

Remarks on the separation of Navier–Stokes flows

Zachary Bradshaw

Department of Mathematical Sciences, 309 SCEN, University of Arkansas,
Fayetteville, AR 72701, United States of America

E-mail: zb002@uark.edu

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Abstract

Recently, strong evidence has accumulated that some solutions to the Navier–Stokes equations in physically meaningful classes are not unique. The primary purpose of this paper is to establish necessary properties for the error of hypothetical non-unique Navier–Stokes flows under conditions motivated by the scaling of the equations. Our first set of results show that some scales are necessarily active—comparable in norm to the full error—as solutions separate. ‘Scale’ is interpreted in several ways, namely via algebraic bounds, the Fourier transform and discrete volume elements. These results include a new type of uniqueness criteria which is stated in terms of the error. The second result is a conditional predictability criteria for the separation of small perturbations. An implication is that the error necessarily activates at larger scales as flows de-correlate. The last result says that the error of the hypothetical non-unique Leray–Hopf solutions of Jia and Šverák locally grows in a self-similar fashion. Consequently, within the Leray–Hopf class, energy can hypothetically de-correlate at a rate which is faster than linear. This contrasts numerical work on predictability which identifies a linear rate. Our results suggest that this discrepancy may be explained by the fact that non-uniqueness might arise from perturbation around a singular flow.

Keywords: Navier–Stokes equations, uniqueness criteria, separation, predictability

Mathematics Subject Classification numbers: 35Q30, 35Q35, 76F99

1. Introduction

We consider the Navier–Stokes equations,

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0; \quad \nabla \cdot u = 0, \quad (\text{NS})$$

which model the motion of a viscous incompressible fluid with velocity u and its associated pressure p . We consider the problem on $\mathbb{R}^3 \times (0, T)$ for a time $T > 0$. A foundational mathematical treatment of the problem was provided by Leray in [46] where global weak solutions were constructed for finite energy data. These solutions are shown to satisfy a global energy inequality, and can therefore be viewed as physically reasonable. Solutions resembling those constructed by Leray are referred to as Leray–Hopf solutions. Although it has been nearly a century since Leray’s original contribution, important questions remain open about (NS). For example, it is not known if Leray–Hopf solutions can possess finite time singularities. It is also unknown if unforced Leray–Hopf solutions are unique. In recent years, evidence has accumulated suggesting negative answers to these questions. In the direction of blow-up, Tao has constructed singular solutions for a nonlinear model replicating certain features of (NS) [52]. Regarding uniqueness, Buckmaster and Vicol have demonstrated non-uniqueness in a class of solutions which is weaker than the Leray–Hopf class using convex integration [18]. These solutions are not known to satisfy the global or local energy inequalities. Under non-physical forcing, non-uniqueness has been shown in the Leray–Hopf class by Albritton, Brué and Colombo [3]. These non-uniqueness results are built upon foundational work on the Euler equations by De Lellis and Székelyhidi [21], which precedes [18] and Vishik [57, 58] (see also [4]), which precedes [3]. For the Leray–Hopf class with no forcing, it is possible that non-uniqueness cannot be studied perturbatively from the Euler equations. Nevertheless, a conjectural research program of Jia and Šverák [33, 34], as well as the numerical work of Guilloid and Šverák [31], provide strong evidence for non-uniqueness.

The possibility that a deterministic PDE gives rise to multiple solutions from a common initial condition is a concern in modelling and forecasting. In light of the mounting evidence for non-uniqueness of solutions to (NS), it is important to understand how non-uniqueness could evolve. Ideally, all possible solutions remain close together, indicating they are predictable from a single flow. This can be viewed as a sort of stability. The possibility that two solutions can separate explosively—meaning, e.g. that time derivatives of the separation rate are unbounded at $t = 0$ —is more concerning from the standpoint of predictability. It is therefore natural to ask: *What are necessary properties for the error of non-unique solutions to (NS)?* One approach to answering this question would be to establish uniqueness criteria *in terms of the error* by shifting the condition from the background flow to the error.

Before discussing this prospect we briefly discuss uniqueness criteria for (NS), noting that the literature is too expansive for a comprehensive review. Classically, uniqueness often comes for free in settings where the nonlinear problem can be viewed as a perturbation of the heat equation—this is the case for the strong solutions of Kato [38], see also [23], where uniqueness within the class of strong solutions follows from the fixed point construction. A more nuanced problem is that of weak-strong uniqueness, where one solution is assumed to be strong but the other can be weak. For example, if $u_0 \in L^2 \cap L^\infty$, then a bounded strong solution is known to exist for a short period of time due to the Kato theory [38] and a global weak solution with bounded energy is known to exist by Leray’s theory [46]. Weak-strong uniqueness implies that these agree, at least on the time-scales of the Kato theory. As the Kato solution is smooth, this is a way to prove regularity for weak solutions. Even more nuanced problems arise at the interface between the perturbative regime and the fully nonlinear regime where the solutions

belong to certain Besov spaces [7, 20, 45], Morrey spaces [16, 43] or have some structure [9]. We refer the reader to [44] for additional discussion of the matter.

The literature cited above does not shed light on the evolution of the error in non-uniqueness scenarios. Importantly, the error can belong to different classes than u or v . So, it is conceivable that neither u nor v belong to a uniqueness class, but the error $w = u - v$ does. In the case of non-unique self-similar solutions (in an appropriate class of weak solutions, see, e.g. [8]), the error w would satisfy $w \in L^\infty(0, \infty; L^3(\mathbb{R}^3))$. If either u or v belong to this class, then $w \equiv 0$. Therefore, it should not in general be expected that the error fails to be in strong integrability classes. This is good news from the perspective of forecasting; if this were not the case, then non-uniqueness would necessarily imply a large error.

It is not obvious how to write uniqueness criteria in terms of the error. To prove uniqueness, one typically needs to manage the following sort of bound:

$$[w\text{-quantity}] \lesssim [\text{scale-invariant quantity}] \cdot [w\text{-quantity}]. \quad (1.1)$$

Classical uniqueness criteria guarantee the scale-invariant quantity, which depends on u and v , is small. So, the right-hand side is absorbed in the left-hand side and the error consequently vanishes. Size conditions on w do not help close the estimate as u and v necessarily appear. Therefore, uniqueness criteria in terms of the error must look different. Finding such criteria is the first goal of this paper. To get around the obstacle in (1.1) we identify conditions on the error which close these estimates based on the *relative smallness* of certain scales. In particular, we pair certain scales in w with the large part of u and require these scales in w to be relatively small. Heuristically, this amounts to refining (1.1) as follows,

$$\begin{aligned} [w\text{-quantity}] &\lesssim [\text{large scale-invariant quantity}] \cdot [\text{relatively small } w\text{-quantity}] \\ &\quad + [\text{small scale-invariant quantity}] \cdot [w\text{-quantity}]. \end{aligned} \quad (1.2)$$

Because smallness appears in both terms on the right-hand side, the estimate can be closed.

A second goal of this paper is to identify additional properties of the error based on the condition that the error separates *en masse* at a scaling invariant rate. In particular, for several interpretations of ‘scale,’ we will show that intermediate scales which are comparable to $t^{1/2}$ are necessarily active as two solutions separate. The second part of this paper examines how small perturbations de-correlate and is motivated by work on predictability [10, 11, 48]. In this direction, we establish a conditional predictability criteria which states that energy de-correlation requires a certain configuration of activity below wavenumbers in the dissipative range. Additionally, we explain how the conjectural existence of non-unique solutions of Jia and Šverák [33] would imply explosive separation within the Leray–Hopf class, suggesting that the linear separation rate simulated in [10, 11] is not universal.

1.1. Properties of the error of Navier–Stokes flows

Many of our ideas are motivated by the scaling of (NS): If (u, p) solves the Navier–Stokes equations, then so does the pair $(u^\lambda := \lambda u(\lambda x, \lambda^2 t), p^\lambda := \lambda^2 p(\lambda x, \lambda^2 t))$ for any $\lambda > 0$. If $u = u^\lambda$ and $p = p^\lambda$ for all λ , then u is said to be self-similar. A quantity is scaling invariant if it is unchanged upon rescaling. For example, since $\|u\|_{L^3} = \|u^\lambda\|_{L^3}$ for every λ , the L^3 norm is said to be scaling invariant.

We will use u and v to denote two flows having a common initial condition, with $w = u - v$ being the error. Before giving the results, the main assumptions are stated. The first condition is that

$$\sup_{0 < t} \sup_{3 < q \leq \infty} \left(t^{1/2-3/(2q)} (\|u\|_{L^q} + \|v\|_{L^q})(t) + \left(\|u\|_{\dot{B}_{q,\infty}^{-1+3/q}} + \|v\|_{\dot{B}_{q,\infty}^{-1+3/q}} \right)(t) \right) \leq c_1, \quad (\text{A1})$$

for some $c_1 > 0$, which can be large. In analogy with the blow-up literature, we think of this as a Type I condition in that the controlled quantities are all scaling invariant; Type II conditions would only assert weaker bounds. A centred Type I condition is also natural, namely

$$|u|(x, t) + |v|(x, t) \leq \frac{c_1}{|x| + \sqrt{t}}. \quad (\text{A1}')$$

These are motivated by the scaling of (NS) and the fact that, roughly speaking, if the flows satisfied stronger upper bounds, then they would agree due to the classical uniqueness theories. Thus, these are some of the strictest scenarios where non-uniqueness is plausible. Note that (A1') implies (A1).

The second type of condition essentially says that w neither converges to 0 asymptotically, nor diverges slower than the critical rate as $t \rightarrow 0^+$. In particular, it says

$$\|w\|_{L^p}(t) t^{1/2-3/(2p)} \geq c_2, \quad (\text{A2})$$

for some $c_2 > 0$ and some $3 < p \leq \infty$. Compared to (A1) and (A1'), it is less clear that (A2) holds in general non-uniqueness scenarios. In fact, proving it would amount to shifting a uniqueness criteria from the background flows to the error, which we discussed above. When our results use (A2), they should be interpreted as statements about which scales are active given the assumption that a scaling invariant quantity for the error is non-vanishing at $t = 0$.

We are now ready to state our main results, which apply to three interpretations of ‘scale.’ We define the different solution classes appearing in these theorems in section 2. Note that for $3 < p \leq \infty$ there is a local well-posedness theory [27, 38] which implies that any initial data $u_0 \in L^p$ produces a unique strong solution u on $[0, T_0]$ where

$$T_0 = \tilde{c}_p \|u_0\|_{L^p}^{2p/(3-p)},$$

satisfying

$$\sup_{0 \leq t \leq T_0} \|u\|_{L^p}(t) \leq 2\|u_0\|_{L^p},$$

where $\tilde{c}_p$ is a constant which only depends on p .

Our first result is motivated by the hypothetical non-uniqueness example of Jia and Šverák [33], which is self-similar. By [13], self-similar solutions satisfy

$$|w(x, t)| \leq \frac{C(u_0) t^{3/2}}{(|x| + \sqrt{t})^4}.$$

See also [14]. Consequently, for $\gamma > 0$

$$|w(x, t)| \chi_{B_{\gamma^{-1}\sqrt{t}}}^{B_{\gamma^{-1}\sqrt{t}}}(x) \lesssim_{u_0} \frac{1}{(\gamma^{-1} + 1)^4 \sqrt{t}}. \quad (1.3)$$

The prefactor can be made small indicating the activity in the region $|x| \geq \gamma^{-1}\sqrt{t}$ becomes smaller as γ^{-1} grows. On the other hand, by self-similar scaling

$$\|w\|_{L^\infty}(t) = \frac{1}{\sqrt{t}} \|w\|_{L^\infty}(1).$$

Hence, for self-similar non-uniqueness, the majority of the error's activity is necessarily contained within some region $|x| \lesssim_{u_0} \sqrt{t}$. Our first theorem demonstrates that this property of self-similar uniqueness extends to general scenarios under assumption (A1'). It additionally states upper and lower bounds on the error.

Theorem 1.1 (algebraic interpretation of 'scale'). *Assume u and v are mild solutions with the same initial data u_0 . The following hold:*

- (1) (Uniqueness criteria) Fix $3 \leq p < \infty$. Under assumption (A1'), there exist $\epsilon_1(c_1, p) > 0$ and $\eta = \epsilon_1/(c_1 - \epsilon_1)$ so that, if there exists T so that

$$\sup_{0 < t < T} \frac{\|w(\cdot, t) \chi_{B_{\eta^{-1}\sqrt{t}}}\|_{L^p}}{\|w(\cdot, t) \chi_{B_{\eta^{-1}\sqrt{t}}^c}\|_{L^p}} < \epsilon_1, \quad (1.4)$$

then $w = 0$.

- (2) (Lower bound on the error) Let $M_0(t) = 2 \max\{\|u\|_\infty(t), \|v\|_\infty(t)\}$. Assume (A2) for some non-negative real number c_2 . Then, for any $t > 0$, $a \geq 0$ and $c_3 > 0$, there exists $b = b(c_2, c_3, \tilde{c}_\infty, a, t, M_0(t)) > 1$ so that any pair of mild solutions u and v to (NS) with the same data do **not** satisfy

$$\sup_{x \in \mathbb{R}^3} |w(x, t)| \frac{(b|x| + \sqrt{t})^{a+1}}{\sqrt{t}^a} < c_3. \quad (1.5)$$

- (3) (Upper bound on the error) Under assumption (A1'),

$$\sup_{x \in \mathbb{R}^3, 0 < t} |w(x, t)| \frac{(|x| + \sqrt{t})^4}{\sqrt{t}^3} \lesssim_{c_1, C_B} 1.$$

where C_B is a universal constant introduced above (2.2).

Combining Items (2) and (3), we conclude that, for any $0 < a < 3$, there exists a sequence $(x_k, t_k) \rightarrow (0, 0)$ so that

$$\frac{c_3 \sqrt{t}^a}{(b|x_k| + \sqrt{t_k})^{a+1}} \leq |w(x_k, t_k)|.$$

Item (1) implies a unique continuation property: If $w = 0$ in a neighbourhood of $(0, 0)$, then $w = 0$ globally. Note that this is automatically satisfied by self-similar solutions due to scaling. It is also implied more generally by real analyticity at positive times.

To our knowledge Item (1) constitutes a new type of uniqueness criteria. We develop similar criteria below in theorems 1.3 and 1.4.

We next consider an analogous result in terms of frequency. The *a priori* algebraic bounds of self-similar solutions provided the context for theorem 1.1. The next proposition provides a similar context in a frequency sense. In particular, it gives an upper bound on the separation rate

of individual Littlewood-Paley frequencies (see section 2 for the definitions of the Littlewood-Paley decomposition, Besov spaces and local energy solutions).

Proposition 1.2 (separation bounds in frequency). *Assume $u_0 \in L^p(\mathbb{R}^3 \setminus \{0\})$ for some $p > 3$, is divergence free and is (-1) -homogeneous. If u and v are self-similar local energy solutions to (NS) with the same data u_0 , then, for every $J \in \mathbb{Z}$,*

$$\|\Delta_{<J} w\|_{L^\infty}(t) \lesssim_{u_0} 2^{4J} t^{3/2}.$$

This proposition states that, within the self-similar class, the error below a specific frequency vanishes as $t \rightarrow 0^+$ and the rate improves at lower scales. Therefore, because the full self-similar error does not vanish (it blows up at the rate $t^{-1/2}$), it must concentrate at smaller and smaller scales as $t \rightarrow 0^+$. Also note that the time exponent matches that in Item (3) of theorem 1.1. The next theorem establishes properties consistent with this for more general classes of solution. It is formatted to replicate the structure and themes of theorem 1.1, but through a different lens. The same comment applies to theorem 1.4 below.

Theorem 1.3 (frequency interpretation of ‘scale’). *Assume u and v are mild solutions with the same initial data u_0 . The following hold:*

- (1) (Uniqueness criteria) *Assume (A1) for some value c_1 . Fix $p \in (3, \infty]$. There exists $\epsilon_2(c_1, p) > 0$ and $J_1(t)$ with $2^{J_1(t)} \sim_{c_1, p} \epsilon_2 t^{-1/2}$, so that, if there exists $T > 0$ with*

$$\sup_{0 < t < T} \frac{\|w_{\geq J_1}\|_{L^p}(t)}{\|w_{< J_1}\|_{L^p}(t)} \leq \epsilon_2,$$

then $w = 0$.

- (2) (Low frequencies are active) *Let $M_0(t) = 2 \max\{\|u\|_{L^p}(t), \|v\|_{L^p}(t)\}$. If w satisfies (A2) for some c_2 and $p \in (3, \infty]$, then there exist $\gamma(t)$ and $J_2(t)$ with*

$$\gamma = \frac{c_2}{4M_0 \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}},$$

and

$$2^{J_2} \sim \frac{4M_0 \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}}{c_2} \left(\frac{c_2}{4M_0^2 C_B \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}} \right)^{\frac{p}{3-p}},$$

so that we have

$$\frac{\|w_{\geq J_2}\|_{L^p}(t)}{\|w_{< J_2}\|_{L^p}(t)} \leq \gamma.$$

- (3) (Intermediate frequencies are active) *If (A1) and (A2) hold for some $p \in (3, \infty]$, then we have $2^{J_2} \sim_{p, c_1, c_2} \sqrt{t}^{-1}$ and there exists $J_3 < J_2$ with $2^{J_3} \sim_{c_1, p} \sqrt{t}^{-1}$ and*

$$\|w_{J_3 \leq j \leq J_2}\|_{L^p}(t) \sim_\gamma \|w\|_{L^p}(t) \sim_{c_1, c_2} t^{-1/2+3/(2p)}. \quad (1.6)$$

The first item can be interpreted as saying that, if $w \neq 0$, then high modes are active to some extent. This complements the statement of the second item. The same comment applies to theorem 1.4 which appears below.

Our third iteration of this theme involves a discretised interpretation of ‘scale’ which we presently introduce. Fix a lattice of cubes $\{Q_i\}$ with disjoint interiors, volumes h^3 , and whose closures cover $\mathbb{R}^3$. Suppose that one cube is centred at the origin. Let

$$J_h u_0(x) = \sum_j \chi_{Q_j}(x) \frac{1}{|Q_j|} \int_{Q_j} u_0(y) dy.$$

This is effectively a discretisation of the flow based on volume elements and is a whole-space version of an interpolant operator which has been used extensively to study the number of degrees of freedom in 2D NS flows [25, 26, 35, 36] and more recently in the Azouni, Olson & Titi data assimilation paradigm [5] and descendent ideas [19].

Theorem 1.4 (discretised interpretation of ‘scale’). *Assume u and v are mild solutions with the same initial data.*

- (1) (Uniqueness criteria) *Assume (A1') for some value c_1 and suppose u and v are $L^{3,\infty}$ -weak solutions with the same data u_0 ². Fix $3 < p \leq \infty$. There exists $\epsilon_3 = \epsilon_3(c_1, p, \|u_0\|_{L^{3,\infty}})$ so that, letting*

$$\bar{h}(t) = \max \left\{ 2 \frac{c_1 - \epsilon_3}{\epsilon_3} \sqrt{t}, \left(t^{3/4} \frac{t^{3/(2p)-1/2}}{\epsilon_3 \|w\|_{L^p}(t)} \right)^{2/3} \right\},$$

if

$$\|w - J_{\bar{h}} w\|_{L^p}(t) \leq \epsilon_3 \|w\|_{L^p}(t),$$

across a time interval $(0, \delta)$ where $\delta > 0$ is arbitrary, then $w = 0$.

- (2) (Large scales are active) *Let $M_0(t) = 2 \max\{\|u\|_{L^p}(t), \|v\|_{L^p}(t)\}$. If w satisfies (A2) for some c_2 and $p \in (3, \infty]$, then there exist $\gamma(t)$ and $h(t)$ with*

$$\gamma = \frac{c_2}{4M_0 \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}},$$

and

$$\frac{1}{h} \sim \frac{4M_0 \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}}{c_2} \left(\frac{c_2}{4M_0^2 C_B \left(4\tilde{c}_p M_0^{2p/(3-p)} + t \right)^{1/2-3/(2p)}} \right)^{\frac{p}{3-p}},$$

so that we have

$$\|J_h w\|_{L^p}(t) \geq \frac{\gamma}{2} \|w\|_{L^p}(t).$$

Note that, unlike in theorems 1.1 and 1.3, the length scale in the first part depends on $\|w(t)\|_{L^p}$.

² Note that (A1') is consistent with the initial data being $O(|x|^{-1}) \in L^{3,\infty}$, so this is a reasonable class for solutions to belong in. We provide a definition of $L^{3,\infty}$ -weak solutions in section 2.

1.2. Discussion of proofs

Each of theorems 1.1, 1.3 and 1.4 are proved using the same basic ideas. For the uniqueness criteria, we choose length scales so that the background flows are necessarily small at large scales. For example, in the algebraic case and under (A1'), if $\sqrt{t} \ll |x|$, then

$$|v(x, t)| \ll \sqrt{t}^{-1}.$$

This depletes the large scale activity of the error. The smallness condition on the relative size of the small-scales in the error depletes the rest of the error. Of course, the nonlinear nature of the problem means that the large scales of the background flows are not only paired with the large scales of the error in terms like $w \cdot \nabla u$, so there is some nuance in pushing these ideas through.

For the secondary conclusions, we employ an idea worked out in [2] which notes that the integral structure of the error,

$$w(x, t) = e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u - v \otimes v) \, ds,$$

has a part which rapidly decays when small scales are dominant in the initial error and a part which grows from zero at $t = 0$. Hence, if sufficiently small scales are dominant at some time, the L^∞ -norm can be pushed below the assumed lower bound (A2) at some later time, which is contradictory. This idea has been used in prior work on regularity, in particular it relates to a ‘sparseness’ technique of Grujić [28] which was re-imagined in [2]. See also [12, 24, 29, 30].

1.3. Predictability

Forecasting in turbulent media is possible despite the fact that turbulence is a highly chaotic fluid state because information about different scales persists in the flow for different periods of time. This can be seen in weather forecasting where the turnover time of the smallest eddies in the atmosphere is on the order of seconds, indicating a rapid onset of chaos at small scales, while weather forecasts are effective for days [11]. In other words, large scale effects remain predictable for a non-negligible amount of time despite small scale instabilities.

Predictability was initially studied by Lorenz [47] and Leith and Kraichnan [40, 41]. There is also a rich modern literature, an incomplete list being [10, 11, 48, 53]. Several definitions of predictability exist and we adopt that from [11]. For two initial data u_0 and v_0 , define the error energy by $E_\Delta(t) = \|u - v\|_2^2(t)$ where u and v evolve from u_0 and v_0 respectively. For initially small perturbations, the flows are said to be *predictable* if $E_\Delta(t) < \frac{\gamma}{2}(\|u\|_2^2 + \|v\|_2^2)(t)$ for some $\gamma \in (0, 1)$ —for uncorrelated flows the left- and right-hand sides are comparable. Numerical experiments show that, for infinitesimal perturbations of turbulent flows, on average $E_\Delta(t)$ initially grows exponentially according to $E_\Delta(t) = E_\Delta(0)e^{Lt}$, where L is a Lyapunov exponent, and then settles into a linear growth rate $E_\Delta(t) \sim t$ [11]. Linear bounds on growth rates should be expected when perturbing around sufficiently bounded flows. This is even the case when perturbing around an Euler flow, as examined in the context of boundary layer separation by Vasseur and Yang [55, 56]. The length scale at which the perturbation is given plays a role in the dynamics. A careful description of this can be found in [48]. See also [22] for a separation bound on Euler flows in Gevrey norms by Drivas, Elgindi and La.

We include two results which connect predictability to the themes explored in this paper. The first can be viewed as a *conditional predictability criteria* which also sheds light on the distribution of the energy below and above scales in the dissipative range as flows de-correlate.

Theorem 1.5 (predictability criteria). Suppose u and v are distributional solutions to (NS) on $\mathbb{R}^d \times (0, T)$ for $d = 2, 3$. Suppose also that $w = u - v \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ satisfies the energy inequality

$$\partial_t \|w\|_{L^2}^2 + 2\|\nabla w\|_{L^2}^2 \leq -2 \int (w \cdot \nabla u) \cdot w \, dx.$$

There exists a universal constant c_5 so that, if, at a given time there exists $J \in \mathbb{Z}$ so that

$$\frac{\|\Delta_{\leq J} w\|_{L^2}^2}{\|\Delta_{> J} w\|_{L^2}^2} \leq \frac{c_5 2^{2J}}{\min \left\{ \|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}} \right\}^2} - 1,$$

then

$$\partial_t \|w\|_{L^2}^2 + \frac{C \min \left\{ \|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}} \right\}^2}{2c_5} \|w\|_{L^2}^2 < 0, \quad (1.7)$$

meaning the flows are not de-correlating at the given time. Note that all norms are spatial and calculated at the given time. Similarly, (1.7) also holds if there exists $h > 0$ so that

$$\frac{\|J_h w\|_{L^2}^2}{\|w - J_h w\|_{L^2}^2} \leq \frac{c_5 h^{-2}}{\min \left\{ \|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}} \right\}^2} - 1.$$

Note that the preceding condition is stated at a single time. If the condition holds across the time interval $(0, T)$ then $\|w\|_2^2$ is exponentially decaying. Although the condition is formulated at individual frequencies, the rate of exponential decay is independent of J and h .

We explicitly assume that w has an energy inequality so that we do not need to impose additional conditions on u or v . Plainly if u and v are both singular at a particular time, then the conditions of the theorem cannot be met. The problem is symmetric in u and v so $\min \left\{ \|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}} \right\}$ can be replaced by $\min \left\{ \|v\|_{L^\infty}, \sqrt{\|\nabla v\|_{L^\infty}} \right\}$.

Henshaw, Kreiss and Reyna identify a factor of $\|\nabla u\|_\infty^{-1/2}$ with the dissipative length scale in turbulence [32]. From this perspective, our result is describing behaviour of the error in the dissipative range. If the initial error occurs at very small scales, i.e. deep within the dissipative range, then due to continuity we must have that the conditions in the theorem are satisfied for a non-vanishing period of time at scales between the perturbation scale and the inertial range. During this time, the error energy would decrease exponentially at a rate which is independent of the scale of the initial perturbation. This would cease once activity builds up at larger scales, at which point the flows would presumably begin to de-correlate, filling scales in the inertial range in an ‘inverse cascade,’ as simulated in [11].

There are similarities between theorem 1.5 and results in data assimilation and determining functionals [5, 25, 26, 35, 36], a fact which is visible in our proof.

Our second result related to predictability explores the universality of the linear separation rate simulated in [11] within the class of Leray–Hopf weak solutions. The linear separation rates are for perturbations around bounded flows, but the Leray–Hopf class includes solutions which are unbounded at $t = 0$. This would be the case for the localised non-unique solutions hypothesised to exist in [31, 33], where we are viewing non-unique solutions as perturbations with initial perturbation zero.

The hypothetical non-unique Leray–Hopf solutions of Jia and Šverák are built by perturbing³ two (hypothetical) self-similar solutions to finite energy solutions. This involves cutting off the tail of the initial data. This should, in principle, not greatly effect the dynamics near the space-time origin, which is where the singularity of the initial data occurs. In that case the solutions should have an error energy separation which saturates the $t^{1/2}$ rate determined by scaling, as is necessarily the case for the self-similar solutions due to their exact scaling property. This intuition can be made rigorous and suggests that a linear error energy separation rate is likely *not* universal within the Leray–Hopf class. The following proposition re-states results from [33] with the addition of a new lower bound on the error energy at small times.

Proposition 1.6. *Suppose there exists a (-1) -homogeneous, divergence free vector field u_0 and a self-similar local energy solution u_1 satisfying [33, Spectral Condition (B)]. Then, there exists a second self-similar local energy solution u_2 with the same initial data u_0 so that $u_1 \neq u_2$. Furthermore, there exist v_0, v_1 and v_2 so that $u_1 - v_1$ and $u_2 - v_2$ are non-unique Leray–Hopf weak solutions with divergence free, compactly supported initial datum $u_0 - v_0 \in L^2$ and, for small enough t ,*

$$t^{1/2} \lesssim_{u_0, v_0} \| (u_1 - v_1) - (u_2 - v_2) \|_{L^2}^2(t).$$

Consequently, if these solutions exist, then there is no universal linear bound on the separation rates of perturbations within the Leray–Hopf class.

1.4. Organisation

Section 2 contains preliminaries including definitions and discussions of mild solutions, local Leray solutions, $L^{3,\infty}$ -weak solutions and the Littlewood–Paley decomposition. Sections 3–5 contain, respectively, the proofs of theorems 1.1, 1.3 and 1.4. The proofs of results on predictability are respectively contained in sections 6 and 7.

2. Preliminaries

2.1. Mild solutions

We denote by $B(\cdot, \cdot)$ the bilinear operator

$$B(u, v) = -\frac{1}{2} \int_0^\tau \underbrace{e^{(\tau-s)\Delta} \mathbb{P}}_{\text{Oseen tensor}} \nabla \cdot (u \otimes v + v \otimes u)(s) \, ds,$$

where u and v are vectors and $\mathbb{P}$ is the Leray projection operator. Dumahel’s formula applied to the projected form of (NS) formally leads to the integral representation

$$\begin{aligned} u(x, t + \tau) &= e^{\tau\Delta} [u(t)](x) - \int_0^\tau e^{(\tau-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(t + s) \, ds \\ &= e^{\tau\Delta} [u(t)](x) + B(u(t + \cdot), u(t + \cdot)). \end{aligned}$$

³ We are using the term ‘perturbing’ a lot. It presently does not refer to the perturbations in the concept of predictability as in the preceding paragraph, but rather to the method by which Jia and Šverák generate Leray–Hopf weak solutions from self-similar solutions.

Convergence of the solution to the data is understood in the sense of distributions. Although primarily used in the context of strong solutions (in the sense of [38]), this formula can be justified rigorously under very general conditions—in particular, it is valid distributionally for many classes of weak solutions [17, 42]. The kernel K of the Oseen tensor satisfies the following pointwise estimates due to Solonikov [50] where α is a multi-index,

$$|D_x^\alpha K(x, t)| \leq C(\alpha) \frac{1}{(|x| + \sqrt{t})^{3+|\alpha|}}. \quad (2.1)$$

Throughout this paper we will use C_B to denote a universal constant coming from bilinear and kernel estimates. As we will only use the above for $|\alpha| = 0, 1$, we replace $C(\alpha)$ by C_B in what follows. For $1 \leq p \leq \infty$,

$$\|B(u, v)\|_{L^p(t+\tau)} \leq C_B \int_0^\tau \frac{1}{(\tau-s)^{\frac{1}{2} + \frac{3}{2}(\frac{1}{q} - \frac{1}{p})}} \|u \otimes v\|_{L^q}(t+s) ds, \quad (2.2)$$

where the value of C_B has been updated. This is from [23, 38] when $1 \leq p < \infty$ and [27, (2.7) and estimates after (3.1)] for $p = \infty$. The mild formulation of the perturbed Navier–Stokes equations, where v is the background term and w is the unknown, is

$$w(t+\tau) = e^{\tau\Delta} w(t) - \int_0^\tau e^{(\tau-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes w + w \otimes v + w \otimes w)(t+s) ds. \quad (2.3)$$

2.2. Local energy solutions

We need to use the properties of local energy solutions in the proof of proposition 1.6. These solutions were introduced by Lemarié-Rieusset, see the treatments in [42, 44], and played an important role in the proof of local smoothing in [34]. Because $L^{3,\infty} \subset L^2_{\text{uloc}}$ ⁴ it is a natural class in which to consider non-uniqueness [31, 33, 34]. Additional properties of this class have been explored in [16, 17, 37, 39].

Definition 2.1 (local energy solutions). A vector field $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, T])$, $0 < T \leq \infty$, is a local energy solution to (NS) with divergence free initial data $u_0 \in L^2_{\text{uloc}}(\mathbb{R}^3)$, denoted as $u \in \mathcal{N}(u_0)$, if:

- (1) for some $p \in L^{\frac{3}{2}}_{\text{loc}}(\mathbb{R}^3 \times [0, T])$, the pair (u, p) is a distributional solution to (NS),
- (2) for any $R > 0$, u satisfies

$$\text{ess sup}_{0 \leq t < R^2 \wedge T} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} \frac{1}{2} |u(x, t)|^2 dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2 \wedge T} \int_{B_R(x_0)} |\nabla u(x, t)|^2 dx dt < \infty,$$

- (3) for any $R > 0$, $x_0 \in \mathbb{R}^3$, and $0 < T' < T$, there exists a function of time $c_{x_0, R} \in L^{\frac{3}{2}}_{T'}$ so that, for every $0 < t < T'$ and $x \in B_{2R}(x_0)$

$$p(x, t) = c_{x_0, R}(t) - \Delta^{-1} \text{div div} [(u \otimes u) \chi_{4R}(x - x_0)] - \int_{\mathbb{R}^3} (K(x - y) - K(x_0 - y)) (u \otimes u)(y, t) (1 - \chi_{4R}(y - x_0)) dy, \quad (2.4)$$

⁴ L^2_{uloc} is the set of all uniformly locally square integrable functions and $L^{3,\infty}$ is weak- L^3 .

- in $L^{\frac{3}{2}}(B_{2R}(x_0) \times (0, T'))$ where $K(x)$ is the kernel of $\Delta^{-1} \operatorname{div} \operatorname{div}$, $K_{ij}(x) = \partial_i \partial_j \frac{-1}{4\pi|x|}$, and $\chi_{4R}(x)$ is the characteristic function for B_{4R} .
- (4) for all compact subsets K of $\mathbb{R}^3$, $u(t) \rightarrow u_0$ in $L^2(K)$ as $t \rightarrow 0^+$,
- (5) u is suitable, i.e. for all cylinders $Q \Subset Q_T$ and all non-negative $\phi \in C_c^\infty(Q)$, we have the *local energy inequality*

$$2 \iint |\nabla u|^2 \phi \, dx \, dt \leq \iint |u|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \iint (|u|^2 + 2p) (u \cdot \nabla \phi) \, dx \, dt, \quad (2.5)$$

(6) the function

$$t \mapsto \int_{\mathbb{R}^3} u(x, t) \cdot w(x) \, dx, \quad (2.6)$$

is continuous in $t \in [0, T)$, for any compactly supported $w \in L^2(\mathbb{R}^3)$.

Local energy solutions are known to satisfy certain *a priori* bounds [42]. For example, in [16, 34], the following *a priori* bound is proven: Let $u_0 \in L^2_{\text{uloc}}$, $\operatorname{div} u_0 = 0$, and assume $u \in \mathcal{N}(u_0)$. For all $r > 0$ we have

$$\operatorname{ess\,sup}_{0 \leq t \leq \sigma r^2} \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{|u|^2}{2} \, dx \, dt + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\sigma r^2} \int_{B_r(x_0)} |\nabla u|^2 \, dx \, dt < CA_0(r), \quad (2.7)$$

where

$$A_0(r) = rN_r^0 = \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} |u_0|^2 \, dx,$$

and

$$\sigma = \sigma(r) = c_0 \min \left\{ (N_r^0)^{-2}, 1 \right\}, \quad (2.8)$$

for a small universal constant $c_0 > 0$. Additionally, local energy solutions are mild [17].

2.3. $L^{3,\infty}$ -weak solutions

Local energy solutions are defined for initial data in L^2_{uloc} . Note that $L^{3,\infty}$ embeds in L^2_{uloc} . Thus, when $u_0 \in L^{3,\infty}$, a local energy solution exists. The scaling of $L^{3,\infty}$ does not, however, show up in the properties of this solution which come from the definition of local energy solutions. The class of $L^{3,\infty}$ -weak solutions provides a notion of solution which is more tailored to the scaling of $L^{3,\infty}$. This class was introduced by Barker, Seregin and Šverák in [8] and extends ideas in [49]. It has since been extended to non-endpoint critical Besov spaces of negative smoothness [1].

Definition (weak $L^{3,\infty}$ -solutions). Let $T > 0$ be finite. Assume $u_0 \in L^{3,\infty}$ is divergence free. We say that u and an associated pressure p comprise a weak $L^{3,\infty}$ -solution if

- (1) (u, p) satisfies (NS) distributionally, (2.6) and the local energy inequality (2.5),
- (2) $\tilde{u} := u - e^{t\Delta}u_0$ satisfies, for all $t \in (0, T)$,

$$\sup_{0 < s < t} \|\tilde{u}\|_{L^2}^2(s) + \int_0^t \|\nabla \tilde{u}\|_{L^2}^2(s) \, ds < \infty, \quad (2.9)$$

and

$$\|\tilde{u}\|_{L^2}^2(t) + 2 \int_0^t \int |\nabla \tilde{u}|^2 \, dx \, ds \leq 2 \int_0^t \int (e^{s\Delta}u_0 \otimes \tilde{u} + e^{s\Delta}u_0 \otimes e^{s\Delta}u_0) : \nabla \tilde{u} \, dx \, ds. \quad (2.10)$$

In [8], weak solutions are constructed which satisfy the above definition for all $T > 0$. Also, due to their spatial decay, weak $L^{3,\infty}$ -solutions are mild and, in view of [17], are local energy solutions.

An important observation in [8] is that the nonlinear part of a weak $L^{3,\infty}$ -solution satisfies a dimensionless energy estimate, namely

$$\sup_{0 < s < t} \|\tilde{u}\|_{L^2}(s) + \left(\int_0^t \|\nabla \tilde{u}\|_{L^2}^2(s) \, ds \right)^{\frac{1}{2}} \lesssim_{u_0} t^{\frac{1}{4}}. \quad (2.11)$$

We emphasise that the energy associated with $\tilde{u}$ vanishes at $t = 0$. This decay property will be essential in our work. It is worth remarking that this estimate implies an upper bound on the energy separation rate of non-unique solutions in the class of weak $L^{3,\infty}$ -solutions. Interestingly, because self-similar solutions exist in this class, examples exist which saturate this bound. This fact plays an important role in our proof of proposition 1.6.

2.4. Littlewood–Paley

We refer the reader to [6] for an in-depth treatment of Littlewood–Paley and Besov spaces. Let $\lambda_j = 2^j$ be an inverse length and let B_r denote the ball of radius r centred at the origin. Fix a non-negative, radial cut-off function $\chi \in C_0^\infty(B_1)$ so that $\chi(\xi) = 1$ for all $\xi \in B_{1/2}$. Let $\phi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$ and $\phi_j(\xi) = \phi(\lambda_j^{-1})(\xi)$. Suppose that u is a vector field of tempered distributions and let $\Delta_j u = \mathcal{F}^{-1} \phi_j * u$ for $j \geq 0$ and $\Delta_{-1} = \mathcal{F}^{-1} \chi * u$. Then, u can be written as

$$u = \sum_{j \geq -1} \Delta_j u.$$

If $\mathcal{F}^{-1} \phi_j * u \rightarrow 0$ as $j \rightarrow -\infty$ in the space of tempered distributions, then we define $\dot{\Delta}_j u = \mathcal{F}^{-1} \phi_j * u$ and have

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u.$$

We additionally define

$$\Delta_{<J}f = \sum_{j < J} \dot{\Delta}_j f; \quad \Delta_{\geq J}f = f - \Delta_{<J}f,$$

with the obvious modifications for $\Delta_{\leq J}$ and $\Delta_{>J}$. If we do not specify that J is in integer, then we use $\chi(\lambda_1^{-1}2^J\xi)$ in the definition of $\Delta_{\leq J}$.

Littlewood–Paley blocks interact nicely with derivatives and, by Young’s inequality, L^p norms. This is illustrated by the Bernstein inequalities which read:

$$\|D^\alpha \dot{\Delta}_j f\|_{L^p} \leq 2^{j|\alpha|} \|\dot{\Delta}_j f\|_{L^p}; \quad \|\dot{\Delta}_j f\|_{L^p} \leq 2^{j(\frac{3}{q} - \frac{3}{p})} \|\dot{\Delta}_j f\|_{L^q}.$$

The Littlewood–Paley formalism is commonly used to define Besov spaces. We are primarily interested in Besov spaces with infinite summability index, the norms of which are

$$\|u\|_{B_{p,\infty}^s} := \sup_{-1 \leq j < \infty} \lambda_j^s \|\Delta_j u\|_{L^p(\mathbb{R}^n)},$$

and

$$\|u\|_{\dot{B}_{p,\infty}^s} := \sup_{-\infty < j < \infty} \lambda_j^s \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^n)}.$$

The critical scale of Besov spaces are $\dot{B}_{p,\infty}^{3/p-1}$. Note that $L^3 \subset L^{3,\infty} \subset \dot{B}_{p,\infty}^{3/p-1}$ for $3 < p$. In particular, $\dot{B}_{p,\infty}^{3/p-1}$ contains functions f satisfying $|f(x)| \lesssim |x|^{-1}$ when $p > 3$.

3. Algebraic scenario

The following definition and lemma appear in [2]. They will be used to prove Item (2) of theorem 1.1.

Definition 3.1 (L^p -sparseness). Let $1 \leq p \leq \infty$, $\varepsilon, \beta \in (0, 1)$, and $\ell > 0$. A vector field $u_0 \in L^p(\mathbb{R}^d)$ is $(\varepsilon, \beta, \ell)$ -sparse in L^p if there exists a measurable set S such that

$$\|u_0\|_{L^p(S^c)} < \beta \|u_0\|_{L^p} \quad (3.1)$$

and

$$\sup_{x_0 \in \mathbb{R}^d} \frac{|S \cap B_\ell(x_0)|}{|B_\ell(x_0)|} \leq \varepsilon. \quad (3.2)$$

Let $G : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Schwartz function and G_t be the convolution operator

$$G_t u_0 := t^{-\frac{d}{2}} G(\cdot/\sqrt{t}) * u_0. \quad (3.3)$$

when $t > 0$. We have in mind that $G = (4\pi)^{-d/2} e^{-|x|^2/4}$ and G_t is the heat semigroup.

Lemma 3.2. Let $p \in (1, \infty]$, $\gamma \in (0, 1)$, and $t > 0$ be fixed. Let $u_0 \in L^p(\mathbb{R}^d)$ be a vector field. Suppose that u_0 is $(\varepsilon, \beta, \bar{\ell}\sqrt{t})$ -sparse, where the dimensionless parameters $\varepsilon, \beta \in (0, 1)$ and $\bar{\ell} > 0$ satisfy

$$\bar{\ell} \geq f(\gamma); \quad \beta \leq \|G\|_{L^1}^{-1} \gamma/3; \quad \varepsilon^{1-\frac{1}{p}} \leq C_0^{-1} \|G\|_{L^\infty}^{-1} \gamma/\bar{\ell}^d \quad (3.4)$$

where f depends on G and satisfies $f(\gamma) \rightarrow +\infty$ as $\gamma \rightarrow 0^+$, and $C_0 > 1$ is an absolute constant depending only on the dimension. Then

$$\|G_t u_0\|_{L^p} \leq \gamma \|u_0\|_{L^p}. \quad (3.5)$$

When $G_t = e^{t\Delta}$, the above requirement on $\bar{\ell}$ can be made more explicit:

$$\bar{\ell}^2 \geq C_0 \ln(C_0/\gamma). \quad (3.6)$$

Proof of theorem 1.1. (Part 1) Note that $\sup_{0 \leq s < t} s^{(p-3)/2p} \|w\|_{L^p}(s) < \infty$ by (A1'). We will shortly specify a value for ϵ_1 . Fix $3 < p < \infty$. From (2.3) with $\tau = 0$, we have

$$\|w\|_{L^p}(t) \leq C_B \int_0^t \frac{1}{(t-s)^{1/2}} (\|vw\|_{L^p} + \|uw\|_{L^p}) \, ds. \quad (3.7)$$

We choose η to be

$$\eta = \frac{\epsilon_1}{c_1 - \epsilon_1}. \quad (3.8)$$

If $|y| \geq \eta^{-1} \sqrt{s}$, then, by (A1') and our requirement on η ,

$$|v(y, s)| \leq \frac{\epsilon_1}{\sqrt{s}}.$$

Hence,

$$\|vw\|_{L^p}(|y| \geq \eta^{-1} \sqrt{s}) \leq \frac{\epsilon_1}{\sqrt{s}} \|w\|_{L^p}(|y| \geq \eta^{-1} \sqrt{s}).$$

Assume T is small enough that

$$\frac{\|w(\cdot, t) \chi_{B_{\eta^{-1}\sqrt{t}}}\|_{L^p}}{\|w(\cdot, t) \chi_{B_{\eta^{-1}\sqrt{t}}^c}\|_{L^p}} < \epsilon_1,$$

for all $0 < t < T$. Using (1.4),

$$\|vw\|_{L^p}(|y| < \eta^{-1} \sqrt{s}) \leq \|v\|_{L^\infty} \|w\|_{L^p}(|y| < \eta^{-1} \sqrt{s}) \leq \epsilon_1 \frac{c_1}{\sqrt{s}} \|w\|_{L^p}(|y| \geq \eta^{-1} \sqrt{s}).$$

The same estimates hold with v replaced by u . Hence, for $t < T$

$$\begin{aligned} \|w\|_{L^p}(t) &\leq C_B \int_0^t \frac{2c_1 \epsilon_1}{(t-s)^{1/2} s^{1/2+(p-3)/2p}} s^{(p-3)/2p} \|w\|_{L^p}(s) \, ds \\ &\leq 2C_B c_1 \epsilon_1 t^{(3-p)/2p} \sup_{0 \leq s < t} s^{(p-3)/2p} \|w\|_{L^p}(s). \end{aligned} \quad (3.9)$$

Taking

$$\epsilon_1 \leq \frac{1}{4C_B c_1},$$

and, after rearranging things, a supremum on the left-hand side, we have

$$\sup_{0 < s < T} s^{(p-3)/2p} \|w\|_{L^p}(s) \leq 0.$$

In light of (1.4) we conclude that $w = 0$ on $\mathbb{R}^3 \times (0, T)$. Global uniqueness follows from the local well-posedness theory and the fact that $u(\cdot, T/2) \in L^\infty(\mathbb{R}^3)$.

If $p = \infty$, then we set up our argument slightly differently⁵, beginning with,

$$\|w\|_{L^\infty}(t) \leq C_B \int_0^t \frac{1}{(t-s)^{1/2+3/(2q)}} \|w\|_{L^\infty} (\|v\|_{L^q} + \|u\|_{L^q}) \, ds, \quad (3.10)$$

and then reason similarly. Ultimately this avoids having to integrate

$$\int_0^t \frac{1}{(t-s)^{1/2} s} \, ds,$$

which diverges.

(Part 2) We assume (A2). Fix t_0 , which plays the role of t in the statement of the theorem. Let $M_0 = 2 \max\{\|u\|_{L^\infty}(t_0), \|v\|_{L^\infty}(t_0)\}$. Then, there exists $T_0 = 4\tilde{c}_\infty M_0^{-2}$ so that $\|u\|_{L^\infty}(t) + \|v\|_{L^\infty}(t) \leq 2(\|u\|_{L^\infty}(t_0) + \|v\|_{L^\infty}(t_0))$ for all $t_0 \leq t \leq t_0 + T_0$. Bilinear estimates imply

$$\|w\|_{L^\infty}(t) \leq \|e^{(t-t_0)\Delta} w(t_0)\|_{L^\infty} + 2C_B (t-t_0)^{1/2} M_0^2.$$

We have

$$2C_B (t-t_0)^{1/2} M_0^2 \leq \frac{c_2}{2\sqrt{t}},$$

if

$$2C_B (t-t_0)^{1/2} M_0^2 \leq \frac{c_2}{2\sqrt{T_0+t_0}}.$$

Choose t to satisfy

$$t = t_0 + \frac{c_2^2}{16C_B^2 (\tilde{c}_\infty M_0^2 + t_0 M_0^4)}. \quad (3.11)$$

We will show that, if (1.5) holds at time t_0 for a choice of parameters to be specified momentarily, then

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^\infty} \leq \frac{c_2}{4\sqrt{t}},$$

leading to a contradiction, namely

$$\|w\|_{L^\infty}(t) \leq \frac{3c_2}{4\sqrt{t}},$$

⁵ To illustrate the details which are omitted here, we pursue this case in the proof of theorem 1.3 below, as it requires similar logic.

as this violates (A2). For this we will use lemma 3.2 with

$$\gamma = \frac{c_2}{4M_0\sqrt{t}}.$$

Define $\bar{\ell}$, β and ϵ according to this choice of γ , the time scale $t - t_0$ and (3.6). If (1.5) holds at time t_0 , then,

$$\begin{aligned} S_t &:= \{x \in \mathbb{R}^3 : |w(x, t_0)| \geq \beta \|w\|_{L^\infty}(t_0)\} \subset \left\{ \frac{c_3 \sqrt{t_0}^a}{(b|x| + \sqrt{t_0})^{a+1}} \geq \beta \frac{c_2}{\sqrt{t_0}} \right\} \\ &\subset \left\{ \left(\frac{c_3}{\beta c_2} \right)^{1/(a+1)} \sqrt{t_0} \geq b|x| \right\}. \end{aligned} \quad (3.12)$$

To ensure $(\epsilon, \beta, \bar{\ell}\sqrt{t-t_0})$ -sparseness we choose b so that

$$\left| \left\{ \left(\frac{c_3}{\beta c_2} \right)^{1/(a+1)} \sqrt{t_0} \geq b|x| \right\} \right| \leq \epsilon |B_{\bar{\ell}\sqrt{t-t_0}}|,$$

namely,

$$\frac{1}{b^3} := \frac{\epsilon \bar{\ell}^3 (t - t_0)^{3/2}}{(c_3 / (\beta c_2))^{3/(a+1)} t_0^{3/2}}.$$

Under this choice of parameters, by lemma 3.2,

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^\infty} \leq \frac{c_2}{4\sqrt{t}},$$

which we already noted is a contradiction. Therefore, we cannot have that (1.5) holds at any time t_0 .

(Part 3) Part 3 follows from an argument appearing in [33] and again in [13, 51, 54]. We will need point-wise bounds for some convolutions which we copy from [51, lemma 2.1]: Let $a, b \in (0, 5)$ and $a + b > 3$. Then,

$$\phi(x, a, b) = \int_0^1 \int_{\mathbb{R}^3} \left(|x - y| + \sqrt{1 - t} \right)^{-a} (|y| + \sqrt{t})^{-b} dy dt, \quad (3.13)$$

is well defined for $x \in \mathbb{R}^3$, and

$$\phi(x, a, b) \lesssim R^{-a} + R^{-b} + R^{3-a-b} [1 + (1_{a=3} + 1_{b=3}) \log R], \quad (3.14)$$

where $R = |x| + 2$. These estimates can be extended to other time intervals by a change of variable. From (2.1) we have

$$|w(x, t)| \leq \int_0^t \int_{\mathbb{R}^3} \frac{C_B}{(|x - y| + \sqrt{t - s})^4} \frac{c_1^2}{(|y| + \sqrt{s})^2} dy ds,$$

which implies

$$|w(x, t)| \leq C_B \frac{c_1^2 \sqrt{t}}{(|x| + \sqrt{t})^2},$$

by the preceding convolution estimates. Applying this argument two more times yields the advertised result. We illustrate the first application. We have using the new bound for $|w|$ and (A1') that

$$\begin{aligned} B(u, w) &\leq \int_0^t \int_{\mathbb{R}^3} \frac{C_B}{(|x-y| + \sqrt{t-s})^4} |w| |u| \, dy \, ds \\ &\leq C_B c_1^3 \sqrt{t} \int_0^t \int_{\mathbb{R}^3} \frac{C_B}{(|x-y| + \sqrt{t-s})^4} \frac{1}{(|y| + \sqrt{s})^3} \, dy \, ds. \end{aligned} \quad (3.15)$$

The pointwise estimates for the time convolution imply this term is bounded by $t/(|x| + \sqrt{t})^3$. The same applies to the other terms in the integral expansion for w . One more repetition results in the bound $t^{3/2}/(|x| + \sqrt{t})^4$. $\square$

4. Frequency scenario

We begin by proving separation rates for individual modes of self-similar solutions.

Proof of proposition 1.2. Observe that

$$\frac{1}{|x|^n} \in \dot{B}_{\infty, \infty}^{-n}.$$

This follows from the fact that $|x|^{-n}$ is $(-n)$ -homogeneous and the relationship this induces on $\|\dot{\Delta}_j(|\cdot|^{-n})\|_{L^\infty}$ compared to $\|\dot{\Delta}_0(|\cdot|^{-n})\|_{L^\infty}$. In [13] it is shown that $w(x, 1) \lesssim (1 + |x|)^4$ (this is why we require the solutions be local energy solutions). Hence, $w(\cdot, 1) \in \dot{B}_{\infty, \infty}^{-4}$. Additionally, the self-similar relationship between modes can be calculated as in [15, (2.5)], implying, for a given j and t and letting $2^{2(j-i)}t = 1$,

$$\|\dot{\Delta}_j w\|_{L^\infty}(t) = \frac{1}{\sqrt{t}} \|\dot{\Delta}_i w(2^{j-i}(\cdot), 1)\|_{L^\infty} \lesssim_{u_0} \frac{1}{\sqrt{t}} 2^{4i},$$

due to membership in $\dot{B}_{\infty, \infty}^{-4}$. Hence,

$$\|\dot{\Delta}_j w\|_{L^\infty}(t) \lesssim_{u_0} 2^{4j} t^{3/2}.$$

The stated conclusion follows after summing over $j < J$. $\square$

Note that we used a global bound on the profile at $t = 1$. For discretely self-similar solutions, these bounds are only available away from the origin—it is not known in general whether or not the solutions can be singular on a ball centred at the origin. Additional work would therefore be needed to check that $w(\cdot, 1) \in \dot{B}_{\infty, \infty}^{-4}$ when u and v are only discretely self-similar.

We now recall a definition of sparseness framed in terms of the Littlewood-Paley decomposition. Compared to physical sparseness, this definition has the advantage of involving fewer parameters. This notion of sparseness encompasses the spatial version, at least within a certain parameter range, as demonstrated in [2].

Definition 4.1 (L^p -sparseness in frequency). Let $\beta \in (0, 1)$ and $J \in \mathbb{R}$. Then a vector field $u_0 \in L^p$ is (β, J) -sparse in frequency in L^p if

$$\|\Delta_{<J} u_0\|_{L^p} \leq \beta \|u_0\|_{L^p}. \quad (4.1)$$

The following lemma is taken from [2].

Lemma 4.2. Fix $1 \leq p \leq \infty$, $t > 0$ and $\gamma > 0$. Let $u_0 \in L^p$. There exists $J \in \mathbb{Z}$ satisfying $2^J \sim \gamma^{-1} t^{-1/2}$ and $\beta = \gamma/2$ so that, if u_0 is (β, J) -sparse in frequency, then

$$\|e^{t\Delta} u_0\|_{L^p} \leq \gamma \|u_0\|_{L^p}. \quad (4.2)$$

We are now ready to prove theorem 1.3.

Proof of theorem 1.3. (Part 1) We first prove the $p = \infty$ case. Applying $\dot{\Delta}_{<J_1}$ to the perturbed Navier–Stokes equations and adopting the abbreviation $f_{<J} = \dot{\Delta}_{<J}$ (as well as similar abbreviations for $\dot{\Delta}_{\leq J}$, $\dot{\Delta}_{\geq J}$ and $\dot{\Delta}_{>J}$) gives

$$(\partial_t - \Delta) w_{<J_1} + \mathbb{P} \nabla \cdot (w \otimes v + v \otimes w + w \otimes w)_{<J_1} = 0.$$

Then,

$$w_{<J_1}(x, t) = - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (w \otimes v + v \otimes w + w \otimes w)_{<J_1}(s) \, ds.$$

Note that, provided $v \in L^\infty(0, T; \dot{B}_{\infty, \infty}^{-1})$, we have

$$\|v_{<J}\|_{L^\infty} \leq \|v\|_{L^\infty(0, T; \dot{B}_{\infty, \infty}^{-1})} 2^J.$$

Also, by support considerations in the Fourier variable, we have

$$(f_{<J_1} g)_{<J_1} = (f_{<J_1} g_{<J_1+2})_{<J_1}.$$

The bilinear terms are all bounded the same, as is illustrated in the following estimate where q is taken in $(3, \infty)$,

$$\begin{aligned} & \left| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (w \otimes v)_{<J_1}(s) \, ds \right| \\ & \leq C_B \int_0^t \frac{1}{(t-s)^{\frac{1}{2} + \frac{3}{2q}}} \|w_{<J_1}\|_{L^\infty} \|v_{<J_1+2}\|_{L^q} \, ds + C_B \int_0^t \frac{1}{(t-s)^{\frac{1}{2} + \frac{3}{2q}}} \|(w_{\geq J_1} \otimes v)_{<J_1}\|_{L^q} \, ds \\ & \leq C_B C(q) \int_0^t \frac{c_1}{(t-s)^{\frac{1}{2} + \frac{3}{2q}} s^{\frac{1}{2}}} s^{\frac{1}{2}} \|w_{<J_1}\|_{L^\infty}(s) 2^{J_1(s)(1-\frac{3}{q})} \, ds \\ & \quad + C C_B \int_0^t \frac{c_1}{(t-s)^{\frac{1}{2} + \frac{3}{2q}} s^{1-\frac{3}{2q}}} s^{\frac{1}{2}} \|w_{\geq J_1}\|_{L^\infty}(s) \, ds \\ & \leq 2 C_B C(q) c_1 \epsilon_2 t^{-1/2} \sup_{0 \leq s \leq t} s^{1/2} \|w_{<J_1}\|_{L^\infty}(s), \end{aligned} \quad (4.3)$$

which holds by the Besov space bounds in (A1) and (2.2), the additional assumption that $2^{J_1(s)(1-3/q)} \sim_{c_1, c_B} \epsilon_2 s^{-(1-3/q)/2}$ (which defines J_1) and the observation in (4.6), from which

we obtain a constant $C(q)$. Since q was arbitrary, we may fix, e.g. $q = 6$ to make this constant universal. We therefore take ϵ_2 to satisfy

$$2CC_Bc_1\epsilon_2t^{-1/2} < \frac{1}{3}t^{-1/2},$$

whence obtaining

$$t^{1/2} \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (w \otimes v)_{<J_1}(s) \, ds \right\|_{L^\infty} < \sup_{0 \leq s \leq t} \frac{1}{3} s^{1/2} \|w_{<J_1}\|_{L^\infty}(s).$$

Repeating this for the other terms in the expansion for $w_{<J_1}$ and taking a time-supremum of the left-hand side of the $w_{<J_1}$ integral expansion implies $w = 0$.

The $3 < p < \infty$ case is similar but we do not need to pass to the L^q norm in our bilinear estimate—this resembles what we did in detail in the proof of theorem 1.1.

(Part 2) We assume (A2). Fix t_0 . Let $M_0 = 2 \max\{\|u\|_{L^p}(t_0), \|v\|_{L^p}(t_0)\}$. Then, there exists $T_0 = 4\tilde{c}_p M_0^{2p/(3-p)}$ so that $\|u\|_{L^p}(t) + \|v\|_{L^p}(t) \leq 2(\|u\|_{L^p}(t_0) + \|v\|_{L^p}(t_0))$ for all $t_0 \leq t \leq t_0 + T_0$. The bilinear estimate (2.2) implies

$$\begin{aligned} \|w\|_{L^p}(t) &\leq \|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} + \int_{t_0}^t \frac{C_B}{(t-s)^{\frac{1}{2} + \frac{3}{2p}}} (\|uw\|_{L^{p/2}}(s) + \|wv\|_{L^{p/2}}(s)) \, ds \\ &\leq \|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} + 2C_B(t-t_0)^{1/2-3/(2p)} M_0^2. \end{aligned} \quad (4.4)$$

We have

$$2C_B(t-t_0)^{1/2-3/(2p)} M_0^2 \leq \frac{c_2}{2t^{1/2-3/(2p)}},$$

if

$$2C_B(t-t_0)^{1/2-3/(2p)} M_0^2 \leq \frac{c_2}{2(T_0+t_0)^{1/2-3/(2p)}}.$$

Choose t to satisfy

$$t = t_0 + \left(\frac{c_2}{4M_0^2 C_B (T_0 + t_0)^{1/2-3/(2p)}} \right)^{\frac{2p}{p-3}} \quad (4.5)$$

We will obtain the contradiction if

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} \leq \frac{c_2}{4t^{1/2-3/(2p)}}.$$

Lemma 4.2 will give us a conclusion like

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} \leq \gamma \|w\|_{L^p}(t_0),$$

which is bounded above by γM_0 . We therefore want to use lemma 4.2 with γ satisfying

$$\gamma M_0 \leq \frac{c_2}{4(T_0+t_0)^{1/2-3/(2p)}}, \text{ and so we fix } \gamma = \frac{c_2}{4M_0(T_0+t_0)^{1/2-3/(2p)}}.$$

Choosing $\beta = \gamma/2$ and, recalling $t - t_0$ is given in (3.11), choosing

$$2^{J_2} \sim \gamma^{-1} (t - t_0)^{-1/2},$$

we see by lemma 4.2 that if $w(t_0)$ is (β, J_2) -sparse in L^p , then

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} \leq \frac{c_2}{4t^{1/2-3/(2p)}}.$$

Therefore, $w(t_0)$ can never be (β, J_2) -sparse in L^p .

(Part 3) Note that (Part 2), (A1) and (A2) together imply

$$\|w_{\leq J_2}\|_{L^p}(t) \sim_{\gamma} t^{3/(2p)-1/2} \sim_{c_1, c_2} \|w\|_{L^p}(t).$$

By (A1),

$$\begin{aligned} \|w_{< J_3}\|_{L^p} &\leq \sum_{j < J_3} \|u_j - v_j\|_{L^p} \leq 2c_1 \sum_{j < J_3} 2^{j(1-\frac{3}{p})} \\ &= 2c_1 2^{J_3(1-\frac{3}{p})} \sum_{j < 0} 2^{j(1-\frac{3}{p})} \lesssim_p 2c_1 2^{(J_3+1)(1-\frac{3}{p})}. \end{aligned} \quad (4.6)$$

We require that

$$2c_1 2^{(J_3+1)(1-3/p)} \lesssim t^{3/(2p)-1/2},$$

where the suppressed constant is chosen small enough that

$$\|w_{< J_3}\|_p \leq \frac{1}{2} \|w_{\leq J_2}\|_{L^p}.$$

It follows that

$$\|w_{J_3 \leq j \leq J_2}\|_{L^p} \geq \frac{1}{2} \|w_{\leq J_2}\|_{L^p},$$

which completes the proof. $\square$

Remark 4.3. It may be interesting to point out that, under the assumptions of theorem 1.3, if (A1) and (A2) hold, the latter for $p = \infty$, then there exists ϵ_4 so that,

$$\inf_{t>0} \|w\|_{\dot{B}_{\infty, \infty}^{-1}}(t) > \epsilon_4.$$

In other words, a condition like (A2) implies other scaling invariant measurements of the error do not vanish at $t = 0$. To prove this, recall (2.3) and observe that, for $t > 0$,

$$\begin{aligned} \int_0^\tau e^{(\tau-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes w)(t+s) \, ds &\leq C_B \sup_{t < s' < t+\tau} \int_0^\tau \frac{1}{(\tau-s)^{1/2}} \frac{c_1^2}{t} \, ds \\ &\leq \frac{C_B c_1^2 \tau^{1/2}}{t} \\ &\leq \frac{c_2}{6\sqrt{t+\tau}}, \end{aligned} \quad (4.7)$$

provided

$$\tau = \frac{c_2^2 t}{72 C_B^2 c_1^4}.$$

Other terms in the bilinear part of (2.3) are handled identically. Therefore,

$$\frac{c_2}{(t + \tau)^{1/2}} \leq \|w\|_{L^\infty}(t + \tau) \leq \|e^{(t+\tau)\Delta} w(t)\|_{L^\infty} + \frac{c_2}{2\sqrt{t + \tau}},$$

implying

$$c_2 \leq \sup_{0 < \tilde{\tau} < \infty} (t + \tilde{\tau})^{1/2} \|e^{(t+\tilde{\tau})\Delta} w(t)\|_{L^\infty} \sim \|w\|_{\dot{B}_{\infty,\infty}^{-1}}(t).$$

5. Discretised scenario

We begin by establishing an analogue to lemmas 3.2 and 4.2 in the context of the discretised projection operator. Recall that we consider a fixed lattice of cubes $\{Q_i\}$ with disjoint interiors, volume h^3 , and whose closures cover $\mathbb{R}^3$. Denote the centre of Q_i by x_i . Let

$$I_{h,t} u_0(x) = \sum_j \chi_{Q_j}(x) \sum_i \frac{1}{t^{3/2}} e^{-|x_j - x_i|^2/(4t)} \int_{Q_i} u_0(y) dy.$$

and let

$$J_h u_0(x) = \sum_j \chi_{Q_j}(x) \frac{1}{|Q_j|} \int_{Q_j} u_0(y) dy.$$

The next lemma should be understood in analogy with lemmas 3.2 and 4.2 but where the interpretation of ‘sparseness’ is understood through the length scale h of the interpolant operator.

Lemma 5.1. Fix $t > 0$, $p \in [1, \infty]$ and $\gamma > 0$. Take $h \lesssim \gamma \sqrt{t}$. If

$$\|J_h u_0\|_{L^p} \leq \gamma/2 \|u_0\|_{L^p},$$

then

$$\|e^{t\Delta} u_0\|_{L^p} \leq \gamma \|u_0\|_{L^p}.$$

Proof. Begin by taking $x \in Q_j$. We have

$$|e^{t\Delta} u_0(x)| \leq |e^{t\Delta} u_0(x) - I_{h,t} u_0(x)| + |I_{h,t} u_0(x)|. \quad (5.1)$$

Expanding the leading term on the right-hand side gives

$$\begin{aligned} |e^{t\Delta} u_0(x) - I_{h,t} u_0(x)| &\lesssim \sum_i \frac{1}{t^{3/2}} \int_{Q_i} e^{-|x_j - x_i|^2/(4t)} \underbrace{\left(e^{-|x - y|^2/(4t) + |x_j - x_i|^2/(4t)} - 1 \right)}_{=: F(x, y, x_j, x_i)} u_0(y) dy \\ &\lesssim \sum_i F(x, y, x_j, x_i) \frac{1}{t^{3/2}} e^{-|x_j - x_i|^2/(4t)} \|u_0\|_{L^p(Q_i)} |Q_i|^{1-1/p}. \end{aligned} \quad (5.2)$$

By the mean value theorem and the fact that the Gaussian is Schwartz, it follows that

$$\sup_{i,j} \sup_{x \in Q_j, y \in Q_i} F(x, y, x_j, x_i) \lesssim \frac{h}{\sqrt{t}}.$$

To elaborate, note that $x_j - x_i \in Q_j - Q_i$ and $x - y \in Q_j - Q_i$. Hence $\|x_j - x_i\| - \|x - y\| \lesssim h$. Viewing $F(x, y, x_j, x_i)$ as $|e^{-(z/(2\sqrt{t}))^2} - e^0|$ where $|z| = \|x_j - x_i\| - \|x - y\| \lesssim h$ and using the mean value theorem as well as the chain rule to evaluate partial derivatives in z of $e^{-|z|^2/4t}$, we find that

$$F(x, y, x_j, x_i) \lesssim h \frac{|\bar{z}|}{t} e^{-|\bar{z}|^2/(4t)},$$

for some $\bar{z}$ satisfying $|\bar{z}| \lesssim h$. The fact that $e^{-|\bar{z}|^2/(4t)} \lesssim \frac{\sqrt{t}}{\bar{z}}$ yields the desired bound on F . Using this bound we find that

$$\begin{aligned} \|e^{t\Delta} u_0 - I_{h,t} u_0\|_{L^p(Q_j)} &\lesssim \frac{h}{\sqrt{t}} |Q_j|^{1/p} \sum_i \frac{1}{t^{3/2}} e^{-|x_j - x_i|^2/(4t)} \|u_0\|_{L^p(Q_i)} |Q_i|^{1-1/p} \\ &\lesssim \frac{h}{\sqrt{t}} \sum_i \frac{h^3}{t^{3/2}} e^{-|x_j - x_i|^2/(4t)} \|u_0\|_{L^p(Q_i)}, \end{aligned} \quad (5.3)$$

which is a discrete convolution. We now apply ℓ^p to the sequence $\{\|e^{t\Delta} u_0 - I_{h,t} u_0\|_{L^p(Q_j)}\}$ and use Young's inequality to obtain

$$\|e^{t\Delta} u_0 - I_{h,t} u_0\|_{L^p(\mathbb{R}^3)} \lesssim \frac{h}{\sqrt{t}} \|u_0\|_{L^p(\mathbb{R}^3)} \sum_i \frac{h^3}{t^{3/2}} e^{-|x_i|^2/(4t)} \lesssim \frac{h}{\sqrt{t}} \|u_0\|_{L^p(\mathbb{R}^3)}.$$

We also observe that

$$|I_{h,t} u_0(x)| = \left| \sum_i \chi_{Q_i}(x) \frac{h^3}{t^{3/2}} e^{-|x_j - x_i|^2/(4t)} \frac{1}{|Q_i|} \int_{Q_i} u_0(y) dy \right| \lesssim |J_h u_0(x)|, \quad (5.4)$$

and, so,

$$\|I_{h,t} u_0\|_{L^p(\mathbb{R}^3)} \lesssim \|J_h u_0\|_{L^p(\mathbb{R}^3)}.$$

Combining the above observations and taking $h \lesssim \gamma\sqrt{t}$, we obtain

$$\|e^{t\Delta} u_0\|_{L^p(\mathbb{R}^3)} \lesssim \frac{h}{\sqrt{t}} \|u_0\|_{L^p(\mathbb{R}^3)} + \|J_h u_0\|_{L^p(\mathbb{R}^3)} \leq \gamma \|u_0\|_{L^p(\mathbb{R}^3)}.$$

□

Proof of theorem 1.4. We include details for $3 < p < \infty$. Our starting point is

$$\|w\|_{L^p}(t) \leq C_B \int_0^t \frac{1}{(t-s)^{1/2}} (\|vw\|_{L^p} + \|uw\|_{L^p}) ds, \quad (5.5)$$

where we take $t < \delta$. We only consider the case $\|vw\|_{L^p}$ as the treatment of u is identical.

Recall from the proof of theorem 1.1 that, choosing η to be

$$\eta = \frac{\epsilon_1}{c_1 - \epsilon_1}, \quad (5.6)$$

we have

$$\|vw\|_{L^p(|y| \geq \eta^{-1}\sqrt{s})} \leq \frac{\epsilon_1}{\sqrt{s}} \|w\|_{L^p(|y| \geq \eta^{-1}\sqrt{s})}.$$

We now choose $\bar{h} = \bar{h}(s)$ so that $B_{\eta^{-1}\sqrt{s}}(0) \subset Q_0$, where Q_0 is the cube centred at the origin with edge-lengths $\bar{h}$ —that is, $\bar{h} \geq 2\eta^{-1}\sqrt{s}$. As ϵ_1 was fixed in the proof of theorem 1.1, we replace ϵ_1 with ϵ_3 and will subsequently adjust its value.

We next write

$$\|vw\|_{L^p(|y| < \eta^{-1}\sqrt{s})} \leq \|vJ_{\bar{h}}w\|_{L^p(|y| < \eta^{-1}\sqrt{s})} + \|v(w - J_{\bar{h}}w)\|_{L^p(|y| < \eta^{-1}\sqrt{s})}.$$

For the first term on the right-hand side we have

$$\|vw\|_{L^p(|y| < \eta^{-1}\sqrt{s})} \leq \left\| v \frac{1}{\bar{h}^3} \int_{Q_0} w(y) \, dy \right\|_{L^p(Q_0)} \leq c_1 c_4 s^{3/(2p)-1/2} s^{1/4} \bar{h}^{-3/2},$$

where, because u and v are $L^{3,\infty}$ -weak solutions, we have $\|w\|_{L^2}(t) \leq c_4 t^{1/4}$, for a constant c_4 depending on $\|u_0\|_{L^{3,\infty}}$. We further restrict $\bar{h}$ so that

$$s^{3/(2p)-1/2} s^{1/4} \bar{h}^{-3/2} \leq \frac{\epsilon_3}{\sqrt{s}} \|w\|_{L^p}(s).$$

For the second term on the right-hand side we have

$$\|v(w - J_{\bar{h}}w)\|_{L^p} \leq \frac{c_1 \epsilon_3}{\sqrt{s}} \|w\|_{L^p}(s),$$

by assumption.

Combining these bounds we obtain

$$\begin{aligned} \|w\|_{L^p}(t) &\lesssim_{c_1, c_4, c_B} \epsilon_3 \int_0^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1-3/(2p)}} s^{1/2-3/(2p)} \|w(s)\|_{L^p} \, ds \\ &\leq \frac{1}{2} t^{3/(2p)-1/2} \sup_{0 < s < t} s^{1/2-3/(2p)} \|w(s)\|_{L^p}, \end{aligned} \quad (5.7)$$

provided ϵ_3 is chosen small compared to c_1 , c_4 and c_B , and

$$\bar{h} = \max \left\{ 2 \frac{c_1 - \epsilon_3}{\epsilon_3} \sqrt{s}, \left(t^{3/4} \frac{t^{3/(2p)-1/2}}{\epsilon_3 \|w\|_{L^p}(t)} \right)^{2/3} \right\}.$$

This is enough to conclude $w \equiv 0$.

As we have seen in the proofs of theorems 1.1 and 1.3, the case $p = \infty$ follows similarly but we need to initialise our argument with the estimate

$$\|w\|_{L^\infty}(t) \leq C_B \int_0^t \frac{1}{(t-s)^{1/2+3/(2q)}} (\|vw\|_{L^q} + \|uw\|_{L^q}) \, ds, \quad (5.8)$$

where q can be any value in $(3, \infty)$.

(Part 2) This proof is essentially identical to the proof of part 2 of theorem 1.3. The only difference is that we replace 2^{J_2} with h^{-1} . We then see by lemma 5.1 that, if

$$\|J_h w\|_{L^p}(t_0) \leq \frac{\gamma}{2} \|w\|_{L^p}(t_0),$$

then

$$\|e^{(t-t_0)\Delta} w(t_0)\|_{L^p} \leq \frac{C_2}{4t^{1/2-3/(2p)}}.$$

This implies that we *cannot* have

$$\|J_h w\|_{L^p}(t_0) \leq \frac{\gamma}{2} \|w\|_{L^p}(t_0).$$

□

6. Conditional predictability criteria

Proof of theorem 1.5. Note that all calculations are done at the fixed time as stated in the theorem, which is suppressed in what follows. We have

$$\partial_t \|w\|_{L^2}^2 + 2\|\nabla w\|_2^2 \leq 2 \int w \cdot \nabla u w \, dx.$$

Note that

$$\|\Delta_{>J} w\|_2 \leq C 2^{-J} \|\nabla w\|_2,$$

which is just a Bernstein inequality. Hence,

$$\partial_t \|w\|_{L^2}^2 + C 2^{2J} \|\Delta_{>J} w\|_2^2 + \|\nabla w\|_2^2 \leq C \|u\|_{L^\infty} \|\nabla w\|_2 \|w\|_2 \leq C^2 \|u\|_{L^\infty}^2 \|w\|_2^2 + \|\nabla w\|_2^2.$$

We alternatively have

$$\partial_t \|w\|_{L^2}^2 + C 2^{2J} \|\Delta_{>J} w\|_2^2 + \|\nabla w\|_2^2 \leq C \|\nabla u\|_{L^\infty} \|w\|_2^2.$$

Assume that at some time (which everything that follows occurs at) and some $K > 0$ that

$$\|\Delta_{\leq J} w\|_{L^2}^2 \leq K \|\Delta_{>J} w\|_{L^2}^2.$$

Then,

$$\|w\|_2^2 \leq \|w_{\leq J}\|_2^2 + \|w_{>J}\|_2^2 \leq (K+1) \|w_{>J}\|_2^2.$$

And, provided

$$C 2^{2J} \|\Delta_{>J} w\|_2^2 \geq 2(K+1) C^2 \|u\|_{L^\infty}^2 \|\Delta_{>J} w\|_2^2,$$

or

$$C 2^{2J} \|\Delta_{>J} w\|_2^2 \geq 2(K+1) C \|\nabla u\|_{L^\infty} \|\Delta_{>J} w\|_2^2,$$

which is implied if $(K+1)^{1/2} \min\{\|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}}\} \lesssim 2^J$ for a sufficiently small suppressed constant, we obtain

$$\partial_t \|w\|_{L^2}^2 + \frac{C}{2} 2^{2J} \|\Delta_{>J} w\|_{L^2}^2 \leq 0.$$

Noting again that $\|w\|_2^2 \leq (K+1) \|w_{\geq J}\|_2^2$ we improve this to

$$\partial_t \|w\|_{L^2}^2 + \frac{C 2^{2J}}{(K+1)} \|w\|_2^2 \leq 0.$$

We now think of J as fixed and define K according to $(K+1)^{1/2} \min\{\|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}}\} = c_5 2^J$ where c_5 is a universal constant consistent with the suppressed constant above. In this case we see that, if

$$\frac{\|\Delta_{\leq J} w\|_{L^2}^2}{\|\Delta_{>J} w\|_{L^2}^2} \leq \left(\frac{c_5 2^{2J}}{\min\{\|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}}\}^2} - 1 \right),$$

then

$$\partial_t \|w\|_{L^2}^2 + \frac{C \min\{\|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}}\}^2}{2c_5} \|w\|_{L^2}^2 \leq \partial_t \|w\|_{L^2}^2 + \frac{C}{2} 2^{2J} \|\Delta_{>J} w\|_{L^2}^2 \leq 0,$$

and the error is non-increasing. The preceding condition makes sense provided $K \geq 0$, which implies

$$2^{2J} \geq \frac{\min\{\|u\|_{L^\infty}, \sqrt{\|\nabla u\|_{L^\infty}}\}^2}{c_5}.$$

The proof for the discretised operator is identical once we observe that, in place of Bernstein's inequality, we have by the Poincaré inequality that

$$\|w - J_h w\|_{L^2} \leq Ch \|\nabla w\|_{L^2}.$$

Then, we just replace 2^J with h^{-1} , $\Delta_{\leq J} w$ with $J_h w$ and $\Delta_{>J} w$ with $w - J_h w$ throughout the proof. $\square$

Remark 6.1. One can use sparseness to get a similar result but it seems sub-optimal compared to the energy methods employed above. In particular, it is possible to prove that for $T > 0$ given, there exist J and h so that $2^{-J} \sim h \sim M^{-1}$ where $M = \sup_{0 < s < T} (\|u\|_{L^\infty} + \|v\|_{L^\infty})(s)$ and, if

$$\|w_{\leq J}\|_{L^2} \leq \frac{1}{4} \|w\|_{L^2} \text{ or } \|J_h w\|_{L^2} \leq \frac{1}{4} \|w\|_{L^2},$$

for all $t \in (0, T)$, then

$$\sup_{0 < s < T} \|w\|_{L^2}(s) \leq 2 \|w\|_{L^2}(0).$$

To execute the sparseness argument using (2.3), we need M to depend on $\|u\|_\infty$ and $\|v\|_\infty$. This is not the case if we use energy methods due to the standard cancellation. This is why only a single non-unique flow field in theorem 1.5 needs to be finite. Furthermore, the decay rate in the proof of theorem 1.5 does not follow obviously from the sparseness argument. So, even though mild solution methods allow us to suppress the L^2 norm, they seem sub-optimal compared to energy methods.

7. The energy separation of Jia and Šverák's hypothetical solutions

In [33], Jia and Šverák demonstrate that, if certain non-unique self-similar solutions exist, then non-unique Leray–Hopf solutions also exist. This is not immediate because self-similar initial data for (NS) is -1 -homogeneous and so does not belong to L^2 . Our proposition 1.6 makes the observation that the error energy of these non-unique Leray–Hopf solutions is bounded below. Jia and Šverák also identify sufficient conditions for non-uniqueness in [33] based on the eigenvalues of a linearised operator, but this is less relevant to the point we are making in proposition 1.6. In this section we will prove proposition 1.6. The proof is fairly short after a close reading of [33]. Before stating the proof, we will recall some of the notation and machinery of [33].

Self-similarity plays a crucial role in [33]. If a solution u to (NS) is self-similar, then it has a stationary profile defined by

$$\frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right) := u(x, t),$$

for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$. The profile U satisfies a stationary system referred to as the Leray equations, namely,

$$\Delta U + \frac{x}{2} \cdot \nabla U + \frac{1}{2}U - U \cdot \nabla U - \nabla P = 0, \quad \nabla \cdot U = 0.$$

In [33], Navier–Stokes solutions involving a second profile, which they denote ϕ , are also considered [33, (1.3)]. However, in the result from [33] which we care about (namely [33, theorem 5.2]), $\phi = 0$ and so we do not discuss it further. In [33], a linearised operator is defined based on the Leray equations. The non-uniqueness result is then conditioned on the eigenvalues of this operator.

We consider the Leray–Hopf weak solutions in [33] which are guaranteed to exist by [33, theorem 5.2] under what Jia and Šverák label spectral condition (B). The proof of [33, theorem 5.2] is not written explicitly, but follows the same logic as the proof of [33, theorem 5.1], for which details are given. The difference between the proof of [33, theorem 5.1] and the proof of [33, theorem 5.2] is that $\phi = 0$ for both u_i in the latter—this is because under spectral condition (B), both u_i are self-similar.

The main ingredient in both proofs is [33, theorem 3.1], which constructs solutions to a perturbed (NS) around self-similar solutions. We presently recall the details with some modifications to tailor the statement to our needs. We do not recall the precise spectral assumption made on a solution to Leray's equation as this is not our main point.

Theorem 7.1 (perturbations of self-similar fields). *Let U satisfy a spectral assumption stated precisely in [33, theorem 3.1]. Suppose also that U satisfies⁶*

$$U \in L^\infty; \quad \nabla \cdot U = 0; \quad \text{ess sup}_x \left((1 + |x|)^{1+|\alpha|} |\nabla^\alpha U(x)| < \infty \text{ for } |\alpha| = 0, 1, 2 \right).$$

Let $u(x, t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$. Given a divergence free vector field $v_0 \in L^4(\mathbb{R}^3)$, there exists a sufficiently small $T = T(U, v_0) > 0$ and solution v to the problem

$$\partial_t v - \Delta v + u \cdot \nabla v + v \cdot \nabla u + v \cdot \nabla v + \nabla \pi = 0; \quad \nabla \cdot v = 0, \quad (7.1)$$

in $\mathbb{R}^3 \times (0, T)$ with initial data $v(\cdot, 0) = v_0$ in the sense that

$$\lim_{t \rightarrow 0^+} \|v(\cdot, t) - v_0\|_{L^4} = 0.$$

Moreover v satisfies

$$v \in L^\infty(0, T; L^4); \quad t^{1/2} \|\nabla v\|_{L^4}(t) \in L^\infty(0, T); \quad \lim_{t \rightarrow 0} t^{1/2} \|\nabla v\|_{L^4}(t) = 0,$$

with

$$\|v\|_{L^\infty(0, T; L^4)} + \|(\cdot)^{1/2} \|\nabla v\|_{L^4(\mathbb{R}^3)}(\cdot)\|_{L^\infty(0, T)} < C(U) \|u_0\|_{L^4}.$$

Furthermore, v is the unique solutions satisfying the above properties.

In [33, theorem 3.1], this theorem is applied to non-unique, local energy self-similar solutions with a common $O(|x|^{-1})$ initial condition u_0 . We label these solutions u_i for $i = 1, 2$ and denote their corresponding similarity profiles by U_i . Note that u_i are bounded at positive times. This and the known integrability conditions on v_i imply they satisfy the local energy inequality (2.5).

Leray–Hopf solutions are generated by theorem 7.1 by applying it to the tail of u_0 , which is in L^4 . We can mean many things by the ‘tail of u_0 ,’ for example we could mean a divergence free vector field which is supported on $\mathbb{R}^3 \setminus B(0, 2)$ and identical to u_0 on $\mathbb{R}^3 \setminus B(0, 3)$ —this sort of construction is well known and can be made rigorous using the Bogovskii map [54, p 45]. theorem 7.1 generates solutions v_i to (7.1) with initial data taken to be this ‘tail.’ The sought Leray–Hopf solutions then turn out to be $V_1 := u_1 - v_1$ and $V_2 := u_2 - v_2$, the initial datum of which is self-similar on $B(0, 2)$, compactly supported and belongs to L^2 . The validity of this procedure is confirmed in [33, theorem 5.2].

The main idea behind the following proof is that, within $B(0, 2)$ and as $t \rightarrow 0$, v_i goes to zero relatively quickly meaning V_i has the same asymptotics as $u_i = V_i + v_i$. This means that V_i is approximately self-similar near the space-time origin. This approximate self-similarity then leads to a lower bound for the error energy of the difference between the Leray–Hopf weak solutions.

Proof of proposition 1.6. In this proof we take u_0 , u_1 , and u_2 to be as described above. Let ψ be a smooth cut-off function so that $\psi = 0$ on $\mathbb{R}^3 \setminus B(0, 3)$ and $\psi = 1$ on $B(0, 2)$. Let $\bar{v}_0 = (1 - \psi)u_0$. Let Φ be the Bogovskii map as defined in [54, section 2.8]. Observe that $\text{div}(\bar{v}_0) = v_0 \text{div}(1 - \psi) \in L^4(B(0, 3) \setminus B(0, 2))$. Hence $\Phi(\bar{v}_0) \in W_0^{1,4}(B(0, 3) \setminus B(0, 2))$,

$$\text{div } \Phi = v_0 \text{div}(1 - \psi),$$

⁶ In [33] these properties define the space Y . Also, u is labelled $\tilde{a}$ and U is labelled a . The term b in [33] is zero for us.

and

$$\|\Phi(\bar{v}_0)\|_{W_0^{1,4}} \lesssim \|v_0\|_{L^4}.$$

Letting

$$v_0 = \bar{v}_0 - \Phi(\bar{v}_0),$$

we see that

$$\text{supp}(v_0) = \mathbb{R}^3 \setminus B(0, 2); \quad v_0 = u_0 \text{ on } \mathbb{R}^3 \setminus B(0, 3); \text{ and } v_0 \in L^4.$$

Let v_i be as defined in theorem 7.1 as perturbations of the background fields u_i and initial data v_0 . Let $V_i = u_i - v_i$ and denote the respective pressures by Π_i . By [33, theorem 5.2], these are Leray–Hopf weak solutions.

We will show that, for any ϵ , there exists T_ϵ so that

$$\|v_i\|_{L^2(B(0,1))}^2(t) = \|V_i - u_i\|_{L^2(B(0,1))}^2(t) \leq \epsilon t^{1/2}, \quad (7.2)$$

for $t \in (0, T_\epsilon]$. This will be enough to prove the proposition after noting that, for $t \leq 1$, the exact scalings of u_i imply that

$$t^{1/2} \|u_1 - u_2\|_{L^2(B(0,1))}^2(1) = \|u_1 - u_2\|_{L^2(B(0,t^{1/2}))}^2(t) \leq \|u_1 - u_2\|_{L^2(B(0,1))}^2(t),$$

from which, and by taking $\sqrt{\epsilon} = \|u_1 - u_2\|_{L^2(B(0,1))}(1)/8$, we infer that

$$\begin{aligned} \|V_1 - V_2\|_{L^2(B(0,1))}(t) &\geq \|u_1 - u_2\|_{L^2(B(0,1))}(t) - \|v_1\|_{L^2(B(0,1))}(t) - \|v_2\|_{L^2(B(0,1))}(t) \\ &\geq t^{1/4} \|u_1 - u_2\|_{L^2(B(0,1))}(1) - t^{1/4} \|u_1 - u_2\|_{L^2(B(0,1))}(1)/4 \\ &= \frac{3}{4} t^{1/4} \|u_1 - u_2\|_{L^2(B(0,1))}(1), \end{aligned} \quad (7.3)$$

for $t \in (0, T_\epsilon)$.

To show (7.2), we use the following version of the local energy inequality:

$$\begin{aligned} &\int |v_i(x, t)|^2 \phi \, dx + 2 \int_0^t \int |\nabla v_i|^2 \phi \, dx \, ds \\ &\leq \int |v_0(x)|^2 \phi \, dx + \int_0^t \int |v_i|^2 \Delta \phi \, dx \, ds + \int_0^t \int (|v_i|^2 + 2\pi_i) (v_i \cdot \nabla \phi) \, dx \, ds \\ &\quad - 2 \int_0^t \int (v_i \cdot \nabla u_i) \cdot v_i \phi \, dx \, ds + \int_0^t \int |v_i|^2 u_i \cdot \nabla \phi \, dx \, ds, \end{aligned} \quad (7.4)$$

which applies to non-negative $\phi \in C_c^\infty(\mathbb{R}^3)$. We have already mentioned why each v_i satisfy the local energy inequality (2.5). To deduce (7.4) from (2.5) we just use the noted convergence property of v_i to v_0 as $t \rightarrow 0^+$. Let $\phi \geq 0$ belong to $C_c^\infty(\mathbb{R}^3)$ evaluate to 1 on $B(0, 1)$ and have support in $B(0, 2)$. Importantly, the support properties of v_0 imply the first term on the right-hand side of (7.4) is zero. We will bound the other terms on the right-hand side of (7.4) term-by-term using the fact that

$$\lim_{t \rightarrow 0^+} t^{1/2} \|\nabla v_i\|_{L^4}(t) = 0 \text{ and } \lim_{t \rightarrow 0^+} \|v_i - v_0\|_{L^4}(t) = 0,$$

where $i = 1, 2$, which is stated in theorem 7.1. Since $v_0 = 0$ on $B(0, 2)$ we also have that $\lim_{t \rightarrow 0^+} \|v_i\|_{L^4(B(0,2))}(t) = 0$.

For the pressure term we have

$$\begin{aligned} \int_0^t \int \pi_i v_i \cdot \nabla \phi \, dx \, ds &\lesssim \int_0^t \|v_i\|_{L^4(B(0,2))} \|\pi_i\|_{L^4} \|\nabla \phi\|_{L^2} \, ds \\ &\lesssim_\phi \left(\sup_{0 < s < t} \|v_i\|_{L^4(B(0,2))} \right) \left(\sup_{0 < s < t} \|v_i\|_{L^4} \right) \int_0^t \|u_i\|_{L^\infty} \, ds \\ &\quad + \left(\sup_{0 < s < t} \|v_i\|_{L^4(B(0,2))} \right) \left(\sup_{0 < s < t} \|v_i\|_{L^4}^{5/4} \right) \int_0^t \|\nabla v_i\|_{L^4}^{3/4} \, ds, \end{aligned} \quad (7.5)$$

Where we have used the Calderon-Zygmund theory, the fact that π_i can be computed from u_i and v_i via Riesz transforms and the Gagliardo-Nirenberg inequality $\|f\|_8 \lesssim \|f\|_4^{5/8} \|\nabla f\|_4^{3/8}$ to obtain the last term. Since u_i are local energy solutions and are self-similar, they satisfy $\|u_i\|_\infty(t) \leq \|U_i\|_\infty t^{-1/2}$. On the other hand, $\|\nabla v_i\|_4^{3/4}(t) \lesssim_{v_0} t^{-3/8}$ and, upon integrating in time, we note $t^{5/8} \leq t^{1/2}$ for $t \leq 1$. Hence,

$$\int_0^t \int \pi_i v_i \cdot \nabla \phi \, dx \, ds \leq C(u_i, v_0, \phi) \underbrace{\left(\sup_{0 < s < t} \|v_i\|_{L^4(B(0,2))} \right)}_{\rightarrow 0 \text{ as } t \rightarrow 0^+} t^{1/2}. \quad (7.6)$$

Plainly by taking t sufficiently small we can bound the above by $\epsilon \sqrt{t}/4$. For the terms involving $|v_i|^2$ we have

$$\begin{aligned} &\int_0^t \int |v_i|^2 \Delta \phi \, dx \, ds + \int_0^t \int |v_i|^2 u_i \cdot \nabla \phi \, dx \, ds + \int_0^t \int |v_i|^2 v_i \cdot \nabla \phi \, dx \, ds \\ &\leq \int_0^t \|v_i\|_{L^4(B(0,2))}^2 \|\Delta \phi\|_{L^2} \, ds + C(u_i) \int_0^t \|v_i\|_{L^4(B(0,2))}^2 \|\nabla \phi\|_{L^2} \frac{1}{\sqrt{s}} \, ds \\ &\quad + C(v_0) \int_0^t \|v_i\|_{L^4(B(0,2))}^2 \|\nabla \phi\|_{L^4} \, ds \leq \frac{\epsilon}{4} \sqrt{t}, \end{aligned} \quad (7.7)$$

for t sufficiently small by the same logic as was applied to the pressure term. For the last term,

$$\int_0^t \int (v_i \cdot \nabla u_i) \cdot v_i \phi \, dx \, ds,$$

we integrate by parts to get a term where the gradient falls on ϕ , which, in view of the preceding estimate, is bounded by $\epsilon \sqrt{t}/4$, and a term which is a multiple of

$$\int_0^t \int (v_i \cdot \nabla v_i) \cdot u_i \phi \, dx \, ds.$$

This term lacks a cancellation and requires the most care. To bound this note that u_i is a local energy solution to (NS) and so satisfies (2.7). Consequently,

$$\begin{aligned} \int_0^t \int (v_i \cdot \nabla v_i) u_i \phi \, dx \, ds &\leq \sup_{0 < s < t} \|u_i\|_{L^2(B(0,3))} \int_0^t \|v_i\|_{L^4(B(0,2))} \|\nabla v_i\|_{L^4} \, ds \\ &\lesssim_{u_0, v_0} \sup_{0 < s < t} \|v_i\|_{L^4} \int_0^t \frac{1}{s^{1/2}} \, ds \leq \epsilon t^{1/2}/4, \end{aligned} \quad (7.8)$$

again by taking t sufficiently small. Since

$$\int |v_0(x)|^2 \phi \, dx = 0,$$

we obtain from (7.4) that

$$\int_{B(0,1)} |v_i|^2(x, t) \, dx \leq \epsilon t^{1/2},$$

for sufficiently small t . □

Data availability statement

No new data were created or analysed in this study.

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