condition (C0) for the pair $(\mathcal{H}, \mathcal{G})$ does not imply the same condition for $(\widetilde{\mathcal{H}}, \mathcal{G})$, since the momentum maps on $\mathcal{X}$ with respect to the left actions do not need to coincide: we have $\rho_\mathcal{X} \colon \mathcal{X} \to \mathcal{H}^{(0)}$ for the left $\mathcal{H}$-action $\triangleright$, while we have $r_\mathcal{X} \colon \mathcal{X} \to \mathcal{X}^{(0)} = \widetilde{\mathcal{H}}^{(0)}$ for the left $\widetilde{\mathcal{H}}$-action $\cdot$ (see Lemma 2.22).

Given a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence $\mathcal{X}$, we have shown in Proposition 3.3 that the orbit space $\mathcal{H}\backslash\mathcal{X}$ and, by extension, $\mathcal{X}/\mathcal{G}$ are groupoids. In Proposition 3.12, we will establish that $\mathcal{H}$ has a self-similar left action on $\mathcal{X}/\mathcal{G}$; similarly, $\mathcal{G}$ has a self-similar right action on $\mathcal{H}\backslash\mathcal{X}$. We can then consider the self-similar product groupoids $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$, as constructed in Definition 2.16. Our main result is that these two self-similar product groupoids are equivalent via their actions on $\mathcal{X}$ in the sense of [22, Definition 2.1] as summed up in the following theorem; this generalizes [17, Lemma 3.2].

Theorem 3.10 *(cf. [17, Lemma 3.2]). Let $\mathcal{H}, \mathcal{G}, \mathcal{X}$ be groupoids, and suppose that $\mathcal{X}$ is a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence in the sense of Definition 3.6, that is,*

- *$s_\mathcal{H}$, $s_\mathcal{G}$, and $s_\mathcal{X}$ are open maps,*
- *$\mathcal{H}$ has a self-similar left action on $\mathcal{X}$ that is free and proper (Definition 2.2),*
- *$\mathcal{G}$ has a self-similar right action on $\mathcal{X}$ that is free and proper (Definition 2.32), and*
- *the two actions are in tune (Definition 3.5).*

Then there is a natural way to turn $\mathcal{X}$ into a groupoid equivalence from $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ to $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$.

For the description of the equivalence structure on $\mathcal{X}$, see Proposition 3.14. Examples of applications of Theorem 3.10 can be found in Subsection 3.2.

Example 3.11. Theorem 3.10 recovers [17, Lemma 3.2]: when H and G are locally compact Hausdorff groups and their free and proper actions on a groupoid $\mathcal{X}$ are actions by automorphisms, then we may let $\mathcal{X}$ act trivially on both H and G, i.e., $h \triangleleft x = h$ and $x \blacktriangleright s = s$. This makes $\mathcal{X}$ a (H, G)-self-similar para-equivalence, and the equivalence structure alluded to in Theorem 3.10 makes $\mathcal{X}$ a groupoid equivalence between $(\mathcal{X}/G) \rtimes H$ and $G \ltimes (H\backslash\mathcal{X})$.

Proposition 3.12. *Suppose $\mathcal{X}$ is a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence. Then $\mathcal{H}$ has a self-similar left action on $\mathcal{X}/\mathcal{G}$: the momentum map is given by $\tilde{\rho}(x \blacktriangleleft G) = \rho_{\mathcal{X}}(x)$, and the actions are defined by*

$$\mathcal{H} \curvearrowright (\mathcal{X}/\mathcal{G}): \qquad \mathcal{H}\,{}_{s}{*}_{\tilde{\rho}}\,(\mathcal{X}/\mathcal{G}) \ni (h, x \blacktriangleleft \mathcal{G}) \mapsto h \circledtriangleright (x \blacktriangleleft \mathcal{G}) := (h \triangleright x) \blacktriangleleft \mathcal{G} \in \mathcal{X}/\mathcal{G}$$
$$\mathcal{H} \looparrowleft (\mathcal{X}/\mathcal{G}): \qquad \mathcal{H}\,{}_{s}{*}_{\tilde{\rho}}\,(\mathcal{X}/\mathcal{G}) \ni (h, x \blacktriangleleft \mathcal{G}) \mapsto h \circledtriangleleft (x \blacktriangleleft \mathcal{G}) := h \triangleleft x \in \mathcal{H}$$

Likewise, $\mathcal{G}$ has a self-similar right action on $\mathcal{H}\backslash\mathcal{X}$: the momentum map is given by $\tilde{\sigma}(\mathcal{H} \triangleright x) = \sigma_{\mathcal{X}}(x)$, and the actions are defined by

$$(\mathcal{H}\backslash\mathcal{X}) \curvearrowleft \mathcal{G}: \qquad (\mathcal{H}\backslash\mathcal{X})\,{}_{\tilde{\sigma}}{*}_{r}\,\mathcal{G} \ni (\mathcal{H} \triangleright x, s) \mapsto (\mathcal{H} \triangleright x) \circledblacktriangleleft s := \mathcal{H} \triangleright (x \blacktriangleleft s) \in \mathcal{H}\backslash\mathcal{X}$$
$$(\mathcal{H}\backslash\mathcal{X}) \looparrowright \mathcal{G}: \qquad (\mathcal{H}\backslash\mathcal{X})\,{}_{\tilde{\sigma}}{*}_{r}\,\mathcal{G} \ni (\mathcal{H} \triangleright x, s) \mapsto (\mathcal{H} \triangleright x) \circledblacktriangleright s := x \blacktriangleright s \in \mathcal{G}$$

Note that, even though $\triangleright$ and $\blacktriangleleft$ are free and proper, the same is not necessarily true for $\circledtriangleright$ or $\circledblacktriangleleft$. This fact prevents us from turning an iterated quotient such as $\mathcal{H}\backslash(\mathcal{X}/\mathcal{G})$ or $(\mathcal{H}\backslash\mathcal{X})/\mathcal{G}$ into a topological groupoid, if we were so inclined. (Luckily, we aren't.)

Proof. The momentum map is well-defined by Condition (C0) and it is surjective because $\rho_{\mathcal{X}}$ is surjective. It remains to check that $\tilde{\rho}$ is continuous. Since $r_{\mathcal{G}}$ is open, we know that the quotient map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ is open by [38, Proposition 2.12]. In particular, if $\{x_i \blacktriangleleft \mathcal{G}\}_{i \in I}$ is a net converging to $x \blacktriangleleft \mathcal{G}$ in $\mathcal{X}/\mathcal{G}$, then Proposition A.1 says that we can find a subnet $\{x_{f(j)} \blacktriangleleft \mathcal{G}\}_{j \in J}$ which allows a convergent lift in $\mathcal{X}$, i.e., there exist $y_j \in x_{f(j)} \blacktriangleleft \mathcal{G}$ for all j with $y_j \to y$ for some $y \in x \blacktriangleleft \mathcal{G}$. Continuity of $\rho_{\mathcal{X}}$ then implies

$$\tilde{\rho}(x_{f(j)} \blacktriangleleft \mathcal{G}) = \rho_{\mathcal{X}}(y_j) \to \rho_{\mathcal{X}}(y) = \tilde{\rho}(x \blacktriangleleft \mathcal{G}).$$

Using Lemma A.2, we conclude that $\tilde{\rho}$ is continuous.

We next verify that $\circledtriangleright$ is well-defined. If $x \blacktriangleleft \mathcal{G} = y \blacktriangleleft \mathcal{G}$, there exists a unique $s \in \mathcal{G}$ such that $x = y \blacktriangleleft s$. Now by the commuting Condition (C1),

$$h \triangleright x = h \triangleright (y \blacktriangleleft s) = (h \triangleright y) \blacktriangleleft s.$$

Therefore, $(h \triangleright x) \blacktriangleleft \mathcal{G} = (h \triangleright y) \blacktriangleleft \mathcal{G}$. Similarly, to show that $\circledtriangleleft$ is well-defined, let x, y, s be as above, and let $h \in \mathcal{H}$ be such that $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$. By Condition (C3), we have

$$h \triangleleft x = h \triangleleft (y \blacktriangleleft s) = h \triangleleft y.$$

To see that this $\mathcal{H}$-action on $\mathcal{X}/\mathcal{G}$ is self-similar, we observe that the $\mathcal{H}$-action on $\mathcal{X}$ passes through the quotient and $(x \blacktriangleleft \mathcal{G})(y \blacktriangleleft \mathcal{G}) = (xy) \blacktriangleleft \mathcal{G}$ whenever $r_{\mathcal{X}}(y) = s_{\mathcal{X}}(x)$. Therefore, the Conditions (L2) through (L6) from the self-similar $\mathcal{H}$-action on $\mathcal{X}$ all pass through to the $\mathcal{H}$-action on $\mathcal{X}/\mathcal{G}$, proving that this $\mathcal{H}$-action on $\mathcal{X}/\mathcal{G}$ is also self-similar.

Lastly, we will check that $\textcircled{$\triangleright$}$ is continuous. So assume we are given $h_i \in \mathcal{H}$ such that $s_{\mathcal{H}}(h_j) = \rho_{\mathcal{X}}(x_i)$ and $h_i \to h$. By continuity of $\triangleright$ and $\triangleleft$, we have $h_{f(j)} \triangleright y_j \to h \triangleright y$ in $\mathcal{X}$ and $h_{f(j)} \triangleleft y_j \to h \triangleleft y$ in $\mathcal{H}$. Continuity of the quotient map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ then implies that

$$h_{f(j)} \textcircled{$\triangleright$} (x_{f(j)} \blacktriangleleft \mathcal{G}) = (h_{f(j)} \triangleright y_j) \blacktriangleleft \mathcal{G} \to (h \triangleright y) \blacktriangleleft \mathcal{G} = h \textcircled{$\triangleright$} (x \blacktriangleleft \mathcal{G}),$$

and likewise we have

$$h_{f(j)} \textcircled{$\triangleleft$} (x_{f(j)} \blacktriangleleft \mathcal{G}) = h_{f(j)} \triangleleft y_j \to h \triangleleft y = h \textcircled{$\triangleleft$} (x \blacktriangleleft \mathcal{G}).$$

Lemma A.2 again implies that $\textcircled{$\triangleright$}$ is continuous.

The claims for $\textcircled{$\blacktriangleleft$}$ and $\textcircled{$\blacktriangleright$}$ follow *mutatis mutandis*. □

Following Definitions 2.16 and 2.34, we obtain two groupoids, $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$. By Remark 2.19, the unit space of the self-similar product groupoid $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ is homeomorphic to the unit space of $\mathcal{X}/\mathcal{G}$. In other words, we have:

$$((\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H})^{(0)} \approx (\mathcal{X}^{(0)})/\mathcal{G} = \{u \blacktriangleleft \mathcal{G} : u \in \mathcal{X}^{(0)}\};$$
$$(\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X}))^{(0)} \approx \mathcal{H}\backslash(\mathcal{X}^{(0)}) = \{\mathcal{H} \triangleright u : u \in \mathcal{X}^{(0)}\}.$$

The following lemma computes the range and source maps explicitly for these two self-similar product groupoids. It follows immediately from Remark 2.19.

Lemma 3.13. *Consider $(\xi, h) \in (\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $(t, \eta) \in \mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$, and let $x \in \xi$ and $y \in \eta$ be arbitrary. We have*

(1) $r(\xi, h) = r_{\mathcal{X}/\mathcal{G}}(\xi) = r_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G}$
(2) $s(\xi, h) = h^{-1} \textcircled{$\triangleright$} s_{\mathcal{X}/\mathcal{G}}(\xi) = (h^{-1} \triangleright s_{\mathcal{X}}(x)) \blacktriangleleft \mathcal{G}$
(3) $r(t, \eta) = r_{\mathcal{H}\backslash\mathcal{X}}(\eta) \textcircled{$\blacktriangleleft$} t^{-1} = \mathcal{H} \triangleright (r_{\mathcal{X}}(y) \blacktriangleleft t^{-1})$
(4) $s(t, \eta) = s_{\mathcal{H}\backslash\mathcal{X}}(\eta) = \mathcal{H} \triangleright s_{\mathcal{X}}(y)$

We now define left and right actions of these groupoids on $\mathcal{X}$.

Proposition 3.14. *Let $\mathcal{X}$ be a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence. Define $\mathfrak{r} \colon \mathcal{X} \to [(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}]^{(0)}$ and $\mathfrak{s} \colon \mathcal{X} \to [\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})]^{(0)}$ by*

$$\mathfrak{r}(x) = r_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G} \quad \textit{and} \quad \mathfrak{s}(x) = \mathcal{H} \triangleright s_{\mathcal{X}}(x).$$

These are well-defined, surjective, continuous, open maps. Using them as momentum maps, we can define a left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$- and a right $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-action via:

$$((\xi, h), y) \mapsto (\xi, h) \cdot y := x(h \triangleright y), \qquad \text{where } x \in \xi \text{ is such that } (x, h \triangleright y) \in \mathcal{X}^{(2)};$$
$$(y, (t, \eta)) \mapsto y \cdot (t, \eta) := (y \blacktriangleleft t)z, \qquad \text{where } z \in \eta \text{ is such that } (y \blacktriangleleft t, z) \in \mathcal{X}^{(2)}.$$

Here, $x(h \triangleright y)$ and $(y \blacktriangleleft g)z$ denote composition in the groupoid $\mathcal{X}$. These actions are free and proper, and they commute.

Proof. We will do everything for the left-hand side; the claims for the right-hand side will follow *mutatis mutandis.*

First, notice that $\mathfrak{r}$ is clearly continuous and surjective since $r_{\mathcal{X}}$ is continuous and surjective. Furthermore, $\mathfrak{r}$ is open as a concatenation of open maps: $r_{\mathcal{X}}$ is open by assumption, and the quotient map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ is open by [38, Proposition 2.12] since $r_{\mathcal{G}}$ is open by assumption.

Next, we verify that the left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$-action is well-defined. Given a pair $((\xi, h), y)$ with $\mathfrak{r}(y) = s(\xi, h)$, it follows from the definition of $\mathfrak{r}$, from Lemma 3.13, and from (L10) that $r_{\mathcal{X}}(h \triangleright y) \blacktriangleleft \mathcal{G} = s_{\mathcal{X}/\mathcal{G}}(\xi)$, where $s_{\mathcal{X}/\mathcal{G}} \colon \mathcal{X}/\mathcal{G} \to (\mathcal{X}^{(0)})/\mathcal{G}$ is as in Equation (3.1). Therefore, there exists $x \in \xi$ such that $s_{\mathcal{X}}(x) = r_{\mathcal{X}}(h \triangleright y)$. Since the action on $\mathcal{X}$ is assumed to be free, we may invoke a $\blacktriangleleft$-version of Lemma 2.15 to conclude that such x must be unique. Therefore, the left action is well-defined.

We now verify that the left action is free. Pick any $y \in \mathcal{X}$ and $(\xi, h) \in (\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ such that $(\xi, h) \cdot y = y$, and let $x \in \xi$ satisfy $r_{\mathcal{X}}(h \triangleright y) = s_{\mathcal{X}}(x)$. By the definition of the left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$-action on $\mathcal{X}$, our assumption $(\xi, h) \cdot y = y$ implies $x(h \triangleright y) = y$. In particular,

$$s_{\mathcal{X}}(y) = s_{\mathcal{X}}(x(h \triangleright y)) = s_{\mathcal{X}}(h \triangleright y) = h \triangleright s_{\mathcal{X}}(y).$$

Since the $\mathcal{H}$-action on $\mathcal{X}$ is free, we have $h \in \mathcal{H}^{(0)}$ and thus $y = xy$. This only happens when $x = r_{\mathcal{X}}(y)$ and thus $(\xi, h) = (r_{\mathcal{X}}(y) \blacktriangleleft \mathcal{G}, h)$ is a unit in $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$.

To see that the left action is continuous, assume that we have nets $\{(\xi_i, h_i)\}_{i\in I}$ in $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\{y_i\}_{i\in I}$ in $\mathcal{X}$ which converge to (ξ, h) and y, respectively, and which satisfy

$$s(\xi_i, h_i) = \mathfrak{r}(y_i), \qquad \text{i.e.,} \qquad s_{\mathcal{X}/\mathcal{G}}(\xi_i) = r_{\mathcal{X}}(h_i \triangleright y_i) \blacktriangleleft \mathcal{G}.$$

If we let $x_i \in \xi_i$ and $x \in \xi$ be the unique elements such that

$$u_i := s_{\mathcal{X}}(x_i) = r_{\mathcal{X}}(h_i \triangleright y_i) \quad \text{and} \quad u := s_{\mathcal{X}}(x) = r_{\mathcal{X}}(h \triangleright y),$$

then by Lemma A.2, it suffices to find a subnet of $\{x_i(h_i \triangleright y_i)\}_{i\in I}$ that converges to $x(h \triangleright y)$. As $(h_i, y_i) \to (h, y)$, we only need to show that a subnet of $\{x_i\}_{i\in I}$ converges to x; furthermore, it gives us that $u_i \to u$. Since $\xi_i \to \xi$ and since $q\colon \mathcal{X} \to \mathcal{X}/\mathcal{G}$ is open, Proposition A.1 then implies that there exists a subnet $\{\xi_j\}_{j\in J}$ of $\{\xi_i\}_{i\in I}$ and lifts $z_i \in \xi_j$ such that $z_i \to x$. As $x_j \in \xi_j$ also, there exist $t_i \in \mathcal{G}$ such that $z_i = x_j \blacktriangleleft t_i$. In particular, by continuity of $s_{\mathcal{X}}$ and by (R10), we have $u_j \blacktriangleleft t_i = s_{\mathcal{X}}(z_i) \to s_{\mathcal{X}}(x) = u$. Since $u_j \to u$, we therefore have that

$$(u_j \blacktriangleleft t_i, u_j) \to (u, u).$$

As the right action of $\mathcal{G}$ on $\mathcal{X}$ is free and proper, it now follows from [38, Corollary 2.26] that $t_i \to \sigma_{\mathcal{X}}(u) = \sigma_{\mathcal{X}}(x)$ by definition of $\sigma_{\mathcal{X}}$. Thus, $x_j = z_i \blacktriangleleft t_i^{-1}$ converges to $x \blacktriangleleft \sigma_{\mathcal{X}}(x)^{-1} = x \blacktriangleleft \sigma_{\mathcal{X}}(x) = x$ by (R2).

To show that the left action is proper, suppose $y_i \to y$ and $(\xi_i, h_i) \cdot y_i \to z$ in $\mathcal{X}$; according to [38, Proposition 2.17], it suffices to show that $\{(\xi_i, h_i)\}_{i\in I}$ has a convergent subnet. As before, let $x_i \in \xi_i$ be the unique element such that $u_i := s_{\mathcal{X}}(x_i) = r_{\mathcal{X}}(h_i \triangleright y_i)$, so that $(\xi_i, h_i) \cdot y_i = x_i(h_i \triangleright y_i) \to z$.

We have $s_{\mathcal{X}}(y_i) \to s_{\mathcal{X}}(y)$ and

$$(h_i \triangleleft y_i) \triangleright s_{\mathcal{X}}(y_i) = s_{\mathcal{X}}(h_i \triangleright y_i) = s_{\mathcal{X}}((\xi_i, h_i) \cdot y_i) \to s_{\mathcal{X}}(z).$$

Since $\triangleright$ is proper, this implies that (a subnet of) $\{h_i \triangleleft y_i\}_{i\in I}$ converges in $\mathcal{H}$; let g be its limit. Note that

$$h_i \triangleright r_{\mathcal{X}}(y_i) = h_i \triangleright (y_i y_i^{-1}) = (h_i \triangleright y_i)\big[(h_i \triangleleft y_i) \triangleright y_i\big] \qquad \text{by (L4).}$$

If we multiply by $(h_i \triangleright y_i)^{-1}$ on the left, we therefore get

$$(h_i \triangleright y_i)^{-1}\left[h_i \triangleright r_{\mathcal{X}}(y_i)\right] = (h_i \triangleleft y_i) \triangleright y_i \to g \triangleright y. \tag{3.3}$$

Since $\mathcal{H}$ leaves $\mathcal{X}^{(0)}$ invariant (Lemma 2.14), we have

$$(h_i \triangleright y_i)^{-1}\big[h_i \triangleright r_{\mathcal{X}}(y_i)\big] = (h_i \triangleright y_i)^{-1},$$

and so it follows from (3.3) that $h_i \triangleright y_i \to (g \triangleright y)^{-1}$. Again, since $y_i \to y$, properness of $\triangleright$ now implies that (a subnet of) h_i converges in $\mathcal{H}$; let h be its limit. Thus

$$x_i = \big[x_i(h_i \triangleright y_i)\big](h_i \triangleright y_i)^{-1} \to z(h \triangleright y)^{-1}.$$

We have shown that (a subnet of) $\{(x_i, h_i)\}_{i\in I}$ converges, namely to $\big(z(h \triangleright y)^{-1}, h\big)$. We conclude that (a subnet of) $\{(\xi_i, h_i)\}_{i\in I}$ converges as well. This concludes our proof of properness.

We now want to verify that these two actions commute. Pick $(\xi, h) \in (\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$, $y \in \mathcal{X}$, and $(t, \eta) \in \mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ with matching range, source, and momentum maps, respectively. Let x be the unique element in ξ such that $s_{\mathcal{X}}(x) = r_{\mathcal{X}}(h \triangleright y)$. We want to argue that we can choose a particular representative of η. We compute

$$\begin{aligned}
\mathfrak{s}(x(h \triangleright y)) &= \mathcal{H} \triangleright s_{\mathcal{X}}(x(h \triangleright y)) && (\text{def'n of } \mathfrak{s}) \\
&= \mathcal{H} \triangleright s_{\mathcal{X}}(h \triangleright y) \\
&= \mathcal{H} \triangleright [(h \triangleleft y) \triangleright s_{\mathcal{X}}(y)] && (\text{by (L10)}) \\
&= \mathcal{H} \triangleright s_{\mathcal{X}}(y) = r(t, \eta) && (\text{def'n of } \mathcal{G} \bowtie (\mathcal{H} \backslash \mathcal{X})).
\end{aligned}$$

This shows that (t, η) can act on the right of $x(h \triangleright y)$. Our previous explanation now implies that there exists a unique representative $z \in \eta$ which has range equal to $s_{\mathcal{X}}([x(h \triangleright y)] \blacktriangleleft t)$. This choice of x and z makes the following computation particularly easy:

$$\begin{aligned}
\big[(\xi, h) \cdot y\big] \cdot (t, \eta) = [x(h \triangleright y)] \cdot (t, \eta) &= \big[\big(x(h \triangleright y)\big) \blacktriangleleft t\big] z \\
&= \big[x \blacktriangleleft \big((h \triangleright y) \blacktriangleright t\big)\big]((h \triangleright y) \blacktriangleleft t) z && (\text{by (R4)}) \\
&= \big[x \blacktriangleleft \big(y \blacktriangleright t\big)\big](h \triangleright (y \blacktriangleleft t)) z && (\text{by (C3) and (C1)}).
\end{aligned}$$

On the other hand, let $z' \in \eta$ satisfy $r_{\mathcal{X}}(z') = s_{\mathcal{X}}(y \blacktriangleleft t)$, and let $x' \in \xi$ be the unique element such that $s_{\mathcal{X}}(x') = r_{\mathcal{X}}(h \triangleright ((y \blacktriangleleft t) z'))$. Then

$$\begin{aligned}
(\xi, h) \cdot \big[y \cdot (t, \eta)\big] &= (\xi, h) \cdot [(y \blacktriangleleft t) z'] && (\text{choice of } z') \\
&= x' \big[h \triangleright ((y \blacktriangleleft t) z')\big] && (\text{choice of } x') \\
&= x' [h \triangleright y \blacktriangleleft t] \big[(h \triangleleft (y \blacktriangleleft t)) \triangleright z'\big] && (\text{by (L4)}) \\
&= x' [h \triangleright y \blacktriangleleft t] \big[(h \triangleleft y) \triangleright z'\big] && (\text{by (C3)}).
\end{aligned}$$

Thus, to prove that $\big[(\xi, h) \cdot y\big] \cdot (t, \eta) = (\xi, h) \cdot \big[y \cdot (t, \eta)\big]$, it suffices to show that

$$x \blacktriangleleft \big(y \blacktriangleright t\big) = x' \quad \text{and} \quad z = (h \triangleleft y) \triangleright z'.$$

For the right equation, we compute the range of the right-hand side as

$$\begin{aligned}
r_{\mathcal{X}}\big((h \triangleleft y) \triangleright z'\big) &= (h \triangleleft y) \triangleright r_{\mathcal{X}}(z') && (\text{by (L10)}) \\
&= (h \triangleleft y) \triangleright s_{\mathcal{X}}(y \blacktriangleleft t) && (\text{choice of } z') \\
&= (h \triangleleft y) \triangleright [s_{\mathcal{X}}(y) \blacktriangleleft t] && (\text{by (R10)}) \\
&= s_{\mathcal{X}}(h \triangleright y) \blacktriangleleft t && (\text{by (L10)}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
r_{\mathcal{X}}(z) &= s_{\mathcal{X}}\big([x(h \triangleright y)] \blacktriangleleft t\big) && (\text{choice of } z) \\
&= s_{\mathcal{X}}(x(h \triangleright y)) \blacktriangleleft t && (\text{by (L10)}) \\
&= s_{\mathcal{X}}(h \triangleright y) \blacktriangleleft t.
\end{aligned}$$

Both combined yield

$$r_{\mathcal{X}}\big((h \triangleright y \blacktriangleleft t) \triangleright z'\big) = r_{\mathcal{X}}(z).$$

Since $\mathcal{H} \triangleright z = \mathcal{H} \triangleright z' = H \triangleright \big((h \triangleleft y) \triangleright z'\big)$ and since the actions are free, it follows from Lemma 2.15 that $z = (h \triangleleft y) \triangleright z'$. A similar argument shows that $x' = x \blacktriangleleft (y \blacktriangleright t)$. We proved that the left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$- and the right $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-actions on $\mathcal{X}$ commute. $\square$

We now prove the first main result (Theorem 3.10) which states that $\mathcal{X}$ is a $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H} - \mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-equivalence.

Proof of Theorem 3.10. According to Proposition 3.14, we have commuting free and proper left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$- and right $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-actions on $\mathcal{X}$. It remains to show that $\mathfrak{r}$ induces a homeomorphism $\tilde{\mathfrak{r}}$ between $\mathcal{X}/(\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X}))$ and $((\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H})^{(0)}$; a similar proof will then show that $\mathfrak{s}$ induces an analogous homeomorphism.

Fix $y \in \mathcal{X}$ and consider any $(t, \eta) \in \mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ with $\mathfrak{s}(y) = r(t, \eta)$. Let $z \in \eta$ be the unique element such that $r_{\mathcal{X}}(z) = s_{\mathcal{X}}(y \blacktriangleleft t)$, so that by definition of the right-$\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-action, $y \cdot (s, \eta) = (y \blacktriangleleft t)z$. Consider its range in $\mathcal{X}$:

$$\begin{aligned} r_{\mathcal{X}}(y \cdot (t, \eta)) &= r_{\mathcal{X}}((y \blacktriangleleft t)z) = r_{\mathcal{X}}(y \blacktriangleleft t) \\ &= r_{\mathcal{X}}(y) \blacktriangleleft (y \blacktriangleright t) \in r_{\mathcal{X}}(y) \blacktriangleleft G. \end{aligned}$$

Therefore, if we write $\overline{y}$ for the equivalence class of y in $\mathcal{X}/(\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X}))$, then $\tilde{\mathfrak{r}}(\overline{y}) = r_{\mathcal{X}}(y) \blacktriangleleft \mathcal{G}$ is well-defined. Surjectivity, continuity, and openness of $\tilde{\mathfrak{r}}$ is trivial, since $\mathfrak{r}$ is surjective, continuous, and open. To see that $\tilde{\mathfrak{r}}$ is injective, take any y, y' with $r_{\mathcal{X}}(y) \blacktriangleleft \mathcal{G} = r_{\mathcal{X}}(y') \blacktriangleleft \mathcal{G}$; we need to find $t \in \mathcal{G}$ and $\eta \in \mathcal{H}\backslash\mathcal{X}$ such that $y \cdot (s, \eta) = y'$. By assumption, there exists $s \in \mathcal{G}$ such that $r_{\mathcal{X}}(y') = r_{\mathcal{X}}(y) \blacktriangleleft s$. Set $t = y^{-1} \blacktriangleright s$ Then $s = y \blacktriangleright t$ and thus by (R10),

$$r_{\mathcal{X}}(y') = r_{\mathcal{X}}(y) \blacktriangleleft (y \blacktriangleright t) = r_{\mathcal{X}}(y \blacktriangleleft t).$$

Since y' and $y \blacktriangleleft t$ have the same range in $\mathcal{X}$, we may let $x = (y \blacktriangleleft t)^{-1} y' \in \mathcal{X}$, so that $y' = (y \blacktriangleleft t)x$, i.e., $y' = y \cdot (t, \mathcal{H} \triangleright x)$. $\square$

Remark 3.15. Let us briefly recap which topological assumption in Theorem 3.10 was needed for which part of the proof. We required the source map of $\mathcal{H}$ to be open in order for the quotient map $q \colon \mathcal{X} \to \mathcal{H}\backslash\mathcal{H}$ to be open which, in turn, we used to show that the momentum map $\tilde{\sigma}$ of the $\mathcal{G}$-action on $\mathcal{H}\backslash\mathcal{X}$ is continuous (see proof of Proposition 3.12). Freeness of the $\mathcal{H}$-action on $\mathcal{X}$ allowed us to turn $\mathcal{H}\backslash\mathcal{X}$ into a groupoid (Lemma 3.2), and its properness plus openness of q was needed to make $\mathcal{H}\backslash\mathcal{X}$ a locally compact Hausdorff groupoid (Proposition 3.3). Lastly, the source map of $\mathcal{X}$ was required to be open in order to prove that the momentum map $\mathfrak{s}$ of the right $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$-action on $\mathcal{X}$

(Proposition 3.14) is open and can therefore induce a homeomorphism of the quotient by the right $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$-action onto the unit space of $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$.

Corollary 3.16 *(cf. [38, Proposition 2.47]). Suppose $\mathcal{H}$ and $\mathcal{X}$ are groupoids and that $\mathcal{H}$ has a self-similar left action on $\mathcal{X}$ that is free and proper. If $s_{\mathcal{X}}$ and $s_{\mathcal{H}}$ are open maps, then the groupoids $\mathcal{X} \bowtie \mathcal{H}$ and $\mathcal{H}\backslash\mathcal{X}$ are equivalent.*

Proof. We have seen in Example 3.8 that $\mathcal{X}$ is a $(\mathcal{H}, \{e\})$-self-similar para-equivalence (modulo switching the roles of $\mathcal{H}$ and $\mathcal{G}$). Theorem 3.10 thus implies that $(\mathcal{X}/\{e\}) \bowtie \mathcal{H}$ and $\{e\} \bowtie (\mathcal{H}\backslash\mathcal{X})$ are equivalent groupoids. By Examples 2.21 and 3.4, we have $\{e\} \bowtie (\mathcal{H}\backslash\mathcal{X}) \cong \mathcal{H}\backslash\mathcal{X}$ and $(\mathcal{X}/\{e\}) \bowtie \mathcal{H} \cong \mathcal{X} \bowtie \mathcal{H}$, respectively. The claim now follows. □

3.2. Applications of Theorem 3.10

Example 3.17 *(continuation of Examples 2.8 and 2.13).* Suppose again that a locally compact Hausdorff group $K = G \bowtie H$ acts on the left on a locally compact Hausdorff space X, denoted by $*$. We let $\mathcal{X} = G \ltimes X$ be the transformation groupoid, and we define the self-similar left action $\triangleright$ and $\triangleleft$ of H on $\mathcal{X}$ as in (2.2). We assume that $*$ is free and proper, so that $\triangleright$ and $\triangleleft$ are free and proper by our computations in Example 2.13. Thus, by Corollary 3.16, we get that $\mathcal{X} \bowtie H$ is equivalent to $H\backslash\mathcal{X}$. (Here, the assumption that the source maps are open is trivially satisfied: the source map of H is constant and the source map of X is the identity map.)

Note that the map

$$\phi\colon (G \ltimes X) \bowtie H \to (G \bowtie H) \ltimes X, \quad ((t, x), h) \mapsto ((t, h), h^{-1} * x),$$

is a groupoid isomorphism $\mathcal{X} \bowtie H \cong K \ltimes X$. Indeed, using the definition of $\bowtie$ in $\mathcal{X} \bowtie H$, we compute the product of two elements of the domain to be

$$\big((t, x), h\big)\,\big((s, y), k\big) = \big((t, x)[h \triangleright (s, y)], [h \triangleleft (s, y)]k\big) = \big((t, x)(h \cdot s, h|_s * y), h|_s k\big).$$

Of the tuple on the far right-hand side, the first component is a product in $\mathcal{X}$; it is defined if and only if the source of (t, x) equals the range of $(h \cdot s, h|_s * y)$. In other words, we must have $x = (h \cdot s) * [h|_s * y] = [(h \cdot s)(h|_s)] * y = [hs] * y$, in which case their product is $(t[h \cdot s], h|_s * y)$. Therefore, the composition in $\mathcal{X} \bowtie H$ can be described succinctly as follows:

$$\big((t, [hs] * y), h\big)\,\big((s, y), k\big) = \big((t[h \cdot s], h|_s * y), h|_s k\big).$$

Applying ϕ, we end up with

$$\phi\Big(\big((t, [hs]{*}y), h\big)\,\big((s, y), k\big)\Big) = \big((t[h{\cdot}s], h|_s k), [h|_s k]^{-1}{*}[h|_s{*}y]\big) = \big((t[h{\cdot}s], h|_s k), k^{-1}{*}y\big).$$

On the other hand, the product of $\phi((t,h),x)$ with $\phi((s,k),y)$ in the codomain $K \ltimes X$ is defined if and only if the source $h^{-1} * x$ of $((t,h),h^{-1}x)$ equals the range $(s,k)*(k^{-1}*y)$ of $((s,k),k^{-1}*y)$. In other words, we get the same necessary condition for composability as above, namely that $x = [hs] * y$, in which case

$$\phi\big((t,[hs]*y),h\big)\,\phi\big((s,y),k\big) = \big((t,h)(s,k),k^{-1}*y\big).$$

In K, we have $(t,h)(s,k) = (t[h\cdot s], h|_s k)$, which shows that indeed

$$\phi\big((t,x),h\big)\,\big((s,y),k\big) = \phi\big((t,x),h\big)\,\phi\big((s,y),k\big).$$

The setup in Example 3.17 arises abundantly in group dynamics.

Example 3.18 *(First special case of Example 3.17).* In the above example, suppose that $G = \{e\}$, so $K = H$ and $\mathcal{X} = X$ is a trivial groupoid (i.e., a space). The action $\triangleleft$ is now trivial and the action $\triangleright$ is exactly the action $*$ of K on X that we started with. If $*$ is free and proper, Example 3.17 shows that the transformation groupoid $K \ltimes X$ is equivalent to $K\backslash X$. Note that the trivial groupoid $K\backslash X$ always admits a Haar system (see, for example, [38, Example 1.22]). Assuming that the two groupoids are second countable, $K \ltimes X$ therefore also admits a Haar system by [37, Theorem 2.1]. We may now apply [22, Theorem 2.8], which states that the C*-algebras of equivalent groupoids with Haar systems are Morita equivalent. In other words, we exhibit the known result that the crossed product $C_0(X) \rtimes K$ is Morita equivalent to $C_0(K\backslash X)$.

The following is a concrete example using a finite group K.

Example 3.19 *(Second special case of Example 3.17).* Consider the symmetric group S_4, which is a group of order 24, and the elements

$$a = (1 \quad 2 \quad 3) \quad \text{and} \quad r = (1 \quad 2 \quad 3 \quad 4)\,, f = (1 \quad 3)\,.$$

Let $G = \langle a\rangle$ and $H = \langle r, f\rangle$; one can verify that G and H are of order 3 and 8 respectively, that neither subgroup is normal, and that $G \cong C_3$ and $H \cong D_4$.

Since $|S_4| = |G|\cdot|H|$ and $|G\cap H| = 1$, we must have $S_4 = G\cdot H$, i.e., each element in S_4 is a unique product of the form th for $t \in G$ and $h \in H$. In other words, $S_4 = K$ is the internal Zappa–Szép product of G and H, and in particular, we get Zappa–Szép actions $G \circlearrowright H$ in such a way that any product ht of $h \in H$ and $t \in G$ in S_4 can be uniquely decomposed as

$$ht = (h\cdot t)(h|_t)$$

where $h \cdot t \in G$ and $h|_t \in H$. Tables 1 and 2 contains an overview of these actions.

Table 1
Action map $h \cdot t$ on S_4.

h \ t	e	a	a^2
e	e	a	a^2
r	e	a^2	a
r^2	e	a	a^2
r^3	e	a^2	a
f	e	a^2	a
rf	e	a	a^2
r^2f	e	a^2	a
r^3f	e	a	a^2

Table 2
Restriction map $h|_t$ on S_4.

h \ t	e	a	a^2
e	e	e	e
r	r	r^2f	r^3
r^2	r^2	rf	r^3f
r^3	r^3	r	r^2f
f	f	f	f
rf	rf	r^3f	r^2
r^2f	r^2f	r^3	r
r^3f	r^3f	r^2	rf

Now let $X = S_4$ and we let S_4 act on X by left translation, so that $K \ltimes X = S_4 {}_{\mathrm{lt}}\!\ltimes S_4$. One can explicitly write out all the orbits in $H\backslash(G \ltimes X)$, and verify that the nine elements in $G \ltimes G \subseteq G \ltimes X$ are in different H-orbits. Since $|H\backslash(G \ltimes X)| = 9$, we have $H\backslash(G \ltimes X) \cong G \ltimes G$. By Example 3.17, we conclude that the groupoids $S_4 {}_{\mathrm{lt}}\!\ltimes S_4$ and $G \ltimes G$ are equivalent. By the Stone–von Neumann Theorem, their groupoid C*-algebras are given by $\mathcal{K}(\ell^2(S_4)) \cong M_{24}(\mathbb{C})$ and $\mathcal{K}(\ell^2(G)) \cong M_3(\mathbb{C})$. Consequently, these C*-algebras are Morita equivalent.

Example 3.20 *(continuation of Example 2.24).* Suppose again that $\mathbf{c}\colon \mathcal{G} \to H$ is a continuous homomorphism from a groupoid to a group. In Example 2.24, we described a self-similar left action of H on the skew-product groupoid $\mathcal{G}(\mathbf{c})$. This action is free and proper. Note that $s_{\mathcal{G}}$ is open if and only if $s_{\mathcal{G}(\mathbf{c})}$ is open, in which case it follows from Corollary 3.16 that $\mathcal{G}(\mathbf{c}) \bowtie H$ is equivalent to $H\backslash\mathcal{G}(\mathbf{c}) \cong \mathcal{G}$.

3.3. Haar systems on quotients

To construct a right Haar systems on $\mathcal{X}/\mathcal{G}$ out of a right Haar system on $\mathcal{X}$, we again require $\blacktriangleleft$-invariance.

Lemma 3.21 *(cf. [17, Prop. A.10]). Suppose $\mathcal{G}$ and $\mathcal{X}$ are locally compact Hausdorff groupoids, that $\mathcal{G}$ has a free and proper self-similar right action on $\mathcal{X}$, and that $\mathcal{X}$ has a $\blacktriangleleft$-invariant right Haar system $\{\lambda_u\}_{u\in\mathcal{X}^{(0)}}$ (Definition 2.35). Then there exists a right Haar system $\{\kappa_{u\blacktriangleleft\mathcal{G}}\}_{u\blacktriangleleft\mathcal{G}}$ on $\mathcal{X}/\mathcal{G}$ given for any $\widehat{f} \in C_c(\mathcal{X}/\mathcal{G})$ by*

$$\int \widehat{f}(x \blacktriangleleft G) \, \mathrm{d}\kappa_{u \blacktriangleleft G}(x \blacktriangleleft G) = \int_{\mathcal{X}} \widehat{f}(x \blacktriangleleft G) \, \mathrm{d}\lambda_u(x).$$

Proof. The argument is verbatim as in the proof of [17, Prop. A.10], only that the range map of $\mathcal{X}$ has to be replaced by its source map. To be precise, we will invoke [29, Lemma 1.3] for $(X, Y, G, \pi) = (\mathcal{X}, \mathcal{X}^{(0)}, \mathcal{G}, s_{\mathcal{X}})$. Since we assumed $\blacktriangleleft$ to be free and proper, $\mathcal{X}$ is a principal $\mathcal{G}$-space. Since $\mathcal{X}$ is assumed to have a Haar system, its continuous source map $s_{\mathcal{X}}$ is open [38, Prop. 1.23]. It is furthermore equivariant by (R10), so that we may apply [29, Lemma 1.3]. The given formula for κ is hence a system for the map $\mathcal{X}/\mathcal{G} \to \mathcal{X}^{(0)}/\mathcal{G}$, $x \blacktriangleleft \mathcal{G} \mapsto s_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G}$, which is the source map of the groupoid $\mathcal{X}/\mathcal{G}$. In other words, κ is a right Haar system for $\mathcal{X}/\mathcal{G}$. □

Lemma 3.22. *Suppose $\mathcal{X}, \mathcal{H}, \mathcal{G}$ are locally compact Hausdorff groupoids and that $\mathcal{X}$ has a left $\mathcal{H}$-action $\triangleright$ and a free and proper right $\mathcal{G}$-action $\blacktriangleleft$. Assume $\{\lambda_u\}_{u \in \mathcal{X}^{(0)}}$ is a $\blacktriangleleft$-invariant right Haar system on $\mathcal{X}$ (Definition 2.35), and let $\{\kappa_{u \blacktriangleleft \mathcal{G}}\}_{u \blacktriangleleft \mathcal{G}}$ be the induced right Haar system on $\mathcal{X}/\mathcal{G}$ (Lemma 3.21). If the left Haar system $\{\lambda^u\}_{u \in \mathcal{X}^{(0)}}$ on $\mathcal{X}$ defined by $\lambda^u(E) = \lambda_u(E^{-1})$ is $\triangleright$-invariant (Definition 2.27), then the left Haar system $\{\kappa^{u \blacktriangleleft \mathcal{G}}\}_{u \blacktriangleleft \mathcal{G}}$ on $\mathcal{X}/\mathcal{G}$ associated to $\{\kappa_{u \blacktriangleleft \mathcal{G}}\}_{u \blacktriangleleft \mathcal{G}}$ is $\circledtriangleright$-invariant.*

Proof. The computation is straightforward: on the one hand,

$$\begin{aligned}
\kappa^{u \blacktriangleleft \mathcal{G}}(h^{-1} \circledtriangleright [E \blacktriangleleft \mathcal{G}]) &= \kappa^{u \blacktriangleleft \mathcal{G}}([h^{-1} \triangleright E] \blacktriangleleft \mathcal{G}) && (\text{def'n of } \circledtriangleright) \\
&= \kappa_{u \blacktriangleleft \mathcal{G}}\big([h^{-1} \triangleright E]^{-1} \blacktriangleleft \mathcal{G}\big) && (\text{def'n of } \kappa^{u \blacktriangleleft \mathcal{G}} \text{ and of } {}^{-1} \text{ on } \mathcal{X}/\mathcal{G}) \\
&= \lambda_u([h^{-1} \triangleright E]^{-1}) && (\text{def'n of } \kappa_{u \blacktriangleleft \mathcal{G}}) \\
&= \lambda^u(h^{-1} \triangleright E) && (\text{def'n of } \lambda^u) \\
&= \lambda^{h \triangleright u}(E) && (\triangleright\text{-invariance of } \lambda^u).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\kappa^{h \circledtriangleright [u \blacktriangleleft \mathcal{G}]}(E \blacktriangleleft \mathcal{G}) &= \kappa_{h \circledtriangleright [u \blacktriangleleft \mathcal{G}]}([E \blacktriangleleft \mathcal{G}]^{-1}) && (\text{def'n of } \kappa^{h \circledtriangleright [u \blacktriangleleft \mathcal{G}]}) \\
&= \kappa_{[h \circledtriangleright u] \blacktriangleleft \mathcal{G}}(E^{-1} \blacktriangleleft \mathcal{G}) && (\text{def'n of } \circledtriangleright \text{ and of } {}^{-1} \text{ on } \mathcal{X}/\mathcal{G}) \\
&= \lambda_{h \triangleright u}(E^{-1}) && (\text{def'n of } \kappa_{[h \circledtriangleright u] \blacktriangleleft \mathcal{G}}) \\
&= \lambda^{h \triangleright u}(E) && (\text{def'n of } \lambda^{h \triangleright u}).
\end{aligned}$$

This shows that $\kappa^{u \blacktriangleleft \mathcal{G}}(h^{-1} \circledtriangleright [E \blacktriangleleft \mathcal{G}]) = \kappa^{h \circledtriangleright [u \blacktriangleleft \mathcal{G}]}(E \blacktriangleleft \mathcal{G})$. □

Corollary 3.23. *Suppose $\mathcal{G}, \mathcal{H}, \mathcal{X}$ are as in Theorem 3.10. Assume that $\mathcal{X}$ has a $\triangleright$-invariant left Haar system (Definition 2.27) whose associated right Haar system is $\blacktriangleleft$-invariant. If $\mathcal{H}$ and $\mathcal{G}$ have Haar systems, then so do $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$, and so their C^*-algebras $C^*((\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H})$ and $C^*(\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X}))$ are Morita equivalent.*

In Corollary 6.8, we will generalize the above result to Fell bundle C*-algebras.

Proof. Since $\mathcal{G}$ acts properly and freely on $\mathcal{X}$, it follows from Lemma 3.21 that the right Haar system on $\mathcal{X}$ induces a right Haar system on $\mathcal{X}/\mathcal{G}$. By Lemma 3.22, the associated left Haar system is ⓑ-invariant. It follows from Proposition 2.28 that $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ has a Haar system. Since our assumptions are symmetric, we likewise get a Haar system on $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$. As the two groupoids are equivalent by Theorem 3.10 and have Haar systems, it follows from [22, Theorem 2.8] that their C*-algebras are Morita equivalent. □

4. Self-similar actions on Fell bundles

Fell bundles were originally introduced by Fell as "C*-algebraic bundles" [10]; they are a powerful tool to study C*-algebras graded by groups or groupoids, and many C*-algebras can be realized as Fell bundle C*-algebras. One may refer to [4,8,19,40] for a more detailed discussion on the subject.

4.1. Self-similar left actions on Fell bundles

We will now extend the notion of self-similar actions to Fell bundles. Similar to the construction of a Zappa–Szép product Fell bundle in [6], this will allow us to construct a self-similar product Fell bundle.

Definition 4.1. Let $\mathscr{B} = (q_{\mathscr{B}} \colon B \to \mathcal{X})$ be a Fell bundle. Suppose $\mathcal{H}$ has a left self-similar action on $\mathcal{X}$ with momentum map $\rho_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{H}^{(0)}$. Define $\rho_{\mathscr{B}} = \rho_{\mathcal{X}} \circ q_{\mathscr{B}}$ and let

$$\mathcal{H}\,{}_{s}{*}_{\rho}\,\mathscr{B} = \{(h, b) \in \mathcal{H} \times \mathscr{B} : s_{\mathcal{H}}(h) = \rho_{\mathscr{B}}(b)\}$$

be equipped with the subspace topology of $\mathcal{H} \times \mathscr{B}$. A *left self-similar $\mathcal{H}$-action on $\mathscr{B}$* is a continuous map

$$\textvisiblespace \vDash \textvisiblespace \colon \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathscr{B} \to \mathscr{B}$$

satisfying the following conditions:

(B1) For any $(h, x) \in \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathcal{X}$, the map $h \vDash \textvisiblespace$ maps $\mathscr{B}_x$ into $\mathscr{B}_{h \triangleright x}$ and is linear.
(B2) For any $(k, h) \in \mathcal{H}^{(2)}$, we have $k \vDash (h \vDash \textvisiblespace) = (kh) \vDash \textvisiblespace$.
(B3) For any $u \in \mathcal{H}^{(0)}$, the map $u \vDash \textvisiblespace$ is the identity.
(B4) For any $(b, c) \in \mathscr{B}^{(2)}$ such that $(h, bc) \in \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathscr{B}$, we have

$$h \vDash (bc) = (h \vDash b)\,[(h \triangleleft q_{\mathscr{B}}(b)) \vDash c]\,.$$

(B5) For any $(h, b) \in \mathcal{H}\,{}_{s}{*}_{\rho}\,\mathscr{B}$, we have

$$(h \vDash b)^* = [h \triangleleft q_{\mathscr{B}}(b)] \vDash b^*.$$

Writing $h \blacktriangleleft b := h \triangleleft q_{\mathscr{B}}(b) \in \mathcal{H}$ for $(h, b) \in \mathcal{H}\,{}_s{*}_\rho\,\mathscr{B}$ highlights the similarities between the above definition and Definition 2.2; compare (L4) and (L9) on the left to (B4) and (B5) on the right:

$$h \triangleright (xy) = (h \triangleright x)[(h \triangleleft x) \triangleright y] \qquad \text{versus} \qquad h \blacktriangleright (bc) = (h \blacktriangleright b)\,[(h \blacktriangleleft b) \blacktriangleright c]\,,$$

$$(h \triangleright x)^{-1} = (h \triangleleft x) \triangleright x^{-1} \qquad \text{versus} \qquad (h \blacktriangleright b)^* = [h \blacktriangleleft b] \blacktriangleright b^*.$$

Remark 4.2. When $\mathcal{X}$ and $\mathcal{H}$ form a matched pair of groupoids, Definition 4.1 coincides with the notion of a $(\mathcal{X}, \mathcal{H})$-compatible $\mathcal{H}$-action [6, Definition 3.1].

In general, we saw in Proposition 2.23 that the self-similar product groupoid $\mathcal{X} \bowtie \mathcal{H}$ is isomorphic to the Zappa–Szép product groupoid $\mathcal{X} \bowtie \widetilde{\mathcal{H}}$. The next proposition proves that a similar result holds in the realm of Fell bundles.

Proposition 4.3. *Suppose $\mathcal{H}$ has a self-similar left action $\blacktriangleright$ on a Fell bundle $\mathscr{B} = (q_{\mathscr{B}} \colon B \to \mathcal{X})$ and write $r_{\mathscr{B}} = r_{\mathcal{X}} \circ q_{\mathscr{B}}$. Let $\widetilde{\mathcal{H}} = \{(u, h) \in \mathcal{X}^{(0)} \times \mathcal{H} : \rho_{\mathcal{X}}(u) = r_{\mathcal{H}}(h)\}$ be the transformation groupoid of the $\mathcal{H}$-action on $\mathcal{X}^{(0)}$ with source map given by $s_{\widetilde{\mathcal{H}}}(u, h) = h^{-1} \triangleright u$. Let*

$$\beta \colon \widetilde{\mathcal{H}}\,{}_s{*}_r\,\mathscr{B} \to \mathscr{B} \quad \text{be defined by} \quad \beta((u, h), b) = h \blacktriangleright b.$$

Then β is a $(\mathcal{X}, \widetilde{\mathcal{H}})$-compatible $\widetilde{\mathcal{H}}$-action on $\mathscr{B}$ in the sense of [6, Definition 3.1].

Proof. To see that β is well-defined, take $(u, h) \in \widetilde{\mathcal{H}}$ and $b \in \mathscr{B}_x$ with $s_{\widetilde{\mathcal{H}}}(u, h) = r_{\mathcal{X}}(x)$. Since $s_{\widetilde{\mathcal{H}}}(u, h) = h^{-1} \triangleright u$, we have

$$\rho_{\mathcal{X}}(x) = \rho_{\mathcal{X}}^{(0)}(r_{\mathcal{X}}(x)) = \rho_{\mathcal{X}}^{(0)}(h^{-1} \triangleright u) \overset{\text{(L1)}}{=} r_{\mathcal{H}}(h^{-1}) = s_{\mathcal{H}}(h).$$

Therefore, $(h, b) \in \mathcal{H}\,{}_s{*}_\rho\,\mathscr{B}$ and β is well-defined. It is routine to check that β is indeed an $(\mathcal{X}, \widetilde{\mathcal{H}})$-compatible $\widetilde{\mathcal{H}}$-action on $\mathscr{B}$. $\square$

One immediate consequence is that, fiberwise, $\blacktriangleright$ shares all the nice properties of an $(\mathcal{X}, \mathcal{H})$-compatible action. For example, the following is a consequence of [6, Corollary 3.3]:

Corollary 4.4. *For each $h \in \mathcal{H}$ and $x \in \mathcal{X}$ with $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(x)$, the map $h \blacktriangleright \_ \colon \mathscr{B}_x \to \mathscr{B}_{h \triangleright x}$ is isometric.*

4.2. The self-similar product Fell bundle

The Zappa–Szép product Fell bundle was first defined in [6, Theorem 3.8] under the assumption that

(1) the underlying groupoids $\mathcal{X}$ and $\mathcal{H}$ form a matched pair and
(2) the underlying groupoids are étale.

Inspired by the construction of semi-crossed product Fell bundles in [16], we now define a similar construction with these two assumptions removed. To be precise, we aim to define the product Fell bundle from a self-similar $\mathcal{H}$-action on a Fell bundle $\mathscr{B}$, where the underlying groupoids are locally compact Hausdorff.

Definition 4.5. Suppose $\mathcal{H}$ has a self-similar left action $\triangleright$ on a Fell bundle $\mathscr{B} = (q_{\mathscr{B}}\colon B \to \mathcal{X})$ (Definition 4.1). Define the *(left) self-similar product Fell bundle $\mathscr{B} \bowtie \mathcal{H}$* to have the total space

$$B \bowtie \mathcal{H} = B\ {}_{\rho_{\mathcal{X}} \circ s_{\mathcal{X}} \circ q_{\mathscr{B}}}{*}_{r_{\mathcal{H}}}\ \mathcal{H} = \{(b,h) \in B \times \mathcal{H} : (q_{\mathscr{B}}(b), h) \in \mathcal{X} \bowtie \mathcal{H}\}$$

with bundle projection $q_{\mathscr{B}\bowtie\mathcal{H}}(b,h) = (q_{\mathscr{B}}(b), h)$, mapping $B \bowtie \mathcal{H}$ to $\mathcal{X} \bowtie \mathcal{H}$. The fiber

$$(\mathscr{B} \bowtie \mathcal{H})_{(x,h)} = \{(b,h) \in B \bowtie \mathcal{H} : q_{\mathscr{B}}(b) = x\}$$

is equipped with the norm $\|(b,h)\| = \|b\|$.

As always, let

$$(\mathscr{B} \bowtie \mathcal{H})^{(2)} := (B \bowtie \mathcal{H})\ {}_{s_{\mathscr{B}\bowtie\mathcal{H}}}{*}_{r_{\mathscr{B}\bowtie\mathcal{H}}}\ (B \bowtie \mathcal{H}),$$

and define multiplication and involution by

$$(a,h)(b,k) = \big(a[h \triangleright b], [h \triangleleft b]k\big) \quad \text{and} \quad (b,h)^* = (h^{-1} \triangleright b^*, h^{-1} \triangleleft b^*).$$

We note that the proof that $\mathscr{B} \bowtie \mathcal{H}$ is a Fell bundle over $\mathcal{X} \bowtie \mathcal{H}$ follows *mutatis mutandis* as in the proof in [6, Section 3].

For the first example, we will need a bit of notation.

Notation 4.6. Let $\mathscr{A} = (q_{\mathscr{A}}\colon A \to \mathcal{K}^{(0)})$ be an upper semi-continuous C*-bundle over the unit space $\mathcal{K}^{(0)}$ of a groupoid $\mathcal{K}$, and let $(\mathscr{A}, \mathcal{K}, \alpha)$ be a groupoid dynamical system (see [24, Definition 4.1] or [11, Chapter 3] for more details). We let $\mathscr{B}(\mathscr{A}, \mathcal{K}, \alpha)$ denote the Fell bundle associated to this dynamical system: as a set, it is given by $\mathscr{A}\ {}_{q}{*}_{r}\mathcal{K}$ with bundle projection $q_{\mathscr{B}}(a,k) = k$. The involution is given by $(a,k)^* := \big(\alpha_{k^{-1}}(a)^*, k^{-1}\big)$, and the product of two elements $(a_i, k_i) \in \mathscr{B}(\mathscr{A}, \mathcal{K}, \alpha)$ with $(k_1, k_2) \in \mathcal{K}^{(2)}$ is given by

$$(a_1, k_1) \cdot (a_2, k_2) := (a_1 \alpha_{k_1}(a_2), k_1 k_2)\,.$$

The C*-algebra of this Fell bundle is exactly the groupoid crossed product $\mathscr{A} \rtimes_\alpha \mathcal{K}$ [23, Example 2.8].

Example 4.7 *(generalization of [6, Example 3.10]).* Suppose $\mathcal{H}$ has a self-similar left action on a groupoid $\mathcal{X}$, and suppose that $(\mathscr{A}, \mathcal{X} \bowtie \mathcal{H}, \alpha)$ is a groupoid dynamical system. Let $\alpha|_{\mathcal{X}}$ be the restriction of α to the subgroupoid $\mathcal{X}$, i.e., $(\alpha|_{\mathcal{X}})_x := \alpha_{(x,\rho_{\mathcal{X}}(x))}$. Then $\mathcal{H}$ has a self-similar left action $\vDash$ on $\mathscr{B} = \mathscr{B}(\mathscr{A}, \mathcal{X}, \alpha|_{\mathcal{X}})$ defined for $h \in \mathcal{H}$ and $(a, x) \in \mathscr{A}\,{}_q{*}_r\,\mathcal{X}$ with $s_{\mathcal{H}}(h) = \rho_{\mathscr{B}}(a, x) = \rho_{\mathcal{X}}(x)$ by

$$h \vDash (a, x) := \big(\alpha_{(r(h),h)}(a), h \triangleright x\big) .$$

One can check that

$$\mathscr{B}(\mathscr{A}, \mathcal{X} \bowtie \mathcal{H}, \alpha) \cong \mathscr{B}(\mathscr{A}, \mathcal{X}, \alpha|_{\mathcal{X}}) \bowtie \mathcal{H}.$$

Remark 4.8. If $\mathcal{H}$ has a self-similar left action $\vDash$ on a Fell bundle $\mathscr{B}$, then

$$(\mathscr{B} \bowtie \mathcal{H})_{(x,h)} \cdot (\mathscr{B} \bowtie \mathcal{H})_{(y,k)} = (\mathscr{B}_x \times \{h\}) \cdot (\mathscr{B}_y \times \{k\}) \subseteq \mathscr{B}_{x(h\triangleright y)} \times \{(h \triangleleft y)k\}.$$

Moreover, our assumptions on $\vDash$ imply that $h \vDash \mathscr{B}_y = \mathscr{B}_{h \vDash y}$, rather than merely a containment of the left-hand side in the right-hand side. Thus, if $\mathscr{B}$ is saturated (meaning that the closed linear span of the $\mathscr{B}$-product of any $\mathscr{B}_{x_1}$ with any compatible $\mathscr{B}_{x_2}$ equals the entire $\mathscr{B}_{x_1x_2}$), then by the above argument, we automatically have that $\mathscr{B} \bowtie \mathcal{H}$ is saturated also.

Similar to the case of a self-similar product groupoid (see Lemma 2.22), we can lift the action $\vDash$ to a $\widetilde{\mathcal{H}}$-action β, where $(\mathcal{X}, \widetilde{\mathcal{H}})$ is a matched pair of groupoids. When the groupoids $\mathcal{X}$ and $\mathcal{H}$ are étale, this construction is closely related to the construction in [6] in the following sense.

Proposition 4.9. *If the groupoids $\mathcal{X}$ and $\mathcal{H}$ are étale, then so is the groupoid $\widetilde{\mathcal{H}}$ from Proposition 4.3 and the self-similar product Fell bundle $\mathscr{B} \bowtie \mathcal{H}$ is isomorphic to the Zappa–Szép product Fell bundle $\mathscr{B} \bowtie_\beta \widetilde{\mathcal{H}}$ constructed in [6].*

Remark 4.10. As always, a similar construction can be done on the other side: if $\mathscr{B}$ carries a *right* self-similar $\mathcal{G}$-action $\dashv$, we can let $\mathcal{G} \bowtie \mathscr{B}$ be given as the bundle with the total space

$$\mathcal{G} \bowtie B = \mathcal{G}\,{}_s{*}_\sigma\, B = \{(s, b) \in \mathcal{G} \times B : (s, q_{\mathscr{B}}(b)) \in \mathcal{G} \bowtie \mathcal{X}\}$$

and the analogous Fell bundle structure.

5. The orbit Fell bundle from self-similar actions

The following is analogous to the construction in [17, Corollary A.12].

Definition 5.1. If $\mathcal{H}$ is a groupoid and a topological space B is a left $\mathcal{H}$-space, where the action is denoted by $\vDash$, we may let $\mathcal{H}\backslash B = \{\mathcal{H} \vDash b : b \in B\}$ be the quotient space which we equip with the quotient topology, i.e., the largest topology making $\pi\colon B \to \mathcal{H}\backslash B$ continuous.

Remark 5.2. We will frequently assume that an acting groupoid $\mathcal{H}$ has open source map, because then [38, Prop. 2.12] implies that the quotient map $\pi\colon B \to \mathcal{H}\backslash B$ is open.

When $\mathcal{H}$ has a self-similar left action on a Fell bundle $\mathscr{B} = (q_{\mathscr{B}}\colon B \to \mathcal{X})$ (Definition 4.1), then B is a left $\mathcal{H}$-space. In this case, since $h \vDash \_$ maps $\mathscr{B}_x$ to $\mathscr{B}_{h \triangleright x}$ by (B1), the map

$$q_{\mathcal{H}\backslash\mathscr{B}}\colon \mathcal{H}\backslash B \to \mathcal{H}\backslash\mathcal{X} \quad \text{given by} \quad \mathcal{H} \vDash b \mapsto \mathcal{H} \triangleright q_{\mathscr{B}}(b) \tag{5.1}$$

is well-defined, and we let $\mathcal{H}\backslash\mathscr{B} := (q_{\mathcal{H}\backslash\mathscr{B}}\colon \mathcal{H}\backslash B \to \mathcal{H}\backslash\mathcal{X})$. The fiber over $\xi \in \mathcal{H}\backslash\mathcal{X}$ of the bundle is therefore given by

$$(\mathcal{H}\backslash\mathscr{B})_\xi = \{\mathcal{H} \vDash b : b \in \mathscr{B} \text{ such that } q_{\mathscr{B}}(b) \in \xi\}.$$

Lemma 5.3. *Suppose the self-similar left action $\triangleright$ of $\mathcal{H}$ on the groupoid $\mathcal{X}$ is free. Let $\xi \in \mathcal{H}\backslash\mathcal{X}$. For Ξ, Θ in the fiber $(\mathcal{H}\backslash\mathscr{B})_\xi$ and for $z \in \mathbb{C}$, we may let*

$$\|\Xi\| := \|b\| \quad \text{and} \quad z\,\Xi = \mathcal{H} \vDash (zb) \quad \text{where } b \in \Xi, \text{ and}$$
$$\Xi + \Theta = \mathcal{H} \vDash \big([h \vDash b] + c\big) \quad \text{where } b \in \Xi, c \in \Theta, h \in \mathcal{H} \text{ such that } q_{\mathscr{B}}(c) = q_{\mathscr{B}}(h \vDash b).$$

With this structure, $(\mathcal{H}\backslash\mathscr{B})_\xi$ is a complex Banach space.

Proof. First note that $\|\cdot\|$ is well-defined: Since $h \vDash \_$ is isometric on each fiber, $\mathcal{H} \vDash a = \mathcal{H} \vDash b$ implies $\|a\| = \|b\|$. Likewise, scalar multiplication is well-defined since each $h \vDash \_$ is $\mathbb{C}$-linear by assumption.

To see that addition is well-defined, we first check that h exists. If we pick any $b \in \Xi, c \in \Theta$, then by definition of the fiber $(\mathcal{H}\backslash\mathscr{B})_\xi$, we have $q_{\mathscr{B}}(b), q_{\mathscr{B}}(c) \in \xi$. In particular, there exists $h \in \mathcal{H}$ such that $q_{\mathscr{B}}(c) = h \triangleright q_{\mathscr{B}}(b) = q_{\mathscr{B}}(h \vDash b)$. This shows that c and $h \vDash b$ are in the same fiber of $\mathscr{B}$, so that $[h \vDash b] + c$ makes sense. It remains to check that $\Xi + \Theta$ does not depend on the choices, so assume that we are given b', c', h' with $q_{\mathscr{B}}(c') = q_{\mathscr{B}}(h' \vDash b')$. As $b, b' \in \Xi$ and $c, c' \in \Theta$, there exist $k, l \in \mathcal{H}$ such that $b' = k \vDash b$ and $c' = l \vDash c$. In particular,

$$\begin{aligned} h' \triangleright q_{\mathscr{B}}(b') &= q_{\mathscr{B}}(c') = q_{\mathscr{B}}(l \vDash c) = l \triangleright q_{\mathscr{B}}(c) = l \triangleright [h \triangleright q_{\mathscr{B}}(b)] \\ &= (lh) \triangleright q_{\mathscr{B}}(k^{-1} \vDash b') = (lhk^{-1}) \triangleright q_{\mathscr{B}}(b'). \end{aligned}$$

Since the $\mathcal{H}$-action on $\mathcal{X}$ is free, we conclude that $h' = lhk^{-1}$, and thus

$$[h' \blacktriangleright b'] + c' = (lhk^{-1}) \blacktriangleright (k \blacktriangleright b) + l \blacktriangleright c = (lh) \blacktriangleright b + l \blacktriangleright c = l \blacktriangleright \big([h \blacktriangleright b] + c\big),$$

which shows that $[h' \blacktriangleright b'] + c'$ and $[h \blacktriangleright b] + c$ represent the same class in $(\mathcal{H}\backslash\mathscr{B})_\xi$.

It is now easy to check that we have a normed vector space. To see that $(\mathcal{H}\backslash\mathscr{B})_\xi$ is complete, let $(\Xi_n)_n$ be a Cauchy sequence. If we pick arbitrary $b_n \in \Xi_n$ for each n, then we can find $h_n \in \mathcal{H}$ such that $c_n := h_n \blacktriangleright b_n$ is in the same fiber as the representative b_1 of ξ_1; say, in $\mathscr{B}_x$. We now have a sequence $(c_n)_n$ in $\mathscr{B}_x$. Note that, by definition of the linear structure on $(\mathcal{H}\backslash\mathscr{B})_\xi$, we have $\Xi_n - \Xi_m = \mathcal{H} \blacktriangleright (c_n - c_m)$, so that

$$\|\Xi_n - \Xi_m\| = \|c_n - c_m\|_{\mathscr{B}_x}.$$

Thus, $(c_n)_n$ is a Cauchy sequence in the Banach space $\mathscr{B}_x$ and hence converges to some element c. As

$$\|\Xi_n - \mathcal{H} \blacktriangleright c\| = \|c_n - c\|_{\mathscr{B}_x},$$

we conclude that $\Xi_n \to \mathcal{H} \blacktriangleright c$ in norm in $(\mathcal{H}\backslash\mathscr{B})_\xi$. $\square$

Corollary 5.4. *Suppose the self-similar left action $\triangleright$ of $\mathcal{H}$ on the groupoid $\mathcal{X}$ is free and $\mathcal{H}$ has open source map. Then $\mathcal{H}\backslash\mathscr{B} = (q_{\mathcal{H}\backslash\mathscr{B}} \colon \mathcal{H}\backslash B \to \mathcal{H}\backslash\mathcal{X})$ is a USC Banach bundle.*

Proof. We will check that we can apply [7, Proposition 6.13] to the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\ \pi\ } & \mathcal{H}\backslash B \\ {\scriptstyle q_{\mathscr{B}}}\downarrow & & \downarrow{\scriptstyle q_{\mathcal{H}\backslash\mathscr{B}}} \\ \mathcal{X} & \longrightarrow & \mathcal{H}\backslash\mathcal{X} \end{array}$$

We have already noted in Lemma 5.3 that the fibers of $\mathcal{H}\backslash\mathscr{B}$ are complex Banach spaces. By definition of the topologies of the spaces on the right-hand side, the vertical maps are quotient maps. Moreover, π is open by Remark 5.2 and $\mathcal{X} \to \mathcal{H}\backslash\mathcal{X}$ is open by [38, Proposition 2.12] since $s_{\mathcal{H}}$ is open. Therefore, Assumption (i) of [7, Proposition 6.13] holds. By definition of the Banach space structure on the fibers of $\mathcal{H}\backslash\mathscr{B}$ (see Lemma 5.3), Assumption (ii) holds.

Lastly, let $\Xi \in \mathcal{H}\backslash B$ and $x \in q_{\mathcal{H}\backslash\mathscr{B}}(\Xi)$ be given, and take any $b \in \Xi \subseteq B$. Since $q_{\mathscr{B}}(b) \in q_{\mathcal{H}\backslash\mathscr{B}}(\Xi)$, there exists $h \in \mathcal{H}$ such that $x = h \triangleright q_{\mathscr{B}}(b) = q_{\mathscr{B}}(h \blacktriangleright b)$. This means that $h \blacktriangleright b \in \mathscr{B}_x$ satisfies $\pi(h \blacktriangleright b) = \Xi$, since $\pi \circ (h \blacktriangleright \textvisiblespace) = \pi$ where both are defined. This proves the final Assumption (iii) of [7, Proposition 6.13]. $\square$

As before, we will write $s_{\mathcal{H}\backslash\mathscr{B}} := s_{\mathcal{H}\backslash\mathcal{X}} \circ q_{\mathcal{H}\backslash\mathscr{B}}$ and $r_{\mathcal{H}\backslash\mathscr{B}} := r_{\mathcal{H}\backslash\mathcal{X}} \circ q_{\mathcal{H}\backslash\mathscr{B}}$.

Proposition 5.5. *Suppose the self-similar left action $\triangleright$ of $\mathcal{H}$ on the groupoid $\mathcal{X}$ is free and proper and $\mathcal{H}$ has open source map. For two elements Ξ, Θ of $\mathcal{H}\backslash\mathscr{B}$ with $s_{\mathcal{H}\backslash\mathscr{B}}(\Xi) = r_{\mathcal{H}\backslash\mathscr{B}}(\Theta)$, define*

$$\Xi\Theta = \mathcal{H} \vdash (bc) \quad \textit{where } b \in \Xi, c \in \Theta \textit{ are such that } (b,c) \in \mathscr{B}^{(2)}.$$

Further, let $(\mathcal{H} \vdash b)^* = \mathcal{H} \vdash b^*$*. With this structure,* $\mathcal{H}\backslash\mathscr{B}$ *is a Fell bundle, which we call the* left quotient bundle of $\mathscr{B}$ by $\mathcal{H}$*.*

Proof. Since $\mathcal{H}$ acts freely and properly on $\mathcal{X}$, the quotient $\mathcal{H}\backslash\mathcal{X}$ is a groupoid by Proposition 3.3. We first verify that $b \in \Xi$ and $c \in \Theta$ exist, so start with two *arbitrary* elements $b \in \Xi$ and $c' \in \Theta$. By construction of $s_{\mathcal{H}\backslash\mathscr{B}}$ and $r_{\mathcal{H}\backslash\mathscr{B}}$, we have $s_{\mathscr{B}}(b) \in s_{\mathcal{H}\backslash\mathscr{B}}(\Xi)$ and $r_{\mathscr{B}}(c') \in r_{\mathcal{H}\backslash\mathscr{B}}(\Theta)$. By assumption, the equivalence classes are the same element in $\mathcal{H}\backslash\mathcal{X}$, so there exists $h \in \mathcal{H}$ such that $s_{\mathscr{B}}(b) = h \triangleright r_{\mathscr{B}}(c')$. We have

$$h \triangleright r_{\mathscr{B}}(c') = h \triangleright r_{\mathcal{X}}(q_{\mathscr{B}}(c')) \overset{\text{(L10)}}{=} r_{\mathcal{X}}(h \triangleright q_{\mathscr{B}}(c')) \overset{\text{(B1)}}{=} r_{\mathcal{X}}(q_{\mathscr{B}}(h \vdash c')).$$

Thus, for the element $c := h \vdash c'$ of Θ, we have shown that $(b,c) \in \mathscr{B}^{(2)}$. Next, we must show that the multiplication does not depend on the choice of $(b,c) \in \mathscr{B}^{(2)}$, so assume that (b_1, c_1) is another composable pair of $\mathscr{B}$ for which $b_1 \in \Xi$ and $c_1 \in \Theta$. Then there exist $k, l \in \mathcal{H}$ such that $b_1 = k \vdash b$ and $c_1 = l \vdash c$. A computation similar to that in the proof of Lemma 3.2 shows that

$$[k \dashv b] \triangleright s_{\mathscr{B}}(b) = s_{\mathscr{B}}(b_1) = r_{\mathscr{B}}(c_1) = l \triangleright r_{\mathscr{B}}(c) = l \triangleright s_{\mathscr{B}}(b).$$

Since the $\mathcal{H}$-action on $\mathcal{X}$ is free, we conclude $l = k \dashv b$, so that (B4) implies

$$b_1 c_1 = [k \vdash b][l \vdash c] = [k \vdash b]\big[(k \dashv b) \vdash c\big] = k \vdash bc.$$

In other words, $b_1 c_1 \in \mathcal{H} \vdash bc$, as claimed.

Now, if $b_1 = k \vdash b$, then $b_1^* = (k \vdash b)^* = [k \dashv b] \vdash b^*$, which shows that $\mathcal{H} \vdash b_1^* = \mathcal{H} \vdash b^*$, i.e., involution is well-defined on $\mathcal{H}\backslash\mathscr{B}$.

As noted in Corollary 5.4, $\mathcal{H}\backslash\mathscr{B}$ is a USC Banach bundle. Moreover, the algebraic and norm-related properties for Fell bundles (that is, (F1)–(F10) in [7, Definition 2.9]) are all swiftly verified and follow from the respective properties of $\mathscr{B}$. For example, to show (F10), take an arbitrary $\Xi \in \mathcal{H}\backslash\mathscr{B}$ and any $b \in \Xi$; let $u := s_{\mathscr{B}}(b)$. Since $\mathscr{B}$ is a Fell bundle, we have $b^* \cdot b = c^* c$ for some $c \in \mathscr{B}_u$. The definition of the multiplication and involution on $\mathcal{H}\backslash\mathscr{B}$ thus implies that

$$\Xi^*\Xi = \mathcal{H} \vdash (b^* \cdot b) = \mathcal{H} \vdash (c^* c) = (\mathcal{H} \vdash c)^*(\mathcal{H} \vdash c).$$

Since $\mathcal{H} \vdash c \in (\mathcal{H}\backslash\mathscr{B})_{\mathcal{H}\triangleright u}$ and $\mathcal{H} \triangleright u = \mathcal{H} \triangleright s_{\mathscr{B}}(b) = s_{\mathcal{H}\backslash\mathscr{B}}(\Xi)$, this proves that $\Xi^*\Xi$ is a positive element of the C*-algebra $(\mathcal{H}\backslash\mathscr{B})_{\mathcal{H}\triangleright u}$, as needed for (F10). $\square$

Remark 5.6. If $\mathscr{B}$ is saturated, then so is $\mathcal{H}\backslash\mathscr{B}$. Indeed, take $(\xi_1, \xi_2) \in (\mathcal{H}\backslash\mathcal{X})^{(2)}$ and let $\Theta \in (\mathcal{H}\backslash\mathscr{B})_{\xi_1\xi_2}$ be arbitrary. By definition of the fiber, there exists $b \in \mathscr{B}$ with

$q_{\mathscr{B}}(b) \in \xi_1\xi_2$ and $\mathcal{H} \vDash b = \Theta$. Since ξ_1 and ξ_2 are composable, we can find $x_i \in \xi_i$ such that $(x_1, x_2) \in \mathcal{X}^{(2)}$. Thus, there exists $h \in \mathcal{H}$ such that (L4) implies $q_{\mathscr{B}}(b) = h \triangleright (x_1x_2) = y_1y_2$, where $y_1 := h \triangleright x_1$ and $y_2 = (h \triangleleft x_1) \triangleright x_2$. Since $\mathscr{B}$ is saturated, we can approximate b by linear combinations of products of elements in $\mathscr{B}_{y_1}$ and in $\mathscr{B}_{y_2}$. Since $y_i \in \mathcal{H} \triangleright x_i = \xi_i$, the images of these elements under π are in $(\mathcal{H}\backslash\mathscr{B})_{\xi_i}$ and, by definition of the linear and topological structure on $\mathcal{H}\backslash\mathscr{B}$, they approximate $\mathcal{H} \vDash b = \Theta$, as claimed.

Example 5.7. Suppose that the self-similar left action $\triangleright$ of $\mathcal{H}$ on the groupoid $\mathcal{X}$ is free and proper, that $\mathcal{H}$ has open source map, and that $(\mathscr{A}, \mathcal{H}\backslash\mathcal{X}, \alpha)$ is a groupoid dynamical system. If we define $\tilde{\alpha}_x := \alpha_{\mathcal{H}\triangleright x}$ for $x \in \mathcal{X}$, then $(\mathscr{A}, \mathcal{X}, \tilde{\alpha})$ is a groupoid dynamical system. Moreover, $\mathcal{H}$ has a self-similar left action on $\mathscr{B}(\mathscr{A}, \mathcal{X}, \tilde{\alpha})$ given by $h \vDash (a, x) := (a, h \triangleright x)$ and the quotient bundle $\mathcal{H}\backslash\mathscr{B}(\mathscr{A}, \mathcal{X}, \tilde{\alpha})$ is exactly $\mathscr{B}(\mathscr{A}, \mathcal{H}\backslash\mathcal{X}, \alpha)$.

Remark 5.8. Analogously to Proposition 5.5, we can define the *right* quotient bundle $\mathscr{B}/\mathcal{G}$ from the right self-similar action $\dashv$ of $\mathcal{G}$ on $\mathscr{B}$. We denote an element of $\mathscr{B}/\mathcal{G}$ by $b \dashv \mathcal{G}$ and let $q_{\mathscr{B}/\mathcal{G}}(b \dashv \mathcal{G}) = q_{\mathscr{B}}(b) \blacktriangleleft \mathcal{G}$.

We next require a Fell bundle analogue of in tune actions.

Assumption 5.9. We assume that

(1) $\mathcal{G}$ and $\mathcal{H}$ are locally compact Hausdorff groupoids;
(2) $\mathcal{X}$ is a $(\mathcal{H}, \mathcal{G})$-self-similar para-equivalence with self-similar actions $\triangleright$ of $\mathcal{H}$ and $\blacktriangleleft$ of $\mathcal{G}$ respectively (Definition 3.6); in particular, the actions are in tune, free, and proper, and the source maps of all three groupoids are open;
(3) $\mathscr{B} = (q_{\mathscr{B}} \colon B \to \mathcal{X})$ is a saturated Fell bundle,
(4) $\mathcal{H}$ and $\mathcal{G}$ act on the left and right of $\mathscr{B}$ by self-similar actions $\vDash$ and $\dashv$, respectively; and
(5) for any $h \in \mathcal{H}$, $b \in \mathscr{B}_x$, $t \in \mathcal{G}$ for which $(h \triangleright x) \blacktriangleleft t$ is well-defined, we have:

$$(h \vDash b) \dashv t = h \vDash (b \dashv t). \tag{BC1}$$

Note that, with the notation introduced after Definition 4.1, we automatically also have

$$(h \vDash b) \vdash t = b \vdash t \tag{BC2}$$

$$h \dashv (b \dashv t) = h \dashv b \tag{BC3}$$

as a consequence of Condition (B1) combined with Condition (C2) and (C3).

We first show that the actions $\vDash$ and $\dashv$ on $\mathscr{B}$ pass to the quotients. We remind the reader of some definitions we made earlier:

$$q_{\mathcal{H}\backslash\mathscr{B}} \colon \mathcal{H}\backslash\mathscr{B} \to \mathcal{H}\backslash\mathcal{X} \quad \text{is defined by} \quad q_{\mathcal{H}\backslash\mathscr{B}}(\mathcal{H} \vdash b) = \mathcal{H} \rhd q_{\mathscr{B}}(b), \text{ and}$$
$$q_{\mathscr{B}/\mathcal{G}} \colon \mathscr{B}/\mathcal{G} \to \mathcal{X}/\mathcal{G} \quad \text{is defined by} \quad q_{\mathscr{B}/\mathcal{G}}(b \dashv \mathcal{G}) = q_{\mathscr{B}}(b) \blacktriangleleft \mathcal{G}.$$

Moreover, in Proposition 3.12, we defined a left self-similar $\mathcal{H}$-action on $\mathcal{X}/\mathcal{G}$ with momentum map $\tilde{\rho}(x\blacktriangleleft G) = \rho_{\mathcal{X}}(x)$, and a right self-similar $\mathcal{G}$-action on $\mathcal{H}\backslash\mathcal{X}$ with momentum map $\tilde{\sigma}(\mathcal{H} \rhd x) = \sigma_{\mathcal{X}}(x)$.

Proposition 5.10. *We assume all conditions in Assumption 5.9. With*

$$\_ \mathbin{\textcircled{$\lhd$}} \_ \colon \mathcal{H}\backslash\mathscr{B} \, {}_{\tilde{\sigma}\circ q}{*}_{r} \, \mathcal{G} \to \mathcal{H}\backslash\mathscr{B}, \qquad \Xi \mathbin{\textcircled{$\lhd$}} s = \mathcal{H} \vdash [b \dashv s] \quad \textit{where } b \in \Xi,$$

$\textcircled{$\lhd$}$ *is a right self-similar $\mathcal{G}$-action on $\mathcal{H}\backslash\mathscr{B}$, and with*

$$\_ \mathbin{\textcircled{$\rhd$}} \_ \colon \mathcal{H} \, {}_{s_{\mathcal{H}}}{*}_{\tilde{\rho}\circ q} \, \mathscr{B}/\mathcal{G} \to \mathscr{B}/\mathcal{G}, \qquad h \mathbin{\textcircled{$\rhd$}} \Xi := [h \vdash b] \dashv \mathcal{G} \quad \textit{where } b \in \Xi,$$

$\textcircled{$\rhd$}$ *is a left self-similar $\mathcal{H}$-action on $\mathscr{B}/\mathcal{G}$.*

Proof. As always, we will focus only on one of the two statements, namely $\textcircled{$\lhd$}$.

To see that $\textcircled{$\lhd$}$ is well-defined, assume $c \in \Xi$, so there exists $h \in \mathcal{H}$ such that $c = h \vdash b$. Therefore, by Equation (BC1) and the definition of $\vdash$,

$$\mathcal{H} \vdash [c \dashv s] = \mathcal{H} \vdash \big[(h \vdash b) \dashv s\big] = \mathcal{H} \vdash \big[h \vdash (b \dashv s)\big] = \mathcal{H} \vdash [b \dashv s].$$

It remains to show that $\textcircled{$\lhd$}$ satisfies all the conditions listed in Definition 4.1. We start with the algebraic properties. For (B1), take an arbitrary element $\xi = \mathcal{H} \rhd x \in \mathcal{H}\backslash\mathcal{X}$ and $\Xi \in (\mathcal{H}\backslash\mathscr{B})_{\xi}$. If $b \in \Xi \cap \mathscr{B}_x$, then

$$\Xi \mathbin{\textcircled{$\lhd$}} s = \mathcal{H} \vdash [b \dashv s] \quad \in \quad (\mathcal{H}\backslash\mathscr{B})_{\mathcal{H}\rhd(x\blacktriangleleft s)} = (\mathcal{H}\backslash\mathscr{B})_{\xi \textcircled{$\lhd$} s}.$$

This is linear as a map $(\mathcal{H}\backslash\mathscr{B})_{\xi} \to (\mathcal{H}\backslash\mathscr{B})_{\xi\textcircled{$\lhd$}s}$ because $\_\dashv s$ is linear as a map $\mathscr{B}_x \to \mathscr{B}_{x\blacktriangleleft s}$ and because of how we defined the linear structure on the fibers of the quotient (see Lemma 5.3).

Both (B2) and (B3) are trivial. For (B4), let $(\Xi, \Theta) \in (\mathcal{H}\backslash\mathscr{B})^{(2)}$. If $b \in \Xi$ and $c \in \Theta$ with $(b, c) \in \mathscr{B}^{(2)}$, then $\Xi\Theta = \mathcal{H} \vdash (bc)$ by our definition in Proposition 5.5. If $x = q_{\mathscr{B}}(b)$, then

$$\begin{aligned}
[\mathcal{H} \vdash bc] \mathbin{\textcircled{$\lhd$}} s &= \mathcal{H} \vdash (bc \dashv s) && \text{(def'n of } \textcircled{$\lhd$}\text{)}\\
&= \mathcal{H} \vdash \big([b \dashv (x \blacktriangleright s)][c \dashv s]\big) && \text{(Property (B4) for } \dashv\text{)}\\
&= \big[\mathcal{H} \vdash \big(b \dashv (x \blacktriangleright s)\big)\big]\,[\mathcal{H} \vdash (c \dashv s)] && \text{(def'n of } \mathcal{H}\backslash\mathscr{B}\text{; see Prop. 5.5)}\\
&= \big[(\mathcal{H} \vdash b) \mathbin{\textcircled{$\lhd$}} (x \blacktriangleright s)\big]\,[(\mathcal{H} \vdash c) \mathbin{\textcircled{$\lhd$}} s] && \text{(def' of } \textcircled{$\lhd$}\text{)}.
\end{aligned}$$

Since $x \blacktriangleright s = (\mathcal{H} \rhd x) \mathbin{\textcircled{$\blacktriangleright$}} s = q_{\mathcal{H}\backslash\mathscr{B}}(\Xi) \mathbin{\textcircled{$\blacktriangleright$}} s$ (see the definition of $\textcircled{$\blacktriangleright$}$ in Proposition 3.12 and that of $q_{\mathcal{H}\backslash\mathscr{B}}$ in Equation (5.1)) and since $\Xi = \mathcal{H} \vdash b$ and $\Theta = \mathcal{H} \vdash c$, this proves that

$$\Xi\Theta \circledast s = \big(\Xi \textcircled{$\blacktriangleleft$} [q_{\mathcal{H}\backslash\mathscr{B}}(\Xi) \textcircled{$\blacktriangleright$} s]\big)\,(\Theta \textcircled{$\blacktriangleleft$} s),$$

as required.

For (B5), we compute

$$\begin{aligned}
\big(\Xi \textcircled{$\blacktriangleleft$} s\big)^* &= \big(\mathcal{H} \vDash [b \blacktriangleleft s]\big)^* && \text{for } b \in \Xi\\
&= \mathcal{H} \vDash [b \blacktriangleleft s]^* && \text{(involution on } \mathcal{H}\backslash\mathscr{B}\text{; see Prop. 5.5)}\\
&= \mathcal{H} \vDash \big[b^* \blacktriangleleft (b \blacktriangleright s)\big] && \text{(Property (B5) for } \blacktriangleleft)\\
&= \big(\mathcal{H} \vDash b^*\big) \textcircled{$\blacktriangleleft$} \big(b \blacktriangleright s\big) && \text{(def'n of } \textcircled{$\blacktriangleleft$})\\
&= \big(\mathcal{H} \vDash b^*\big) \textcircled{$\blacktriangleleft$} \big(q_{\mathcal{H}\backslash\mathscr{B}}(\Xi) \textcircled{$\blacktriangleright$} s\big) && \text{(def'n of } \textcircled{$\blacktriangleright$}\text{; see Prop. 3.12)}\\
&= \Xi^* \textcircled{$\blacktriangleleft$} \big(q_{\mathcal{H}\backslash\mathscr{B}}(\Xi) \textcircled{$\blacktriangleright$} s\big) && \text{(involution on } \mathcal{H}\backslash\mathscr{B}\text{; see Prop. 5.5),}
\end{aligned}$$

as required.

Lastly, we have to check that $\textcircled{$\blacktriangleleft$}$ is continuous, so let $\{(\Xi_i, s_i)\}_{i\in I}$ be a net in $\mathcal{H}\backslash\mathscr{B}\ {}_{\tilde\sigma\circ q}{*}_{r_{\mathcal{G}}}\ \mathcal{G}$ that converges to (Ξ, s). Since the quotient map $\mathscr{B} \to \mathcal{H}\backslash\mathscr{B}$ is open by Remark 5.2, there exists a subnet $\{\Xi_j\}_{j\in J}$ of $\{\Xi_i\}_{i\in I}$ and lifts $b_\mu \in \Xi_j$, $b \in \Xi$ such that $b_j \to b$ in $\mathscr{B}$. Since $\blacktriangleleft$ is continuous, it follows that $b_j \blacktriangleleft s_j \to b \blacktriangleleft s$, so that

$$\Xi_j \textcircled{$\blacktriangleleft$} s_j = \mathcal{H} \vDash [b_\mu \blacktriangleleft s_j] \to \mathcal{H} \vDash [b \blacktriangleleft s] = \Xi \textcircled{$\blacktriangleleft$} s.$$

By Lemma A.2, this suffices to conclude that $\textcircled{$\blacktriangleleft$}$ is continuous. $\square$

6. The symmetric imprimitivity theorem for self-similar actions

Let us rehash what the conditions in Assumption 5.9 imply. By Proposition 5.5, we get two quotient Fell bundles: the right quotient $\mathscr{B}/\mathcal{G}$ over the groupoid $\mathcal{X}/\mathcal{G}$ and the left quotient $\mathcal{H}\backslash\mathscr{B}$ over $\mathcal{H}\backslash\mathcal{X}$. These are saturated by Remark 5.6, since $\mathscr{B}$ is assumed to be saturated. We have seen in Proposition 5.10 that $\mathcal{H}\backslash\mathscr{B}$ carries a right self-similar $\mathcal{G}$-action $\textcircled{$\blacktriangleleft$}$, and likewise, $\mathscr{B}/\mathcal{G}$ carries a left self-similar $\mathcal{H}$-action $\textcircled{$\triangleright$}$. We can therefore take two self-similar products, as explained in Definition 4.5 and respectively, Remark 4.10:

- the product $(\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$ of $\mathscr{B}/\mathcal{G}$ with $\mathcal{H}$ is a bundle over $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and will be denoted $q_{\mathscr{A}}\colon \mathscr{A} \to (\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$, while
- the product $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathscr{B})$ of $\mathcal{H}\backslash\mathscr{B}$ with $\mathcal{G}$ is a bundle over $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ and will be denoted $q_{\mathscr{C}}\colon \mathscr{C} \to \mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$.

These self-similar product Fell bundles are saturated by Remark 4.8.

We now prove that $\mathscr{A}$ and $\mathscr{C}$ are equivalent via the bundle $\mathscr{B}$ in the sense of [23, Definition 6.1]. Recall that $\mathcal{X}$ is a groupoid equivalence between $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ by Theorem 3.10 when equipped with the structure defined in Proposition 3.14. We remind the reader that $\mathfrak{r}\colon \mathcal{X} \to [(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}]^{(0)}$ denotes the momentum map of $\mathcal{X}$ for that left action and $\mathfrak{s}\colon \mathcal{X} \to [\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})]^{(0)}$ the momentum map for that right-action. Consequently, we will write $\mathfrak{r}_{\mathscr{B}} := \mathfrak{r}\circ q_{\mathscr{B}}$, not to be confused with $r_{\mathscr{B}} = r_{\mathcal{X}}\circ q_{\mathscr{B}}\colon \mathscr{B} \to \mathcal{X}^{(0)}$; we likewise let $\mathfrak{s}_{\mathscr{B}} := \mathfrak{s} \circ q_{\mathscr{B}}$.

Theorem 6.1 *(cf. [17, Theorem 3.1]). We assume all conditions in Assumption 5.9. Then $\mathscr{B}$ is a Fell bundle equivalence between $\mathscr{A} = (\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathscr{C} = \mathcal{G} \bowtie (\mathcal{H}\backslash\mathscr{B})$ in the following way:*

(1) *$\mathscr{A}$ acts on the left of $\mathscr{B}$: whenever $(\Theta, h) \in \mathscr{A}$ and $b \in \mathscr{B}$ are such that $s_{\mathscr{A}}(\Theta, h) = \mathfrak{r}_{\mathscr{B}}(b)$ in $[(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}]^{(0)}$, we let*

$$(\Theta, h) \cdot b = a[h \vDash b], \text{ where } a \in \Theta \text{ is such that } s_{\mathscr{B}}(a) = r_{\mathscr{B}}(h \vDash b).$$

(2) *$\mathscr{C}$ acts on the right of $\mathscr{B}$: whenever $b \in \mathscr{B}$ and $(t, \Xi) \in \mathscr{C}$ are such that $\mathfrak{s}_{\mathscr{B}}(b) = r_{\mathscr{C}}(t, \Xi)$ in $[\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})]^{(0)}$, we let*

$$b \cdot (t, \Xi) = [b \dashv t]c, \text{ where } c \in \Xi \text{ is such that } s_{\mathscr{B}}(b \dashv t) = r_{\mathscr{B}}(c).$$

(3) *The left $\mathscr{A}$-valued inner product defined on $\mathscr{B}\,{}_{\mathfrak{s}}{*}_{\mathfrak{s}}\,\mathscr{B}$ is given by*

$$_{\mathscr{A}}\langle a \mid b\rangle = \big([a(h \vDash b^*)] \dashv \mathcal{G}, h \dashv b^*\big),$$

where h is the unique element of $\mathcal{H}$ such that $s_{\mathscr{B}}(a) = h \triangleright s_{\mathscr{B}}(b)$.

(4) *The right $\mathscr{C}$-valued inner product defined on $\mathscr{B}\,{}_{\mathfrak{r}}{*}_{\mathfrak{r}}\,\mathscr{B}$ is given by*

$$\langle a \mid b\rangle_{\mathscr{C}} = \big(a^* \vDash t, \mathcal{H} \vDash [(a^* \dashv t)b]\big),$$

where t is the unique element of $\mathcal{G}$ such that $r_{\mathscr{B}}(a) \blacktriangleleft t = r_{\mathscr{B}}(b)$.

Example 6.2 *(see also Example 3.11).* Theorem 6.1 recovers [17, Theorem 3.1]: suppose G and H are locally compact Hausdorff groups with commuting actions on a Fell bundle $\mathscr{B} = (q_{\mathscr{B}}\colon B \to \mathcal{X})$ by Fell bundle automorphisms, where $\mathcal{X}$ is a locally compact Hausdorff groupoid. The induced actions of G and H on $\mathcal{X}$ are then by groupoid automorphisms, and so (with $\mathcal{X}$ acting trivially on H and G) they are self-similar actions on $\mathcal{X}$. If the actions are free and proper, then $\mathscr{B}$ as described in Theorem 6.1 is a Fell bundle equivalence between the semi-direct product bundles $(\mathscr{B}/G) \rtimes H$ and $G \ltimes (H\backslash\mathscr{B})$ as considered in [17].

We will do the proof in pieces.

Lemma 6.3. *The formulas in (1) and (2) of Theorem 6.1 define actions on the USC Banach bundle $\mathscr{B}$ in the sense of [7, Definition 2.10].*

Proof. We will follow similar ideas as in the proof of Proposition 3.14 and we will only do the proof for the left action; the other one follows *mutatis mutandis*. We will denote the source and range map of $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ merely by r and s, respectively.

First, let us check that the condition $s_{\mathscr{A}}(\Theta, h) = \mathfrak{r}_{\mathscr{B}}(b)$ implies that there indeed exists $a \in \Theta$ with $s_{\mathscr{B}}(a) = r_{\mathscr{B}}(h \vDash b)$, so that $a[h \vDash b]$ makes sense. If a_0 is *any* element of Θ, then

$$\begin{aligned}
s_{\mathscr{A}}(\Theta, h) &= s\big(q_{\mathscr{A}}(\Theta, h)\big) = s\big(q_{\mathscr{B}/\mathcal{G}}(\Theta), h\big) && \text{(def'n of } q_{\mathscr{A}} \text{ in Definition 4.5)}\\
&= s\big(q_{\mathscr{B}}(a_0) \blacktriangleleft \mathcal{G}, h\big) && \text{(def'n of } q_{\mathscr{B}/\mathcal{G}}\text{; cf. (5.1) on p. 42)}\\
&= h^{-1} \circledtriangleright s_{\mathcal{X}/\mathcal{G}}(q_{\mathscr{B}}(a_0) \blacktriangleleft \mathcal{G}) && \text{(Rmk 2.19 and def'n of } s\text{; cf. Definition 2.16)}\\
&= h^{-1} \circledtriangleright [s_{\mathscr{B}}(a_0) \blacktriangleleft \mathcal{G}] && \text{(def'n of } s_{\mathcal{X}/\mathcal{G}}\text{; cf. Lemma 3.1)}\\
&= [h^{-1} \triangleright s_{\mathscr{B}}(a_0)] \blacktriangleleft \mathcal{G} && \text{(def'n of } \circledtriangleright\text{; see Proposition 3.12)}
\end{aligned}$$

On the other hand, $\mathfrak{r}_{\mathscr{B}}(b) = r_{\mathscr{B}}(b) \blacktriangleleft \mathcal{G}$, and so our assumption $s_{\mathscr{A}}(\Theta, h) = \mathfrak{r}_{\mathscr{B}}(b)$ implies that there exists $t \in \mathcal{G}$ such that

$$r_{\mathscr{B}}(b) = [h^{-1} \triangleright s_{\mathscr{B}}(a_0)] \blacktriangleleft t \overset{\text{(C1)}}{=} h^{-1} \triangleright [s_{\mathscr{B}}(a_0) \blacktriangleleft t],$$

i.e., $h \triangleright r_{\mathcal{X}}(q_{\mathscr{B}}(b)) = s_{\mathcal{X}}(q_{\mathscr{B}}(a_0)) \blacktriangleleft t$. Since

$$h \triangleright r_{\mathcal{X}}(q_{\mathscr{B}}(b)) \overset{\text{(L10)}}{=} r_{\mathcal{X}}(h \triangleright q_{\mathscr{B}}(b)) \overset{\text{(B1)}}{=} r_{\mathcal{X}}(q_{\mathscr{B}}(h \vDash b))$$

and likewise, $s_{\mathscr{B}}(a_0) \blacktriangleleft t = s_{\mathscr{B}}(a_0 \dashv t)$, we may thus let $a := a_0 \dashv t$, which is the required element of $a_0 \dashv \mathcal{G} = \Theta$.

Note that this chosen representative $a \in \Theta$ is unique, since the $\mathcal{G}$-action on $\mathcal{X}$ is free: if $a \dashv s$ also satisfies $s_{\mathscr{B}}(a \dashv s) = r_{\mathscr{B}}(h \vDash b)$, then

$$s_{\mathscr{B}}(a) \blacktriangleleft s \overset{\text{(R10)}}{=} s_{\mathscr{B}}(a \dashv s) = s_{\mathscr{B}}(a), \text{ so } s \in \mathcal{G}^{(0)}.$$

To see that the left action is continuous, assume that we have a net $\{(\Theta_i, h_i, b_i)\}_{i \in I}$ in $\mathscr{A} {}_{s}{*}_{\mathfrak{r}} \mathscr{B}$ that converges to (Θ, h, b). For each i, let $a_i \in \Theta_i$ be the unique element such that $u_i := s_{\mathscr{B}}(a_i) = r_{\mathscr{B}}(h_i \vDash b_i)$. By Lemma A.2, it suffices to check that a subnet of $a_i[h_i \vDash b_i]$ converges to $a[h \vDash b]$. Since $\vDash$ is continuous, we already know that $\{h_i \vDash b_i\}_{i \in I}$ converges to $h \vDash b$; and so in particular $u_i \to u := s_{\mathscr{B}}(a)$ in $\mathcal{X}^{(0)}$, and since multiplication on $\mathscr{B}$ is continuous, it suffices to show that a subnet of $\{a_i\}_i$ converges to a.

Since the quotient map $\mathscr{B} \to \mathscr{B}/\mathcal{G}$ is open (cf. Remark 5.2) and since $\Theta_i \to \Theta$, Proposition A.1 implies that there exists a subnet $\{\Theta_j\}_{j \in J}$ and lifts $c_j \in \Theta_j$ such that $c_j \to a$ in $\mathscr{B}$. Since $a_j \in \Theta_j$ also, there exist $t_j \in \mathcal{G}$ such that $a_j \dashv t_j = c_j$. In particular, by continuity of $s_{\mathscr{B}}$, we have

$$u_j \blacktriangleleft t_j = s_{\mathscr{B}}(a_j) \blacktriangleleft t_j \overset{\text{(R10)}}{=} s_{\mathcal{X}}(q_{\mathscr{B}}(a_j) \blacktriangleleft t_j) \overset{\text{(B1)}}{=} s_{\mathcal{X}}(q_{\mathscr{B}}(a_j \dashv t_j)) = s_{\mathscr{B}}(c_j) \to s_{\mathscr{B}}(a) = u,$$

so that

$$(u_j \blacktriangleleft t_j, u_j) \to (u, u) \quad \text{in } \mathcal{X}^{(0)} \times \mathcal{X}^{(0)}.$$

As the right action of $\mathcal{G}$ on $\mathcal{X}^{(0)}$ is proper, it now follows from [38, Corollary 2.26] that t_j converges; since the action is free, (R2) implies that it must converge to $\sigma_{\mathcal{X}}(u) \in \mathcal{G}^{(0)}$. This, in turn, implies that

$$a_j = c_j \prec t_j^{-1} \to a \prec \sigma_{\mathcal{X}}(u)^{-1} \overset{\text{(B3)}}{=} a,$$

as needed.

To see that (FA1) of [7, Definition 2.10] holds, we must check that $q_{\mathscr{B}}((a \prec \mathcal{G}, h) \cdot b) = q_{\mathscr{A}}(a \prec \mathcal{G}, h) \cdot q_{\mathscr{B}}(b)$, where $\cdot$ is the left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$-action on $\mathcal{X}$ as defined in Proposition 3.14. Let $q_{\mathscr{B}}(a) = x$ and $q_{\mathscr{B}}(b) = y$, so that $s_{\mathscr{B}}(a) = s_{\mathcal{X}}(x)$ equals $r_{\mathscr{B}}(h \vdash b) = r_{\mathcal{X}}(h \triangleright y)$ and

$$q_{\mathscr{B}}((a \prec \mathcal{G}, h) \cdot b) = q_{\mathscr{B}}(a[h \vdash b]) = x[h \triangleright y].$$

On the other hand, $q_{\mathscr{A}}(a \prec \mathcal{G}, h) = (x \blacktriangleleft \mathcal{G}, h)$. By Proposition 3.14, since $s_{\mathcal{X}}(x) = r_{\mathcal{X}}(h \triangleright y)$, we know that $(x \blacktriangleleft \mathcal{G}, h)$ can act on the left of y and we get

$$q_{\mathscr{A}}(a \prec \mathcal{G}, h) \cdot q_{\mathscr{B}}(b) = (x \blacktriangleleft \mathcal{G}, h) \cdot y = x[h \triangleright y].$$

This proves (FA1).

Next, we must show that (FA2) of [7, Definition 2.10] holds, i.e., associativity, so for $i = 1, 2$ pick $a_i \in \mathscr{B}_{x_i}, b \in \mathscr{B}$ and $h_i \in \mathcal{H}$ with appropriate range and sources such that

$$(a_1 \prec \mathcal{G}, h_1)(a_2 \prec \mathcal{G}, h_2) \quad \text{and} \quad (a_2 \prec \mathcal{G}, h_2) \cdot b$$

make sense; we have to show

$$\big[(a_1 \prec \mathcal{G}, h_1)(a_2 \prec \mathcal{G}, h_2)\big] \cdot b = (a_1 \prec \mathcal{G}, h_1) \cdot \big[(a_2 \prec \mathcal{G}, h_2) \cdot b\big]. \tag{6.1}$$

In $\mathscr{A} = (\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$, we have

$$\begin{aligned}
&(a_1 \prec \mathcal{G}, h_1)(a_2 \prec \mathcal{G}, h_2) \\
&= \big([a_1 \prec \mathcal{G}](h_1 \circledtriangleright [a_2 \prec \mathcal{G}]), [h_1 \circledtriangleleft q_{\mathscr{A}}(a_2 \prec \mathcal{G})]h_2\big) && \text{(Definition 4.5)} \\
&= \big([a_1 \prec \mathcal{G}][(h_1 \vdash a_2) \prec \mathcal{G}], [h_1 \circledtriangleleft (x_2 \blacktriangleleft \mathcal{G})]h_2\big) && \text{(def'n of } \circledtriangleright \text{ and } q_{\mathscr{A}} \text{ in Prop. 5.10)} \\
&= \big([a_1(h_1 \vdash a_2)] \prec \mathcal{G}, [h_1 \triangleleft x_2]h_2\big) && \text{(Prop. 5.5 for } \mathscr{B}/\mathcal{G}; \circledtriangleleft \text{ in Prop. 3.12).}
\end{aligned}$$

Therefore, we get

$$\big[(a_1 \prec \mathcal{G}, h_1)(a_2 \prec \mathcal{G}, h_2)\big] \cdot b = a_1(h_1 \vDash a_2)\big([(h_1 \lhd x_2)h_2] \vDash b\big).$$

On the other hand,

$$\begin{aligned}(a_1 \prec \mathcal{G}, h_1) \cdot \big[(a_2 \prec \mathcal{G}, h_2) \cdot b\big] &= (a_1 \prec \mathcal{G}, h_1) \cdot (a_2[h_2 \vDash b]) \\ &= a_1\Big[h_1 \vDash \big(a_2[h_2 \vDash b]\big)\Big] \\ &= a_1(h_1 \vDash a_2)\big([(h_1 \lhd x_2)h_2] \vDash b\big) \qquad \text{(by (B4) for } \mathscr{B}),\end{aligned}$$

so we have shown Equation (6.1).

For (FA3) of [7, Definition 2.10], recall from Lemma 4.4 that $h \vDash \_$ is isometric and $\|(b \prec \mathcal{G}, h)\| = \|b\|$. Therefore,

$$\|(a \prec \mathcal{G}, h) \cdot b\| = \|a(h \vDash b)\| \leq \|a\|\|b\| = \|(a \prec \mathcal{G}, h)\|\|b\|,$$

as needed. □

One by one, we will now check that Properties (FE1)–(FE3) of [7, Definition 2.11] are satisfied.

Lemma 6.4. *The left and right actions commute.*

Proof. Let $(\Theta, h) \in \mathscr{A}$, $b \in \mathscr{B}$, and $(t, \Xi) \in \mathscr{C}$ be such that $s_{\mathscr{A}}(\Theta, h) = \mathfrak{r}_{\mathscr{B}}(b)$ and $\mathfrak{s}_{\mathscr{B}}(b) = r_{\mathscr{C}}(t, \Xi)$; we have to confirm that $[(\Theta, h) \cdot b] \cdot (t, \Xi) = (\Theta, h) \cdot [b \cdot (t, \Xi)]$. For the left-hand side, we let a be the (unique) element of Θ with $s_{\mathscr{B}}(a) = r_{\mathscr{B}}(h \vDash b)$, so that $(\Theta, h) \cdot b = a[h \vDash b]$; then let c be the (unique) element in Ξ with $s_{\mathscr{B}}((a[h \vDash b]) \prec t) = r_{\mathscr{B}}(c)$, so that

$$\begin{aligned}[(\Theta, h) \cdot b] \cdot (t, \Xi) &= \big[(a[h \vDash b]) \prec t\big] c \\ &= [a \prec ([h \vDash b] \succ t)]\ ([h \vDash b] \prec t)\, c \qquad \text{(by (B4) for } \prec) \\ &= [a \prec (b \succ t)]\ ([h \vDash b] \prec t)\, c \qquad \text{(by (BC2))}.\end{aligned}$$

On the other hand, let c' be the (unique) element in Ξ with $s_{\mathscr{B}}(b \prec t) = r_{\mathscr{B}}(c')$, so that $b \cdot (t, \Xi) = [b \prec t]c'$; then let a' be the (unique) element in Θ with $s_{\mathscr{B}}(a') = r_{\mathscr{B}}(h \vDash ([b \prec t]c'))$, so that

$$\begin{aligned}(\Theta, h) \cdot [b \cdot (t, \Xi)] &= a'\big[h \vDash ([b \prec t]c')\big] \\ &= a'\ (h \vDash [b \prec t])\ [(h \lhd [b \prec t]) \vDash c'] \qquad \text{(by (B4))} \\ &= a'\ (h \vDash [b \prec t])\ [(h \lhd b) \vDash c'] \qquad \text{(by (BC3))}.\end{aligned}$$

Since $[h \vDash b] \prec t = h \vDash [b \prec t]$ by (BC1), we see that it suffices to check that

$$a' = a \mathbin{\prec} (b \vdash t) \quad \text{and} \quad (h \dashv b) \vdash c' = c.$$

Note that the second equation is the $\vdash$-version of the first equation, so by symmetry, it suffices to check the first equation. We have $a \mathbin{\prec} (b \vdash t) \in a \mathbin{\prec} \mathcal{G} = \Theta$, so by uniqueness of a', we only need to check that $s_{\mathscr{B}}(a \mathbin{\prec} (b \vdash t)) = r_{\mathscr{B}}(h \vdash ([b \mathbin{\prec} t]c'))$.

Since

$$q_{\mathscr{B}}(a \mathbin{\prec} (b \vdash t)) = q_{\mathscr{B}}(a) \blacktriangleleft [b \vdash t],$$

we have

$$\begin{aligned} s_{\mathscr{B}}\big(a \mathbin{\prec} (b \vdash t)\big) &= s_{\mathcal{X}}\big(q_{\mathscr{B}}(a) \blacktriangleleft [b \vdash t]\big) \\ &= s_{\mathscr{B}}(a) \blacktriangleleft [b \vdash t] && \text{(by (R10))} \\ &= r_{\mathscr{B}}(h \vdash b) \blacktriangleleft [b \vdash t] && \text{(by choice of } a) \\ &= \big(h \triangleright r_{\mathscr{B}}(b)\big) \blacktriangleleft [b \vdash t] && \text{(by (L10))} \\ &= h \triangleright \big(r_{\mathscr{B}}(b) \blacktriangleleft [b \vdash t]\big) && \text{(by (C1))} \\ &= h \triangleright r_{\mathcal{X}}\big(q_{\mathscr{B}}(b) \blacktriangleleft t\big) && \text{(by (R10))}. \end{aligned}$$

On the other hand, since $q_{\mathscr{B}}(b'c') = q_{\mathscr{B}}(b')q_{\mathscr{B}}(c')$ (Property (F1) in [7, Definition 2.9]), we have

$$\begin{aligned} q_{\mathscr{B}}(h \vdash ([b \mathbin{\prec} t]c')) &\overset{\text{(B1)}}{=} h \triangleright q_{\mathscr{B}}([b \mathbin{\prec} t]c') \overset{\text{(F1)}}{=} h \triangleright [q_{\mathscr{B}}(b \mathbin{\prec} t)q_{\mathscr{B}}(c')] \\ &\overset{\text{(L4)}}{=} [h \triangleright q_{\mathscr{B}}(b \mathbin{\prec} t)] \Big([h \triangleleft q_{\mathscr{B}}(b \mathbin{\prec} t)] \triangleright q_{\mathscr{B}}(c')\Big), \end{aligned}$$

so that it follows from (L10) and (B1) for $\mathbin{\prec}$ that

$$r_{\mathscr{B}}(h \vdash ([b \mathbin{\prec} t]c')) = r_{\mathcal{X}}\big(h \triangleright q_{\mathscr{B}}(b \mathbin{\prec} t)\big) = h \triangleright r_{\mathcal{X}}\big(q_{\mathscr{B}}(b) \blacktriangleleft t\big).$$

Our earlier computation therefore shows that $s_{\mathscr{B}}\big(a \mathbin{\prec} (b \vdash t)\big) = r_{\mathscr{B}}(h \vdash ([b \mathbin{\prec} t]c'))$, as needed. This shows that the left and right actions commute. □

Lemma 6.5. *The formulas in (3) and (4) of Theorem 6.1 define inner products on the USC Banach bundle $\mathscr{B}$ in the sense of [7, Definition 2.11],* (FE2.a)–(FE2.c).

Proof. We will do the proof for the left inner product; the other one follows *mutatis mutandis.*

First, we verify that the inner product is well-defined. As $\mathfrak{s}_{\mathscr{B}}(a) = \mathfrak{s}_{\mathscr{B}}(b)$, the definition of $\mathfrak{s}_{\mathscr{B}} = \mathfrak{s} \circ q_{\mathscr{B}}$ implies the existence of h satisfying $h \triangleright s_{\mathscr{B}}(b) = s_{\mathscr{B}}(a)$. As this implies $s_{\mathcal{H}}(h) = \rho_{\mathcal{X}}(s_{\mathscr{B}}(b))$, we therefore have

$$\rho_{\mathscr{B}}(b^*) = \rho_{\mathcal{X}}(r_{\mathscr{B}}(b^*)) = \rho_{\mathcal{X}}(s_{\mathscr{B}}(b)) = s_{\mathcal{H}}(h),$$

and so $h \vdash b^*$ and $h \dashv b^* = h \triangleleft q_{\mathscr{B}}(b^*)$ make sense. Now $q_{\mathscr{B}}(h \vdash b^*) = h \triangleright q_{\mathscr{B}}(b^*)$, and so by (L10), we thus have

$$r_{\mathscr{B}}(h \vdash b^*) = h \vdash r_{\mathscr{B}}(b^*) = h \triangleright s_{\mathscr{B}}(b) = s_{\mathscr{B}}(a),$$

so that $a[h \vdash b^*]$ makes sense.

To see that

$${}_{\mathscr{A}}\langle a \mid b\rangle = \big([a(h \vdash b^*)] \dashv \mathcal{G}, h \dashv b^*\big)$$

is an element of $\mathscr{A} = (\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$, we have to verify that

$$(\rho_{\mathcal{X}/\mathcal{G}} \circ s_{\mathcal{X}/\mathcal{G}} \circ q_{\mathscr{B}/\mathcal{G}})\big([a(h \vdash b^*)] \dashv \mathcal{G}\big) = r_{\mathcal{H}}(h \dashv b^*).$$

Recall from a $\dashv$-version of Equation (5.1) that

$$q_{\mathscr{B}/\mathcal{G}}\big([a(h \vdash b^*)] \dashv \mathcal{G}\big) = q_{\mathscr{B}}\big(a(h \vdash b^*)\big) \blacktriangleleft \mathcal{G}.$$

Moreover, $s_{\mathcal{X}/\mathcal{G}}(x \blacktriangleleft \mathcal{G}) = s_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G}$ (cf. the definition before Lemma 3.1) and $\rho_{\mathcal{X}/\mathcal{G}}(x \blacktriangleleft \mathcal{G}) = \rho_{\mathcal{X}}(x)$ by the definition in Proposition 3.12. Thus

$$(\rho_{\mathcal{X}/\mathcal{G}} \circ s_{\mathcal{X}/\mathcal{G}} \circ q_{\mathscr{B}/\mathcal{G}})\big([a(h \vdash b^*)] \dashv \mathcal{G}\big) = \rho_{\mathcal{X}}\big(s_{\mathscr{B}}(a[h \vdash b^*])\big) = \rho_{\mathcal{X}}\big(s_{\mathscr{B}}(h \vdash b^*)\big).$$

On the other hand, we have

$$r_{\mathcal{H}}(h \dashv b^*) = r_{\mathcal{H}}(h \triangleleft q_{\mathscr{B}}(b^*)) \overset{\text{(L1)}}{=} \rho_{\mathcal{X}}\big(s_{\mathcal{X}}(h \triangleright q_{\mathscr{B}}(b^*))\big) \overset{\text{(B1)}}{=} \rho_{\mathcal{X}}\big(s_{\mathscr{B}}(h \vdash b^*)\big),$$

as required. The inner product is thus well-defined and lands in the right space.

Since multiplication on $\mathscr{B}$, $\vdash$, and $\dashv$ are linear, we see that ${}_{\mathscr{A}}\langle \cdot \mid \cdot\rangle$ is linear in the first and conjugate linear in the second coordinate. To check that it satisfies the other required properties, let $x := q_{\mathscr{A}}(a)$ and $y := q_{\mathscr{A}}(b)$ and $h \in \mathcal{H}$ be as above.

For (FE2.a), we must check that, when $q_{\mathscr{A}}\big({}_{\mathscr{A}}\langle a \mid b\rangle\big) \in (\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ acts on the left of y, it yields x. By the definition of $q_{\mathscr{A}}$ (see Definition 4.5) and our computations above, we have

$$\begin{aligned} q_{\mathscr{A}}\big({}_{\mathscr{A}}\langle a \mid b\rangle\big) &= \big(q_{\mathscr{B}/\mathcal{G}}\big([a(h \vdash b^*)] \dashv \mathcal{G}\big), h \dashv b^*\big) = \big(q_{\mathscr{B}}\big(a(h \vdash b^*)\big) \blacktriangleleft \mathcal{G}, h \dashv b^*\big) \\ &= \big([x(h \triangleright y^{-1})] \blacktriangleleft \mathcal{G}, h \triangleleft y^{-1}\big). \end{aligned}$$

By the definition of the left $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$-action on $\mathcal{X}$ (Proposition 3.14),

$$\begin{aligned} &\big([x(h \triangleright y^{-1})] \blacktriangleleft \mathcal{G}, h \triangleleft y^{-1}\big) \cdot y \\ &= x(h \triangleright y^{-1})[(h \triangleleft y^{-1}) \triangleright y] = x(h \triangleright (y^{-1}y)) && \text{(by (L4))} \\ &= x(h \triangleright s_{\mathcal{X}}(y)) = x s_{\mathcal{X}}(x) = x && \text{(by choice of } h\text{).} \end{aligned}$$

To show (FE2.b), we must prove that ${}_{\mathscr{A}}\langle a \mid b\rangle^* = {}_{\mathscr{A}}\langle b \mid a\rangle$ Since $s_{\mathcal{X}}(y) = h^{-1} \triangleright s_{\mathcal{X}}(x)$, we have

$$ {}_{\mathscr{A}}\langle b \mid a\rangle = \big(\big[b(h^{-1} \vDash a^*)\big] \dashv \mathcal{G}, h^{-1} \triangleleft x^{-1}\big). $$

Using the definition of the involution on $\mathscr{A}$ (see Definition 4.5), we can compute the adjoint of

$$ {}_{\mathscr{A}}\langle a \mid b\rangle = \big([a(h \vDash b^*)] \dashv \mathcal{G}, h \dashv b^*\big). $$

Its $\mathscr{B}/\mathcal{G}$-component

$$ \Big[(h \triangleleft y^{-1})^{-1} \vDash \big(a[h \vDash b^*]\big)^*\Big] \dashv \mathcal{G} \quad \text{has to equal} \quad \big[b(h^{-1} \vDash a^*)\big] \dashv \mathcal{G} \tag{6.2} $$

and that its $\mathcal{H}$-component

$$ (h \triangleleft y^{-1})^{-1} \triangleleft \big(x[h \triangleright y^{-1}]\big)^{-1} \quad \text{has to equal} \quad h^{-1} \triangleleft x^{-1}. \tag{6.3} $$

If $z := h \triangleright y^{-1}$ and $k := h \triangleleft y^{-1}$, then by (L9), we have $k^{-1} = h^{-1} \triangleleft z$. Thus, the asserted equality in (6.3) is easily seen:

$$ \begin{aligned} (h \triangleleft y^{-1})^{-1} \triangleleft (x[h \triangleright y^{-1}])^{-1} &= k^{-1} \triangleleft (xz)^{-1} \\ &= (h^{-1} \triangleleft z) \triangleleft (xz)^{-1} \overset{\text{(L3)}}{=} h^{-1} \triangleleft x^{-1}. \end{aligned} $$

For the asserted equality in (6.2), we compute

$$ \big(a[h \vDash b^*]\big)^* = [h \vDash b^*]^* a^* \overset{\text{(B5)}}{=} \big[(h \dashv b^*) \vDash b\big] a^*. $$

If $c := (h \dashv b^*) \vDash b = k \vDash b$, then we have for the left-hand side of (6.2)

$$ (h \triangleleft y^{-1})^{-1} \vDash \big(a[h \vDash b^*]\big)^* = k^{-1} \vDash (ca^*) \overset{\text{(B4)}}{=} (k^{-1} \vDash c)\big[(k^{-1} \dashv c) \vDash a^*\big]. \tag{6.4} $$

Since $k = h \triangleleft y^{-1} = h \dashv b^*$, we have

$$ k^{-1} \vDash c = k^{-1} \vDash [(h \dashv b^*) \vDash b] \overset{\text{(B2)}}{=} \big[k^{-1}(h \dashv b^*)\big] \vDash b \overset{\text{(B3)}}{=} b. $$

On the other hand, $q_{\mathscr{B}}(c) = k \triangleright y$ by (BC1), so that

$$ k^{-1} \dashv c = k^{-1} \triangleleft (k \triangleright y) \overset{\text{(L9)}}{=} (k \triangleleft y)^{-1} = \big([h \triangleleft y^{-1}] \triangleleft y\big)^{-1} \overset{\text{(L3)}}{=} h^{-1}. $$

Plugging the results of our last computations back into Equation (6.4), we get

$$(h \triangleleft y^{-1})^{-1} \vdash \big(a[h \vdash b^*]\big)^* = b\big(h^{-1} \vdash a^*\big),$$

which is, on the nose, what we needed for (6.2).

Lastly, for (FE2.c), we need that the inner product is $\mathscr{A}$-linear in the first component, so let (Θ, k) be an arbitrary element of $\mathscr{A}$ with $s_{\mathscr{A}}(\Theta, k) = \mathfrak{r}_{\mathscr{B}}(a)$. If $c \in \Theta$ is such that $s_{\mathscr{B}}(c) = r_{\mathscr{B}}(k \vdash a)$, then our definition of the left $\mathscr{A}$-action on $\mathscr{B}$ (see 6.1(1)) yields $(\Theta, k) \cdot a = c[k \vdash a]$. Note that $m := (k \triangleleft x)h$ is the unique element of $\mathcal{H}$ such that $s_{\mathscr{B}}(c[k \vdash a]) = m \triangleright s_{\mathscr{B}}(b)$, since

$$s_{\mathscr{B}}(c[k \vdash a]) = s_{\mathscr{B}}(k \vdash a) \overset{\text{(L10)}}{=} (k \triangleleft x) \triangleright s_{\mathscr{B}}(a) = (k \triangleleft x) \triangleright [h \triangleright s_{\mathscr{B}}(b)].$$

We have

$$_{\mathscr{A}}\langle (\Theta, k) \cdot a \mid b \rangle = \Big(\big[(c[k \vdash a])(m \vdash b^*)\big] \dashv \mathcal{G}, m \dashv b^* \Big). \tag{6.5}$$

On the other hand, according to Definition 4.5, the product of

$$(\Theta, k)\,_{\mathscr{A}}\langle a \mid b \rangle = (\Theta, k)\,\big([a(h \vdash b^*)] \dashv \mathcal{G}, h \dashv b^*\big)$$

in $\mathscr{A}$ has $\mathscr{B}/\mathcal{G}$-component

$$\Theta\Big[k \circledtriangleright \Big(\big[a(h \vdash b^*)\big] \dashv \mathcal{G}\Big)\Big] = \Theta\Big[\Big(k \vdash \big[a(h \vdash b^*)\big]\Big) \dashv \mathcal{G}\Big]. \tag{6.6}$$

We compute

$$\begin{aligned} k \vdash \big[a(h \vdash b^*)\big] &= (k \vdash a)\big[(k \dashv a) \vdash (h \vdash b^*)\big] && \text{(by (B4))} \\ &= (k \vdash a)(m \vdash b^*) && \text{(by (B2))}. \end{aligned}$$

Note that $c \in \Theta$ was chosen such that $s_{\mathscr{B}}(c) = r_{\mathscr{B}}(k \vdash a)$, so that the above computation together with the definition of the multiplication in $\mathscr{B}/\mathcal{G}$ (cf. Proposition 5.5) shows that the $\mathscr{B}/\mathcal{G}$-component of $(\Theta, k)\,_{\mathscr{A}}\langle a \mid b \rangle$ is

$$\Theta\Big[\big((k \vdash a)(m \vdash b^*)\big) \dashv \mathcal{G}\Big] = \Big[c[(k \vdash a)(m \vdash b^*)]\Big] \dashv \mathcal{G},$$

which, by associativity of the multiplication on $\mathscr{B}$, is exactly the $\mathscr{B}/\mathcal{G}$-component of $_{\mathscr{A}}\langle (\Theta, k) \cdot a \mid b \rangle$; see (6.5).

Similarly, the $\mathcal{H}$-component of $(\Theta, k)\,_{\mathscr{A}}\langle a \mid b \rangle$ is given by

$$\begin{aligned} &\Big[k \circledtriangleleft q_{\mathscr{B}/\mathcal{G}}\Big(\big[a(h \vdash b^*)\big] \dashv \mathcal{G}\Big)\Big](h \dashv b^*) \\ &\quad = \Big[k \circledtriangleleft \Big(q_{\mathscr{B}}\big(a(h \vdash b^*)\big) \dashv \mathcal{G}\Big)\Big](h \dashv b^*) && \text{(def'n of } q_{\mathscr{B}/\mathcal{G}}) \end{aligned}$$

$$
\begin{aligned}
&= \Big[k \triangleleft q_{\mathscr{B}}\big(a(h \vDash b^*)\big)\Big](h \dashv b^*) && \text{(def'n of } \circledcirc\text{)}\\
&= \Big[(k \triangleleft x) \triangleleft q_{\mathscr{B}}(h \vDash b^*)\Big](h \dashv b^*) && \text{(by (L3) and (F1))}\\
&= \Big[(k \triangleleft x) \triangleleft (h \triangleright y^{-1})\Big](h \triangleleft y^{-1}) && \text{(by (BC1) and def'n of } y)\\
&= [(k \triangleleft x)h] \triangleleft y^{-1} && \text{(by (L6))}
\end{aligned}
$$

which is exactly $m \dashv b^*$, as needed. □

Lemma 6.6 *(Regarding* (FE2.d)*). The inner products on the USC Banach bundle $\mathscr{B}$ satisfy* (FE2.d)*, i.e., ${}_{\mathscr{A}}\langle a \mid b\rangle \cdot c = a \cdot \langle b \mid c\rangle_{\mathscr{C}}$ whenever both inner products make sense.*

Proof. Let $a \in \mathscr{B}_x$, $b \in \mathscr{B}_y$, and $c \in \mathscr{B}_z$. For the inner products to be defined, we require $\mathfrak{s}_{\mathscr{B}}(a) = \mathfrak{s}_{\mathscr{B}}(b)$ and $\mathfrak{r}_{\mathscr{B}}(b) = \mathfrak{r}_{\mathscr{B}}(c)$, so there exist $h \in \mathcal{H}$ and $t \in \mathcal{G}$ such that $s_{\mathcal{X}}(x) = h \triangleright s_{\mathcal{X}}(y)$ and $r_{\mathcal{X}}(y) \blacktriangleleft t = r_{\mathcal{X}}(z)$, so that

$$
{}_{\mathscr{A}}\langle a \mid b\rangle = \big([a(h \vDash b^*)] \dashv \mathcal{G}, h \dashv b^*\big) \quad \text{and} \quad \langle b \mid c\rangle_{\mathscr{C}} = \big(b^* \vDash t, \mathcal{H} \vDash [(b^* \dashv t)c]\big).
$$

If we let $\Theta := [a(h \vDash b^*)] \dashv \mathcal{G}$, then

$$
\begin{aligned}
s_{\mathscr{A}}\big({}_{\mathscr{A}}\langle a \mid b\rangle\big) &= (h \dashv b^*)^{-1} \circledR\, s_{\mathscr{B}/\mathcal{G}}(\Theta) && \text{(cf. Def. 2.16 and Rmk. 2.19)}\\
&= (h \triangleleft y^{-1})^{-1} \circledR\, s_{\mathcal{X}/\mathcal{G}}\big([x(h \triangleright y^{-1})] \blacktriangleleft \mathcal{G}\big)\\
&= \big[(h \triangleleft y^{-1})^{-1} \triangleright s_{\mathcal{X}}(x[h \triangleright y^{-1}])\big] \blacktriangleleft \mathcal{G} && \text{(def'n of } s_{\mathcal{X}/\mathcal{G}} \text{ and } \circledR).
\end{aligned}
$$

Since

$$
s_{\mathcal{X}}\big(x[h \triangleright y^{-1}]\big) = s_{\mathcal{X}}(h \triangleright y^{-1}) \overset{\text{(L10)}}{=} [h \triangleleft y^{-1}] \triangleright r_{\mathcal{X}}(y),
$$

it follows that

$$
s_{\mathscr{A}}\big({}_{\mathscr{A}}\langle a \mid b\rangle\big) = r_{\mathcal{X}}(y) \blacktriangleleft \mathcal{G} = r_{\mathscr{B}}(c) \blacktriangleleft \mathcal{G} = \mathfrak{r}_{\mathscr{B}}(c),
$$

so that ${}_{\mathscr{A}}\langle a \mid b\rangle \cdot c$ is indeed defined. Moreover, we see that t can act on the right of $a(h \vDash b^*)$ and that

$$
\begin{aligned}
s_{\mathscr{B}}\big([a(h \vDash b^*)] \dashv t\big) = s_{\mathcal{X}}\big(x[h \triangleright y^{-1}]\big) \blacktriangleleft t &\overset{\text{(C1)}}{=} [h \triangleleft y^{-1}] \triangleright [r_{\mathcal{X}}(y) \blacktriangleleft t]\\
&= [h \triangleleft y^{-1}] \triangleright r_{\mathcal{X}}(z) \overset{\text{(L10)}}{=} r_{\mathcal{X}}\big([h \triangleleft y^{-1}] \triangleright z\big).
\end{aligned}
$$

Thus, $[a(h \vDash b^*)] \dashv t$ is the (unique) element of Θ whose image under $s_{\mathscr{B}}$ equals $r_{\mathscr{B}}([h \dashv b^*] \vDash c)$, so that

$$ {}_{\mathscr{A}}\langle a \mid b\rangle \cdot c = \Big(\big[a(h \vDash b^*)\big] \prec t \Big) \big([h \dashv b^*] \vDash c \big). \tag{6.7} $$

A similar argument shows that $a \cdot \langle b \mid c\rangle_{\mathscr{C}}$ is well-defined and that

$$ a \cdot \langle b \mid c\rangle_{\mathscr{C}} = \big(a \prec [b^* \succ t]\big) \Big(h \vDash \big[(b^* \prec t)c\big] \Big). \tag{6.8} $$

We compute the first element of the product in (6.7) to be

$$ \begin{aligned} \big[a(h \vDash b^*)\big] \prec t &= \Big(a \prec \big[(h \vDash b^*) \succ t\big] \Big) \big[(h \vDash b^*) \prec t\big] && \text{(by (R4))} \\ &= \big(a \prec [b^* \succ t]\big) \big[h \vDash (b^* \prec t)\big] && \text{(by (C2) and (BC1))} \end{aligned} $$

and its second element to be

$$ [h \dashv b^*] \vDash c = \big[h \dashv (b^* \prec t)\big] \vDash c \qquad \text{(by (C3))}, $$

so that it follows from (6.8) that

$$ \begin{aligned} {}_{\mathscr{A}}\langle a \mid b\rangle \cdot c &= \big(a \prec [b^* \succ t]\big) \big[h \vDash (b^* \prec t)\big] \Big(\big[h \dashv (b^* \prec t)\big] \vDash c \Big) \\ &= \big(a \prec [b^* \succ t]\big) \Big(h \vDash \big[(b^* \prec t)c\big] \Big) = a \cdot \langle b \mid c\rangle_{\mathscr{C}} \qquad \text{(by (B4))}. \quad \square \end{aligned} $$

Lemma 6.7. *With the induced actions, each $B(x)$ is a $A\big(\mathfrak{r}(x)\big) - C\big(\mathfrak{s}(x)\big)$-imprimitivity bimodule.*

Proof. For $x \in \mathcal{X}$, we have (see the definitions of $\mathfrak{s}$ and $\mathfrak{r}$ in Proposition 3.14):

$$ \mathfrak{r}(x) = r_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G} \quad \text{and} \quad \mathfrak{s}(x) = \mathcal{H} \triangleright s_{\mathcal{X}}(x). $$

Recall that here, we have identified the unit spaces of $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ with those of $\mathcal{X}/\mathcal{G}$ and $\mathcal{H}\backslash\mathcal{X}$, respectively; cf. Remark 2.19. Thus, if we want to think of $\mathfrak{r}(x)$ and $\mathfrak{s}(x)$ in $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$, we must write

$$ \mathfrak{r}(x) = \big(r_{\mathcal{X}}(x) \blacktriangleleft \mathcal{G}, \rho_{\mathcal{X}}(x)\big) \quad \text{and} \quad \mathfrak{s}(x) = \big(\sigma_{\mathcal{X}}(x), \mathcal{H} \triangleright s_{\mathcal{X}}(x)\big), $$

where we have used that $\rho_{\mathcal{X}}^{(0)} \circ r_{\mathcal{X}} = \rho_{\mathcal{X}}$ and $\sigma_{\mathcal{X}}^{(0)} \circ s_{\mathcal{X}} = \sigma_{\mathcal{X}}$ by definition of the right-hand sides.

Now, recall that $\mathscr{B}$ is a Fell bundle equivalence between $\mathscr{B}$ and itself; in particular, we know that each $B(x)$ is a $B\big(r_{\mathcal{X}}(x)\big) - B\big(s_{\mathcal{X}}(x)\big)$-imprimitivity bimodule. We claim that the fiber $A(\mathfrak{r}(x))$ is isomorphic to $B(r_{\mathcal{X}}(x))$ and likewise that $C(\mathfrak{s}(x))$ is isomorphic to $B(s_{\mathcal{X}}(x))$, and that these isomorphisms turn the canonical $B(r(x)) - B(s(x))$-imprimitivity bimodule $B(x)$ into our bi-Hilbertian $A(\mathfrak{r}(x)) - C(\mathfrak{s}(x))$-module $B(x)$, proving that the latter is an imprimitivity bimodule also. We will do so for the fiber $A(\mathfrak{r}(x))$ of $\mathscr{A} = (\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$.

Define

$$\psi\colon B(r_{\mathcal{X}}(x)) \to A(\mathfrak{r}(x)) \quad \text{by} \quad \psi(a) = (a \blacktriangleleft \mathcal{G}, \rho_{\mathcal{X}}(x)).$$

This map is clearly linear, $*$-preserving, surjective, and injective, since the norm on $\mathscr{A}_{\mathfrak{r}(x)}$ is inherited from $\mathscr{B}_{r(x)}$. Therefore, ψ defines an isomorphism of C*-algebras.

Notice that this isomorphism indeed turns the left $A(\mathfrak{r}(x))$-action on $B(x)$ into the left $B(r_{\mathcal{X}}(x))$-multiplication on $B(x)$: if $b \in B(x)$ and $a \in \mathscr{B}_{r_{\mathcal{X}}(x)}$, then

$$s_{\mathscr{B}}(a) = s_{\mathcal{X}}(r_{\mathcal{X}}(x)) = r_{\mathcal{X}}(x) = r_{\mathscr{B}}(b) \overset{\text{(BC3)}}{=} r_{\mathscr{B}}(\rho_{\mathcal{X}}(x) \triangleright b),$$

proving that a is the unique element in $a \blacktriangleleft \mathcal{G}$ such that $s_{\mathscr{B}}(a) = r_{\mathscr{B}}(\rho_{\mathcal{X}}(x) \triangleright b)$, so that the definition of the left $\mathscr{A}$-action on $\mathscr{B}$ implies

$$\psi(a) \cdot b = (a \blacktriangleleft \mathcal{G}, \rho_{\mathcal{X}}(x)) \cdot b = ab,$$

as claimed. □

Proof of Theorem 6.1. The groupoids $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ are locally compact Hausdorff: In Proposition 3.3, we have seen that the quotient of locally compact Hausdorff groupoids is again locally compact Hausdorff, and clearly so is the self-similar product of such groupoids.

We have seen that $\mathcal{X}$ is a groupoid equivalence between $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ (Theorem 3.10 and Proposition 3.14), and that $\mathscr{A}$ and $\mathscr{C}$ are Fell bundles by Definition 4.5 and Remark 4.10. Moreover, as $\mathscr{B}$ is assumed to be saturated, it follows from Remark 5.6 that $\mathcal{H}\backslash\mathscr{B}$ and $\mathscr{B}/\mathcal{G}$ are saturated also. Consequently, it follows from Remark 4.8 that $\mathscr{A}$ and $\mathscr{C}$ are saturated, and so we are dealing with the right ingredients.

We have then checked that all conditions in [7, Definition 2.11] are satisfied. Indeed,

- Re (FE1): Lemma 6.3 shows that the formulas in 6.1(1) and 6.1(2) define actions in the sense of [7, Definition 2.10], and Lemma 6.4 shows that they commute.
- Re (FE2): Lemma 6.5 shows that the formulas in 6.1(3) and 6.1(4) define inner products, while Lemma 6.6 shows that they satisfy the imprimitivity condition (FE2.d), and finally
- Re (FE3): Lemma 6.7 shows that each $B(x)$ is an imprimitivity bimodule. □

Corollary 6.8. *We assume all conditions in Assumption 5.9. Assume that $\mathcal{X}$ has a $\triangleright$-invariant left Haar system (Definition 2.27) whose associated right Haar system is $\blacktriangleleft$-invariant, and that $\mathcal{H}$ and $\mathcal{G}$ also have Haar systems. Then the Fell bundle C*-algebras $C^*((\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H})$ and $C^*(\mathcal{G} \bowtie (\mathcal{H}\backslash\mathscr{B}))$ are Morita equivalent.*

Recall from Corollary 2.31 that the assumptions regarding Haar systems are satisfied if $\mathcal{X}, \mathcal{H}, \mathcal{G}$ are étale.

Proof. All Fell bundles in sight are saturated, since $\mathscr{B}$ is saturated. By Theorem 6.1, the Fell bundles $(\mathscr{B}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathscr{B})$ are equivalent. Recall from Corollary 3.23 that both groupoids $(\mathcal{X}/\mathcal{G}) \bowtie \mathcal{H}$ and $\mathcal{G} \bowtie (\mathcal{H}\backslash\mathcal{X})$ allow Haar systems, so that the claim now follows from an application of [23, Theorem 6.4]. $\square$

One immediate application is the one-sided imprimitivity theorem by setting $\mathcal{G} = \{e\}$.

Corollary 6.9. *Let $\mathcal{X}$ be a groupoid and $\mathscr{B}$ be a Fell bundle over $\mathcal{X}$. Suppose $\mathcal{H}$ has a self-similar left action on the Fell bundle $\mathscr{B}$, and that the action of $\mathcal{H}$ on $\mathcal{X}$ is free and proper. Then $\mathscr{B}$ is a Fell bundle equivalence between $\mathscr{B} \bowtie \mathcal{H}$ and $\mathcal{H}\backslash\mathscr{B}$. In particular, if $\mathcal{X}$ has a $\triangleright$-invariant Haar system and $\mathcal{H}$ admits any Haar system, then $C^*(\mathscr{B} \bowtie \mathcal{H})$ and $C^*(\mathcal{H}\backslash\mathscr{B})$ are Morita equivalent.*

Example 6.10 *(combination of previous examples).* Suppose that the self-similar left action $\triangleright$ of $\mathcal{H}$ on the groupoid $\mathcal{X}$ is free and proper, that $\mathcal{H}$ has open source map, and that $(\mathscr{A}, \mathcal{H}\backslash\mathcal{X}, \alpha)$ is a groupoid dynamical system. We have stated in Example 5.7 that

$$\mathcal{H}\backslash\mathscr{B}(\mathscr{A}, \mathcal{X}, \tilde{\alpha}) \cong \mathscr{B}(\mathscr{A}, \mathcal{H}\backslash\mathcal{X}, \alpha), \tag{6.9}$$

where $\tilde{\alpha} = \alpha \circ q$ for q the quotient map. On the other hand, if we let $p\colon \mathcal{X} \bowtie \mathcal{H} \to \mathcal{X}$ be the projection onto the first component, then $(\mathscr{A}, \mathcal{X} \bowtie \mathcal{H}, \tilde{\alpha} \circ p)$ is a groupoid dynamical system on $\mathscr{A}$ whose restriction to $\mathcal{X}$ is $\tilde{\alpha}$. By Example 4.7, $\mathcal{H}$ thus has a self-similar left action on $\mathscr{B}(\mathscr{A}, \mathcal{X}, \tilde{\alpha})$ given by $h \blacktriangleright (a, x) := (a, h \triangleright x)$, and we have

$$\mathscr{B}(\mathscr{A}, \mathcal{X}, \tilde{\alpha}) \bowtie \mathcal{H} \cong \mathscr{B}(\mathscr{A}, \mathcal{X} \bowtie \mathcal{H}, \tilde{\alpha} \circ p). \tag{6.10}$$

By Corollary 6.9, the Fell bundles on the left-hand sides of (6.9) and (6.10) are equivalent, so that $\mathscr{B}(\mathscr{A}, \mathcal{H}\backslash\mathcal{X}, \alpha)$ and $\mathscr{B}(\mathscr{A}, \mathcal{X} \bowtie \mathcal{H}, \tilde{\alpha} \circ p)$ are also equivalent. If the groupoids have appropriate Haar systems (for example, if they are étale), then this implies that the groupoid crossed product $\mathscr{A} \rtimes_{\alpha} (\mathcal{H}\backslash\mathcal{X})$ is Morita equivalent to $\mathscr{A} \rtimes_{\tilde{\alpha}\circ p} (\mathcal{X} \bowtie \mathcal{H})$.

7. Examples on Deaconu–Renault groupoids

One interesting class of self-similar actions arises from Deaconu–Renault groupoids [34, Section 3], and we devote the last section to examples from this class of groupoids. It is observed in [2, Proposition 5.1] that a Deaconu–Renault groupoid generated by a pair of $*$-commuting endomorphisms has a Zappa–Szép product structure. We will describe this as self-similar product in more detail, and apply our main result on equivalent groupoids (Theorem 3.10) in this context.

We first give a brief overview of Deaconu–Renault groupoids. For Y a topological space, we say a map $\sigma\colon Y \to Y$ is an *endomorphism* if it is a surjective local homeomorphism, and we denote the collection of all endomorphisms on Y by $\mathrm{End}(Y)$. We note that an endomorphism may not be injective. Suppose $\theta\colon \mathbb{N}^k \to \mathrm{End}(Y)$ is a semigroup action on Y by endomorphisms. The Deaconu–Renault groupoid, denoted $Y \rtimes_\theta \mathbb{N}^k$, is defined as

$$Y \rtimes_\theta \mathbb{N}^k = \big\{(x, p-q, y) \in Y \times \mathbb{Z}^k \times Y : \theta_p(x) = \theta_q(y)\big\}$$

with multiplication and inverse given by

$$(x, p-q, y)(y, m-n, z) = (x, (p+m)-(q+n), z),$$
$$(x, p-q, y)^{-1} = (y, q-p, x).$$

Its range and source maps are therefore given by

$$r(x, p-q, y) = (x, 0, x),$$
$$s(x, p-q, y) = (y, 0, y),$$

and its unit space is identified as $\{(x,0,x) : x \in Y\} \approx Y$. We give $Y \rtimes_\theta \mathbb{N}^k$ the topology induced by the basic open sets $Z_\theta(U, m, n, V)$, defined for open subsets $U, V \subseteq Y$ and vectors $m, n \in \mathbb{N}^k$ by

$$Z_\theta(U, m, n, V) := \{(x, m-n, y) : x \in U, y \in V \text{ and } \theta_m x = \theta_n y\}.$$

This makes $Y \rtimes_\theta \mathbb{N}^k$ a locally compact Hausdorff étale groupoid [34, Lemma 3.1.].

To two commuting elements $S, T \in \mathrm{End}(Y)$, we can naturally associate an $\mathbb{N}^2$-action on Y given by $\theta_{p,m}(x) = T^p S^m x$. We let $\mathcal{K} = Y \rtimes_\theta \mathbb{N}^2$ be the corresponding Deaconu–Renault groupoid. Each of the endomorphisms S and T corresponds to an $\mathbb{N}$-action on Y, so we can define their respective Deaconu–Renault groupoid as

$$\mathcal{H} = Y \rtimes_T \mathbb{N} = \{(x, p-q, y) \in Y \times \mathbb{Z} \times Y : T^p x = T^q y\},$$
$$\mathcal{X} = Y \rtimes_S \mathbb{N} = \{(x, m-n, y) \in Y \times \mathbb{Z} \times Y : S^m x = S^n y\}.$$

From now on, we fix S and T and further assume that they $*$-*commute*: not only do we have $ST = TS$, but whenever $Sx = Ty$ for some $x, y \in Y$, then there exists a unique $z \in Y$ such that $Tz = x$ and $Sz = y$. Note that, for all integers $p, q \geq 1$, S^p, T^q are also $*$-commuting. It was observed in [2, Proposition 5.1] that, in this setting, $\mathcal{K}$ can be realized as the Zappa–Szép product groupoid $\mathcal{X} \bowtie \mathcal{H}$. The proof uses a unique decomposition property but does not describe the actions of $\mathcal{X}$ and $\mathcal{H}$ on each other explicitly, so we start by giving such a description.

Lemma 7.1. *Let $\mathcal{H}$ and $\mathcal{X}$ be the Deaconu–Renault groupoids described above. Then the following maps define a self-similar left action of $\mathcal{H}$ on $\mathcal{X}$, where $w \in Y$ is the unique element that satisfies $S^n w = S^m x$ and $T^p w = T^q z$:*

$$\mathcal{H} \curvearrowright \mathcal{X}: \qquad (x, p-q, y) \triangleright (y, m-n, z) = (x, m-n, w) \in \mathcal{X}$$
$$\mathcal{H} \circlearrowleft \mathcal{X}: \qquad (x, p-q, y) \triangleleft (y, m-n, z) = (w, p-q, z) \in \mathcal{H}$$

Here, $\mathcal{H}^{(0)} = \mathcal{X}^{(0)} = Y$, so that we can use id_Y as the continuous surjection $\mathcal{X}^{(0)} \to \mathcal{H}^{(0)}$.

Proof. First, the element $w \in Y$ exists because

$$T^p(S^m x) = S^m(T^p x) = S^m(T^q y) = T^q(S^m y) = T^q(S^n z) = S^n(T^q z).$$

We apply the $*$-commuting condition for T^p and S^n to obtain the desired w.

From [2, Proposition 5.1], $Y \rtimes_\theta \mathbb{N}^2$ is an internal Zappa–Szép product of the groupoids $\mathcal{H}$ and $\mathcal{X}$. Here, we embed $\mathcal{H}$ and $\mathcal{X}$ as subgroupoids of $Y \rtimes_\theta \mathbb{N}^2$ by identifying $(x, k, y) \in \mathcal{H}$ and $(y, \ell, z) \in \mathcal{X}$ as $(x, (k, 0), y)$ and $(y, (0, \ell), z)$ in $Y \rtimes_\theta \mathbb{N}^2$, respectively.

It follows from [2, Proposition 3.4] that the corresponding self-similar actions are uniquely determined by the equation

$$gh = (h \triangleright g)(h \triangleleft g), \quad h \in \mathcal{H}, g \in \mathcal{X}.$$

Therefore, it suffices to verify that the self-similar left action of $\mathcal{H}$ on $\mathcal{X}$ satisfies this equation.

Pick any $x, y, z \in Y$ and $p, q, m, n \in \mathbb{Z}$ such that

$$(x, (p-q, 0), y) \in \mathcal{H} \subseteq Y \rtimes_\theta \mathbb{N}^2 \quad \text{and} \quad (y, (0, m-n), z) \in \mathcal{G} \subseteq Y \rtimes_\theta \mathbb{N}^2.$$

If $w \in Y$ is the unique element that satisfies $S^n w = S^m x$ and $T^p w = T^q z$, then

$$\begin{aligned}(x, (p-q, 0), y)(y, (0, m-n), z) &= (x, (p-q, m-n), z) \\ &= (x, (0, m-n), w)(w, (p-q, 0), z). \quad \square\end{aligned}$$

For a map $T: Y \to Y$, we say that $x \in Y$ is a *periodic point* for T if $T^k x = x$ for some $k \in \mathbb{N}^\times$. If no such x exists, we call T *non-periodic.*

Lemma 7.2. *The self-similar left action $\triangleright$ defined in Lemma 7.1 is free if and only if T is non-periodic.*

Proof. Suppose $\triangleright$ is not free, so there exists x, y, z and $p \neq q$ such that $(x, p-q, y) \triangleright (y, m-n, z) = (y, m-n, z)$. By definition of $\triangleright$, this equality forces $x = y$. Since $(x, p-q, x) \in \mathcal{H}$ by assumption, this implies $T^p x = T^q x$, so since $p \neq q$, T has a periodic point.

Conversely, assume T has a periodic point x, so there exists $k > 0$ with $T^k x = x$. In this case, $(x, k, x) \in \mathcal{H} \setminus \mathcal{H}^{(0)}$ and $(x, 0, x) \in \mathcal{X}$. One can easily verify that $(x, k, x) \triangleright (x, 0, x) = (x, 0, x)$. $\square$

While the action $\triangleright$ in Lemma 7.1 may not be a proper map in general, the examples on certain classes of 2-graphs that we shall consider later satisfy this property. With properness, Corollary 3.16 implies that the self-similar product groupoid $\mathcal{H} \bowtie \mathcal{X} \cong Y \rtimes \mathbb{N}^2$ is equivalent to the quotient groupoid $\mathcal{H} \backslash \mathcal{X}$, which we conjecture is another Deaconu–Renault groupoid.

Conjecture 7.3. *Partition Y into the equivalence classes given by $[z]_T = \cup_{p,q \in \mathbb{N}} \{w \in Y : T^p w = T^q z\}$. On the quotient space Y_T, define $\widehat{S}\colon Y_T \to Y_T$ by $\widehat{S}([z]_T) = [Sz]_T$. If T is non-periodic, then the map*

$$\Phi\colon \mathcal{H} \backslash \mathcal{X} \to Y_T \rtimes_{\widehat{S}} \mathbb{N}, \quad \mathcal{H} \triangleright (y, k, z) \mapsto ([y]_T, k, [z]_T), \tag{7.1}$$

is an (algebraic) isomorphism of groupoids. If, furthermore, the self-similar left action $\triangleright$ defined in Lemma 7.1 is proper and $\widehat{S}$ is locally injective (so that both groupoids are locally compact Hausdorff), then Φ is a homeomorphism.

While it is easy to show that Φ is a continuous bijection that preserves the groupoid structure, we found no reason for Φ to be open. We are furthermore unsure under which circumstances $\triangleright$ is proper or $\widehat{S}$ locally injective. If the conjecture is true, then it would follow from Corollary 3.16 that the Deaconu–Renault groupoids $Y \rtimes_\theta \mathbb{N}^2$ and $Y_T \rtimes_{\widehat{S}} \mathbb{N}$ are equivalent.

Appendix A. Exercises in topology

Above, the most frequently used topological fact is Fell's Criterion, which we repeat here for convenience.

Proposition A.1 *(Fell's Criterion; [38, Prop. 1.1]). Let $f\colon X \to Y$ be a surjective map between topological spaces. Then f is open if and only if, whenever $\{y_i\}_{i \in I}$ is a net in Y that converges to some $f(x)$, there exists a subnet $\{y_j\}_{j \in J}$ which allows a lift $\{x_j\}_{j \in J}$ in X under f that converges to x.*

The next lemma is an immediate consequence of (2) $\implies$ (1) in [25, Theorem 18.1.].

Lemma A.2. *If $f\colon X \to Y$ is a function, then the following are equivalent.*

(1) *f is continuous.*
(2) *If $\{x_i\}_{i \in I}$ is a net in X which converges to x, then there exists a subnet $\{f(x_j)\}_{j \in J}$ of $\{f(x_i)\}_{i \in I}$ which converges to $f(x)$.*

Data availability

No data was used for the research described in the article.

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