POSITROIDS, KNOTS, AND q,t-CATALAN NUMBERS

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Abstract

We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over $\mathbb{C}$ and point counts over $\mathbb{F}_q$) to Khovanov–Rozansky homology of associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational q,t-Catalan numbers. Via the curious Lefschetz property of cluster varieties, this implies the q,t-symmetry and unimodality properties of rational q,t-Catalan numbers. We show that the q,t-symmetry phenomenon is a manifestation of Koszul duality for category $\mathcal{O}$, and discuss relations with open Richardson varieties and extension groups of Verma modules.

Contents

Introduction . . . 2117
1. Main results . . . 2119
2. Point count and the HOMFLY polynomial . . . 2135
3. Soergel bimodules, Rouquier complexes, and Khovanov–Rozansky homology . . . 2138
4. Cohomology of positroid and Richardson varieties . . . 2150
5. Equivariant derived categories of flag varieties . . . 2164
6. Proof of Proposition 5.3 . . . 2168
7. Proof of Proposition 5.4 . . . 2171
8. Ordinary cohomology, Koszul duality, and Verma modules . . . 2180
9. Catalan numbers associated to positroid varieties . . . 2183
References . . . 2187

Introduction

The binomial coefficients $\binom{n}{k}$ have natural q-analogues $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_q$, known as *Gaussian polynomials*. On the other hand, the rational Catalan numbers $C_{k,n-k} := \frac{1}{n}\binom{n}{k}$

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(defined for $\gcd(k,n)=1$) have two different well-studied q-analogues: the area generating function $\sum_{P\in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}$ of rational Dyck paths (see [18]), and the polynomial $\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$ going back to [88].

The Poincaré polynomial of the complex Grassmannian $\mathrm{Gr}(k,n)$, and the number of points of $\mathrm{Gr}(k,n)$ over a finite field $\mathbb{F}_q$, are both well known to be given by $\begin{bmatrix} n \\ k \end{bmatrix}_q$. We give a Catalan analogue of this statement by considering the top-dimensional positroid variety $\Pi^\circ_{k,n} \subset \mathrm{Gr}(k,n)$, introduced in [77] building on the results of [103]. The space $\Pi^\circ_{k,n}$ is the subspace of $\mathrm{Gr}(k,n)$ where all cyclically consecutive Plücker coordinates are nonvanishing. We show that, up to a simple factor, the mixed Hodge polynomial $\mathcal{P}(\Pi^\circ_{k,n};q,t)$ coincides with the rational q,t-Catalan number $C_{k,n-k}(q,t)$ introduced in [86] in the study of Macdonald polynomials (see [50], [63]). It follows that the Poincaré polynomial of $\Pi^\circ_{k,n}$ equals $\sum_{P\in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}$, while the point count $\#\Pi^\circ_{k,n}(\mathbb{F}_q)$ equals $\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$, both up to a simple factor.

The coincidence of the Poincaré polynomial and the point count of $\mathrm{Gr}(k,n)$ is reflected in the purity of the mixed Hodge structure on the cohomology of $\mathrm{Gr}(k,n)$. Purity holds for many spaces of interest in combinatorics, for example, for complements of hyperplane arrangements. By contrast, the mixed Hodge structure on $H^\bullet(\Pi^\circ_{k,n})$ is not pure, and simultaneously yields both of the natural q-analogues of rational Catalan numbers discussed above.

Our proof proceeds via relating both sides to Khovanov–Rozansky knot homology (see [74]–[76]). Our main result connects the cohomology of arbitrary open positroid varieties, and more generally open Richardson varieties in generalized flag varieties, to knot homology.

Connections between knot invariants and Macdonald theory have received an enormous amount of attention in recent years (see, e.g., [22], [34], [60], [61], [64], [69], [92], [98]). In particular, Khovanov–Rozansky homology of torus knots and links was computed in [34], [68], [69], and [92]. For torus knots, the answer is given by the rational q,t-Catalan numbers.

Our main results are described in detail in the next section. We start by highlighting some consequences of our approach from several points of view.

Combinatorics

The coefficients of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are well known to form a unimodal palindromic sequence. A geometric explanation for this phenomenon is the hard Lefschetz theorem for the cohomology of $\mathrm{Gr}(k,n)$. It follows from the results of [46] and [83] that the cohomology of $\Pi^\circ_{k,n}$ satisfies the curious Lefschetz property which, combined with our main result, yields a geometric proof that $C_{k,n-k}(q,t)$ is q,t-symmetric and unimodal. Furthermore, our work produces a whole family of q,t-symmetric and unimodal polynomials, which includes $C_{k,n-k}(q,t)$ as a special case.

We discuss their $q = t = 1$ specialization in Section 9 where we obtain a new combinatorial interpretation for rational Catalan numbers in terms of certain kinds of pipe dreams (see Figure 5).

Knot theory

We introduce a class of *Richardson links*, which are closures of braids of the form $\beta(w) \cdot \beta(v)^{-1}$ for pairs of permutations $v, w \in S_n$ such that $v \leq w$ in the Bruhat order. We give a geometric interpretation of the top a-degree coefficient[1] of Khovanov–Rozansky (KR) homology and of the HOMFLY polynomial (see [42], [104]) for such links. When a Richardson link is a knot, we show that the associated q,t-polynomial is q,t-symmetric. Our investigations suggest that KR homology may have hitherto unstudied unimodality and Lefschetz-type properties. Our results generalize equally well to other Dynkin types.

Representation theory

We show that the q,t-symmetry property is a consequence of the Koszul duality phenomenon (see [11], [15]) in the derived category of the flag variety.

The computation of the extension groups $\mathrm{Ext}^\bullet(M_v, M_w)$ between Verma modules in the principal block $\mathcal{O}_0$ of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ (see, e.g., [70]) is a classical, still open problem. We show that these extension groups are isomorphic to knot-homology groups. Along the same vein, we show that the R-polynomials of Kazhdan and Lusztig [72], [73] are certain coefficients of the HOMFLY polynomial.

Algebraic geometry

Our results provide evidence for a $P = W$ conjecture relating the weight filtration of $\Pi^\circ_{k,n}$ with the perverse filtration of the compactified Jacobian $J_{k,n-k}$ (see Section 1.12.2).

1. Main results

We give a detailed description of our main results. The historical context and motivation for our work is delayed to Section 1.12. We give the full background on the objects below in the main body of the paper.

[1]The top a-degree coefficient encodes the zeroth Hochschild cohomology (1.17) which sometimes corresponds to the *bottom* a-degree in the literature. Our conventions are chosen so that the a-degree in KR homology matches the a-degree in the HOMFLY polynomial.

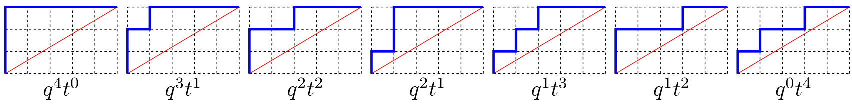


Figure 1. (Color online) Computing the rational q,t-Catalan number $C_{3,5}(q,t)$.

1.1. *Rational q,t-Catalan numbers*

Let a and b be coprime positive integers. The *rational q,t-Catalan number* $C_{a,b}(q,t) \in \mathbb{N}[q,t]$ was introduced by Loehr and Warrington [86] (see also [58], [59]), generalizing the work of Garsia and Haiman [50]. It is defined as

$$C_{a,b}(q,t) := \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}, \tag{1.1}$$

where $\mathrm{Dyck}_{a,b}$ is the set of lattice paths P inside a rectangle of height a and width b that stay above the diagonal, $\mathrm{area}(P)$ is the number of unit squares fully contained between P and the diagonal, and $\mathrm{dinv}(P)$ is the number of pairs (h,v) satisfying the following conditions: h is a horizontal step of P, v is a vertical step of P that appears to the right of h, and there exists a line of slope a/b (parallel to the diagonal) intersecting both h and v. For example,

$$C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + q^2t + qt^3 + qt^2 + t^4, \tag{1.2}$$

as shown in Figure 1.

1.2. *Positroid varieties in the Grassmannian*

The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of k-dimensional linear subspaces of $\mathbb{C}^n$. Building on Postnikov's cell decomposition in [103] of its totally nonnegative part, Knutson, Lam, and Speyer [77] constructed a stratification

$$\mathrm{Gr}(k,n) = \bigsqcup_{f \in \mathbf{B}_{k,n}} \Pi^\circ_f, \tag{1.3}$$

where the *(open) positroid varieties* Π°_f are defined as the nonempty intersections of cyclic rotations of n Schubert cells. These varieties also arise in Poisson geometry (see [16]) and in the study of scattering amplitudes (see [5]). Open positroid varieties are indexed by a finite set $\mathbf{B}_{k,n}$ of *bounded affine permutations*, and for $f \in \mathbf{B}_{k,n}$ the reduction of f modulo n is a permutation $\bar{f} \in S_n$. (See Section 4.1 for further background.)

Let $f_{k,n} \in \mathbf{B}_{k,n}$ be the bounded affine permutation given by $f_{k,n}(i) = i + k$. The positroid stratification (1.3) contains a unique open stratum, the *top-dimensional* *positroid variety* $\Pi^\circ_{k,n} := \Pi^\circ_{f_{k,n}}$, which can be described explicitly as

$$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{1,2,\dots,k}(V), \Delta_{2,3,\dots,k+1}(V), \dots, \Delta_{n,1,\dots,k-1}(V) \neq 0\}, \tag{1.4}$$

consisting of subspaces whose cyclically consecutive Plücker coordinates are nonvanishing.

For each $f \in \mathbf{B}_{k,n}$, the space Π°_f is a smooth algebraic variety. Two basic questions one can ask are:

(1) What is the number of points in $\Pi^\circ_f(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ with q elements?

(2) What are the Betti numbers of Π°_f considered as a complex manifold?

These two questions are related by the *mixed Hodge structure* (see [29]) on cohomology. The cohomology ring $H^\bullet(\Pi^\circ_f) = H^\bullet(\Pi^\circ_f, \mathbb{C})$ of an open positroid variety is of Hodge–Tate type, and we have a *Deligne splitting*

$$H^k(\Pi^\circ_f, \mathbb{C}) \cong \bigoplus_{p\in\mathbb{Z}} H^{k,(p,p)}(\Pi^\circ_f, \mathbb{C}). \tag{1.5}$$

Since Π°_f is smooth, we have that $H^{k,(p,p)}$ vanishes unless $k/2 \le p \le k$. We view (1.5) as a bigrading on $H^\bullet(\Pi^\circ_f)$ and let $\mathcal{P}(\Pi^\circ_f; q,t)$ be the suitably renormalized (see (4.6)) Poincaré polynomial of this bigraded vector space, called the *mixed Hodge polynomial*.

We are ready to state the most important special case of our main result.

THEOREM 1.1
Assume that $\gcd(k,n) = 1$. *Then*

$$\mathcal{P}(\Pi^\circ_{k,n}; q,t) = (q^{\frac{1}{2}} + t^{\frac{1}{2}})^{n-1} C_{k,n-k}(q,t). \tag{1.6}$$

The equality (1.6) arises as a conjecture from the works [114] and [115], and we thank Vivek Shende for drawing our attention to the conjecture (see Section 1.12.2 for further discussion). We generalize Theorem 1.1 to all positroid varieties in Theorem 1.17 below.

Let us discuss the specializations of Theorem 1.1 that give answers to questions (1) and (2) above. Denote

$$[n]_q := 1 + q + \cdots + q^{n-1}, \qquad [n]_q! := [1]_q[2]_q \cdots [n]_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

COROLLARY 1.2

The Poincaré polynomial and point count of $\Pi^\circ_{k,n}$ are

$$\mathcal{P}(\Pi^\circ_{k,n};q) = (q+1)^{n-1} \cdot C_{k,n-k}(q^2,1), \tag{1.7}$$

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{1.8}$$

Our proof of Theorem 1.1 involves a number of ingredients, including Khovanov–Rozansky knot homology and derived categories of flag varieties. The point count specialization (1.8) requires less advanced machinery and we give a quicker elementary proof in Section 2. Associating a *link* $\hat{\beta}_f$ to each positroid variety Π°_f (Section 1.5), we compare the point count $\#\Pi^\circ_f(\mathbb{F}_q)$ to the HOMFLY polynomial of $\hat{\beta}_f$ (Section 1.6). The HOMFLY polynomial is categorified by Khovanov–Rozansky knot homology, and our proof of Theorem 1.1 may be considered a "categorification" of the point count computation.

We have the following elegant but baffling corollary.

COROLLARY 1.3

Let $\gcd(k,n) = 1$. *Then the probability that a uniformly random k-dimensional subspace of* $(\mathbb{F}_q)^n$ *belongs to* $\Pi^\circ_{k,n}(\mathbb{F}_q)$ *is given by*

$$\operatorname{Prob}\big(V \in \Pi^\circ_{k,n}(\mathbb{F}_q)\big) = \frac{(q-1)^n}{q^n-1}.$$

The probability $\frac{(q-1)^n}{q^n-1}$ does not depend on k. We do not have a direct explanation for this phenomenon.

1.3. Cluster structure and the curious Lefschetz theorem

Since the work of Scott [111], positroid varieties have been expected to admit a natural *cluster algebra* (see [40]) structure arising from Postnikov diagrams. We recently proved this conjecture building on the results of [85], [97], and [112].

THEOREM 1.4 ([46])

The coordinate ring of each positroid variety Π°_f is isomorphic to the associated cluster algebra.

This result allows one to study Π°_f as a *cluster variety*, and for such spaces the mixed Hodge structure can be explored using the machinery developed by Lam and Speyer [83], whose work implies the following properties of the mixed Hodge polynomials $\mathcal{P}(\Pi^\circ_f;q,t)$.

THEOREM 1.5 ([46], [83])
For each $f \in \mathbf{B}_{k,n}$, *the* mixed Hodge polynomial $\mathcal{P}(\Pi_f^\circ; q,t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ *has the following properties:*

(i) q,t*-symmetry:* $\mathcal{P}(\Pi_f^\circ; q,t) = \mathcal{P}(\Pi_f^\circ; t,q)$*;*

(ii) q,t*-unimodality: for each* d, *the coefficients of* $\mathcal{P}(\Pi_f^\circ; q,t)$ *at* $q^d t^0$, $q^{d-1}t^1$, $\dots$, $q^0 t^d$ *form a unimodal sequence;*

(iii) $\mathcal{P}(\Pi_f^\circ; 1, q^2)$ *equals the Poincaré polynomial of* Π_f° *(considered as a variety over* $\mathbb{C}$*);*

(iv) $q^{\frac{1}{2}\dim \Pi_f^\circ} \cdot \mathcal{P}(\Pi_f^\circ; q,t)|_{t^{\frac{1}{2}} = -q^{-\frac{1}{2}}}$ *equals the point count* $\#\Pi_f^\circ(\mathbb{F}_q)$*.*

See Example 1.9 below.

Parts (i) and (ii) are consequences of the *curious Lefschetz property*, formalized in [67] and proved to hold for certain cluster varieties in [83] (see Section 4.3).

1.4. The Catalan variety

Let $T \cong (\mathbb{C}^*)^{n-1}$ be the group of diagonal matrices in $\mathrm{PGL}_n(\mathbb{C})$: it is the quotient of the group of diagonal $n \times n$ matrices by the group of scalar matrices. The group T acts on $\mathrm{Gr}(k,n)$ preserving the positroid stratification. For $u \in S_n$, let

$$c(u) := \text{ the number of cycles of } u, \tag{1.9}$$

and let $\mathbf{B}_{k,n}^{c=1} := \{f \in \mathbf{B}_{k,n} \mid c(\bar{f}) = 1\}$. The following observation is proved in Section 4.2.

PROPOSITION 1.6
The action of T *on* Π_f° *is free if and only if the permutation* $\bar{f}$ *is a single cycle.*

For $f \in \mathbf{B}_{k,n}^{c=1}$, the quotient $\mathcal{X}_f^\circ := \Pi_f^\circ / T$ is again a smooth affine variety that we call a *positroid configuration space* (see also [6]). It is a cluster variety (with no frozen variables, since the T-action on the frozen variables of Π_f° is free), and thus Theorem 1.5 applies to it.

PROPOSITION 1.7
For $f \in \mathbf{B}_{k,n}^{c=1}$, *the mixed Hodge polynomials of* Π_f° *and* $\mathcal{X}_f^\circ$ *are related by:*

$$\mathcal{P}(\Pi_f^\circ; q,t) = (q^{\frac{1}{2}} + t^{\frac{1}{2}})^{n-1} \cdot \mathcal{P}(\mathcal{X}_f^\circ; q,t). \tag{1.10}$$

When $\gcd(k,n) = 1$, we have $f_{k,n} \in \mathbf{B}_{k,n}^{c=1}$. The quotient $\mathcal{X}_{k,n}^\circ := \Pi_{k,n}^\circ / T$ satisfies

$$\mathcal{P}(\mathcal{X}^\circ_{k,n};q,t) = C_{k,n-k}(q,t), \tag{1.11}$$

and we refer to $\mathcal{X}^\circ_{k,n}$ as *the Catalan variety*. Let $d_{k,n} := (k-1)(n-k-1) = \dim(\mathcal{X}^\circ_{k,n})$. We obtain the following as a consequence of Theorem 1.5.

COROLLARY 1.8
Assume that $\gcd(k,n)=1$. *We have:*
(i) *q,t-symmetry: $C_{k,n-k}(q,t) = C_{k,n-k}(t,q)$;*
(ii) *q,t-unimodality: for each d, the coefficients of $C_{k,n-k}(q,t)$ at $q^d t^0$, $q^{d-1}t^1$, $\dots$, $q^0 t^d$ form a unimodal sequence;*
(iii) *the Poincaré polynomial of $\mathcal{X}^\circ_{k,n}$ is given by*

$$\sum_d q^{\frac{d}{2}} \dim H^{d_{k,n}-d}(\mathcal{X}^\circ_{k,n}) = C_{k,n-k}(q,1)$$
$$= \sum_{P\in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}; \tag{1.12}$$

(iv) *the number of $\mathbb{F}_q$-points of $\mathcal{X}^\circ_{k,n}$ is given by*

$$\#\mathcal{X}^\circ_{k,n}(\mathbb{F}_q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{\frac{1}{2}d_{k,n}} \cdot C_{k,n-k}(q,1/q). \tag{1.13}$$

While part (i) is known, the remaining parts of Corollary 1.8 appear to be new (see Section 1.12.1). Note also that the odd Betti numbers of $\mathcal{X}^\circ_{k,n}$ vanish, a phenomenon that we do not have a direct explanation for. Parts (iii)–(iv) may be deduced directly from Corollary 1.2 using Proposition 1.6.

Example 1.9
Let $k=3$ and $n=8$. The coordinate ring of $\mathcal{X}^\circ_{3,8}$ is a cluster algebra of type E_8 (with no frozen variables). The associated *mixed Hodge table* is given in Table 1 (see [83, Table 5]). The grading conventions (4.6) are chosen so that the first row contributes $q^4+q^3t+q^2t^2+qt^3+t^4$ while the second row contributes q^2t+qt^2 to $\mathcal{P}(\mathcal{X}^\circ_{3,8};q,t)$. Note that all odd cohomology groups vanish, which is why all

Table 1. The mixed Hodge table recording the dimensions of $H^{k,(p,p)}(\mathcal{X}^\circ_{3,8})$ for the cluster algebra of type E_8 (see [83, Table 5]). The dimensions agree with the coefficients of $C_{3,5}(q,t)$ (see Example 1.9).

H^k	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	0	1		

monomials have integer powers of q and t. Comparing the result with (1.2), we find $\mathcal{P}(\mathcal{X}^{\circ}_{3,8}; q, t) = C_{3,5}(q,t)$, in agreement with Theorem 1.1.

The polynomial $C_{3,5}(q,t)$ given in (1.2) is indeed q,t-symmetric. It is also q,t-unimodal: fixing the total degree of q and t, it splits into polynomials $q^4 + q^3t + q^2t^2 + qt^3 + t^4$ and $q^2t + qt^2$, both of which have unimodal coefficient sequences, corresponding to the rows of Table 1. We also have $C_{3,5}(q,1) = q^4 + q^3 + 2q^2 + 2q + 1$; the coefficient of $q^{d/2}$ is equal to $\dim H^{d_{k,n}-d}(\Pi^{\circ}_{3,8})$ for each d (these coefficients are column sums in Table 1). This agrees with (1.12).

1.5. Links associated to positroid varieties

Let us say that a permutation $w \in S_n$ is k-*Grassmannian* if $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k)$ and $w^{-1}(k+1) < \cdots < w^{-1}(n)$. We denote by "$\leq$" the (strong) Bruhat order on S_n. Let $\mathbf{Q}_{k,n}$ denote the set of pairs (v,w) of permutations such that $v \leq w$ and w is k-Grassmannian. The following result is well known (see Proposition 4.2).

PROPOSITION 1.10 ([77])
There exists a bijection $(v,w) \mapsto f_{v,w}$ between $\mathbf{Q}_{k,n}$ and $\mathbf{B}_{k,n}$ such that for every $f = f_{v,w} \in \mathbf{B}_{k,n}$, we have $\bar{f} = wv^{-1}$.

For example, when $f = f_{k,n}$, we have $v = \mathrm{id}$ and the permutation $w = \bar{f}$ sends $i \mapsto i + k$ modulo n for all $i \in [n]$. The dimension of Π°_f equals $\ell_{v,w} := \ell(w) - \ell(v)$, where $\ell(u)$ is the number of inversions of $u \in S_n$.

The group S_n is generated by simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Similarly, let $\mathcal{B}_n$ be the *braid group* on n strands, generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$. Connecting the corresponding endpoints of a braid β gives rise to a *link* called the *closure* $\hat{\beta}$ of β (see Figure 2).

For each element $u \in S_n$, let $\beta(u)$ denote the corresponding braid, obtained by choosing a reduced word $u = s_{i_1}s_{i_2}\cdots s_{i_{\ell(u)}}$ for u and then replacing each s_i with σ_i.

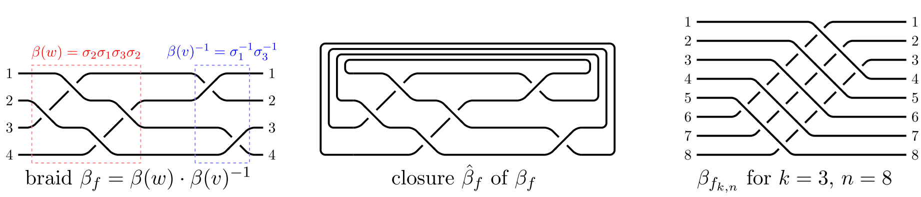


Figure 2. (Color online) Braids and links associated to positroid varieties.

Definition 1.11
For $f = f_{v,w} \in \mathbf{B}_{k,n}$, we set

$$\beta_f := \beta(w) \cdot \beta(v)^{-1}. \tag{1.14}$$

We refer to the closure $\hat{\beta}_f$ as *the link associated to* f. See Figure 2 for an example.

The link $\hat{\beta}_f$ is a *knot* (i.e., has one connected component) if and only if $f \in \mathbf{B}_{k,n}^{c=1}$ (see Proposition 1.6).

1.6. HOMFLY polynomial

The *HOMFLY polynomial* $P(L) = P(L;a,z)$ of an (oriented) link L is defined by the skein relation

$$aP(L_+) - a^{-1}P(L_-) = zP(L_0) \qquad \text{and} \qquad P(\bigcirc) = 1, \tag{1.15}$$

where $\bigcirc$ denotes the unknot and L_+, L_-, L_0 are three links whose planar diagrams locally differ as follows:

L_+ L_- L_0

Example 1.12
For $n = 2$, we may take L_+ to be the closure of σ_1, in which case L_- is the closure of σ_1^{-1} and $L_0 = \bigcirc\bigcirc$ is the 2-component unlink. Applying (1.15), we find $P(L_0) = \frac{a-a^{-1}}{z}$.

Surprisingly, the HOMFLY polynomial computes the number of $\mathbb{F}_q$-points of *any* positroid variety.

THEOREM 1.13
For all $f \in \mathbf{B}_{k,n}$, let $P_f^{\mathrm{top}}(q)$ be obtained from the top a-degree term of $P(\hat{\beta}_f;a,z)$ by substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Then

$$\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot P_f^{\mathrm{top}}(q). \tag{1.16}$$

Remark 1.14
When $\gcd(k,n) = 1$, we have $f_{k,n} \in \mathbf{B}_{k,n}^{c=1}$, and the associated knot $\hat{\beta}_{f_{k,n}}$ is the $(k, n-k)$*-torus knot* (see Figure 2 (right)). The value of $P(\hat{\beta}_{f_{k,n}};a,z)$ was computed in [71], and its relationship with Catalan numbers was clarified in [54]. Thus, (1.8) follows from Theorem 1.13 as a direct corollary.

Example 1.15
For $k=3$, $n=8$, one calculates (e.g., using Sage[2]) that

$$P(\hat{\beta}_{f_{k,n}};a,z)=\frac{z^8+8z^6+21z^4+21z^2+7}{a^8}-\frac{z^6+7z^4+14z^2+8}{a^{10}}+\frac{z^2+2}{a^{12}}.$$

Substituting $a:=q^{-\frac{1}{2}}$ and $z:=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$ into $\frac{z^8+8z^6+21z^4+21z^2+7}{a^8}$, we get

$$P_f^{\mathrm{top}}(q)=q^8+q^6+q^5+q^4+q^3+q^2+1=q^4\cdot C_{3,5}(q,1/q).$$

This agrees with (1.13) and (1.16).

1.7. Richardson varieties

Let G be a complex semisimple algebraic group of adjoint type, and choose a pair $B,B_-\subset G$ of opposite Borel subgroups. Let $T:=B\cap B_-$ be the maximal torus, and let $W:=N_G(T)/T$ be the associated Weyl group. We have Bruhat decompositions $G=\bigsqcup_{w\in W}BwB=\bigsqcup_{v\in W}B_-vB$, and the intersection $BwB\cap B_-vB$ is nonempty if and only if $v\le w$ in the Bruhat order on W. For $v\le w$, we denote by $R^\circ_{v,w}:=(BwB\cap B_-vB)/B$ an *open Richardson variety* inside the *(generalized) complete flag variety* G/B. The varieties $R^\circ_{v,w}$ form a stratification of G/B.

Now suppose that $G=\mathrm{PGL}_n(\mathbb{C})$. We have $W=S_n$, the subgroups $B,B_-\subset G$ consist of upper and lower triangular matrices, and $T\cong(\mathbb{C}^*)^{n-1}$ is the group of diagonal matrices modulo scalar matrices. In this case, we denote the generalized flag variety G/B by $\mathrm{Fl}(n)$. By Proposition 1.10, positroid varieties correspond to pairs $v\le w$ of permutations such that w is k-Grassmannian. The projection map $\mathrm{Fl}(n)\to\mathrm{Gr}(k,n)$ restricts to an isomorphism $R^\circ_{v,w}\cong\Pi^\circ_f$ for every permutation $f=f_{v,w}\in\mathbf{B}_{k,n}$ (see Proposition 4.3). Thus, positroid varieties are special cases of Richardson varieties. One can similarly associate a braid $\beta_{v,w}:=\beta(w)\cdot\beta(v)^{-1}$ to any pair $v\le w$ in S_n and consider its closure $\hat{\beta}_{v,w}$. We refer to links of the form $\hat{\beta}_{v,w}$ as *Richardson links*.

The point count $\#R^\circ_{v,w}(\mathbb{F}_q)$ is given by the *Kazhdan–Lusztig R-polynomial* (see [72], [73]), and both the statement and the proof of Theorem 1.13 generalize to this setting (for G of arbitrary type); see Theorems 2.1 and 2.3.

1.8. Main result: Ordinary cohomology

Our results for the positroid variety $\Pi^\circ_{k,n}$ are special cases of a statement which applies to open Richardson varieties in arbitrary Dynkin type. This includes all positroid varieties Π°_f for $f\in\mathbf{B}_{k,n}$, where the number of cycles $c(\bar{f})$ can be arbitrary. We start with the nonequivariant version of our result.

[2] https://doc.sagemath.org/html/en/reference/knots/sage/knots/link.html.

Let $\mathfrak{h} := \mathrm{Lie}(T)$ be the Cartan subalgebra of $\mathrm{Lie}(G)$ corresponding to T, and denote $R := \mathbb{C}[\mathfrak{h}] = \mathrm{Sym}_{\mathbb{C}}\mathfrak{h}^*$. The ring R is graded so that the elements of $\mathfrak{h}^* \subset R$ have polynomial degree 2. For $G = \mathrm{PGL}_n(\mathbb{C})$, $R = \mathbb{C}[y_1, \dots, y_{n-1}]$ is the polynomial ring. Since W is a Coxeter group, we can consider the category $\mathbb{S}\mathrm{Bim}$ of *Soergel bimodules* (see [35], [118]). Each object $B \in \mathbb{S}\mathrm{Bim}$ is a graded R-bimodule, and we will be interested in its R-invariants, which by definition form the *zeroth Hochschild cohomology of* B:

$$HH^0(B) := \{b \in B \mid rb = br \text{ for all } r \in R\}. \tag{1.17}$$

Thus, $HH^0(B)$ is a graded R-module. Denote

$$HH^0_{\mathbb{C}}(B) := HH^0(B) \otimes_R \mathbb{C}, \tag{1.18}$$

where $\mathbb{C} = R/(\mathfrak{h}^*)$ is the R-module on which $\mathfrak{h}^*$ acts by zero. While the functor HH^0 involves Soergel bimodules, the functor $HH^0_{\mathbb{C}}$ involves *Soergel modules* instead (see Corollary 3.6).

To any element $u \in W$, Rouquier [110] associates two cochain complexes $F^\bullet(u)$ and $F^\bullet(u)^{-1}$ of Soergel bimodules. For a braid $\beta_{v,w} = \beta(w) \cdot \beta(v)^{-1}$, we set $F^\bullet_{v,w} := F^\bullet(w) \otimes_R F^\bullet(v)^{-1}$. Applying the functor $HH^0_{\mathbb{C}}$ to each term of this complex yields a complex $HH^0_{\mathbb{C}}(F^\bullet_{v,w})$ of graded R-modules. Taking its cohomology

$$HHH^0_{\mathbb{C}}(F^\bullet_{v,w}) := H^\bullet\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big), \tag{1.19}$$

we get a bigraded vector space. Explicitly, we have

$$HHH^0_{\mathbb{C}}(F^\bullet_{v,w}) = \bigoplus_{k,p \in \mathbb{Z}} H^{k,(p)}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big),$$

where $H^{k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$ is the polynomial degree-$2p$ part of $H^k(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$.

Recall from (1.5) that we have a bigrading on $H^\bullet(R^\circ_{v,w})$ coming from the Deligne splitting.

THEOREM 1.16
For all $v \le w \in W$ and $k, p \in \mathbb{Z}$, we have

$$H^{k,(p,p)}(R^\circ_{v,w}) \cong H^{-k,(p)}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big). \tag{1.20}$$

See Tables 2 and 3 for examples.

1.9. Main result: Equivariant cohomology

The spaces $HHH^0(F^\bullet_{v,w})$ and $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ are closely related. By Theorem 1.16, $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ yields the cohomology of $R^\circ_{v,w}$. It turns out that $HHH^0(F^\bullet_{v,w})$ yields the *torus-equivariant* cohomology of $R^\circ_{v,w}$.

Table 2. Summary of examples computed in Sections 3.6 and 4.6, illustrating Theorems 1.16 and 1.17. The last four columns list all values of k, p for which the corresponding bigraded component is nonzero (in each case, it is 1-dimensional). We have $f_{2,4} = s_2 s_1 s_3 s_2$ and $f_{2,5} = s_3 s_2 s_1 s_4 s_3 s_2$.

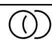
Title	n	v	w	$\ell_{v,w}$	$\hat{\beta}_{v,w}$	$R^\circ_{v,w}$	$H^{k,(p,p)}(R^\circ_{v,w})$	$H^{k,(p,p)}_{T,c}(R^\circ_{v,w})$	$H^{k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$	$H^{k,(p)}(HH^0(F^\bullet_{v,w}))$
Unknot-I	1	id	id	0		pt	$k=0,\ p=0$	$k=0,\ p=0$	$k=0,\ p=0$	$k=0,\ p=0$
Unknot-II	2	id	s_1	1		$\Pi^\circ_{1,2}$	$k=0,\ p=0$ $k=1,\ p=1$	$k=1,\ p=0$	$k=0,\ p=0$ $k=-1,\ p=1$	$k=0,\ p=0$
2-cpt. unlink	2	s_1	s_1	0		pt	$k=0,\ p=0$	$k=2p,\ p\in\mathbb{Z}_{\geq 0}$	$k=0,\ p=0$	$k=0,\ p\in\mathbb{Z}_{\geq 0}$
Hopf link	4	id	$f_{2,4}$	4		$\Pi^\circ_{2,4}$	Table 3	$k=4,\ p=0$ $k=2+2p,\ p\in\mathbb{Z}_{\geq 2}$	Table 3	$k=0,\ p=0$ $k=-2,\ p\in\mathbb{Z}_{\geq 2}$
Trefoil knot	5	id	$f_{2,5}$	6		$\Pi^\circ_{2,5}$	Table 3	$k=6,\ p=0$ $k=8,\ p=2$	Table 3	$k=0,\ p=0$ $k=-2,\ p=2$

Table 3. Comparing the mixed Hodge tables of $\Pi^\circ_{2,4}$ and $\Pi^\circ_{2,5}$ (top) to $HHH^0_{\mathbb{C}}$ of the associated links (bottom).

H^k	H^0	H^1	H^2	H^3	H^4
$k-p=0$	1	3	4	3	1

$H^{k,(p,p)}(\Pi^\circ_{2,4})$

H^k	H^0	H^1	H^2	H^3	H^4	H^5	H^6
$k-p=0$	1	4	7	8	7	4	1

$H^{k,(p,p)}(\Pi^\circ_{2,5})$

H^k	H^{-4}	H^{-3}	H^{-2}	H^{-1}	H^0
$k+p=0$	1	3	4	3	1

$H^{k,(p)}HH^0_{\mathbb{C}}(F^\bullet(\beta_{f_{2,4}}))$

H^k	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	H^0
$k+p=0$	1	4	7	8	7	4	1

$H^{k,(p)}HH^0_{\mathbb{C}}(F^\bullet(\beta_{f_{2,5}}))$

The algebraic torus T acts on each Richardson variety $R^\circ_{v,w}$, and thus we can consider its T*-equivariant cohomology with compact support*, denoted $H^\bullet_{T,c}(R^\circ_{v,w})$. It is equipped with an action of the ring $H^\bullet_T(\mathrm{pt}) \cong R$. Similarly to (1.5), $H^\bullet_{T,c}(R^\circ_{v,w})$ admits a second grading via the mixed Hodge structure and is therefore a bigraded R-module.

THEOREM 1.17
For all $v \leq w \in W$, we have an isomorphism of bigraded R-modules

$$H^\bullet_{T,c}(R^\circ_{v,w}) \cong HHH^0(F^\bullet_{v,w}). \tag{1.21}$$

For each $k, p \in \mathbb{Z}$, it restricts to a vector space isomorphism

$$H^{\ell_{v,w}+2p+k,(p,p)}_{T,c}(R^\circ_{v,w}) \cong H^{k,(p)}\big(HH^0(F^\bullet_{v,w})\big), \tag{1.22}$$

where $\ell_{v,w} = \ell(w) - \ell(v) = \dim R^\circ_{v,w}$.

See Table 2 for examples. We explain how Theorems 1.1 and 1.16 follow from Theorem 1.17 in Sections 4.4 and 8.1, respectively.

Observe that the grading conventions in (1.20) and (1.22) are quite different. In fact, the two statements are related by an application of the q,t-symmetry (1.23), as we now explain.

1.10. Koszul duality and q,t-symmetry

For any $f \in \mathbf{B}_{k,n}^{c=1}$, the positroid variety $\mathcal{X}_f^\circ$ is a cluster variety (see [46]), so the polynomial $\mathcal{P}(\Pi_f^\circ; q,t)$ is q,t-symmetric by Corollary 1.8(i). Richardson varieties are not yet known to admit cluster structures (see [85]), and in particular, the curious Lefschetz theorem of [83] cannot yet be applied to conclude that $\mathcal{P}(R_{v,w}^\circ; q,t)$ is q,t-symmetric for arbitrary $v \leq w \in S_n$. In Section 8.2, we show that the q,t-symmetry phenomenon for positroid and Richardson varieties is a manifestation of *Koszul duality* for the derived category of Schubert-constructible sheaves on the flag variety (see [4], [11], [15]).

THEOREM 1.18

For all $v \leq w \in W$ and $k, p \in \mathbb{Z}$, we have an isomorphism

$$H^{k,(p,p)}(R_{v,w}^\circ, \mathbb{C}) \cong H^{\ell_{v,w}+k-2p,(\ell_{v,w}-p,\ell_{v,w}-p)}(R_{v,w}^\circ, \mathbb{C}) \tag{1.23}$$

of vector spaces. In other words, the polynomial $\mathcal{P}(R_{v,w}^\circ; q,t)$ is q,t-symmetric.

This gives a new proof of the q,t-symmetry of $C_{k,n-k}(q,t)$ for $\gcd(k,n) = 1$.

We now explain the connection to link invariants. Given a Richardson link $\hat{\beta}_{v,w}$, one can consider the bigraded vector spaces $HHH^0(F^\bullet(\beta_{v,w}))$ and $HHH_{\mathbb{C}}^0(F^\bullet(\beta_{v,w}))$, and their suitably renormalized bigraded Hilbert series, denoted $\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta_{v,w}; q,t)$ and $\mathcal{P}_{\mathrm{KR};\mathbb{C}}^{\mathrm{top}}(\beta_{v,w}; q,t)$, respectively (see (3.15)–(3.16)).

The polynomial $\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta_{v,w}; q,t)$ is the top a-degree coefficient (see the footnote in the introduction) of the celebrated *Khovanov–Rozansky homology* (see [74]–[76]) of $\hat{\beta}_{v,w}$, which is a link invariant, that is, depends only on the closure $\hat{\beta}_{v,w}$ of $\beta_{v,w}$ (see Section 3.5).

Our results imply (see Section 8.1) that when $\hat{\beta}_{v,w}$ is a knot, we have

$$\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\hat{\beta}_{v,w}; q,t) = \mathcal{P}_{\mathrm{KR};\mathbb{C}}^{\mathrm{top}}(\beta_{v,w}; q,t) \qquad \text{and} \tag{1.24}$$

$$\mathcal{P}(R_{v,w}^\circ/T; q,t) = (q^{\frac{1}{2}} t^{\frac{1}{2}})^{\chi(\beta_{v,w})} \mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\hat{\beta}_{v,w}; q,t), \tag{1.25}$$

where $\chi(\beta_{v,w}) := \frac{\ell_{v,w}-n+c(f)}{2}$ (see (3.12)). More generally, Theorem 1.16 implies that for any $v \leq w \in S_n$, one can relate $\mathcal{P}_{\mathrm{KR};\mathbb{C}}^{\mathrm{top}}(\beta_{v,w}; q,t)$ to $\mathcal{P}(R_{v,w}^\circ; q,t)$ by a simple transformation. As we show in Corollary 3.13, (1.24) holds more generally for arbitrary knots $\hat{\beta}$.

For a general link $\hat{\beta}$, the question of whether $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta};q,t)$ is q,t-symmetric has been a major open problem[3] going back to [33]. For Richardson links, we show that it also amounts to applying Koszul duality.

COROLLARY 1.19
For any $v \le w \in S_n$, we have

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w};q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w};t,q).$$

Consequently, by (1.24), if $\hat{\beta}_{v,w}$ is a knot, then

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};t,q).$$

1.11. Extensions of Verma modules

The above theorems have representation-theoretic applications to *Verma modules*, which are certain infinite-dimensional modules over the Lie algebra $\mathfrak{g}$ of G. Consider the principal block $\mathcal{O}_0$ of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$, and let M_w be the Verma module with highest weight $w(\rho)-\rho$, where ρ is half the sum of positive roots of the root system of $\mathfrak{g}$. We also denote by L_w the corresponding simple module. The graded dimensions of $\mathrm{Ext}^\bullet(M_v, L_w)$ famously coincide with the coefficients of the *Kazhdan–Lusztig P-polynomials* $P_{v,w}(q)$ (see, e.g., [11, Theorem 3.11.4]). On the other hand, computing extension groups $\mathrm{Ext}^\bullet(M_v, M_w)$ is an important open problem (see, e.g., [32], [90]).

A *graded* version of $\mathcal{O}_0$ was introduced by Beilinson, Ginzburg, and Soergel [11]. They constructed the (essentially unique) graded lifts of Verma modules M_w (see also [124]), thus endowing the space $\mathrm{Ext}^\bullet(M_v, M_w)$ with a second grading:

$$\mathrm{Ext}^\bullet(M_v, M_w) = \bigoplus_{k,r\in\mathbb{Z}} \mathrm{Ext}^{k,(r/2)}(M_v, M_w).$$

These Ext-groups can be related to the cohomology of open Richardson varieties using the *localization theorem* of [9] and [17]. In the case of Kazhdan–Lusztig polynomials, the groups $\mathrm{Ext}^\bullet(M_v, L_w)$ are "pure": the two gradings agree. On the other hand, the bigrading on $\mathrm{Ext}^\bullet(M_v, M_w)$ turns out to be quite nontrivial. As a corollary to our approach, we obtain the following result.[4]

[3] At the final stages of the preparation of this manuscript, we learned that the q,t-symmetry of $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta};q,t)$ for the case when $\hat{\beta}$ is a knot has been established in a very recent preprint [99]. (Note added in 2023: see also [56].).

[4] We remark that Soergel's original work [117] directly relates Ext-groups in category $\mathcal{O}$ and Hom-groups in $\mathbb{S}\mathrm{Bim}$ (see also (3.9)).

THEOREM 1.20
For all $v \le w \in W$ and $k, r \in \mathbb{Z}$, we have

$$\dim \operatorname{Ext}^{k,((r-\ell_{v,w})/2)}(M_v, M_w) = \dim H^{k-r,(r/2)}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big).$$

(In particular, both sides are zero for odd r.)

Thus, while $\operatorname{Ext}^\bullet(M_v, L_w)$ gives the Kazhdan–Lusztig polynomials, $\operatorname{Ext}^\bullet(M_v, M_w)$ gives the rational q,t-Catalan numbers for $v = \mathrm{id}$ and $w = f_{k,n}$.

1.12. Notes

We collect the historical background and several remarks on the above results.

1.12.1. Symmetry and unimodality

The symmetry and unimodality of the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are consequences of the hard Lefschetz theorem for $H^\bullet(\mathrm{Gr}(k,n))$. Whereas symmetry is apparent from the combinatorial definition of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see [122, Proposition 1.7.3]), unimodality is notoriously difficult to see combinatorially. Unimodality was first proved by Sylvester [126], the relation to hard Lefschetz observed by Stanley [121], and a combinatorial proof given by O'Hara [100].

When $a = n$ and $b = n+1$, $C_{a,b}(q,t)$ recovers the famous *q,t-Catalan numbers* $C_n(q,t)$ of Garsia and Haiman [50] studied extensively in, for example, [49], [58], [59], [62], and [63]. The fact that $C_n(q,t)$ is q,t-symmetric and q,t-unimodal follows from the results of Haiman [65], [66]. For arbitrary a, b, an explanation for the q,t-symmetry property was given by the *rational shuffle conjecture* of [60], proved recently in [19] and [91]. The specialization $q^{\frac{1}{2}d_{k,n}} C_{k,n-k}(q, 1/q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$ in (1.13) is also a consequence of the rational shuffle conjecture. To our knowledge, the q,t-unimodality of $C_{k,n-k}(q,t)$ in Corollary 1.8(ii) is a new result. See also [125, Section 2.2], which includes a specialization of our unimodality result.

1.12.2. Compactified Jacobians and the $P = W$ conjecture

We explain the original motivation coming from the results of [114] and [115] that led to the statement of Theorem 1.1. The compactified Jacobian $J_{a,b}$ of the plane curve singularity $x^a = y^b$ (with $\gcd(a,b) = 1$) is a compact, singular variety with a long history of connections to Catalan theory. Beauville [7] showed that the Euler characteristic of $J_{a,b}$ is the rational Catalan number $C_{a,b}$ and Piontkowski [102] (see also Lusztig and Smelt [87]) showed that the Poincaré polynomial and point count are given by the q-analogue $\sum_{P \in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{dinv}(P)}$. Gorsky and Mazin [58], [59] first suggested the relation between $J_{a,b}$ and q,t-Catalan numbers and since then there has been an explosion of developments relating compactified Jacobians and knot invariants (see, e.g., [24], [60], [61]). Our work provides evidence for the following conjecture, arising from the works [114] and [115] (see [27] for the original $P = W$ conjecture).

CONJECTURE 1.21
There is a deformation retraction from the torus quotient $\mathcal{X}^\circ_{k,n}$ to the compactified Jacobian $J_{k,n-k}$ sending the weight filtration of $H^\bullet(\mathcal{X}^\circ_{k,n})$ to the perverse filtration of $H^\bullet(J_{k,n-k})$ (see [89], [94]).

Conjecture 1.21 is motivated by the isomorphism, discovered in [114], between open positroid varieties and moduli spaces of constructible sheaves associated to Legendrian knots (see [115]). We thank Vivek Shende for explaining a conjectural *wild* non-abelian Hodge correspondence in that setting.

More generally, when a Richardson link is algebraic (i.e., arising as the link of a singularity), one may expect a statement similar to Conjecture 1.21 for the compactified Jacobian of the singularity. See Remark 4.24 for related discussion.

Remark 1.22
After discovering the proof of (1.8) via the HOMFLY polynomial, we found that it can also be deduced from the results of [114] and [115]. Our proof is new and yields a generalization (Theorem 1.13) of (1.8) to arbitrary open positroid varieties, and more generally to open Richardson varieties in generalized flag varieties.

1.12.3. Plabic graph links
In Section 1.5, we associated a link $\hat{\beta}_f$ to each positroid variety Π°_f. Two other (more complicated) ways of assigning a Legendrian/transverse link to a positroid variety have appeared recently in [37] and [114], stated in the language of Postnikov's *plabic graphs* (see [103]). Conjecturally, the links of [37] and [114] are isotopic to our links $\hat{\beta}_f$. We hope to return to this question in future work [47] (see also [21]).

1.12.4. Geometric interpretations and other Dynkin types
A geometric interpretation of the full triply-graded KR homology was given by Webster and Williamson [129]. Our approach yields a different geometric interpretation of the (doubly-graded) top a-degree part of KR homology. Our geometric interpretation in addition holds for Dynkin types outside type A. The analogue of the HOMFLY polynomial in other Dynkin types (as a trace on the Hecke algebra; cf. Section 2.2) was introduced in [52] (see also [109]). For related discussion of knot invariants in other types, see, for example, [127] and [128]; see also [14], [20], [25], [26], and [93] for related results.

1.12.5. Odd cohomology vanishing

It follows from the results of [69] and [92] that KR homology of any (positive) torus knot or link is concentrated in even degrees. It is tempting to conjecture that the same property holds for all Richardson knots or links. However, this is not the case: see Examples 4.21, 4.22, and 4.23. (See Remark 4.24 for a discussion of the closely related class of *algebraic knots*.)

1.12.6. Complements of hyperplane arrangements

The top positroid variety $\Pi^{\circ}_{k,n}$ may be considered "the complement of a hyperplane arrangement in the Grassmannian": it is obtained from $\mathrm{Gr}(k,n)$ by removing n hypersurfaces, each given by a linear equation in the Plücker coordinates on $\mathrm{Gr}(k,n)$. More general "Grassmannian hyperplane arrangements" appear naturally in the study of amplituhedra and Grassmann polytopes (see [45], [81]).

The cohomology of complements of hyperplane arrangements in projective space is very well studied: both the Poincaré polynomial and the point count are simple specializations of the characteristic polynomial. The coincidence is a manifestation of the purity of the mixed Hodge structure (see [113]), a property that also holds for the Grassmannian $\mathrm{Gr}(k,n)$.

1.12.7. Recurrence relations

Our results associate a q,t-polynomial to each positroid variety. One possible advantage of this approach is a recurrence for these polynomials, arising from the recurrence for positroid varieties developed by Muller and Speyer [97]. For instance, their results allow one to compute the point counts recursively (cf. [47]). To compute the Poincaré or the mixed Hodge polynomials, the recurrence of [97] yields a long exact sequence for the cohomology. It seems plausible that in favorable cases (e.g., when the odd cohomology vanishes), this sequence may be used to calculate the mixed Hodge polynomials of special families of links as was done in [34], [69], and [92]. We remark that the latter recurrences pass through complexes of Soergel bimodules which do not come from any braids; an interesting open problem is to understand the positroid/Richardson interpretation of such complexes.

Structure of the paper

In Section 2, we study the relationship between the point count and the HOMFLY polynomial, and prove Theorem 1.13 and its generalization (Theorem 2.3) to open Richardson varieties. In Sections 3 and 4, we discuss background on KR homology and cohomology of positroid varieties, respectively. We deduce Theorem 1.1 from Theorem 1.17 in Section 4.4. In Section 5, we recast our results in the language of equivariant derived categories, and split the main result (Theorem 1.17) into two statements, Propositions 5.3 and 5.4. These statements are proved in Sections 6 and 7, respectively, thereby completing the proof of Theorem 1.17. In Section 8, we deduce the rest of our results (Theorems 1.16, 1.18, and 1.20) from Theorem 1.17. Finally, in Section 9, we study analogues of Catalan numbers associated to arbitrary positroid varieties.

2. Point count and the HOMFLY polynomial

2.1. Type A

Let $W = S_n$ and $G = \mathrm{PGL}_n(\mathbb{C})$. Recall from [72, Lemmas A3 and A4] that the number of $\mathbb{F}_q$-points of a Richardson variety $R^\circ_{v,w}$ is given by the *Kazhdan–Lusztig R-polynomial* $R_{v,w}(q)$. When $v \not\leq w$, we have $R_{v,w}(q) = 0$ and $R^\circ_{v,w} = \emptyset$, and for $v = w$, we have $R_{v,w}(q) = 1$ and $R^\circ_{v,w} = \mathrm{pt}$. For $v \leq w \in W$, $R_{v,w}(q)$ can then be computed by a recurrence relation (see [72, Section 2]):

$$R_{v,w}(q) = \begin{cases} R_{sv,sw}(q) & \text{if } sv < v \text{ and } sw < w, \\ (q-1)R_{sv,w}(q) + qR_{sv,sw}(q) & \text{if } sv > v \text{ and } sw < w. \end{cases} \tag{2.1}$$

Here, $s = s_i$ for some $1 \leq i \leq n-1$ is a simple transposition satisfying $sw < w$.

Recall from Section 1.7 that we associate an n-strand braid $\beta_{v,w} := \beta(w) \cdot \beta(v)^{-1}$ to any pair $v, w \in S_n$ of permutations. For a Laurent polynomial $P = P(a,z)$, we denote by $\deg_a^{\mathrm{top}}(P) \in \mathbb{Z}$ the maximal degree of a in P, and for $\kappa \in \mathbb{Z}$, we let $[a^\kappa]P \in \mathbb{C}[z^{\pm 1}]$ be the coefficient of a^κ in P. For $v, w \in S_n$, recall that we set $\ell_{v,w} := \ell(w) - \ell(v)$. Denote

$$\kappa_{v,w} := n - 1 - \ell_{v,w} \qquad \text{and} \qquad P_{v,w} = P_{v,w}(a,z) := P(\hat{\beta}_{v,w}; a, z), \tag{2.2}$$

where $P(\hat{\beta}_{v,w}; a, z)$ is the HOMFLY polynomial defined in Section 1.6. The goal of this section is to show the following strengthening of Theorem 1.13.

THEOREM 2.1

Let $v, w \in S_n$.

(i) *If $v \not\leq w$, then* $\deg_a^{\mathrm{top}}(P_{v,w}) < \kappa_{v,w}$.

(ii) *If $v \leq w$, then* $\deg_a^{\mathrm{top}}(P_{v,w}) = \kappa_{v,w}$.

(iii) *For any $v, w \in S_n$, let $P^{\mathrm{top}}_{v,w}(q)$ be obtained from $a^{\kappa_{v,w}} \cdot ([a^{\kappa_{v,w}}]P_{v,w})$ by substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Then*

$$R_{v,w}(q) = (q-1)^{n-1} \cdot P^{\mathrm{top}}_{v,w}(q). \tag{2.3}$$

Proof
We start by recalling the following result, which states that the *(lower) Morton–Franks–Williams inequality* (see [41], [96]) is not sharp for negative braids. It may be alternatively deduced from (2.10) below.

LEMMA 2.2 ([53, Proposition 2.1])
Let $v \in S_n$ be a nonidentity permutation, and let $\beta := \beta(v)^{-1}$ be the associated negative braid. Then[5]

$$\deg_a^{\mathrm{top}}\big(P(\hat{\beta}; a, z)\big) < n - 1 + \ell(v). \tag{2.4}$$

We now prove all parts of Theorem 2.1 by induction on $\ell(w)$. Consider the base case $\ell(w) = 0$. Then (i) is the content of (2.4). For (ii), we observe that $\ell(w) = 0$ implies $v = w = \mathrm{id}$, and iterating Example 1.12, we get $P_{v,w} = (\frac{a-a^{-1}}{z})^{n-1}$. Thus, $\deg_a^{\mathrm{top}}(P_{v,w}) = n - 1 = \kappa_{v,w}$ for $v = w = \mathrm{id}$. For (iii), if $v \not\leq w$, then by (i), we get $[a^{\kappa_{v,w}}]P_{v,w} = 0$, so $P^{\mathrm{top}}_{v,w}(q) = 0$, in agreement with $R_{v,w}(q) = 0$. If $v \leq w$, then $v = w = \mathrm{id}$, $a^{\kappa_{v,w}} \cdot [a^{\kappa_{v,w}}]P_{v,w} = (a/z)^{n-1}$, so $P^{\mathrm{top}}_{v,w}(q) = (q-1)^{-(n-1)}$, in agreement with (2.3). We have shown the base case for each part.

For the induction step, suppose that $\ell(w) > 0$. Choose some $1 \leq i \leq n-1$ such that $s_i w < w$, and let $s := s_i$ and $\sigma := \sigma_i$. If $sv < v$, then the links $\hat{\beta}_{v,w}$ and $\hat{\beta}_{sv,sw}$ are isotopic since $\beta_{v,w} = \sigma\beta_{sv,sw}\sigma^{-1}$, and thus $P_{v,w} = P_{sv,sw}$. We also have $\kappa_{v,w} = \kappa_{sv,sw}$, and thus $P^{\mathrm{top}}_{v,w}(q) = P^{\mathrm{top}}_{sv,sw}(q)$. By (2.1), $R_{v,w}(q) = R_{sv,sw}(q)$. So in the case $sv < v$, the induction step holds trivially for each of the three parts.

Assume now that we have $sw < w$ and $sv > v$. In this case, we have $\beta_{v,w} = \sigma\beta(sw)\beta(sv)^{-1}\sigma \sim \beta(sw)\beta(sv)^{-1}\sigma^2$, $\beta_{v,sw} = \beta(sw)\beta(sv)^{-1}\sigma$, and $\beta_{sv,sw} = \beta(sw)\beta(sv)^{-1}$, where $\sim$ relates conjugate braids. Applying (1.15) with

$$L_+ := \hat{\beta}_{v,w}, \qquad L_0 := \hat{\beta}_{v,sw}, \qquad L_- := \hat{\beta}_{sv,sw},$$

we get $aP_{v,w} - a^{-1}P_{sv,sw} = zP_{v,sw}$, and thus

$$P_{v,w} = \frac{z}{a}P_{v,sw} + a^{-2}P_{sv,sw}. \tag{2.5}$$

Note that $\kappa_{v,sw} = \kappa_{v,w} + 1$ and $\kappa_{sv,sw} = \kappa_{v,w} + 2$. Let us show (i). We have $v \not\leq w$, $sw < w$, and $sv > v$, and thus clearly $v \not\leq sw$ and $sv \not\leq sw$. By the induction hypothesis, we have $\deg_a^{\mathrm{top}}(P_{v,sw}) < \kappa_{v,sw}$ and $\deg_a^{\mathrm{top}}(P_{sv,sw}) < \kappa_{sv,sw}$. By (2.5), we get $\deg_a^{\mathrm{top}}(P_{v,w}) < \kappa_{v,w}$, finishing the proof of (i). In particular, we have shown that (2.3) holds for all $v \not\leq w$.

Now assume that $v \leq w$. We show (ii) and (iii) simultaneously. By the induction hypothesis, we have $\deg_a^{\mathrm{top}}(P_{v,sw}) \leq \kappa_{v,sw}$, $\deg_a^{\mathrm{top}}(P_{sv,sw}) \leq \kappa_{sv,sw}$ (whether

[5]Our conventions for $P(\hat{\beta}; a, z)$ differ from those of [53] by changing $a \mapsto a^{-1}$.

the equality holds depends on whether $v \le sw$ and $sv \le sw$). Thus, by (2.5), $\deg_a^{\mathrm{top}}(P_{v,w}) \le \kappa_{v,w}$. The links $L_0 = \hat{\beta}_{v,sw}$ and $\hat{\beta}_{sv,w}$ are isotopic since $\beta_{sv,w} = \sigma\beta(sw)\beta(sv)^{-1} \sim \beta(sw)\beta(sv)^{-1}\sigma = \beta_{v,sw}$, so $P_{v,sw} = P_{sv,w}$. Applying the map $P \mapsto a^{\kappa_{v,w}} \cdot ([a^{\kappa_{v,w}}]P)$ to both sides of (2.5) and substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, we get

$$P^{\mathrm{top}}_{v,w}(q) = (q-1)P^{\mathrm{top}}_{sv,w}(q) + qP^{\mathrm{top}}_{sv,sw}(q).$$

Combining this with the induction hypothesis and (2.1), we get $R_{v,w}(q) = (q-1)^{n-1} \cdot P^{\mathrm{top}}_{v,w}(q)$. In particular, the coefficient of $a^{\kappa_{v,w}}$ in $P_{v,w}$ is nonzero, so $\deg_a^{\mathrm{top}}(P_{v,w}) = \kappa_{v,w}$. Thus, we have completed the induction step for both (ii) and (iii). □

2.2. *Arbitrary type*

The above connection between point counts and the HOMFLY polynomial can be generalized to arbitrary Weyl groups as follows. Let $\mathcal{H}$ be the *Hecke algebra* of W: it is generated over $\mathbb{C}[q^{\pm 1}]$ by the elements $\{T_s\}_{s\in S}$ satisfying the braid relations as well as the Hecke relation

$$(T_s + q)(T_s - 1) = 0 \quad \text{for } s \in S. \tag{2.6}$$

The algebra $\mathcal{H}$ admits a linear basis $\{T_w\}_{w\in W}$ indexed by the elements of W: we set $T_w := T_{s_1}T_{s_2}\cdots T_{s_{\ell(w)}}$ for any reduced word $w = s_1s_2\cdots s_{\ell(w)}$. The *standard trace* $\epsilon : \mathcal{H} \to \mathbb{C}[q^{\pm 1}]$ is the $\mathbb{C}[q^{\pm 1}]$-linear map defined by

$$\epsilon(T_w) := \begin{cases} 1 & \text{if } w = \mathrm{id}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

THEOREM 2.3
For any $v, w \in W$, we have

$$R_{v,w}(q) = q^{\ell_{v,w}}\epsilon(T_w^{-1}T_v). \tag{2.8}$$

For $W = S_n$, in view of the well-known relation between traces and the HOMFLY polynomial (going back to [71]), Theorem 2.3 specializes to Theorem 2.1.

Proof
First, we state a simple consequence of (2.6): for any $v \in W$ and $s \in S$, we have

$$T_sT_v = \begin{cases} T_{sv} & \text{if } sv > v, \\ (1-q)T_v + qT_{sv} & \text{if } sv < v. \end{cases} \tag{2.9}$$

Next, we claim that for any $u, v \in W$, we have

$$\epsilon(T_u T_v) = \begin{cases} q^{\ell(v)} & \text{if } u = v^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

We prove (2.10) by induction on $\ell(u)$. The base case $\ell(u) = 0$ is clear. Otherwise, choose $s \in S$ such that $u = xs$ with $x < xs$. If $v < sv$, then $T_u T_v = T_x T_{sv}$ and we are done by induction. Thus, assume that $v > sv$. By (2.9), we get

$$\epsilon(T_u T_v) = (1-q)\epsilon(T_x T_v) + q\epsilon(T_x T_{sv}). \tag{2.11}$$

We have $u = v^{-1}$ if and only if $x = (sv)^{-1}$, in which case by induction we find $\epsilon(T_u T_v) = q^{\ell(v)}$. If $u \neq v^{-1}$, then the right-hand side of (2.11) is zero unless $x = v^{-1}$. But $x = v^{-1}$ contradicts our assumptions $x < xs$ and $v > sv$. This completes the proof of (2.10).

It is well known that $T_w^{-1} \in \mathrm{Span}\{T_u\}_{u \leq w}$. Thus, by (2.10), $\epsilon(T_w^{-1} T_v) = 0$ unless $v \leq w$. We now proceed to prove (2.8) by induction on $\ell_{v,w}$. For $v = w$, the result again follows from (2.10). For $v < w$, we choose $s \in S$ such that $sw < w$ and then calculate using $T_s^{-1} = q^{-1} T_s + (1 - q^{-1})$ that

$$T_w^{-1} T_v = \begin{cases} T_{sw}^{-1} T_{sv} & \text{if } sv < v \text{ and } sw < w, \\ q^{-1} T_{sw}^{-1} T_{sv} + (1-q^{-1}) T_{sw}^{-1} T_v & \text{if } sv > v \text{ and } sw < w. \end{cases}$$

Applying ϵ and multiplying both sides by $q^{\ell_{v,w}}$, the result matches perfectly with (2.1). □

Remark 2.4
Theorem 2.3 may also be deduced from Theorem 1.17 by taking the Euler characteristic: Soergel bimodules categorify the Hecke algebra, with Rouquier complexes $F^\bullet(w)$ corresponding to the elements T_w, and the zeroth Hochschild cohomology functor HH^0 categorifies the trace ϵ.

3. Soergel bimodules, Rouquier complexes, and Khovanov–Rozansky homology

In this section, we review Khovanov–Rozansky (KR) link homology. A friendly introduction to most of this material can be found in the excellent recent book [35].

3.1. *Soergel bimodules*

Let $R := \mathbb{C}[\mathfrak{h}]$ be as in Section 1.9. It is a graded ring where we set $\deg(y) = 2$ for $y \in \mathfrak{h}^*$. We refer to $\deg(y)$ as the *polynomial degree* (as opposed to the cohomological degree introduced later on). The Weyl group W acts naturally on R. Denote by I the indexing set of simple roots of R, and thus W is generated by the simple reflections

$S = \{s_i\}_{i\in I}$. When $G = \mathrm{PGL}_n(\mathbb{C})$, recall that $R = \mathbb{C}[y_1,\dots,y_{n-1}]$ is the polynomial ring and $S = \{s_1,\dots,s_{n-1}\}$ is the set of simple transpositions in $W = S_n$. The action of S_n on R is obtained by restricting the permutation action on $\mathbb{C}[x_1,x_2,\dots,x_n]$ to $R \subset \mathbb{C}[x_1,x_2,\dots,x_n]$, where we identify $y_i = x_i - x_{i+1}$ for $1 \le i \le n-1$. For example,

$$s_1(y_1) = -y_1, \qquad s_2(y_1) = y_1 + y_2, \qquad s_3(y_1) = \cdots = s_{n-1}(y_1) = y_1. \quad (3.1)$$

Soergel bimodules are special kinds of *graded R-bimodules*, that is, graded $\mathbb{C}$-vector spaces equipped with a left and a right graded action of R. For a graded R-bimodule $B = \bigoplus_i B^i$ and $m \in \mathbb{Z}$, we denote by $B\{m/2\} := \bigoplus_i B^{i-m}$ the polynomial grading shift by m on B. Thus, $y \in \mathfrak{h}^*$ has degree 2 as an element of R but has degree 0 as an element of $R\{-1\}$.

Let us introduce the "building blocks" of Soergel bimodules.

Definition 3.1
For $s \in S$, let $R^s := \{r \in R \mid sr = r\}$. Define

$$B_s := R \otimes_{R^s} R. \quad (3.2)$$

For a sequence $\underline{u} = (s_{i_1}, s_{i_2},\dots,s_{i_m})$ of elements of S, let

$$B_{\underline{u}} := B_{s_{i_1}} \otimes_R B_{s_{i_2}} \otimes_R \cdots \otimes_R B_{s_{i_m}} = R \otimes_{R^{s_{i_1}}} R \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_m}}} R. \quad (3.3)$$

Both B_s and $B_{\underline{u}}$ are naturally graded R-bimodules, called *Bott–Samelson bimodules*, where R acts on the leftmost and the rightmost terms of the tensor product by multiplication.

We let $\mathbb{S}\mathrm{Bim}$ denote the *category of Soergel bimodules*. By definition, its objects are graded shifts of direct summands of Bott–Samelson bimodules $B_{\underline{u}}$, where $\underline{u}$ runs over all finite sequences of elements in S. The morphisms in $\mathbb{S}\mathrm{Bim}$ are given by degree-0 maps of R-bimodules. The indecomposable objects $\{S_w\}_{w\in W}$ of $\mathbb{S}\mathrm{Bim}$ are indexed by the elements of W: for each $w \in W$ and any reduced word $\underline{w}$ for w, $B_{\underline{w}}$ contains a unique indecomposable summand S_w that is not contained in $B_{\underline{v}}$ for any $v < u$ and any reduced word $\underline{v}$ for v. Up to isomorphism, the bimodule S_w depends only on w and not on the choice of $\underline{w}$.

3.2. *Rouquier complexes*

We let $\mathrm{K^b}\mathbb{S}\mathrm{Bim}$ denote the homotopy category of $\mathbb{S}\mathrm{Bim}$. Its objects are bounded cochain complexes $C^\bullet = (\cdots \to C^{-1} \to \underline{C^0} \to C^1 \to \cdots)$ of Soergel bimodules, and morphisms are homotopy classes of maps of complexes. When depicting a complex, we usually omit some of the zeros and underline the object that is in cohomological degree 0. For example, a complex with only two nonzero entries may be written as $(\underline{C^0} \to C^1)$ or as $(0 \to \underline{C^0} \to C^1)$. We denote by $[m]$ the *cohomological shift* on $\mathrm{K^b}\mathbb{S}\mathrm{Bim}$. It shifts each cochain complex m steps to the left: $C^\bullet[1] = (\cdots \to C^{-1} \to C^0 \to \underline{C^1} \to \cdots)$.

The tensor product $C^\bullet \otimes_R D^\bullet$ of two cochain complexes is the total complex of a double complex whose entries are $C^i \otimes_R D^j$ for $i, j \in \mathbb{Z}$. The sign of the map $C^i \otimes_R D^j \to C^i \otimes_R D^{j+1}$ is the negation of the obvious one for all even i; the differential of the resulting total complex squares to zero.

Since (W, S) is a Coxeter system, we can consider the associated Artin braid group $\mathcal{B}_W$ generated by $\{\sigma_i\}_{i\in I}$. The following construction is due to Rouquier [110].

Definition 3.2
For $s = s_i \in S$ and $\sigma = \sigma_i$, define the *Rouquier complexes*

$$F^\bullet(\sigma) := (B_s \to \underline{R}), \qquad F^\bullet(\sigma^{-1}) := \big(\underline{R} \to B_s\{-1\}\big), \tag{3.4}$$

where the first map sends $f \otimes g \mapsto fg$ and the second map sends $1 \mapsto (\alpha_s \otimes 1 + 1 \otimes \alpha_s)$. Here $\alpha_s \in \mathfrak{h}^*$ is the simple root corresponding to s. Note that both $1 \in R$ and $(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in B_s\{-1\}$ have polynomial degree 0. For a braid $\beta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m} \in \mathcal{B}_W$, we set

$$F^\bullet(\beta) := F^\bullet(\sigma_{i_1}) \otimes_R F^\bullet(\sigma_{i_2}) \otimes_R \cdots \otimes_R F^\bullet(\sigma_{i_m}),$$
$$F^\bullet(\beta^{-1}) \;=\; F^\bullet(\beta)^{-1} := F^\bullet(\sigma_{i_m}^{-1}) \otimes_R \cdots \otimes_R F^\bullet(\sigma_{i_2}^{-1}) \otimes_R F^\bullet(\sigma_{i_1}^{-1}).$$

We also let $F^\bullet(\mathrm{id}) := (0 \to \underline{R} \to 0)$.

A priori, the complex $F^\bullet(\beta)$ depends on the choice of the word $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_m})$. However, modulo homotopy, it does not.

PROPOSITION 3.3 ([110, Section 3])
If $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m} = \sigma_{i'_1}\sigma_{i'_2}\cdots\sigma_{i'_m}$ *in* $\mathcal{B}_W$, *then*

$$F^\bullet(\sigma_{i_1}) \otimes_R F^\bullet(\sigma_{i_2}) \otimes_R \cdots \otimes_R F^\bullet(\sigma_{i_m})$$
$$\cong F^\bullet(\sigma_{i'_1}) \otimes_R F^\bullet(\sigma_{i'_2}) \otimes_R \cdots \otimes_R F^\bullet(\sigma_{i'_m})$$

in $\mathrm{K^b}\mathbb{S}\mathrm{Bim}$.

For example, one can check that $F^\bullet(\sigma_i) \otimes_R F^\bullet(\sigma_i^{-1}) \cong F^\bullet(\mathrm{id})$. It follows that the functors $(-) \otimes_R F^\bullet(\sigma_i)$ and $(-) \otimes_R F^\bullet(\sigma_i^{-1})$ are mutually inverse biadjoint equivalences of categories: for complexes $C^\bullet, D^\bullet \in \mathrm{K^b}\mathbb{S}\mathrm{Bim}$, we have

$$\mathrm{Hom}_{\mathrm{K^b}\mathbb{S}\mathrm{Bim}}\big(C^\bullet, D^\bullet \otimes_R F^\bullet(\sigma_i)\big) \cong \mathrm{Hom}_{\mathrm{K^b}\mathbb{S}\mathrm{Bim}}\big(C^\bullet \otimes_R F^\bullet(\sigma_i^{-1}), D^\bullet\big), \tag{3.5}$$

$$\mathrm{Hom}_{\mathrm{K^b}\mathbb{S}\mathrm{Bim}}\big(C^\bullet, D^\bullet \otimes_R F^\bullet(\sigma_i^{-1})\big) \cong \mathrm{Hom}_{\mathrm{K^b}\mathbb{S}\mathrm{Bim}}\big(C^\bullet \otimes_R F^\bullet(\sigma_i), D^\bullet\big). \tag{3.6}$$

Proposition 3.3 allows one to define $F^\bullet(\beta) \in \mathrm{K^b}\mathbb{S}\mathrm{Bim}$ unambiguously for any braid $\beta \in \mathcal{B}_W$. Recall that we are interested in the braid $\beta_{v,w} = \beta(w)\cdot\beta(v)^{-1}$, which corresponds to the complex

$$F^\bullet_{v,w} := F^\bullet(\beta_{v,w}) = F^\bullet\big(\beta(w)\big) \otimes_R F^\bullet\big(\beta(v)^{-1}\big). \tag{3.7}$$

3.3. *KR homology*

Recall from (1.17) that the functor HH^0 sends a graded R-bimodule B to the graded R-module $HH^0(B)$ of its R-invariants. Alternatively, it can be expressed as

$$HH^0(B) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}\big(R, B\{-r/2\}\big). \tag{3.8}$$

Remark 3.4

The R-module $HH^0(B)$ is free for any Bott–Samelson bimodule B. One can make explicit combinatorial computations with this R-module (including finding a basis and computing the maps in the Rouquier complexes) using the *diagrammatic calculus* developed by Elias and Williamson [36].

Remark 3.5

Higher Hochschild cohomology functors HH^h, which give the full (triply-graded) KR homology, are the right derived functors of HH^0. They can be computed using a Koszul resolution of R (see, e.g., [74], [92]).

Applying the functor HH^0 to a complex $C^\bullet$ of Soergel bimodules yields a complex $HH^0(C^\bullet)$ of graded R-modules. In particular, for each $k \in \mathbb{Z}$, the cohomology $H^k(HH^0(C^\bullet))$ of this complex is a graded R-module. For $r \in \mathbb{Z}$, we denote by $H^{k,(r/2)}(HH^0(C^\bullet))$ its graded piece of polynomial degree r. It is not hard to check that we have

$$H^{k,(r/2)}\big(HH^0(C^\bullet)\big) \cong \mathrm{Hom}_{\mathrm{K^b}\mathbb{S}\mathrm{Bim}}\big(R, C^\bullet[k]\{-r/2\}\big). \tag{3.9}$$

For any $\beta \in \mathcal{B}_W$, $F^\bullet(\beta)$ is concentrated in even polynomial degrees, and thus $H^{k,(r/2)}(HH^0(F^\bullet(\beta)))$ vanishes when r is odd. Similarly to (1.19), we denote

$$HHH^0\big(F^\bullet(\beta)\big) := \bigoplus_{k,p\in\mathbb{Z}} H^{k,(p)}\big(HH^0\big(F^\bullet(\beta)\big)\big). \tag{3.10}$$

The complex $F^{\bullet}_{v,w}$ is concentrated in cohomological degrees $-\ell(w), -\ell(w)+1, \dots, \ell(v)$, and thus $H^{k,(p)}(HH^0(F^{\bullet}_{v,w})) = 0$ unless $-\ell(w) \le k \le \ell(v)$, and the index $p \in \mathbb{Z}$ is bounded from below.

The functor $HH^0_{\mathbb{C}}$ admits a similar description. Recall that $\mathbb{C}$ is considered an R-bimodule on which $\mathfrak{h}^*$ acts by zero on both sides. Any Soergel bimodule $B \in \mathbb{S}\mathrm{Bim}$ gives rise to a *Soergel module* $B \otimes_R \mathbb{C}$, which is a graded R-module. We let $\mathbb{S}\mathrm{Mod}$ denote the category of Soergel modules (with morphisms being maps of polynomial degree 0). By a result of Soergel [118] (see [35, Proposition 15.27]), for any $B, B' \in \mathbb{S}\mathrm{Bim}$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(B, B') \otimes_R \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{S}\mathrm{Mod}}(B \otimes_R \mathbb{C}, B' \otimes_R \mathbb{C}).$$

Applying this to the case $B = R$, we get the following result.

COROLLARY 3.6
For any Soergel bimodule $B \in \mathbb{S}\mathrm{Bim}$, we have

$$HH^0_{\mathbb{C}}(B) \cong \mathrm{Hom}_{\mathbb{S}\mathrm{Mod}}(\mathbb{C}, B \otimes_R \mathbb{C}).$$

3.4. Link components and R-module structure

For this section, we assume that $W = S_n$. Let β be a braid, and let $u \in S_n$ be the image of β. Let $\tilde{R}$ denote the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$, and let $\tilde{F}^{\bullet}(\beta)$ denote the Rouquier complex using $\tilde{R}$ instead of R (cf. (3.1)). Thus, $F^{\bullet}(\beta)$ is a reduced version of $\tilde{F}^{\bullet}(\beta)$. The complex $\tilde{F}^{\bullet}(\beta)$ is an $(\tilde{R} \otimes \tilde{R})$-module, and it is known (see [55], [105]) that the action of $x_i \otimes 1$ is homotopic to the action of $1 \otimes x_{u(i)}$. Indeed, it follows from [55, Proposition 2.11, Theorem 2.18] that there exist cochain maps ξ_i (of polynomial degree 2 and cohomological degree -1) called *dot-sliding homotopies* such that

(1) $d\xi_i + \xi_i d = x_i \otimes 1 - 1 \otimes x_{u(i)}$ for $i = 1, 2, \dots, n$, and
(2) $\xi_i \xi_j + \xi_j \xi_i = 0$ for $i, j = 1, 2, \dots, n$.

In $HH^0(\tilde{F}^{\bullet}(\beta))$, the two R-actions are equalized, so $d\xi_i + \xi_i d = x_i - x_{u(i)}$. Thus, the actions of x_i and x_j on $HHH^0(\tilde{F}^{\bullet}(\beta))$ agree when i and j belong to the same component of the link $\hat{\beta}$. Working instead with the smaller polynomial ring $R = \mathbb{C}[x_1 - x_2, \dots, x_{n-1} - x_n] \subset \tilde{R}$, we deduce that $x_i - x_j$ acts as zero on $HHH^0(F^{\bullet}(\beta))$ when i, j belong to the same component of the link $\hat{\beta}$. In particular, if $\hat{\beta}$ is a knot, then the action of R on $HHH^0(F^{\bullet}(\beta))$ factors through the natural map $R \to \mathbb{C}$.

Suppose now that $\hat{\beta}$ is a knot. Denote $z_i := x_i - x_{u(i)}$ for $i = 1, 2, \dots, n$. Thus, $R = \mathbb{C}[z_1, \dots, z_{n-1}]$. Recall that $HH^0(F^{\bullet}(\beta))$ is a complex of graded, free (cf. Remark 3.4) R-modules. Let $a \in HH^0(F^{\bullet}(\beta))$ be a nonzero element satisfying $d(a) = 0$. Using the relation $d\xi_i + \xi_i d = z_i$, one can show by induction on $k = 0, 1, \dots, n-1$ that for all $1 \le i_1 < i_2 < \cdots < i_k \le n-1$, we have

$$d\xi_{i_1 i_2 \cdots i_k}(a) = \sum_{j=1}^{k} (-1)^{j-1} z_{i_j} \xi_{i_1 \cdots \hat{i}_j \cdots i_k}(a), \tag{3.11}$$

where $\xi_{i_1 i_2 \cdots i_k} := \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}$, and $\hat{i}_j$ denotes omission of ξ_{i_j}. We therefore obtain a subcomplex $K^\bullet(a)$ of $HH^0(F^\bullet(\beta))$ given by

$$R \cdot \xi_{12\cdots n-1}(a) \to \cdots \to \sum_{1 \le i_1 < \cdots < i_k \le n-1} R \cdot \xi_{i_1 \cdots i_k}(a) \to \cdots$$
$$\to \sum_{1 \le i \le n-1} R \cdot \xi_i(a) \to R \cdot a.$$

Definition 3.7
Let $a \in HH^0(F^\bullet(\beta))$ be such that $d(a) = 0$. We say that $K^\bullet(a)$ is a *Koszul subcomplex* if the set $\{\xi_{i_1 \cdots i_k}(a) \mid 0 \le k \le n-1, 1 \le i_1 < \cdots < i_k \le n-1\}$ can be extended to a free R-module basis of $HH^0(F^\bullet(\beta))$.

It follows from (3.11) that we have a natural cochain map $\bigotimes_{i=1}^{n-1}(R \xrightarrow{z_i} R) \to HH^0(F^\bullet(\beta))$ with image $K^\bullet(a)$, and this map is an isomorphism when $K^\bullet(a)$ is a Koszul subcomplex.

Our next goal is to show that $HH^0(F^\bullet(\beta))$ admits a filtration by Koszul subcomplexes and contractible subcomplexes of the form $R \xrightarrow{\sim} R$.

Definition 3.8
We say that a complex $(C^\bullet, d)$ of finite rank, free, graded R-modules *admits a $\bigwedge$-action* if there exist endomorphisms $\xi_1, \xi_2, \ldots, \xi_{n-1}$ of cohomological degree -1 and polynomial degree 2 satisfying $d\xi_i + \xi_i d = z_i$ and $\xi_i \xi_j + \xi_j \xi_i = 0$, for all $i, j = 1, 2, \ldots, n-1$.

We thank the anonymous referee for suggesting to us that the following statement may be deduced from the results of [55].

PROPOSITION 3.9
Suppose that $(C^\bullet, d)$ admits a $\bigwedge$-action. Then $C^\bullet$ has a filtration by Koszul complexes and trivial complexes $R \cong R$. That is, there exists a family of subcomplexes $0 = F_0^\bullet \subset F_1^\bullet \subset \cdots \subset F_t^\bullet = C^\bullet$ such that for all $j = 1, 2, \ldots, t$, $C^\bullet / F_{j-1}^\bullet$ is free, admits a $\bigwedge$-action, and $F_j^\bullet / F_{j-1}^\bullet$ is either a Koszul subcomplex of $C^\bullet / F_{j-1}^\bullet$ or a trivial subcomplex isomorphic to $R \xrightarrow{\sim} R$.

Proof
Let $C^{k,(p/2)}$ denote the subspace of $C^\bullet$ of cohomological degree k and polynomial degree p. Assuming that $C^\bullet$ is nonzero, let $D^\bullet \subset C^\bullet$ be the sum of those nonzero pieces $C^{k,(p/2)}$ where $k + p/2$ is minimal.

Suppose that $a \in D^\bullet$ satisfies $d(a) = 0$. Then for any $i_1, \dots, i_k$, we have that $\xi_{i_1 \cdots i_k}(a) \in D^\bullet$. Any linearly independent (over $\mathbb{C}$) elements in $D^\bullet$ can be extended to a free R-module basis of $C^\bullet$. Thus, using (3.11), one can show by induction on $k = 0, 1, \dots, n-1$ that the elements $\{\xi_{i_1 \cdots i_k}(a) \mid 1 \le i_1 < \cdots < i_k \le n-1\}$ are linearly independent. It follows that $K^\bullet(a)$ is a Koszul subcomplex of $C^\bullet$, and furthermore that the quotient by this complex is again free and admits a $\bigwedge$-action.

Repeating this, we may assume that $d|_{D^\bullet}$ is injective. We claim that for any nonzero element $b \in D^\bullet$, $d(b)$ may be completed to a free basis of $C^\bullet$. Suppose that $b \in D^{k-1}$. We proceed by inverse induction on k. For the base case, if $D^k = 0$, then $d(b)$ is a $\mathbb{C}$-linear combination of free basis elements, so the statement follows. For the induction step, suppose that $b \in D^{k-1}$ satisfies $d(b) \in R \cdot D^k$. Write $d(b) = \sum_{i=1}^{n-1} y_i e_i$, and let i be such that $e_i \neq 0$. By the induction hypothesis, $f := d(e_i)$ may be extended to a free basis of C^{k+1}, and we find that the coefficient of $y_i f$ in $d^2(b)$ is nonzero, which is a contradiction. Thus, $d(b) \notin R \cdot D^k$. Comparing the polynomial degree of $d(b)$ to that of D^k, we get that $d(b)$ can be extended to an R-basis of C^k.

A subcomplex $R \xrightarrow{\sim} R$ in $C^\bullet$ is called *splittable* if the quotient by this subcomplex again consists of free R-modules. Let k be the smallest index such that $D^k \neq 0$, and let $a_1 \in D^k$ be a nonzero element. We have shown above that $a_0 := d(a_1)$ may be completed to a free basis of $C^\bullet$, and thus a_1 and a_0 generate a splittable subcomplex $S^\bullet \cong (R \xrightarrow{\sim} R)$. For any $1 \le i \le n-1$, we have $\xi_i(a_1) \in D^{k-1} = 0$. Using $d\xi_i + \xi_i d = z_i$, we get $z_i a_1 = \xi_i a_0$. Thus, the subcomplex $S^\bullet$ is closed under the action of $\xi_1, \dots, \xi_{n-1}$. Using Gaussian elimination (see [35, Exercise 19.12]), one can check that in this case, the quotient complex $C^\bullet / S^\bullet$ admits a $\bigwedge$-action.

Repeating the above procedure, we obtain the desired filtration. □

Recall that the Koszul resolution of the R-module $\mathbb{C}$ by free R-modules yields a 2^{n-1}-dimensional complex $\mathrm{Tor}_\bullet^R(\mathbb{C}, \mathbb{C}) \cong (\mathbb{C} \xrightarrow{0} \underline{\mathbb{C}})^{\otimes(n-1)}$.

COROLLARY 3.10
Suppose that $(C^\bullet, d)$ admits a $\bigwedge$-action. Let $(C^\bullet_{\mathbb{C}}, d_{\mathbb{C}})$ be the complex of $\mathbb{C}$-vector spaces obtained by setting $y_1 = y_2 = \cdots = y_{n-1} = 0$. Thus, $C^\bullet_{\mathbb{C}} := C^\bullet \otimes_R \mathbb{C}$ as in (1.18). Then

$$H^\bullet(C^\bullet_{\mathbb{C}}) \cong \mathrm{Tor}_\bullet^R\big(\mathbb{C}, H^\bullet(C^\bullet)\big) \cong H^\bullet(C^\bullet) \otimes (\mathbb{C} \xrightarrow{0} \underline{\mathbb{C}})^{\otimes(n-1)}$$

(as complexes of graded $\mathbb{C}$-vector spaces with zero differentials).

Proof

It is clear that the filtration constructed in Proposition 3.9 induces an injection $H^\bullet(F_j^\bullet/F_{j-1}^\bullet) \hookrightarrow H^\bullet(C^\bullet/F_{j-1}^\bullet)$, and a similar statement holds after setting $y_1 = y_2 = \cdots = y_{n-1} = 0$. Thus, each Koszul complex $K^\bullet(a)$ appearing in the filtration contributes a 1-dimensional subcomplex to $H^\bullet(C^\bullet)$. In view of (3.11), $K^\bullet(a)$ contributes to $H^\bullet(C_{\mathbb{C}}^\bullet)$ a 2^{n-1}-dimensional subcomplex isomorphic to $(\mathbb{C} \xrightarrow{0} \underline{\mathbb{C}})^{\otimes(n-1)}$. □

3.5. *Link invariant*

For this section, we continue to assume that $W = S_n$. The above construction may be turned into a link invariant as we now explain. We follow the conventions of [69].

For a braid $\beta \in \mathcal{B}_{S_n} = \mathcal{B}_n$, let $e(\beta)$ denote the *exponent sum* of β:

$$\beta = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\cdots\sigma_{i_m}^{\epsilon_m} \quad \Longrightarrow \quad e(\beta) := \epsilon_1 + \epsilon_2 + \cdots + \epsilon_m.$$

Thus, $e(\beta_{v,w}) = \ell_{v,w} = \ell(w) - \ell(v)$. Next, define

$$\chi(\beta) := \frac{e(\beta) - n + c(\beta)}{2}, \tag{3.12}$$

where $c(\beta)$ is the number of components of the link $\hat{\beta}_{v,w}$, which equals the number of cycles of the corresponding permutation (obtained from the group homomorphism $\mathcal{B}_{S_n} \to S_n$ sending $\sigma_i \mapsto s_i$ for each $1 \le i \le n-1$). It is easy to check that $\chi(\beta)$ is always an integer. Define

$$\mathcal{P}_{\mathrm{KR}}(\beta; a, q, t) := (1-t)^{c(\beta)-1}(q^{\frac{1}{2}}t^{-\frac{1}{2}}a^{-2})^{\chi(\beta)} \times \sum_{k,p,h\in\mathbb{Z}} (-1)^h q^{\frac{k}{2}} t^{p+\frac{k}{2}+h} a^{-2h} \dim H^{k,(p)}\big(HH^h\big(F^\bullet(\beta)\big)\big).$$

Let $\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta; q, t)$ be its top a-degree coefficient:

$$\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta; q, t) := [a^{\deg_a^{\mathrm{top}}(\mathcal{P}_{\mathrm{KR}}(\beta))}]\mathcal{P}_{\mathrm{KR}}(\beta). \tag{3.13}$$

THEOREM 3.11 ([74])

$\mathcal{P}_{\mathrm{KR}}$ and $\mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}$ are link invariants: if $\beta \in \mathcal{B}_{S_n}$, $\beta' \in \mathcal{B}_{S_{n'}}$ are two braids such that the corresponding links $\hat{\beta} \cong \hat{\beta}'$ are isotopic, then

$$\mathcal{P}_{\mathrm{KR}}(\beta; a, q, t) = \mathcal{P}_{\mathrm{KR}}(\beta'; a, q, t) \qquad \text{and} \qquad \mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta; q, t) = \mathcal{P}_{\mathrm{KR}}^{\mathrm{top}}(\beta'; q, t).$$

Thus, it makes sense to write $\mathcal{P}_{\mathrm{KR}}(\hat{\beta}; a, q, t) := \mathcal{P}_{\mathrm{KR}}(\beta; a, q, t)$ and $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}; q, t) := \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta; q, t)$. Both $\mathcal{P}_{\mathrm{KR}}(\hat{\beta}; a, q, t)$ and $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}; q, t)$ have been recently shown to be q, t-symmetric when $\hat{\beta}$ is a knot (see [99]).

It is well known that $\mathcal{P}_{\mathrm{KR}}(\hat{\beta}; a, q, t)$ specializes to the HOMFLY polynomial of $\hat{\beta}$:

$$\mathcal{P}_{\mathrm{KR}}(\hat{\beta})\big|_{t^{\frac{1}{2}}=-q^{-\frac{1}{2}}} = (-1)^{\chi(\beta)}(z/a)^{c(\beta)-1} P(\hat{\beta}; a, z)\big|_{z=q^{\frac{1}{2}}-q^{-\frac{1}{2}}}. \tag{3.14}$$

Note that $\chi(\beta)$ is *not* a link invariant, but $(-1)^{\chi(\beta)}$ and $c(\beta)$ are link invariants.

Clearly, for any braid β, we have $\deg^{\mathrm{top}}_a(\mathcal{P}_{\mathrm{KR}}(\beta)) \le -2\chi(\beta)$. Let $v \le w \in S_n$. Comparing (3.12) with (2.2), we find $\kappa_{v,w} = -2\chi(\beta_{v,w}) + c(\beta_{v,w}) - 1$, and thus the coefficient of $a^{-2\chi(\beta_{v,w})}$ in $\mathcal{P}_{\mathrm{KR}}(\beta_{v,w})$ is nonzero, by (3.14) combined with Theorem (ii). Therefore $\deg^{\mathrm{top}}_a(\mathcal{P}_{\mathrm{KR}}(\beta_{v,w})) = -2\chi(\beta_{v,w})$, and we get the following result.

PROPOSITION 3.12
For $v \le w \in S_n$, $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}) = [a^{-2\chi(\beta_{v,w})}]\mathcal{P}_{\mathrm{KR}}(\hat{\beta}_{v,w})$ is given by

$$\begin{aligned}\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}; q, t) &= (1-t)^{c(\beta_{v,w})-1}(q^{\frac{1}{2}}t^{-\frac{1}{2}})^{\chi(\beta_{v,w})} \\ &\quad \times \sum_{k,p\in\mathbb{Z}} q^{\frac{k}{2}} t^{p+\frac{k}{2}} \dim H^{k,(p)}\big(HH^0\big(F^\bullet(\beta_{v,w})\big)\big).\end{aligned} \tag{3.15}$$

Let us also define the analogous polynomial in the nonequivariant case (cf. Section 1.8). For $\beta \in \mathcal{B}_{S_n}$, set

$$\begin{aligned}\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta; q, t) &= \frac{(q^{\frac{1}{2}}t^{-\frac{1}{2}})^{\chi(\beta)}}{(1+q^{-\frac{1}{2}}t^{\frac{1}{2}})^{n-c(\beta)}} \\ &\quad \times \sum_{k,p\in\mathbb{Z}} q^{\frac{k}{2}} t^{p+\frac{k}{2}} \dim H^{k,(p)}\big(HH^0_{\mathbb{C}}\big(F^\bullet(\beta)\big)\big).\end{aligned} \tag{3.16}$$

The denominator $(1+q^{-\frac{1}{2}}t^{\frac{1}{2}})^{n-c(\beta)}$ in (3.16) is chosen in view of the discussion in Section 3.4: when β is a link with $c(\beta)$ components, a filtration analogous to the one in Proposition 3.9 would involve complexes with $2^{n-c(\beta)}$ terms.

The following result is a consequence of Corollary 3.10.

COROLLARY 3.13
Assume that $\hat{\beta}$ is a knot such that $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta; q, t) \ne 0$. Then

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta; q, t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta; q, t).$$

In Section 8.1, we give an alternative proof for knots of the form $\hat{\beta}_{v,w}$ for $v \le w \in S_n$.

3.6. Examples

We compute $HHH^0(F^\bullet(\beta))$ and $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta};q,t)$, as well as $HHH^0_{\mathbb{C}}(F^\bullet(\beta))$ and $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta;q,t)$, for a few braids β. Throughout, we assume that $G = \mathrm{PGL}_n(\mathbb{C})$, in which case recall that $R = \mathbb{C}[y_1,\dots,y_{n-1}]$ is a polynomial ring. These examples are summarized in Table 2. We abbreviate $\otimes_R$ by $\otimes$.

Example 3.14 (Unknot-I)

Let $n = 1$, $v = \mathrm{id}$, $w = \mathrm{id}$, and thus $c(\beta_{v,w}) = 1$ and $\chi(\beta_{v,w}) = 0$. We have $F^\bullet(\beta_{v,w}) = (0 \to \underline{R} \to 0)$ and $R = \mathbb{C}$. Thus, the only nonzero term is $H^{0,(0)}(HH^0(F^\bullet_{v,w})) \cong H^{0,(0)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})) \cong \mathbb{C}$. We have

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\hat{\beta}_{v,w};q,t) = 1.$$

Note that any $(1,b)$-torus knot is isotopic to the unknot, and we have $C_{1,b}(q,t) = 1$.

Example 3.15 (Unknot-II)

Let $n = 2$, $v = \mathrm{id}$, $w = s_1$, and thus $c(\beta_{v,w}) = 1$ and $\chi(\beta_{v,w}) = 0$. We have $F^\bullet_{v,w} = (B_{s_1} \to \underline{R})$. It is easy to see that $HH^0(B_{s_1})$ is a free R-module spanned by $(y_1 \otimes 1 + 1 \otimes y_1)$, and thus $HH^0(B_{s_1}) \cong R\{1\}$, and $HH^0(F^\bullet_{v,w}) = (R\{1\} \to \underline{R})$, with the map sending $1 \mapsto 2y_1$. The only nonzero term is $H^{0,(0)}(HH^0(F^\bullet_{v,w})) \cong \mathbb{C}$. Tensoring with $\mathbb{C}$, we get $HH^0_{\mathbb{C}}(F^\bullet_{v,w}) = (\mathbb{C}\{1\} \xrightarrow{0} \underline{\mathbb{C}})$, so there are two nonzero terms: $H^{0,(0)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})) \cong H^{-1,(1)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})) \cong \mathbb{C}$. Therefore,

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};q,t) = 1 \qquad \text{and} \qquad \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\hat{\beta}_{v,w};q,t) = \frac{1}{1+q^{-\frac{1}{2}}t^{\frac{1}{2}}}(1+q^{-\frac{1}{2}}t^{\frac{1}{2}}) = 1.$$

Let us also consider an example of $\hat{\beta}_{v,w}$ for $v \not\leq w$.

Example 3.16 (Unknot-III)

Let $n = 2$, $v = s_1$, $w = \mathrm{id}$, and thus $c(wv^{-1}) = 1$ and $\chi(\beta_{v,w}) = -1$. We have $F^\bullet_{v,w} = (\underline{R} \to B_{s_1}\{-1\})$, $HH^0(F^\bullet_{v,w}) = (\underline{R} \xrightarrow{\sim} R)$, and $HH^0_{\mathbb{C}}(F^\bullet_{v,w}) = (\underline{\mathbb{C}} \xrightarrow{\sim} \mathbb{C})$ so the right-hand sides of (3.15) and (3.16) are zero:

$$[a^2]\mathcal{P}_{\mathrm{KR}}(\hat{\beta}_{v,w};a,q,t) = 0 \qquad \text{and} \qquad \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w};q,t) = 0.$$

This is consistent with the fact that $\mathcal{P}_{\mathrm{KR}}(\hat{\beta}_{v,w})$ is a link invariant satisfying $\mathcal{P}_{\mathrm{KR}}(\bigcirc) = 1$; therefore, by (3.13), we have $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}) = 1$. (For a computation of $HH^1(F^\bullet(\beta_{v,w}))$, see [92].)

In the next two examples, we have $c(\beta) > 1$. We start with the case of the 2-*component unlink* . It is the closure of $\mathrm{id} \in \mathcal{B}_{S_2}$, but we consider the representative $\beta_{s_1,s_1} = \sigma_1\sigma_1^{-1}$ instead.

Example 3.17 (2-component unlink)
Let $n = 2$, $v = s_1$, $w = s_1$, and thus $c(wv^{-1}) = 2$ and $\chi(\beta_{v,w}) = 0$. We have

$$F^\bullet(\beta_{v,w}) = \big(B_{s_1} \to \underline{\big(R \oplus \big(B_{s_1} \otimes B_{s_1}\{-1\}\big)\big)} \to B_{s_1}\{-1\}\big).$$

We apply the well-known (see, e.g., [57, Example 3.12]) Soergel bimodule isomorphism $B_{s_1} \otimes B_{s_1} \cong B_{s_1}\{1\} \oplus B_{s_1}$, sending $1 \otimes 1 \otimes 1 \mapsto (0, 1 \otimes 1)$ and $1 \otimes y_1 \otimes 1 \mapsto (1 \otimes 1, 0)$. Next, we use Gaussian elimination (see [35, Exercise 19.12]) to obtain $F^\bullet(\beta_{v,w}) \cong F^\bullet(\mathrm{id})$ in $\mathrm{K^b}\mathbb{S}\mathrm{Bim}$, in agreement with Proposition 3.3. We have $R = \mathbb{C}[y_1]$, $HH^0(R) \cong R$, and $HH^0_{\mathbb{C}}(R) \cong \mathbb{C}$. (More generally, recall from Remark 3.4 that the R-module $HH^0(B)$ is always free.) Therefore, the only nonzero terms are $H^{0,(p)}(HH^0(F^\bullet_{v,w})) \cong \mathbb{C}$ for $p = 0, 1, 2, \dots$, and $H^{0,(0)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})) \cong \mathbb{C}$. We find

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}; q, t) = (1 - t)(1 + t + t^2 + \cdots) = 1 \qquad \text{and}$$
$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w}; q, t) = 1.$$

The *Hopf link* $\hat{\beta} =$  consists of two linked unknots. It is isotopic to $\hat{\beta}_{f_{2,4}}$, as well as to the closure of $(\sigma_1)^2 \in \mathcal{B}_{S_2}$.

Example 3.18 (Hopf link)
Let $n = 2$, $\beta = (\sigma_1)^2$, and thus $c(\beta) = 2$ and $\chi(\beta) = 1$. We have

$$F^\bullet(\beta) = (B_{s_1} \otimes B_{s_1} \to B_{s_1} \oplus B_{s_1} \to \underline{R}).$$

Using Gaussian elimination as in Example 3.17, we obtain

$$F^\bullet(\beta) \cong \big(B_{s_1}\{1\} \to B_{s_1} \to \underline{R}\big).$$

Here, the first map sends $1 \otimes 1 \mapsto y_1 \otimes 1 - 1 \otimes y_1$, and the second map sends $1 \otimes 1 \mapsto 1$. Taking R-invariants (cf. Remark 3.4), we find

$$HH^0\big(F^\bullet(\beta)\big) = \big(R\{2\} \xrightarrow{0} R\{1\} \xrightarrow{2y_1} \underline{R}\big). \tag{3.17}$$

We get $H^{0,(0)}(HH^0(F^\bullet(\beta))) \cong \mathbb{C}$ and $H^{-2}(HH^0(F^\bullet(\beta))) \cong R\{2\}$. In other words,

$$H^{0,(0)}\big(HH^0\big(F^\bullet(\beta)\big)\big) \cong \mathbb{C},\ H^{-2,(p)}\big(HH^0\big(F^\bullet(\beta)\big)\big) \cong \mathbb{C} \quad \text{for } p = 2, 3, 4, \dots. \tag{3.18}$$

Sending $y_1 \to 0$ in (3.17), we find

$$H^{k,(p)}\big(HH^0_{\mathbb{C}}\big(F^\bullet(\beta)\big)\big) \cong \begin{cases} \mathbb{C} & \text{if } (k,p) \in \{(0,0),(-1,1),(-2,2)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} \mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta};q,t) &= (1-t)q^{\frac{1}{2}}t^{-\frac{1}{2}}\big(1+t/q(1+t+t^2+\cdots)\big) \\ &= q^{\frac{1}{2}}t^{-\frac{1}{2}} - q^{\frac{1}{2}}t^{\frac{1}{2}} + q^{-\frac{1}{2}}t^{\frac{1}{2}}, \\ \mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta;q,t) &= q^{\frac{1}{2}}t^{-\frac{1}{2}}(1+q^{-\frac{1}{2}}t^{\frac{1}{2}}+t/q) = q^{\frac{1}{2}}t^{-\frac{1}{2}} + 1 + q^{-\frac{1}{2}}t^{\frac{1}{2}}. \end{aligned}$$

Remark 3.19
Since $\mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta};q,t)$ is a link invariant and $\hat{\beta} \cong \hat{\beta}_{2,4}$ (with $\chi(\hat{\beta}) = \chi(\hat{\beta}_{2,4})$), we see that $HHH^0(F^\bullet(\beta)) \cong HHH^0(F^\bullet(\beta_{2,4}))$ as bigraded vector spaces. Using an elaborate computation, one can also check that $\mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta;q,t) = \mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta_{2,4};q,t)$. However, observe that (3.16) has $(1+q^{-\frac{1}{2}}t^{\frac{1}{2}})^{n-c(\beta)}$ in the denominator, where $n=2$ for β and $n=4$ for $\beta_{2,4}$. Thus, $HHH^0_{\mathbb{C}}(F^\bullet(\beta_{2,4}))$ differs from $HHH^0_{\mathbb{C}}(F^\bullet(\beta))$ by "multiplication by $(1+q^{-\frac{1}{2}}t^{\frac{1}{2}})^2$," and the actual bigraded dimensions of $HHH^0_{\mathbb{C}}(F^\bullet(\beta_{2,4}))$ are given in Table 3 (bottom left).

Remark 3.20
We have a resolution of $\mathbb{C}$ by free R-modules: $0 \to R \xrightarrow{y_1} R \to \mathbb{C} \to 0$. Thus, $\operatorname{Tor}^R_\bullet(\mathbb{C},\mathbb{C}) = (\mathbb{C} \xrightarrow{0} \mathbb{C})$. Noting that $\operatorname{Tor}^R_\bullet(\mathbb{C},R) = \mathbb{C}$, we see that $HHH^0_{\mathbb{C}}(F^\bullet(\beta)) \cong \operatorname{Tor}^R_\bullet(\mathbb{C}, HHH^0(F^\bullet(\beta)))$. We conjecture that this holds more generally for all links (see (8.7)).

As we explained in Remark 1.14, for $k=2$ and $n=5$, $\hat{\beta}_{f_{k,n}}$ is the $(2,3)$-torus knot, which is isotopic to the trefoil knot: $\hat{\beta}_{f_{k,n}} \cong$ . It can be alternatively obtained as the closure of the braid $(\sigma_1)^3 \in \mathcal{B}_2$.

Example 3.21 (Trefoil knot)
Let $n=2$, $\beta = (\sigma_1)^3$, and thus $c(\beta)=1$ and $\chi(\beta)=1$. We have

$$F^\bullet(\beta) = (B_{s_1}^{\otimes 3} \to 3B_{s_1}^{\otimes 2} \to 3B_{s_1} \to \underline{R}).$$

Here $3B_{s_1}^{\otimes 2}$ denotes the direct sum of three copies of $B_{s_1} \otimes B_{s_1}$, and so on. Applying Gaussian elimination as in Example 3.17, we arrive at a simplified complex

$$F^\bullet(\beta) \cong \big(B_{s_1}\{2\} \to B_{s_1}\{1\} \to B_{s_1} \to \underline{R}\big)$$

with the three maps given by $1\otimes 1\mapsto y_1\otimes 1+1\otimes y_1$, $1\otimes 1\mapsto y_1\otimes 1-1\otimes y_1$, and $1\otimes 1\mapsto 1$, respectively. Taking R-invariants, we find

$$HH^0\big(F^\bullet(\beta)\big)=\big(R\{3\}\xrightarrow{2y_1}R\{2\}\xrightarrow{0}R\{1\}\xrightarrow{2y_1}\underline{R}\big).$$

We find that the only nonzero terms are

$$H^{0,(0)}\big(HH^0\big(F^\bullet(\beta)\big)\big)\cong H^{-2,(2)}\big(HH^0\big(F^\bullet(\beta)\big)\big)\cong\mathbb{C}.$$

Sending $y_1\to 0$, we also compute $H^{k,(p)}(HH^0_{\mathbb{C}}(F^\bullet(\beta)))$, which leads to

$$\mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta};q,t)=\mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta;q,t)=q^{\frac12}t^{-\frac12}+q^{-\frac12}t^{\frac12}.$$

The corresponding q,t-Catalan number is $C_{2,3}(q,t)=q+t=q^{\frac12}t^{\frac12}\cdot\mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta};q,t)$, in agreement with (4.9).

Remark 3.22
We have $\hat{\beta}\cong\hat{\beta}_{v,w}$ for $v=\text{id}$, $w=f_{2,5}\in S_5$, and $\chi(\beta)=\chi(\beta_{v,w})=1$; thus, $\mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta})=\mathcal{P}^{\text{top}}_{\text{KR}}(\hat{\beta}_{v,w})$. Similarly to Remark 3.19, we may compute that $\mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta;q,t)=\mathcal{P}^{\text{top}}_{\text{KR};\mathbb{C}}(\beta_{2,5};q,t)$, and thus $HHH^0_{\mathbb{C}}(F^\bullet(\beta_{2,5}))$ is given in Table 3 (bottom right).

4. Cohomology of positroid and Richardson varieties

We briefly review background on positroid varieties, Richardson varieties, and the various versions of cohomology that we will be using.

4.1. Positroid varieties

Recall from Section 1.2 that the Grassmannian $\text{Gr}(k,n)$ is identified with the space of $k\times n$ matrices modulo row operations. Given a $k\times n$ matrix A, we let $\text{RowSpan}(A)\in\text{Gr}(k,n)$ denote its row span and let $A_1,A_2,\dots,A_n$ be its columns. We extend this to a sequence $(A_j)_{j\in\mathbb{Z}}$ by requiring

$$A_{j+n}=A_j\quad\text{for all } j\in\mathbb{Z}.$$

Definition 4.1 ([77])
A bijection $f:\mathbb{Z}\to\mathbb{Z}$ is called a (k,n)-*bounded affine permutation* if it satisfies

- $f(j+n)=f(j)+n$ for all $j\in\mathbb{Z}$,
- $\sum_{j=1}^n(f(j)-j)=kn$, and
- $j\le f(j)\le j+n$ for all $j\in\mathbb{Z}$.

Alternatively, the second condition can be replaced with $k=\#\{j\in[n]\mid f(j)>n\}$.

We let $\mathbf{B}_{k,n}$ denote the (finite) set of (k,n)-bounded affine permutations. For a full rank $k \times n$ matrix A, we let $f_A : \mathbb{Z} \to \mathbb{Z}$ be given by

$$f_A(i) = \min\{j \geq i \mid A_i \in \mathrm{Span}(A_{i+1}, A_{i+2}, \dots, A_j)\} \quad \text{for } i \in \mathbb{Z}. \tag{4.1}$$

For example, if A_i is a zero column, then $f_A(i) = i$, and if A_i is not in the span of other columns, then $f_A(i) = i + n$. It is known (see [77]) that f_A is a (k,n)-bounded affine permutation which depends only on the row span of A. The *positroid stratification* of $\mathrm{Gr}(k,n)$ is given by

$$\mathrm{Gr}(k,n) = \bigsqcup_{f \in \mathbf{B}_{k,n}} \Pi^\circ_f, \quad \text{where } \Pi^\circ_f := \{\mathrm{RowSpan}(A) \in \mathrm{Gr}(k,n) \mid f_A = f\}.$$

We extend any permutation $u \in S_n$ to a bijection $\tilde{u} : \mathbb{Z} \to \mathbb{Z}$ satisfying $\tilde{u}(j+n) = \tilde{u}(j) + n$ for all n. We introduce a (k,n)-bounded affine permutation $\tau_{k,n} : \mathbb{Z} \to \mathbb{Z}$, determined by

$$\tau_{k,n}(j) = \begin{cases} j + n & \text{if } 1 \leq j \leq k, \\ j & \text{if } k+1 \leq j \leq n. \end{cases}$$

Recall that $\mathbf{Q}_{k,n} := \{(v,w) \in S_n \times S_n \mid v \leq w \text{ and } w \text{ is } k\text{-Grassmannian}\}$. The following result explains the bijection $(v,w) \mapsto f_{v,w}$ introduced in Proposition 1.10.

PROPOSITION 4.2 ([77, Proposition 3.15])
For every $f \in \mathbf{B}_{k,n}$, there exists a unique pair $(v,w) \in \mathbf{Q}_{k,n}$ such that $f = \tilde{w} \circ \tau_{k,n} \circ \tilde{v}^{-1}$.

Here, "$\circ$" denotes the usual composition of bijections $\mathbb{Z} \to \mathbb{Z}$. We thus define $f_{v,w} := \tilde{w} \circ \tau_{k,n} \circ \tilde{v}^{-1}$. Furthermore, we have the following relationship between positroid and Richardson varieties.

PROPOSITION 4.3 ([77, Theorem 5.9])
Let $G = \mathrm{PGL}_n(\mathbb{C})$. For each $f = f_{v,w} \in \mathbf{B}_{k,n}$, the natural projection map $\mathrm{Fl}(n) \to \mathrm{Gr}(k,n)$ restricts to an isomorphism $R^\circ_{v,w} \cong \Pi^\circ_f$. Thus, open positroid varieties are special cases of open Richardson varieties.

4.2. *Torus action and Richardson varieties*

The goal of this section is to prove Proposition 1.6. We start by generalizing one direction to Richardson varieties of arbitrary type; the type A specialization is discussed below. Let G be a complex semisimple algebraic group of adjoint type and of rank r, and let $\dot{G}$ denote the simply-connected group of the same Dynkin type. We use the notation $\dot{T}$, $\dot{B}$, $\dot{U}$ for the corresponding subgroups of $\dot{G}$.

Let us give a convenient well-known description (see [16, Theorem 2.3], [85, Lemma 2.2], or [46, Lemma 3.1]) of $R^\circ_{v,w}$ as an explicit affine variety. For $v \in W = N_{\dot{G}}(\dot{T})/\dot{T}$, let $\dot{v} \in \dot{G}$ denote an arbitrary fixed representative. For $v \le w$, denote

$$N^\circ_{v,w} := \dot{v}\dot{U}_- \cap \dot{U}_-\dot{v} \cap \dot{B}w\dot{B}. \tag{4.2}$$

Observe that the set $\dot{B}w\dot{B}$ does not depend on the choice of a representative for w.

LEMMA 4.4
The map $g \mapsto g\dot{B}/\dot{B}$ provides an isomorphism

$$N^\circ_{v,w} \xrightarrow{\sim} R^\circ_{v,w}. \tag{4.3}$$

For $u \in W$, let $\mu(u)$ denote the dimension of the eigenspace with eigenvalue 1 for u acting on $\mathfrak{h}^*$. We say that $u \in W$ is *elliptic* if $\mu(u) = 0$.

Let P denote the weight lattice of the root system of G (i.e., the character lattice of $\dot{G}$), and let $Q \subset P$ denote the root lattice (i.e., the character lattice of G). We let $\omega_1, \dots, \omega_r \in P$ denote the fundamental weights. For $u \in W$, we note that $(u - \mathrm{id})P \subseteq Q$.

For $\gamma, \delta \in P$, let $\Delta_{\gamma,\delta}$ denote the corresponding *generalized minor* (see [12], [39]) for $\dot{G}$. The condition that $g \in \dot{B}w\dot{B}$ implies that

$$\Delta_{w\omega_i,\omega_i}(g) \neq 0 \quad \text{for } i = 1, 2, \dots, r. \tag{4.4}$$

Indeed, by [39, Definition 1.4], we have $\Delta_{w\omega_i,\omega_i}(g) = \Delta_{\omega_i,\omega_i}(\overline{\overline{w^{-1}}}g)$ for a certain representative $\overline{\overline{w^{-1}}} \in G$ of w^{-1}, so (4.4) follows from [39, Proposition 2.9, Corollary 2.5].

PROPOSITION 4.5
Suppose that vw^{-1} is elliptic and $(vw^{-1} - \mathrm{id})P = Q$. Then T acts freely on $R^\circ_{v,w}$ and we have a T-equivariant isomorphism

$$R^\circ_{v,w} \cong (R^\circ_{v,w}/T) \times T.$$

Proof
Define

$$N^{\circ,\Delta=1}_{v,w} := \left\{x \in N^\circ_{v,w} \;\middle|\; \Delta_{\omega_i,w\omega_i}(g) = 1 \text{ for } i = 1, 2, \dots, r\right\}.$$

We will show that $N^{\circ,\Delta=1}_{v,w} \cong R^\circ_{v,w}/T$ and $R^\circ_{v,w} \cong N^{\circ,\Delta=1}_{v,w} \times T$.

The action of $\dot{T}$ on $R^\circ_{v,w}$ corresponds to the action $t \cdot g := tg\dot{v}^{-1}t^{-1}\dot{v}$ for $t \in \dot{T}$ and $g \in N^\circ_{v,w}$. Let $\chi_\omega = \Delta_{\omega,\omega} : \dot{T} \to \mathbb{C}$ denote the character corresponding to a weight $\omega \in P$. Then

$$\Delta_{w\omega_i,\omega_i}(tg\dot{v}^{-1}t^{-1}\dot{v}) = \chi_{w\omega_i}(t)\chi_{\omega_i}(\dot{v}^{-1}t^{-1}\dot{v})\Delta_{w\omega_i,\omega_i}(g).$$

Since $\chi_{\omega_i}(\dot{v}^{-1}t^{-1}\dot{v}) = \chi_{v\omega_i}(t^{-1})$, the weight of this generalized minor is $(w - v)\omega_i = -v(\mathrm{id} - v^{-1}w)\omega_i$. The condition $\mu(vw^{-1}) = 0$ implies that $\mathrm{id} - v^{-1}w$ is invertible. The condition $(vw^{-1} - \mathrm{id})P = Q$ implies that $(\mathrm{id} - v^{-1}w)\omega_1, (\mathrm{id} - v^{-1}w)\omega_2, \dots, (\mathrm{id} - v^{-1}w)\omega_r$ form a $\mathbb{Z}$-basis of Q. Since Q is the character lattice of T, it follows that the action of T on $N^\circ_{v,w}$ is free and that the functions $\Delta_{w\omega_i,\omega_i}$, $i = 1, 2, \dots, r$, can be simultaneously set to 1 by a unique element $t \in T$. It follows that $N^{\circ,\Delta=1}_{v,w} \cong R^\circ_{v,w}/T$ and $R^\circ_{v,w} \cong N^{\circ,\Delta=1}_{v,w} \times T$. □

Remark 4.6

It would be interesting to classify elliptic elements $u \in W$ satisfying the condition $(u - \mathrm{id})P = Q$, which depends only on the conjugacy class of u. It is not satisfied for all elliptic elements. For example, in type D_4, the longest element w_0 acts by $-\mathrm{id}$ on $\mathfrak{h}^*$, but we have $2P \subsetneq Q$ since $\frac{\alpha_i}{2} \notin P$ for any $i \in I$.

PROPOSITION 4.7

Suppose that $c \in W$ is a Coxeter element. Then c is elliptic and satisfies $(c - \mathrm{id})P = Q$.

Proof

We may assume that c is a standard Coxeter element. That is, $c = s_1s_2\cdots s_r$ where s_i are simple generators corresponding to positive simple roots $\alpha_1, \dots, \alpha_r$. Define roots $\beta_1 = \alpha_1$, $\beta_2 = s_1\alpha_2, \dots, \beta_r = s_1s_2\cdots s_{r-1}\alpha_r$. By [82, Lemma 10.2], c is elliptic. By [82, Proposition 10.5], we have $(\mathrm{id} - c)\omega_i = \beta_i \in \alpha_i + \sum_{j<i}\mathbb{Z}\alpha_j$. It follows that $(c - \mathrm{id})P = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i = Q$. □

Let us now consider the type A case, where $G = \mathrm{PGL}_n(\mathbb{C})$ and $\dot{G} = \mathrm{SL}_n(\mathbb{C})$. For a permutation $u \in S_n$, any of the following conditions are equivalent: (i) u is a single cycle, (ii) u is an elliptic element, and (iii) u is a Coxeter element. Each generalized minor $\Delta_{\omega_i,u\omega_i} : \dot{G} \to \mathbb{C}$ is the usual matrix minor with row set $\{1, 2, \dots, i\}$ and column set $\{u(1), u(2), \dots, u(i)\}$. The representative $\dot{v}$ in (4.2) may be chosen to be a signed permutation matrix of v. In the next result, $c(\cdot)$ denotes the number of cycles of a permutation (cf. (1.9); as opposed the Coxeter element c considered above).

COROLLARY 4.8

For all $v \le w \in S_n$ such that $c(wv^{-1}) = 1$, the T-action on $R^\circ_{v,w}$ is free and we have a T-equivariant isomorphism

$$R^\circ_{v,w} \cong (R^\circ_{v,w}/T) \times T.$$

Remark 4.9
When $R^\circ_{v,w} \cong \Pi^\circ_f$ (see Proposition 4.3), the functions $\Delta_{[1,i],w[1,i]}$ on $N^\circ_{v,w} \cong R^\circ_{v,w}$ coincide with the Plücker coordinates on Π°_f corresponding to the *Grassmann necklace* of f (see the proof of [46, Lemma 4.7]). In particular, for $f = f_{k,n}$, these are the cyclically consecutive maximal minors as in (1.4).

Proof of Proposition 1.6
If $c(\bar{f}) = 1$, then T acts freely on Π°_f by Corollary 4.8. We prove the converse. For $f \in \mathbf{B}_{k,n}$, let us construct a particular representative $X^{\min}_f \in \Pi^\circ_f$. If $f(i) = i$ for some $i \in \mathbb{Z}$, then the corresponding column is zero, so we may assume that $f(i) \neq i$ for all i. Let us write $\bar{f}$ in cycle notation:

$$\bar{f} = (a_1 a_2 \cdots a_{m_1})(a_{m_1+1} a_{m_1+2} \cdots a_{m_2}) \cdots (a_{m_r+1} \cdots a_n)$$

so that the minimal index of each cycle comes first. We label these indices left to right: set $\lambda(a_1) := 1$, and for $i = 1, 2, \ldots, n-1$, set $\lambda(a_{i+1}) = \lambda(a_i)$ if $a_i < a_{i+1}$ and they belong to the same cycle, and $\lambda(a_{i+1}) = \lambda(a_i) + 1$ otherwise. It is easy to check that $\lambda(a_n) = k$. The element $X^{\min}_f$ is the row span of the $k \times n$ matrix $M = (m_{i,j})$ whose only nonzero entries are $m_{\lambda(a_i),a_i} = 1$. One checks using (4.1) that $X^{\min}_f \in \Pi^\circ_f$. Furthermore, rescaling all columns that belong to a single cycle of $\bar{f}$ by the same value preserves the element $X^{\min}_f$. Therefore when $c(\bar{f}) > 1$, the T-action on Π°_f is not free. $\square$

4.3. *Mixed Hodge structure*

We follow the conventions of [83] (see [29] and [101] for further background). The results of [83] apply to *cluster varieties*. It was shown in [46] that open positroid varieties Π°_f, $f \in \mathbf{B}_{k,n}$ are cluster varieties. By [46, Lemma 3.6], setting the functions $\Delta_{w\omega_i,\omega_i}$ to 1 as we did in the proof of Proposition 4.5 corresponds to setting the frozen variables to 1 in the cluster structure on Π°_f. Thus, for $f \in \mathbf{B}^{c=1}_{k,n}$, Π°_f / T is a cluster variety with no frozen variables.

Consider a smooth complex algebraic variety Y of dimension d. By [101, Lemma-Definition 3.4], the cohomology $H^k(Y,\mathbb{C})$ and the compactly supported cohomology $H^k_c(Y,\mathbb{C})$ are endowed with a *Deligne splitting*

$$H^k(Y,\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}} H^{k,(p,q)}(Y,\mathbb{C}) \qquad \text{and} \qquad H^k_c(Y,\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}} H^{k,(p,q)}_c(Y,\mathbb{C}).$$

This splitting is functorial and satisfies the Poincaré duality (see [101, Theorem 6.23]):

$$H^{k,(p,q)}(Y,\mathbb{C}) \cong H^{2d-k,(d-p,d-q)}_c(Y,\mathbb{C}) \quad \text{for all } k, p, q \in \mathbb{Z}. \tag{4.5}$$

We say that the cohomology of Y is of *Hodge–Tate type* if $H^{k,(p,q)}(Y,\mathbb{C}) = 0$ whenever $p \neq q$. The notions of mixed Hodge structure and Hodge–Tate type also apply to equivariant cohomology.

Definition 4.10
Let Y be a d-dimensional complex variety whose cohomology is of Hodge–Tate type. Define its *mixed Hodge polynomial* $\mathcal{P}(Y;q,t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ by

$$\mathcal{P}(Y;q,t) := \sum_{k,p\in\mathbb{Z}} q^{p-\frac{k}{2}} t^{\frac{d-k}{2}} \dim H^{k,(p,p)}(Y,\mathbb{C}). \tag{4.6}$$

We have $H^{k,(p,p)}(Y,\mathbb{C}) = 0$ for $p > k$. By convention, we set $H^{k,(r,r)}(Y,\mathbb{C}) := 0$ for $r \notin \mathbb{Z}$.

It is convenient to record the dimensions of the spaces $H^{k,(p,p)}(Y,\mathbb{C})$ in a *mixed Hodge table*: the columns are labeled by $H^0, H^1, \dots, H^d$, while the rows are labeled by $k-p = 0,1,2,\dots$. Thus, an entry in a column labeled by H^k and in a row labeled by $k-p$ encodes the dimension of $H^{k,(p,p)}(Y,\mathbb{C})$. Examples of mixed Hodge tables are given in Tables 1 and 3, and in [83, Section 6]. For instance, we see from (4.6) that the two rows of Table 1 yield

$$\mathcal{P}(\Pi^{\circ}_{3,8}/T;q,t) = (q^4 + q^3 t + q^2 t^2 + q t^3 + t^4) + (q^2 t + q t^2),$$

confirming the computation in Example 1.9.

We say that a polynomial $\mathcal{P}(q,t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ is *q,t-symmetric* if $\mathcal{P}(q,t) = \mathcal{P}(t,q)$. We say that $\mathcal{P}$ is *q, t-unimodal* if for each $a,b \in \frac{1}{2}\mathbb{Z}$, the coefficients $([q^{a-k}t^{b+k}]\mathcal{P})_{k\in\mathbb{Z}}$ form a unimodal sequence.[6] Recall that Theorem 1.18, proved in Section 8.2, states that the polynomial $\mathcal{P}(R^{\circ}_{v,w};q,t)$ is q,t-symmetric for all $v \le w \in W$.

A special case of Theorem 1.18 for positroid varieties follows from the *curious Lefschetz theorem* proved in [83, Theorem 8.3]. We say that a $2d$-dimensional complex algebraic variety Y of Hodge–Tate type satisfies the curious Lefschetz theorem if there is a class $\gamma \in H^{2,(2,2)}(Y,\mathbb{C})$ inducing isomorphisms

$$\smile \gamma^{d-p} : H^{p+s,(p,p)}(Y,\mathbb{C}) \cong H^{2d-p+s,(2d-p,2d-p)}(Y,\mathbb{C}). \tag{4.7}$$

As explained in [83], for cluster varieties, one can choose γ to be the Gekhtman–Shapiro–Vainshtein form in [51]. In the case that Π°_f is odd-dimensional, the product $\Pi^{\circ}_f \times \mathbb{C}^*$ will satisfy (4.7), and using the Künneth theorem, unimodality and symmetry for $\mathcal{P}(\Pi^{\circ}_f;q,t)$ can be deduced.

[6] This property is sometimes called *parity unimodality* since the terms with integer degrees and with half-integer degrees are required to form separate unimodal sequences.

The curious Lefschetz theorem for positroid varieties relies on the fact that they admit cluster structures (see [46]), which is not yet available for Richardson varieties. Another consequence of the curious Lefschetz property is that $\mathcal{P}(\Pi^\circ_f; q,t)$ is q,t-unimodal. The question of whether $\mathcal{P}(R^\circ_{v,w}; q,t)$ is q,t-unimodal for arbitrary $v \leq w \in W$ remains open.

4.4. *Proof of Theorem 1.1 from Theorem 1.17*

Let $G = \mathrm{PGL}_n(\mathbb{C})$ and $f = f_{v,w} \in \mathbf{B}^{c=1}_{k,n}$; thus, we have $R^\circ_{v,w} = \Pi^\circ_f$. We set

$$\begin{aligned} \ell_{v,w} &= \ell(w) - \ell(v) = \dim(R^\circ_{v,w}) = \dim(\Pi^\circ_f), \\ d_f &:= \dim(\mathcal{X}^\circ_f) = \ell_{v,w} - n + 1. \end{aligned}$$

Since $c(f) = 1$, we have $d_f = 2\chi(\beta_f)$ by (3.12), and the T-action on Π°_f is free by Proposition 1.6. In this case, we have

$$H^{k+n-1,(p,p)}_{T,c}(\Pi^\circ_f) \cong H^{k,(p,p)}_c(\mathcal{X}^\circ_f) \cong H^{2d_f-k,(d_f-p,d_f-p)}(\mathcal{X}^\circ_f), \tag{4.8}$$

where the first isomorphism is Lemma 4.14 below, and the second isomorphism is the Poincaré duality (4.5).

Therefore, (1.22) yields

$$\mathcal{P}(\mathcal{X}^\circ_f; q,t) = (q^{\frac{1}{2}} t^{\frac{1}{2}})^{\chi(\beta_f)} \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_f; q,t). \tag{4.9}$$

This is a special case of (1.24), which is proved in a similar way in Section 8.1. In the case $f = f_{k,n}$ and $\gcd(k,n) = 1$, Remark 1.14 implies that $\hat{\beta}_f$ is a torus knot, for which the right-hand side of (4.9) was shown by Mellit [92] to coincide with $C_{k,n-k}(q,t)$. Thus, Theorem 1.1 follows from Theorem 1.17.

4.5. *Mixed Hodge structures of open Richardson varieties*

THEOREM 4.11
For any $v \leq w \in W$, the cohomology $H^\bullet(R^\circ_{v,w}, \mathbb{C})$, the compactly supported cohomology $H^\bullet_c(R^\circ_{v,w}, \mathbb{C})$, and the compactly supported T-equivariant cohomology $H^\bullet_{T,c}(R^\circ_{v,w}, \mathbb{C})$ of $R^\circ_{v,w}$ are of Hodge–Tate type.

We have omitted equivariant cohomology $H^\bullet_T(R^\circ_{v,w}, \mathbb{C})$ from Theorem 4.11 because the statement of Poincaré duality in the equivariant setting is considerably more complicated than for ordinary cohomology.

LEMMA 4.12
Let Y be a complex algebraic variety, let $U \subset Y$ be an open subvariety, and let $Z := Y \setminus U$. Suppose that the compactly supported cohomologies of U and Z are of Hodge–Tate type. Then the same is true for Y.

The same statement holds for compactly supported T-equivariant cohomology with the assumption that U, Z are T-stable.

Proof
We have a Gysin long exact sequence for the triple (Y, Z, U) (see, e.g., [101, (B-15)]):

$$\cdots \to H_c^k(U,\mathbb{C}) \to H_c^k(Y,\mathbb{C}) \to H_c^k(Z,\mathbb{C}) \to H_c^{k+1}(U,\mathbb{C}) \to \cdots, \tag{4.10}$$

the maps of which respect the mixed Hodge structure (see [28, Theorem 4.1]).

Taking the (p,q) piece of the Deligne splitting, we have

$$\cdots \to H_c^{k,(p,q)}(U,\mathbb{C}) \to H_c^{k,(p,q)}(Y,\mathbb{C}) \to H_c^{k,(p,q)}(Z,\mathbb{C}) \to \cdots.$$

By assumption, when $p \neq q$, we have $H_c^{k,(p,q)}(U,\mathbb{C}) = 0 = H_c^{k,(p,q)}(Z,\mathbb{C})$. Thus, $H_c^{k,(p,q)}(Y,\mathbb{C}) = 0$. The same proof applies in compactly supported equivariant cohomology. □

Proof of Theorem 4.11
Since $R^\circ_{v,w}$ is smooth, by (4.5) it suffices to show that the compactly supported cohomology and the compactly supported equivariant cohomology are of Hodge–Tate type.

We will prove the statement by induction on $\ell_{v,w} = \dim(R^\circ_{v,w})$. The statement clearly holds if $\ell_{v,w} = 0$, for then $R^\circ_{v,w}$ is a point. We will use a recursion for the varieties $R^\circ_{v,w}$ from [106]. By [106, Lemma 4.3.1, Proof of Proposition 4.3.6], for any open Richardson $Y = R^\circ_{v,w}$ with $w > v$, we can find a decomposition $Y = U \sqcup Z$, where $U \subset Y$ is open and $Z \subset Y$ is closed, and we have isomorphisms

$$Z \cong Y' \times \mathbb{C} U \cong Y'' \times \mathbb{C}^*, \tag{4.11}$$

where Y', Y'' are open Richardson varieties of lower dimension, and Z is possibly empty. Furthermore, U, Z are T-stable and the isomorphisms (4.11) are torus-equivariant, for certain linear actions of T on $\mathbb{C}$, $\mathbb{C}^*$. By the Künneth formula, the Hodge–Tate type property is preserved under products. By the inductive hypothesis, the compactly supported (equivariant) cohomology of U and of Z (when nonempty) are therefore of Hodge–Tate type. It follows that the same statement holds for Y. □

The following corollary of Theorem 4.11 also follows from combining [83, Theorem 8.3] and [46].

COROLLARY 4.13
For all $f \in \mathbf{B}_{k,n}$, the cohomology of Π°_f is of Hodge–Tate type. If $c(\bar{f}) = 1$, then the cohomology of Π°_f / T is also of Hodge–Tate type.

Proof
For the second statement, we note that T acts freely on Π°_f by Corollary 4.8. Thus, $H^\bullet_T(\Pi^\circ_f, \mathbb{C}) \cong H^\bullet(\Pi^\circ_f / T, \mathbb{C})$ is of Hodge–Tate type by Theorem 4.11. ☐

When computing examples in the next section, we shall repeatedly use the following result.

LEMMA 4.14
Suppose that T acts freely on a complex algebraic variety Y, and let $d := \dim_{\mathbb{C}}(T)$. Then we have an isomorphism of mixed Hodge structures

$$H^{k+d,(p,q)}_{T,c}(Y, \mathbb{C}) \cong H^{k,(p,q)}_c(Y/T, \mathbb{C}) \quad \textit{for all } k, p, q \in \mathbb{Z},$$

and the action of $H^\bullet_T(\mathrm{pt}, \mathbb{C})$ on $H^\bullet_{T,c}(Y, \mathbb{C})$ is trivial (i.e., factors through the map $H^\bullet_T(\mathrm{pt}, \mathbb{C}) \to H^0_T(\mathrm{pt}, \mathbb{C}) \cong \mathbb{C}$).

Proof
Suppose that $T = (\mathbb{C}^*)^d$, and let $E_m := (\mathbb{C}^m \setminus \{0\})^d$ be a finite-dimensional approximation to a contractible space ET where T acts freely. Then by definition

$$\begin{aligned} H^k_{T,c}(Y) &= \lim_{m\to\infty} H^k_c(Y \times_T E_m) \\ &= \lim_{m\to\infty} H^k_c(Y/T \times E_m) \\ &= \lim_{m\to\infty} \bigoplus_j H^{k-j}_c(Y/T) \otimes H^j_c(E_m) \quad \text{by the Künneth formula} \\ &= H^{k-d}_c(Y/T). \end{aligned}$$

We have used the fact that $H^\bullet_c(\mathbb{C}^m \setminus \{0\})$ is $\mathbb{C}$ in dimensions 1 and $2m$ and vanishes in other dimensions. Since $H^1_c(\mathbb{C}^m \setminus \{0\})$ is of type $(0,0)$, the isomorphism is compatible with mixed Hodge structures. ☐

4.6. Examples
We compute the compactly supported torus-equivariant cohomology of some open Richardson varieties, corresponding to the examples computed in Section 3.6. As in Section 3.6, we label each example by the link associated to the Richardson variety. Using Table 2, one can compare the examples below with the ones in Section 3.6

and observe that the computations agree with our main results, Theorems 1.16 and 1.17.

Recall that we say that the action of $R = \mathbb{C}[\mathfrak{h}]$ is *trivial* if $\mathfrak{h}^*$ acts by zero. We identify R with $H_T^\bullet(\mathrm{pt}, \mathbb{C})$.

Example 4.15 (Unknot-I)
Let $n = 1$ and $v = w = \mathrm{id}$, and thus $\ell_{v,w} = 0$. Then $R_{v,w}^\circ \cong T \cong \mathrm{pt}$, and the T-action is free. The only nonzero terms in the cohomology are

$$H^{0,(0,0)}(\mathrm{pt}) \cong H_c^{0,(0,0)}(\mathrm{pt}) \cong H_{T,c}^{0,(0,0)}(\mathrm{pt}) \cong \mathbb{C}. \tag{4.12}$$

As in Example 3.14, the $R = \mathbb{C}$-module structure on $H_{T,c}^\bullet(\mathrm{pt})$ is trivial (since T acts freely).

Example 4.16 (Unknot-II)
Let $n = 2$, $v = \mathrm{id}$, and $w = s_1$, and thus $\ell_{v,w} = 1$. Then $R_{v,w}^\circ \cong \Pi_{1,2}^\circ \subset \mathrm{Gr}(1,2)$ and we have $R_{v,w}^\circ \cong T \cong \mathbb{C}^*$. The T-action is free and $R_{v,w}^\circ/T \cong \mathrm{pt}$. Using Poincaré duality (4.5) and Lemma 4.14, we find that the only nonzero terms are

$$\begin{aligned} &H^{0,(0,0)}(\mathbb{C}^*) \cong H^{1,(1,1)}(\mathbb{C}^*) \cong \mathbb{C}, \\ &H_c^{2,(1,1)}(\mathbb{C}^*) \cong H_c^{1,(0,0)}(\mathbb{C}^*) \cong \mathbb{C}, \qquad \text{and} \\ &H_{\mathbb{C}^*,c}^{1,(0,0)}(\mathbb{C}^*) \cong \mathbb{C}. \end{aligned} \tag{4.13}$$

As in Example 3.15, the $R = \mathbb{C}[y_1]$-module structure on $H_{T,c}^\bullet(R_{v,w}^\circ)$ is trivial (since T acts freely).

Example 4.17 (2-component unlink)
Let $n = 2$, $v = s_1$, and $w = s_1$, and thus $\ell_{v,w} = 0$. We have $R_{v,w}^\circ \cong \mathrm{pt}$ and $T \cong \mathbb{C}^*$. The T-action is not free. We have already computed $H^\bullet(\mathrm{pt})$ and $H_{T,c}^\bullet(\mathrm{pt})$ in (4.12). The nonzero terms of $H_{T,c}^\bullet(R_{v,w}^\circ) \cong H_{\mathbb{C}^*,c}^\bullet(\mathrm{pt})$ are given by

$$H_{T,c}^{2p,(p,p)}(\mathrm{pt}) \cong \mathbb{C} \quad \text{for } p = 0, 1, 2, \dots.$$

As in Example 3.17, we have $H_{T,c}^\bullet(R_{v,w}^\circ) \cong H_{T,c}^\bullet(\mathrm{pt}) \cong R$ as an R-module.

Example 4.18 (Hopf link)
Let $n = 4$, $v = \mathrm{id}$, $w = f_{2,4} = s_2 s_1 s_3 s_2$, and thus $\ell_{v,w} = 4$. We have $Y := R_{v,w}^\circ \cong \Pi_{2,4}^\circ \subset \mathrm{Gr}(2,4)$, an open positroid variety of dimension 4. It is isomorphic to

$$Y \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| (a,b,c,d) \in \mathbb{C}^4 : a \neq 0, d \neq 0, ad - bc \neq 0 \right\}.$$

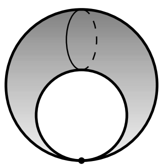
The action of $T \cong (\mathbb{C}^*)^3$ on Y is not free: T acts by rescaling rows and columns in such a way that the first two columns form the identity matrix. There exists a 2-dimensional torus $T' \subset T$ that acts freely on Y; for example, one can always rescale columns 3 and 4 uniquely (since $a, d \neq 0$) to force the minors $\Delta_{2,3}$ and $\Delta_{1,4}$ to be equal to 1:

$$Y/T' \cong \left\{ \begin{pmatrix} 1 & 0 & -1 & -y \\ 0 & 1 & x & 1 \end{pmatrix} \middle| (x,y) \in \mathbb{C}^2 : xy \neq 1 \right\}.$$

The quotient Y/T' can be identified with the 2-dimensional A_1-cluster variety (with one frozen variable; cf. [83, Section 6.1]). We denote it by

$$U := Y/T' \cong \{(x,y) \in \mathbb{C}^2 \mid xy \neq 1\} \subset \mathbb{C}^2.$$

The action of T/T' can be identified with the action of $\mathbb{C}^*$ on U with $\lambda \cdot (x,y) = (\lambda x, \lambda^{-1} y)$ for $\lambda \in \mathbb{C}^*$ and $(x,y) \in \mathbb{C}^2$. Forgetting this torus action, it was shown in [83, Corollary 7.2] that U is homotopy equivalent[7] to a pinched torus:



Therefore the Betti numbers of U are $(1,1,1)$, and moreover we have (see [83, Section 6.1])

$$H^{0,(0,0)}(U) \cong H^{1,(1,1)}(U) \cong H^{2,(2,2)}(U) \cong \mathbb{C}.$$

We have $Y \cong U \times (\mathbb{C}^*)^2$, which corresponds to multiplying the mixed Hodge polynomial of U by $(q^{\frac{1}{2}} + t^{\frac{1}{2}})^2$. The resulting mixed Hodge table of Y, whose sole row contains the coefficients of the polynomial $(q + q^{\frac{1}{2}} t^{\frac{1}{2}} + t) \cdot (q^{\frac{1}{2}} + t^{\frac{1}{2}})^2$, is given in Table 3 (top left).

Let us return to computing the $\mathbb{C}^*$-equivariant cohomology of U. Denote $W := \mathbb{C}^2$, and let $Z := W \setminus U$ be the hyperbola $\{(x,y) \mid xy = 1\}$. We first compute the compactly supported equivariant cohomologies $H^\bullet_{\mathbb{C}^*,c}(Y)$ and $H^\bullet_{\mathbb{C}^*,c}(Z)$. Set $R := H^\bullet_{\mathbb{C}^*}(\mathrm{pt})$.

First, suppose that $\mathbb{C}^*$ acts on $\mathbb{C}^m$ linearly in any way. Then it follows directly from the definitions that $H^\bullet_{\mathbb{C}^*,c}(\mathbb{C}^m)$ is a free R-module with generator in degree $2m$. Specifically, all nonzero terms are given by

[7]We caution the reader that neither the compactly supported cohomology nor the mixed Hodge structure are preserved by homotopy equivalences.

$$H^{2(m+k),(m+k,m+k)}_{\mathbb{C}^*,c}(\mathbb{C}^m) \cong \mathbb{C} \quad \text{for } k = 0,1,2,\dots. \tag{4.14}$$

Next, observe that the variety Z is $\mathbb{C}^*$-equivariantly isomorphic to the variety $\mathbb{C}^*$ on which $\mathbb{C}^*$ acts freely. By (4.14),

$$H^{1,(0,0)}_{\mathbb{C}^*,c}(Z) \cong \mathbb{C}. \tag{4.15}$$

Now we compute $H^{\bullet}_{\mathbb{C}^*,c}(U)$. The Gysin sequence (4.10) for the triple (W,Z,U) gives

$$\begin{aligned} 0 &\to H^0_{\mathbb{C}^*,c}(U) \to H^0_{\mathbb{C}^*,c}(W) \to H^0_{\mathbb{C}^*,c}(Z) \\ &\to H^1_{\mathbb{C}^*,c}(U) \to H^1_{\mathbb{C}^*,c}(W) \to H^1_{\mathbb{C}^*,c}(Z) \to \cdots. \end{aligned}$$

Applying (4.14)–(4.15), we get

$$\begin{aligned} 0 &\to H^0_{\mathbb{C}^*,c}(U) \to 0 \to 0 \to H^1_{\mathbb{C}^*,c}(U) \to 0 \to \mathbb{C} \to H^2_{\mathbb{C}^*,c}(U) \to 0 \to 0 \\ &\to H^3_{\mathbb{C}^*,c}(U) \to 0 \to 0 \to H^4_{\mathbb{C}^*,c}(U) \to \mathbb{C} \to 0 \to H^5_{\mathbb{C}^*,c}(U) \to 0 \to 0 \to \cdots. \end{aligned}$$

We conclude that the nonzero terms of $H^{\bullet}_{\mathbb{C}^*,c}(U)$ are given by

$$H^{2,(0,0)}_{\mathbb{C}^*,c}(U) \cong \mathbb{C} \qquad \text{and} \qquad H^{4+2k,(2+k,2+k)}_{\mathbb{C}^*,c}(U) \cong \mathbb{C} \quad \text{for } k = 0,1,2,\dots.$$

By the same computation as in the proof of Lemma 4.14, we have $H^{k+2,(p,q)}_{T,c}(Y) \cong H^{k,(p,q)}_{\mathbb{C}^*,c}(U)$. Thus, the nonzero terms of $H^{\bullet}_{T,c}(Y)$ are

$$\begin{aligned} H^{4,(0,0)}_{T,c}(Y) &\cong \mathbb{C} \qquad \text{and} \\ H^{6+2k,(2+k,2+k)}_{T,c}(Y) &\cong \mathbb{C} \quad \text{for } k = 0,1,2,\dots. \end{aligned} \tag{4.16}$$

Recall that $\ell_{v,w} = 4$. In view of (1.22), the dimensions in (4.16) match perfectly with those computed in (3.18) from the Soergel bimodule perspective.

Remark 4.19
We observe that the $R = \mathbb{C}[y_1]$-module structure on $H^{\bullet}_{T,c}(Y)$ also agrees with that computed in (3.18). More generally, for $W = S_n$, Corollary 4.8 can be extended to arbitrary open Richardson varieties $R^\circ_{v,w}$: we have a subtorus $T' \subset T$ of dimension $n - c(\beta)$ acting freely on $R^\circ_{v,w}$, and $H^{\bullet}_{T,c}(R^\circ_{v,w}) \cong H^{\bullet}_{T/T',c}(R^\circ_{v,w}/T')$. Recall from Section 3.4 that we may therefore view both sides of (1.21) as graded modules over a polynomial ring R in $c(\beta) - 1$ variables, and we expect that these R-modules are isomorphic under the grading change (1.22).

It would be interesting to combine Example 4.18 with [83] to obtain a description of the equivariant cohomology of more general cluster varieties.

Example 4.20 (Trefoil knot)
Let $n = 5$, $v = \mathrm{id}$, $w = f_{2,5} = s_3s_2s_1s_4s_3s_2$, and thus $\ell_{v,w} = 6$. We have $R^\circ_{v,w} \cong \Pi^\circ_{2,5}$, on which the torus $T \cong (\mathbb{C}^*)^5$ acts freely. As explained in Remark 4.9, the quotient is obtained by fixing the cyclically consecutive maximal minors to 1. An explicit parameterization can be chosen as follows:

$$\Pi^\circ_{2,5}/T \cong \left\{ \begin{pmatrix} 1 & 0 & -1 & -y & \frac{1+y}{1-xy} \\ 0 & 1 & x & xy-1 & 1 \end{pmatrix} \middle| (x,y) \in \mathbb{C}^2 : xy \neq 1 \right\}$$
$$\sqcup \left\{ \begin{pmatrix} 1 & 0 & -1 & 1 & z \\ 0 & 1 & -1 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{C} \right\}.$$

Observe that the point count therefore equals $(q^2 - q + 1) + q = q^2 + 1 = q \cdot C_{2,3}(q, 1/q)$. The variety $\Pi^\circ_{2,5}/T$ is a 2-dimensional cluster variety of type A_2 with no frozen variables. Its cohomology was computed in [83, Section 6.2]: the nonzero terms are

$$H^{0,(0,0)}(\Pi^\circ_{2,5}/T) \cong H^{2,(2,2)}(\Pi^\circ_{2,5}/T) \cong \mathbb{C}.$$

Multiplying the mixed Hodge polynomial by $(q^{\frac{1}{2}} + t^{\frac{1}{2}})^4$, we see that the mixed Hodge table of $\Pi^\circ_{2,5}$ is given in Table 3 (top right). By (4.5) and Lemma 4.14, we find

$$H_c^{2,(0,0)}(\Pi^\circ_{2,5}) \cong H_c^{4,(2,2)}(\Pi^\circ_{2,5}) \cong \mathbb{C} \qquad \text{and}$$
$$H_{T,c}^{6,(0,0)}(\Pi^\circ_{2,5}) \cong H_{T,c}^{8,(2,2)}(\Pi^\circ_{2,5}) \cong \mathbb{C}.$$

As in Example 3.21, the R-module structure on $H^\bullet_{T,c}(R^\circ_{v,w})$ is trivial (since T acts freely).

We now give three examples of Richardson and positroid knots with nonvanishing odd cohomology, as promised in Section 1.12.5. Here by a *positroid knot* we mean a knot of the form $\hat{\beta}_f$ for $f \in \mathbf{B}^{c=1}_{k,n}$.

Example 4.21 (Odd cohomology-I)
Let $n = 5$, $v = s_3$, and $w = s_2s_3s_4s_3s_1s_2s_1$, and thus $\ell_{v,w} = 6$ and $c(wv^{-1}) = 1$. The Richardson knot $\beta_{v,w}$ is the 3-*twist knot*, listed as 5_2 in Rolfsen's table (see [108]). By (2.3), the point count is given by

$$\#(R^\circ_{v,w}/T)(\mathbb{F}_q) = q^2 - q + 1.$$

The appearance of $-q$ implies that the cohomology of $R^\circ_{v,w}/T$ *cannot* be concentrated in even degrees.

The following two examples of *positroid* knots were discovered by David Speyer jointly with the second author.

Example 4.22 (Odd cohomology-II)
Let $k = 7$, $n = 14$, and let $f = f_{v,w} \in \mathbf{B}_{k,n}^{c=1}$ and $v \le w \in S_n$ be given by

$$f = [3, 8, 9, 16, 7, 14, 15, 20, 12, 18, 13, 24, 19, 25],$$
$$v = (2, 1, 8, 4, 6, 3, 11, 9, 10, 5, 7, 13, 14, 12),$$
$$w = (8, 9, 10, 1, 11, 12, 2, 3, 13, 4, 14, 5, 6, 7).$$

Here v and w are given in one-line notation, and $f = [f(1), f(2), \dots, f(n)]$ is given in *window notation*. The permutation w is 7-Grassmannian, and we have $c(\bar{f}) = 1$, $\ell(v) = 19$, $\ell(w) = 40$, and $d_f = 8$. The mixed Hodge table of $\mathcal{X}_f^\circ$ is given by

H^k	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	1	1		

In particular, H^5 is nonvanishing.

Example 4.23 (Odd cohomology-III)
Let $k = 7$, $n = 14$, and let $f = f_{v,w} \in \mathbf{B}_{k,n}^{c=1}$ and $v \le w \in S_n$ be given by

$$f = [7, 4, 15, 13, 11, 8, 19, 16, 14, 12, 23, 20, 17, 24],$$
$$v = (1, 3, 6, 9, 2, 5, 8, 12, 4, 7, 11, 14, 10, 13),$$
$$w = (8, 9, 1, 10, 11, 12, 2, 3, 13, 14, 4, 5, 6, 7).$$

Similarly, w is 7-Grassmannian, $c(\bar{f}) = 1$, $\ell(v) = 19$, $\ell(w) = 40$, and $d_f = 8$. One can easily compute (e.g., using Theorem 1.13) that the point count is given by

$$\#\mathcal{X}_f^\circ(\mathbb{F}_q) = q^8 + q^6 + 3q^5 - q^4 + 3q^3 + q^2 + 1.$$

Similarly to Example 4.21, this polynomial has a negative term, and therefore the odd cohomology does not vanish.

Remark 4.24
Ivan Cherednik has suggested to us that one might expect odd cohomology vanishing for algebraic knots (see [102] and [23, Section 3.4]). The knot 5_2 in Example 4.21 is not algebraic. We thank the anonymous referee for pointing out to us that the positroid knot in Example 4.22 is also not algebraic.

5. Equivariant derived categories of flag varieties

Rather than working with mixed Hodge structures, our proof will be mostly stated in the language of weights (see [30]) on étale cohomology.[8] We assume that the reader is familiar with derived categories of ℓ-adic sheaves in the equivariant setting (see, e.g., [10], [13], [84] for relevant background).

5.1. Conventions

Fix a prime power q. We shall consider schemes over the finite field $\mathbb{F}_q$ and its algebraic closure $\overline{\mathbb{F}}_q$. For an $\mathbb{F}_q$-scheme Y, let $Y_{\overline{\mathbb{F}}_q} := Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ denote the base change to an $\overline{\mathbb{F}}_q$-scheme. We have the Frobenius automorphism $\mathrm{Fr} : Y_{\overline{\mathbb{F}}_q} \to Y_{\overline{\mathbb{F}}_q}$ whose fixed points are exactly the points of $Y_{\overline{\mathbb{F}}_q}$ defined over $\mathbb{F}_q$.

Fix a prime number ℓ different from the characteristic of $\mathbb{F}_q$, and let $\Bbbk := \overline{\mathbb{Q}}_\ell$. Let H be an algebraic group acting on an $\mathbb{F}_q$-variety Y. We consider the bounded derived category $\mathrm{D}^{\mathrm{b}}_{(H)}(Y, \Bbbk)$ of mixed H-constructible (i.e., constructible along H-orbits) $\Bbbk$-sheaves on Y, as well as the corresponding category $\mathrm{D}^{\mathrm{b}}_{(H)}(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$ of H-constructible $\Bbbk$-sheaves on $Y_{\overline{\mathbb{F}}_q}$ (see [10]). We also let $\mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk)$ and $\mathrm{D}^{\mathrm{b}}_H(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$ denote the corresponding H-equivariant bounded derived categories as in [13] (see also [130] for a discussion of mixed derived categories in the equivariant setting). There are functors $\mathrm{For} : \mathrm{D}^{\mathrm{b}}_H(Y_{\overline{\mathbb{F}}_q}, \Bbbk) \to \mathrm{D}^{\mathrm{b}}_{(H)}(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$ and $\mathrm{For} : \mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk) \to \mathrm{D}^{\mathrm{b}}_{(H)}(Y, \Bbbk)$ forgetting the equivariant structure. There are also functors $\omega : \mathrm{D}^{\mathrm{b}}_{(H)}(Y, \Bbbk) \to \mathrm{D}^{\mathrm{b}}_{(H)}(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$ and $\omega : \mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk) \to \mathrm{D}^{\mathrm{b}}_H(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$ obtained by extension of scalars from $\mathbb{F}_q$ to $\overline{\mathbb{F}}_q$.

The language of algebraic stacks [84] allows us to switch between ordinary and equivariant derived categories at our convenience. For example, $\mathrm{D}^{\mathrm{b}}([Y/H], \Bbbk) \cong \mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk)$.

We denote by $[m]$ the cohomological shift m steps to the left in a derived category as in Section 3.2. For $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk)$ and $k \in \mathbb{Z}$, let

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^k_Y(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_H(Y_{\overline{\mathbb{F}}_q}, \Bbbk)}\big(\omega\mathcal{F}, \omega\mathcal{G}[k]\big) \quad \text{for } k \in \mathbb{Z}. \tag{5.1}$$

The space $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_H(Y_{\overline{\mathbb{F}}_q}, \Bbbk)}(\omega\mathcal{F}, \omega\mathcal{G}[k])$ has a natural action of the Frobenius Fr. Therefore, $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G})$ is a graded $H^\bullet_H(\mathrm{pt}, \Bbbk)$-module equipped with an action of Fr, or in other words, an $(H^\bullet_H(\mathrm{pt}, \Bbbk), \mathrm{Fr})$-module. The actual extension groups in $\mathrm{D}^{\mathrm{b}}_H(Y, \Bbbk)$ are denoted by $\mathrm{ext}^k(\mathcal{F}, \mathcal{G})$, and are related to $\mathrm{Ext}^k(\mathcal{F}, \mathcal{G})$ by the exact sequence (see [10, (5.1.2.5)])

$$0 \to \mathrm{Ext}^{i-1}(\mathcal{F}, \mathcal{G})_{\mathrm{Fr}} \to \mathrm{ext}^i(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^i(\mathcal{F}, \mathcal{G})^{\mathrm{Fr}} \to 0, \tag{5.2}$$

[8] As pointed out to us by Wolfgang Soergel, our results could also be formulated using the language of equivariant mixed Tate motives (see [120]).

where $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})^{\mathrm{Fr}}$ and $\mathrm{Ext}^{i-1}(\mathcal{F},\mathcal{G})_{\mathrm{Fr}}$ denote Frobenius invariants and coinvariants, respectively. We denote by $\hom(\mathcal{F},\mathcal{G}) = \mathrm{ext}^0(\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)}(\mathcal{F},\mathcal{G})$ the Hom groups in $\mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$.

We fix an isomorphism $\Bbbk \cong \mathbb{C}$ and denote by $|\lambda|$ the norm of $\lambda \in \Bbbk$ considered as an element of $\mathbb{C}$. If M is an Fr-module, then the *weights* of Fr on M are the real numbers $2\log(\lambda)/\log(q)$ for λ an eigenvalue of the action of Fr. All weights we consider will be integers: the cohomology sheaves of an object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{(H)}(Y,\Bbbk)$ are required to have punctual integer weights (see Section [10, 5.1.5]). We fix a square root $(1/2)$ of the Tate twist, and for $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$ and $r \in \mathbb{Z}$, we denote by $\mathcal{F}(r/2)$ the corresponding Tate twist of $\mathcal{F}$.

Recall (see [10, Section 5.1]) that $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$ has weights at most r if for each i the sheaf $H^i(\mathcal{F})$ has mixed punctual weights at most $r+i$. We say that $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$ has weights at least r if the Verdier dual $\mathbb{D}\mathcal{F}$ has mixed punctual weights at most $-r$. Finally, we say that $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$ is *pure of weight* r if it has weights at most r and weights at least r. If $\mathcal{F}$ is pure of weight r, then $\mathcal{F}[1]$ is pure of weight $r+1$, while $\mathcal{F}(1/2)$ is pure of weight $r-1$.

For an integer $r \in \mathbb{Z}$, we denote by $\mathrm{Ext}^{k,(r/2)}(\mathcal{F},\mathcal{G}) \subset \mathrm{Ext}^k(\mathcal{F},\mathcal{G})$ the generalized eigenspace for Fr of weight r. Thus,

$$\mathrm{Ext}(\mathcal{F},\mathcal{G}) = \bigoplus_{k,r\in\mathbb{Z}} \mathrm{Ext}^{k,(r/2)}(\mathcal{F},\mathcal{G}). \tag{5.3}$$

For all $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$ and $k,k',r,r' \in \mathbb{Z}$, we have

$$\begin{aligned}\mathrm{Ext}^{k,(r/2)}\big(\mathcal{F}[-k'](-r'/2),\mathcal{G}\big) &\cong \mathrm{Ext}^{k,(r/2)}\big(\mathcal{F},\mathcal{G}[k'](r'/2)\big)\\ &\cong \mathrm{Ext}^{k+k',((r+r')/2)}(\mathcal{F},\mathcal{G}).\end{aligned} \tag{5.4}$$

5.2. *Equivariant cohomology*

For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_H(Y,\Bbbk)$, the equivariant hypercohomology $\mathbb{H}^\bullet_H(\mathcal{F}) = \mathbb{H}^\bullet_H(Y,\mathcal{F})$ is defined by

$$\mathbb{H}^\bullet_H(\mathcal{F}) := \mathrm{Ext}^\bullet(\underline{\Bbbk}_Y,\mathcal{F}). \tag{5.5}$$

In particular, we have $\mathbb{H}^{k,(r/2)}_H(\mathcal{F}[k'](r'/2)) = \mathbb{H}^{k+k',((r+r')/2)}_H(\mathcal{F})$.

Let $\pi : Y \to \mathrm{pt} := \mathrm{Spec}(\mathbb{F}_q)$ be the projection to a point. By definition, the H-equivariant cohomology $H^\bullet_H(Y,\Bbbk)$ and the compactly supported H-equivariant cohomology $H^\bullet_{H,c}(Y,\Bbbk)$ of Y are given by

$$\begin{aligned}H^\bullet_H(Y,\Bbbk) &:= \mathbb{H}^\bullet_H(Y,\underline{\Bbbk}_Y) = \mathbb{H}^\bullet_H(\mathrm{pt},\pi_*\underline{\Bbbk}_Y) \qquad \text{and}\\ H^\bullet_{H,c}(Y,\Bbbk) &:= \mathbb{H}^\bullet_H(\mathrm{pt},\pi_!\underline{\Bbbk}_Y).\end{aligned} \tag{5.6}$$

Both $H^\bullet_H(Y,\Bbbk)$ and $H^\bullet_{H,c}(Y,\Bbbk)$ are graded $(H^\bullet_H(\mathrm{pt},\Bbbk),\mathrm{Fr})$-modules.

5.3. Flag varieties

We fix a semisimple algebraic group G, split over $\mathbb{F}_q$, and a maximal torus and a Borel subgroup $T \subset B \subset G$. Let $X = G/B$ be the flag variety over $\mathbb{F}_q$, and let $X_{\overline{\mathbb{F}}_q} := X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ be obtained by extending scalars. The variety X is stratified by B-orbits $\mathring{X}_w := BwB/B$ (known as *Schubert cells*):

$$X = \bigsqcup_{w \in W} \mathring{X}_w.$$

Let $R = H_B^\bullet(\mathrm{pt}, \Bbbk) \cong \Bbbk[\mathfrak{h}]$.

Remark 5.1

We switch from working over $R = \mathbb{C}[\mathfrak{h}]$ to $R = \Bbbk[\mathfrak{h}]$. The results in Section 3 do not depend on the field as long as it is of characteristic zero. Therefore on the Soergel bimodule side, one can freely switch between working over $\mathbb{C}$ and over $\Bbbk$.

For $w \in W$, we let $i_w : \mathring{X}_w \to X$ be the inclusion map. Introduce the *standard* and *costandard sheaves* $\Delta_w, \nabla_w \in \mathrm{D}_B^{\mathrm{b}}(X, \Bbbk)$ defined by

$$\Delta_w := i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}\big[\ell(w)\big]\big(\ell(w)/2\big) \qquad \text{and} \qquad \nabla_w := i_{w,*}\underline{\Bbbk}_{\mathring{X}_w}\big[\ell(w)\big]\big(\ell(w)/2\big).$$

Here, $\underline{\Bbbk}_{\mathring{X}_w}$ denotes the constant sheaf on $\mathring{X}_w$. The *intersection cohomology sheaves* $\mathcal{IC}_w \in \mathrm{D}_B^{\mathrm{b}}(X, \Bbbk)$ are defined using the intermediate extension functor $i_{w,!*}$:

$$\mathcal{IC}_w := i_{w,!*}\underline{\Bbbk}_{\mathring{X}_w}\big[\ell(w)\big]\big(\ell(w)/2\big).$$

Since we have $\mathrm{D}_B^{\mathrm{b}}(X, \Bbbk) \cong \mathrm{D}_{B\times B}^{\mathrm{b}}(G, \Bbbk)$, the equivariant cohomology $\mathbb{H}_B^\bullet(\mathcal{F})$ is a graded $(R \otimes R, \mathrm{Fr})$-module. Furthermore, there is a restriction functor $\mathrm{Res}_{T,B} : \mathrm{D}_B^{\mathrm{b}}(X, \Bbbk) \to \mathrm{D}_T^{\mathrm{b}}(X, \Bbbk)$ (see [13]), and we also have the hypercohomology functor $\mathbb{H}_T^\bullet : \mathrm{D}_T^{\mathrm{b}}(X, \Bbbk) \to H_T^\bullet(\mathrm{pt}) - \mathrm{mod}$. It is well known that $H_B^\bullet(\mathrm{pt}, \Bbbk) \cong R \cong H_T^\bullet(\mathrm{pt}, \Bbbk)$, and furthermore we have

$$\begin{aligned} \mathrm{Res}_{T,B} : \mathrm{D}_B^{\mathrm{b}}(\mathrm{pt}, \Bbbk) \cong \mathrm{D}_T^{\mathrm{b}}(\mathrm{pt}, \Bbbk) \qquad \text{and} \\ \mathbb{H}_B^\bullet(X, \mathcal{F}) \cong \mathbb{H}_T^\bullet(X, \mathrm{Res}_{T,B}\,\mathcal{F}) \end{aligned} \tag{5.7}$$

for $\mathcal{F} \in \mathrm{D}_B^{\mathrm{b}}(X, \Bbbk)$.

5.4. Equivariant cohomology of open Richardson varieties

We split the proof of our main result into two parts. We will focus on the equivariant case (Theorem 1.17). The proof of its nonequivariant version (Theorem 1.16) will follow as a byproduct, and will be discussed in Section 8.1.

Recall from Section 1.9 that our goal is to show the following result.

THEOREM 5.2
For all $v \le w \in W$, we have an isomorphism of bigraded R-modules

$$HHH^0(F^\bullet_{v,w}) \cong H^\bullet_{T,c}(R^\circ_{v,w}).$$

For all $k, p \in \mathbb{Z}$, it restricts to an isomorphism

$$H^{k,(p)}\big(HH^0(F^\bullet_{v,w})\big) \cong H^{\ell_{v,w}+2p+k,(p,p)}_{T,c}(R^\circ_{v,w})$$

of vector spaces.

We will accomplish this in two steps. The first one is an equivariant version of [106, Proposition 4.2.1].

PROPOSITION 5.3
For all $v \le w \in W$, we have an isomorphism of bigraded R-modules

$$H^\bullet_{T,c}(R^\circ_{v,w}) \cong \mathrm{Ext}^\bullet(\Delta_v, \Delta_w).$$

For all $m, r \in \mathbb{Z}$, it restricts to an isomorphism

$$H^{\ell_{v,w}+m,(r/2,r/2)}_{T,c}(R^\circ_{v,w}) \cong \mathrm{Ext}^{m,((r-\ell_{v,w})/2)}(\Delta_v, \Delta_w)$$

of vector spaces. (In particular, both sides are zero for odd r.)

The second step passes through the *mixed equivariant derived category* of [3] and [4] and involves the *degrading functor* of [8] (see also [107]).

PROPOSITION 5.4
For all $v \le w \in W$, we have an isomorphism of bigraded R-modules

$$HHH^0(F^\bullet_{v,w}) \cong \mathrm{Ext}^\bullet(\Delta_v, \Delta_w).$$

For all $k, r \in \mathbb{Z}$, it restricts to an isomorphism

$$H^{k,(r/2)}\big(HH^0(F^\bullet_{v,w})\big) \cong \mathrm{Ext}^{r+k,((r-\ell_{v,w})/2)}(\Delta_v, \Delta_w) \tag{5.8}$$

of vector spaces. (In particular, both sides are zero for odd r.)

5.5. *From weights to the mixed Hodge numbers*

We briefly explain the standard relation between the mixed Hodge structure of the complex variety $Y_{\mathbb{C}} = (R^\circ_{v,w})_{\mathbb{C}}$ and the étale cohomology of the variety $Y_{\overline{\mathbb{F}}_q} = (R^\circ_{v,w})_{\overline{\mathbb{F}}_q}$. First, by the comparison theorem (see [95, Theorem 21.1]) between Betti cohomology and étale cohomology of $Y_{\mathbb{C}}$, we have an isomorphism $H^k_{\mathrm{Betti}}(Y_{\mathbb{C}}, \Bbbk) \cong H^k_{et}(Y_{\mathbb{C}}, \Bbbk)$ which preserves the weight filtration of both sides. Next, we find a discrete valuation ring $S \subset \mathbb{C}$ with residue field $\overline{\mathbb{F}}_q$ and construct the Richardson variety Y_S over S. Then $Y_{\mathbb{C}}$ and $Y_{\overline{\mathbb{F}}_q}$ are obtained from Y_S via base change, and we obtain isomorphisms between the étale cohomologies $H^\bullet(Y_{\mathbb{C}}, \Bbbk) \cong H^\bullet(Y_S, \Bbbk) \cong H^\bullet(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$, compatibly with the weight filtrations (see [95, Section 20]). For $H^\bullet(Y_{\overline{\mathbb{F}}_q}, \Bbbk)$, the weight filtration is obtained by taking sums of generalized eigenspaces of the Frobenius Fr. Finally, the cohomology of $Y_{\mathbb{C}}$ is of Hodge–Tate type, so the weight filtration is simply given by $W^{2r}(H^k(Y_{\mathbb{C}}, \mathbb{C})) = \bigoplus_{p \le r} H^{k,(p,p)}(Y_{\mathbb{C}}, \mathbb{C})$. (All these statements hold also equivariantly.) Summing up, we have the following.

PROPOSITION 5.5
For all $v \le w \in W$ and $k, p \in \mathbb{Z}$, we have the equalities

$$\dim_{\mathbb{C}} H^{k,(p,p)}\big((R^\circ_{v,w})_{\mathbb{C}}, \mathbb{C}\big) = \dim_{\Bbbk} H^{k,(p)}\big((R^\circ_{v,w})_{\overline{\mathbb{F}}_q}, \Bbbk\big),$$
$$\dim_{\mathbb{C}} H^{k,(p,p)}_{T,c}\big((R^\circ_{v,w})_{\mathbb{C}}, \mathbb{C}\big) = \dim_{\Bbbk} H^{k,(p)}_{T,c}\big((R^\circ_{v,w})_{\overline{\mathbb{F}}_q}, \Bbbk\big),$$

where $H^{k,(p)}_{T,c}((R^\circ_{v,w})_{\overline{\mathbb{F}}_q}, \Bbbk) = \mathbb{H}^{k,(p)}_T(\mathrm{pt}, \pi_! \underline{\Bbbk}_{R^\circ_{v,w}}) = \mathrm{Ext}^{k,(p)}(\underline{\Bbbk}_{\mathrm{pt}}, \pi_! \underline{\Bbbk}_{R^\circ_{v,w}})$ *as in (5.6).*

See also [106, Remark 7.1.4], where a comparison between derived categories of flag varieties is given.

6. Proof of Proposition 5.3

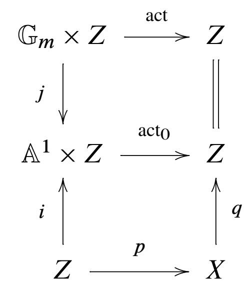
We follow the steps in the proof of [106, Proposition 4.2.1]. Using (5.4) and the adjunction $(i_{v,!}, i_v^!)$, we find

$$\begin{aligned}\mathrm{Ext}^{m,((r-\ell_{v,w})/2)}(\Delta_v, \Delta_w) &\cong \mathrm{Ext}^{m,((r-\ell_{v,w})/2)}\big(i_{v,!}\underline{\Bbbk}_{\mathring{X}_v}, i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}[\ell_{v,w}](\ell_{v,w}/2)\big)\\ &\cong \mathrm{Ext}^{m+\ell_{v,w},(r/2)}(i_{v,!}\underline{\Bbbk}_{\mathring{X}_v}, i_{w,!}\underline{\Bbbk}_{\mathring{X}_w})\\ &\cong \mathrm{Ext}^{m+\ell_{v,w},(r/2)}(\underline{\Bbbk}_{\mathring{X}_v}, i_v^! i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}).\end{aligned}$$

Note that $i_v^! i_{w,!}\underline{\Bbbk}_{\mathring{X}_w} \in D^b_B(\mathring{X}_v, \Bbbk)$. By (5.5) and (5.7), we have

$$\mathrm{Ext}^\bullet(\underline{\Bbbk}_{\mathring{X}_v}, i_v^! i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}) \cong \mathbb{H}^\bullet_B(\mathring{X}_v, i_v^! i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}) \cong \mathbb{H}^\bullet_T(\mathring{X}_v, i_v^! i_{w,!}\underline{\Bbbk}_{\mathring{X}_w}).$$

We now switch to working with T-equivariant derived categories. First, we state a T-equivariant version of [116, Proposition 1]. Let Z be a T-variety, and let $q : X \hookrightarrow Z$ be an inclusion of a closed T-subvariety. An action $\mathbb{G}_m \times Z \to Z$ *contracts Z to X* if there is a commutative diagram



where i, j are the obvious maps and $p \circ q = \mathrm{id}$, and all arrows are T-equivariant, with T acting trivially on $\mathbb{G}_m$ and $\mathbb{A}^1$. Let $\pi : \mathbb{G}_m \times Z \to Z$ be the projection, and suppose that $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_T(Z)$ is $\mathbb{G}_m$-equivariant, that is, satisfies $\mathrm{act}^*(\mathcal{F}) \cong \pi^*(\mathcal{F}) \in \mathrm{D}^{\mathrm{b}}_T(\mathbb{G}_m \times Z)$. There is a natural morphism $q^! \to p_!$ of functors (obtained by composing the adjunction morphism $q_! q^! \to \mathrm{id}$ with $p_!$) and we have the following.

PROPOSITION 6.1 (cf. [116, Proposition 1])
The map $q^!\mathcal{F} \to p_!\mathcal{F}$ is an isomorphism.

Let $a : \mathring{X}_v \to \mathrm{pt}$ and $b : R^\circ_{v,w} \to \mathrm{pt}$ be the projections (cf. Figure 3 (right)). Our next goal is to manipulate the object $i_v^! i_{w,!} \underline{\Bbbk}_{\mathring{X}_w}$, in order to establish the following result.

LEMMA 6.2
We have $i_v^! i_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong a^* b_! \underline{\Bbbk}_{R^\circ_{v,w}}$ in $\mathrm{D}^{\mathrm{b}}_T(\mathring{X}_v)$.

Proof
Recall that $\mathring{X}_w := (BwB)/B$. Denote $\mathring{X}^-_v := (B_- vB)/B$, and thus $R^\circ_{v,w} = \mathring{X}_w \cap \mathring{X}^-_v$. Let $X_w := \bigsqcup_{u \le w} \mathring{X}_u$ be the closure of $\mathring{X}_w$, and denote $R^\circ_{v,\overline{w}} := X_w \cap \mathring{X}^-_v$. A diagram of the various inclusions between the spaces $\mathring{X}_w$, $\mathring{X}_v$, X_w, and $X = G/B$ is

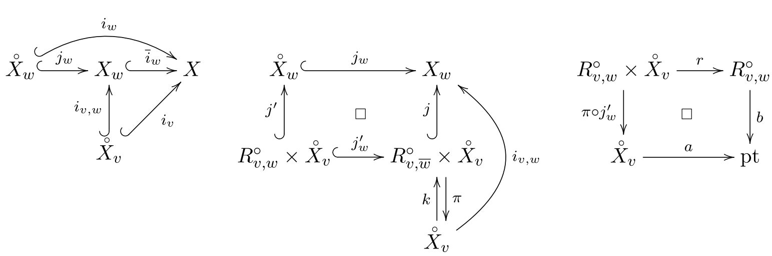


Figure 3. Three commutative diagrams from [106].

given in Figure 3 (left). With this notation, we have isomorphisms

$$i_v^! i_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong i_{v,w}^! \bar{i}_w^! \bar{i}_{w,!} j_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong i_{v,w}^! j_{w,!} \underline{\Bbbk}_{\mathring{X}_w}.$$

The first isomorphism follows from the usual composition rules for sheaf operations (see, e.g., [13, Section 1.4.2]). The second isomorphism follows from $\bar{i}_w^! \bar{i}_{w,!} \cong \mathrm{id}$.

Consider the commutative diagram in Figure 3 (middle). The map k is given by $k(zB/B) = (vB/B, zB/B)$ and π is the obvious projection map. The map j'_w has two components: the first one is the inclusion $R^\circ_{v,w} \hookrightarrow R^\circ_{v,\overline{w}}$, and the second one is the identity map $\mathring{X}_v \xrightarrow{\sim} \mathring{X}_v$. The map $j_w : \mathring{X}_w \hookrightarrow X_w$ is the inclusion map as above. It remains to define the maps j and j'. They have been considered in [73, Section 1.4] (see, e.g., [38] and [78] for further details). Observe that $R^\circ_{v,w} \subset R^\circ_{v,\overline{w}} \subset \mathring{X}_v^-$. The maps j, j' are the restrictions of a map $\phi_v : \mathring{X}_v^- \times \mathring{X}_v \hookrightarrow X$ defined as follows. Recall that $U \subset B$ and $U_- \subset B_-$ are the unipotent radicals of B and B_-. Any element of $\mathring{X}_v^-$ can be written uniquely as xvB/B for an element $x \in U_- \cap vU_-v^{-1}$. (These objects do not depend on the choice of representative $\dot{v}$ of v.) Similarly, any element of $\mathring{X}_v$ can be written uniquely as yvB/B for an element $y \in U \cap vU_-v^{-1}$. We then define

$$\phi_v(xvB/B, yvB/B) := yxvB/B.$$

It is not hard to see (using *Gaussian decomposition*) that the map ϕ_v is injective and yields an isomorphism $\phi_v : \mathring{X}_v^- \times \mathring{X}_v \xrightarrow{\sim} vB_-B/B$. Moreover, if $xvB/B \in \mathring{X}_u$ for some $u \in W$, then we have $yxvB/B \in \mathring{X}_u$ since $y \in U \subset B$. Thus, ϕ_v restricts to an inclusion $R^\circ_{v,u} \times \mathring{X}_v \hookrightarrow \mathring{X}_u$ for each $u \in W$. The map j' is this inclusion for the special case $u := w$. The map j is the restriction of ϕ_v to the union of $R^\circ_{v,u} \times \mathring{X}_v$ over all $u \in W$ satisfying $v \leq u \leq w$.

The torus T acts on each space in each commutative diagram in Figure 3. The action on the direct products $R^\circ_{v,w} \times \mathring{X}_v$ and $R^\circ_{v,\overline{w}} \times \mathring{X}_v$ is given by $t \cdot (aB/B, bB/B) = (taB/B, tbB/B)$ for $t \in T$. Notice that conjugation by t preserves each of the subgroups U, U_-, and vU_-v^{-1}. Therefore for $x \in U_- \cap vU_-v^{-1}$, we have $txvB/B = txt^{-1}vB/B$, where $txt^{-1} \in U_- \cap vU_-v^{-1}$, and similarly for $y \in U \cap vU_-v^{-1}$. Thus, the map ϕ_v is T-equivariant:

$$\phi_v(txvB/B, tyvB/B) = tyxvB/B.$$

We conclude that all maps in Figure 3 are T-equivariant.

The maps j, j', j_w, j'_w in Figure 3 (middle) form a Cartesian square. We get the following isomorphisms:

$$i_{v,w}^! j_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong k^! j^! j_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong k^! j^* j_{w,!} \underline{\Bbbk}_{\mathring{X}_w} \cong k^! j'_{w,!} j'^* \underline{\Bbbk}_{\mathring{X}_w} \cong k^! j'_{w,!} \underline{\Bbbk}_{R^\circ_{v,w} \times \mathring{X}_v}.$$

The first and the last isomorphisms are trivial. The second isomorphism follows from the fact that j is an open embedding (see [13, Section 1.4.5]). The third isomorphism is the base change theorem (see [13, Section 1.4.6]).

We now apply Proposition 6.1 with $q = k$, and $p = \pi$, and $\mathcal{F} = j'_{w,!}\underline{\Bbbk}_{R^\circ_{v,w}\times\mathring{X}_v}$. The $\mathbb{G}_m$-action is the composition of the T-action with the cocharacter $\chi^\vee_\rho : \mathbb{G}_m \to T$, where ρ is a strictly dominant coweight satisfying $\langle\rho,\alpha\rangle > 0$ for any positive root α. Since ρ is strictly dominant, this $\mathbb{G}_m$-action extends to an $\mathbb{A}^1$-action by the same argument as in [44, Section 8.2]. The $\mathbb{A}^1$-action is obviously compatible with the T-action, and thus Proposition 6.1 applies.

We obtain

$$k^! j'_{w,!}\underline{\Bbbk}_{R^\circ_{v,w}\times\mathring{X}_v} \cong \pi_! j'_{w,!}\underline{\Bbbk}_{R^\circ_{v,w}\times\mathring{X}_v}.$$

Applying base change to the Cartesian diagram in Figure 3 (right), we get

$$\pi_! j'_{w,!}\underline{\Bbbk}_{R^\circ_{v,w}\times\mathring{X}_v} \cong (\pi\circ j'_w)_!\underline{\Bbbk}_{R^\circ_{v,w}\times\mathring{X}_v} \cong (\pi\circ j'_w)_! r^*\underline{\Bbbk}_{R^\circ_{v,w}} \cong a^* b_!\underline{\Bbbk}_{R^\circ_{v,w}}. \qquad \square$$

So far we have constructed an isomorphism

$$\begin{aligned}\operatorname{Ext}^{m,((r-\ell_{v,w})/2)}(\Delta_v,\Delta_w) &\cong \mathbb{H}_T^{m+\ell_{v,w},(r/2)}(\mathring{X}_v, a^* b_!\underline{\Bbbk}_{R^\circ_{v,w}})\\ &\cong \mathbb{H}_T^{m+\ell_{v,w},(r/2)}(\mathrm{pt}, a_* a^* b_!\underline{\Bbbk}_{R^\circ_{v,w}}).\end{aligned} \tag{6.1}$$

Since $\mathring{X}_v$ is an affine space that is T-equivariantly homotopy equivalent to a point (cf. [116, Lemma 1]), the adjunction (a^*, a_*) induces an isomorphism of functors $a_*a^* \to \mathrm{id}$, and thus

$$\mathbb{H}_T^{m+\ell_{v,w},(r/2)}(\mathrm{pt}, a_* a^* b_!\underline{\Bbbk}_{R^\circ_{v,w}}) \cong \mathbb{H}_T^{m+\ell_{v,w},(r/2)}(\mathrm{pt}, b_!\underline{\Bbbk}_{R^\circ_{v,w}}), \tag{6.2}$$

which equals to $H_{T,c}^{m+\ell_{v,w},(r/2,r/2)}(R^\circ_{v,w})$ by (5.6). All the isomorphisms are natural (coming from sheaf operations) and thus (6.2) is compatible with the action of R on both sides. This completes the proof of Proposition 5.3. $\square$

7. Proof of Proposition 5.4

Recall from Remark 5.1 that we switch to working with Soergel bimodules over $\Bbbk$, so that for example $R = \Bbbk[\mathfrak{h}]$. By definition, given a (graded) Soergel bimodule $B = \bigoplus_r B^r$, Fr acts diagonally on each B^r by multiplication by q^r.

Equivariant derived categories are identified with categories of dg-modules in the work of Bernstein and Lunts [13]. By using the formalism of Yun [15, Appendix B], we avoid explicit mention of dg-modules in the situation of interest to us.

An (R,Fr)-module is an R-module M equipped with an action of $\mathbb{Z}\mathrm{Fr}$ such that $\mathrm{Fr}(r\cdot x) = \mathrm{Fr}(r)\cdot\mathrm{Fr}(x)$ for $x \in M$ and $r \in R$. The twist functor $\{m/2\}$

sends a module M to the module $M\{m/2\}$ where the action of Fr has been multiplied by $q^{m/2}$. Let $\mathrm{D}_{\mathrm{perf}}(R,\mathrm{Fr})$ denote the full triangulated subcategory of the derived category of (R,Fr)-modules generated by half-integer twists of R. According to [15, Corollary B.4.1], we have an equivalence of triangulated categories $\mathrm{D}^{\mathrm{b}}_{B}(\mathrm{pt},\Bbbk) \cong \mathrm{D}_{\mathrm{perf}}(R,\mathrm{Fr})$. Similarly, we define $\mathrm{D}_{\mathrm{perf}}(R\otimes R,\mathrm{Fr})$ and have an equivalence $\mathrm{D}^{\mathrm{b}}_{B\times B}(\mathrm{pt},\Bbbk) \cong \mathrm{D}_{\mathrm{perf}}(R\otimes R,\mathrm{Fr})$.

Recall that $\mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}$ includes only bimodule morphisms of degree zero. We let

$$\mathrm{Hom}_{R\otimes R}(B,B') := \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}\big(B, B'\{-r/2\}\big)$$

denote the space of morphisms of arbitrary degree. Thus, $\mathrm{Hom}_{R\otimes R}(B,B')$ is an (R,Fr)-module. Given a complex $C^\bullet \in \mathrm{K}^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$, we regard $\mathrm{Hom}_{R\otimes R}(R,C^\bullet)$ (obtained by applying $\mathrm{Hom}_{R\otimes R}(R,-)$ termwise) as a complex of (R,Fr)-modules, treated as an element of $\mathrm{D}_{\mathrm{perf}}(R,\mathrm{Fr})$.

Now, for $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{B}(X,\Bbbk)$, let $\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G}) \in \mathrm{D}^{\mathrm{b}}_{B}(X,\Bbbk)$ denote the internal derived hom. With $\pi : G \to \mathrm{pt}$, we define

$$\mathrm{RHom}(\mathcal{F},\mathcal{G}) = \mathrm{RHom}_X(\mathcal{F},\mathcal{G}) := \pi_* \mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G}) \in \mathrm{D}^{\mathrm{b}}_{B}(\mathrm{pt},\Bbbk).$$

Thus, $\mathbb{H}^\bullet_B(\mathrm{RHom}(\mathcal{F},\mathcal{G})) = \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{G})$. We shall establish the following strengthening of Proposition 5.4.

PROPOSITION 7.1
We have

$$\mathrm{RHom}_X\big(\Delta_v\big(-\ell(v)/2\big), \Delta_w\big(-\ell(w)/2\big)\big) \cong \mathrm{Hom}_{R\otimes R}(R, F^\bullet_{v,w})$$

inside $\mathrm{D}_{\mathrm{perf}}(R,\mathrm{Fr})$.

Proposition 5.4 follows from Proposition 7.1 by taking cohomology $\mathbb{H}_B : \mathrm{D}^{\mathrm{b}}_{B}(\mathrm{pt},\Bbbk) \to (R_{\mathrm{gr}},\mathrm{Fr}) - \mathrm{mod}$, where $(R_{\mathrm{gr}},\mathrm{Fr}) - \mathrm{mod}$ denotes (cohomologically) graded R-modules equipped with an Fr-action. The cohomological degree $r+k$ on the right-hand side of (5.8) appears since the functor $\mathbb{H} : \mathrm{D}_{\mathrm{perf}}(R,\mathrm{Fr}) \to (R_{\mathrm{gr}},\mathrm{Fr}) - \mathrm{mod}$ sends the sum of the two gradings to the cohomological one (see [15, Corollary B.4.1(1)]).

7.1. Realization functors

We record two results on realization functors taken from [1] (see also [8], [107]). For a definition of a filtered version of a triangulated category, see [8, Definition A.1].

PROPOSITION 7.2 ([1, Proposition 2.2])
Let $\mathcal{T}$ be a triangulated category that admits a filtered version $\tilde{\mathcal{T}}$, and let $\mathcal{A} \subset \mathcal{T}$ be a full additive subcategory that admits no negative self-extensions. Then there is a functor of triangulated categories

$$\mathrm{real} : \mathrm{K}^{\mathrm{b}}\mathcal{A} \to \mathcal{T}$$

whose restriction to $\mathcal{A}$ is the inclusion functor.

PROPOSITION 7.3 ([1, Proposition 2.3])
Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories admitting a filtered version, and let $\mathcal{A}_1 \subset \mathcal{T}_1$ and $\mathcal{A}_2 \subset \mathcal{T}_2$ be two full additive subcategories admitting no negative self-extensions. Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor that restricts to an additive functor $F_0 : \mathcal{A}_1 \to \mathcal{A}_2$. If F lifts to a functor $\tilde{F} : \tilde{\mathcal{T}}_1 \to \tilde{\mathcal{T}}_2$, then the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{b}}\mathcal{A}_1 & \xrightarrow{\mathrm{real}} & \mathcal{T}_1 \\ {\scriptstyle \mathrm{K}^{\mathrm{b}}F_0}\downarrow & & \downarrow{\scriptstyle F} \\ \mathrm{K}^{\mathrm{b}}\mathcal{A}_2 & \xrightarrow{\mathrm{real}} & \mathcal{T}_2 \end{array}$$

7.2. Semisimple complexes
Let $\mathrm{Semis}_B(X) \subset \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk)$ denote the additive subcategory generated by semisimple complexes pure of weight 0. Thus, an object of $\mathrm{Semis}_B(X)$ is a direct sum of the twisted intersection cohomology sheaves $\mathcal{IC}_w[n](n/2)$ for $w \in W$ and $n \in \mathbb{Z}$.

Recall from Section 3.1 that $S_w \subset B_{\underline{w}}$ denotes the indecomposable Soergel bimodule indexed by $w \in W$.

LEMMA 7.4
For $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_B(X)$, the Ext*-group* $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ *is pure of weight i for all $i \in \mathbb{Z}$.*

Proof
This follows from [15, Lemma 3.1.5]. □

PROPOSITION 7.5
The hypercohomology functor induces an equivalence of additive categories

$$\mathbb{H}_B : \mathrm{Semis}_B(X) \to \mathbb{S}\mathrm{Bim},$$

enriched over $R \otimes R$, and sending $\mathcal{IC}_w$ to the shifted Soergel bimodule $S_w\{-\ell(w)\}$ and the twist $[n](n/2)$ to the change of grading $\{-n/2\}$.

Proof
By a well-known result of Soergel [119] (see also [36, Section 1.3] or [35, Section 16.1]), we have $\mathbb{H}_B(\mathcal{IC}_w) \cong S_w\{-\ell(w)/2\}$ as a graded $(R \otimes R)$-module. By Lemma 7.4, the cohomological and weight gradings on $\mathbb{H}_B(\mathcal{IC}_w)$ agree, and furthermore $\mathcal{IC}_w[n](n/2)$ is sent to $S_w\{-(\ell(w)+n)/2\}$. The result can then be deduced from [15, Proposition 3.1.6]. □

The following result is well known (see [2] and [15, Lemma B.1.1]).

LEMMA 7.6
For $\mathcal{F}, \mathcal{G} \in \mathrm{Semis}_B(X)$, the action of Fr *on* $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{G})$ *is semisimple. Furthermore, we have*

$$\hom_{\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{0,(0)}(\mathcal{F}, \mathcal{G}) \tag{7.1}$$

and $\mathrm{ext}^i_{\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)}(\mathcal{F}, \mathcal{G}) = 0$ *for* $i < 0$.

7.3. The mixed derived category
Following [4], we define the *mixed derived category*

$$\mathrm{D}^{\mathrm{mix}}_B(X) := \mathrm{K}^{\mathrm{b}}\big(\mathrm{Semis}_B(X)\big)$$

to be the homotopy category of cochain complexes in $\mathrm{Semis}_B(X)$. Define the *Tate twist* of $\mathrm{D}^{\mathrm{mix}}_B(X)$ by

$$\langle n \rangle := \{-n/2\}[-n],$$

where $[n] : \mathrm{K}^{\mathrm{b}}(\mathrm{Semis}_B(X)) \to \mathrm{K}^{\mathrm{b}}(\mathrm{Semis}_B(X))$ is the cohomological shift functor, and $\{n/2\} : \mathrm{Semis}_B(X) \to \mathrm{Semis}_B(X)$ is the autoequivalence $\mathcal{F} \mapsto \mathcal{F}[-n](-n/2)$.

By Proposition 7.5, we have an equivalence of triangulated categories

$$\mathrm{K}^{\mathrm{b}}\mathbb{H}_B : \mathrm{D}^{\mathrm{mix}}_B(X) \to \mathrm{K}^{\mathrm{b}}\mathbb{S}\mathrm{Bim}.$$

THEOREM 7.7
There exists a triangulated realization functor

$$\mathrm{real} : \mathrm{D}^{\mathrm{mix}}_B(X) \to \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk),$$

restricting to the inclusion on $\mathrm{Semis}_B(X)$*, sending*

$$[n] \to [n], \qquad \langle n \rangle \to (n/2), \qquad \{n/2\} \to [-n](-n/2),$$

such that the composition

$$\kappa := \omega \circ \mathrm{real}$$

satisfies

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}_B(X)}\big(\mathcal{F}, \mathcal{G}\langle n\rangle\big) \cong \mathrm{Ext}^{0,(n/2)}(\mathrm{real}\mathcal{F}, \mathrm{real}\mathcal{G}), \tag{7.2}$$

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{mix}}_B(X)}\big(\mathcal{F}, \mathcal{G}\langle n\rangle\big) \cong \mathrm{Ext}^{0}(\mathrm{real}\mathcal{F}, \mathrm{real}\mathcal{G}) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)}(\kappa\mathcal{F}, \kappa\mathcal{G}). \tag{7.3}$$

Furthermore, all functors are compatible with the $(R \otimes R)$*-action on the corresponding* $\mathrm{Ext}^\bullet$*-groups.*

Proof
By Lemma 7.6, $\mathrm{Semis}_B(G/B)$ has no negative self-extensions. Since $\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ admits a filtered version (see [8]), we may apply Proposition 7.2 to obtain a realization functor

$$\mathrm{real} : \mathrm{D}^{\mathrm{mix}}_B(X) \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$$

that restricts to the inclusion on $\mathrm{Semis}_B(G/B)$. For $\mathcal{F},\mathcal{G} \in \mathrm{Semis}_B(X)$, the isomorphism (7.3) follows from (7.1) while (7.2) follows from (7.1) and (7.3), since $\langle n\rangle : \mathrm{D}^{\mathrm{mix}}_B(X) \to \mathrm{D}^{\mathrm{mix}}_B(X)$ is sent to $(n/2) : \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk) \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ by the realization functor real. For $\mathcal{F},\mathcal{G} \in \mathrm{K}^{\mathrm{b}}\,\mathrm{Semis}_B(X)$, (7.2)–(7.3) are proved by double induction on the lengths of chain complexes representing $\mathcal{F}$, $\mathcal{G}$. (See [107, Section 4.1] for a detailed argument.) □

7.4. *Standard objects and Rouquier complexes*

PROPOSITION 7.8
The composition $\mathrm{real} \circ \mathbb{H}_B^{-1} : \mathrm{K}^{\mathrm{b}}\mathbb{S}\mathrm{Bim} \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ *takes* $F^\bullet(w)$ *to* $\Delta_w(-\ell(w)/2)$ *and* $F^\bullet(v)^{-1}$ *to* $\nabla_v(\ell(v)/2)$.

Note that Proposition 5.4 follows nearly immediately from Proposition 7.8 and (7.2)–(7.3): since the realization functor sends $[k]\{-r/2\}$ to $[k+r](r/2)$, we have

$$\begin{aligned} H^{k,(r/2)}\big(HH^0(F^\bullet_{v,w})\big) &\cong \mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}\mathbb{S}\mathrm{Bim}}\big(R, F^\bullet_{v,w}[k]\{-r/2\}\big)\big) \\ &\cong \mathrm{Ext}^{r+k,(r/2)}\big(\Delta_e, \Delta_w\big(-\ell(w)/2\big) \star \nabla_v\big(\ell(v)/2\big)\big) \\ &\cong \mathrm{Ext}^{r+k,(r/2)}\big(\Delta_v\big(-\ell(v)/2\big), \Delta_w\big(-\ell(w)/2\big)\big) \\ &\cong \mathrm{Ext}^{r+k,((r-\ell_{v,w})/2)}(\Delta_v, \Delta_w). \end{aligned}$$

The second isomorphism above is obtained from the adjointness of the convolution functors $(-) \star \Delta_v(-\ell(v/2))$ and $(-) \star \nabla_v(\ell(v)/2))$ to be presently explained (see

Lemma 7.10 for a stronger statement; see also (3.5)–(3.6) for a similar statement on the Soergel bimodule side).

Proof of Proposition 7.8
We prove the claim for $F^\bullet(w)$. The claim for $F^\bullet(v)^{-1}$ is similar.

There is a monoidal structure $\star : \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk) \times \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk) \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ obtained by convolution (see, e.g., [15, Section 3.2] or [4, Section 4.3]). By [15, Proposition 3.2.5], the additive subcategory $\mathrm{Semis}_B(X) \subset \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ is preserved by convolution. According to [15, Proposition 3.2.1], convolution $\star$ is sent by $\mathbb{H}_B$ to the tensor product operation on $\mathbb{S}\mathrm{Bim}$. Note that the *derived* tensor product in [15] reduces to the tensor product on $\mathbb{S}\mathrm{Bim}$ since all Soergel bimodules are free as left (or right) R-modules.

For a simple generator $s \in S$, let $\pi_s : X = G/B \to G/P_s$ denote the projection to the partial flag variety, where $P_s \supset B$ denotes a minimal parabolic subgroup, and let $\theta_s : \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk) \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ denote the composition $\theta = \pi_s^*\pi_{s,*}$. By [15, Lemma 3.2.7] (see also [4, Lemma 4.3]), we have a natural isomorphism of functors $\theta_s \cong (-) \star \mathcal{IC}_s[-1](-1/2)$, and θ_s restricts to an endofunctor $\theta_s : \mathrm{Semis}_B(X) \to \mathrm{Semis}_B(X)$. It is well known (see [117, Korollar 2]) that the equivalence $\mathbb{H}_B : \mathrm{Semis}_B(X) \to \mathbb{S}\mathrm{Bim}$ takes the functor θ_s to the functor $R \otimes_{R^s} (-)$.

Now, there is a natural morphism of functors $\theta_s \to \mathrm{id}$ arising from the adjunction of π_s^* and $\pi_{s,*}$. The map $B_s \to R$ in (3.4) arises by an analogous adjunction (see [110, Section 3]). Now, $\mathcal{IC}_e \cong \Delta_e$ and $\mathbb{H}_B(\mathcal{IC}_e) = R$, and the morphism $\theta_s \to \mathrm{id}$ applied to $\mathcal{IC}_e$ fits into the distinguished triangle

$$\mathcal{IC}_s[-1](-1/2) \to \mathcal{IC}_e \to \Delta_s(-1/2)$$

in $\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ (see, e.g., [15, (C.4)] or [4, Lemma 4.1]). It follows that we have

$$\mathrm{real} \circ \mathbb{H}_B^{-1}\big(F^\bullet(s)\big) = \Delta_s(-1/2). \tag{7.4}$$

(See also [110, Proposition 5.3].) This establishes Proposition 7.8 in the case $\ell(w) = 1$. We then obtain a natural isomorphism of functors

$$\big((-) \star \Delta_s(-1/2)\big) \circ \mathrm{real} \circ \mathbb{H}_B^{-1} \cong \mathrm{real} \circ \mathbb{H}_B^{-1} \circ \big((-) \otimes F(s)\big) \tag{7.5}$$

from $\mathrm{K}^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$ to $\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$.

On the other hand, it is known (see [15, Lemma 3.2.2]) that if $w = s_1 s_2 \cdots s_l$ is a reduced decomposition, then

$$\Delta_w \cong \Delta_{s_1} \star \Delta_{s_2} \star \cdots \star \Delta_{s_l}$$

is in $\mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$. Combining (7.4) with (7.5), we find

$$\begin{aligned}\mathrm{real} \circ \mathbb{H}_B^{-1}\big(F^\bullet(w)\big) &\cong \mathrm{real} \circ \mathbb{H}_B^{-1}\big(F^\bullet(s_1) \otimes \cdots \otimes F^\bullet(s_\ell)\big) \\ &\cong \Delta_{s_1}(-1/2) \star \cdots \star \Delta_{s_\ell}(-1/2) \cong \Delta_w\big(-\ell(w)/2\big). \qquad \square\end{aligned}$$

7.5. *Proof of Proposition 7.1*

7.5.1

A functor $F : D^b(Y,\Bbbk) \to D^b(Z,\Bbbk)$ is called *geometric* (see [2, Definition 6.6]) if there is a natural transformation

$$\mathrm{RHom}_Y(\mathcal{F},\mathcal{G}) \to \mathrm{RHom}_Z(F\mathcal{F}, F\mathcal{G})$$

for $\mathcal{F},\mathcal{G} \in D^b(Y,\Bbbk)$. We shall apply the notion of a geometric functor for B-equivariant derived categories.

LEMMA 7.9

The endofunctors $(-) \star \Delta_s(-1/2), (-) \star \nabla_s(1/2) : \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk) \to \mathrm{D}^{\mathrm{b}}_B(X,\Bbbk)$ *are geometric.*

Proof

This is stated for the affine Grassmannian case in [2, Proposition 12.2]. The same proof applies in the flag variety case. $\square$

LEMMA 7.10

We have

$$\mathrm{RHom}(\Delta_v, \Delta_w) = \mathrm{RHom}(\Delta_e, \Delta_w \star \nabla_v)$$

inside $\mathrm{D}^{\mathrm{b}}_B(\mathrm{pt},\Bbbk)$.

Proof

By Lemma 7.9, for $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_B(X,k)$, we have a map

$$\mathrm{RHom}(\mathcal{F},\mathcal{G}) \to \mathrm{RHom}(\mathcal{F} \star \nabla_s, \mathcal{G} \star \nabla_s) \tag{7.6}$$

and a map

$$\mathrm{RHom}(\mathcal{F} \star \nabla_s, \mathcal{G} \star \nabla_s) \to \mathrm{RHom}(\mathcal{F} \star \nabla_s \star \Delta_s, \mathcal{G} \star \nabla_s \star \Delta_s) \tag{7.7}$$

inside $\mathrm{D}^{\mathrm{b}}_B(\mathrm{pt},\Bbbk)$. Convolution is associative and $\nabla_s \star \Delta_s \cong \Delta_e$ ([4, Proposition 4.4]), and also $(-) \star \Delta_e$ is the identity functor. So composing (7.6) and (7.7), we get an automorphism $\mathrm{RHom}(\mathcal{F},\mathcal{G}) \cong \mathrm{RHom}(\mathcal{F},\mathcal{G})$ inside $\mathrm{D}^{\mathrm{b}}_B(\mathrm{pt},\Bbbk)$. It follows that $\mathrm{RHom}(\mathcal{F},\mathcal{G}) \cong \mathrm{RHom}(\mathcal{F} \star \nabla_s, \mathcal{G} \star \nabla_s)$. Choosing $\mathcal{F} = \Delta_v$ and $\mathcal{G} = \Delta_w$ and repeatedly applying this, we obtain the required statement. $\square$

We remark that we could have defined $\mathrm{RHom}(\Delta_v, \Delta_w)$ and $\mathrm{RHom}(\Delta_e, \Delta_w \star \nabla_v)$ as objects in $\mathrm{D}^{\mathrm{b}}_{B\times B}(\mathrm{pt}, \Bbbk)$, but Lemma 7.10 would not hold. The functors $(-) \star \Delta_s$ and $(-) \star \nabla_s$ "commute" with only one of the B-actions.

7.5.2

Let $\iota_e : B \hookrightarrow G$ be the inclusion, and let $\pi : G/B \to \{e\} = \mathrm{pt}$ be the projection. These maps are B-equivariant. Now, the sheaf Δ_e is supported on a single point $\{e\} \subset G/B$. Thus, for any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk)$, the object $\mathrm{R}\mathcal{H}om(\Delta_e, \mathcal{F}) \in \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk)$ is also supported on $\{e\}$. The pullback ι_e^* and pushforward $\iota_{e,*}$ are inverse equivalences of categories between $D^b_B(\{e\}, \Bbbk)$ and the full subcategory of $\mathrm{D}^{\mathrm{b}}_B(X, \Bbbk)$ consisting of objects whose cohomology sheaves are supported on $\{e\}$. Inside $\mathrm{D}^{\mathrm{b}}_B(\{e\}, \Bbbk) = \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$, we thus have

$$\begin{aligned}\mathrm{RHom}_X(\Delta_e, \mathcal{F}) &\cong \pi_* \mathrm{R}\mathcal{H}om_X(\Delta_e, \mathcal{F}) \\ &\cong \iota_e^* \mathrm{R}\mathcal{H}om_X(\Delta_e, \mathcal{F}) = \mathrm{R}\mathcal{H}om_{\{e\}}(\Bbbk, \iota_e^* \mathcal{F}) = \iota_e^* \mathcal{F}.\end{aligned} \tag{7.8}$$

7.5.3

The equivalence $\mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk) \cong \mathrm{D}_{\mathrm{perf}}(R, \mathrm{Fr})$ (resp., $\mathrm{D}^{\mathrm{b}}_{B\times B}(\mathrm{pt}, \Bbbk) \cong \mathrm{D}_{\mathrm{perf}}(R \otimes R, \mathrm{Fr})$) sends objects pure of weight 0 to objects in $\mathrm{D}_{\mathrm{perf}}(R, \mathrm{Fr})$ (resp., $\mathrm{D}_{\mathrm{perf}}(R \otimes R, \mathrm{Fr})$) concentrated in cohomological degree 0. For $\mathcal{F} \in \mathrm{Semis}_B(X)$, we thus view the Soergel bimodule $\mathbb{H}_B(\mathcal{F})$ as sitting inside $\mathrm{D}_{\mathrm{perf}}(R \otimes R, \mathrm{Fr})$ in cohomological degree 0, that is, as an object in $\mathrm{Mod}(R \otimes R, \mathrm{Fr})$.

LEMMA 7.11

For $\mathcal{F} \in \mathrm{Semis}_B(G/B)$, *we have an isomorphism*

$$\iota_e^* \mathcal{F} \cong \mathrm{Hom}_{R\otimes R}\big(R, \mathbb{H}_B(\mathcal{F})\big)$$

inside $\mathrm{Mod}(R, \mathrm{Fr})$.

Proof

For $\mathcal{F} \in \mathrm{Semis}_B(G/B)$, we have that $\iota_e^* \mathcal{F} = \iota_e^! \mathcal{F}$ is again pure of weight 0. Thus, $\iota_e^* \mathcal{F} \in \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$ can be identified with an element of $\mathrm{Mod}(R, \mathrm{Fr})$. By [15, Proposition 3.1.6] and (7.8), we have

$$\mathrm{Hom}_{R\otimes R}\big(R, \mathbb{H}_B(\mathcal{F})\big) \cong \mathbb{H}_B\big(\mathrm{RHom}(\Delta_e, \mathcal{F})\big) \cong \mathbb{H}_B(\iota_e^* \mathcal{F}).$$

(Since $\iota_e^* \mathcal{F}$ is pure of weight 0, $\mathbb{H}_B(\iota_e^* \mathcal{F})$ is simply the corresponding object in $\mathrm{Mod}(R, \mathrm{Fr})$.) We conclude that $\iota_e^* \mathcal{F} \cong \mathrm{Hom}_{R\otimes R}(R, \mathbb{H}_B(\mathcal{F}))$ inside $\mathrm{Mod}(R, \mathrm{Fr})$ and the result follows. □

We remark that $\mathrm{Hom}_{R\otimes R}(R, \mathbb{H}_B(\mathcal{F}))$ is free as an R-module (cf. [15, Lemma 3.1.5]; see also Remark 3.4).

7.5.4

Recall that we have a realization functor $\mathrm{real} : \mathrm{K^b\,Semis}_B(X) \to \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk)$. We now have two functors from $\mathrm{K^b\,Semis}_B(X)$ to $\mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$. The functor

$$\iota_e^* \circ \mathrm{real} : \mathrm{K^b\,Semis}_B(X) \to \mathrm{D}^{\mathrm{b}}_B(X, \Bbbk) \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk) \tag{7.9}$$

and the functor

$$\mathrm{K^b\,Hom}_{R\otimes R}(R, -) \circ \mathrm{K^b}\mathbb{H}_B : \mathrm{K^b\,Semis}_B(X) \to \mathrm{K^b}\mathbb{S}\mathrm{Bim} \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk). \tag{7.10}$$

We explain the last functor $\mathrm{K^b\,Hom}_{R\otimes R}(R, -) : \mathrm{K^b}\mathbb{S}\mathrm{Bim} \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$. Let $\mathrm{Free}(R, \mathrm{Fr})$ denote the category of finitely generated free R-modules equipped with an action of Fr. The functor $\mathrm{Hom}_{R\otimes R}(R, -)$ takes $\mathbb{S}\mathrm{Bim}$ to $\mathrm{Free}(R, \mathrm{Fr})$, and $\mathrm{K^b\,Hom}_{R\otimes R}(R, -)$ takes $\mathrm{K^b}\mathbb{S}\mathrm{Bim}$ to $\mathrm{K^b\,Free}(R, \mathrm{Fr})$. We have an inclusion $\mathrm{Free}(R, \mathrm{Fr}) \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$. Applying Proposition 7.2, we obtain a triangulated functor $\mathrm{real} : \mathrm{K^b}(\mathrm{Free}(R, \mathrm{Fr})) \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$. Composing $\mathrm{K^b\,Hom}_{R\otimes R}(R, -)$ with real, we obtain $\mathrm{K^b\,Hom}_{R\otimes R}(R, -) : \mathrm{K^b}\mathbb{S}\mathrm{Bim} \to \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$.

By Lemma 7.11, the two triangulated functors (7.9) and (7.10) agree on the subcategory $\mathrm{Semis}_B(X) \subset \mathrm{K^b\,Semis}_B(X)$, sending $\mathrm{Semis}_B(X)$ to $\mathrm{Free}(R, \mathrm{Fr}) \subset \mathrm{D_{perf}}(R, \mathrm{Fr}) \cong \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk)$. Denoting this restriction by $L : \mathrm{Semis}_B(X) \to \mathrm{Free}(R, \mathrm{Fr})$, we apply Proposition 7.3 to deduce that both triangulated functors are isomorphic to

$$\mathrm{real} \circ \mathrm{K^b} L : \mathrm{K^b\,Semis}_B(X) \to \mathrm{K^b\,Free}(R, \mathrm{Fr}) \to \mathrm{D_{perf}}(R, \mathrm{Fr}) \cong \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk).$$

Thus,

$$\iota_e^* \circ \mathrm{real} \cong \mathrm{K^b\,Hom}_{R\otimes R}(R, -) \circ \mathrm{K^b}\mathbb{H}_B. \tag{7.11}$$

Remark 7.12

The essential image of the realization functor $\mathrm{K^b\,Free}(R, \mathrm{Fr}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(R, \mathrm{Fr})$ is a subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(R, \mathrm{Fr})$ equivalent to the infinitesimal extension of $\mathrm{K^b\,Free}(R, \mathrm{Fr})$, in the sense of [2].

7.5.5. Conclusion

By (7.11) and Proposition 7.8, we have

$$\iota_e^*\big(\Delta_w\big(-\ell(w)/2\big) \star \nabla_v\big(\ell(v)/2\big)\big) \cong \mathrm{Hom}_{R\otimes R}(R, F^\bullet_{v,w})$$

inside $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(R, \mathrm{Fr})$. By Lemma 7.10 and (7.8), we find

$$\iota_e^*\big(\Delta_w\big(-\ell(w)/2\big) \star \nabla_v\big(\ell(v)/2\big)\big) \cong \mathrm{RHom}_X\big(\Delta_v\big(-\ell(v)/2\big), \Delta_w\big(-\ell(w)/2\big)\big).$$

This finishes the proof of Proposition 7.1, as well as of Proposition 5.4 and Theorem 1.17. □
□
□

8. Ordinary cohomology, Koszul duality, and Verma modules

The goal of this section is to prove Theorems 1.16, 1.18, and 1.20.

8.1. Ordinary cohomology

We have a forgetful functor $\mathrm{For} : \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk) \to \mathrm{D}^{\mathrm{b}}(\mathrm{pt}, \Bbbk)$, and a commutative diagram (see [13] and [15, Proposition B.3.1])

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_B(\mathrm{pt}, \Bbbk) & \xrightarrow{\mathrm{For}} & \mathrm{D}^{\mathrm{b}}(\mathrm{pt}, \Bbbk) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(R, \mathrm{Fr}) & \xrightarrow{\otimes^L_R \Bbbk} & \mathrm{D}^{\mathrm{b}}(\Bbbk, \mathrm{Fr}) \end{array} \tag{8.1}$$

Here, $\mathrm{D}^{\mathrm{b}}(\Bbbk, \mathrm{Fr})$ is the derived category of finite-dimensional $\Bbbk$-vector spaces equipped with an Fr-action with integer weights. Applying For to Proposition 7.1, we obtain the following.

PROPOSITION 8.1
We have

$$\mathrm{RHom}_{\mathrm{D}^{\mathrm{b}}(X,\Bbbk)}\big(\Delta_v^{(B)}\big(-\ell(v)/2\big), \Delta_w^{(B)}\big(-\ell(w)/2\big)\big) \cong \mathrm{Hom}_{R\otimes R}(R, F^\bullet_{v,w}) \otimes_R \Bbbk$$

inside $\mathrm{D}^{\mathrm{b}}(\Bbbk, \mathrm{Fr})$.

Here, $\Delta_v^{(B)} = \mathrm{For}(\Delta_v) \in \mathrm{D}^{\mathrm{b}}_{(B)}(X, \Bbbk)$ denotes the ordinary standard object in the Borel-constructible derived category, and the derived tensor product $\otimes^L_R \Bbbk$ is replaced by the usual tensor product since $\mathrm{Hom}_{R\otimes R}(R, F^\bullet_{v,w})$ is free as an R-module. Taking the hypercohomology of both sides of Proposition 8.1, we obtain the following.

COROLLARY 8.2
For all $v \le w \in W$, and all $k, r \in \mathbb{Z}$, we have an isomorphism

$$\mathrm{Ext}^{r+k,((r-\ell_{v,w})/2)}(\Delta_v^{(B)}, \Delta_w^{(B)}) \cong H^{k,(r/2)}\big(HH^0_\Bbbk(F^\bullet_{v,w})\big) \tag{8.2}$$

of $\Bbbk$-vector spaces. (In particular, both sides are zero for odd r.)

Here we set $HH^0_{\Bbbk}(B) := HH^0(B) \otimes_R \Bbbk$, similarly to (1.18). For the shift in cohomological degree, see the discussion after Proposition 7.1.

Proof of Theorem 1.16
The nonequivariant version of Proposition 5.3 is given in [106, Proposition 4.2.1]. Similarly to (6.1)–(6.2), we get

$$\mathrm{Ext}^{m,((r-\ell_{v,w})/2)}(\Delta_v^{(B)}, \Delta_w^{(B)}) \cong H_c^{m+\ell_{v,w},(r/2)}(R^\circ_{v,w}, \Bbbk) \quad \text{for all } m, r \in \mathbb{Z}.$$

Poincaré duality (4.5) allows one to translate the compactly supported cohomology into the ordinary cohomology:

$$\mathrm{Ext}^{m,((r-\ell_{v,w})/2)}(\Delta_v^{(B)}, \Delta_w^{(B)}) \cong H^{\ell_{v,w}-m,(\ell_{v,w}-r/2)}(R^\circ_{v,w}, \Bbbk). \tag{8.3}$$

Combining (8.2)–(8.3) with Koszul duality (1.23) proved in the next section, and switching from working over $\Bbbk$ to working over $\mathbb{C}$ via Remark 5.1 and Proposition 5.5, we get

$$\begin{aligned}\dim_{\mathbb{C}} H^{k,(r/2)}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big) &= \dim_{\mathbb{C}} H^{\ell_{v,w}-k-r,(\ell_{v,w}-r/2,\ell_{v,w}-r/2)}(R^\circ_{v,w}, \mathbb{C}) \\ &= \dim_{\mathbb{C}} H^{-k,(r/2,r/2)}(R^\circ_{v,w}, \mathbb{C}). \qquad \square\end{aligned}$$

Proof of (1.24)
By the Künneth formula and Corollary 4.8, we have

$$H^\bullet(R^\circ_{v,w}) \cong H^\bullet(R^\circ_{v,w}/T) \otimes_{\mathbb{C}} H^\bullet(T).$$

The space $H^\bullet(T)$ is 2^{n-1}-dimensional, and the mixed Hodge polynomials are related as

$$\mathcal{P}(R^\circ_{v,w}; q, t) = (q^{\frac{1}{2}} + t^{\frac{1}{2}})^{n-1} \cdot \mathcal{P}(R^\circ_{v,w}/T; q, t). \tag{8.4}$$

This implies (1.10). Next, we claim that

$$\mathcal{P}(R^\circ_{v,w}/T; q, t) = (q^{\frac{1}{2}} t^{\frac{1}{2}})^{\chi(\beta_{v,w})} \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\hat{\beta}_{v,w}; q, t). \tag{8.5}$$

First, combining Theorem 1.16 with Koszul duality (1.23) (to be proved below in Section 8.2), we find

$$H^{k,(p,p)}(R^\circ_{v,w}, \mathbb{C}) \cong H^{-\ell_{v,w}-k+2p,(\ell_{v,w}-p)}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big). \tag{8.6}$$

Setting $k' := -\ell_{v,w} - k + 2p$ and $p' := \ell_{v,w} - p$, we find $p = \ell_{v,w} - p'$ and $k = \ell_{v,w} - k' - 2p'$. Plugging this into (4.6) and applying (8.6), we get

$$\mathcal{P}(R^\circ_{v,w};q,t) = \sum_{k,p\in\mathbb{Z}} q^{p-\frac{k}{2}} t^{\frac{\ell_{v,w}-k}{2}} \dim H^{k,(p,p)}(R^\circ_{v,w},\mathbb{C})$$
$$= \sum_{k',p'\in\mathbb{Z}} q^{\frac{\ell_{v,w}+k'}{2}} t^{p'+\frac{k'}{2}} H^{k',(p')}\big(HH^0_{\mathbb{C}}(F^\bullet_{v,w})\big).$$

On the other hand, rewriting (3.16), we see that the right-hand side of (8.5) is given by

$$(q^{\frac{1}{2}}+t^{\frac{1}{2}})^{1-n} \sum_{k',p'\in\mathbb{Z}} q^{\frac{\ell_{v,w}+k'}{2}} t^{p'+\frac{k'}{2}} \dim H^{k',(p')}\big(HH^0_{\mathbb{C}}\big(F^\bullet(\beta)\big)\big).$$

Together with (8.4), this finishes the proof of (8.5). Finally, (1.24) follows by comparing (8.5) with (the Richardson version of) (4.9). □

Since the R-action on $H^\bullet_{T,c}(R^\circ_{v,w})$ is trivial (i.e., $\mathfrak{h}^*$ acts by zero), by Theorem 1.17, the R-action on $HHH^0(F^\bullet_{v,w})$ is also trivial. Alternatively, for $W = S_n$ and any knot $\hat{\beta}$, the R-action on $HHH^0(F^\bullet(\beta))$ is trivial by Corollary 3.10. It thus follows that we have an isomorphism of bigraded $\mathbb{C}$-modules

$$HHH^0_{\mathbb{C}}\big(F^\bullet(\beta)\big) \cong \operatorname{Tor}^R_\bullet\big(\mathbb{C}, HHH^0\big(F^\bullet(\beta)\big)\big). \tag{8.7}$$

We conjecture that (8.7) holds for all W and all $\beta \in \mathcal{B}_W$. This would follow from (8.1) if $HH^0(F^\bullet(\beta))$ and $HHH^0(F^\bullet(\beta))$ were known to be equivalent in $\mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(R,\mathrm{Fr})$.

8.2. *Koszul duality and q,t-symmetry*

We prove Theorem 1.18. By [15, (5.2), Theorem 5.3.1, and Remark 5.3.2], for $k, r \in \mathbb{Z}$ and $v \le w \in W$, we get an isomorphism

$$\operatorname{Ext}^{k,(r/2)}(\Delta^{(B)}_v, \Delta^{(B)}_w) \cong \operatorname{Ext}^{k-r,(-r/2)}(\Delta^{(B)}_{v^{-1}}, \Delta^{(B)}_{w^{-1}})$$

of vector spaces. By (8.3) and Proposition 5.5, this implies that

$$H^{k,(r/2,r/2)}(R^\circ_{v,w},\mathbb{C}) \cong H^{\ell_{v,w}+k-r,(\ell_{v,w}-r/2,\ell_{v,w}-r/2)}(R^\circ_{v^{-1},w^{-1}},\mathbb{C}). \tag{8.8}$$

The only difference between (8.8) and the desired result (1.23) is the appearance of v^{-1} and w^{-1} on the right-hand side. In fact, it is not hard to see that the Richardson varieties $R^\circ_{v,w}$ and $R^\circ_{v^{-1},w^{-1}}$ are isomorphic. Indeed, recall from Lemma 4.4 that we have an isomorphism $N^\circ_{v,w} \cong R^\circ_{v,w}$. The map $g \mapsto g^{-1}$ restricts to an isomorphism $N^\circ_{v,w} \cong N^\circ_{v^{-1},w^{-1}}$ (choosing $\dot{v}^{-1}$ as the representative for v^{-1}). By (4.3), we get an isomorphism $R^\circ_{v,w} \xrightarrow{\sim} R^\circ_{v^{-1},w^{-1}}$.

8.3. Extensions of Verma modules

We prove Theorem 1.20. First, we explain the bigrading on $\mathrm{Ext}^\bullet(M_v, M_w)$. Out of the several equivalent descriptions listed in [11], the most convenient one for us is given in [11, Section 4.4]; the bigraded vector spaces $\mathrm{Ext}^\bullet(M_v, M_w)$ and $\mathrm{Ext}^\bullet(\Delta_v^{(B)}, \Delta_w^{(B)})$ are isomorphic (after changing the coefficients from $\mathbb{C}$ to $\mathbb{k}$), and the bigrading on $\mathrm{Ext}^\bullet(M_v, M_w)$ comes from the bigrading on $\mathrm{Ext}^\bullet(\Delta_v^{(B)}, \Delta_w^{(B)})$ via Frobenius weights (5.3):

$$\mathrm{Ext}^{k,(r/2)}(M_v, M_w) := \mathrm{Ext}^{k,(r/2)}(\Delta_v^{(B)}, \Delta_w^{(B)}).$$

See also [106, Equation (1.1.1)].

The result follows by combining (8.3) with Koszul duality (1.23).

9. Catalan numbers associated to positroid varieties

Our results give an embedding of the rational q,t-Catalan numbers $C_{k,n-k}(q,t)$ into a family of q,t-polynomials $\mathcal{P}(\mathcal{X}_f^\circ; q,t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ (all of which are q,t-symmetric and q,t-unimodal), indexed by $f \in \mathbf{B}_{k,n}^{c=1}$. The goal of this section is to give a combinatorial interpretation for a specialization of $\mathcal{P}(\mathcal{X}_f^\circ; q,t)$.

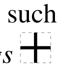
Definition 9.1

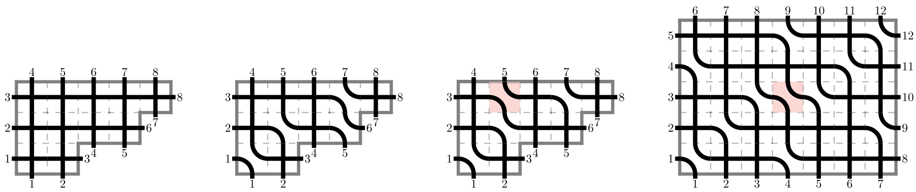
For $f \in \mathbf{B}_{k,n}^{c=1}$, define the *$f$-Catalan number C_f* $\in \mathbb{Z}$ as the specialization

$$C_f := \mathcal{P}(\mathcal{X}_f^\circ; q,t)|_{q^{\frac{1}{2}}=1, t^{\frac{1}{2}}=-1}.$$

Alternatively, C_f is the $q = 1$ specialization of the point count polynomial $\#\mathcal{X}_f^\circ(\mathbb{F}_q)$, and we also have $C_f = P_f^{\mathrm{top}}(1)$, where the polynomial $P_f^{\mathrm{top}}(q)$ is defined in Theorem 1.13.

In particular, $C_{f_{k,n}} = C_{k,n-k}(1,1) = \#\mathrm{Dyck}_{k,(n-k)}$ is the usual rational Catalan number when $\gcd(k,n) = 1$.

Recall from Proposition 1.10 that each $f = f_{v,w} \in \mathbf{B}_{k,n}$ corresponds to a pair $v \leq w \in S_n$ such that w is k-Grassmannian. The set of k-Grassmannian permutations in S_n is well known to be in bijection with the set of Young diagrams that fit inside a $(k \times (n-k))$-rectangle. Let λ be such a Young diagram. We are going to consider fillings of boxes of λ with *crossings*  and *elbows* . An example is given in Figure 4. Each such filling D gives rise to a permutation u_D, obtained as follows. Consider paths labeled by $1, 2, \ldots, n$ entering from the southeast boundary of λ, where the labels increase in the northeast direction. The paths follow crossings and elbows until they exit through the northwest boundary of λ. Recording the positions of outgoing edges, one obtains the permutation u_D (cf. Figure 4).



$w = 45167283$ $v = 12465387$ not distinguished not distinguished

Figure 4. (Color online) For the two fillings on the left, we have $\lambda = (5,4,2)$ and $f = wv^{-1} = 35148276$. The two fillings on the right do not satisfy the distinguished condition: the specific elbow violating the condition is shaded.

Definition 9.2
Let λ be a Young diagram fitting in a $(k \times (n-k))$-rectangle. A *Deogram* (short for *Deodhar diagram*[9]) of shape λ is a filling D of the boxes of λ with crossings and elbows satisfying the following *distinguished condition* (see [31]): for any elbow in D, the label of its bottom-left path is less than the label of its top-right path. In other words, once two paths have crossed an odd number of times, they cannot form an elbow (see Figure 4).

For example, any filling that consists either entirely of crossings or entirely of elbows satisfies the distinguished condition. Observe that when a Deogram D of shape λ consists entirely of crossings, the permutation $u_D = w$ indeed is k-Grassmannian: we have $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k)$ and $w^{-1}(k+1) < \cdots < w^{-1}(n)$. We denote this correspondence by $\lambda_w := \lambda$.

Definition 9.3
Let $f = f_{v,w} \in \mathbf{B}_{k,n}$. An *$f$-Deogram* is a Deogram D of shape λ_w satisfying $u_D = v$. A *maximal f-Deogram* is an f-Deogram with the maximal possible number of crossings among all f-Deograms.

We denote by Deo_f (resp., $\mathrm{Deo}_f^{\max}$) the set of all (resp., maximal) f-Deograms.

Remark 9.4
It is easy to see that any f-Deogram must have at least $n - c(\bar{f})$ elbows. One can also check that for each $f \in \mathbf{B}_{k,n}$, there exist f-Deograms with exactly $n - c(\bar{f})$ elbows.[10]

[9]The terminology *Deodhar diagram* is borrowed from [79].
[10]The same statement does not hold for Richardson varieties: for $w = s_1s_2s_3s_2s_1$ and $v = s_2$ in S_4, there are no subexpressions for v inside w skipping exactly $n - c(wv^{-1}) = 2$ indices.

PROPOSITION 9.5
Let $f \in \mathbf{B}_{k,n}^{c=1}$. Then C_f equals the number of maximal f-Deograms:

$$C_f = \#\mathrm{Deo}_f^{\max}. \tag{9.1}$$

Proof
This is a simple consequence of the results of Deodhar [31]. Let v, w be such that $f = f_{v,w}$. By Proposition 4.3, we have $\#\Pi_f^\circ(\mathbb{F}_q) = \#R_{v,w}^\circ(\mathbb{F}_q)$. Deodhar expressed $\#R_{v,w}^\circ(\mathbb{F}_q)$ as a certain sum over *distinguished subexpressions* for v inside a reduced word $w = s_{i_1}s_{i_2}\cdots s_{i_l}$, where $l = \ell(w)$. Here, a *subexpression* for v is a way to write v as a product $v_1v_2\cdots v_l$, where $v_j \in \{s_{i_j}, \mathrm{id}\}$ for all $j = 1,2,\dots,l$. A subexpression is *distinguished* if for all j such that $\ell(v_1\cdots v_{j-1}s_{i_j}) < \ell(v_1\cdots v_{j-1})$, we have $v_j = s_{i_j}$. Since w is k-Grassmannian, the terms in the product $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ correspond to the boxes of λ_w. Each Deogram $D \in \mathrm{Deo}_f$ gives rise to a distinguished subexpression for v, so that the indices j such that $v_j = s_{i_j}$ correspond to the crossings in D. It is easy to see that this correspondence is bijective. Thus, the results of [31] imply that

$$\#\Pi_f^\circ(\mathbb{F}_q) = \sum_{D\in\mathrm{Deo}_f} (q-1)^{\mathrm{elb}(D)} q^{(\mathrm{xing}(D)-\ell(v))/2}, \tag{9.2}$$

where $\mathrm{elb}(D)$ and $\mathrm{xing}(D)$ denote the number of elbows and crossings in D. By Remark 9.4, each maximal f-Deogram contributes $(q-1)^{n-1}q^{(\ell_{v,w}-n+1)/2}$ to the right-hand side of (9.2). (Note that $\mathrm{xing}(D) + \mathrm{elb}(D) = \ell(w)$ is constant.) It remains to note that $\#\mathcal{X}_f^\circ(\mathbb{F}_q)$ is obtained by dividing $\#\Pi_f^\circ(\mathbb{F}_q)$ by $\#T(\mathbb{F}_q) = (q-1)^{n-1}$, and that C_f is by definition the $q = 1$ specialization of $\#\mathcal{X}_f^\circ(\mathbb{F}_q)$. □

Let us focus on the case $f = f_{k,n}$ with $\gcd(k,n) = 1$. Explicitly, a maximal $f_{k,n}$-Deogram is a way of placing $n-1$ elbows in a $(k \times (n-k))$ rectangle and filling the rest with crossings so that (i) the resulting permutation obtained by following the paths is the identity, and (ii) the distinguished condition in Definition 9.2 is satisfied. By Proposition 9.5, the number $\#\mathrm{Deo}_{f_{k,n}}^{\max}$ of such objects equals the number $\#\mathrm{Dyck}_{k,(n-k)}$ of Dyck paths inside a $(k \times (n-k))$-rectangle.

Problem 9.6
Give a bijective proof of Proposition 9.5. That is, find a bijection between $\mathrm{Deo}_{f_{k,n}}^{\max}$ and $\mathrm{Dyck}_{k,(n-k)}$ for the case $\gcd(k,n) = 1$.

For instance, Figures 1 and 5 both have 7 objects in them, but it is unclear which objects correspond to which. It would also be interesting to understand the area and dinv statistics in the language of $f_{k,n}$-Deograms.

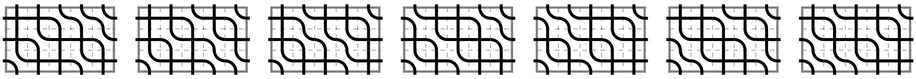


Figure 5. The sets $\mathrm{Deo}^{\max}_{f_{k,n}}$ and $\mathrm{Dyck}_{k,n-k}$ have the same cardinality by Proposition 9.5. Compare with Figure 1.

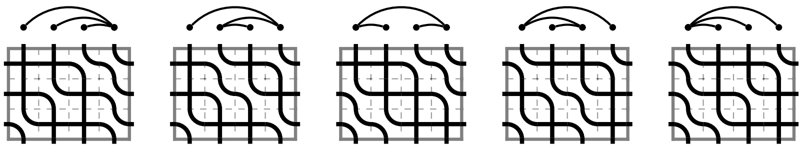


Figure 6. For $n = 2k + 1$, maximal $f_{k,n}$-Deograms are in bijection with noncrossing alternating trees (see Remark 9.7).

Remark 9.7
For the case $n = 2k+1$ of the standard Catalan numbers, the maximal $f_{k,n}$-Deograms are easily seen to be in bijection with *noncrossing alternating trees on* $k + 1$ *vertices* (item 62 in [123]). Explicitly, given $D \in \mathrm{Deo}^{\max}_{f_{k,n}}$, we assign a vertex to each of the $k + 1$ columns of D. One can show that every row of D must contain exactly two elbows, and connecting the two corresponding vertices by an edge for each of the k rows, one obtains a noncrossing alternating tree. The case $k = 3$, $n = 7$ is illustrated in Figure 6.

Remark 9.8
A recursive proof of Proposition 9.5 for the case $n = dk \pm 1$ ($d \geq 2$) was found by David Speyer (private communication). It appears that when $n = dk \pm 1$, the distinguished condition is automatically satisfied for any maximal $f_{k,n}$-Deogram. However, this is not the case for instance when $k = 5$ and $n = 12$ (see Figure 4 (right)). We were able to find a recursive proof of (9.1) for arbitrary k, n. This and some other enumerative consequences of our results will appear in a separate paper [48].

Remark 9.9
A probabilistic interpretation of f-Deograms and their weights in (9.2) in terms of the *stochastic colored six-vertex model* (see [80]) was recently discovered in [43]. In particular, a result closely related to Theorem 2.3 appears in [43, Lemma 7.1 and Proposition 7.3].

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