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IFAC PapersOnLine 58-21 (2024) 179–183

Local Halanay's Inequality for Local Exponential Stabilization

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Abstract: We announce a local version of a method for proving asymptotic stability based on Halanay's inequality. Our approach can be applied to nonlinear systems containing input and state delays. It provides robustness estimates for dynamics that contain actuator uncertainty, in the sense of input-to-state stability. Our numerical example illustrates how our method leads to useful bounds on the allowable uncertainties and on the basin of attraction.

Keywords: Stability of delay systems, input-to-state stability, time-delay systems, nonlinear time-delay systems, robustness analysis

1. INTRODUCTION

Hanalay's inequality was introduced in the celebrated work of Halanay (1966), and has since been generalized in multiple ways that have played important roles in proofs of stability properties for significant classes of nonlinear systems containing input and state delays. Notable contributions have included works by Grifa and Pepe (2021), Mazenc et al. (2022), Grifa and Pepe (2020), and Pepe (2022), which cover continuous-time and discrete-time systems. In its most basic form (as presented, e.g., in (Fridman, 2014, Lemma 4.2, p. 138)), Halanay's inequality requires finding nonnegative valued differentiable functions v and constants $a > 0$, $b \in (0, a)$, and $T > 0$ that satisfy

$$\dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-T,t]} v(\ell) \tag{1}$$

for all $t \geq T$, and then concludes that $v(t)$ exponentially converges to 0 as $t \to +\infty$. Generalizations include works by Grifa and Pepe (2021); Ruan et al. (2020), where instead of a and b in (1) being constants, the a and b are allowed to depend on the time variable t, including situations where $b(t) > a(t)$ for some choices of t, and these works provide advantages of using Halanay's inequality approaches in stability analyses instead of using standard Lyapunov function approaches. However, the preceding works are only directly applicable to globally exponentially stable systems. This is an obstacle, because nonlinear systems are often only locally exponentially stable, and in such situations, one cannot use global Halanay's results to prove global stability. In addition, to the best of the authors' knowledge, there is no general result in the literature that ensures that a nonlinear delayed system (with input or state constraints) whose linear approximation at the origin is exponentially stabilizable enjoys a locally exponentially stability property. More generally, local stability or stabilization of delayed systems is an under-studied topic, which strongly motivates the present work.

Here we present our local version of the Halanay's inequality based stability result for functions satisfying a nonlinear differential inequality, in a local sense. We apply it to systems containing small bounded disturbances that can represent actuator uncertainty, leading to our proof of input-to-state-stability (or ISS) inequalities. The results are amenable to nonlinear systems that contain disturbances and delays, and provide estimates of corresponding basins of attraction. We begin by stating and proving our local Halanay's inequality result in Section 2, which we use to prove a local exponential stabilization theorem for systems with state feedback in Section 3. For generality, we cover nonlinear systems with time-varying and distributed delays, and we illustrate our findings in Section 4 using a controlled version of van der Pol's equation, whose structure precludes using earlier methods to prove global stabilization results, but which are covered by our local Halanay's inequality approach. This provides significantly new estimates for basins of attractions, and sufficient conditions on bounds for the uncertainties for our local stabilization estimates to hold. Here we include summarized proofs; see Malisoff and Mazenc (2024) for complete proofs of all of the results in this paper, and for an extension for systems with outputs that is amenable to systems with saturations.

We use standard notation which is simplified when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise, and $|\cdot|$ is the usual Euclidean vector norm and also denotes the corresponding matrix operator norm. Given matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, the notation $A \leq B$ means that $B - A$ is nonnegative definite, and I is the identity

⋆ The research of the second author was supported by US National Science Foundation Division of Mathematical Sciences Applied Mathematics Program Grants 2009659 and 2308282.

10.1016/j.ifacol.2024.10.175

matrix. When r is a time variable, we use the usual notation $g_r(\ell) = g(r+\ell)$ for functions g and all $\ell \le 0$ and $r \ge 0$ for which $r+\ell$ is in the domain of g. We also use the usual family of functions $\mathcal{K}_\infty$ the usual definitions of ISS from works by Khalil (2002); Sontag (2001), and the controllability definitions from Sontag (1998).

2. LOCAL ISS HANALAY'S RESULTS

Let $\alpha : [0,+\infty) \to [0,+\infty)$ be a continuous, nonnegative valued and nondecreasing function such that there are two constants $v_\star > 0$ and $a > 0$ such that

$$\alpha(v_\star) = a. \tag{2}$$

Let $t_\star \ge 0$ and $b > 0$ be two constants. Let $\zeta : [t_\star,+\infty) \to [0,+\infty)$ be a nondecreasing function such that

$$\zeta(t) < bv_\star \tag{3}$$

holds for all $t \ge t_\star$. Let $\mathcal{L}_0 \ge 0$ be a given constant and $\tau \in (t_\star,+\infty)$ or $\tau = +\infty$. Throughout this section, we let $V : [t_\star - \mathcal{L}_0, \tau) \to [0,+\infty)$ be a C^1 function such that

$$\sup_{m\in[t_\star-\mathcal{L}_0,t_\star]} V(m) < v_\star \tag{4}$$

and whose time derivative $\dot V$ is such that

$$\begin{aligned}\dot V(t) \le\ & -(a+b)V(t) \\ & + \alpha\left(\sup_{m\in[t-\mathcal{L}_0,t]} V(m)\right) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t)\end{aligned} \tag{5}$$

holds for all $t \in [t_\star, \tau)$. We then use this technical result:

Lemma 1. The inequality $V(t) < v_\star$ holds for all $t \in [t_\star - \mathcal{L}_0, \tau)$.

Proof. We prove this lemma by contradiction. Suppose that there existed a $t_c \in [t_\star - \mathcal{L}_0, \tau)$ such that $V(t_c) = v_\star$ and $V(t) < v_\star$ for all $t \in [t_\star - \mathcal{L}_0, t_c)$. Then (4) gives $t_c > t_\star$, and (2), (3), and (5) give

$$\begin{aligned}\dot V(t_c) &< -(a+b)v_\star + \alpha(v_\star)v_\star + bv_\star \\ &= -av_\star + av_\star = 0.\end{aligned} \tag{6}$$

From the inequality $\dot V(t_c) < 0$ and the continuity of V, we can find a $t_d \in (t_\star, t_c)$ such that $V(t_d) > v_\star$. This contradicts the definition of t_c, so the lemma holds. □

Since $a > 0$ and $b > 0$, there is a unique $\lambda > 0$ such that

$$\lambda = a + b - ae^{\lambda\mathcal{L}_0}. \tag{7}$$

Using this λ, we next state and prove:

Theorem 1. Let V satisfy the requirements of Lemma 1 in the case where $\tau = +\infty$. Then

$$V(t) \le \sup_{m\in[s-\mathcal{L}_0,s]} V(m)e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)}\zeta(m)\mathrm{d}m \tag{8}$$

holds for all $t \ge s$ and for all $s \ge t_\star$.

Proof. Since α is nondecreasing, Lemma 1 and (5) give

$$\dot V(t) \le -(a+b)V(t) + \alpha(v_\star) \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{9}$$

for all $t \ge t_\star$. From (2), we deduce that

$$\dot V(t) \le -(a+b)V(t) + a \sup_{m\in[t-\mathcal{L}_0,t]} V(m) + \zeta(t) \tag{10}$$

holds for all $t \ge t_\star$. Since $a > 0$ and $b > 0$, we can then apply Lemma 3 in the appendix below, to conclude. □

Remark 1. The inequality (8) gives an ISS exponential inequality, by upper bounding its right side integral term by $(1/b)\sup_{\ell\in[s,t]}|\zeta(\ell)|$ for all $t \ge s$ and all $s \ge t_\star$.

3. LOCAL EXPONENTIAL STABILIZATION RESULT

We use Theorem 1 to solve a local stabilization problem for a class of nonlinear systems.

3.1 Studied system and preliminary result

Let $h : [0,+\infty) \to [0,+\infty)$ be continuous function for which there is a constant $\bar h > 0$ such that $0 \le h(t) \le \bar h$ for all $t \ge 0$. Let $\delta : [0,+\infty) \to \mathbb{R}^n$ be a continuous function that admits a constant $\bar\Delta$ such that

$$|\delta(t)| \le \bar\Delta \tag{11}$$

for all $t \ge 0$. Consider the system

$$\dot x(t) = Ax(t) + Bu(t-h(t)) + \mathcal{F}(t,x_t) + \delta(t) \tag{12}$$

where x is valued in $\mathbb{R}^n$, the input u is valued in $\mathbb{R}^p$, and $\mathcal{F}$ is a locally Lipschitz continuous function. In all of what follows, we assume that the dynamics satisfy standard forward completeness and existence and uniqueness properties of solutions. Let $t_0 \ge 0$. Consider initial functions $x_0 : [t_0 - \bar h, t_0] \to \mathbb{R}^n$, and we introduce three assumptions:

Assumption 1. The pair (A, B) is controllable.

Assumption 2. There is a continuous nondecreasing function $\rho : [0,+\infty) \to [0,+\infty)$ that is not identically equal to zero such that

$$|\mathcal{F}(t,\phi)| \le \sup_{m\in[-\bar h,0]} |\phi(m)|^2\rho(|\phi(m)|) \tag{13}$$

holds for all functions $\phi : [-\bar h, 0] \to \mathbb{R}^n$ and all $t \ge 0$.

It is well known that Assumption 1 provides a matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix $H = A + BK$ is Hurwitz, and so also a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and constants $c > 0$ and $\bar p > 0$ such that

$$PH + H^\top P \le -cP,\ I \le P, \text{ and } |P| \le \bar p \tag{14}$$

e.g., by first using the Pole-Shifting Theorem from Sontag (1998) to find K, then solving the Riccati equation $PH + H^\top P = -I$ for P. We fix choices of ρ, K, P, and $\bar p$ satisfying the preceding requirements, and assume that $BK \ne 0$. Our last assumption is the following smallness condition on either $\bar h$ or $\bar\Delta$; see Section 4 for an example illustrating how we can easily satisfy all of our assumptions:

Assumption 3. There is a real value $s_\star > 0$ such that

$$\omega_0 = \left(2|A| + 2|BK| + 1 + 2\sqrt{s_\star}\,\rho\left(\sqrt{s_\star}\right)\right)\bar p \tag{15}$$

is such that the inequality

$$\left(e^{2.1\omega_0\bar h} - 1\right)\frac{\bar p\bar\Delta^2}{\omega_0} < s_\star \tag{16}$$

is satisfied.

In terms of the preceding notation and the function

$$W(x) = x^\top P x, \tag{17}$$

we start with a technical lemma, where Assumption 3 ensures that the (18) is satisfied when the initial function is valued in a small enough neighborhood of the origin:

Lemma 2. Let (12) satisfy Assumptions 1-3. Consider (12) in closed-loop with the feedback $u(t-h(t)) = Kx(t-h(t))$. Let x be a solution of this system such that

$$\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0} < s_\star. \tag{18}$$

Then x is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$ and

$$\begin{aligned}\sup_{m\in[t_0-\bar{h},t_0+2\bar{h}]} W(x(m)) \leq & \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} \\ & + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0}\end{aligned} \tag{19}$$

is satisfied.

Proof. (Summary) Pick any maximal solution $x(t)$ of this closed-loop system from the lemma such that (18) holds. Let $[t_0 - \bar{h}, t_0 + t_\infty)$ be the domain of definition of $x(t)$. Then $0 < t_\infty < +\infty$ or $t_\infty = +\infty$. The time derivative of (17) along $x(t)$ satisfies

$$\begin{aligned}\dot{W}(t) \leq\ & 2\overline{p}|x(t)|\left[|A||x(t)| + |BK||x(t-h(t))|\right. \\ & \left. + \sup_{m\in[t-\bar{h},t]} |x(m)|^2\rho(|x(m)|)\right] + 2|x(t)|\overline{p}\bar{\Delta}\end{aligned} \tag{20}$$

for all $t \in (t_0, t_0 + t_\infty)$, by (11), (13), and (14). Since ρ from Assumption 2 is nondecreasing, (14) gives

$$\begin{aligned}\dot{W}(t) \leq\ & \overline{p}\left(2|A| + 2|BK| + 1\right) \sup_{m\in[t-\bar{h},t]} W(x(m)) \\ & + 2\overline{p}\sup_{m\in[t-\bar{h},t]} W^{3/2}(x(m)) \\ & \times\rho\left(\sqrt{\sup_{m\in[t-\bar{h},t]} W(x(m))}\right) + \overline{p}\bar{\Delta}^2\end{aligned} \tag{21}$$

using the triangle inequality to upper bound the last right term in (20) by $2|x(t)|\overline{p}\bar{\Delta} \leq \overline{p}|x(t)|^2 + \overline{p}\bar{\Delta}^2$. Setting $\bar{\omega}(s) = \overline{p}\left(2|A| + 2|BK| + 1\right)s + 2s^{3/2}\overline{p}\rho\left(\sqrt{s}\right)$ then gives

$$\dot{W}(t) \leq \bar{\omega}\left(\sup_{m\in[t-\bar{h},t]} W(x(m))\right) + \overline{p}\bar{\Delta}^2 \tag{22}$$

for all $t \in (t_0, t_0 + t_\infty)$ and, by the definition of ω_0 in (15), we have $\bar{\omega}(s) \leq \omega_0 s$ for all $s \in [0, s_\star]$. We now apply Lemma 4 in the appendix below, with $W(x(t))$, $\bar{\omega}$, ω_0, $\overline{p}^2\bar{\Delta}^2$, t_0, t_∞, $2.1\bar{h}$, $\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))$, $s_\star$, and $\bar{h}$ as the choices of $Z(t)$, Ψ, Ψ_0, $\bar{\Delta}$, t_a, τ, q, $\bar{Z}$, ω and T in the lemma, respectively. Assumption 3 ensures that Assumption A.1 from the appendix below (which is needed to apply Lemma 4 from the appendix) holds. Also, (A.6) holds, since $W(x(t)) \leq \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))$ for all $t \in [t_0 - \bar{h}, t_0]$. Then (18) ensures that (A.7) holds. Hence, Lemma 4 implies that for all $t \in [t_0 - \bar{h}, t_0 + \min\{t_\infty, 2.1\bar{h}\})$, we get

$$\begin{aligned}W(x(t)) \leq & \sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} \\ & + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0}.\end{aligned} \tag{23}$$

Therefore, the finite escape time phenomenon does not occur over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, so $t_\infty > 2\bar{h}$. □

3.2 ISS result

Using the notation from Section 3.1, we use the function

$$\begin{aligned}\beta(m) =\ & 2\bar{h}|PBK|\left(|A| + |BK|\right) \\ & + 2\left(\bar{h}|PBK| + |P|\right)m^{1/2}\rho\left(\sqrt{m}\right).\end{aligned} \tag{24}$$

We now add the following assumption, which can again be regarded as a smallness condition on $\bar{h}$, where the constant $c > 0$ is from (14):

Assumption 4. The bound $2\bar{h}|PBK|\left(|A| + |BK|\right) < c/4$ is satisfied.

It follows that there is a $w_\star > 0$ such that

$$\beta(w_\star) = \tfrac{c}{4} \tag{25}$$

and we fix a $w_\star$ satisfying the preceding requirement in the rest of this subsection. We also assume:

Assumption 5. The inequality

$$\tfrac{4}{c}\left(|PBK|^2\bar{h}^2 + |P|^2\right)\bar{\Delta}^2 < \tfrac{cw_\star}{4} \tag{26}$$

holds.

Assumption 5 can be viewed as a smallness condition on $\bar{\Delta}$. Let $\gamma > 0$ be the constant such that

$$\gamma = \tfrac{c}{2} - \tfrac{c}{4}e^{2\gamma\bar{h}}. \tag{27}$$

We are ready to state and prove the following result:

Theorem 2. Let (12) satisfy Assumptions 1-5. Then, with the notation from the preceding subsection, consider (12) in closed-loop with $u(t - h(t)) = Kx(t - h(t))$. Consider any maximal solution $x(t)$ of the closed-loop system such that

$$\begin{aligned}&\sup_{m\in[t_0-\bar{h},t_0]} W(x(m))e^{2.1\omega_0\bar{h}} + \left(e^{2.1\omega_0\bar{h}} - 1\right)\frac{\overline{p}\bar{\Delta}^2}{\omega_0} \\ &\quad < \min\{s_\star, w_\star\}\end{aligned} \tag{28}$$

holds. Then, for each $s \geq t_0 + \bar{h}$, and with the choice

$$\mathcal{H}(m) = \tfrac{4|PBK|^2\bar{h}}{c}\int_0^m |\delta(r)|^2\,\mathrm{d}r + \tfrac{4|P|^2}{c}\sup_{\ell\in[\bar{h},m]} |\delta(\ell)|^2, \tag{29}$$

the inequality

$$\begin{aligned}&|x(t)| \leq \\ &\sqrt{\overline{p}\sup_{m\in[s-2\bar{h},s]} |x(m)|^2 e^{-\gamma(t-s)} + \int_s^t e^{\frac{c}{4}(m-t)}\mathcal{H}(m)\mathrm{d}m}\end{aligned} \tag{30}$$

holds for all $t \geq s$.

Proof. (Summary) We consider a trajectory $x(t)$ of the closed-loop system satisfying the conditions of Theorem 2. Let $[t_0 - \bar{h}, t_\infty)$ be the largest domain of definition of x. By Lemma 2, $x(t)$ is defined over $[t_0 - \bar{h}, t_0 + 2\bar{h}]$, and (19) holds for all $t \in [t_0 - \bar{h}, t_0 + 2\bar{h}]$. Then necessarily, $t_\infty > t_0 + 2\bar{h}$. From the definition of $H = A + BK$, we deduce that

$$\dot{x}(t) = Hx(t) - BK\int_{t-h(t)}^t \dot{x}(m)\mathrm{d}m + \mathcal{F}(t, x_t) + \delta(t) \tag{31}$$

for all $t \in [t_0 + \bar{h}, t_\infty)$, since the integral in (31) is $x(t) - x(t - h(t))$. According to (14), the time derivative of W along (31) satisfies the following for all $t \in [t_0 + \bar{h}, t_\infty)$:

$$\begin{aligned}\dot{W}(t) \leq\ & -cW(x(t)) - 2x(t)^\top PBK\int_{t-h(t)}^t \dot{x}(m)\mathrm{d}m \\ & + 2x(t)^\top P\mathcal{F}(t, x_t) + 2x(t)^\top P\delta(t)\end{aligned} \tag{32}$$

Since $t_\infty > 2\bar{h}$, it follows that with the choice $\mathcal{H}(m) = |Ax(m) + BKx(m-h(m)) + \mathcal{F}(m, x_m) + \delta(m)|$, we have

$$\begin{aligned}\dot{W}(t) \leq & -cW(x(t)) + 2|x(t)||PBK|\int_{t-h(t)}^t \mathcal{H}(m)\mathrm{d}m \\ & +2x(t)^\top P\mathcal{F}(t,x_t) + 2x(t)^\top P\delta(t)\end{aligned} \quad (33)$$

for all $t \in [t_0+\bar{h}, t_\infty)$. Consequently, from our upper bound $\bar{h}$ on h, Assumption 2, and (14), we deduce that

$$\begin{aligned}\dot{W}(t) \leq & -cW(x(t)) + 2\bar{h}|PBK||A| \sup_{m\in[t-\bar{h},t]} W(x(m)) \\ & +2\bar{h}|PBK||BK| \sup_{m\in[t-2\bar{h},t]} W(x(m)) \\ & +2|PBK||x(t)|\int_{t-h(t)}^t \sup_{r\in[m-\bar{h},m]} |x(r)|^2\rho(|x(r)|)\mathrm{d}m \\ & +2|x(t)||P| \sup_{m\in[t-\bar{h},t]} |x(m)|^2\rho(|x(m)|) \\ & +2|PBK||x(t)|\int_{t-h(t)}^t |\delta(m)|\,\mathrm{d}m + 2|P||x(t)||\delta(t)|\end{aligned} \quad (34)$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. Since ρ is nondecreasing, we get

$$\begin{aligned}\dot{W}(t) \leq & \ 2\bar{h}|PBK|\left(|A| + |BK|\right) \sup_{m\in[t-2\bar{h},t]} W(x(m)) \\ & + 2\bar{h}|PBK|\rho\left(\sup_{m\in[t-2\bar{h},t]} \sqrt{W(x(m))}\right)\mathrm{d}m \\ & \times \sup_{m\in[t-2\bar{h},t]} W(x(m))^{3/2} + 2|P||x(t)||\delta(t)| \\ & +2|P| \sup_{m\in[t-2\bar{h},t]} W(x(m))^{3/2} \\ & \times\rho\left(\sup_{m\in[t-2\bar{h},t]} \sqrt{W(x(m))}\right) \\ & + 2|PBK||x(t)|\int_{t-h(t)}^t |\delta(m)|\,\mathrm{d}m - cW(x(t))\end{aligned} \quad (35)$$

for all $t \in [t_0 + \bar{h}, t_\infty)$. We next use the triangle inequality, Jensen's inequality, and (14). to get

$$\begin{aligned}& 2|PBK||x(t)|\int_{t-h(t)}^t |\delta(m)|\mathrm{d}m \\ & \leq \tfrac{c}{4}W(x(t)) + \tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^t |\delta(m)|^2\mathrm{d}m \\ & \text{and } 2|P||x(t)||\delta(t)| \leq \tfrac{c}{4}W(x(t)) + \tfrac{4}{c}|P|^2|\delta(t)|^2.\end{aligned} \quad (36)$$

It follows from (35) that

$$\begin{aligned}\dot{W}(t) \leq & -\tfrac{c}{2}W(x(t)) + \tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^t |\delta(m)|^2\,\mathrm{d}m \\ & + 2\bar{h}|PBK|\left(|A| + |BK|\right) \sup_{m\in[t-2\bar{h},t]} W(x(m)) \\ & +2\left(\bar{h}|PBK| + |P|\right) \sup_{m\in[t-2\bar{h},t]} W(x(m))^{3/2} \\ & \times\rho\left(\sup_{m\in[t-2\bar{h},t]} \sqrt{W(x(m))}\right) + \tfrac{4}{c}|P|^2|\delta(t)|^2\end{aligned} \quad (37)$$

holds for all $t \in [t_0 + \bar{h}, t_\infty)$. Therefore, we have

$$\begin{aligned}& \dot{W}(t) \leq -\tfrac{c}{2}W(x(t)) \\ & + \beta\left(\sup_{m\in[t-2\bar{h},t]} W(x(m))\right) \sup_{m\in[t-2\bar{h},t]} W(x(m)) + \delta_\sharp(t)\end{aligned} \quad (38)$$

for all $t \in [t_0 + \bar{h}, t_\infty)$, where β was defined in (24) and

$$\delta_\sharp(t) = \tfrac{4}{c}|PBK|^2\bar{h}\int_{t-h(t)}^t |\delta(m)|^2\,\mathrm{d}m + \tfrac{4}{c}|P|^2|\delta(t)|^2. \quad (39)$$

Note that (11) and (26) give

$$|\delta_\sharp(t)| < \tfrac{cw_\star}{4} \quad (40)$$

for all $t \geq \bar{h}$. Then let us recall that (19) holds, by (28). Consequently (28) ensures that

$$\sup_{m\in[t_0-\bar{h},t_0+2\bar{h}]} W(x(m)) < w_\star \quad (41)$$

We can now apply Theorem 1 with $\lambda = \gamma$, and with

$$\begin{aligned}& V(t) = W(x(t)),\ a = b = c/4,\ \alpha = \beta, \\ & \zeta(t) = \sup_{\ell\in[\bar{h},t]} |\delta_\sharp(\ell)|,\ \mathcal{L}_0 = 2\bar{h},\ v_\star = w_\star,\ \tau = t_\infty,\end{aligned} \quad (42)$$

and $t_\star = t_0 + \bar{h}$. Then (25) ensures that (2) is satisfied. Then (40)-(41) imply (3)-(4). Using Lemma 1, we can prove that the finite escape time phenomenon does not occur, so $t_\infty = +\infty$, and Theorem 1 gives

$$\begin{aligned}W(x(t)) \leq & \sup_{m\in[s-2\bar{h},s]} W(x(m))e^{-\gamma(t-s)} \\ & + \int_s^t e^{c(m-t)/4} \sup_{\ell\in[0,m]} |\delta_\sharp(\ell)|\mathrm{d}m\end{aligned} \quad (43)$$

when $t \geq s \geq t_0 + \bar{h}$ where γ is the constant defined in (27), and where the sup was needed in (42) and in the integrand in (43) because Theorem 1 requires its function ζ to be nondecreasing. Hence, (14) gives

$$\begin{aligned}|x(t)|^2 \leq & \ \overline{p} \sup_{m\in[s-2\bar{h},s]} |x(m)|^2 e^{-\gamma(t-s)} \\ & + \int_s^t e^{c(m-t)/4} \sup_{\ell\in[\bar{h},m]} |\delta_\sharp(\ell)|\mathrm{d}m\end{aligned} \quad (44)$$

when $t \geq s \geq t_0 + \bar{h}$. This allows us to conclude. □

4. ILLUSTRATION OF THEOREM 2

Consider the controlled van der Pol equation

$$\begin{cases}\dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + \epsilon(1-x_1^2)x_2 + u(t-h(t))\end{cases} \quad (45)$$

for constants $\epsilon > 0$ and a continuous delay $h(t)$; see, e.g., (Khalil, 2002, Section 13.2) for simpler cases with no delays. The dynamics represent oscillations in vacuum tube circuits, and provide a fundamental equation for nonlinear oscillation. The system has the form (12) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathcal{F}(t, x_t) = \begin{bmatrix} 0 \\ -\epsilon x_1^2(t)x_2(t) \end{bmatrix}, \quad (46)$$

and $\delta = 0$. Using the computer program Mathematica (2015), we can check that Assumption 1-2 hold with $\bar{\Delta} = 0$, $K = [-1.25, -2]$, $\rho(s) = \epsilon s$, and

$$P = \begin{bmatrix} 4.09112 & 0.722222 \\ 0.722222 & 1.17951 \end{bmatrix} \quad (47)$$

when $\epsilon = 0.01$, where P was found by first solving for a positive definite symmetric matrix $P_1 \in \mathbb{R}^{2\times 2}$ such that $P_1H + H^\top P_1 = -I$ holds with $H = A + BK$, then choosing $c = 0.75$ in order to satisfy $cP_1 \leq I$, and then multiplying P_1 by 3.25 to satisfy the requirement that $P \geq I$ with the choice $P = 3.25P_1$. Also, since $\bar{\Delta} = 0$, Assumptions 3 and 5 hold for any $s_* > 0$. We can then also use Mathematica to compute the basin of attraction from Theorem 2. For instance, when the delay h is the zero function, we can check that we can satisfy the requirements of Theorem 2 with $w_\star = s_* = 2.20049$ and all initial functions that are bounded by 0.718677. If we instead use the delay bound $\bar{h} = 0.008$ and keep all other values the same as before, then the basin of attraction consists of all initial functions whose norms are bounded by 0.137212. This illustrates the trade-off that increasing the bound $\bar{h}$ on the allowable input delays $h(t)$ can reduce the basin of attraction.

5. CONCLUSION

We provided a local version of Halanay's inequality to prove local asymptotic stability for nonlinear systems that contain state or input delays and uncertainties. Our new results are significant, because of the well-known benefits of using global versions of Halanay's inequality to prove global asymptotic stability for systems with unknown delays, and because many significant systems are only locally asymptotically stable and so are beyond the scope of global versions of Halanay's inequality. Another significant benefit of our work is that we allow the dynamics to contain unknown nonlinearities that violate the standard linear growth conditions and that can contain distributed delays. We illustrated how our methods provide new estimates for basins of attraction for a controlled van der Pol equation.

KEY LEMMAS

We first provide a key lemma from Malisoff and Mazenc (2024) that we used in our proof of Theorem 1; see Malisoff and Mazenc (2024) for the proof of the lemma. First let $t_\star \geq 0$ and $\mathcal{L}_0 \geq 0$ be given constants. Consider a C^1 function $V : [t_\star - \mathcal{L}_0, +\infty) \to [0, +\infty)$, a nonnegative valued nondecreasing continuous function ζ, and constants $a > 0$ and $b > 0$ such that

$$\dot{V}(t) \leq -(a+b)V(t) + a \sup_{m \in [t-\mathcal{L}_0, t]} V(m) + \zeta(t) \quad \text{(A.1)}$$

for all $t \geq t_\star$. Let $\lambda > 0$ be the constant in (7). We can then prove the following:

Lemma 3. The inequality

$$V(t) \leq \sup_{m \in [s-\mathcal{L}_0, s]} V(m) e^{-\lambda(t-s)} + \int_s^t e^{b(m-t)} \zeta(m) \mathrm{d}m \quad \text{(A.2)}$$

holds for all $t \geq s$ and all $s \geq t_\star$.

We used the next lemma in the proof of Lemma 2. We use constants $T > 0$, $q > 0$, $\Psi_0 > 0$, $\omega > 0$, $\tau > 0$, $\bar{\Delta} \geq 0$ and $t_a \geq 0$ and a continuous, nondecreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that

$$\Psi(\ell) \leq \Psi_0 \ell \quad \text{(A.3)}$$

for all $\ell \in [0, \omega]$. Let $Z : [t_a - T, t_a + \tau) \to [0, +\infty)$ be a nonnegative valued function of class C^1 such that

$$\dot{Z}(t) \leq \Psi\left(\sup_{\ell \in [t-T, t]} Z(\ell) \right) + \bar{\Delta} \quad \text{(A.4)}$$

for all $t \in [t_a, t_a + \tau)$. We use the following assumption:

Assumption A.1. The inequality

$$\left(e^{\Psi_0 q} - 1\right) \frac{\bar{\Delta}}{\Psi_0} < \omega \quad \text{(A.5)}$$

is satisfied.

In the following lemma, the existence of values $\bar{Z} > 0$ such that (A.7) is satisfied follows from (A.5):

Lemma 4. Let Assumption A.1 hold. Let Z be such that

$$Z(\ell) \leq \bar{Z} \text{ for all } \ell \in [t_a - T, t_a] \quad \text{(A.6)}$$

where $\bar{Z} \in \mathbb{R}$ is such that

$$\bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \frac{\bar{\Delta}}{\Psi_0} < \omega. \quad \text{(A.7)}$$

Then

$$Z(t) \leq \bar{Z} e^{\Psi_0 q} + \left(e^{\Psi_0 q} - 1\right) \frac{\bar{\Delta}}{\Psi_0} \quad \text{(A.8)}$$

holds for all $t \in [t_a - T, t_a + \min\{\tau, q\})$.

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