Mathematical Control and Related Fields
Vol. 15, No. 2, June 2025, pp. 570-589
doi:10.3934/mcrf.2024026

LOCAL HALANAY'S INEQUALITY WITH APPLICATION TO FEEDBACK STABILIZATION

Michael Malisoff[1] and Frederic Mazenc[*2]

[1]Louisiana State University, USA

[2]INRIA EPI DISCO, L2S-CNRS-CentraleSupélec, France

(Communicated by Lars Grüne)

Abstract. We provide a local version of an approach to proving asymptotic stability that is based on Halanay's inequality. Our results are applicable to a family of nonlinear systems that contain state and input delays. We determine input-to-state stability inequalities when the systems contain additive uncertainty. We combine the results with an observer and a Gramian approach, to solve an output feedback stabilization problem. Our numerical examples illustrate how our theorems lead to new basin of attraction estimates.

1. **Introduction.** Since its introduction in [6], Halanay's inequality and its generalizations have been very useful in many cases to establish asymptotic stability properties for families of nonlinear systems. These results have been developed in several contributions, including [5], [7], [12], [13], [15], [16], and [17] for both continuous-time and discrete-time systems. In its basic form (e.g., [4, Lemma 4.2, p. 138]), Halanay's inequality calls for finding nonnegative valued differentiable functions v and constants $a > 0$, $b \in (0, a)$, and $T > 0$ such that

$$\dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-T,t]} v(\ell) \tag{1}$$

holds for all $t \geq T$, in order to prove that $v(t)$ exponentially converges to 0 as $t \to +\infty$. See also works such as [20] for generalizations where, instead of being constants, the a and b in (1) can depend on t, which include cases where there are t values such that $b(t) > a(t)$, and which explain the advantages of using Halanay's inequality approaches for stability analysis instead of standard Lyapunov function methods. However, to the best of our knowledge, they only apply to globally exponentially stable systems. On the other hand, in many cases, nonlinear systems are only locally exponentially stable, and in those cases, the global Halanay's results cannot be applied to establish global stability results. Moreover, surprisingly perhaps, to the best of our knowledge, we think that the local stability or stabilization of delayed systems is an under-studied topic even if [1], [3], and [9] present results on this subject. This motivates the present work.

2020 *Mathematics Subject Classification.* Primary: 93C23, 93D05, 93D20, 93D25; Secondary: 39B72, 34D23, 34D05, 26A46, 26A48.

Key words and phrases. Stabilization, delayed systems, feedback control.

The first author is supported by US National Science Foundation Grants 2009659 and 2308282.

*Corresponding author: Frederic Mazenc.

We establish a local version of the Halanay's inequality based stability result for functions that satisfy a nonlinear differential inequality in a suitable local sense. We apply it to systems that contain small bounded additive uncertainties, and we determine input-to-state-stability (or ISS) inequalities; see below for the definition of ISS. The results can be applied to nonlinear systems with delays, and enable us to estimate basins of attraction. In Section 4, we apply this result to a local output feedback exponential stabilization problem using an observer and an invertible Gramian approach. We revisit the main result of the paper [11], by considering systems with poorly known nonlinear terms that violate the usual linear growth conditions.

We state and prove our local Halanay's inequality result in Section 2, which we use to prove our local exponential stabilization result for state feedback in Section 3, and our generalization for systems with outputs in Section 4. Our work covers nonlinear systems with distributed state delays and time-varying input delays. We illustrate our results in Section 5 using a controlled version of van der Pol's equation, and a system with an output and a saturation, whose structures preclude using previous methods to prove global stabilization, but which are amenable to our results. This leads to our estimates for basins of attraction and sufficient conditions on the bounds for the uncertainties for our local stabilization estimates to hold. Then in Section 6, we provide our suggestions for future research.

We use standard notation which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless indicated otherwise, and $|\cdot|$ is the usual Euclidean vector norm and corresponding matrix operator norm. For matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, we write $A \leq B$ provided $B - A$ is nonnegative definite, and I denotes the identity matrix. When r is a time variable, we use the standard notation $g_r(\ell) = g(r + \ell)$ for functions g and all $\ell \leq 0$ and $r \geq 0$ such that $r + \ell$ is in the domain of g. We also use the standard family of functions $\mathcal{K}_\infty$ and the standard definitions of ISS [8, 19] and controllability [18]. Let us recall the definition of ISS systems. A system of the form

$$\dot{X}(t) = f(t, X_t, \delta(t))$$

with initial conditions in the set $C^0([-\bar{\tau}, 0])$ of all $\mathbb{R}^n$-valued continuous functions that are defined in the interval $[-\bar{\tau}, 0]$ and a delay $\tau(t)$ valued in $[0, \bar{\tau}]$ for all $t \geq 0$ for a given constant $\bar{\tau} \geq 0$ is ISS with respect to δ, where δ is a locally bounded piecewise continuous function, provided there are a function β of class $\mathcal{KL}$ and a function α of class $\mathcal{K}$ such that all the solutions of the system are such that

$$|X(t)| \leq \beta\left(|X_s|, t-s\right) + \alpha\left(\sup_{m\in[s,t]} |\delta(m)|\right)$$

for all $t \geq s$ and all $s \geq 0$.

2. Local ISS Hanalay's results.

Let $\alpha : [0, +\infty) \to [0, +\infty)$ be a continuous, nonnegative valued and nondecreasing function such that there are two constants $v_\star > 0$ and $a > 0$ such that

$$\alpha(v_\star) = a. \tag{2}$$

Let $t_\star \geq 0$ and $b > 0$ be two constants. Let $\zeta : [t_\star, +\infty) \to [0, +\infty)$ be a nondecreasing function such that

$$\zeta(t) < bv_\star \tag{3}$$

holds for all $t \geq t_\star$. Let $\mathcal{L}_0 \geq 0$ be a constant, and $\tau \in (t_\star, +\infty)$ or $\tau = +\infty$. Throughout this section, we let $V : [t_\star - \mathcal{L}_0, \tau) \to [0, +\infty)$ be a C^1 function such that

$$\sup_{m\in[t_\star-\mathcal{L}_0,t_\star]} V(m) < v_\star \tag{4}$$

and such that its time derivative $\dot{V}$ satisfies the inequality

$$\dot{V}(t) \quad \leq \quad -(a+b)V(t) + \alpha\left(\sup_{m\in[t-\mathcal{L}_0,t]} V(m)\right) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{5}$$

for all $t \in [t_\star, \tau)$. Notice that if α is a positive constant, then the inequality (5) is a standard Halanay's inequality. With the preceding notation, we then state and prove a result which is instrumental in establishing our main result:

Lemma 2.1. *Consider the function V introduced above. Then the inequality $V(t) < v_\star$ is satisfied for all $t \in [t_\star - \mathcal{L}_0, \tau)$.*

Proof. We prove this result by contradiction. Suppose that there were a $t_c \in [t_\star - \mathcal{L}_0, \tau)$ such that

$$V(t_c) = v_\star \tag{6}$$

and $V(t) < v_\star$ for all $t \in [t_\star - \mathcal{L}_0, t_c)$. Then (4) gives $t_c > t_\star$, and (2), (3), and (5) give

$$\dot{V}(t_c) < -(a+b)v_\star + \alpha\left(v_\star\right) v_\star + bv_\star = -av_\star + av_\star = 0. \tag{7}$$

From the inequality $\dot{V}(t_c) < 0$ and the continuity of V, we deduce that there is $t_d \in (t_\star, t_c)$ such that $V(t_d) > v_\star$. This contradicts the definition of t_c, so the lemma holds. □

Since $a > 0$ and $b > 0$, there is a unique value $\lambda > 0$ such that

$$\lambda = a + b - ae^{\lambda\mathcal{L}_0}. \tag{8}$$

Using this λ, we next state and prove the main result of this section.

Theorem 2.2. *Consider the function V introduced above, in the case where $\tau = +\infty$. Then*

$$V(t) \leq \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)}\zeta(m)\mathrm{d}m \tag{9}$$

holds for all $t \geq s$ and for all $s \geq t_\star$.

Remark 2.3. Two key differences between Theorem 2.2 and Theorem 3.2 in [15] are that (i) a disturbance is taken into account in Theorem 2.2 and (ii) Theorem 2.2 gives exponential stability when the disturbance is not present, whereas the [15, Theorem 3.2] establishes asymptotic stability. The specifics of the nonlinear functions involved in the nonlinear Halanay's inequality considered in [15] makes it possible to prove asymptotic stability of systems which are not exponentially stable. Proving a local ISS result for classes of systems that we will study below that are locally asymptotically stable but not exponentially stable when no disturbance is present is an interesting open problem. On the other hand, see [14] for results for systems of the form $\dot{x}(t) = Ax(t) + f(x(t), u(t))$ having Banach spaces as the state spaces and having controls u that are valued in a normed space, where A is the generator of a C_0-semigroup. However, this class of systems from [14] do not include our main class of systems, where x_t can enter in a nonlinear way and which therefore include nonlinear delay systems; see (15) below. Moreover, we believe that the methods of [14] do not lend themselves to being adapted or extended to cover our general systems, because of the nonlinearity of our systems in x_t. See also [2, Theorem 21] for time invariant cases, for which asymptotic stability is shown to imply local ISS.

Proof. Since the function α is nondecreasing, Lemma 2.1 and (5) ensure that

$$\dot{V}(t) \quad \leq \quad -(a+b)V(t) + \alpha(v_\star) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{10}$$

holds for all $t \geq t_\star$. From (2), we deduce that

$$\dot{V}(t) \quad \leq \quad -(a+b)V(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{11}$$

holds for all $t \geq t_\star$. Since $a > 0$ and $b > 0$, we can then apply Lemma A.1.1 in Appendix A.1 below, to conclude. □

Remark 2.4. The inequality (9) gives the ISS inequality

$$V(t) \leq \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \tfrac{1}{b} \sup_{m\in[s,t]} \zeta(m) \tag{12}$$

for all $t \geq s$ and all $s \geq t_\star$.

3. Local exponential stabilization result.

We use Theorem 2.2 to solve a local stabilization problem for a class of nonlinear systems containing a small time-varying delay in the control law.

3.1. Studied system and preliminary result.

Let $h : [0,+\infty) \to [0,+\infty)$ be a continuous function for which there is a constant $\bar{h} > 0$ such that

$$0 \leq h(t) \leq \bar{h} \tag{13}$$

for all $t \geq 0$. Let $\delta : [0,+\infty) \to \mathbb{R}^n$ be a continuous function that admits a constant $\bar{\Delta}$ such that

$$|\delta(t)| \leq \bar{\Delta} \tag{14}$$

for all $t \geq 0$. Consider the system

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t-h(t)) + \mathcal{F}(t,x_t) + \delta(t) \tag{15}$$

where x is valued in $\mathbb{R}^n$, the input u is valued in $\mathbb{R}^p$, and $\mathcal{F}$ is locally Lipschitz with respect to its second argument and piecewise continuous with respect to the first.

Throughout this paper, we assume that the dynamics satisfy the usual forward completeness properties, with standard existence and uniqueness properties of solutions. Let $t_0 \geq 0$. We consider initial functions $x_0 : [t_0 - \bar{h}, t_0] \to \mathbb{R}^n$, and we introduce three assumptions:

Assumption 3.1. *The pair (A, B) is controllable.*

Assumption 3.2. *There is a continuous nondecreasing function $\rho : [0,+\infty) \to [0,+\infty)$ that is not identically equal to zero such that*

$$|\mathcal{F}(t,\phi)| \leq \sup_{m\in[-\bar{h},0]} |\phi(m)|^2 \rho(|\phi(m)|) \tag{16}$$

holds for all functions $\phi : [-\bar{h}, 0] \to \mathbb{R}^n$ and all $t \geq 0$.

It is well known that Assumption 3.1 provides a matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix

$$H = A + BK \tag{17}$$

is Hurwitz, and so also a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and constants $c > 0$ and $\overline{p} > 0$ such that

$$PH + H^\top P \leq -cP \tag{18}$$

and

$$I \leq P \;\text{ and }\; |P| \leq \overline{p}, \tag{19}$$

e.g., by using the Pole-Shifting Theorem (as stated in [18, p.186]) to find K, and by then solving the Riccati equation $PH + H^\top P = -I$ for P, and then choosing $c > 0$ small enough such that $cP \leq I$ [18], and then scaling P by a big enough positive constant if needed to satisfy our additional requirement that $P \geq I$. In what follows, we fix ρ, K, P, and $\bar{p}$ satisfying the preceding requirements, and we assume that $BK \neq 0$. Our last assumption is the following, which can be viewed as a smallness condition on $\bar{h}$ or on $\bar{\Delta}$:

Assumption 3.3. *There is a real value $s_\star > 0$ such that with the choice*

$$\omega_0 = \big(2|A| + 2|BK| + 1 + 2\sqrt{s_\star}\,\rho\,(\sqrt{s_\star})\big)\overline{p}, \tag{20}$$

the inequality

$$\left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0} < s_\star \tag{21}$$

is satisfied.

In terms of the preceding notation and the positive definite quadratic function

$$W(x) = x^\top P x, \tag{22}$$

where x is valued in $\mathbb{R}^n$, we start with a technical lemma, where Assumption 3.3 ensures that (24) is satisfied when the initial function is valued in a small enough neighborhood of the origin:

Lemma 3.4. *Let the system (15) satisfy Assumptions 3.1-3.3. With the preceding notation, consider (15) in closed-loop with the feedback*

$$u(t - h(t)) = Kx(t - h(t)) \tag{23}$$

for all $t \geq t_0$. Let x be a solution of this system such that

$$\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0} < s_\star. \tag{24}$$

Then x is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$ and

$$\sup_{m\in[t_0-\bar{h},t_0+2\bar{h}]} W(x(m)) \leq \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0} \tag{25}$$

is satisfied.

Proof. Consider a solution $x(t)$ of this closed-loop system from the lemma such that (24) holds. Let $[t_0 - \bar{h}, t_0 + t_\infty)$ be the domain of definition of $x(t)$. Then

$0 < t_\infty < +\infty$ or $t_\infty = +\infty$. The time derivative of W defined in (22) along this solution satisfies

$$\begin{aligned} \dot{W}(t) &= 2x(t)^\top P[Ax(t) + BKx(t-h(t)) + \mathcal{F}(t,x_t) + \delta(t)] \\ &\leq 2\overline{p}|x(t)|\Bigg[|A||x(t)| + |BK||x(t-h(t))| \\ &\quad + \sup_{m\in[t-\bar{h},t]} |x(m)|^2\rho(|x(m)|)\Bigg] \\ &\quad + 2|x(t)|\overline{p}\bar{\Delta} \end{aligned} \tag{26}$$

for all $t \in [t_0, t_0 + t_\infty)$, by (14), (16), (19), and the continuity of δ. Since ρ from Assumption 3.2 is nondecreasing, it follows from (19) that

$$\begin{aligned} \dot{W}(t) &\leq \overline{p}\left(2|A| + 2|BK| + 1\right) \sup_{m\in[t-\bar{h},t]} W(x(m)) \\ &\quad + 2\overline{p} \sup_{m\in[t-\bar{h},t]} W(x(m))^{3/2}\rho\left(\sqrt{\sup_{m\in[t-\bar{h},t]} W(x(m))}\right) + \overline{p}\bar{\Delta}^2 \end{aligned} \tag{27}$$

by applying the triangle inequality to upper bound the last right side term in (26), in order to get $2|x(t)|\overline{p}\bar{\Delta} \leq \overline{p}|x(t)|^2 + \overline{p}\bar{\Delta}^2$. In terms of the function

$$\bar{\omega}(s) = \overline{p}\left(2|A| + 2|BK| + 1\right)s + 2s^{3/2}\overline{p}\rho\left(\sqrt{s}\right), \tag{28}$$

we therefore have

$$\dot{W}(t) \leq \bar{\omega}\left(\sup_{m\in[t-\bar{h},t]} W(x(m))\right) + \overline{p}\bar{\Delta}^2 \tag{29}$$

for all $t \in [t_0, t_0 + t_\infty)$ and, according to the definition of ω_0 in (20), we have

$$\bar{\omega}(s) \leq \omega_0 s \tag{30}$$

for all $s \in [0, s_\star]$. We apply now Lemma A.2.2 in Appendix A.2 below, with the choices

$$W(x(t)),\ \bar{\omega},\ \omega_0,\ \overline{p}^2\bar{\Delta}^2,\ t_0,\ t_\infty,\ 2.1\bar{h},\ \sup_{m\in[t_0-\bar{h},t_0]} W(x(m)),\ s_\star,\ \text{and}\ \bar{h} \tag{31}$$

playing the roles of $Z(t)$, Ψ, Ψ_0, $\bar{\Delta}$, t_a, τ, q, $\bar{Z}$, ω and T, respectively. Assumption 3.3 ensures that Assumption A.2.1 from Appendix A.2 (which is needed to apply Lemma A.2.2) is satisfied. Also, (A.2.4) holds, since

$$W(x(t)) \leq \sup_{m\in[t_0-\bar{h},t_0]} W(x(m)) \tag{32}$$

for all $t \in [t_0 - \bar{h}, t_0]$. Then inequality (24) ensures that the inequality (A.2.5) is satisfied. Therefore, according to Lemma A.2.2, it follows that for all $t \in [t_0 - \bar{h}, t_0 + \min\{t_\infty, 2.1\bar{h}\})$, we have

$$W(x(t)) \leq \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\tfrac{\overline{p}\bar{\Delta}^2}{\omega_0}. \tag{33}$$

Therefore, the finite escape time phenomenon does not occur over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, which implies that $t_\infty > 2\bar{h}$. □

3.2. **ISS result.** Using the notation from Section 3.1, let us introduce the function

$$\beta(m) = 2\bar{h}|PBK|\left(|A| + |BK|\right) + 2\sqrt{m}\left(\bar{h}|PBK| + |P|\right)\rho\left(\sqrt{m}\right). \tag{34}$$

We now add the following assumption, which can again be regarded as a smallness condition on $\bar{h}$, where $c > 0$ is the positive constant from (18):

Assumption 3.5. *The bound* $2\bar{h}|PBK|\left(|A| + |BK|\right) < c/4$ *is satisfied.*

It follows that there is a constant $w_\star > 0$ such that

$$\beta(w_\star) = \tfrac{c}{4} \tag{35}$$

and we fix a $w_\star$ satisfying the preceding requirement in the rest of this subsection. We also assume:

Assumption 3.6. *The inequality*

$$\tfrac{4}{c}\left(|PBK|^2\bar{h}^2 + |P|^2\right)\bar{\Delta}^2 < \tfrac{cw_\star}{4} \tag{36}$$

holds.

Assumption 3.6 can be viewed as a smallness condition on $\bar{\Delta}$. Let $\gamma > 0$ be the constant such that

$$\gamma = \tfrac{c}{2} - \tfrac{c}{4}e^{2\gamma\bar{h}}. \tag{37}$$

We are ready to state and prove the following result:

Theorem 3.7. *Let the system (15) satisfy Assumptions 3.1-3.6. Then, with the notation from the preceding subsection, consider* (15) *in closed-loop with the control*

$$u(t - h(t)) = Kx(t - h(t)). \tag{38}$$

Consider any maximal solution $x(t)$ of the closed-loop system such that

$$\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\tfrac{\bar{p}\bar{\Delta}^2}{\omega_0} < \min\{s_\star, w_\star\} \tag{39}$$

holds. Then, for each $s \geq t_0 + \bar{h}$, and with the choice

$$\mathcal{H}(m) = \frac{4}{c}|PBK|^2\bar{h}\int_0^m |\delta(r)|^2\,\mathrm{d}r + \frac{4}{c}|P|^2 \sup_{\ell\in[\bar{h},m]}|\delta(\ell)|^2, \tag{40}$$

the inequality

$$|x(t)| \leq \sqrt{\bar{p}\sup_{m\in[s-2\bar{h},s]}|x(m)|^2 e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)}\mathcal{H}(m)\mathrm{d}m} \tag{41}$$

holds for all $t \geq s$.

Proof. We consider a trajectory x of the closed-loop system satisfying the conditions of Theorem 3.7. Let $[t_0 - \bar{h}, t_\infty)$ be the largest domain of definition of x. Notice for later use that it follows from Lemma 3.4 that the solution is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, and that the inequality (25) holds. Then necessarily, $t_\infty > t_0 + 2\bar{h}$. From the definition of H in (17), we deduce that

$$\dot{x}(t) \quad = \quad Hx(t) - BK\int_{t-h(t)}^t \dot{x}(m)\mathrm{d}m + \mathcal{F}(t, x_t) + \delta(t) \tag{42}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$, since the integral in (42) is $x(t) - x(t - h(t))$. According to (18), the time derivative of W along (42) satisfies

$$\begin{aligned} \dot{W}(t) \quad &\leq \quad -cW(x(t)) - 2x(t)^\top PBK \int_{t-h(t)}^{t} \dot{x}(m)\mathrm{d}m \\ &\quad + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t) \end{aligned} \tag{43}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$.

Since $t_\infty > 2\bar{h}$, it follows that

$$\begin{aligned} \dot{W}(t) \quad \leq \quad & -cW(x(t)) \\ & + 2|x(t)||PBK| \textstyle\int_{t-h(t)}^{t} |Ax(m) + BKx(m - h(m)) \\ & + \mathcal{F}(m, x_m) + \delta(m)| \,\mathrm{d}m \\ & + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t) \end{aligned} \tag{44}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. Consequently,

$$\begin{aligned} \dot{W}(t) \quad \leq \quad & -cW(x(t)) + 2|PBK||x(t)| \int_{t-h(t)}^{t} |Ax(m)| \,\mathrm{d}m \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} |BKx(m - h(m))| \,\mathrm{d}m \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} |\mathcal{F}(m, x_m)| \,\mathrm{d}m + 2x(t)^\top P\mathcal{F}(t, x_t) \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} |\delta(m)| \,\mathrm{d}m + 2x(t)^\top P\delta(t) \end{aligned} \tag{45}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. From our bound $\bar{h}$ on h from (13), Assumption 3.2, and (19), we deduce that

$$\begin{aligned} \dot{W}(t) \quad \leq \quad & -cW(x(t)) + 2\bar{h}|PBK||A| \sup_{m \in [t-\bar{h}, t]} W(x(m)) \\ & + 2\bar{h}|PBK||BK| \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} \sup_{r \in [m-\bar{h}, m]} |x(r)|^2 \rho(|x(r)|)\mathrm{d}m \\ & + 2|x(t)||P| \sup_{m \in [t-\bar{h}, t]} |x(m)|^2 \rho(|x(m)|) \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} |\delta(m)| \,\mathrm{d}m + 2|P||x(t)||\delta(t)| \end{aligned} \tag{46}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. Since ρ is nondecreasing, we get

$$\begin{aligned} \dot{W}(t) \quad \leq \quad & -cW(x(t)) + 2\bar{h}|PBK| \left(|A| + |BK|\right) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\ & + 2\bar{h}|PBK|\rho\left(\sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))}\right) \mathrm{d}m \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \\ & + 2|P||x(t)||\delta(t)| \\ & + 2|P| \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \rho\left(\sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))}\right) \\ & + 2|PBK||x(t)| \int_{t-h(t)}^{t} |\delta(m)| \,\mathrm{d}m \end{aligned} \tag{47}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. We next use the triangle inequality, Jensen's inequality, and (19), to get

$$\begin{array}{l} 2|PBK||x(t)| \int_{t-h(t)}^{t} |\delta(m)| \mathrm{d}m \\ \leq \frac{c}{4} W(x(t)) + \frac{4}{c}|PBK|^2 \bar{h} \int_{t-h(t)}^{t} |\delta(m)|^2 \mathrm{d}m \text{ and} \\ 2|P||x(t)||\delta(t)| \leq \frac{c}{4} W(x(t)) + \frac{4}{c}|P|^2|\delta(t)|^2. \end{array} \tag{48}$$

It follows from (47) that

$$\begin{array}{rcl} \dot{W}(t) & \leq & -\frac{c}{2} W(x(t)) + 2\bar{h}|PBK| \left(|A| + |BK|\right) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\ & & + \frac{4}{c}|PBK|^2 \bar{h} \int_{t-h(t)}^{t} |\delta(m)|^2 \, \mathrm{d}m \\ & & + 2\left(\bar{h}|PBK| + |P|\right) \sup_{m \in [t-2\bar{h}, t]} W(x(m))^{\frac{3}{2}} \\ & & \times \rho\left(\sup_{m \in [t-2\bar{h}, t]} \sqrt{W(x(m))}\right) + \frac{4}{c}|P|^2|\delta(t)|^2 \end{array} \tag{49}$$

holds for all $t \in [t_0 + \bar{h}, t_\infty)$. Therefore, we have

$$\begin{array}{rcl} \dot{W}(t) & \leq & -\frac{c}{2} W(x(t)) + \beta\left(\sup_{m \in [t-2\bar{h}, t]} W(x(m))\right) \sup_{m \in [t-2\bar{h}, t]} W(x(m)) \\ & & + \delta_\sharp(t) \end{array} \tag{50}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$, where β was defined in (34) and

$$\delta_\sharp(t) = \frac{4}{c}|PBK|^2 \bar{h} \int_{t-h(t)}^{t} |\delta(m)|^2 \, \mathrm{d}m + \frac{4}{c}|P|^2|\delta(t)|^2. \tag{51}$$

Note that (14) implies that $|\delta_\sharp(t)| \leq \frac{4}{c}\left(|PBK|^2 \bar{h}^2 + |P|^2\right) \bar{\Delta}^2$ for all $t \geq \bar{h}$. Hence, (36) gives

$$|\delta_\sharp(t)| < \frac{c w_\star}{4} \tag{52}$$

for all $t \geq \bar{h}$. Then let us recall that (25) holds, by (39). Consequently (39) ensures that

$$\sup_{m \in [t_0 - \bar{h}, t_0 + 2\bar{h}]} W(x(m)) < w_\star \tag{53}$$

We can now apply Theorem 2.2 with $\lambda = \gamma$, and with

$$\begin{array}{l} V(t) = W(x(t)), \ a = b = c/4, \ \alpha = \beta, \ \zeta(t) = \sup_{\ell \in [\bar{h}, t]} |\delta_\sharp(\ell)|, \\ \mathcal{L}_0 = 2\bar{h}, \ v_\star = w_\star, \ \tau = t_\infty, \text{ and } t_\star = t_0 + \bar{h}. \end{array} \tag{54}$$

Note that (35) ensures that (2) is satisfied. Then (52)-(53) ensure that (3)-(4) are satisfied. Using Lemma 2.1, we can prove that the finite escape time phenomenon does not occur, so $t_\infty = +\infty$, and Theorem 2.2 gives

$$W(x(t)) \leq \sup_{m \in [s-2\bar{h}, s]} W(x(m)) e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell \in [\bar{h}, m]} |\delta_\sharp(\ell)| \mathrm{d}m \tag{55}$$

when $t \geq s \geq t_0 + \bar{h}$ where γ is the constant defined in (37), and where the sup was needed in (54) and in the integrand in (55) because Theorem 2.2 requires its function ζ to be nondecreasing. Hence, (19) gives

$$|x(t)|^2 \leq \overline{p} \sup_{m\in[s-2\bar{h},s]} |x(m)|^2 e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell\in[\bar{h},m]} |\delta_\sharp(\ell)| \mathrm{d}m \tag{56}$$

for all $t \geq s \geq t_0 + \bar{h}$. This allows us to conclude. □

4. **Output feedback local stabilization.**

4.1. **Statement of result.**

Consider the system

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + \mathcal{G}(x(t)) + \delta(t) \\ y(t) &= Cx(t) \end{cases} \tag{57}$$

with x valued in $\mathbb{R}^n$, the input u is valued in $\mathbb{R}^p$, the output y valued in $\mathbb{R}^q$, $\mathcal{G}$ being a locally Lipschitz nonlinear function and δ being continuous. We introduce two assumptions:

Assumption 4.1. *The pair (A, C) is observable, the pair (A, B) is controllable, and the system (57) is forward complete for each continuous choice of $u : [0, +\infty) \to \mathbb{R}^p$.*

Assumption 4.2. *There is a continuous, nondecreasing function $\kappa : [0, +\infty) \to [0, +\infty)$ that is not identically equal to the zero function such that*

$$|\mathcal{G}(x)| \leq |x|^2 \kappa(|x|) \tag{58}$$

holds for all $x \in \mathbb{R}^n$.

Since (A, B) is controllable, we can argue as in Section 3.1 to find a matrix $K \in \mathbb{R}^{p\times n}$, a symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$, and constants $c > 0$ and $\bar{p} > 0$ such that

$$PH + H^\top P \leq -cP \tag{59}$$

where

$$H = A + BK \tag{60}$$

and

$$I \leq P \text{ and } |P| \leq \overline{p}. \tag{61}$$

Let $T > 0$ be a constant and set

$$E = \int_{-T}^{0} e^{A^\top s} C^\top C e^{As} \mathrm{d}s. \tag{62}$$

Assumption 4.1 implies that the matrix E is invertible, e.g., by [18, Section 6.3]. We also use the functions $\hat{x}$ and μ that are defined by

$$\begin{aligned} \hat{x}(t) \quad &= \quad E^{-1} \int_{t-T}^{t} e^{A^\top (s-t)} C^\top y(s) \mathrm{d}s \\ &\quad + E^{-1} \int_{t-T}^{t} e^{A^\top (s-t)} C^\top C \int_s^t e^{A(s-m)} Bu(m) \mathrm{d}m \mathrm{d}s \end{aligned} \tag{63}$$

for all $t \geq T$ and $\hat{x}(t) = 0$ for all $t \in [0, T)$, and

$$\mu(x_t) = E^{-1} \int_{t-T}^{t} e^{A^\top (s-t)} C^\top C \int_s^t e^{A(s-m)} \mathcal{G}(x(m)) \mathrm{d}m \mathrm{d}s, \tag{64}$$

which is defined for all $t \geq T$. We also fix any constant g such that

$$g \geq \left(\int_{-T}^{0} \int_{s}^{0} \left| BKE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} \right| \mathrm{d}m \mathrm{d}s \right)^2, \tag{65}$$

and, in terms of the function κ from Assumption 4.2, we use the function

$$\gamma_0(u) = \tfrac{8}{c}\overline{p}(1+g)\kappa^2\left(\sqrt{u}\right)u. \tag{66}$$

From our assumption on κ, we then fix a constant $u_\star > 0$ such that

$$\gamma_0(u_\star) = \tfrac{c}{4}. \tag{67}$$

Using the preceding notation, our final assumption is the following smallness condition on the δ_i's:

Assumption 4.3. *There are a vector $M = [m_1, \ldots, m_n]^\top \in [0, +\infty)^n$, a constant $\bar{\Delta} > 0$, and a constant $d > 0$ such that*

$$d \geq \left(1 + \int_{-T}^{0}\int_{s}^{0}\left|BKE^{-1}e^{A^\top s}C^\top Ce^{A(s-m)}M\right|\mathrm{d}m\mathrm{d}s\right)|M| \tag{68}$$

for which the conditions

$$\sup_{t\geq 0}|\delta_i(t)| \leq m_i\bar{\Delta}\ \text{ for } i = 1, \ldots, n \tag{69}$$

and

$$\bar{\Delta} < \tfrac{c}{4d}\sqrt{\tfrac{u_\star}{\overline{p}}} \tag{70}$$

are satisfied.

We also use the function $\delta^\sharp : [T, +\infty) \to \mathbb{R}^n$ that is defined by

$$\delta^\sharp(t) = \delta(t) - BKE^{-1}\int_{t-T}^{t}e^{A^\top(s-t)}C^\top C\int_{s}^{t}e^{A(s-m)}\delta(m)\mathrm{d}m\mathrm{d}s. \tag{71}$$

Let $\nu > 0$ be the constant such that

$$\nu = \tfrac{c}{2} - \tfrac{c}{4}e^{\nu T}. \tag{72}$$

In terms of the preceding notation, our main result of this subsection is:

Theorem 4.4. *Consider the system (57) in closed-loop with the output feedback*

$$u(t) = K\hat{x}(t). \tag{73}$$

Let Assumptions 4.1-4.3 hold. Consider any maximal solution $x(t)$ of this system such that

$$\sup_{\ell\in[0,T]} x(\ell)^\top Px(\ell) < u_\star. \tag{74}$$

Then, for each $s \geq T$, the inequality

$$|x(t)| \leq \sqrt{\overline{p}}\sqrt{\sup_{m\in[s-T,s]}|x(m)|^2e^{-\nu(t-s)} + \tfrac{4}{c}\int_{s}^{t}e^{\frac{c}{4}(m-t)}\sup_{\ell\in[T,m]}|\delta^\sharp(\ell)|^2\mathrm{d}m} \tag{75}$$

holds for all $t \geq s$.

Remark 4.5. From the inequality (75) and the subadditivity of the square root, we can deduce a standard ISS inequality. A crucial difference between our result and the one of [11] is that we do not assume that the function $\mathcal{G}$ is known. Instead, we assume that the function κ in Assumption 4.2 is known. Note that the observer $\hat{x}$ is not present in the inequality (75) and the rate of convergence of $x(t)$ depends only on the choices of K and T. We have assumed that (57) is forward complete. This is a technical assumption, but we conjecture that it can be relaxed. Since $u_\star > 0$, one can often use continuous dependence arguments to check (74), e.g., when $\delta = 0$ because then the closed loop system admits a 0 equilibrium, which allows us to satisfy (74) in cases where $|\delta(t)|$ is known to be small enough for all values $t \in [0,T]$.

4.2. **Proof of theorem 4.4.** The system (57) in closed loop with (73) is

$$\dot{x}(t) = Ax(t) + BK\hat{x}(t) + \mathcal{G}(x(t)) + \delta(t). \tag{76}$$

By applying the method of variation of parameters to (76) on the interval $[s,t]$ for any $s \in [t-T,t]$, then left multiplying the result by $e^{A^\top(s-t)}C^\top Ce^{A(s-t)}$ and then integrating the result over all $s \in [t-T,t]$, it follows that

$$x(t) = \hat{x}(t) + \mu(x_t) + E^{-1}\int_{t-T}^{t} e^{A^\top(s-t)}C^\top C\int_s^t e^{A(s-m)}\delta(m)\mathrm{d}m\mathrm{d}s \tag{77}$$

for all $t \geq T$. By solving for $\hat{x}$ in (77) and substituting the result into (76), it follows that

$$\begin{aligned}\dot{x}(t) = Ax(t) + BK\Big(&x(t) - \mu(x_t)\\ &-E^{-1}\textstyle\int_{t-T}^{t} e^{A^\top(s-t)}C^\top C\int_s^t e^{A(s-m)}\delta(m)\mathrm{d}m\mathrm{d}s\Big)\\ &+\mathcal{G}(x(t)) + \delta(t)\end{aligned} \tag{78}$$

for all $t \geq T$. From the definitions of $H = A + BK$ from (60) and the $\delta^\sharp$ formula in (71), it follows that for all $t \geq T$, we obtain

$$\dot{x}(t) = Hx(t) - BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t). \tag{79}$$

Notice for later use that Assumption 4.3 and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}|\delta^\sharp(t)| &\leq |M|\bar{\Delta} + |M|\textstyle\int_{t-T}^{t}\int_s^t \left|BKE^{-1}e^{A^\top(s-t)}C^\top Ce^{A(s-m)}M\right|\mathrm{d}m\mathrm{d}s\bar{\Delta}\\ &\leq d\bar{\Delta}\end{aligned} \tag{80}$$

for all $t \geq T$, where d was introduced in (68). Now, we introduce the Lyapunov function:

$$U(x) = x^\top Px. \tag{81}$$

Its time derivative along all trajectories of (79) satisfies

$$\dot{U}(t) = 2x(t)^\top P[Hx(t) - BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)] \tag{82}$$

for all $t \geq T$. From (59), it follows that

$$\begin{aligned}\dot{U}(t) &\leq -cU(x(t)) + 2x(t)^\top P[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)]\\ &\leq -cU(x(t))\\ &\quad +2\left\{x(t)^\top\sqrt{P}\right\}\left\{\sqrt{P}[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)]\right\}\\ &\leq -\tfrac{c}{2}U(x(t)) + \tfrac{2}{c}[-BK\mu(x_t) + \mathcal{G}(x(t)) + \delta^\sharp(t)]^\top P[-BK\mu(x_t)\\ &\quad +\mathcal{G}(x(t)) + \delta^\sharp(t)]\end{aligned} \tag{83}$$

where the last inequality came from applying Young's inequality to get $2ab \leq \frac{c}{2}|a|^2 + \frac{2}{c}|b|^2$, where a and b are the terms in curly braces in (83). Consequently, the Cauchy-Schwarz inequality gives

$$\begin{aligned}\dot{U}(t) &\leq -\tfrac{c}{2}U(x(t)) + \tfrac{4}{c}\overline{p}\left[|-BK\mu(x_t) + \mathcal{G}(x(t))|^2 + |\delta^\sharp(t)|^2\right]\\ &\leq -\tfrac{c}{2}U(x(t)) + \tfrac{8}{c}\overline{p}\left[|BK\mu(x_t)|^2 + |\mathcal{G}(x(t))|^2\right] + \tfrac{4\overline{p}}{c}|\delta^\sharp(t)|^2\end{aligned} \tag{84}$$

for all $t \geq T$. We can then use our choice (64) of μ to obtain

$$\begin{aligned} |BK\mu(x_t)| &= \left| \int_{t-T}^{t} \int_{s}^{t} BKE^{-1} e^{A^\top (s-t)} C^\top C e^{A(s-m)} \mathcal{G}(x(m)) \mathrm{d}m \mathrm{d}s \right| \\ &\leq \int_{t-T}^{t} \int_{s}^{t} \left| BKE^{-1} e^{A^\top (s-t)} C^\top C e^{A(s-m)} \right| |x(m)|^2 \kappa(|x(m)|) \mathrm{d}m \mathrm{d}s \end{aligned} \tag{85}$$

where the last inequality is a consequence of (58). Since κ is nondecreasing, it follows that

$$\begin{aligned} &|BK\mu(x_t)| \\ &\leq \int_{t-T}^{t} \int_{s}^{t} \left| BKE^{-1} e^{A^\top (s-t)} C^\top C e^{A(s-m)} \right| \mathrm{d}m \mathrm{d}s \\ &\quad \times \sup_{r \in [t-T,t]} |x(r)|^2 \kappa \left(\sup_{r \in [t-T,t]} |x(r)| \right) \\ &= \int_{-T}^{0} \int_{s}^{0} \left| BKE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} \right| \mathrm{d}m \mathrm{d}s \\ &\quad \times \sup_{r \in [t-T,t]} |x(r)|^2 \kappa \left(\sup_{r \in [t-T,t]} |x(r)| \right) \\ &\leq \int_{-T}^{0} \int_{s}^{0} \left| BKE^{-1} e^{A^\top s} C^\top C e^{A(s-m)} \right| \mathrm{d}m \mathrm{d}s \\ &\quad \times \sup_{r \in [t-T,t]} U(x(r)) \kappa \left(\sup_{r \in [t-T,t]} \sqrt{U(x(r))} \right) \end{aligned} \tag{86}$$

where the last inequality is a consequence of (61). We deduce that

$$|BK\mu(x_t)|^2 \quad \leq \quad g \sup_{m \in [t-T,t]} U^2(x(m)) \kappa^2 \left(\sup_{m \in [t-T,t]} \sqrt{U(x(m))} \right) \tag{87}$$

with g defined in (65). Similarly, we have

$$|\mathcal{G}(x)|^2 \leq U^2(x) \kappa^2 \left(\sqrt{U(x)} \right) \tag{88}$$

for all $x \in \mathbb{R}^n$. It follows that

$$\begin{aligned} &|BK\mu(x_t)|^2 + |\mathcal{G}(x(t))|^2 \\ &\leq (1+g) \sup_{m \in [t-T,t]} U^2(x(m)) \kappa^2 \left(\sup_{m \in [t-T,t]} \sqrt{U(x(m))} \right). \end{aligned} \tag{89}$$

Consequently, for all $t \geq T$, (84) gives

$$\begin{aligned} \dot{U}(t) \quad \leq \quad & -\tfrac{c}{2} U(x(t)) + \gamma_0 \left(\sup_{m \in [t-T,t]} U(x(m)) \right) \sup_{m \in [t-T,t]} U(x(m)) \\ & + \tfrac{4\overline{p}}{c} |\delta^\sharp(t)|^2 \end{aligned} \tag{90}$$

with γ_0 defined in (66).

Now, let us apply Theorem 2.2 with $U(x(t))$ playing the role of $V(t)$, γ_0 playing the role of α, $v_\star = u_\star$, $\mathcal{L}_0 = t_\star = T$, $\tau = +\infty$ and $a = b = \frac{c}{4}$. The inequality (70) gives

$$\bar{\Delta}^2 < \tfrac{c^2}{16 d^2 \overline{p}} u_\star. \tag{91}$$

Then, according to (80),

$$\tfrac{4\overline{p}}{c} \sup_{\ell \in [0,t]} |\delta^\sharp(\ell)|^2 < \tfrac{c}{4} u_\star \tag{92}$$

for all $t \geq T$. Consider any maximal solution $x(t)$ of the closed loop system satisfying (74). Since (72) and (92) also hold, we deduce from Theorem 2.2 that $U(x(t))$ satisfies the ISS inequality

$$U(x(t)) \leq \sup_{m\in[s-T,s]} U(x(m))e^{-\nu(t-s)} + \frac{4\overline{p}}{c}\int_s^t e^{\frac{c}{4}(m-t)} \sup_{\ell\in[T,m]} |\delta^\sharp(\ell)|^2 \mathrm{d}m \tag{93}$$

if $t \geq s \geq T$. This inequality and (61) allow us to conclude.

5. Illustrations.

5.1. Illustration of Theorem 3.7.

Consider the controlled van der Pol equation

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + \epsilon(1-x_1^2)x_2 + u(t-h(t)) \end{cases} \tag{94}$$

for constants $\epsilon > 0$ and a continuous choice $h(t)$ of the delay; see, e.g., [8, Section 13.2] for simpler cases with no delays. The dynamics are used to represent oscillations in vacuum tube circuits, and provide a fundamental equation in nonlinear oscillation theory. The system has the form (15) with the choices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \text{and}\ \mathcal{F}(t,x_t) = \begin{bmatrix} 0 \\ -\epsilon x_1^2(t)x_2(t) \end{bmatrix} \tag{95}$$

with $\delta = 0$. Using Mathematica [10], we can check that Assumption 3.1-3.2 hold with $\bar{\Delta} = 0$, $K = [-1.25, -2]$, $\rho(s) = \epsilon s$,

$$P = \begin{bmatrix} 4.09112 & 0.722222 \\ 0.722222 & 1.17951 \end{bmatrix}, \tag{96}$$

and $\epsilon = 0.01$, where P was found by first solving for a positive definite symmetric matrix $P_1 \in \mathbb{R}^{2\times 2}$ such that $P_1H + H^\top P_1 = -I$ holds with $H = A + BK$, then choosing $c = 0.75$ in order to satisfy $cP_1 \leq I$, and then multiplying P_1 by 3.25 to satisfy the requirement that $P \geq I$ with the choice $P = 3.25P_1$. Also, since $\bar{\Delta} = 0$, Assumptions 3.3 and 3.6 hold for any $s_\star > 0$. We can then also use Mathematica to compute the basin of attraction from Theorem 3.7. For instance, when the delay h is the zero function, we can check that we can satisfy the requirements of Theorem 3.7 with $w_\star = s_\star = 2.20049$ and all initial functions whose norms are bounded by 0.718677. If we instead use the delay bound $\bar{h} = 0.008$ and keep all other parameter values the same as before, then the basin of attraction consists of all initial functions that are bounded by 0.137212. This illustrates the trade-off that increasing the bound $\bar{h}$ on the allowable input delays $h(t)$ can reduce the basin of attraction.

5.2. Illustration of Theorem 4.4.

Consider the two dimensional system

$$\begin{cases} \dot{x}_1(t) &= \mathrm{sat}(x_2(t)) + \delta_1(t) \\ \dot{x}_2(t) &= u(t) + \delta_2(t) \\ y(t) &= x_1(t) \end{cases} \tag{97}$$

where sat is the standard saturation that is defined by $\mathrm{sat}(s) = s$ when $|s| \leq 1$, $\mathrm{sat}(s) = 1$ if $s > 1$, and $\mathrm{sat}(s) = -1$ if $s < -1$. We illustrate Theorem 4.4. To satisfy Assumption 4.2 with

$$\mathcal{G}(x) = [\mathrm{sat}(x_2) - x_2, 0]^\top, \tag{98}$$

we prove that

$$|\mathcal{G}(x)| \leq |x|^2\kappa(|x|) \tag{99}$$

for all $x \in \mathbb{R}^2$, where κ is defined by

$$\kappa(r) = \max\{r-1, 0\}, \tag{100}$$

by considering two cases:
1) If $|x| \le 1$, then $|x_2| \le 1$. It follows that $\mathcal{G}(x) = 0$. Consequently, (99) is satisfied.
2) If $|x| > 1$ and $|x_2| \le 1$, then $\mathcal{G}(x) = 0$ which implies that (99) is satisfied. Next, consider the case where $x_2 > 1$. Then $|\mathcal{G}(x)| = x_2 - 1$. Thus (99) is satisfied if and only if $x_2 - 1 \le |x|^2\kappa(|x|)$, which is satisfied because $x_2 - 1 \le x_2^2(x_2-1) = x_2^2\kappa(x_2)$. If $x_2 < -1$, then we get $|\mathcal{G}(x)| = -x_2 - 1$ and we obtain a similar result.

It follows that Assumptions 4.1-4.2 are satisfied, once we choose

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \text{and } C = [1 \quad 0]. \tag{101}$$

Since the preceding choices give

$$e^{As} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{102}$$

for all $s \in \mathbb{R}$, our formulas (62)-(63) give

$$E = \int_{-T}^{0} \begin{bmatrix} 1 & s \\ s & s^2 \end{bmatrix} \mathrm{d}s = \begin{bmatrix} T & -\frac{T^2}{2} \\ -\frac{T^2}{2} & \frac{T^3}{3} \end{bmatrix} \tag{103}$$

and therefore also

$$\begin{aligned} \hat{x}(t) &= \tfrac{12}{T^3} \begin{bmatrix} \frac{T^2}{3} & \frac{T}{2} \\ \frac{T}{2} & 1 \end{bmatrix} \int_{t-T}^{t} \begin{bmatrix} 1 \\ s-t \end{bmatrix} y(s)\mathrm{d}s \\ &+ \tfrac{12}{T^3} \begin{bmatrix} \frac{T^2}{3} & \frac{T}{2} \\ \frac{T}{2} & 1 \end{bmatrix} \int_{t-T}^{t} \begin{bmatrix} 1 \\ s-t \end{bmatrix} \int_{s}^{t} (s-m)u(m)\mathrm{d}m\mathrm{d}s \end{aligned} \tag{104}$$

for all $t \ge T$. Let us choose

$$K = [-1, -2]. \tag{105}$$

We can then use Mathematica to check that our requirements are met with

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \tag{106}$$

$c = 0.845$, $T = 0.1$, $g = 9.3218$, $\gamma_0(s) = 293.164\kappa^2(\sqrt{s})s$, and $u_\star = 1.053$. Using the inequality

$$2|x|^2 \le x^\top P x \tag{107}$$

we obtain the value 0.592453 for the bound on the norm of the initial function from Theorem 4.4. Moreover, the maximum allowable upper bound on $\bar{\delta}$ is 0.0309026 when $M = [1,0]^\top$, and the maximum allowable bound on $\bar{\delta}$ is 0.112342 when $M = [0,1]^\top$.

6. **Conclusion.** We advanced the state of the art for stability analysis of nonlinear systems, by providing a local version of Halanay's inequality that is conducive to proving local asymptotic stability properties for nonlinear systems that contain state or input delays and uncertainties. Our new results are significant, because of the well-known benefits of using global versions of Halanay's inequality to prove global asymptotic stability for systems with unknown delays, and because many significant systems are only locally asymptotically stable and therefore are beyond

the scope of earlier global versions of Halanay's inequality. Another significant benefit of our work is that we allow the dynamics to contain unknown nonlinearities that violate the standard linear growth conditions and that can contain distributed state delays. We used our approach to prove a local feedback stabilization result for a large class of nonlinear systems with outputs. We illustrated how our methods provide new estimates for basins of attraction for a controlled van der Pol equation and other cases that contain input delays and outputs. We hope to find analogs for discrete-time systems, and vector versions providing local analogs of the corresponding continuous-time Halanay's inequality results from [12].

Appendices: Proofs of key lemmas.

A.1. **ISS inequality.** We prove a key lemma that we used in our proof of Theorem 2.2. To this end, first let $t_\star \geq 0$ and $\mathcal{L}_0 \geq 0$ be given constants. Consider a C^1 function $V : [t_\star - \mathcal{L}_0, +\infty) \to [0, +\infty)$, a nonnegative valued nondecreasing continuous function ζ, and constants $a > 0$ and $b > 0$ such that

$$\dot{V}(t) \leq -(a+b)V(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{A.1.1}$$

holds for all $t \geq t_\star$. Let $\lambda > 0$ be the constant defined as is (8). We then have the following result, where we can use the inequality (A.1.2) to easily deduce an ISS inequality:

Lemma A.1.1. *Consider the function V introduced in Section 2 above. The inequality*

$$V(t) \leq \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)}\zeta(m)\mathrm{d}m \tag{A.1.2}$$

holds for all $t \geq s$ and all $s \geq t_\star$.

Proof. Let $s \geq t_\star$ be a constant. Let us introduce the function

$$\theta(t) = \int_s^{\max\{t,s\}} e^{b(m-t)}\zeta(m)\mathrm{d}m. \tag{A.1.3}$$

Notice that $\dot{\theta}(t) = -b\theta(t) + \zeta(t)$ for all $t > s$. Using the nondecreasing property of ζ to get

$$\theta(t) \leq \int_s^t e^{b(m-t)}\mathrm{d}m\zeta(t) \leq \frac{1}{b}\zeta(t) \tag{A.1.4}$$

for all $t \geq s$, it follows that θ is nondecreasing over $[s, +\infty)$. Moreover, $\theta(t) = 0$ for all $t \in [s - \mathcal{L}_0, s]$. Hence,

$$\dot{\theta}(t) = -(a+b)\theta(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) \tag{A.1.5}$$

holds for all $t > s$. We next choose

$$v_l = \sup_{m\in[s-\mathcal{L}_0,s]} V(m) \text{ and } \chi_\epsilon(t) = \theta(t) + v_l e^{-\lambda(t-s)} + \epsilon, \tag{A.1.6}$$

where $\epsilon > 0$ is a constant. Then

$$\chi_\epsilon(t) > \theta(t) + v_l e^{-\lambda(t-s)} \geq v_l = \sup_{m\in[s-\mathcal{L}_0,s]} V(m) \tag{A.1.7}$$

holds for all $t \in [s - \mathcal{L}_0, s]$. Therefore,

$$\chi_\epsilon(t) > V(t) \text{ for all } t \in [s - \mathcal{L}_0, s]. \tag{A.1.8}$$

We next prove that

$$\chi_\epsilon(t) > V(t) \tag{A.1.9}$$

holds for all $t \geq s - \mathcal{L}_0$. Let us start to prove this by observing that

$$\dot{\chi}_\epsilon(t) = -(a+b)\theta(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) - \lambda v_l e^{-\lambda(t-s)} \tag{A.1.10}$$

for all $t > s$. From (8), we deduce that

$$\begin{array}{rcl} \dot{\chi}_\epsilon(t) & = & -(a+b)\theta(t) + a \sup\limits_{m\in[t-\mathcal{L}_0,t]} \theta(m) + \zeta(t) \\ & & -\,(a+b-ae^{\lambda\mathcal{L}_0})v_l e^{-\lambda(t-s)} \\ & = & -(a+b)\left(\theta(t) + v_l e^{-\lambda(t-s)}\right) \\ & & +a\left(\sup\limits_{m\in[t-\mathcal{L}_0,t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)}\right) + \zeta(t). \end{array} \tag{A.1.11}$$

Now, observe that

$$\sup_{m\in[t-\mathcal{L}_0,t]} \left(\theta(m) + v_l e^{-\lambda(m-s)}\right) \leq \sup_{m\in[t-\mathcal{L}_0,t]} \theta(m) + v_l e^{-\lambda(t-s-\mathcal{L}_0)}. \tag{A.1.12}$$

As an immediate consequence,

$$\begin{array}{rcl} \dot{\chi}_\epsilon(t) & \geq & -(a+b)\left(\theta(t) + v_l e^{-\lambda(t-s)}\right) \\ & & +a \sup\limits_{m\in[t-\mathcal{L}_0,t]} \left(\theta(m) + v_l e^{-\lambda(m-s)}\right) + \zeta(t) \end{array} \tag{A.1.13}$$

for all $t > s$. Using the definition of χ_ϵ, we obtain

$$\begin{array}{rcl} \dot{\chi}_\epsilon(t) & \geq & -(a+b)\left(\chi_\epsilon(t) - \epsilon\right) + a \sup\limits_{m\in[t-\mathcal{L}_0,t]} \left(\chi_\epsilon(m) - \epsilon\right) + \zeta(t) \\ & = & -(a+b)\chi_\epsilon(t) + a \sup\limits_{m\in[t-\mathcal{L}_0,t]} \chi_\epsilon(m) + \zeta(t) + b\epsilon. \end{array} \tag{A.1.14}$$

We next proceed by contradiction. Bearing in mind (A.1.8), suppose that there were a $t_c > s$ such that $\chi_\epsilon(t) > V(t)$ for all $t \in [s - \mathcal{L}_0, t_c)$ and $\chi_\epsilon(t_c) = V(t_c)$. Then (A.1.14) gives

$$\dot{\chi}_\epsilon(t_c) \quad \geq \quad -(a+b)V(t_c) + a \sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) + \zeta(t_c) + b\epsilon. \tag{A.1.15}$$

On the other hand (A.1.1) gives

$$-\dot{V}(t_c) \geq (a+b)V(t_c) - a \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m) - \zeta(t_c). \tag{A.1.16}$$

By adding (A.1.15) and (A.1.16), we obtain

$$\dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) \quad \geq \quad a\left[\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) - \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m)\right] + b\epsilon. \tag{A.1.17}$$

The definition of t_c ensures that $\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) \geq V(t)$ for all $t \in [t_c - \mathcal{L}_0, t_c]$. It follows that

$$\sup_{m\in[t_c-\mathcal{L}_0,t_c]} \chi_\epsilon(m) - \sup_{m\in[t_c-\mathcal{L}_0,t_c]} V(m) \geq 0.$$

We deduce from (A.1.17) that $\dot{\chi}_\epsilon(t_c) - \dot{V}(t_c) > 0$. Since $\chi_\epsilon(t_c) - V(t_c) = 0$, we deduce that there is $t_d \in (s, t_c)$ such that $\chi_\epsilon(t_d) - V(t_d) < 0$. This contradicts the definition of t_c. Hence, $\chi_\epsilon(t) > V(t)$ holds for all $t \geq s - \mathcal{L}_0$. Since $\epsilon > 0$ is arbitrary, we deduce that $\theta(t) + v_l e^{-\lambda(t-s)} \geq V(t)$ for all $t \geq s - \mathcal{L}_0$. Therefore, the conclusion of the lemma follows.

A.2. **Technical result.** We prove the key lemma that we used in the proof of Lemma 3.4. We use constants $T > 0$, $q > 0$, $\Psi_0 > 0$, $\omega > 0$, $\tau > 0$, $\bar{\Delta} \geq 0$ and $t_a \geq 0$ and a continuous, nondecreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that

$$\Psi(\ell) \leq \Psi_0 \ell \tag{A.2.1}$$

for all $\ell \in [0, \omega]$. Let $Z : [t_a - T, t_a + \tau) \to [0, +\infty)$ be a nonnegative valued function of class C^1 such that

$$\dot{Z}(t) \leq \Psi\left(\sup_{\ell \in [t-T,t]} Z(\ell)\right) + \bar{\Delta} \tag{A.2.2}$$

for all $t \in [t_a, t_a + \tau)$. We use the following assumption:

Assumption A.2.1. *The inequality*

$$\left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} < \omega \tag{A.2.3}$$

is satisfied.

In the following lemma, the existence of values $\bar{Z} > 0$ such that (A.2.5) is satisfied follows from (A.2.3):

Lemma A.2.2. *Let Assumption A.2.1 be satisfied. Let Z be such that*

$$Z(\ell) \leq \bar{Z} \text{ for all } \ell \in [t_a - T, t_a] \tag{A.2.4}$$

where $\bar{Z} \in \mathbb{R}$ is such that

$$\bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} < \omega. \tag{A.2.5}$$

Then

$$Z(t) \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.2.6}$$

holds for all $t \in [t_a - T, t_a + \min\{\tau, q\})$.

Proof. We first prove that $Z(t) < \omega$ for all $t \in [t_a - T, t_a + \min\{\tau, q\})$. To prove this, we proceed by contradiction. Note that (A.2.4)-(A.2.5) imply that $Z(t) < \omega$ for all $t \in [t_a - T, t_a]$, and let us suppose that there were a $t_c \in [0, \min\{\tau, q\})$ such that $Z(t) < \omega$ for all $t \in [t_a - T, t_a + t_c)$ and $Z(t_a + t_c) = \omega$. Then

$$\dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T,t]} Z(\ell) + \bar{\Delta} \tag{A.2.7}$$

for all $t \in [t_a, t_a + t_c]$, by the bounds (A.2.1) and (A.2.2). Now, let us introduce the function

$$\xi_\epsilon(t) = (\bar{Z} + \epsilon) e^{(\Psi_0 + \epsilon)(t - t_a)} + \left[e^{(\Psi_0 + \epsilon)(t - t_a)} - 1\right] \tfrac{\bar{\Delta}}{\Psi_0 + \epsilon} \tag{A.2.8}$$

with $\epsilon > 0$. Let us observe that $Z(t_a) \leq \bar{Z} < \bar{Z} + \epsilon \leq \xi_\epsilon(t_a)$. We next show that $Z(t) < \xi_\epsilon(t)$ for all $t \in [t_a, t_a + t_c]$. We argue by contradiction. Suppose there were a $t_f \in [t_a, t_a + t_c]$ such that

$$Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_f) \text{ and } Z(t_f) = \xi_\epsilon(t_f). \tag{A.2.9}$$

Simple calculations based on the formula (A.2.8) then give

$$\begin{aligned} \dot{\xi}_\epsilon(t) &= (\Psi_0 + \epsilon)(\bar{Z} + \epsilon) e^{(\Psi_0 + \epsilon)(t - t_a)} + e^{(\Psi_0 + \epsilon)(t - t_a)} \bar{\Delta} \\ &= (\Psi_0 + \epsilon) \left[\xi_\epsilon(t) + \left(-e^{(\Psi_0 + \epsilon)(t - t_a)} + 1\right) \tfrac{\bar{\Delta}}{\Psi_0 + \epsilon}\right] \\ &\quad + e^{(\Psi_0 + \epsilon)(t - t_a)} \bar{\Delta} \\ &= (\Psi_0 + \epsilon) \xi_\epsilon(t) + \bar{\Delta} \end{aligned} \tag{A.2.10}$$

for all $t > t_a$. On the other hand, (A.2.7) and (A.2.9) give $\dot{Z}(t_f) \leq \Psi_0 \xi_\epsilon(t_f) + \bar{\Delta}$, by our choice of ξ_ϵ. Consequently, (A.2.10) gives

$$\dot{Z}(t_f) < \dot{\xi}_\epsilon(t_f), \tag{A.2.11}$$

which we can combine with (A.2.9) to deduce that there is a $t_g \in [t_a, t_f)$ such that

$$\xi_\epsilon(t_g) < Z(t_g). \tag{A.2.12}$$

This yields a contradiction with the definition of t_f. We deduce that

$$Z(t) < \xi_\epsilon(t) \text{ for all } t \in [t_a, t_a + t_c]. \tag{A.2.13}$$

Since ϵ is an arbitrary positive number, we deduce that

$$Z(t) \leq \bar{Z} e^{\Psi_0(t-t_a)} + \left(e^{\Psi_0(t-t_a)} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \text{ for all } t \in [t_a, t_a + t_c]. \tag{A.2.14}$$

Since $t_c < q$, it follows that

$$Z(t_a + t_c) \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0}. \tag{A.2.15}$$

Since $Z(t_a + t_c) = \omega$, we obtain

$$\omega \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0}. \tag{A.2.16}$$

This contradicts (A.2.5). Hence,

$$Z(t) < \omega \text{ for all } \in [t_a - T, t_a + \min\{\tau, q\}). \tag{A.2.17}$$

It follows from (A.2.1) that

$$\dot{Z}(t) \leq \Psi_0 \sup_{\ell \in [t-T, t]} Z(\ell) + \bar{\Delta} \tag{A.2.18}$$

for all $t \in [t_a, t_a + \min\{\tau, q\})$. Arguing as in (A.2.7)-(A.2.13) except with t_c replaced by $\min\{\tau, q\}$, we obtain $Z(t) \leq \xi_\epsilon(t)$ for all $t \in [t_a, t_a + \min\{\tau, q\})$ which implies that

$$Z(t) \leq \bar{Z} e^{\Psi_0(t-t_a)} + \left(e^{\Psi_0(t-t_a)} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.2.19}$$

for all $t \in [t_a, t_a + \min\{\tau, q\})$. Since

$$Z(t) \leq \bar{Z} \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \tfrac{\bar{\Delta}}{\Psi_0} \tag{A.2.20}$$

for all $t \in [t_a - T, t_a]$, we can conclude. □

Acknowledgments. The authors thank the anonymous reviewers, whose helpful comments helped us improve this paper.

REFERENCES

[1] E. Beretta and D. Breda, Discrete or distributed delay? Effects on stability of population growth, *Mathematical Biosciences and Engineering*, **13** (2016), 19-41.

[2] A. Chaillet, I. Karafyllis, P. Pepe and Y. Wang, The ISS framework for time-delay systems: A survey, *Mathematics of Control, Signals, and Systems*, **35** (2023), 237-306.

[3] Y. Chen, Z. Wang, B. Shen and Q.-L. Han, Local stabilization for multiple input-delay systems subject to saturating actuators: The continuous-time case, *IEEE Transactions on Automatic Control*, **67** (2022), 3090-3097.

[4] E. Fridman, *Introduction to Time-Delay Systems*, Birkhäuser, Cham, 2014.

[5] M. Grifa and P. Pepe, On stability analysis of discrete-time systems with constrained time-delays via nonlinear Halanay-type inequality, *IEEE Control Systems Letters*, **5** (2021), 869-874.

[6] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, NY, 1966.

[7] R. Katz and E. Fridman, Sampled-data finite-dimensional boundary control of 1D parabolic PDEs under point measurement via a novel ISS Halanay's inequality, *Automatica*, **135** (2022), 14 pp.

[8] H. Khalil, *Nonlinear Systems*, 3^{rd} edition, Prentice Hall, Upper Saddle River, NJ, 2002.

[9] H. Lhachemi and R. Shorten, On the derivation of stability properties for time-delay systems without constraint on the time-derivative of the initial condition, *IEEE Transactions on Automatic Control*, **66** (2021), 5401-5406.

[10] Mathematica, The world's definitive system for modern technical computing, Wolfram Research, 2015, http://www.wolfram.com/mathematica/.

[11] F. Mazenc and M. Malisoff, Almost finite-time observers for a family of nonlinear continuous-time systems, *IEEE Control Systems Letters*, **6** (2022), 2593-2598.

[12] F. Mazenc, M. Malisoff and M. Krstic, Vector extensions of Halanay's inequality, *IEEE Transactions on Automatic Control*, **67** (2022), 1453-1459.

[13] F. Mazenc, M. Malisoff and H. Özbay, Stability and robustness analysis for switched systems with time-varying delays, *SIAM Journal on Control and Optimization*, **56** (2018), 158-182.

[14] A. Mironchenko, Local input-to-state stability: Characterizations and counterexamples, *Systems Control Letters*, **87** (2016), 23-28.

[15] P. Pepe, A nonlinear version of Halanay's inequality for the uniform convergence to the origin, *Mathematical Control and Related Fields*, **12** (2022), 789-811.

[16] P. Pepe and E. Fridman, On global exponential stability preservation under sampling for globally Lipschitz time-delay systems, *Automatica*, **82** (2017), 295-300.

[17] A. Selivanov and E. Fridman, Distributed event-triggered control of transport-reaction systems, *IFAC-PapersOnLine*, **48** (2015), 593-597.

[18] E. Sontag, *Mathematical Control Theory*, 2^{nd} edition, Springer, New York, 1998.

[19] E. Sontag, The ISS philosophy as a unifying framework for stability-like behavior, In *Nonlinear Control in the Year 2000*, **2** (2001), 443-467.

[20] X. Zhao and F. Deng, Time-varying Halanay inequalities with application to stability and control of delayed stochastic systems, *IEEE Transactions on Automatic Control*, **67** (2022), 1226-1240.

Received January 2024; 1st revision April 2024; 2nd revision June 2024; early access June 2024.