THE MIRROR CONJECTURE FOR MINUSCULE FLAG VARIETIES

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Abstract

We prove Rietsch's mirror conjecture that the Dubrovin quantum connection for minuscule flag varieties is isomorphic to the character D-module of the Berenstein–Kazhdan geometric crystal. The idea is to recognize the quantum connection as Galois and the geometric crystal as automorphic. We reveal surprising relations with the works of Frenkel and Gross; Heinloth, Ngô, and Yun; and Zhu on Kloosterman sheaves. The isomorphism comes from global rigidity results where Hecke eigensheaves are determined by their local ramification. As corollaries, we obtain combinatorial identities for counts of rational curves and the Peterson variety presentation of the small quantum cohomology ring.

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1. Introduction

Let G be a complex, almost simple, algebraic group, let $B \subset G$ be a Borel subgroup, and let $P \subset G$ be a parabolic subgroup containing B. Let $B^\vee \subset P^\vee \subset G^\vee$ denote the Langlands duals. In the case that $P^\vee$ is a minuscule maximal parabolic subgroup, we prove the mirror theorem that the *quantum connection* of the partial flag variety $G^\vee/P^\vee$ is isomorphic to the *character D-module* of the geometric crystal associated to (G, P). This isomorphism is the top row of the diagram of D-modules of Figure 1, where the bottom row is an instance of the geometric Langlands program.

In fact, our main result is stronger. It concerns the *equivariant* quantum cohomology of $G^\vee/P^\vee$, and moreover adds a *parameter* $\hbar \in \mathbb{A}^1$. This is Theorem 11.14. Specializing $\hbar = 1$ is the equivariant version of the above mirror theorem which was conjectured by Rietsch [107], and taking the semiclassical limit ($\hbar \to 0$) yields the equivariant Peterson isomorphism which was stated in the as yet unpublished lectures of Peterson [100].

We now discuss the diagram of Figure 1 in detail.

1.1. *Quantum cohomology and mirror symmetry for flag varieties*

The study of the topology of flag varieties $G^\vee/B^\vee$ has a storied history. Borel found the cohomology rings $H^*(G^\vee/B^\vee, \mathbb{C})$ to be isomorphic to the coinvariant algebras of the Weyl group W acting on the natural reflection representations. This result is continued by the works of Chevalley, Bernstein, Gelfand, and Gelfand, Demazure, Lascoux, and Schützenberger, and many others on the Schubert calculus of flag varieties.

Much progress was made on the quantum cohomology of flag varieties in the last two decades. Givental and Kim [56] and Ciocan-Fontanine [28] (for $G^\vee$ of type A), and Kim [80] (for general $G^\vee$) identified the quantum cohomology rings

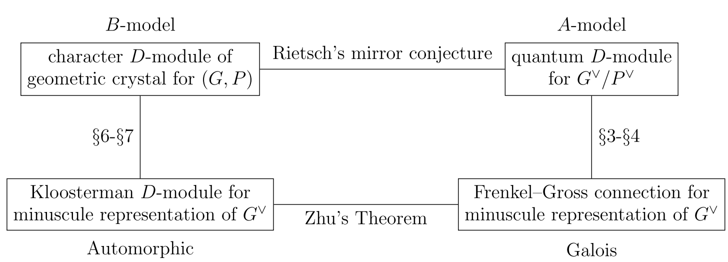

Figure 1. The four D-modules in this work.

$QH^*(G^\vee/B^\vee,\mathbb{C})$ with the ring of regular functions on the nilpotent leaf of the Toda lattice of G. Subsequently, Givental [54] formulated a mirror conjecture that oscillatory integrals over a middle-dimensional cycle inside the mirror manifold should be solutions to the quantum D-module, and established this result for $G^\vee$ of type A (see also [37]). This mirror theorem was extended to general $G^\vee/B^\vee$ by Rietsch in [107] and [108]. These oscillatory integrals gave new integral formulas for Whittaker functions.

By contrast, our understanding of mirror symmetry for partial flag varieties $G^\vee/P^\vee$ is much more limited. Peterson [100] discovered a uniform geometric description of the quantum cohomology rings $QH^*(G^\vee/P^\vee,\mathbb{C})$, but this work remains unpublished (see, however, [25], [87], [105], [107]). The quantum D-modules of $G^\vee/P^\vee$ have remained largely unstudied in full generality. Batyrev et al. [5] proposed a mirror conjecture for $\mathrm{SL}(n)/P^\vee$, and Rietsch formulated a mirror conjecture for arbitrary $G^\vee/P^\vee$, in the style of Givental.

One of the main aims of this work is to establish Rietsch's mirror conjecture in the case that $P^\vee$ is minuscule (see Section 2.4). This class of spaces includes projective spaces, Grassmannians, and orthogonal Grassmannians (see Figure 2 for the full list). Even for the case of Grassmannians, whose quantum cohomology rings are well studied in [11], [113], and [123] and a large part of the mirror conjecture established in [93], our results are new.

1.2. Small quantum D-module

We now let $P^\vee$ be a minuscule (maximal) parabolic subgroup. The small quantum cohomology ring $QH^*(G^\vee/P^\vee)$ is isomorphic to $\mathbb{C}[q,q^{-1}]\otimes H^*(G^\vee/P^\vee)$ as a vector space,[1] with quantum multiplication denoted by $*_q$.

Let $\mathbb{C}_q^\times=\mathrm{Spec}(\mathbb{C}[q,q^{-1}])$ be the 1-dimensional torus with coordinate q. The small quantum D-module (at $\hbar=1$) (see [36]) is the connection on the trivial $H^*(G^\vee/P^\vee)$-bundle over $\mathbb{C}_q^\times$ given by

$$\mathcal{Q}^{G^\vee/P^\vee}:=d+(\sigma *_q)\frac{dq}{q}, \tag{1.2.1}$$

where $\sigma\in H^2(G^\vee/P^\vee,\mathbb{Z})$ is the effective divisor class, and we consider

$$\sigma *_q\in \mathrm{End}\big(H^*(G^\vee/P^\vee)\big)\otimes\mathbb{C}[q]. \tag{1.2.2}$$

In [26], Chevalley gave a combinatorial formula for the cup product in $H^*(G^\vee/P^\vee)$ with the divisor class σ, that is, for (1.2.2) at $q=0$. A quantum Chevalley formula (see Theorem 4.3) evaluating (1.2.2) in terms of Schubert classes for general

[1] In this paper, cohomologies and quantum cohomologies are all taken with $\mathbb{C}$ coefficients.

flag varieties was stated by Peterson [100] and proved by Fulton and Woodward [46]. This formula has been extended to the equivariant case by Mihalcea [94] and to the cotangent bundle of partial flag varieties by Su [119]. For recent developments in the minuscule case, see [19].

In the sequel, we also let $\mathcal{Q}^{G^\vee/P^\vee}$ denote the corresponding algebraic D-module, where $D = D_{\mathbb{C}_q^\times} = \mathbb{C}[q,q^{-1}]\langle \partial_q \rangle$ is the ring of differential operators on $\mathbb{C}_q^\times$, and $\partial_q := \frac{d}{dq}$.

1.3. The character D-module of a geometric crystal

Berenstein and Kazhdan [8], based on previous works by Lusztig and of Berenstein and Zelevinsky [10], have constructed geometric crystals which are certain complex algebraic varieties equipped with rational maps. The motivation of the construction was the birational lifting of the combinatorics of Lusztig's canonical bases and Kashiwara's crystal bases.

Fix opposite Borel subgroups B and B_- of G with unipotent subgroups U and U_-, and let $T = B \cap B_-$. Let R denote the root system, and let $R^\pm$ denote the subsets of positive and negative roots. Let $\psi : U \to \mathbb{G}_a$ be a *nondegenerate* character in the sense that ψ is nontrivial on every simple root space when composed with the exponential isomorphism $\bigoplus_{i\in I} \mathbb{A}^1 \cong U/[U,U]$.

For a parabolic subgroup $P \subset G$, let $W_P \subset W$ be the Weyl group of the Levi subgroup L_P, and let $I_P \subset I$ be the corresponding subset of the Dynkin diagram. There is a unique set $W^P \subset W$ of minimal length coset representatives for the quotient W/W_P. Define $w_P^{-1} \in W$ to be the longest element in W^P. The *(parabolic) geometric crystal* $X = X_{(G,P)}$ is the smooth subvariety

$$X = UZ(L_P)\dot{w}_P U \cap B_- \subset G,$$

where $Z(L_P)$ denotes the center of the Levi subgroup L_P, and $\dot{w}_P \in G$ is a representative of $w_P \in W$, equipped (see [8, Section 2.2]) with geometric crystal actions $\mathbb{G}_m \times X \to X$ (which are rational maps, defined on a dense open subset) and three regular maps of importance to us:

$$f : X \to \mathbb{A}^1, \qquad u_1 t \dot{w}_P u_2 \mapsto \psi(u_1) + \psi(u_2) \qquad \text{called the } \textit{decoration function},$$

$$\gamma : X \to T, \qquad x \mapsto x \mod U_- \in B_-/U_- \cong T \qquad \text{called the } \textit{weight function},$$

$$\pi : X \to Z(L_P), \qquad u_1 t \dot{w}_P u_2 \mapsto t \qquad \text{called the } \textit{highest weight function}.$$

The fiber $X_t := \pi^{-1}(t)$ for $t \in Z(L_P)$ is called the *geometric crystal with highest weight* t. For any $t \in Z(L_P)$, X_t is a log Calabi–Yau variety isomorphic to the open projected Richardson variety $\mathring{G/P} \subset G/P$ (see [81]), the complement in G/P

of a particular anticanonical divisor $\partial_{G/P}$. (The boundary divisor $\partial_{G/P}$ is not normal crossing in general. There is a Bott–Samelson resolution that provides an explicit compactification of $\mathring{G/P}$ with normal crossing divisors; see [81, Appendix], [84, Section 4.2].) The affine variety $\mathring{G/P}$ has a distinguished holomorphic volume form ω (see Section 6.6), with simple poles along the boundary divisor $\partial_{G/P}$.

On $\mathbb{A}^1$ we consider the cyclic D-module $\mathsf{E} := D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial_x - 1)$ with generator the exponential function, where $D_{\mathbb{A}^1} = \mathbb{C}[x]\langle\partial_x\rangle$. The pullback $f^*\mathsf{E}$ is a D-module on X. Note that one can identify the D-module E with the connection $d - dx$ on the trivial line bundle on $\mathbb{A}^1$. The pullback $f^*\mathsf{E}$ can be identified with the connection $d - df$ on the trivial line bundle on X. We define the *character D-module* of the geometric crystal X by

$$\mathrm{Cr}_{(G,P)} := R\pi_* f^*\mathsf{E}, \tag{1.3.1}$$

which is a D-module on $Z(L_P)$. For $P = B$, the geometric crystal X tropicalizes to Kashiwara's combinatorial crystals (see [8, Section 6]). As explained in Lam [86] and Chhaibi [27], the tropicalization of (1.3.1) is an irreducible character of $G^\vee$.

A priori $\mathrm{Cr}_{(G,P)}$ is a complex of D-modules, but we show in Theorem 1.8 (= Theorem 7.10) that it is just a D-module. Our proof is via the left-hand side of Figure 1, which enables us to recognize this statement as the Ramanujan property, in the context of the geometric Langlands program, for a certain cuspidal automorphic D-module $A_{\mathfrak{g}}$ (see Section 1.7 below and [71, Theorem 1]).

This article seems to be the first time the properties of the character D-module $\mathrm{Cr}_{(G,P)}$ are studied. There are other geometric crystals, and as we shall see below, other families of Landau–Ginzburg models that one could apply this construction to. We also note that automorphic D-modules with wild ramification, and geometric analogues of Arthur conjectures—both of which play an important role in our study—are themes which have been largely unexplored at the present time.

1.4. Rietsch's Landau–Ginzburg model

In [107], Rietsch constructed conjectural Landau–Ginzburg mirror partners of all partial flag varieties G/P. Her construction was motivated by earlier works of Givental [54], Joe and Kim [77], and Batyrev et al. [5] for type A flag varieties, and also by the Peterson presentation of $QH^*(G^\vee/P^\vee)$.

Rietsch's mirror construction are families of varieties fibered over $q \in \mathrm{Spec}(\mathbb{C}[q_i^{\pm 1} \mid i \notin I_P])$, equipped with holomorphic superpotentials f_q, and holomorphic volume forms ω_q.

It was observed by Lam [86] and Chhaibi [27] that Rietsch's mirror construction could be obtained from the group geometry of geometric crystals. Thus, after identifying $\mathrm{Spec}(\mathbb{C}[q_i^{\pm 1} \mid i \notin I_P])$ with $Z(L_P)$, Rietsch's mirror family can be identified with the highest weight function $\pi : X \to Z(L_P)$ of Section 1.3, and Rietsch's superpotential becomes the decoration function $f_t := f|_{X_t} : X_t \to \mathbb{A}^1$; henceforth we will use f_q or f_t interchangeably (q being a point in $\mathrm{Spec}(\mathbb{C}[q_i^{\pm 1} \mid i \notin I_P])$ and t a point in $Z(L_P)$).

Earlier mirror Landau–Ginzburg models for various partial flag varieties (see, e.g., [5], [37], [54]) were Laurent polynomial superpotentials defined on an algebraic torus. These Landau–Ginzburg models arose from toric degenerations of $G^\vee/P^\vee$. In contrast, Rietsch's candidate mirror Landau–Ginzburg model is defined (see [98], [99]) on a partial compactification of a torus, and is intrinsically related to the group geometry of G/P (and not to any toric degeneration). In the literature, this distinction also appears in the form of "strong mirror" versus "weak mirror."

Stated informally our main goal in this paper is to show:

If $P^\vee$ is minuscule, then $G^\vee/P^\vee$ and $(\overset{\circ}{G/P}, f_q)$ form a Fano type mirror pair.

On the A-model side $G^\vee/P^\vee$ is a projective Fano variety, and on the B-model side $\overset{\circ}{G/P}$ is a log Calabi–Yau variety (see [79] for general expectations for Fano type mirror pairs). We show that some of the mirror symmetry expectations hold.

One expectation is a relationship between the Gromov–Witten invariants of $G^\vee/P^\vee$ and the coefficients of the Laurent series expansion of the potential f_q restricted to a torus of $\overset{\circ}{G/P}$. If $P^\vee$ is minuscule, then we shall establish such a relationship in the form of an integral representation of the quantum period of $G^\vee/P^\vee$ which was previously known for Grassmannians in [93] and for quadrics in [99] (see Section 13 for details).

1.5. The mirror isomorphism

The following is a simple version of the main result of this paper and establishes the top row of Figure 1.

THEOREM 1.6 (Theorem 8.3)
Suppose that $P^\vee$ is minuscule. The geometric crystal D-module $\mathrm{Cr}_{(G,P)}$ and the quantum cohomology D-module $\mathcal{Q}^{G^\vee/P^\vee}$ for $G^\vee/P^\vee$ are isomorphic.

For $G^\vee/P^\vee$ a projective space $\mathbb{P}^n$, the result is well known (see [53], [78]). The homological mirror symmetry version is established in [1] and [40]. Our approach gives an original perspective in terms of hyper-Kloosterman sheaves studied in SGA $4\frac{1}{2}$ (see [33]).

For $G^\vee/P^\vee$ a Grassmannian $\mathrm{Gr}(k,n)$, the result is already new. Partial results are obtained by Marsh and Rietsch [93], notably a canonical injection of $\mathcal{Q}^{G^\vee/P^\vee}$ into $\mathrm{Cr}_{(G,P)}$, who establish as a consequence a conjecture of Batyrev et al. [4, Conjecture 5.2.3]. Our Theorem 1.6 is stronger. Indeed, it establishes the conjecture of [93, Section 3] that the canonical injection is bijective, and thereby also another conjecture of Batyrev et al. [5, Conjecture 5.1.1].

For $G^\vee/P^\vee$ an even-dimensional quadric, the injection of $\mathcal{Q}^{G^\vee/P^\vee}$ into $\mathrm{Cr}_{(G,P)}$ is obtained by Pech, Rietsch, and Williams [99], and our Theorem 1.6 establishes a conjecture of [99, Section 4].

Although both sides of Theorem 1.6 are described explicitly, this does not lead to a way of establishing the isomorphism. Indeed our proof will follow a lengthy path, where the isomorphism will eventually arise from Langlands reciprocity for the automorphic D-module $A_{\mathcal{E}}$ of [71, Section 2.5] over the rational function field $\mathbb{C}(t)$.

While the Langlands program has integrated for a long time adjacent areas of mathematics into the solution of some of its conjectures, in the other direction, there are yet rather few applications of the Langlands program to problems outside of number theory. Interestingly, in this paper we shall apply recent advances in the geometric Langlands program to establish results on the geometry of flag varieties.

1.7. Kloosterman sums, Kloosterman sheaves, and Kloosterman D-modules

For a prime p and a finite field $\mathbb{F}_q$, $q = p^m$, define the two maps

$$f : (\mathbb{F}_q^\times)^n \to \mathbb{F}_q \qquad (x_1, x_2, \dots, x_n) \mapsto x_1 + x_2 + \cdots + x_n,$$
$$\pi : (\mathbb{F}_q^\times)^n \to \mathbb{F}_q^\times \qquad (x_1, x_2, \dots, x_n) \mapsto x_1 x_2 \cdots x_n.$$

The *(hyper-)Kloosterman sum* in $(n-1)$-variables is

$$\mathrm{Kl}_n(a; q) := (-1)^{n-1} \sum_{x \in \pi^{-1}(a)} \exp\Big(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} f(x)\Big), \tag{1.7.1}$$

where $a \in \mathbb{F}_q^\times$. Deligne [33] defines the *(hyper-)Kloosterman sheaf* to be the ℓ-adic sheaf on $\mathbb{F}_p^\times$ given by

$$\mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell} := R\pi_! f^* \mathrm{AS}_\psi[n-1], \tag{1.7.2}$$

where AS_ψ is the Artin–Schreier sheaf on $\mathbb{A}^1$ corresponding to a nontrivial character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell$. For an appropriate embedding $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, the Kloosterman sum (1.7.1) is identified as the Frobenius trace of the Kloosterman sheaf (1.7.2): $\mathrm{Kl}_n(a; q) = \iota(\mathrm{Tr}(\mathrm{Frob}_a, \mathrm{Kl}_n^{\overline{\mathbb{Q}}_\ell}))$. The *Kloosterman D-module* is defined (see [78]) by replacing the Artin–Schreier sheaf with the exponential D-module:

$$\mathrm{Kl}_n := R\pi_! f^* \mathsf{E}. \tag{1.7.3}$$

The pair $(\pi : (\mathbb{F}_q^\times)^n \to \mathbb{F}_q^\times, f)$ and (1.7.2) should be compared with the geometric crystal mirror family $(\pi : X \to Z(L_P), f)$ and (1.3.1).

Heinloth, Ngô, and Yun [71] generalize Kloosterman sheaves and D-modules. More precisely, for a representation V of $G^\vee$, they define a generalized Kloosterman D-module $\mathrm{Kl}_{(G^\vee,V)}$ on $\mathbb{C}^\times$. Their construction uses the moduli stack $\mathrm{Bun}_{\mathcal{G}}$ of $\mathcal{G}$-bundles on $\mathbb{P}^1$, where $\mathcal{G}$ is a particular nonconstant group scheme over $\mathbb{P}^1$ (see Section 7); equivalently, $\mathrm{Bun}_{\mathcal{G}}$ classifies G-bundles with specified ramification behavior. Heinloth, Ngô, and Yun construct an automorphic Hecke eigen-D-module $A_{\mathcal{G}}$ on the Hecke stack over $\mathrm{Bun}_{\mathcal{G}}$. The generalized Kloosterman D-module $\mathrm{Kl}_{(G^\vee,V)}$ is defined to be the Hecke eigenvalue of $A_{\mathcal{G}}$. The projection and superpotential maps π and f are replaced in this setting by the projection maps of the Hecke moduli stack.

A remarkable feature of the automorphic D-module $A_{\mathcal{G}}$ is that it is *rigid*: it can be characterized uniquely by its local components. Indeed, the existence of the rigid local systems constructed by Heinloth, Ngô, and Yun was predicted by Gross, who constructed the trace function of $A_{\mathcal{G}}$ over finite fields via the stable trace formula. (We refer to [126] for a comprehensive survey of rigid automorphic forms.)

The following result gives an automorphic interpretation of geometric crystals.

THEOREM 1.8 (Theorem 7.10)
Let $P \subset G$ be a cominuscule parabolic, and let V be the corresponding minuscule representation of $G^\vee$. The character D-module $\mathrm{Cr}_{(G,P)}$ is isomorphic to the Kloosterman D-module $\mathrm{Kl}_{(G^\vee,V)}$ defined as the V-Hecke eigenvalue of the automorphic D-module $A_{\mathcal{G}}$.

The proof of Theorem 1.8 is by a comparison of the geometry of the Hecke moduli stack and that of parabolic geometric crystals.

The above suggests a new parallel between exponential sums over finite fields and Landau–Ginzburg models. Recall that *arithmetic mirror symmetry* has been studied for mirror Calabi–Yau varieties (see [122, Section 3]). The present work leads us to suggest that arithmetic mirror symmetry could conjecturally extend to Fano varieties and their mirror Landau–Ginzburg models. Although we do not pursue this direction in the present paper, we observe for example the precise compatibility between the recent conjecture of Katzarkov, Kontsevich, and Pantev [79, (3.1.5)], specialized to $G^\vee/P^\vee = \mathbb{P}^n$, and the classical theorem of Sperber [116] on the Newton polygon of the Kloosterman sums $\mathrm{Kl}_n(a;q)$. We remark that the Hodge theory of Kloosterman connections was studied in [44].

We believe that the purity property of an exponential sum (viewed from a mirror Landau–Ginzburg model perspective) should mirror the Hodge–Tate property of the cohomology of a Fano variety. In the context of Theorem 1.8, the Kloosterman sum $\mathrm{Kl}_{(G^\vee,V)}$ is pointwise pure (see [71]), and the partial flag variety G/P has cohomology of Hodge–Tate type. We speculate that the slope multiplicities of the Newton polygon of $\mathrm{Kl}_{(G^\vee,V)}$ should coincide with the Betti numbers $\dim H^i(G/P)$ (which are well known). The same construction applied to other mirror families (see, e.g., [30]) produces new ℓ-adic sheaves and overconvergent F-isocrystals.

1.9. Frenkel–Gross rigid connection

In [43], Frenkel and Gross study a rigid irregular connection on the trivial $G^\vee$-bundle on $\mathbb{P}^1$ given by the formula

$$\nabla^{G^\vee} := d + y_p \frac{dq}{q} + x_\theta \, dq, \tag{1.9.1}$$

where $y_p \in \mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$ is a principal nilpotent, and $x_\theta \in \mathfrak{g}^\vee_\theta$ lives in the highest root space. For any $G^\vee$-representation V, we have an associated connection $\nabla^{(G^\vee,V)}$.

When V is the minuscule representation of $G^\vee$ corresponding to parabolic $P^\vee$, we have a natural isomorphism $L : H^*(G^\vee/P^\vee) \cong V$.

THEOREM 1.10 (Theorem 4.14)
Under the isomorphism $L : H^*(G^\vee/P^\vee) \cong V$, the quantum connection $\mathcal{Q}^{G^\vee/P^\vee}$ is isomorphic to the connection $\nabla^{(G^\vee,V)}$.

The isomorphism L sends the Schubert basis of $H^*(G^\vee/P^\vee)$ to the canonical basis of V. The proof of Theorem 1.10 is via a direct comparison of the Frenkel–Gross connection in the canonical basis with the quantum Chevalley formula.

1.11. Zhu's theorem

Beilinson and Drinfeld have introduced a class of connections called *opers*, extending earlier work of Drinfeld and Sokolov. They use opers to construct (part of) the Galois-to-automorphic direction of the geometric Langlands correspondence. Frenkel and Gross [43] have observed that (1.9.1) can be put into oper form after a gauge transformation.

Zhu [128] has extended Beilinson and Drinfeld's construction to allow certain nonconstant group schemes, or equivalently to allow specified ramifications. He thereby confirms the conjecture of [71] that the Kloosterman D-module $\mathrm{Kl}_{G^\vee}$ is isomorphic to the Frenkel–Gross connection $\nabla^{G^\vee}$.

Theorem 1.6 is obtained by composing the isomorphisms of Theorems 1.8 and 1.10, and Zhu's theorem.

This concludes our discussion of Figure 1. We continue this introduction by explaining the stronger Theorem 1.15 and how it relates to Rietsch's equivariant mirror conjecture and to the equivariant Peterson isomorphism.

1.12. Equivariant quantum cohomology and weighted geometric crystals
We may replace the quantum cohomology ring $QH^*(G^\vee/P^\vee)$ by the $T^\vee$-equivariant quantum cohomology ring $QH^*_{T^\vee}(G^\vee/P^\vee)$. We briefly discuss the new features.

The corresponding equivariant quantum connection $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ is a connection over $\mathbb{C}_q^\times \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$, where $\mathfrak{t} = \mathrm{Lie}(T)$ and instead of $\sigma *_q$ in (1.2.1), we have the operator $c_1^T(O(1)) *_q^{T^\vee}$ of equivariant quantum multiplication in $QH^*_{T^\vee}(G^\vee/P^\vee)$ (see Section 10). Identifying $S = H^*_{T^\vee}(\mathrm{pt})$ with the coordinate $\mathbb{C}$-algebra $\mathrm{Sym}(\mathfrak{t}) \cong \mathbb{C}[\mathfrak{t}^*]$, we may equivalently consider the family of connections $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee} \otimes_h \mathbb{C}$ indexed by $h \in \mathfrak{t}^*$ viewed as algebra morphism $h : S \to \mathbb{C}$. These specialize again to connections on the trivial $H^*(G^\vee/P^\vee)$-bundle over $\mathbb{C}_q^\times$.

Let us now discuss the *weighted character D-module* of the geometric crystal X, equipped with the weight function $\gamma : X \to T$. The character D-module (1.3.1) can be weighted with a parameter $h \in \mathfrak{t}^*$, to give

$$\mathrm{WCr}_{(G,P)} := R\pi_*(\gamma^* \mathsf{M}_T \otimes f^* \mathsf{E}), \tag{1.12.1}$$

where M_T denotes the $(D \otimes S)$-module on T defined in Section 9.2.

The D-module version of Rietsch's equivariant mirror conjecture in [107] states that there is an isomorphism

$$\mathrm{WCr}_{(G,P)} \simeq \mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}.$$

1.13. A deformation of Zhu's theorem
The *twisted Kloosterman D-module* TKl_n arises in (1.7.2) by replacing the Artin–Schreier sheaf by the tensor product of an Artin–Schreier sheaf and a Kummer sheaf. The automorphic D-module $A_{\mathfrak{g}}$ can similarly be deformed to an automorphic D-module which further depends on the choice of a character of T (see [71]). We thus obtain the twisted Kloosterman D-module, denoted $\mathrm{TKl}_{G^\vee}$.

The corresponding deformation of the Frenkel–Gross connection (1.9.1) has not appeared in the literature as far as we know. We define the *deformed Frenkel–Gross connection* to be

$$\widetilde{\nabla}^{G^\vee} := d + (y_p + h)\frac{dq}{q} + x_\theta \, dq, \tag{1.13.1}$$

for $h \in \mathfrak{t}^\vee$ an element of the Cartan subalgebra.

With these modifications, Figure 1 and Theorems 1.6, 1.8, and 1.10 all hold with their equivariant and deformed counterparts. In particular, we deduce Rietsch's equivariant mirror conjecture for minuscule flag varieties. With some mild variation, we also generalize Zhu's theorem to the deformed setting: the twisted Kloosterman D-module $\mathrm{TKl}_{G^\vee}$ and the deformed Frenkel–Gross connection $\widetilde{\nabla}^{G^\vee}$ are isomorphic.

1.14. $D_\hbar$-module generalization

The passage from the quantum connection $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ to the quantum cohomology ring $QH_{T^\vee}^*(G^\vee/P^\vee)$ itself is obtained by taking a semiclassical limit. A framework to rigorously formalize the semiclassical limit is to extend the mirror theorem to an isomorphism of $D_\hbar$-modules, where $D_\hbar := \mathbb{C}[q, q^{-1}, \hbar]\langle \hbar\partial_q\rangle$ and $\hbar$ is an additional parameter. In (1.12.1), the D-module E is replaced by the $D_\hbar$-module $\mathsf{E}^{1/\hbar}$ defined in Section 11.8, the $(D \otimes S)$-module M_T is replaced by the $(D_\hbar \otimes S)$-module $\mathsf{M}^{1/\hbar}$ defined in Section 11.1, and we obtain the $(D_\hbar \otimes S)$-module

$$\mathrm{WCr}_{(G,P)}^{1/\hbar} := R\pi_*(\gamma^* \mathsf{M}_T^{1/\hbar} \otimes f^* \mathsf{E}^{1/\hbar}). \tag{1.14.1}$$

Similarly, one obtains an $\hbar$-deformation $\mathrm{TKl}_{G^\vee}^{1/\hbar}$ of the twisted Kloosterman D-module. The equivariant quantum connection and deformed Frenkel–Gross connection (1.13.1) also possess a further $\hbar$-deformation

$$\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee} := \hbar d + c_1^T\big(O(1)\big) *_q \frac{dq}{q},$$

$$\widetilde{\nabla}_\hbar^G := \hbar d + (y_p + h)\frac{dq}{q} + x_\theta\, dq.$$

These two formulas define $\hbar$-connections over $\mathbb{C}_q^\times \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$, hence in particular $(D_\hbar \otimes S)$-modules.

We are now able to state the main result of this paper, which includes the equivariant $\hbar$-mirror isomorphism.

THEOREM 1.15 (Theorems 11.6, 11.9, 11.12, 11.14)
Suppose that $P^\vee$ is minuscule. The four $(D_\hbar \otimes S)$-modules $\mathrm{WCr}_{(G,P)}^{1/\hbar}$, $\widetilde{\nabla}_\hbar^{G^\vee}$, $\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee}$, and $\mathrm{TKl}_{G^\vee}^{1/\hbar}$ are isomorphic.

Specializing both $h = 0 \in \mathfrak{t}^*$ and $\hbar = 1$ yields the earlier mirror Theorem 8.3. Specializing $\hbar = 0$ establishes the equivariant Peterson isomorphism, which we explain in the next subsection. Theorem 1.15 is obtained by exploiting the grading of the quantum product on one side, and the homogeneity of the potential f_q on the other side.

1.16. Application: Proof of the Peterson isomorphism

Given a regular function on an algebraic variety, one can consider the sheaf of Jacobian ideals generated by all the first-order derivatives. Its quotient ring defines a subscheme, possibly nonreduced, of critical points of the function. Since $G\overset{\circ}{/}P$ is affine, applying this construction to the weighted potential, we obtain a (relative) Jacobian ring $\mathrm{Jac}(G\overset{\circ}{/}P, f_q, \gamma)$, which has the structure of a $\mathbb{C}[\mathfrak{t}^*, q, q^{-1}]$-algebra.

THEOREM 1.17 (Theorem 12.4)
If $P^\vee$ is minuscule, then we have an isomorphism of $\mathbb{C}[\mathfrak{t}^*, q, q^{-1}]$-algebras $QH^*_{T^\vee}(G^\vee/P^\vee) \cong \mathrm{Jac}(\mathring{G/P}, f_q, \gamma)$.

Specializing to nonequivariant cohomology, we obtain the mirror isomorphism of $\mathbb{C}[q, q^{-1}]$-algebras $QH^*(G^\vee/P^\vee) \cong \mathrm{Jac}(\mathring{G/P}, f_q)$. The same isomorphism is expected to hold for every Fano mirror dual pair. It is established for (possibly orbifold) toric Fano varieties in [3], [31], [45], [60], and [97].

The equivariant Peterson variety $\mathcal{Y}$ is the closed subvariety of $G/B_- \times \mathfrak{t}^*$ defined by

$$\mathcal{Y} := \{(gB_-, h) \in (G/B_-) \times \mathfrak{t}^* \mid g^{-1} \cdot (F - h) \text{ vanishes on } [\mathfrak{u}_-, \mathfrak{u}_-]\},$$

where $F \in \mathfrak{g}^*$ is a principal nilpotent defined as in (1.9.1) (see [107, Section 3.2]), and $\mathfrak{u}_- := \mathrm{Lie}(U_-)$, and $g^{-1} \cdot (-)$ denotes the coadjoint action. It contains an open subscheme

$$\mathcal{Y}^* := \mathcal{Y} \cap B_- w_0 B_-/B_-,$$

obtained by intersecting with the open Schubert cell $B_- w_0 B_-/B_-$, where w_0 denotes the longest element of W. The intersection of $\mathcal{Y}^*$ with the opposite Schubert stratification $\{BwB_-/B_-\}$ gives the $2^{\mathrm{rk}(\mathfrak{g})}$ strata

$$\mathcal{Y}^*_P := \mathcal{Y}^* \cap B w_0^P B_-/B_-, \tag{1.17.1}$$

where w_0^P is the longest element of $W_P \subset W$ and the intersections are to be taken scheme-theoretically. In [100], Peterson announced the isomorphism $\mathcal{Y}^*_P \cong \mathrm{Spec}(QH^*_{T^\vee}(G^\vee/P^\vee))$.

Rietsch [107] has proved that $\mathrm{Jac}(\mathring{G/P}, f_q, \gamma)$ is isomorphic to $\mathbb{C}[\mathcal{Y}^*_P]$ as $\mathbb{C}[\mathfrak{t}^*, q, q^{-1}]$-algebras. We thus obtain the following corollary.

COROLLARY 1.18 (Equivariant Peterson isomorphism; see Corollary 12.5)
If $P^\vee$ is minuscule, then we have an isomorphism of $\mathbb{C}[\mathfrak{t}^*, q, q^{-1}]$-algebras $QH^*_{T^\vee}(G^\vee/P^\vee) \cong \mathbb{C}[\mathcal{Y}^*_P]$.

The Peterson isomorphism has been established directly for Grassmannians by Rietsch [104], for quadrics by Pech, Rietsch, and Williams [99], and for Lagrangian and orthogonal Grassmannians by Cheong [25], all in the nonequivariant case (i.e., specializing $h \in \mathfrak{t}^*$ to zero). In the equivariant case, the results of [104] and [93] can be combined to also obtain Corollary 1.18 for Grassmannians (see [93, Section 5]). For some other works on the spectrum of classical equivariant cohomology rings, which correspond to the specialization $q = 0$, see [61] and [62].

Theorem 1.17 is proved by passing to the semiclassical limit ($\hbar \to 0$) in the equivariant $\hbar$-mirror isomorphism of Theorem 1.15.

1.19. Mirror pairs of Fano type and towards mirror symmetry for Richardson varieties

In our mirror theorem, the A-model $G^\vee/P^\vee$ and the B-model (X_t, f_t) play distinctly different roles. On the other hand, the geometry of G/P features prominently in the construction of X_t. This suggests a more symmetric mirror conjecture should exist.

One such setting could be the mirror pairs of compactified Landau–Ginzburg models studied in [79], and one might speculate on the mirror symmetry of the pair of compactified Landau–Ginzburg models

$$(G/P, g, \omega_{G/P}, f_{G/P}) \qquad \text{and} \qquad (G^\vee/P^\vee, g^\vee, \omega_{G^\vee/P^\vee}, f_{G^\vee/P^\vee}),$$

where g is a Kähler form, $\omega_{G/P}$ denotes the volume form of [81], and $f_{G/P}$ denotes the potential function on $G\overset{\circ}{/}P$ discussed above. If such a mirror theorem holds, we would expect a matching of the cohomologies of the log Calabi–Yau manifolds $G\overset{\circ}{/}P$ and $G^\vee\overset{\circ}{/}P^\vee$. Indeed, the equality $H^*(G\overset{\circ}{/}P) \cong H^*(G^\vee\overset{\circ}{/}P^\vee)$ holds more generally for open Richardson varieties.

Namely, we identify the Weyl group of G and of $G^\vee$, and denote it by W. For $v, w \in W$ with $v \leq w$, the open Richardson variety $\mathcal{R}_v^w \subset G/B$ is the intersection of the Schubert cell $B_-\dot{v}B/B$ with the opposite Schubert cell $B\dot{w}B/B$. We denote by $\check{\mathcal{R}}_v^w \subset G^\vee/B^\vee$ the open Richardson variety attached to $G^\vee$. Then we have the equality $H^*(\mathcal{R}_v^w) \cong H^*(\check{\mathcal{R}}_v^w)$ (see Proposition 14.6). We are thus led to the question: Can our mirror theorems be generalized to Richardson varieties?

Let us also comment that the open Richardson varieties $\mathcal{R}_v^w$ are expected to be cluster varieties (see [89]). We refer to [59], [65], and [66] for recent results on canonical bases on log Calabi–Yau varieties assuming the existence of a cluster structure. For a discussion of the cluster structure of $G\overset{\circ}{/}P$, see [7, Section 2].

1.20. Other related works

Witten [124] has related Langlands reciprocity for connections with possibly irregular singularities and mirror symmetry of Hitchin moduli spaces of Higgs bundles (see also [70] and [35] in the absence of singularities). The present work may perhaps be seen as an instance of this relation in the case of rigid connections, although we are considering rather the moduli spaces of *holomorphic* bundles. Another important difference is that automorphic D-modules appear in [124] as the A-side, as opposed to the B-side in the present work.

Recently, the rigid connections of [43] and [71] have been generalized by Yun and Chen to parahoric structures and Yun considered rigid automorphic forms ramified at three points. See also [13], [17], and [111] for recent advances on wild character varieties.

Quantum multiplication by divisor classes in the equivariant quantum cohomology ring $QH^*_{T^\vee \times \mathbb{C}^\times}(T^*G^\vee/P^\vee)$ of the cotangent bundle has been recently computed for any partial flag variety by Su [119], extending work of Braverman, Maulik, and Okounkov [16] for the cotangent bundle $T^*G^\vee/B^\vee$ of the full flag variety. Specializing the $\mathbb{C}^\times$-equivariant parameter to zero, it recovers the $T^\vee$-equivariant quantum Chevalley formula of Mihalcea [94]. It would be interesting to investigate generalizations of Rietsch's mirror conjecture to this setting. See [120] for work in this direction.

A different approach to mirror phenomena for partial flag varieties is the study of period integrals of hypersurfaces in G/P by Lian, Song, and Yau [90]. Their "tautological system" is further studied in [75], where geometry such as the open projected Richardson $\mathring{G/P}$ also makes an appearance.

Since $QH^*(G^\vee/P^\vee)$ is known (see [23]) to be semisimple for minuscule $P^\vee$, the Dubrovin conjecture concerning full exceptional collections of vector bundles on $G^\vee/P^\vee$ and the Stokes matrix of $\mathcal{Q}^{G^\vee/P^\vee}$ at $q = \infty$ is expected to hold. It has been established for projective spaces by Guzzetti [68], and more generally for Grassmannians by Ueda [121]. Recent works on exceptional collections for projective homogeneous spaces include [85] for $G^\vee$ classical and [39] for $G^\vee$ of type E_6.

Related to the Dubrovin conjecture are the Gamma conjectures in [48]. The relation with mirror symmetry is discussed in [49] and [79]. The conjectures are known for Grassmannians (see [48]), for certain toric varieties (see [49]), and are compatible with taking hyperplane sections (see [49]). Also, [58] establishes the Gamma conjecture I for Fano 3-folds with Picard rank 1, exploiting notably the modularity of the quantum differential equation which holds for 15 of the 17 families from the Iskovskikh classification. See [41] and [74] for further recent progress.

2. Preliminaries

Notation in Sections 2.1–2.4 will be used frequently in the article. The results of Section 2.5 will be used in Sections 3 and 4.

2.1. Root systems and Weyl groups

Let G denote a complex almost simple algebraic group, let $T \subset G$ be a maximal torus, and let B, B_- be opposed Borel subgroups. The Lie algebras are denoted $\mathfrak{g}$, $\mathfrak{t}$, $\mathfrak{b}$, $\mathfrak{b}_-$, respectively. Let R denote the root system of G, and let I denote the vertex set of the Dynkin diagram. The simple roots are denoted $\alpha_i \in \mathfrak{t}^*$, and the simple coroots are denoted $\alpha_i^\vee \in \mathfrak{t}$. The pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ is denoted by $\langle .,. \rangle$. Thus $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ are the entries of the Cartan matrix. Let $R^+, R^- \subset R$ denote the subsets of positive and negative roots. Let $\theta \in R^+$ denote the highest root, and let $\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ be the half-sum of positive roots.

We let W denote the Weyl group, and let s_i, $i \in I$ denote the simple generators. For a root $\alpha \in R$, we let $s_\alpha \in W$ denote the corresponding reflection. The length of w is denoted $\ell(w)$. For $w \in W$, we let $\mathrm{Inv}(w) := \{\alpha \in R^+ \mid w\alpha \in R^-\}$ denote the inversion set of w. Thus $|\mathrm{Inv}(w)| = \ell(w)$. Let "$\leq$" denote the Bruhat order on W, and let "$\lessdot$" denote a cover relation (i.e., $w \lessdot v$ if $w < v$ and $\ell(w) = \ell(v) - 1$).

Let $P \subset G$ denote the standard parabolic subgroup associated to a subset $I_P \subset I$. Let $W_P \subset W$ be the subgroup generated by $s_i, i \in I_P$. Let W^P be the set of minimal length coset representatives for W/W_P. Let $\pi_P : W \to W^P$ denote the composition of the natural map $W \to W/W_P$ with the bijection $W/W_P \cong W^P$. Let $R_P \subset R$ denote the root system of the Levi subgroup of P. Let $\rho_P := \frac{1}{2}\sum_{\alpha \in R_P^+} \alpha$.

For a weight λ of $\mathfrak{g}$, we let V_λ denote the irreducible highest weight representation of $\mathfrak{g}$ with highest weight λ.

2.2. *Root vectors and Weyl group representatives*

We pick a generator x_α for the weight space $\mathfrak{g}_\alpha$ for each root α. Write y_j for $x_{-\alpha_j}$, x_j for x_{α_j}, and y_α for $x_{-\alpha}$. Define

$$\dot{s}_j := \exp(-x_j)\exp(y_j)\exp(-x_j) \in G.$$

Then if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, the group element $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_\ell}$ does not depend on the choice of reduced expression.

We assume that the root vectors x_α have been chosen to satisfy:

(1) $\dot{w} \cdot x_\alpha = \pm x_{w\alpha}$,

(2) $[x_\alpha, y_\alpha] = \alpha^\vee$,

where $\dot{w} \cdot x_\alpha$ denotes the adjoint action (see [117, Chapter 3]). In (3.9.1), we will make a choice of sign for x_θ.

2.3. *Quantum roots*

If G is simply laced, then we consider all roots to be long roots. Otherwise, we have both long and short roots. Let $\tilde{R}^+ \subseteq R^+$ be the subset of positive roots defined by

$$\tilde{R}^+ = \{\beta \in R^+ \mid \ell(s_\beta) = \langle 2\rho, \beta^\vee \rangle - 1\}.$$

A root $\beta \in \tilde{R}^+$ is called a *quantum root* (terminology to be explained in Section 4.2). If G is simply laced, then $\tilde{R}^+ = R^+$. Otherwise, it is a proper subset. A root $\alpha \in R^+$ belongs to $\tilde{R}^+$ if one of the following is satisfied (see [16]):

(1) α is a long root, or

(2) no long simple roots α_i appear in the expansion of α in terms of simple roots.

R	Dynkin Diagram	V_{ϖ_i}	$\dim V$	Flag variety	$\dim G/P$
A_n	$1\ 2\ \cdots\ k\ \cdots\ n$	$\Lambda^k\mathbb{C}^{n+1}$	$\binom{n+1}{k}$	Grassmannian $\mathrm{Gr}_{k,n+1}$	$k(n+1-k)$
$B_n\ (n \geq 2)$	$1\ 2\ \cdots\ \cdots\ n$	spinor	2^n	odd orthogonal Grassmannian	$n(n+1)/2$
$C_n\ (n \geq 2)$	$1\ 2\ \cdots\ \cdots\ n$	$\mathbb{C}^{2n}$	$2n$	projective space	$2n-1$
$D_n\ (n \geq 4)$	$1\ 2\ \cdots\ \cdots\ n{-}1,\ n$	$\mathbb{C}^{2n}$	$2n$	even dimensional quadric	$2n-2$
$D_n\ (n \geq 4)$	$1\ 2\ \cdots\ \cdots\ n{-}1,\ n$ $1\ 2\ \cdots\ \cdots\ n{-}1,\ n$	spinor	2^{n-1}	even orthogonal Grassmannian	$n(n-1)/2$
E_6	$2;\ 1\ 3\ 4\ 5\ 6$ $2;\ 1\ 3\ 4\ 5\ 6$		27	Cayley plane	16
E_7	$2;\ 1\ 3\ 4\ 5\ 6\ 7$		56	Freudenthal variety	27

Figure 2. The minuscule parabolic quotients. In the second column, the possible minuscule nodes are indicated with a black vertex.

If G is of type B_n, then $\tilde{R}^+$ is the union of the long positive roots with the short simple root α_n. If G is of type C_n, then $\tilde{R}^+$ is the union of the long positive roots with the short roots of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, where $1 \leq i \leq j \leq n-1$.

2.4. *Minuscule weights*

Let $\mathrm{i} \in I$, and let ϖ_i denote the corresponding fundamental weight. We call i, or ϖ_i, *minuscule*, if the weights of $V = V_{\varpi_\mathrm{i}}$ are exactly the set $W \cdot \lambda$. Equivalently, $\mathrm{i} \in I$ is minuscule if the coefficient of $\alpha_\mathrm{i}^\vee$ in every coroot $\alpha^\vee$ is at most 1.

Let $P = P_\mathrm{i} \subset G$ be the parabolic subgroup associated to $I_P = I \setminus \{\mathrm{i}\}$. Then W_P is the stabilizer of ϖ_i. We have natural bijections between W^P, W/W_P, and $W \cdot \varpi_\mathrm{i}$. We have that $\alpha \in R_P$ if the simple root α_i does not occur in α. We say that $P = P_\mathrm{i}$ is a *minuscule parabolic* if i is a minuscule weight.

The minuscule nodes for each irreducible root system are listed in Figure 2. Our conventions follow the Bourbaki numbering (see [15, Chapter VIII, Section 7.4, Proposition 8]).

If G is simply laced, then a minuscule node is also cominuscule. Thus the coefficient of α_i in every root $\alpha \in R^+$ is at most 1. This means that the nilradical of P is abelian; hence by Borel–de Siebenthal theory, G/P is a compact Hermitian symmetric space (see, e.g., [64]).

2.5. A remarkable quantum root

Fix a minuscule node $\mathfrak{i}$ and corresponding parabolic $P = P_{\mathfrak{i}}$. Define the long root $\kappa = \kappa(\mathfrak{i}) \in R$ by:

$$\kappa := \begin{cases} \alpha_{\mathfrak{i}} & \text{if } G \text{ is simply laced,} \\ \alpha_{n-1} + 2\alpha_n & \text{if } G \text{ is of type } B_n \text{ (and thus } \mathfrak{i} = n), \\ 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n = \theta & \text{if } G \text{ is of type } C_n \text{ (and thus } \mathfrak{i} = 1). \end{cases}$$

Since κ is a long root, it is also a quantum root. The coroot $\kappa^\vee$ is the (unique) "Peterson–Woodward lift" of $\alpha_{\mathfrak{i}}^\vee + Q_P^\vee \in Q^\vee / Q_P^\vee$, where $Q^\vee$, $Q_P^\vee$ denote coroot lattices (see Section 15).

Let $I_Q = \{j \in I_P \mid \langle \alpha_j, \kappa^\vee \rangle = 0\} = \{j \in I_P \mid \langle \kappa, \alpha_j^\vee \rangle = 0\}$. Then $\alpha \in R_Q$ if no simple root α_j with $j \notin I_Q$ occurs in α. If G is simply laced, then I_Q is the set of nodes in I not adjacent to $\mathfrak{i}$. If G is of type B_n, then $I_Q = \{1, 2, \ldots, n-3, n-1\}$. If G is of type C_n, then $I_Q = \{2, 3, \ldots, n\} = I_P$.

LEMMA 2.6

The root κ has the following properties:

(1) $\langle \varpi_{\mathfrak{i}}, \kappa^\vee \rangle = 1$, *and*

(2) $\langle \alpha, \kappa^\vee \rangle = -1$ *for* $\alpha \in R_P^+ \setminus R_Q^+$.

Proof

This is by direct check. □

It turns out that the root κ can be characterized in a number of ways.

PROPOSITION 2.7

Let $\beta \in R^+ \setminus R_P^+$. Then the following are equivalent:

(1) $\beta = \kappa$;

(2) *we have* $\langle \alpha, \beta^\vee \rangle \in \{-1, 0\}$ *for all* $\alpha \in R_P^+$;

(3) *there exists* $w \in W^P$ *such that* $\beta = -w^{-1}(\theta)$.

Define $W(\kappa) := \{w \in W^P \mid w\kappa = -\theta\}$. Let $w_{P/Q} \in W_P$ be the longest element that is a minimal length coset representative in $w_{P/Q} W_Q$. Note that $\mathrm{Inv}(w_{P/Q}) = R_P^+ \setminus R_Q^+$ (see Lemma 15.3). Denote $s'_\kappa := s_\kappa w_{P/Q}^{-1}$.

PROPOSITION 2.8

Suppose that $w \in W(\kappa)$. Then:

(1) $\ell(ws_\kappa) = \ell(w) - \ell(s_\kappa)$,

(2) $\ell(ws'_\kappa) = \ell(w) - \ell(s_\kappa) - \ell(w_{P/Q}^{-1}) = \ell(w) - \ell(s'_\kappa)$,

(3) $ws'_\kappa = \pi_P(ws_\kappa)$,

(4) *there is a unique length-additive factorization* $w = uw'$, *where* $u \in W_J$ *and* $w' \in W(\kappa)$ *is the minimal length element in the double coset* $W_J w W_P$*; here,* W_J *is a standard parabolic subgroup all of whose generators stabilize* θ.

Conversely, suppose that $w \in W^P$ *satisfies (1) and (2). Then* $w \in W(\kappa)$.

The proofs of Propositions 2.7 and 2.8 are given in Section 15.

3. Frenkel–Gross connection

We caution the reader that the roles of G and $G^\vee$ are reversed in Sections 3–5 compared to the rest of the paper.

3.1. *Principal* $\mathfrak{sl}_2$

Let $y_p := \sum_{i\in I} y_i$ which is a principal nilpotent in $\mathfrak{b}_-$. Let $2\rho^\vee = \sum_{\alpha\in R^+}\alpha^\vee$, viewed as an element of $\mathfrak{t}$. We have

$$2\rho^\vee = 2\sum_{i\in I}\varpi_i^\vee = \sum_{i\in I} c_i\alpha_i^\vee,$$

where the c_i are positive integers. Let $x_p := \sum_{i\in I} c_i x_i \in \mathfrak{b}$. Then $(x_p, 2\rho^\vee, y_p)$ is a principal $\mathfrak{sl}_2$-triple (see [63] and [15, Chapter VIII, Section 11, no. 4]).

Let $\mathfrak{z}(y_p)$ be the centralizer of y_p which is an abelian subalgebra of dimension equal to the rank of $\mathfrak{g}$. The adjoint action of $2\rho^\vee$ preserves $\mathfrak{z}(y_p)$ and the eigenvalues are nonnegative even integers. We denote the eigenspaces by $\mathfrak{z}(y_p)_{2m}$ with $m \geq 0$. Thus $\mathfrak{z}(y_p)_0 = \mathfrak{z}(\mathfrak{g})$. The integers $m \geq 1$ counted with multiplicity $\dim \mathfrak{z}(y_p)_{2m}$ coincide with the exponents $m_1 \leq \cdots \leq m_r$ of the root system R. Kostant has shown that $\mathfrak{g} = \oplus_{m\geq 0}\mathrm{Sym}^{2m}(\mathbb{C}^2)\otimes \mathfrak{z}(y_p)_{2m}$ as a representation of the principal $\mathfrak{sl}_2$. It implies that twice the sum of exponents is equal to the number of roots $|R|$.

The first exponent is $m_1 = 1$ since $\mathfrak{z}(y_p)_2$ contains y_p. The last exponent is $m_r = c - 1$ which is the height of the highest root θ because $x_{-\theta} \in \mathfrak{z}(y_p)$. In fact, $m_i + m_{r+1-i} = c$ for any i.

3.2. *Rigid irregular connection*

Frenkel and Gross [43] construct a meromorphic connection ∇^G on the trivial G-bundle on $\mathbb{P}^1\backslash\{0,\infty\}$ by the formula

$$\nabla^G := d + y_p\frac{dq}{q} + x_\theta\, dq. \tag{3.2.1}$$

Here d is the trivial connection and $y_p\frac{dq}{q} + x_\theta\, dq$ is the $\mathfrak{g}$-valued connection 1-form attached to the trivialization $G\times\mathbb{P}^1\backslash\{0,\infty\}$. For any finite-dimensional G-module

V, it induces a meromorphic flat connection $\nabla^{(G,V)}$ on the trivial vector bundle $V \times \mathbb{P}^1\backslash\{0,\infty\}$. If V_λ is an irreducible highest module, then we also write $\nabla^{(G,\lambda)}$ for $\nabla^{(G,V_\lambda)}$.

The formula (3.2.1) is in oper form (see [43]) because ∇^G is everywhere transversal to the trivial B-bundle $B \times \mathbb{P}^1\backslash\{0,\infty\}$ inside $G \times \mathbb{P}^1\backslash\{0,\infty\}$. The connection ∇^G has a regular singularity at the point 0 with monodromy generated by the principal unipotent $\exp(2i\pi y_p)$. It has an irregular singularity at the point ∞, and it is shown in [43] that the slope is $1/c$, where c is the Coxeter number of G. One of the main results of [43] is that the connection is rigid in the sense of the vanishing of the cohomology of the intermediate extension to $\mathbb{P}^1$ of $\nabla^{(G,\mathrm{Ad})}$, viewed as a holonomic D-module on $\operatorname{Spec}\mathbb{C}[q,q^{-1}] = \mathbb{P}^1\backslash\{0,\infty\}$. Here, Ad is the adjoint representation of G on $\mathfrak{g}$.

3.3. Outer automorphisms

In certain cases, the connection ∇^G admits a reduction of the structure group. This is related to outer automorphisms of G, and thus to automorphisms of the Dynkin diagram. If G is of type A_{2n-1}, then ∇^G can be reduced to type C_n. If G is of type E_6, then ∇^G can be reduced to type F_4. If G is of type D_{n+1} with $n \geq 4$, then ∇^G can be reduced to type B_n. In particular, there is a reduction from type D_4 to type B_3. In fact, by using the full group S_3 of automorphisms of the Dynkin diagram, if G is of type D_4, then ∇^G can be reduced to type G_2. As a consequence, there is also a reduction of ∇^G from type B_3 to type G_2 even though B_3 has no outer automorphism. It follows from Frenkel and Gross [43, Sections 6 and 13], who determine the differential Galois group of ∇^G for every G, that the above is a complete list of possible reductions.

3.4. Homogeneity

We make the observation that the connection ∇^G is compatible with the natural grading on $\mathfrak{g}$ induced by the adjoint action of the cocharacter subgroup $\rho^\vee : \mathbb{G}_m \to G$. Precisely, we have a $\mathbb{G}_m$-action on $\mathfrak{g}$ induced by $\zeta \mapsto \mathrm{Ad}(\rho^\vee(\zeta))$, where $\zeta \in \mathbb{G}_m$. Consider also the $\mathbb{G}_m$-action on $\mathbb{P}^1\backslash\{0,\infty\}$ given by $\zeta \cdot q = \zeta^c q$, where we recall that c is the Coxeter number of $\mathfrak{g}$. It induces a natural $\mathbb{G}_m$-equivariant linear action on $T^*\mathbb{P}^1\backslash\{0,\infty\}$ and also on the bundle $\mathfrak{g} \otimes T^*\mathbb{P}^1\backslash\{0,\infty\}$.

LEMMA 3.5

The connection 1*-form* $y_p \frac{dq}{q} + x_\theta\, dq$ *in* $\Omega^1(\mathbb{P}^1\backslash\{0,\infty\},\mathfrak{g})$ *is homogeneous of degree* 1 *under the above* $\mathbb{G}_m$*-equivariant action.*

Proof
We have seen in Section 3.1 that y_p has degree $m_1 = 1$ and x_θ has degree $-m_r = 1 - c$, which implies the assertion. □

3.6. Frenkel–Gross operator acting on the minuscule representation
Let $\mathfrak{i}$ be a minuscule node, and let $V_{\varpi_\mathfrak{i}}$ denote the minuscule representation. In this section, we explicitly compute $\nabla^{(G,\varpi_\mathfrak{i})}$. We shall use the canonical basis of $V_{\varpi_\mathfrak{i}}$, constructed in [50].

There is a basis $\{v_w \mid w \in W^P\}$ of $V_{\varpi_\mathfrak{i}}$ characterized by the properties

$$x_j(v_w) = \begin{cases} v_{s_j w} & \text{if } \langle w\varpi_\mathfrak{i}, \alpha_j^\vee\rangle = -1, \\ 0 & \text{otherwise,} \end{cases} \qquad y_j(v_w) = \begin{cases} v_{s_j w} & \text{if } \langle w\varpi_\mathfrak{i}, \alpha_j^\vee\rangle = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the condition that v_w has weight $w\varpi_\mathfrak{i}$. Note that in the formulas above, $s_j w$ always lies in W^P. (For example, $\langle w\varpi_\mathfrak{i}, \alpha_j^\vee\rangle = 1$ implies that $s_j w > w$ and $s_j w W_P \neq w W_P$. Together with $w \in W^P$, we have that $s_j w \in W^P$.)

The following result follows from [50, Lemma 3.1] and the discussion after [50, Lemma 3.3]. We caution that our $\dot{s}_j$ is equal to Geck's $n_j(-1)$.

LEMMA 3.7
(1) *For $w \in W^P$, we have $\dot{w} v_e = v_w$. For $u \in W$ and $w \in W^P$, we have $\dot{u} v_w = \pm v_{\pi_P(uw)}$.*
(2) *For $\alpha \in R^+$ and $w \in W^P$, we have*

$$x_\alpha(v_w) = \begin{cases} \pm v_{s_\alpha w} & \textit{if } \langle w\varpi_\mathfrak{i}, \alpha^\vee\rangle = -1, \\ 0 & \textit{otherwise,} \end{cases}$$

$$x_{-\alpha}(v_w) = \begin{cases} \pm v_{s_\alpha w} & \textit{if } \langle w\varpi_\mathfrak{i}, \alpha^\vee\rangle = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

LEMMA 3.8
Let $j \in I$ and $w \in W^P$. Then

$$y_j v_w = \begin{cases} v_{w s_\beta} & \textit{if } \beta = w^{-1}(\alpha_j) \in R^+ \setminus R_P^+, \\ 0 & \textit{otherwise.} \end{cases}$$

In the first case, we automatically have $w s_\beta \gtrdot w$ and $w s_\beta \in W^P$.

Proof
Let $\beta = w^{-1}(\alpha_j)$. The condition that $\beta > 0$ is equivalent to $s_j w > w$. In this case,

$\mathrm{Inv}(s_j w) = \mathrm{Inv}(w) \cup \{\beta\}$, so the condition that $s_j w \in W^P$ is equivalent to $\beta \notin R_P$. The condition $\langle w\varpi_{\mathrm{i}}, \alpha_j^\vee \rangle = 1$ is thus equivalent to $\beta \in R^+ \setminus R_P^+$. □

Recall that we have defined a distinguished root $\kappa = \kappa(\mathrm{i}) \in R^+$ and a subset $W(\kappa) \subset W$ in Section 2.5.

LEMMA 3.9
There is a sign $\varepsilon \in \{+1, -1\}$, not depending on $w \in W^P$, such that

$$\varepsilon x_\theta v_w = \begin{cases} v_{\pi_P(w s_\kappa)} & \text{if } w \in W(\kappa), \\ 0 & \text{otherwise.} \end{cases}$$

Proof
Let $\beta = -w^{-1}(\theta)$. By Lemma 3.7(2), $x_\theta v_w \neq 0$ if and only if $\langle w\varpi_{\mathrm{i}}, \theta^\vee \rangle = -1$. By a similar argument to the proof of Lemma 3.8, this holds if and only if $\beta \in R^+ \setminus R_P^+$. By Proposition 2.7, we have $x_\theta v_w \neq 0$ if and only if $\beta = \kappa$, that is, $w \in W(\kappa)$.

Suppose that $w \in W(\kappa)$. By Proposition 2.8(4), we have $w = uw'$, where $u \in W_J$ is an element of a standard parabolic subgroup stabilizing θ, and the product uw' is length-additive. Then we have

$$x_\theta v_w = x_\theta \dot{u} v_{w'} = \varepsilon' \dot{u} \dot{w}' x_{-\kappa} (\dot{w}')^{-1} v_{w'} = \varepsilon \dot{u} v_{w' s_\kappa'} = \varepsilon v_{w s_\kappa'},$$

where ε', ε are signs not depending on w. For the first equality, we have used Lemma 3.7(1). For the second equality, we used that $\dot{u}$ is a product of elements $\dot{s}_j$, where $s_j\theta = \theta$ and thus $\dot{s}_j$ commutes with x_θ. For the third equality, we used $x_{-\kappa} v_e = v_{s_\kappa}$ which follows from Lemma 3.7(2) and Lemma 2.6(1). In the last two equalities, we used Proposition 2.8(2) applied to $w, w' \in W(\kappa)$. In the last equality, we also used that $\ell(w s_\kappa') = \ell(u) + \ell(w' s_\kappa')$. □

From now on, we make the assumption that

$$x_\theta \in \mathfrak{g}_\theta \text{ is chosen so that } \varepsilon = 1 \text{ in Lemma 3.9.} \tag{3.9.1}$$

4. Quantum cohomology connection

4.1. Quantum cohomology of partial flag varieties

Let $P \subset G$ be an arbitrary standard parabolic subgroup. Let $QH^*(G/P)$ denote the small quantum cohomology ring of G/P. It is an algebra over $\mathbb{C}[q_i \mid i \notin I_P]$, where we write q_i for $q_{\alpha_i^\vee}$. For $w \in W^P$, let $\sigma_w \in QH^*(G/P)$ denote the quantum Schubert class. For each $i \in I$, let $\sigma_i := \sigma_{s_{\alpha_i}}$. Then we have

$$QH^*(G/P) \cong \bigoplus_{w \in W^P} \mathbb{C}[q_i \mid i \notin I_P] \cdot \sigma_w.$$

4.2. Quantum Chevalley formula

The quantum Chevalley formula for a general G/P is due to Fulton and Woodward [46] and Peterson [100]. Let $\eta_P : Q^\vee \to Q^\vee/Q_P^\vee$ be the quotient map, where $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ (resp., $Q_P^\vee = \bigoplus_{i \in I_P} \mathbb{Z}\alpha_i^\vee$) is the coroot lattice. Recall that $\rho_P = \frac{1}{2}\sum_{\alpha \in R_P^+} \alpha$. The following version of the quantum Chevalley rule is from [87, Theorem 10.14, Lemma 10.18].

THEOREM 4.3

For $w \in W^P$, we have

$$\sigma_i *_q \sigma_w = \sum_\beta \langle \varpi_i, \beta^\vee \rangle \sigma_{ws_\beta} + \sum_\nu \langle \varpi_i, \nu^\vee \rangle q_{\eta_P(\nu^\vee)} \sigma_{\pi_P(ws_\nu)},$$

where the first summation is over $\beta \in R^+ \setminus R_P^+$ such that $ws_\beta > w$ and $ws_\beta \in W^P$, and the second summation is over $\nu \in R^+ \setminus R_P^+$ such that

$$\ell(ws_\nu) = \ell(w) - \ell(s_\nu) \qquad \textit{and} \tag{4.3.1}$$

$$\ell\big(\pi_P(ws_\nu)\big) = \ell(w) + 1 - \big\langle 2(\rho - \rho_P), \nu^\vee \big\rangle. \tag{4.3.2}$$

4.4. Degrees

The quantum cohomology ring $QH^*(G/P)$ is a graded ring. The degree of σ_w is equal to $2\ell(w)$. The degrees of the quantum parameters $q_i = q_{\alpha_i^\vee}$ for $i \in I \setminus I_P$ are given by

$$\deg(q_i) = 2\int_{G/P} c_1(T_{G/P}) \cdot \sigma_{\pi_P(w_0 s_i)} = \big\langle 4(\rho - \rho_P), \alpha_i^\vee \big\rangle.$$

The second equality is [46, Lemma 3.5]. Indeed, the first Chern class of G/P satisfies

$$c_1(T_{G/P}) = \sum_{i \in I \setminus I_P} \big\langle 2(\rho - \rho_P), \alpha_i^\vee \big\rangle \sigma_i. \tag{4.4.1}$$

We verify that the quantum multiplication $\sigma_i *_q$ is homogeneous of degree 2 directly from Theorem 4.3. Indeed, σ_{ws_β} has degree $2\ell(w) + 2$, and

$$\begin{aligned}&\deg q_{\eta_P(\nu^\vee)} + 2\ell\big(\pi_P(ws_\nu)\big)\\ &\quad = \big\langle 4(\rho - \rho_P), \eta_P(\nu^\vee)\big\rangle + 2\ell(w) + 2 - 2\big\langle 2(\rho - \rho_P), \nu^\vee \big\rangle = 2\ell(w) + 2,\end{aligned}$$

where the second equality follows because $\rho - \rho_P$ is orthogonal to $Q_P^\vee$.

4.5. Quantum connection and quantum D-module

We let $\mathbb{C}_q$ be the complex points of $\operatorname{Spec}\mathbb{C}[q_i \mid i \notin I_P]$ and let $\mathbb{C}_q^\times$ be the complex points of $\operatorname{Spec}\mathbb{C}[q_i^{\pm 1} \mid i \notin I_P]$. We can attach a quantum connection $\mathcal{Q}^{G/P}$ on the trivial bundle $\mathbb{C}_q^\times \times H^*(G/P)$ over $\mathbb{C}_q^\times$ as follows. For each $i \in I \backslash I_P$, the connection $\mathcal{Q}^{G/P}$ in the direction of q_i is given by

$$q_i \frac{\partial}{\partial q_i} + \sigma_i *_q,$$

where $*_q$ is quantum multiplication with quantum parameter q. The connection is integrable, which is equivalent to the associativity of the quantum product. The associated connection 1-form is

$$\sum_{i \in I \backslash I_P} (\sigma_i *_q) \frac{dq_i}{q_i} \in \Omega^1\big(\mathbb{C}_q^\times, \operatorname{End}\big(H^*(G/P)\big)\big). \tag{4.5.1}$$

Define a $\mathbb{C}^\times$-action on $H^*(G^\vee/P^\vee)$ by $\zeta \cdot \sigma = \zeta^i \sigma$ for $\zeta \in \mathbb{C}^\times$ and $\sigma \in H^{2i}(G^\vee/P^\vee)$. Also define a $\mathbb{C}^\times$-action on $\mathbb{C}_q^\times$ by $\zeta \cdot q_i = \zeta^{\deg(q_i)/2} q_i$ for $i \notin I_P$. Then it is clear from the previous Section 4.4 that the connection 1-form (4.5.1) is homogeneous of degree 1 for the action of $\mathbb{C}^\times$.

It follows from Theorem 4.3 that quantum multiplication is a Laurent polynomial; hence $\mathcal{Q}^{G/P}$ is an *algebraic* connection.

Remark 4.6

We may identify the universal cover of $\mathbb{C}_q^\times$ with $H^2(G/P)$ and define a flat connection on $H^2(G/P)$ instead, which would correspond to the general framework of Frobenius manifolds (see [36], [72], [92]). Viewing $\{\sigma_i \mid i \notin I_P\}$ as a basis of $H^2(G/P)$, the link is the change of parameters given by $(q_i \mid i \notin I_P) \mapsto \sum_{i \in I \backslash I_P} \log(q_i)\sigma_i \in H^2(G/P)$ (see, e.g., [76, Section 2.2]). Intrinsically, $\mathbb{C}_q^\times$ is identified with the quotient $H^2(G/P)/2i\pi H^2(G/P,\mathbb{Z})$ (see also Lemma 8.2 below).

4.7. Minuscule case

For minuscule G/P, with $I_P = I \backslash \{\mathrm{i}\}$, where i is a minuscule node, we shall simplify Theorem 4.3. The Schubert divisor class $\sigma_{\mathrm{i}} \in H^2(G/P)$ is a generator of $\operatorname{Pic}(G/P)$. It defines a minimal homogeneous embedding $G/P \subset \mathbb{P}(V)$, and G/P is realized as the closed orbit of the highest weight vector $v_e \in V = V_{\varpi_{\mathrm{i}}}$. The hyperplane class of $\mathbb{P}(V)$ restricts to σ. The following is established in [21] and [114].

LEMMA 4.8

If $P = P_{\mathrm{i}}$ is a minuscule parabolic, then $\langle 2(\rho - \rho_P), \alpha_{\mathrm{i}}^\vee \rangle = c$, the Coxeter number of G.

It then follows from (4.4.1) that the first Chern class $c_1(T_{G/P})$ is equal to $c\sigma$. There is only one quantum parameter $q = q_{\mathrm{i}} = q_{\alpha_{\mathrm{i}}^\vee}$ which has degree $2c$.

PROPOSITION 4.9
Let $\kappa = \kappa(\mathrm{i})$ be the long root of Section 2.5. Then for $w \in W^P$, we have

$$\sigma_{\mathrm{i}} *_q \sigma_w = \sum_\beta \sigma_{ws_\beta} + \boldsymbol{\chi}(w) q \sigma_{\pi_P(ws_\kappa)},$$

where the first summation is over $\beta \in R^+ \setminus R_P^+$ such that $ws_\beta \gtrdot w$ and $ws_\beta \in W^P$, and $\boldsymbol{\chi}(w)$ equals 1 or 0 depending on whether $w \in W(\kappa)$ or not.

Proof
For $\beta^\vee \in R^+$, the coefficient $\langle \varpi_{\mathrm{i}}, \beta^\vee \rangle$ is either 0 or 1, and it is equal to 1 if $\beta^\vee \in R^+ \setminus R_P^+$. This explains the first summation.

Suppose that $\nu \in R^+ \setminus R_P^+$ and we have (4.3.1) and (4.3.2). Define $I_{Q'} := \{j \in I_P \mid \langle \alpha_j, \nu^\vee \rangle = 0\} = \{j \in I_P \mid \langle \nu, \alpha_j^\vee \rangle = 0\}$. We have $\mathrm{Inv}(ws_\nu) \cap R_{Q'}^+ = \emptyset$ and thus $\mathrm{Inv}(ws_\nu) \cap R_P^+ \subseteq (R_P^+ \setminus R_{Q'}^+)$. Now, (4.3.1) implies that $s_\nu \in W^P$ and thus $\langle \alpha_i, \nu^\vee \rangle \leq 0$ for $i \in I_P$. It follows from our definition of $I_{Q'}$ that $\langle \alpha_i, \nu^\vee \rangle < 0$ for $i \in I_P$ and $\langle \alpha_i, \nu^\vee \rangle = 0$ for $i \in I_{Q'}$. Thus $\langle \alpha, \nu^\vee \rangle < 0$ for $\alpha \in R_P^+ \setminus R_{Q'}^+$, so

$$|R_P^+ \setminus R_{Q'}^+| \leq -\sum_{\alpha \in R_P^+ \setminus R_{Q'}^+} \langle \alpha, \nu^\vee \rangle = -\sum_{\alpha \in R_P^+} \langle \alpha, \nu^\vee \rangle = -\langle 2\rho_P, \nu^\vee \rangle.$$

Condition (4.3.2) guarantees that we have equality and hence that $\langle \alpha, \nu^\vee \rangle = -1$ for $\alpha \in R_P^+ \setminus R_{Q'}^+$. By Proposition 2.7, we conclude that $\nu = \kappa$. It follows from the last sentence of Proposition 2.8 that $w \in W(\kappa)$. □

Example 4.10
Suppose that $G/P = \mathrm{Gr}(n-1, n) = \mathbb{CP}^{n-1}$. The minimal representative permutations $w \in W^P$ are determined by the value $w(n) \in [1, n]$, or equivalently by a Young diagram which is a single column of length $w(n) - 1$. Denote the Schubert classes by $\sigma_\emptyset = 1$, $\sigma_1 = \sigma_{s_{\alpha_1}}$, $\sigma_2, \ldots, \sigma_{n-1}$. Then $\sigma_1^{*j} = \sigma_j$ for $1 \leq j \leq n-1$ and $\sigma_1 *_q \sigma_{n-1} = q$. The quantum cohomology ring has presentation $\mathbb{C}[\sigma_1, q]/(\sigma_1^n - q)$.

Chaput, Manivel, and Perrin [21]–[23] study the quantum cohomology of minuscule and cominuscule flag varieties. In particular, they obtain a combinatorial description in terms of certain quivers (see [21, Proposition 24]), which may be compared with Proposition 4.9 above.

4.11. Minuscule representation
Define the linear isomorphism

$$L : H^*(G/P) \to V, \qquad \sigma_w \mapsto v_w \quad \text{for } w \in W^P. \tag{4.11.1}$$

Recall the principal $\mathfrak{sl}_2$-triple $(x_p, 2\rho^\vee, y_p)$.

PROPOSITION 4.12 (Gross [64])
The isomorphism L intertwines the action of the Lefschetz $\mathfrak{sl}_2$ on $H^*(G/P)$ and the action of the principal $\mathfrak{sl}_2$ on V.

Proof
If the term σ_{ws_β} occurs in $\sigma_{\mathrm{i}} *_0 \sigma_w$, then $w\beta = \alpha_j$ for some j (see [118]). It then follows from Lemma 3.8 that $L(\sigma_{\mathrm{i}} *_0 \sigma_w) = y_p v_w = y_p \circ L(\sigma_w)$. On the other hand, we have $\dim(G/P) = \langle \varpi_{\mathrm{i}}, 2\rho^\vee \rangle$, and $\ell(w) = \langle \varpi_{\mathrm{i}}, \rho^\vee \rangle - \langle w\varpi_{\mathrm{i}}, \rho^\vee \rangle$ (see [64, Section 6]). Since $L(\sigma_w) = v_w$ has weight $w\varpi_{\mathrm{i}}$ (see Section 3.6), for every $d \in [0, 2\dim(G/P)]$, the image $L(H^d(G/P))$ is equal to the $2\rho^\vee$-eigenspace of V of eigenvalue $\dim(G/P) - d$. □

Consider quantum multiplication $\sigma_{\mathrm{i}} *_q$ as an operator on $H^*(G/P)$ with coefficients in $\mathbb{C}[q]$.

PROPOSITION 4.13
We have $L \circ \sigma_{\mathrm{i}} *_q = (y_p + q x_\theta) \circ L$.

Proof
Let $\sigma_{\mathrm{i}} *_q = D_1 + D_2$, where D_1 and D_2 correspond to the two terms of Proposition 4.9. We have seen in the proof of Proposition 4.12 that $L \circ D_1 = y_p \circ L$. Assumption (3.9.1) and Proposition 4.9 show that $L \circ D_2 = q x_\theta \circ L$. □

Golyshev and Manivel [57] study "quantum corrections" to the geometric Satake correspondence. Their main result is closely related to our Proposition 4.13 for the simply laced cases.

Recall from Section 4.5 that the quantum connection on $\mathbb{C}_q^\times$ is given by

$$\mathcal{Q}^{G/P} = d + \sigma_{\mathrm{i}} *_q \frac{dq}{q}. \tag{4.13.1}$$

THEOREM 4.14
If $P \subset G$ is minuscule with minuscule representation V, then under the isomorphism $L : H^*(G/P) \to V$, the quantum connection $\mathcal{Q}^{G/P}$ is isomorphic to the rigid connection $\nabla^{(G,V)}$. Moreover, the isomorphism is graded with respect to the gradings in Sections 3.4 and 4.5.

4.15. Automorphism groups

The connected automorphism group of a projective homogeneous space H/P is of the same Dynkin type as H except in the following three exceptional cases (see [2, Section 3.3]):

- If $H = \mathrm{Sp}(2n)$ is of type C_n, $n \geq 2$, and $\mathfrak{i} = 1$ is the unique minuscule node, then H/P_1 is isomorphic to projective space $\mathbb{P}^{2n-1}$. Thus it is homogeneous under the bigger automorphism group $G = \mathrm{PGL}(2n)$.
- If $H = \mathrm{SO}(2n+1)$ is of type B_n, $n \geq 2$, and $\mathfrak{i} = n$ is the unique minuscule node, then the odd orthogonal Grassmannian $\mathrm{SO}(2n+1)/P_n$ is isomorphic to the even orthogonal Grassmannian $\mathrm{SO}(2n+2)/P_{n+1}$.
- If H is of type G_2 and $\mathfrak{i} = 1$, then H/P_1 is a 5-dimensional quadric which is also isomorphic to $\mathrm{SO}(7)/P_1$. In this case, $\mathfrak{i}$ corresponds to the unique short root, which is therefore also the shortest highest root, and thus H/P_1 is quasiminuscule and coadjoint, but it is neither minuscule nor cominuscule. On the other hand, the 5-dimensional quadric is cominuscule as a homogeneous space under $G = \mathrm{SO}(7)$.

In each of the above cases the quantum cohomology rings coincide, and hence the quantum connections also coincide. In the first two cases we can apply Theorem 4.14 to deduce that the corresponding rigid connections associated to a minuscule representation V coincide. In view of Section 3.3, we conclude that if there is a minuscule Grassmannian H/P whose connected automorphic group is G, then ∇^G can be reduced to ∇^H.

4.16. Quantum period solution

The connection $\mathcal{Q}^{G/P}$ has regular singularities at $q = 0$. Let $S(q)$ be the horizontal section of the dual connection that is asymptotic to $\sigma_{w_0 w_0^P}$ as $q \to 0$. Here, $w_0 w_0^P$ (resp., w_0 and w_0^P) is the longest element of W^P (resp., W and W_P). The quantum period of G/P is $\langle S(q), 1\rangle$. Here, $\langle \cdot, \cdot \rangle$ denotes the intersection pairing on $H^*(G/P)$, so $\langle S(q), 1\rangle$ is equal to the coefficient of $\sigma_{w_0 w_0^P}$ in the Schubert expansion of $S(q)$. The quantum period $\langle S(q), 1\rangle$ has a power series expansion in q with nonnegative coefficients, which one can determine using the Frobenius method. We determine the first term in the q-expansion in the following.

LEMMA 4.17

As $q \to 0$,

$$\langle S(q), 1\rangle = 1 + q \int_{G/P} \sigma_{\mathfrak{i}}^{c-1} \sigma_{\pi_P(w_0 w_0^P s_\kappa)} + O(q^2).$$

The integral above is the number of paths in Bruhat order inside W^P from $\pi_P(w_0 w_0^P s_\kappa)$ to $w_0 w_0^P$. It is a positive integer.

Proof

We write $S(q) = \sigma_{w_0 w_0^P} + qv + O(q^2)$, where $v \in H^*(G/P)$. Since S is a horizontal section of the connection dual to $\mathcal{Q}^{G/P}$, we have $\frac{dS}{dq} = \sigma_{\mathrm{i}} *_q S(q)$. Using the quantum Chevalley formula in Proposition 4.9 and $\frac{dS}{dq} = v + O(q)$, this implies that

$$v = \sigma_{\mathrm{i}} *_0 v + \sigma_{\pi_P(w_0 w_0^P s_\kappa)}.$$

Since $\sigma_{\mathrm{i}} *_0$ is nilpotent, this equation uniquely determines v.

We have $\ell(w_0 w_0^P) = \dim(G/P)$, and

$$\ell\big(\pi_P(w_0 w_0^P s_\kappa)\big) = \dim(G/P) + 1 - c,$$

where $c = \langle 2(\rho - \rho_P), \kappa^\vee \rangle$ is the Coxeter number of G by Lemma 4.8. Hence we find that $\langle v, 1 \rangle$ is as stated in the lemma.

The interpretation as counting paths in Bruhat order follows from the classical Chevalley formula for the cup product with σ_{i}. It is a general fact that the Bruhat order of any W^P is a directed poset with maximal element $w_0 w_0^P$. In particular, there exists always a path in Bruhat order from any element to the top, and the count is positive. □

5. Examples

5.1. Grassmannians

Let $G = \mathrm{PGL}_n$. Then G/P is the Grassmannian $\mathrm{Gr}(k, n)$ for $1 \leq k \leq n-1$. The Weyl group $W = S_n$, and the simple root $\alpha_{\mathrm{i}} = \kappa$ corresponds to the transposition of k and $k+1$. We have $\langle 2(\rho - \rho_P), \kappa^\vee \rangle = n$. The maximal parabolic subgroup is $W_P = S_k \times S_{n-k}$. The minimal representatives $w \in W^P$ are the permutations such that $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$. Any such permutation can be identified with a Young diagram that fits inside a $k \times (n-k)$ rectangle, and $\ell(w)$ is the number of boxes in the diagram. The projection $\pi_P : W \to W^P$ consists in reordering the values $w(1), \ldots, w(k)$ in increasing order and similarly for $w(k+1), \ldots, w(n)$.

In the quantum Chevalley formula of Proposition 4.9, the condition that $ws_\beta \gtrdot w$ means that $\beta \in R^+ \setminus R_P^+$ is the transposition of $l \in [1, k]$ and $m \in (n-k, n]$ with $w(m) = w(l) + 1$. Equivalently, the Young diagram of ws_β has one additional box on the $(k - l + 1)$th row. In the second term of the quantum Chevalley formula, the condition $\ell(\pi_P(ws_\kappa)) = \ell(w) + 1 - n$ is equivalent to $w\kappa = -\theta$, which is in turn equivalent to $w(k) = n$ and $w(k+1) = 1$. This can also be seen from the fact that the element $\pi_P(ws_\kappa)$ has Young diagram obtained by deleting the rim of the diagram of w (see [12]). A presentation for the quantum cohomology ring of Grassmannians is given in [18], [20], and [113].

The first term in the q-expansion in Lemma 4.17 is $\binom{n-2}{k-1}$. Indeed, $\pi_P(w_0 w_0^P s_\kappa)$ has Young diagram the $(k-1)\times(n-k-1)$ rectangle. The number of paths in Bruhat order is equal to the number of ways to sequentially add boxes to form the $k \times n$ rectangle which corresponds to the maximal element $w_0 w_0^P$ of W^P. This is consistent with the q-expansion of the quantum period in terms of binomial coefficients in [5, Theorem 5.1.6] and [93, Corollary 4.7].

The fundamental representation $V = V_{\varpi_{\mathrm{i}}}$ is the exterior product $\Lambda^k \mathbb{C}^n$. The highest weight vector is $v_e = e_1 \wedge \cdots \wedge e_k$. For every $w \in W^P$, the basis vector is $v_w = \dot{w} \cdot v_e = e_{w(1)} \wedge \cdots \wedge e_{w(k)}$. The Schubert class σ_w is the B-orbit closure of $\mathrm{Span}(e_{w(1)}, \ldots, e_{w(k)})$ inside $\mathrm{Gr}(k,n)$.

Example 5.2

Assume that $k = 2$ and $n = 4$. Denote the Schubert classes by $\sigma_\emptyset = 1$, $\sigma_1 = \sigma_{s_{\alpha_{\mathrm{i}}}}$, $\sigma_{11}, \sigma_2, \sigma_{21}$, and σ_{22}. The quantum Chevalley formula gives the identities $\sigma_1 *_q \sigma_1 = \sigma_{11} + \sigma_2$, $\sigma_1 *_q \sigma_{11} = \sigma_1 *_q \sigma_2 = \sigma_{21}$, $\sigma_1 *_q \sigma_{21} = \sigma_{22} + q$, and $\sigma_1 *_q \sigma_{22} = q\sigma_{21}$.

5.3. *Type D*

If $G = \mathrm{SO}(2n)$ is of type D_n, $n \geq 4$, and $\mathrm{i} = 1$, then G/P_1 is a quadric of dimension $2n-2$ in $\mathbb{P}^{2n}$. The quantum cohomology ring is described in [22] and [99].

The two minuscule nodes $\mathrm{i} = n$ and $\mathrm{i} = n-1$ are equivalent, and then G/P_n is isomorphic to one connected component of the orthogonal Grassmannian of maximal isotropic subspaces in $\mathbb{C}^{2n}$. A presentation for the quantum cohomology ring is given in [83].

5.4. *Exceptional cases*

A presentation of the quantum cohomology ring of the Cayley plane E_6/P_6 (resp., the Freudenthal variety E_7/P_7) is given in [21, Theorem 31] (resp., [21, Theorem 34]). The quantum corrections in the quantum Chevalley formula are also described in terms of the respective Hasse diagram. There are 6 (resp., 12) correction terms for E_6/P_6 (resp., E_7/P_7).

5.5. *Six-dimensional quadric, triality of D_4*

A case of special interest is $G = \mathrm{SO}(8)$ of type D_4 where all minuscule nodes 1, 3, 4 are equivalent. The homogeneous space G/P_1 is a 6-dimensional quadric. It also coincides with the Grassmannian $\mathrm{SO}(7)/P_3$ of isotropic spaces of dimension 3 inside $\mathbb{C}^7$.

The quadric is minuscule both as an SO(8)-homogeneous space and as an SO(7)-homogeneous space. Theorem 4.14 applies in both cases so that $\mathcal{Q}^{SO(8)/P_1} \simeq \mathcal{Q}^{\mathrm{SO}(7)/P_3}$ is isomorphic to the Frenkel–Gross connection $\nabla^{(G,V)}$ for both $G = \mathrm{SO}(8)$

and $G = \mathrm{SO}(7)$. Here, the representation V is either the standard representation of $\mathrm{SO}(8)$, or its restriction to $\mathrm{SO}(7)$ which is the direct sum of the trivial representation plus the standard representation.

PROPOSITION 5.6

(i) *The quantum connection* $\mathcal{Q}^{\mathrm{SO}(8)/P_1}$ *of the* 6*-dimensional quadric is the direct sum of two irreducible constituents of dimensions* 1 *and* 7*, respectively.*

(ii) *The differential Galois group is* G_2*.*

Proof

We have seen in Section 3.3 that ∇^G for G of type D_4 reduces to ∇^G for G of type G_2. Thus it suffices to observe that the standard representation V of $\mathrm{SO}(8)$ when restricted to G_2 decomposes into the trivial representation plus the irreducible representation of dimension 7. This holds because the restriction of the standard representation of $\mathrm{SO}(7)$ is the 7-dimensional representation of G_2. □

A presentation of the quantum cohomology ring of the homogeneous space G/P_1 is given in [22]. It is also given in [83] as a particular case of Grassmannians of isotropic spaces and in [99] as a particular case of even-dimensional quadrics. From either of these presentations or from the quantum Chevalley formula, we find the quantum multiplication by $\sigma = \sigma_{\mathrm{i}}$ in the Schubert basis; thus

$$\mathcal{Q}^{G/P_1} = q\frac{d}{dq} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The middle cohomology $H^6(G/P_1)$ is 2-dimensional, spanned by $\{\sigma_3^+, \sigma_3^-\}$. Since $\sigma *_q \sigma_3^+ = \sigma *_q \sigma_3^-$, the subspace $\mathbb{C}(\sigma_3^+ - \sigma_3^-)$ is in the kernel of σ and in particular is a stable 1-dimensional subspace of the connection. The other stable subspace, denoted $H^\#(G/P_1)$ following [67], has dimension 7 and is spanned by $\sigma_3^+ + \sigma_3^-$ and all the cohomology in the remaining degrees. This is consistent with Proposition 5.6(i).

The rank-7 subspace $H^\#(G/P_1)$ is generated as an algebra by $H^2(G/P_1)$, and moreover the vector 1 is cyclic for the multiplication by σ. The quantum D-module $\mathcal{Q}^{G/P_1}$ is then given in scalar form as D/DL, where

$$L := \left(q\frac{d}{dq}\right)^7 + 4q^2\frac{d}{dq} + 2q.$$

The differential Galois group of L on $\mathbb{P}^1\backslash\{0,\infty\} \simeq \mathbb{C}_q^\times$ is equal to G_2 according to Proposition 5.6(ii). Recall from [43] that ultimately the reason for the differential Galois group to be G_2 is the triality of D_4 and the invariance of the Frenkel–Gross connection $\nabla^{\mathrm{SO}(8)}$ under outer automorphisms which reduces it to ∇^{G_2}.

After rescaling L by $q \mapsto -q/4$, the D-module D/DL becomes isomorphic to the hypergeometric D-module ${}_1F_6\left({}_{1\ 1\ 1\ 1\ 1\ 1}^{\quad 1/2}\right)$ studied in [78] with the notation $\mathcal{H}(0,0,0,0,0,0,0;1/2)$. Katz proved in [78, Theorem 4.1.5] that the differential Galois group is G_2, which is consistent with Proposition 5.6(ii).

Our work gives a new interpretation of D/DL studied by Katz and by Frenkel and Gross as the quantum connection $\mathcal{Q}^{G/P_1}$. Hence we have the following question.

Question 5.7
Is it possible to see a priori that the differential Galois group of the quantum connection of the 6-dimensional quadric is G_2?

The question seems subtle because, for example, the quantum connection of the 5-dimensional quadric, which is homogeneous under G_2, has rank 6 (see Section 5.8 below), and thus its differential Galois group is unrelated to the group G_2.

5.8. *Odd-dimensional quadrics*

More generally, let $G = \mathrm{SO}(2n+1)$ be of type B_n with $n \geq 3$. Then G/P_1 is a $(2n-1)$-dimensional quadric and is cominuscule. The cohomology has total dimension $2n$. There is one Schubert class σ_k in each even degree $2k \leq 4m-2$. The quantum product is determined in [22, Section 4.1.2]. In particular, $\sigma_1 *_q \sigma_{k-1} = \sigma_k$ for $1 \leq k \leq n-1$ and $n+1 \leq k \leq 2n-2$, $\sigma_1 *_q \sigma_{n-1} = 2\sigma_n$, $\sigma_1 *_q \sigma_{2n-2} = \sigma_{2n-1} + q$, and $\sigma_1 *_q \sigma_{2n-1} = q\sigma_1$, which also follows from the quantum Chevalley formula. The relation between the quantum connection and hypergeometric D-modules is studied in detail by Pech, Rietsch, and Williams [99].

The space G_2/P_1 is a 5-dimensional quadric. Its connected automorphism group is SO(7) by Section 4.15. It is coadjoint as a G_2-homogeneous space and cominuscule as an SO(7)-homogeneous space. The cohomology has total dimension 6. A presentation of the quantum cohomology ring is $\mathbb{C}[\sigma,q]/(\sigma^6 - 4\sigma q)$ (see [24, Section 5.1]).

6. Character D-module of a geometric crystal

In this section, we introduce the character D-module for the geometric crystal of Berenstein and Kazhdan [8]. The roles of G and $G^\vee$ are interchanged relative to Sections 3–5.

6.1. Double Bruhat cells

Let $U \subset B$ and $U_- \subset B_-$ be opposite maximal unipotent subgroups. For each $w \in W$, define

$$B_-^w := U\dot{w}U \cap B_-,$$
$$U^w := U \cap B_-\dot{w}B_-.$$

LEMMA 6.2

Let $U(w) := U \cap \dot{w}U_-\dot{w}^{-1}$. For $u \in U^{w^{-1}}$, there is a unique $\eta(u) \in B_-^w$ and a unique $\tau(u) \in U(w)$ such that

$$\eta(u) = \tau(u)\dot{w}u.$$

The twist map $\eta : U^w \to B_-^w$ is a biregular isomorphism and $\tau : U^w \hookrightarrow U(w)$ is an injection.

Proof

This is [9, Propositions 5.1 and 5.2] (see also [10, Theorem 4.7] and [8, Claim 3.25]). Since our conventions differ from those in [8] and [10] slightly, we provide a proof.

If we define the subgroup $U'(w) := U_- \cap \dot{w}U_-\dot{w}^{-1}$, then the multiplication maps $U(w) \times U'(w) \to \dot{w}U_-\dot{w}^{-1}$ and $U'(w) \times U(w) \to \dot{w}U_-\dot{w}^{-1}$ are bijective. In particular, $B_-\dot{w}B_-\dot{w}^{-1} = B_-\dot{w}U_-\dot{w}^{-1} = B_-U'(w)U(w) = B_-U(w)$.

We have $u^{-1} \in U^w$. Thus $u^{-1}\dot{w}^{-1} \in B_-U(w) \subset B_-U$. Since $B_- \cap U = 1$, the factorization $u^{-1}\dot{w}^{-1} = \eta(u)^{-1}\tau(u)$ with $\eta(u) \in B_-$ and $\tau(u) \in U(w)$ is unique. Moreover, $\eta(u) \in B_-^w$. Since $\tau(u)w \in B_-u^{-1}$, it follows similarly that $u \mapsto \tau(u)$ is injective.

Conversely, $\dot{w}^{-1}U\dot{w}U = (\dot{w}^{-1}U\dot{w} \cap U_-)U = \dot{w}^{-1}U(w)\dot{w}U$. Hence, given $x \in B_-^w$ we have $\dot{w}^{-1}x \in \dot{w}^{-1}U(w)\dot{w}U$, which provides by factorization an inverse element $\eta^{-1}(x) \in U^w$. □

LEMMA 6.3

For $t \in T$, let $s := \dot{w}t\dot{w}^{-1}$. Each of U^w and $U(w)$ is $\mathrm{Ad}(T)$-stable, and for $u \in U^w$,

$$\tau(tut^{-1}) = s\tau(u)s^{-1}, \qquad \eta(tut^{-1}) = s\eta(u)t^{-1}.$$

Proof

Since $s\dot{w} = \dot{w}t$, we have $s\eta(u)t^{-1} = s\tau(u)s^{-1}\dot{w}tut^{-1}$; hence the assertion follows. □

6.4. Geometric crystals

Fix an arbitrary standard parabolic subgroup $P \subset G$. Let $w_0 \in W$ be the longest element of W, and let $w_0^P \in W_P$ be the longest element of W_P. Define $w_P := w_0^P w_0$ so that w_P^{-1} is the longest element in W^P. In this case, the subgroup $U_P := U(w_P)$ is the unipotent radical of P. The *parabolic geometric crystal* associated to (G, P) is

$$X := UZ(L_P)\dot{w}_P U \cap B_- = Z(L_P)B_-^{w_P}.$$

We now define three maps π, γ, f on X, called the highest weight map, the weight map, and the decoration or superpotential.

The *highest weight map* is given by

$$\pi : X \to Z(L_P) \qquad x = u_1 t \dot{w}_P u_2 \mapsto t.$$

Let $X_t = \pi^{-1}(t) = \{u_1 t \dot{w}_P u_2 \in B_-\}$ be the fiber of X over t. We call X_t the *geometric crystal with highest weight* t. Since the product map $Z(L_P) \times B_-^{w_P} \to X$ is an isomorphism, we have a natural isomorphism $X_t \cong B_-^{w_P}$. Geometrically, we think of X as a family of open Calabi–Yau manifolds fibered over $Z(L_P)$.

The *weight map* is given by

$$\gamma : X \to T \qquad x \mapsto x \mod U_- \in B_-/U_- \cong T. \tag{6.4.1}$$

For $i \in I$, let $\chi_i : U \to \mathbb{A}^1$ be the additive character uniquely determined by

$$\chi_i\big(\exp(t x_j)\big) = \delta_{ij} t,$$

where the elements x_j for $j \in I$ are given in Section 2.2. Let the *standard additive character* be $\psi := \sum_{i \in I} \chi_i$. The *decoration* (or *superpotential*) is given by

$$f : X \to \mathbb{A}^1 \qquad x = u_1 t \dot{w}_P u_2 \mapsto \psi(u_1) + \psi(u_2).$$

It follows from [107, Lemma 5.2] that f agrees with Rietsch's superpotential. (This reference has conventions Langlands dual to ours. Our $Z(L_P)$ and w_0^P are denoted $(T^\vee)^{W_P}$ and w_P in [107], while our $\dot{s}_i$ is inverse to the corresponding notation there.)

Set $\psi_t(u) := \psi(tut^{-1})$ for $t \in T$ and $u \in U$. For $t \in Z(L_P)$, the potential can be expressed as a function of $u \in U^{w_P^{-1}}$ as follows:

$$f_t(u) := f\big(t\eta(u)\big) = \psi_t\big(\tau(u)\big) + \psi(u). \tag{6.4.2}$$

Equivalently, the potential is expressed on $B_-^{w_P} = U\dot{w}_P U \cap B_-$ by

$$f_t(u_1 \dot{w}_P u_2) = \psi_t(u_1) + \psi(u_2).$$

Example 6.5
Let $G = \mathrm{SL}(2)$ and $P = B$. With the parameterizations

$$u_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix},$$

$$\dot{w}_P = \dot{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & t^2/a \\ 0 & 1 \end{pmatrix},$$

the geometric crystal X is the set of matrices

$$X = \left\{ \begin{pmatrix} a/t & 0 \\ 1/t & t/a \end{pmatrix} \,\middle|\, a, t \in \mathbb{C}^\times \right\} \subset \mathrm{SL}(2),$$

equipped with the functions

$$f(x) = a + t^2/a, \qquad \pi(x) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \qquad \text{and} \qquad \gamma(x) = \begin{pmatrix} a/t & 0 \\ 0 & t/a \end{pmatrix}.$$

6.6. Open projected Richardson varieties
For $v, w \in W$ with $v \leq w$, the open Richardson variety $\mathcal{R}_v^w \subset G/B$ is the intersection of the Schubert cell $B_-\dot{v}B/B$ with the opposite Schubert cell $B\dot{w}B/B$. The map $u \mapsto u\dot{w}_0 \pmod{B}$ induces an isomorphism $U^w \xrightarrow{\sim} \mathcal{R}_{ww_0}^{w_0}$. For every $t \in Z(L_P)$, we have a sequence of isomorphisms

$$X_t \cong B_-^{w_P} \cong U^{w_P} \cong \mathcal{R}_{w_0^P}^{w_0}, \tag{6.6.1}$$

given by $x = tu_1\dot{w}_P u_2 \mapsto u_1\dot{w}_P u_2 \mapsto u_2^{-1} \mapsto u_2^{-1}\dot{w}_0 B$, where in the factorization we assume that $u_1 \in U(w_P)$. We describe directly the composition of these isomorphisms as follows.

LEMMA 6.7
For every $x = u_1 t\dot{w}_P u_2 \in X_t$ with $u_1 \in U_P = U(w_P)$, we have $x^{-1}\dot{w}_0^P B = u_2^{-1}\dot{w}_0 B$.

Proof
We have $x^{-1}\dot{w}_0^P B = u_2^{-1}\dot{w}_P^{-1}u_1\dot{w}_P\dot{w}_0 B$. It suffices to observe that $\dot{w}_0\dot{w}_P^{-1} \times u_1\dot{w}_P\dot{w}_0 \in U$ since $u_1 \in U(w_P)$. □

The projection $p : G/B \to G/P$ induces an isomorphism of $\mathcal{R}_{w_0^P}^{w_0}$ onto its image $\mathring{G/P}$, the *open projected Richardson variety* of G/P. The complement of $\mathring{G/P}$ in G/P is the divisor $\partial_{G/P}$ in G/P, the multiplicity-free union of the irreducible codimension-1 subvarieties D^i, $i \in I$ and D_i, $i \notin I_P$, where

$$D^i := \overline{p(\mathcal{R}^{w_0 s_i}_{w_0^P})} \qquad \text{and} \qquad D_i := \overline{p(\mathcal{R}^{w_0}_{s_i w_0^P})}.$$

By [81, Lemma 5.4], $\partial_{G/P}$ is anticanonical in G/P, or in other words, the canonical bundle is given by $\omega_{G/P} = \mathcal{O}_{G/P}(\partial_{G/P})$. There is thus, up to scalar, a unique meromorphic form ω on G/P that has no zeros, and simple poles along $\partial_{G/P}$. We let $1/\omega$ denote the section of the anticanonical bundle inverse to ω.

6.8. Explicit formula for superpotential

We now give an explicit formula for the superpotential f_t as a function on $U^{w_P^{-1}}$. Given $g = b_- v$, where $b_- \in B_-$ and $v \in U$, we set $\pi_+(g) = v$. Also let $g \mapsto g^T$ denote the transpose antiautomorphism of G (see, e.g., [42]). Let $g \mapsto g^{-T}$ denote the composition of the inverse and transpose antiautomorphisms (which commute). There is an involution $\star : I \to I$ determined by $w_0 \cdot \alpha_i = -\alpha_{i^\star}$. We let $P^\star$ be the standard parabolic subgroup determined by $I_{P^\star} = (I_P)^\star$.

LEMMA 6.9
For $u \in U^{w_P^{-1}}$, we have

$$\pi_+\big((\dot{w}_0)^{-1} u^T \dot{w}_0^{P^\star}\big) = (\dot{w}_0)^{-1} \tau(u)^{-T} \dot{w}_0.$$

Proof
Let $v = \tau(u)$. Then $x = v\dot{w}_P u \in B_-$ and $u = (\dot{w}_P)^{-1} v^{-1} x$. Noting that $(\dot{w})^T = (\dot{w})^{-1}$, we have

$$(\dot{w}_0)^{-1} u^T = (\dot{w}_0)^{-1} x^T v^{-T} \dot{w}_P = \big[(\dot{w}_0)^{-1} x^T \dot{w}_0\big]\big[(\dot{w}_0)^{-1} v^{-T} \dot{w}_0\big]\big[(\dot{w}_0)^{-1} \dot{w}_P\big].$$

We note that $[(\dot{w}_0)^{-1} x^T \dot{w}_0] \in B_-$ and $[(\dot{w}_0)^{-1} v^{-T} \dot{w}_0] \in U$, so the claim follows from the equality

$$(\dot{w}_0)^{-1} \dot{w}_P = (\dot{w}_0^{P^\star})^{-1}.$$

We first argue that $(\dot{w}_0)^{-1} \dot{s}_i \dot{w}_0 = \dot{s}_{i^\star}$. Write $\alpha^\vee(t)$ for the cocharacter $\mathbb{G}_m \to T$. Then $\alpha_i^\vee(-1) = (\dot{s}_i)^2 \in T$ and $\alpha_i^\vee(-1)^2 = 1$. Let $w' = s_i w_0 = w_0 s_{i^\star}$, and compute

$$(\dot{w}_0)^{-1} \dot{s}_i \dot{w}_0 = (\dot{w}_0)^{-1} \alpha_i^\vee(-1) \dot{w}' = (\dot{w}_0)^{-1} \alpha_i^\vee(-1) \dot{w}_0 \alpha_{i^\star}^\vee(-1) \dot{s}_{i^\star} = \dot{s}_{i^\star},$$

where we have used $(\dot{w}_0)^{-1} \alpha_i^\vee(t) \dot{w}_0 = w_0 \cdot \alpha_i(t) = \alpha_{i^\star}(t^{-1})$. It follows that

$$(\dot{w}_0)^{-1} \dot{w}_P = (\dot{w}_0)^{-1} \dot{w}_P \dot{w}_0 (\dot{w}_0)^{-1} = \dot{w}_{P^\star} (\dot{w}_0)^{-1} = (\dot{w}_0^{P^\star})^{-1},$$

as required. ☐

In the following, we shall assume that G is simply connected. Since the partial flag variety G/P depends only on the type of G, we lose no generality.

For a fundamental weight ϖ_i and elements $u, w \in W$, there is a generalized minor $\Delta_{u\varpi_i, w\varpi_i} : G \to \mathbb{A}^1$, defined in [42]. This function is equal to the matrix coefficient $g \mapsto \langle g \cdot v_{w\varpi_i}, v_{u\varpi_i} \rangle$ of G acting on the irreducible representation V_{ϖ_i}, with respect to extremal weight vectors $v_{w\varpi_i} := \dot{w} \cdot v_{\varpi_i}$ and $v_{u\varpi_i} := \dot{u} \cdot v_{\varpi_i}$ with weights $w\varpi_i$ and $u\varpi_i$, respectively. Here v_{ϖ_i} denotes a fixed highest weight vector with weight ϖ_i.

LEMMA 6.10
For $u \in U^{w_P^{-1}}$, we have

$$\psi\big(\tau(u)\big) = \sum_{i \in I \setminus I_P^\star} \frac{\Delta_{w_0^{P^\star} s_i \varpi_i, w_0 \varpi_i}(u)}{\Delta_{\varpi_i, w_0 \varpi_i}(u)}.$$

Thus

$$f_t(u) = \psi(u) + \sum_{i \in I \setminus I_P^\star} \alpha_{i^\star}(t) \frac{\Delta_{w_0^{P^\star} s_i \varpi_i, w_0 \varpi_i}(u)}{\Delta_{\varpi_i, w_0 \varpi_i}(u)}. \tag{6.10.1}$$

Proof
First note that $\chi_i(\pi_+(g)) = \frac{\Delta_{\varpi_i, s_i\varpi_i}(g)}{\Delta_{\varpi_i,\varpi_i}(g)}$. We have $\psi(\tau(u)) = \sum_{i \notin I_P} \chi_i(\tau(u))$. Since $(\dot{w}_0)^{-1} \exp(t y_i) \dot{w}_0 = \exp(-t x_{i^\star})$, we have $\chi_i(\tau(u)) = \chi_{i^\star}((\dot{w}_0)^{-1} \tau(u)^{-T} \times \dot{w}_0)$. By Lemma 6.9, we have for $i^\star \notin I_P$, the equalities

$$\begin{aligned}\chi_{i^\star}\big(\tau(u)\big) = \chi_i\big(\pi_+\big((\dot{w}_0)^{-1} u^T \dot{w}_0^{P^\star}\big)\big) &= \frac{\Delta_{\varpi_i, s_i\varpi_i}((\dot{w}_0)^{-1} u^T \dot{w}_0^{P^\star})}{\Delta_{\varpi_i,\varpi_i}((\dot{w}_0)^{-1} u^T \dot{w}_0^{P^\star})} \\ &= \frac{\Delta_{w_0\varpi_i, w_0^{P^\star} s_i \varpi_i}(u^T)}{\Delta_{w_0\varpi_i, w_0^{P^\star}\varpi_i}(u^T)} = \frac{\Delta_{w_0^{P^\star} s_i\varpi_i, w_0\varpi_i}(u)}{\Delta_{w_0^{P^\star}\varpi_i, w_0\varpi_i}(u)}.\end{aligned}$$

Finally, observe that we have $w_0^{P^\star}\varpi_i = \varpi_i$ whenever $i \in I \setminus I_P^\star$. For the last formula, we note that for $t \in Z(L_P)$, we have $\chi_i(t\tau(u)t^{-1}) = \alpha_i(t)\chi_i(\tau(u))$. ☐

Fix a reduced word $\mathbf{i} = i_1 i_2 \cdots i_\ell$ of w_P^{-1}. We have the Lusztig rational parameterization $\mathbb{G}_m^\ell \to U^{w_P^{-1}}$ given by

$$\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \longmapsto x_{\mathbf{i}}(\mathbf{a}) = x_{i_1}(a_1) x_{i_2}(a_2) \cdots x_{i_\ell}(a_\ell),$$

where $x_i(t) := \exp(t x_i)$ denotes a one-parameter subgroup of G.

COROLLARY 6.11
In the Lusztig parameterization, the superpotential $f_t|_{\mathbb{G}_m^\ell} : \mathbb{G}_m^\ell \to \mathbb{A}^1$ *is given by the function*

$$f_t(a_1, a_2, \ldots, a_\ell) = a_1 + a_2 + \cdots + a_\ell + \sum_{i \in I \setminus I_P} \alpha_i(t), P_i,$$

where P_i *is a Laurent polynomial in* $a_1, a_2, \ldots, a_\ell$ *with positive coefficients.*

Proof
We may assume that G is simply connected and apply Lemma 6.10. We have $\psi(x_{\mathbf{i}}(\mathbf{a})) = a_1 + a_2 + \cdots + a_\ell$. Now, for any $i \in I \setminus I_P^\star$, the generalized minor $\Delta_{w_0^{P^\star} s_i \varpi_i, w_0 \varpi_i}(x_{\mathrm{i}}(\mathbf{a}))$ is a polynomial in $a_1, a_2, \ldots, a_\ell$ with positive coefficients by [10, Theorem 5.8] and $\Delta_{\varpi_i, w_0 \varpi_i}(x_{\mathbf{i}}(\mathbf{a}))$ is a monomial in $a_1, a_2, \ldots, a_\ell$ by [10, Corollary 9.5]. □

Corollary 6.11 generalizes [27, Theorem 5.6] to the parabolic setting.

Example 6.12
If P is minuscule, then $I \setminus I_P = \{\mathrm{i}\}$ consists of a single minuscule node. It follows from the proofs of [10, Theorem 5.8, Corollary 9.5] that $\Delta_{\varpi_{\mathrm{i}^\star}, w_0 \varpi_{\mathrm{i}^\star}}(x_{\mathbf{i}}(\mathbf{a})) = a_1 a_2 \cdots a_\ell$ and that $\Delta_{w_0^{P^\star} s_{\mathrm{i}^\star} \varpi_{\mathrm{i}^\star}, w_0 \varpi_{\mathrm{i}^\star}}(x_{\mathrm{i}}(\mathbf{a}))$ is a square-free polynomial in $a_1, a_2, \ldots, a_\ell$ with positive integer coefficients.

Example 6.13
Let us pick $G = \mathrm{SL}(4)$ and $\mathrm{i} = 2$. A reduced word for $w_P = w_P^{-1}$ is 2312. We obtain the parameterization

$$(a_1, a_2, a_3, a_4) \longmapsto u = x_{\mathbf{i}}(\mathbf{a}) = \begin{pmatrix} 1 & a_3 & a_3 a_4 & 0 \\ 0 & 1 & a_1 + a_4 & a_1 a_2 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau(u) = \begin{pmatrix} 1 & 0 & \frac{1}{a_1 a_3} & -\frac{1}{a_1} \\ 0 & 1 & \frac{a_1 + a_4}{a_1 a_2 a_3 a_4} & -\frac{1}{a_1 a_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $\psi(\tau(u)) = (a_1 + a_4)/a_1 a_2 a_3 a_4$. This is equal to the ratio $\Delta_{24,34}(u)/\Delta_{12,34}(u)$, agreeing with Lemma 6.10. Here, $\Delta_{I,J}$ denotes the minor using rows I and columns J. Thus the superpotential is given by

$$f_t|_{\mathbb{G}_m^4} = a_1 + a_2 + a_3 + a_4 + q\frac{a_1 + a_4}{a_1a_2a_3a_4},$$

where $q = \alpha_\mathfrak{i}(t)$. For generic q, the function f_t has four critical points in the chart $\{(a_1, a_2, a_3, a_4) \in \mathbb{G}_m^4\}$, which for $q = 1$ are given by

$$(-1/\sqrt{2}, -\sqrt{2}, -\sqrt{2}, -1/\sqrt{2}), \qquad (-i/\sqrt{2}, -i\sqrt{2}, -i\sqrt{2}, -i/\sqrt{2}),$$
$$(i/\sqrt{2}, i\sqrt{2}, i\sqrt{2}, i/\sqrt{2}), \qquad (1/\sqrt{2}, \sqrt{2}, \sqrt{2}, 1/\sqrt{2}).$$

We have $4 < 6 = \dim(H^*(\mathrm{Gr}(2,4)))$. So the Laurent polynomial $f_t|_{\mathbb{G}_m^4}$ is a "weak mirror": the missing critical points lie outside of this toric chart inside X_t, consistent with the discussion in [106, Section 9].

Example 6.14
Let us pick $G = \mathrm{SL}(5)$ and $\mathfrak{i} = 2$. A reduced word for w_P is 234123. Using the reversed reduced word for w_P^{-1}, we obtain the parameterization

$$(a_1, a_2, a_3, a_4, a_5, a_6) \longmapsto u = x_\mathbf{i}(\mathbf{a}) = \begin{pmatrix} 1 & a_3 & a_3a_6 & 0 & 0 \\ 0 & 1 & a_2 + a_6 & a_2a_5 & 0 \\ 0 & 0 & 1 & a_1 + a_5 & a_1a_4 \\ 0 & 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau(u) = \begin{pmatrix} 1 & 0 & \frac{1}{a_1a_2a_3} & -\frac{1}{a_1a_2} & \frac{1}{a_1} \\ 0 & 1 & \frac{a_1a_2+a_1a_6+a_5a_6}{a_1a_2a_3a_4a_5a_6} & -\frac{a_1+a_5}{a_1a_2a_4a_5} & \frac{1}{a_1a_4} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $\psi(\tau(u)) = (a_1a_2 + a_1a_6 + a_5a_6)/a_1a_2a_3a_4a_5a_6$. This is equal to the ratio $\Delta_{235,345}(u)/\Delta_{123,345}(u)$, agreeing with Lemma 6.10. Thus the superpotential is given by

$$f_t|_{\mathbb{G}_m^6} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + q\frac{a_1a_2 + a_1a_6 + a_5a_6}{a_1a_2a_3a_4a_5a_6},$$

where $q = \alpha_\mathfrak{i}(t)$. In this case, $f_t|_{\mathbb{G}_m^6}$ has $10 = \dim H^*(\mathrm{Gr}(2,5))$ critical points in the toric chart, so there are no "missing critical points."

6.15. Polar divisor of the superpotential
By construction, the potential f_t can be identified with a rational function on G/P. We now show that the polar divisor of f_t is equal to the anticanonical divisor $\partial_{G/P} \subset G/P$.

PROPOSITION 6.16
The potential f_t, viewed as a rational function on G/P, has polar divisor $\partial_{G/P}$. It can thus be written as the ratio $(1/\eta_t)/(1/\omega)$ of two holomorphic anticanonical sections on G/P, where $1/\omega$ is the holomorphic anticanonical section of Section 6.6.

Proof
Let $x = tu_1\dot{w}_P u_2 \in X_t$, where $u_1 \in U(w_P)$. Under (6.6.1), we have that x is sent to $x^{-1}\dot{w}_0^P P = u_2^{-1}\dot{w}_0 P$.

Given $y \in B_-$, and $i \in I$, we have

$$\Delta_{w_0\varpi_i,\varpi_i}(y) = 0 \iff y \in \overline{B\dot{w}_0\dot{s}_i B}. \tag{6.16.1}$$

Indeed, writing $y = b_1 v b_2$, we have

$$\begin{aligned}\langle y \cdot v_{\varpi_i}, v_{w_0\varpi_i}\rangle = 0 &\iff \langle b_1 \cdot v_{v\varpi_i}, v_{w_0\varpi_i}\rangle = 0 \iff v \le w_0 s_i \\ &\iff y \in \overline{B\dot{w}_0\dot{s}_i B}.\end{aligned}$$

Working in the open affine chart (the big cell) $(B_- \cap Uw_P^{-1}U)P/P \subset G/P$, the divisor D^i is thus cut out by the single equation $\Delta_{w_0\varpi_i,\varpi_i}(y) = 0$.

Now, we take $y = x^{-1}$. By Lemma 6.7, the vanishing of $\Delta_{w_0\varpi_i,\varpi_i}(x^{-1})$ is equivalent to the vanishing of $\Delta_{w_0\varpi_i,\varpi_i}(u_2^{-1}\dot{w}_0)$ (noting that $w_0^P\varpi_i = \varpi_i$). On the other hand, by [42, Proposition 2.6] and [8, (1.8)],

$$\chi_i(u_2) = -\chi_i(u_2^{-1}) = -\langle u_2^{-1} v_{w_0\varpi_i}, v_{w_0 s_i\varpi_i}\rangle = -\frac{\Delta_{w_0 s_i\varpi_i, w_0\varpi_i}(u_2^{-1})}{\Delta_{w_0\varpi_i,w_0\varpi_i}(u_2^{-1})},$$

which has a pole along the zero locus of $\Delta_{w_0\varpi_i,w_0\varpi_i}(u_2^{-1}) = \Delta_{w_0\varpi_i,\varpi_i}(u_2^{-1}w_0)$. Thus the function $\chi_i(u_2)$ has a simple pole along D^i and the function $\psi(u_2)$ has simple poles along all the $D^i, i \in I$. In a similar manner using Lemma 6.10, we find that $\psi(u_1)$ has simple poles along the divisors $D_i, i \notin I_P$. Since all the divisors D^i and D_i are distinct, the rational function f_t has polar divisor exactly $\partial_{G/P}$. □

Remark 6.17
Proposition 6.16 is one manifestation of mirror symmetry of Fano manifolds. For example, the potentials of mirrors of toric Fano varieties are constructed in [55] and the same property can be seen to hold. In general, it is explained in Katzarkov, Kontsevich, and Pantev [79, Remark 2.5(ii)] by the fact that the cup product by $c_1(K_{G^\vee/P^\vee})$ on the cohomology of the mirror manifold $G^\vee/P^\vee$ is a nilpotent endomorphism.

Remark 6.18
The zero divisor of $1/\omega$ and the zero divisor of $1/\eta_t$ may intersect, so f_t has

points of indeterminacy. Indeed, this happens in the example of $\mathbb{P}^2$ (see also [79, Remark 2.5(i)]).

6.19. The character D-module of a geometric crystal

Let $\mathsf{E} := D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial_x - 1)$ be the exponential D-module on $\mathbb{A}^1$. Let $\mathsf{E}^f = f^*\mathsf{E}$ be the pullback D-module on X. Finally, define the character D-module of the geometric crystal X by

$$\mathrm{Cr}_{(G,P)} := R\pi_*\mathsf{E}^f \quad \text{on } Z(L_P). \tag{6.19.1}$$

A priori $\mathrm{Cr}_{(G,P)}$ lies in the derived category of D-modules on $Z(L_P)$. But in Theorem 7.10 we shall see that $R^i\pi_*\mathsf{E}^f = 0$ for $i \neq 0$, and thus $\mathrm{Cr}_{(G,P)}$ is just a D-module.

Remark 6.20

Our conventions for D-modules follow those in [73]. All the D-modules we study in this paper are integrable connections, so in particular they are holonomic. We will not need the formalism of derived categories of D-modules, since we are always just handling D-modules. For the six functors for holonomic D-modules, we refer the reader to [73, Section 3]. Our Rf_* and $Rf_!$ are the same as $\int_f$ and $\int_{f!}$ there. Our f_* and $f_!$ are the degree-0 parts $\int_f^0$ and $\int_{f!}^0$ there. Additionally, the reader may consult [44] and the references therein for more background on exponential D-modules of the form E^f.

6.21. Homogeneity

Recall that $\rho_P = \frac{1}{2}\sum_{\alpha\in R_P^+}\alpha$ and that $w_P = w_0^P w_0$ is the inverse of the longest element of W^P.

LEMMA 6.22

We have $w_P(\rho) = -\rho + 2\rho_P$.

Proof

The element w_0^P sends R_P^+ to R_P^- and permutes the elements of $R^+ \setminus R_P^+$. We compute

$$\begin{aligned} w_P(\rho) &= w_0^P w_0(\rho) = w_0^P(-\rho) = -w_0^P(\rho - \rho_P) - w_0^P(\rho_P) \\ &= -(\rho - \rho_P) + \rho_P = -\rho + 2\rho_P. \end{aligned}$$ □

We view $2\rho^\vee$ as a cocharacter $\mathbb{G}_m \to T$. Similarly, we view $2\rho^\vee - 2\rho_P^\vee$ as a cocharacter $\mathbb{G}_m \to Z(L_P)$.

LEMMA 6.23
For any $u \in U^{w_P^{-1}}$, $t \in Z(L_P)$, and $\zeta \in \mathbb{G}_m$,

$$f_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}\big(\mathrm{Ad}\rho^\vee(\zeta)(u)\big) = \zeta f_t(u).$$

Proof
We have $\psi_{\rho^\vee(\zeta)}(u) = \zeta\psi(u)$ for any $u \in U$ and $\zeta \in \mathbb{G}_m$. Thus in view of (6.4.2), we have

$$f_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}\big(\mathrm{Ad}\rho^\vee(\zeta)(u)\big) = \psi_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}\big(\tau\big(\mathrm{Ad}\rho^\vee(\zeta)(u)\big)\big) - \zeta\psi(u),$$

and we are now reduced to treating the first term. It follows from Lemma 6.3 that

$$\tau\big(\mathrm{Ad}\rho^\vee(\zeta)(u)\big) = \mathrm{Ad}\big(w_P(\rho^\vee)\big)(\zeta)\big(\tau(u)\big),$$

and therefore the first term above is equal to

$$\begin{aligned}\psi_{(w_P(\rho^\vee))(\zeta)(\rho^\vee - \rho_P^\vee)(\zeta^2)t}\big(\tau(u)\big) &= \psi_{(w_P(\rho^\vee)+2\rho^\vee - 2\rho_P^\vee)(\zeta)t}\big(\tau(u)\big) \\ &= \psi_{\rho^\vee(\zeta)t}\big(\tau(u)\big) = \zeta\psi_t\big(\tau(u)\big).\end{aligned}$$

In the second line we have used Lemma 6.22, but with $\rho^\vee$ and $\rho_P^\vee$ instead of ρ and ρ_P. □

We define the following $\mathbb{G}_m$-actions on X, $Z(L_P)$, and $\mathbb{A}^1$. For $\zeta \in \mathbb{G}_m$, we have

$$\begin{aligned}&\zeta \cdot x = \rho^\vee(\zeta)x\rho^\vee(\zeta)^{-1} \quad \text{for } x \in X,\\ &\zeta \cdot t = (2\rho^\vee - 2\rho_P^\vee)(\zeta)t \quad \text{for } t \in Z(L_P),\\ &\zeta \cdot a = \zeta a \quad \text{for } a \in \mathbb{A}^1.\end{aligned}$$

Also equip T with the trivial $\mathbb{G}_m$-action.

PROPOSITION 6.24
The maps $\pi : X \to Z(L_P)$, $f : X \to \mathbb{A}^1$, and $\gamma : X \to T$ are $\mathbb{G}_m$-equivariant.

Proof
We have $\zeta \cdot x = (2\rho^\vee - 2\rho_P^\vee)(\zeta)w_P(\rho^\vee)(\zeta)x\rho^\vee(\zeta)^{-1}$. We verify using Lemma 6.3 that $x \mapsto w_P(\rho^\vee)(\zeta)x\rho^\vee(\zeta)^{-1}$ is an automorphism of $B_-^{w_P}$. This shows that $\pi(\zeta \cdot x) = \zeta \cdot \pi(x)$. The second claim follows from Lemma 6.23. The last claim is immediate from the definitions. □

COROLLARY 6.25
For any $\lambda \in \mathbb{C}^\times$, we have

$$R\pi_*\mathsf{E}^{f/\lambda} \cong \big[t \mapsto (2\rho^\vee - 2\rho_P^\vee)(\lambda)t\big]_* \mathrm{Cr}_{(G,P)}.$$

Proof
By definition, the left-hand side is equal to $R\pi_* f^*[a \mapsto a/\lambda]^*\mathsf{E}$. In view of Proposition 6.24, it is isomorphic to

$$\begin{aligned} R\pi_*(x \mapsto \hbar \cdot x)_* f^*\mathsf{E} &= \big[t \mapsto (2\rho^\vee - 2\rho_P^\vee)(\hbar)t\big]_* R\pi_* f^*\mathsf{E} \\ &= \big[t \mapsto (2\rho^\vee - 2\rho_P^\vee)(\hbar)t\big]_* \mathrm{Cr}_{(G,P)}, \end{aligned}$$

which concludes the proof. □

We record the following lemma which will be needed in Section 13 in the context of rapid decay cycles.

LEMMA 6.26
The meromorphic form ω on G/P with simple poles along the anticanonical divisor $\partial_{G/P}$ is preserved under the T-action.

Proof
First we observe that each irreducible component of the divisor $\partial_{G/P}$ is T-invariant and thus the sections cutting them out are T-weight vectors. Now for $P = B$, the sections cutting out the $2|I|$ divisor components D_i and D^i have T-weights $\varpi_1, \dots, \varpi_n$ and $w_0\varpi_1, \dots, w_0\varpi_n$. The sum of these weights is zero, so the form $\omega_{G/B}$ must be T-invariant. Each open Richardson variety $\mathcal{R}_v^w$ has its own canonical form $\omega_{\mathcal{R}_v^w}$ which is obtained from $\omega_{G/B}$ by taking residues, so again these forms are T-invariant. Finally, for each parabolic P, the projection map $p : G/B \to G/P$ induces an isomorphism of $\mathcal{R}_{w_0^P}^{w_0}$ onto its image $\mathring{G/P}$. Since p is T-equivariant, the result follows. □

6.27. Convention for affine Weyl groups
Let $w\tau^\lambda \in W_{\mathrm{af}} = W \ltimes P^\vee$ denote an element of the affine Weyl group, and let δ denote the null root of the affine root system. Then for $\mu \in P$,

$$w\tau^\lambda \cdot (\mu + n\delta) = w\mu + \big(n - \langle \mu, \lambda \rangle\big)\delta. \tag{6.27.1}$$

6.28. Cominuscule case
We now assume that G is simple and of adjoint type. Fix a cominuscule node $\mathfrak{i}$ of G, which is also a minuscule node of $G^\vee$. Let $P = P_{\mathfrak{i}}$ be the corresponding (maximal) parabolic, and identify $Z(L_P)$ with $\mathbb{G}_m \cong \mathbb{P}^1 \backslash \{0, \infty\}$ via the simple root $\alpha_{\mathfrak{i}}$.

LEMMA 6.29
If P is a cominuscule parabolic, then the composition of $(2\rho^\vee - 2\rho_P^\vee) : \mathbb{G}_m \to Z(L_P)$ with $\alpha_\mathrm{i} : Z(L_P) \cong \mathbb{G}_m$, is the character $q \mapsto q^c$, where c is the Coxeter number of G.

Proof
We have $\rho_G^\vee - \rho_P^\vee = \rho_{G^\vee} - \rho_{P^\vee}$ for the dual minuscule parabolic group $P^\vee$ of $G^\vee$. Since α_i is a simple coroot of $G^\vee$, it follows from Lemma 4.8 that $\langle 2(\rho_G^\vee - \rho_P^\vee), \alpha_\mathrm{i}\rangle = c$, where c is the Coxeter number of $G^\vee$ which is also the Coxeter number of G. ☐

Let Ω be the quotient of the coweight lattice of G by the coroot lattice. Thus Ω is isomorphic to the center of $G^\vee$. Let $\kappa \in \Omega$ be the element corresponding to the cominuscule node i. Namely, $\kappa \equiv -\varpi_\mathrm{i}^\vee$ under this identification (see [87, Section 11.2]); we have $\kappa = \tau^{-\varpi_\mathrm{i}^\vee} w_P$. For a coweight λ of G, we abuse notation by letting $\tau^\lambda \in G((\tau))$ denote the corresponding element, which is a lift of the translation element $\tau^\lambda \in W_{\mathrm{af}}$. Our choice is uniquely determined by the following property: $\tau^\lambda U_\alpha \tau^{-\lambda} = U_{\tau^\lambda \cdot \alpha}$, where $U_\alpha \subset G((\tau))$ denotes the one-parameter subgroup indexed by the affine root α.

Let $\dot\kappa = \tau^{-\varpi_\mathrm{i}^\vee} \dot w_P = \dot w_P \tau^{\varpi_{\mathrm{i}^\star}^\vee}$. Then $\dot\kappa \in G((\tau))$ is a lift of κ to the loop group. Note that $\dot\kappa|_{\tau^{-1}=1} = \dot w_P$. Let $G[\tau^{-1}]_1 := \ker(G[\tau^{-1}] \xrightarrow{\mathrm{ev}_\infty} G)$, where ev_∞ is given by $\tau^{-1} = 0$.

LEMMA 6.30
(a) *Let $\alpha \in R$. Then*

$$\kappa^{-1} \cdot \alpha \in \begin{cases} R - \delta & \text{for } \alpha \in R^+ \setminus R_P^+, \\ R + \mathbb{Z}_{\geq 0}\delta & \text{for } \alpha \notin R^+ \setminus R_P^+. \end{cases}$$

(b) *For $u \in U_P$, we have $\dot\kappa^{-1} u \dot\kappa \in G[\tau^{-1}]_1$.*
(c) *We have $w_P^{-1}\alpha_\mathrm{i} = -\theta$ and $\kappa^{-1}\alpha_\mathrm{i} = -\theta - \delta$.*

Proof
(a) We first note that $\mathrm{Inv}(w_P^{-1}) = R^+ \setminus R_P^+$ and $w_P = w_{P^\star}^{-1}$ (see Section 6.8). Thus w_P^{-1} acts as a bijection from $R^+ \setminus R_P^+$ to $-(R^+ \setminus R_{P^\star}^+)$. In particular, $w_P^{-1} \cdot R^+ \setminus R_P^+ = -(R^+ \setminus R_{P^\star}^+)$. For $\alpha \in R$, we compute by (6.27.1)

$$\kappa^{-1} \cdot \alpha = \tau^{-\varpi_{\mathrm{i}^\star}^\vee} \cdot w_P^{-1} \cdot \alpha = \tau^{-\varpi_{\mathrm{i}^\star}^\vee} \cdot w_P^{-1}(\alpha) = w_P^{-1}\alpha - \langle w_P^{-1}\alpha, -\varpi_{\mathrm{i}^\star}^\vee\rangle\delta, \quad (6.30.1)$$

where δ, the null root of the affine root system, is the weight of τ. If $\alpha \in R^+ \setminus R_P^+$, then (6.30.1) shows that $\kappa^{-1} \cdot \alpha \in R_- - \delta$, using that i cominuscule implies $\mathrm{i}^\star$ cominuscule implies $\langle \beta, \varpi_{\mathrm{i}^\star}^\vee\rangle = 1$ for all $\beta \in R^+ \setminus R_{P^\star}^+$. If $\alpha \notin R^+ \setminus R_P^+$, then $\langle w_P^{-1}\alpha, -\varpi_{\mathrm{i}^\star}^\vee\rangle \geq 0$ and $\kappa^{-1} \cdot \alpha \in R + \mathbb{Z}_{\geq 0}\delta$. This proves (a). Statement (b) follows immediately from (a).

We prove (c). Since w_P^{-1} sends $R^+ \setminus R_P^+$ to $-(R^+ \setminus R_{P^\star}^+)$, we see that $-\theta \in w_P^{-1}(R^+ \setminus R_P^+)$. Since i is cominuscule, every root $\beta \in R^+ \setminus R_P^+$ is of the form $\beta = \alpha_\mathrm{i}$ mod $\sum_{j\in I_P} \mathbb{Z}_{\geq 0}\alpha_j$. But $w_P^{-1}\alpha_j > 0$ for all $j \in I_P$, so $w_P^{-1}(\alpha_\mathrm{i} + \sum_{j\in I_P} \mathbb{Z}_{\geq 0}\alpha_j) \subset w_P^{-1}(\alpha_\mathrm{i}) + \sum_{j\in I} \mathbb{Z}_{\geq 0}\alpha_j$. We deduce that $w_P^{-1}\alpha_\mathrm{i} = -\theta$. The second statement follows from (6.30.1). □

Thus we obtain an inclusion

$$\iota_t : X_t \longrightarrow G[\tau^{-1}]_1 \tag{6.30.2}$$

$$x = u_1 t \dot{w}_P u_2 \longmapsto \dot{\kappa}^{-1} t^{-1} u_1 t \dot{\kappa} \in G[\tau^{-1}]_1, \tag{6.30.3}$$

where $u_1 \in U_P$ and $u_2 \in U^{w_P^{-1}}$.

6.31. Embedding the geometric crystal into the affine Grassmannian
We interpret the inclusion ι_t via the affine Grassmannian.

Let $\mathrm{Gr} = G((\tau))/G[[\tau]]$ denote the affine Grassmannian of G. The connected components Gr^κ of Gr are indexed by $\kappa \in \Omega$. For a dominant weight λ, let $\mathrm{Gr}_\lambda := G[[\tau]]\tau^{-\lambda} \subset \mathrm{Gr}$ denote the $G[[\tau]]$-orbit. The closure of Gr_λ is a spherical Schubert variety inside Gr. (The minus sign in $\tau^{-\lambda}$ is chosen to match the convention (6.27.1) and our choice of τ^λ.)

For $\lambda = \varpi_\mathrm{i}^\vee$, we have that $\mathrm{Gr}_{\varpi_\mathrm{i}^\vee} \cong G/P$ is already closed in Gr. Indeed, the map $G \to \mathrm{Gr}$ given by $g \mapsto g\tau^{\varpi_{\mathrm{i}^\star}^\vee} \mod G[[\tau]]$ has stabilizer $P^\star$, giving a closed embedding $G/P^\star \cong \mathrm{Gr}_{\varpi_\mathrm{i}^\vee} \hookrightarrow \mathrm{Gr}^\kappa$. Note that $\varpi_{\mathrm{i}^\star}^\vee \in W \cdot (-\varpi_\mathrm{i}^\vee)$, so $\tau^{\varpi_{\mathrm{i}^\star}^\vee} \in G[[\tau]] \cdot \tau^{-\varpi_\mathrm{i}^\vee}$, and $G/P \cong G/P^\star$.

Since $\dot{w}_P^{-1} U_P \dot{w}_P \cap P^\star = \{e\}$, we have an inclusion $X_t \hookrightarrow G/P^\star$ given by $x = u_1 t \dot{w}_P u_2 \mapsto t^{-1} u_1^{-1} t \dot{w}_P \mod P^\star$, where $u_1 \in U_P$ and $u_2 \in U^{w_P^{-1}}$. Composed with the isomorphism $G/P^\star \xrightarrow{\sim} \mathrm{Gr}_{\varpi_\mathrm{i}^\vee}$, we obtain an inclusion

$$X_t \longrightarrow \mathrm{Gr}_{\varpi_\mathrm{i}^\vee}, \tag{6.31.1}$$

$$x = u_1 t \dot{w}_P u_2 \longmapsto t^{-1} u_1 t \dot{w}_P \tau^{\varpi_{\mathrm{i}^\star}^\vee} = t^{-1} u_1 t \cdot \dot{\kappa} = \dot{\kappa}\, \iota_t(x). \tag{6.31.2}$$

7. Heinloth, Ngô, and Yun's Kloosterman D-module

While the article [71] works over a finite field, we will work over the complex numbers $\mathbb{C}$ (cf. [71, Section 2.6]). In this section, we assume that G is simple and of adjoint type.

7.1. A group scheme over $\mathbb{P}^1$
Take t to be the coordinate on $\mathbb{P}^1$, and set $s = t^{-1}$. Let

$$I(0) = I_\infty(0) := \{g \in G[[s]] \mid g(0) \in B\},$$
$$I(1) = I_\infty(1) := \{g \in G[[s]] \mid g(0) \in U\}.$$

Here $G[[s]]$ is a shorthand for $G(\mathbb{C}[[s]])$. Similarly, define

$$I_0^{\mathrm{opp}}(0) := \{g \in G[[t]] \mid g(0) \in B_-\},$$
$$I_0^{\mathrm{opp}}(1) := \{g \in G[[t]] \mid g(0) \in U_-\}.$$

Also let $I_\infty(2) = I(2) = [I(1), I(1)]$ be the commutator subgroup of $I(1)$. One verifies that the Lie algebra of $I(1)/I(2)$ has weights given by the set I_{af} of simple affine roots, and in particular our definition of $I(2)$ agrees with that in [71, Section 1.2]. Via the exponential map, we obtain an isomorphism

$$I(1)/I(2) \cong \bigoplus_{i \in I_{\mathrm{af}}} \mathbb{A}^1.$$

Let $\phi : I(1)/I(2) \to \mathbb{A}^1$ be the standard affine character. Precisely, we fix root vectors $x_i = x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and define ϕ by $\phi(\exp(t x_i)) = -t$ for all $i \in I_{\mathrm{af}}$. The choice of x_i for $i \in I$ is already fixed in Section 2.2. Since $\mathfrak{g}_{\alpha_0}$ can be identified with $s\mathfrak{g}_{-\theta}$, the choice of $x_0 \in \mathfrak{g}_{\alpha_0}$ is equivalent to a choice of $x_{-\theta} \in \mathfrak{g}_{-\theta}$. The choice of $x_{-\theta}$ satisfying the compatibilities (1) and (2) of Section 2.2 is equivalent to a choice of a sign, which will be fixed in (7.8.2). We have that ϕ is a *generic affine character*, meaning that it takes nonzero values on each $\exp(x_i)$, for $i \in I_{\mathrm{af}}$.

Denote by $\mathcal{G}(1,2)$ the group scheme over $\mathbb{P}^1$ in [71, Section 1.2], satisfying

$$\mathcal{G}(1,2)|_{\mathbb{P}^1 \backslash \{0,\infty\}} \cong G \times \mathbb{P}^1 \backslash \{0,\infty\},$$
$$\mathcal{G}(1,2)(\mathcal{O}_0) = I_0^{\mathrm{opp}}(1),$$
$$\mathcal{G}(1,2)(\mathcal{O}_\infty) = I(2).$$

Here $\mathcal{O}_0 \cong \mathbb{C}[[s]]$ (resp., $\mathcal{O}_\infty \cong \mathbb{C}[[t]]$) is the completed local ring at 0 (resp., ∞). The group scheme $\mathcal{G}(1,2)$ is constructed by *dilatation* (see [14, Section 3.2]). First, the group scheme $\mathcal{G}(1,1)$ is the dilatation of the constant group scheme $G \times \mathbb{P}^1$ along $U_- \times \{0\} \subset G \times \{0\}$ and along $U \times \{\infty\} \subset G \times \{\infty\}$. Then $\mathcal{G}(1,2)$ is the dilatation of $\mathcal{G}(1,1)$ which is an isomorphism away from ∞ and at ∞ induces $\mathcal{G}(1,2)(\mathcal{O}_\infty) = I(2) \subset I(1) = \mathcal{G}(1,1)(\mathcal{O}_\infty)$.

7.2. Hecke modifications

Recall that Ω is the quotient of the coweight lattice of G by the coroot lattice. Let $\mathrm{Bun}_{\mathcal{G}} = \mathrm{Bun}_{\mathcal{G}(1,2)}$ denote the moduli stack of $\mathcal{G}(1,2)$-bundles on $\mathbb{P}^1$ defined in [71, Section 1.4]. The stack $\mathrm{Bun}_{\mathcal{G}}$ is the union $\mathrm{Bun}_{\mathcal{G}} = \bigsqcup_{\kappa\in\Omega} \mathrm{Bun}_{\mathcal{G}}^{\kappa}$ of connected components indexed by $\kappa \in \Omega$. We let $\star_\kappa$ denote the basepoint of $\mathrm{Bun}_{\mathcal{G}}^{\kappa}$. Under the isomorphism $\mathrm{Bun}_{\mathcal{G}}^{0} \cong \mathrm{Bun}_{\mathcal{G}}^{\kappa}$ of [71, Corollary 1.2], the basepoint $\star_\kappa$ is the image of the point corresponding to the trivial bundle $\star$.

The stack of Hecke modifications is the stack which on a $\mathbb{C}$-scheme S takes value the groupoid

$$\mathrm{Hecke}_{\mathcal{G}}(S) := \big\{(\mathcal{E}_1, \mathcal{E}_2, x, \phi) \bigm| \mathcal{E}_i \in \mathrm{Bun}_{\mathcal{G}}(S), x : S \to \mathbb{P}^1\backslash\{0,\infty\},$$
$$\phi : \mathcal{E}_1|_{(\mathbb{P}^1\backslash\{x\})\times S} \xrightarrow{\cong} \mathcal{E}_2|_{(\mathbb{P}^1\backslash\{x\})\times S}\big\}.$$

It has two natural forgetful maps

$$\begin{array}{ccc} & \mathrm{Hecke}_{\mathcal{G}} & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathrm{Bun}_{\mathcal{G}} & & \mathrm{Bun}_{\mathcal{G}} \times \mathbb{P}^1\backslash\{0,\infty\} \end{array} \tag{7.2.1}$$

given by $\mathrm{pr}_1(\mathcal{E}_1, \mathcal{E}_2, x, \phi) = \mathcal{E}_1$ and $\mathrm{pr}_2(\mathcal{E}_1, \mathcal{E}_2, x, \phi) = (\mathcal{E}_2, x)$. The geometric fibers of pr_2 over $\mathrm{Bun}_{\mathcal{G}} \times \mathbb{P}^1\backslash\{0,\infty\}$ are isomorphic to the affine Grassmannian $\mathrm{Gr}_G = G((\tau))/G[[\tau]]$, where τ is a local coordinate at x.

Let λ be an integral coweight of G, and assume that Gr_λ lies in the κ-component of Gr. The $G[[\tau]]$-orbits Gr_λ (and their closures $\overline{\mathrm{Gr}_\lambda}$) in Gr_G define substacks $\mathrm{Hecke}_\lambda \subset \mathrm{Hecke}_{\mathcal{G}}$ (and $\overline{\mathrm{Hecke}_\lambda}$); see [71, p. 259].

7.3. Parameterization

Assume now that (G, P) are as in Section 6.28, that is, G is simple and of adjoint type, and $P = P_{\mathrm{i}}$ is the maximal parabolic subgroup corresponding to the cominuscule node i. In particular, $Z(L_P)$ is 1-dimensional. We now fix the isomorphism

$$\alpha_{\mathrm{i}} : Z(L_P) \cong \mathbb{P}^1\backslash\{0,\infty\} \qquad z \mapsto \alpha_{\mathrm{i}}(z). \tag{7.3.1}$$

Via (7.3.1), we may use "t" both as a coordinate on $\mathbb{P}^1$ and as a coordinate on $Z(L_P)$.

We follow [71, Section 5.2] in the remainder of this subsection. Let Hk be the restriction of $\mathrm{pr}_2 : \mathrm{Hecke}_{\mathcal{G}} \to \mathrm{Bun}_{\mathcal{G}} \times \mathbb{P}^1\backslash\{0,\infty\}$ to $\star_\kappa \times \mathbb{P}^1\backslash\{0,\infty\} \subset \mathrm{Bun}_{\mathcal{G}} \times \mathbb{P}^1\backslash\{0,\infty\}$ and for $q \in \mathbb{P}^1\backslash\{0,\infty\}$, let Hk_q denote the restriction to $\star_\kappa \times \{q\}$.

By [71, Corollary 1.3], $\mathrm{Bun}_{\mathcal{G}}^{\kappa}$ contains an affine open substack isomorphic to $T \times I(1)/I(2)$, called the *big cell*. Let $\mathrm{Hk}^\circ \subset \mathrm{Hk}$ denote the inverse image of the big cell $T \times I(1)/I(2) \subset \mathrm{Bun}_{\mathcal{G}}^{0}$ under pr_1, and similarly define Hk_q°. Denote the map

$$\mathrm{Hk}^\circ \to T \times I(1)/I(2) \simeq T \times U_{-\theta} \times U/[U,U]$$

by the following triple:

$$(f_T, f_0, f_+) : \mathrm{Hk}^\circ \longrightarrow T \times U_{-\theta} \times U/[U,U].$$

Our aim is to parameterize Hk° and compute f_T, f_0, f_+.

Let $\mathcal{E}_0 = G \times \mathbb{P}^1$ be the trivial bundle, and let $\mathcal{E}_\kappa$ be the $\mathcal{G}$-bundle corresponding to the basepoint $\star_\kappa \in \mathrm{Bun}_{\mathcal{G}}^\kappa$. The bundle $\mathcal{E}_\kappa$ is obtained by gluing the trivial bundle on $\mathbb{P}^1 \backslash \{\infty\}$ with the trivial bundle on the formal disk around ∞ via the transition function $\kappa(t^{-1}) = \tau^{-\varpi_i^\vee} \dot{w}_P$. We fix once and for all trivializations of $\mathcal{E}_0$ over $\mathbb{P}^1$ and of $\mathcal{E}_\kappa$ over $\mathbb{P}^1 \backslash \{\infty\}$.

We use the local parameter $\tau = 1 - t/q$ at q. Thus $\tau = 0, 1, \infty$ (or $\tau^{-1} = \infty, 1, 0$) corresponds to $t = q, 0, \infty$, respectively. Let

$$\kappa(\tau^{-1}) = \dot{\kappa} = \tau^{-\varpi_i^\vee} \dot{w}_P \in G[\tau, \tau^{-1}], \quad \text{so that } \kappa(\tau^{-1})|_{\tau^{-1}=1} = \dot{w}_P. \tag{7.3.2}$$

We view $\kappa(\tau^{-1})$ as an isomorphism

$$\kappa(\tau^{-1}) : \mathcal{E}_0|_{\mathbb{P}^1 \backslash \{q,\infty\}} \longrightarrow \mathcal{E}_\kappa|_{\mathbb{P}^1 \backslash \{q,\infty\}},$$

using the trivializations of $\mathcal{E}_0$ and $\mathcal{E}_\kappa$ over $\mathbb{P}^1 \backslash \{\infty\}$. Since

$$\tau^{-1} = -qt^{-1} + O\big((t^{-1})^2\big), \tag{7.3.3}$$

the Laurent expansions of $\kappa(\tau^{-1})$ and $\kappa(t^{-1})$ in t^{-1} differ by an element of $G[[t^{-1}]]$. Thus $\kappa(\tau^{-1})$ extends to an isomorphism

$$\kappa(\tau^{-1}) : \mathcal{E}_0|_{\mathbb{P}^1 \backslash \{q\}} \longrightarrow \mathcal{E}_\kappa|_{\mathbb{P}^1 \backslash \{q\}}. \tag{7.3.4}$$

Any point in Hk_q° can be obtained by precomposing $\kappa(\tau^{-1})$ with an element of

$$\mathrm{Aut}(\mathcal{E}_0|_{\mathbb{P}^1 \backslash \{q\}}) \cong G[\tau^{-1}].$$

From the definition of $\mathcal{G}$, a bundle $\mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}$ is equipped with *level structures* at 0 and at ∞. Let $\kappa(\tau^{-1}) g(\tau^{-1})$ represent a point $\mathcal{E} \in \mathrm{Hk}_q^\circ$ under our parameterization. We define the level structure (at 0 and at ∞) associated to $\mathcal{E}$ to be the pair

$$\big(\mathrm{ev}_{t=0}\big[\kappa(\tau^{-1}) g(\tau^{-1})\big)^{-1}\big], \mathrm{ev}_{t=\infty}\big[g(\tau^{-1})^{-1}\big]\big) = \big(g(1)^{-1} \dot{w}_P^{-1}, g(0)^{-1}\big) \in G \times G$$

of elements of G. (The isomorphism (7.3.4) preserves the level structure at ∞, and $\star_\kappa$ has the trivial level structure at ∞, hence the formula $\mathrm{ev}_{t=\infty}[g(\tau^{-1})^{-1}]$ for the level structure at ∞.)

The big cell $T \times I(1)/I(2) \subset \mathrm{Bun}_{\mathcal{G}}^0$ is the orbit of $\mathcal{E}_0$ under the action of the group $I_0^{\mathrm{opp}}(0) \times I(1)$ (recall that $T \cong I_0^{\mathrm{opp}}(0)/I_0^{\mathrm{opp}}(1)$, $I(1) = I_\infty(1)$, and $I(2) = I_\infty(2)$). It follows that $\kappa(\tau^{-1}) g(\tau^{-1})$ projects under pr_1 to the big cell $\mathrm{Bun}_{\mathcal{G}}^0$ if and only if

$$g(0)^{-1} \in U \Leftrightarrow g(0) \in U,$$

$$g(1)^{-1} \dot{w}_P^{-1} \in B_- \Leftrightarrow \dot{w}_P g(1) \in B_-.$$

We have a natural evaluation map $\mathrm{ev}_q : \mathrm{Hk}_q^\circ \to \mathrm{Gr}_q$ given by considering $\kappa(\tau^{-1})g(\tau^{-1})$ as an element of $G((\tau))/G[[\tau]] \cong \mathrm{Gr}_q$. The image $\mathrm{ev}_q(\mathrm{Hk}_q^\circ)$ is denoted Gr_q°. We may further rigidify the moduli problem by precomposing with an element of $\mathrm{Aut}(\mathcal{E}_0) = \mathrm{Aut}(G \times \mathbb{P}^1) = G$ to obtain an isomorphism

$$\kappa(\tau^{-1})g(\tau^{-1})g(0)^{-1} : \mathcal{E}_0|_{\mathbb{P}^1\backslash\{q\}} \longrightarrow \mathcal{E}_\kappa|_{\mathbb{P}^1\backslash\{q\}},$$

which is the identity at ∞. Set $h(\tau^{-1}) := g(\tau^{-1})g(0)^{-1} \in G[\tau^{-1}]_1$, where we recall that $G[\tau^{-1}]_1 = \ker(G[\tau^{-1}] \xrightarrow{\tau^{-1}=0} G)$. This gives the parameterization

$$\mathrm{Hk}_q^\circ \cong \left\{h(\tau^{-1}) \in G[\tau^{-1}]_1 \;\middle|\; h(1) \in \dot{w}_P^{-1} B_- U\right\}.$$

Varying q, this gives the parameterization

$$\mathrm{Hk}^\circ \cong \left\{h(\tau^{-1}) \in G[\tau^{-1}]_1 \;\middle|\; h(1) \in \dot{w}_P^{-1} B_- U\right\} \times \mathbb{P}^1\backslash\{0,\infty\}. \tag{7.3.5}$$

Under this parameterization, the image of $h(\tau^{-1})$ in $\mathrm{Gr}_q \cong G((\tau))/G[[\tau]]$ is equal to $\dot{\kappa}h(\tau^{-1})$.

LEMMA 7.4
Under the parameterization (7.3.5), write $h(1) = \dot{w}_P^{-1} b_- u$ for $u \in U$ and $b_- \in B_-$. Then we have

$$\begin{aligned}
f_T(h,q) &= b_-^{-1} \mod U_- \in B_-/U_- \cong T,\\
f_+(h,q) &= u \mod [U,U] \in U/[U,U],\\
f_0(h,q) &= q a_{-\theta}(h) \in U_{-\theta} \cong \mathfrak{g}_{-\theta},
\end{aligned}$$

where $a_{-\theta}(h)$ denotes the $\mathfrak{g}_{-\theta}$-part of the tangent vector $.\frac{dh(\tau^{-1})}{d(\tau^{-1})}|_{\tau^{-1}=0} \in \mathfrak{g}$.

Proof
The formulas for f_T and f_+ follow from the parameterization (7.3.5). The function $f_0(h,q)$ is obtained by expanding $h(\tau^{-1})^{-1}$ at $t = \infty$ using the local parameter t^{-1}. By (7.3.3), we have

$$\begin{aligned}
\frac{dh(\tau^{-1})^{-1}}{d(t^{-1})}\Big|_{t^{-1}=0} &= \frac{d\tau^{-1}}{d(t^{-1})}\frac{dh(\tau^{-1})^{-1}}{d(\tau^{-1})}\Big|_{\tau^{-1}=0} = -q\frac{dh(\tau^{-1})^{-1}}{d(\tau^{-1})}\Big|_{\tau^{-1}=0}\\
&= q\frac{dh(\tau^{-1})}{d(\tau^{-1})}\Big|_{\tau^{-1}=0} \in \mathfrak{g},
\end{aligned}$$

where for the last equality we have used the condition $h(0) = 1 \in G$: if $h(\tau^{-1})^{-1} = 1 + h_1\tau^{-1} + O(\tau^{-2})$, then $h(\tau^{-1}) = 1 - h_1\tau^{-1} + O(\tau^{-2})$. ☐

7.5. Kloosterman D-module

Let λ be an integral coweight for G. Let $\overline{\mathrm{Hk}}_\lambda$ be the restriction of $\overline{\mathrm{Hecke}}_\lambda$ to $\star_\kappa \times \mathbb{P}^1\backslash\{0,\infty\}$. Define $\mathrm{Hk}^\circ_\lambda$ and $\mathrm{Hk}^\circ_{q,\lambda}$ by intersecting with Hk° and Hk°_q the substack $\mathrm{Hecke}_\lambda \subset \mathrm{Hecke}_{\mathcal{G}}$.

Let $\mathcal{O}_\lambda$ denote the structure sheaf of Gr_λ, considered as a D_{Gr_λ}-module. Denote the minimal extension of $\mathcal{O}_\lambda$ under the inclusion $j : \mathrm{Gr}_\lambda \hookrightarrow \overline{\mathrm{Gr}}_\lambda$ by D_λ. Abusing notation, also denote by D_λ the holonomic D-module on $\overline{\mathrm{Hk}}_\lambda$ obtained via the isomorphism $\overline{\mathrm{Hk}}_\lambda \cong \overline{\mathrm{Gr}}_\lambda$. We consider the following diagram, where we recall that the maps f_+, f_0, pr_2 have been defined at the beginning of Section 7.3:

$$\begin{array}{ccc} & \mathrm{Hk}^\circ_\lambda \hookrightarrow \overline{\mathrm{Hk}}_\lambda & \\ {\scriptstyle (f_+,f_0)} \swarrow & & \searrow {\scriptstyle \mathrm{pr}_2} \\ U/[U,U] \times U_{-\theta} & & \star_\kappa \times \mathbb{P}^1\backslash\{0,\infty\} \end{array} \tag{7.5.1}$$

We write

$$(\phi_+,\phi_0) : U/[U,U] \times U_{-\theta} \cong I(1)/I(2) \xrightarrow{\phi} \mathbb{A}^1.$$

Recall our D-module conventions from Remark 6.20, and recall that $\mathsf{E} = D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial_x - 1)$ denotes the exponential D-module on $\mathbb{A}^1$. We write $\mathsf{E}^{\phi_+} := \phi_+^*\mathsf{E}$ and $\mathsf{E}^{\phi_0} := \phi_0^*\mathsf{E}$. Define (see [71, (5.8)]) the *Kloosterman D-module* by

$$\mathrm{Kl}_{(G^\vee,\lambda)} := \mathrm{pr}_{2,!}(f_+^*\mathsf{E}^{\phi_+} \otimes f_0^*\mathsf{E}^{\phi_0} \otimes D_\lambda). \tag{7.5.2}$$

Remark 7.6

In [71, Theorem 1], there is another definition of $\mathrm{Kl}_{(G^\vee,\lambda)}$ as the λ-Hecke eigenvalue of an automorphic D-module $A_{\mathcal{G}}$. We shall discuss $A_{\mathcal{G}}$ in Section 9 below. In view of the results of [71, Section 5.2], the two definitions agree.

7.7. Comparison

The inclusion $\iota_t : X_t \to G[\tau^{-1}]_1$ of (6.30.2) can be extended to an inclusion

$$\tilde{\iota} = (\iota,\pi) : X \longrightarrow G[\tau^{-1}] \times \mathbb{P}^1\backslash\{0,\infty\},$$

where for $x = u_1 t \dot{w}_P u_2$ with $u_1 \in U_P$ and $u_2 \in U^{w_P^{-1}}$, we have via (7.3.1),

$$\begin{aligned} \iota(x) &= \iota_t(x) = \dot{\kappa}^{-1} t^{-1} u_1 t \dot{\kappa} \in G[\tau^{-1}] \qquad \text{and} \\ \pi(x) &= \alpha_{\mathrm{i}}(t) \in \mathbb{P}^1\backslash\{0,\infty\}. \end{aligned} \tag{7.7.1}$$

LEMMA 7.8

Under the identification (7.3.5), we have an isomorphism $\tilde{\iota} : X \cong \mathrm{Hk}^\circ_{\varpi_{\mathrm{i}}^\vee}$.

Proof

Fix $t \in Z(L_P)$, and let $q = \alpha_{\mathrm{i}}(t) \in \mathbb{P}^1 \backslash \{0, \infty\}$. We show that $\iota_t : X_t \cong \mathrm{Hk}^\circ_{q,\varpi_{\mathrm{i}}^\vee} = \mathrm{Hk}^\circ_q \cap \mathrm{Hk}_\lambda$. (In this proof, we explicitly distinguish t and q for clarity.)

Let $x = u_1 t \dot{w}_P u_2 \in X$. Then by (7.3.2), we have

$$\dot{w}_P (\dot{\kappa}^{-1} t^{-1} u_1 t \dot{\kappa})|_{\tau^{-1}=1} = t^{-1} u_1 t \dot{w}_P = (t^{-1} x)(u_2^{-1}) \in B_- U, \tag{7.8.1}$$

so $\iota(x) \in \mathrm{Hk}^\circ_q$. The map $x \mapsto \dot{\kappa}\iota_t(x)$ is an inclusion $X_t \hookrightarrow \mathrm{Gr}_{\varpi_{\mathrm{i}}^\vee}$ (see (6.31.1)). It follows that $\iota_t(X_t) \subset \mathrm{Hk}^\circ_{q,\varpi_{\mathrm{i}}^\vee}$ and the map ι_t is injective.

Under our identification (7.3.5), any element of $\mathrm{Hk}_{q,\varpi_{\mathrm{i}}^\vee}$ is represented by $h(\tau^{-1}) = \dot{\kappa}^{-1} g \dot{\kappa}$, where $g \in G$ is constant. It follows from Lemma 6.30(a) that $h(\tau^{-1}) \in G[\tau^{-1}]_1$ is equivalent to $g \in U_P$. The condition that $h(1) \in \dot{w}_P^{-1} B_- U$ is then equivalent to $h(\tau^{-1}) \in \iota_t(X_t)$. Thus, ι_t is surjective. We are done. □

By Lemma 6.30(c), we have $w_P^{-1}\alpha_{\mathrm{i}} = -\theta$. Henceforth, we assume that

$$x_{-\theta} \in \mathfrak{g}_{-\theta} \text{ is equal to } -\dot{w}_P^{-1} x_{\mathrm{i}} \in \mathfrak{g}_{-\theta}. \tag{7.8.2}$$

Note that the choice (7.8.2) is independent of (3.9.1), which in the notation of this section is an assumption on the root vectors of $\mathfrak{g}^\vee$ (rather than $\mathfrak{g}$).

PROPOSITION 7.9

We have

$$\begin{aligned} f_T\big(\tilde{\iota}(x)\big) &= t\gamma(x)^{-1} \in B_-/U_- \cong T, \\ f_+\big(\tilde{\iota}(x)\big) &= u_2^{-1} \mod [U,U] \in U/[U,U], \\ f_0\big(\tilde{\iota}(x)\big) &= -\psi(u_1) x_{-\theta} \in U_{-\theta} \cong \mathfrak{g}_{-\theta}, \end{aligned}$$

where $\psi : U \to \mathbb{A}^1$ denotes the standard additive character from Section 6.4.

Proof

Let $\iota(x) = h(\tau^{-1})$. Then by (7.8.1), we have

$$\dot{w}_P h(1) = (t^{-1}x)(u_2^{-1}).$$

By Lemma 7.4 and the definition (6.4.1) of the weight map γ, we have

$$f_T\big(\tilde{\iota}(x)\big) = x^{-1}t \mod U_- = \gamma(x)^{-1} t = t\gamma(x)^{-1} \in T$$

and $f_+(\tilde{\iota}(x)) = u_2^{-1} \mod [U,U]$.

It remains to compute $f_0(\tilde{\iota}(x))$. Let $\mathfrak{u}_P = \mathrm{Lie}(U_P)$. Then $\mathfrak{u}_P = \bigoplus_{\alpha \in R^+ \backslash R_P^+} \mathfrak{g}_\alpha$. Since i is cominuscule, α_{i} occurs in every $\alpha \in R^+ \setminus R_P^+$ with coefficient 1. It follows that $\alpha+\beta$ is never a root for $\alpha,\beta\in R^+\setminus R_P^+$. In particular, $\mathfrak{u}_P$ is an abelian Lie algebra and U_P is an abelian algebraic group, and so is $\dot{\kappa}^{-1}U_P\dot{\kappa}$. Let $\exp:\mathfrak{u}_P\to U_P$ be the exponential map, which is an isomorphism since U_P is unipotent.

Now, $\dot{\kappa}^{-1}\mathfrak{g}_\alpha\kappa$ can be identified with the root space $\hat{\mathfrak{g}}_{\kappa^{-1}\cdot\alpha}$, where $\hat{\mathfrak{g}}=\mathfrak{g}[\tau^{\pm 1}]$ is the loop algebra, and $\kappa^{-1}\cdot\alpha$ is now an affine root. By Lemma 6.30(c), we have $w_P^{-1}\alpha_{\mathrm{i}}=-\theta$, and $\kappa^{-1}\alpha_{\mathrm{i}}=-\theta-\delta$, where δ denotes the null root. It follows that

$$a_{-\theta}(h)=\dot{w}_P^{-1}\cdot\exp^{-1}(u)_{\alpha_{\mathrm{i}}}=-\psi(u)x_{-\theta}, \tag{7.9.1}$$

where $h(\tau^{-1})=\dot{\kappa}^{-1}u\dot{\kappa}\in\dot{\kappa}^{-1}U_P\dot{\kappa}$ for $u\in U_P$, and $\exp^{-1}(u)_{\alpha_{\mathrm{i}}}$ denotes the component of $\exp^{-1}(u)\in\mathfrak{u}_P$ lying in the root space $\mathfrak{g}_{\alpha_{\mathrm{i}}}$.

We compute

$$\begin{aligned} f_0\big(\tilde{\iota}(x)\big)&=qa_{-\theta}(h)\quad\text{(by Lemma 7.4)}\\ &=-\alpha_{\mathrm{i}}(t)\psi(t^{-1}u_1t)x_{-\theta}\quad\text{(by (7.3.1), (7.7.1), and (7.9.1))}\\ &=-\alpha_{\mathrm{i}}(t)\alpha_{\mathrm{i}}(t^{-1})\psi(u_1)x_{-\theta}=-\psi(u_1)x_{-\theta}\in\mathfrak{g}_{-\theta},\end{aligned}$$

as claimed. □

We can now prove the main result of this section.

THEOREM 7.10
The character D-module $\mathrm{Cr}_{(G,P)}$ is a flat connection (smooth and concentrated in one degree) and is isomorphic to the Kloosterman D-module $\mathrm{Kl}_{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$.

Proof
Recall (see (7.5.2)) that by definition,

$$\mathrm{Kl}_{(G^\vee,\varpi_{\mathrm{i}}^\vee)}:=\mathrm{pr}_{2,!}(f_+^*\mathsf{E}^{\phi_+}\otimes f_0^*\mathsf{E}^{\phi_0}\otimes D_{\varpi_{\mathrm{i}}^\vee}).$$

The translation element $\tau^{\varpi_{\mathrm{i}}^\vee}$ is minimal in the Bruhat order of W_{af}/W. Thus $\mathrm{Gr}_{\varpi_{\mathrm{i}}^\vee}=\overline{\mathrm{Gr}}_{\varpi_{\mathrm{i}}^\vee}$, and we have $D_{\varpi_{\mathrm{i}}^\vee}\cong\mathcal{O}_{\mathrm{Hk}_{\varpi_{\mathrm{i}}^\vee}}$.

Thus we may restrict ourselves to considering the diagram

$$\begin{array}{ccc} & \mathrm{Hk}^\circ_{\varpi_{\mathrm{i}}^\vee} & \\ {}^{(f_+,f_0)}\swarrow & & \searrow^{\mathrm{pr}_2} \\ U/[U,U]\times U_{-\theta} & & \star_\kappa\times\mathbb{P}^1\setminus\{0,\infty\} \end{array} \tag{7.10.1}$$

Recall that $U_{-\theta}\cong\mathfrak{g}_{-\theta}$ is identified with $\mathbb{A}^1$ via the root vector $x_{-\theta}$. We then have $\mathrm{Kl}_{(G^\vee,\varpi_{\mathrm{i}}^\vee)}=\mathrm{pr}_{2,!}(f_+^*\mathsf{E}^{\phi_+}\otimes f_0^*\mathsf{E}^{\phi_0})$. By Lemma 7.8 and Proposition 7.9, diagram (7.10.1) gets identified with the diagram

$$\begin{array}{ccc} & X & \\ {}^{\Theta:\, u_1 t \dot{w}_P u_2 \mapsto (u_2^{-1}, -\psi(u_1))} \swarrow & & \searrow^{\pi} \\ U/[U,U] \times \mathbb{A}^1 & & Z(L_P) \end{array} \tag{7.10.2}$$

Setting $\phi := (\phi^+, \phi_0)$, it follows that $\phi(u,a) = -\psi(u) - a$ for $(u,a) \in U/[U,U] \times \mathbb{A}^1$. The definition of the character D-module can then be written as

$$\mathrm{Cr}_{(G,P)} = R\pi_* \mathsf{E}^f = R\pi_*(\Theta^* \mathsf{E}^{-\phi}) = R\mathrm{pr}_{2,*}\big((f_+, f_0)^* \mathsf{E}^{\phi}\big) \xrightarrow{\sim} \mathrm{Kl}_{(G^\vee, \varpi_i^\vee)},$$

where Θ is the left arrow in (7.10.2). The last isomorphism is due to [71, p. 269]; in the D-module setting it also follows from the main result of [128]. It also follows from this calculation that $\mathrm{Cr}_{(G,P)}$ is a D-module, rather than a complex of D-modules. □

Remark 7.11

Similarly, over a finite field $\mathbb{F}_q$ equipped with a nontrivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$, we can define the Artin–Schreier ℓ-adic sheaf $\mathcal{L}_{\psi(f)} := f^* \mathcal{L}_\psi$ on X and a geometric crystal ℓ-adic sheaf $R\pi_* \mathcal{L}_{\psi(f)}$ on $Z(L_P)$. The comparison with generalized Kloosterman ℓ-adic sheaves is the same.

7.12. Homogeneity

In [127, Section 2.6.4], a $\mathbb{G}_m$-action is defined on Hk°. Under the parameterization (7.3.5), $\zeta \in \mathbb{G}_m$ acts by conjugation by $\rho^\vee(\zeta)$ on the first factor $G[\tau^{-1}]_1$ and by $q \mapsto \zeta^c q$ on the second factor $\mathbb{P}^1 \backslash \{0, \infty\}$, where c is the Coxeter number. The map $(f_+, f_0) : \mathrm{Hk}^\circ \to I(1)/I(2)$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $I(1)/I(2)$ by scalar multiplication in every affine simple root space.

The $\mathbb{G}_m$-action on Hk° preserves $\mathrm{Hk}^\circ_{\omega_i^\vee}$, and under the isomorphism of the diagrams (7.10.1) and (7.10.2), this $\mathbb{G}_m$-action is identified with the one in Section 6.21.

8. The mirror isomorphism for minuscule flag varieties

8.1. D-module mirror theorem

Assume as before that G is of adjoint type and $G^\vee$ is simply connected. Let $P \subset G$ be a parabolic subgroup, and let $P^\vee \subset G^\vee$ be the corresponding parabolic of the dual group.

LEMMA 8.2

There is a canonical exact sequence

$$2i\pi H^2(G^\vee/P^\vee, \mathbb{Z}) \to H^2(G^\vee/P^\vee) \to Z(L_P).$$

Proof
By Borel's theorem, there is a canonical isomorphism $H^2(G^\vee/P^\vee) \cong \mathfrak{t}^{W_P}$. We have $Z(L_P) = T^{W_P}$ and thus it only remains to apply the exponential map. $\square$

Recall that the character D-module $\mathrm{Cr}_{(G,P)}$ attached to the Berenstein–Kazhdan parabolic geometric crystal has been constructed in Section 6, and that the quantum connection $\mathcal{Q}^{G^\vee/P^\vee}$ for the projective homogeneous space $G^\vee/P^\vee$ has been described in Section 4 in terms of the quantum Chevalley formula. The base of $\mathrm{Cr}_{(G,P)}$ is $Z(L_P)$, and the base $\mathcal{Q}^{G^\vee/P^\vee}$ is $\mathbb{C}_q^\times$. Since $\mathbb{C}_q^\times \cong H^2(G^\vee/P^\vee)/2i\pi H^2(G^\vee/P^\vee,\mathbb{Z})$ by Remark 4.6, via

$$(q_i \mid i \notin I_P) \mapsto \sum_{i\in I\setminus I_P} \log(q_i)\sigma_i,$$

the above Lemma 8.2 shows that the two base tori are canonically isomorphic.

THEOREM 8.3
Suppose that P is a cominuscule parabolic subgroup of G, and let $P^\vee$ be the dual minuscule parabolic subgroup of $G^\vee$. The geometric crystal D-module $\mathrm{Cr}_{(G,P)}$ and the quantum connection $\mathcal{Q}^{G^\vee/P^\vee}$ for $G^\vee/P^\vee$ are isomorphic.

Proof
Let i be the minuscule node corresponding to P. We claim that the isomorphism $Z(L_P) \xrightarrow{\sim} H^2(G^\vee/P^\vee)/2i\pi H^2(G^\vee/P^\vee,\mathbb{Z})$ of Lemma 8.2 factors as the composition

$$Z(L_P) \xrightarrow{\alpha_\mathrm{i}} \mathbb{P}^1\setminus\{0,\infty\} = \mathbb{C}_q^\times \xrightarrow{\log} \mathbb{C}/2i\pi\mathbb{Z} \xrightarrow{\sigma_\mathrm{i}} H^2(G^\vee/P^\vee)/2i\pi H^2(G^\vee/P^\vee,\mathbb{Z}).$$

Indeed, the Schubert class $\sigma_\mathrm{i} \in H^2(G^\vee/P^\vee,\mathbb{C})$ corresponds to the fundamental coweight $\varpi_\mathrm{i}^\vee \in \mathfrak{t}^{W_P}$ under Borel's isomorphism. Thus composing with the exponential map, we see that the isomorphism

$$\mathbb{C}_q^\times \xrightarrow{\log} \mathbb{C}/2i\pi\mathbb{Z} \xrightarrow{\sigma_\mathrm{i}} H^2(G^\vee/P^\vee)/2i\pi H^2(G^\vee/P^\vee,\mathbb{Z}) \xrightarrow{\sim} Z(L_P) = T^{W_P}$$

is given by the cocharacter $q \mapsto \varpi_\mathrm{i}^\vee(q)$. Composing with α_i the claim follows from $\langle \alpha_\mathrm{i}, \varpi_\mathrm{i}^\vee\rangle = 1$.

The proof of the theorem follows by combining the following three results:

- Theorem 7.10 says that $\mathrm{Cr}_{(G,P)}$ is isomorphic to the Kloosterman D-module $\mathrm{Kl}_{(G^\vee,\varpi_\mathrm{i}^\vee)}$ if we identify the respective bases $Z(L_P)$ and $\mathbb{P}^1\setminus\{0,\infty\}$ via α_i.
- Zhu [128] proved that $\mathrm{Kl}_{(G^\vee,\varpi_\mathrm{i}^\vee)}$ is isomorphic to the Frenkel–Gross connection $\nabla^{(G^\vee,\varpi_\mathrm{i}^\vee)}$ (see also Theorem 9.6 below).

- Theorem 4.14 says that $\nabla^{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$ is isomorphic to $\mathcal{Q}^{G^\vee/P^\vee}$, if we identify the bases via $\mathbb{P}^1\backslash\{0,\infty\} = \mathbb{C}_q^\times$.

In Zhu's isomorphism the choice of affine generic character ϕ in the definition of $\mathrm{Kl}_{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$ matches with a particular choice of highest root vector in the definition of $\nabla^{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$ (see Theorem 9.6). All our sign choices lead to a single overall sign, which is equivalent to an isomorphism $q \mapsto \pm q$ of the curve $\mathbb{P}^1\backslash\{0,\infty\}$.

To determine this sign and conclude that $\mathrm{Cr}_{(G,P)}$ is isomorphic to $\mathcal{Q}^{G^\vee/P^\vee}$, we consider the quantum period solution $\langle S(q), 1\rangle$ of $\mathcal{Q}^{G^\vee/P^\vee}$. From Lemma 4.17, we know that the first term in the q-expansion is positive. On the other hand, the corresponding solution of $\mathrm{Cr}_{(G,P)}$ is

$$\frac{1}{(2i\pi)^\ell}\oint e^{f_t(x)}\omega = \frac{1}{(2i\pi)^\ell}\oint e^{a_1+\cdots+a_\ell+\alpha_{\mathrm{i}}(t)P_{\mathrm{i}}}\frac{da_1}{a_1}\cdots\frac{da_\ell}{a_\ell}, \tag{8.3.1}$$

where we use the expression of the superpotential from Corollary 6.11. Since P_{i} is a Laurent polynomial with positive coefficients, and $\alpha_{\mathrm{i}}(t) = q$, we deduce from Cauchy's residue theorem that the first term in the q-expansion of the above integral is also positive. □

If G is of type A_n, this proves a conjecture of Marsh and Rietsch [93, Section 3], and if G is of type D_n, a conjecture of Pech, Rietsch, and Williams [99, Section 4]. They construct in both cases a D-module homomorphism $\mathcal{Q}^{G^\vee/P^\vee} \to \mathrm{Cr}_{(G,P)}$ and show that it is injective. The conjecture remained open whether it is an isomorphism, or equivalently whether the dimension of $H^*(G^\vee/P^\vee)$ is equal to the rank of $\mathrm{Cr}_{(G,P)}$. This is Theorem 8.3.

COROLLARY 8.4

Suppose that P is a cominuscule parabolic of G. The number of paths in Bruhat order inside W^P from $\pi_P(w_0 w_0^P s_\kappa)$ to $w_0 w_0^P$ is equal to $P_{\mathrm{i}}(1,1,\ldots,1)$, where P_{i} is the Laurent polynomial of Corollary 6.11.

Proof

Lemma 4.17 gives that the first term in the q-expansion of the quantum period is given by the above number of paths. In Example 6.12, we have seen that $P_{\mathrm{i}}(a_1,\ldots,a_\ell)$ is the ratio of a square-free polynomial by the product $a_1\cdots a_\ell$. Hence (8.3.1) evaluates to $P_{\mathrm{i}}(1,1,\ldots,1)$. The corollary follows from Theorem 8.3. □

9. Generalization of Zhu's theorem

In this section, we explain how Zhu's results in [128] establish Theorem 9.6 below. We assume the reader is familiar with [128].

9.1. Deformation of the Frenkel–Gross connection

We use the notation from Section 3, except that G and $G^\vee$ are swapped. We define a deformation of the Frenkel–Gross connection parameterized by $h \in \mathfrak{t}^*$ as follows:

$$\widetilde{\nabla}^{G^\vee} := d + (y_p + h)\frac{dq}{q} + x_\theta \, dq. \tag{9.1.1}$$

This is a connection on the trivial $G^\vee$-bundle over $\mathbb{P}^1 \backslash \{0, \infty\} \times \mathfrak{t}^*$ *relative* to $\mathfrak{t}^*$ (i.e., we view the connection 1-form $(y_p + h)\frac{dq}{q} + x_\theta \, dq$ as a relative differential on $\mathbb{P}^1 \backslash \{0, \infty\} \times \mathfrak{t}^*$ over $\mathfrak{t}^*$ for the projection morphism $\mathbb{P}^1 \backslash \{0, \infty\} \times \mathfrak{t}^* \to \mathfrak{t}^*$). Thus $\widetilde{\nabla}^{G^\vee}$ specialized to $0 \in \mathfrak{t}^*$ is the connection $\nabla^{G^\vee}$ of [43] considered in Section 3. As before, the connection (9.1.1) depends on a choice of basis vector x_θ, but this choice is suppressed from the notation. If the choice of $x_\theta \in \mathfrak{g}^\vee_\theta$ is not mentioned, then by default we will use (3.9.1). As before, we also have the associated vector bundle with connection $\widetilde{\nabla}^{(G^\vee, \lambda)} = \widetilde{\nabla}^{(G^\vee, V_\lambda)}$ over $\mathbb{P}^1 \backslash \{0, \infty\} \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$.

9.2. Rigid automorphic D-module

Recall from Section 7.1 the standard affine character $\phi : I(1)/I(2) \to \mathbb{A}^1$. Recall that $\mathsf{E} = D_{\mathbb{A}^1}/(\partial_x - 1)$ denotes the exponential D-module on $\mathbb{A}^1$, and that we write $\mathsf{E}^\phi := \phi^* \mathsf{E}$ for the pullback D-module on $I(1)/I(2)$. Let $S = \mathrm{Sym}(\mathfrak{t})$, and identify the complex points of $\mathrm{Spec}(S)$ with $\mathfrak{t}^*$. Define the $(D_T \otimes S)$-module M_T as the free rank-1 $(\mathcal{O}_T \otimes S)$-module with basis element "x^h," for $h \in \mathrm{Spec}(S) = \mathfrak{t}^*$, with the action of the differential operator $\partial_k \in D_T \otimes S$ along $k \in \mathfrak{t} \subset \mathrm{Fun}(\mathfrak{t}^*)$ given by

$$\partial_k \cdot x^h := k \, x^h.$$

Equivalently, M_T is a rank-1 connection on the trivial line bundle over $T \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$ (i.e., for every $h \in \mathfrak{t}^*$, and locally for x in an open simply connected open subset of T the horizontal sections of M_T specialized at h are given by constant multiples of the function $x \mapsto x^h$).

Recall our D-module conventions from Remark 6.20. Let $j_\kappa : T \times I(1)/I(2) \hookrightarrow \mathrm{Bun}^\kappa_{\mathcal{G}}$ denote the inclusion of the big cell into the κ-component of $\mathrm{Bun}_{\mathcal{G}}$. By [71, Corollary 1.3], j_κ is an affine open embedding. For an affine map f, we have $Rf_* = f_*$, and it follows that $Rj_{\kappa,*} = j_{\kappa,*}$ and $Rj_{\kappa,!} = j_{\kappa,!}$ (see [73, p. 95]).

Now consider the $(D_{T \times I(1)/I(2)} \otimes S)$-module $\mathsf{M}_T \boxtimes \mathsf{E}^\phi$ on $T \times I(1)/I(2)$.

LEMMA 9.3
We have $j_{\kappa,!}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi) \xrightarrow{\sim} j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)$.

Proof
This is the D-module version of [71, Lemma 2.3]. We repeat the argument. For $w \in$

$W_{\mathrm{af}} - \Omega$, let P_w denote the $\mathcal{G}(1,2)$-bundle defined by a lift of w. Pick $\alpha \in \mathrm{Inv}(w)$, and let $U_\alpha \subset I(1)$ denote the corresponding one-parameter subgroup. We have an inclusion $U_\alpha \hookrightarrow \mathrm{Aut}(P_w)$ and a commutative diagram

$$\begin{array}{ccc} U_\alpha \times \mathrm{pt} & \xrightarrow{\pi} & \mathrm{pt} \\ {\scriptstyle \mathrm{id}\times P_w}\downarrow & & \downarrow{\scriptstyle P_w} \\ U_\alpha \times \mathrm{Bun}_{\mathcal{G}} & \xrightarrow{\mathrm{act}} & \mathrm{Bun}_{\mathcal{G}} \end{array}$$

Pulling back $j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)$ in two ways, we obtain

$$\mathsf{E}^\phi|_{U_\alpha} \boxtimes j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)|_{P_w} \cong \pi^* j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)|_{P_w}.$$

Since $\mathsf{E}^\phi|_{U_\alpha}$ is defined by a nontrivial character of U_α, it follows that the stalk of $j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)$ vanishes at P_w. Similarly, the stalk of $j_{\kappa,!}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)$ vanishes at P_w. Since this holds for all $w \in W_{\mathrm{af}} - \Omega$, the result follows. ☐

Following Heinloth, Ngô, and Yun [71, Definition 2.4, p. 269], we make the following definition.

Definition 9.4
Define $A_{\mathcal{G},T}$ to be the $(D_{\mathrm{Bun}_{\mathcal{G}}} \otimes S)$-module on $\mathrm{Bun}_{\mathcal{G}}$ given by $j_{\kappa,!}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi) = j_{\kappa,*}(\mathsf{M}_T \boxtimes \mathsf{E}^\phi)$ on each connected component $\mathrm{Bun}_{\mathcal{G}}^\kappa$.

Thus $A_{\mathcal{G},T}$ is the intermediate, or minimal, extension of D-modules (see, e.g., [73, Section 3.4]).

9.5. *Twisted Kloosterman D-modules and statement of the main result*

It is established in [71, Theorem 1] that $A_{\mathcal{G},T}$ has the Hecke eigenproperty. Let $\mathrm{TKl}_{G^\vee}$ denote the corresponding $G^\vee$-Hecke eigenvalue which is a $G^\vee$-connection on $\mathbb{P}^1 \backslash \{0,\infty\} \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$. Our basis element x^h for $h \in \mathfrak{t}^* = \mathrm{Spec}(S)$ corresponds to the multiplicative character χ in [71, Remark 2.5] and the D-module on $\mathrm{Bun}_{\mathcal{G}}$ given by $A_{\mathcal{G},T} \otimes_S h$ is denoted $A_{\phi,\chi}$ in [71, Theorem 1]. The $G^\vee$-connection on $\mathbb{P}^1 \backslash \{0,\infty\}$ given by $\mathrm{TKl}_{G^\vee} \otimes_S h$ is denoted $\mathrm{Kl}_{{}^L\mathcal{G}}(\phi,\chi)$ in [71, Theorem 1].

THEOREM 9.6
There is a choice of basis vector $x_\theta \in \mathfrak{g}_\theta^\vee$, which is compatible up to sign with Section 2.2, such that the above $G^\vee$-connections are isomorphic:

$$\mathrm{TKl}_{G^\vee} \cong \widetilde{\nabla}^{G^\vee}.$$

Specialized to $0 \in \mathfrak{t}^*$, Theorem 9.6 reduces to $\mathrm{Kl}_{G^\vee} \cong \nabla^{G^\vee}$. The proof of Theorem 9.6 occupies the rest of this section. We will assume the reader is familiar with [128].

9.7. Classical Hitchin map

We use notation from Section 7.1, and we let $\mathfrak{v} := I(1)/I(2)$ in the rest of this section.

LEMMA 9.8

The stack $\mathrm{Bun}_{\mathcal{G}}$ *is good in the sense of Beilinson and Drinfeld; that is, we have* $\dim T^*\mathrm{Bun}_{\mathcal{G}} = 2\dim \mathrm{Bun}_{\mathcal{G}}$.

Proof

In [128, Lemma 17], it is shown that $\mathrm{Bun}_{\mathcal{G}(0,1)}$ is good, where $\mathcal{G}(0,1)$ is the group scheme over $\mathbb{P}^1$ obtained from the dilatation of the constant group scheme $G \times \mathbb{P}^1$ along $U \times \{\infty\} \subset G \times \{\infty\}$. The lemma follows after noting that $\mathrm{Bun}_{\mathcal{G}} = \mathrm{Bun}_{\mathcal{G}(1,2)}$ is a principal bundle over $\mathrm{Bun}_{\mathcal{G}(0,1)}$ under the group $I_0^{\mathrm{opp}}(0)/I_0^{\mathrm{opp}}(1) \times I(1)/I(2) \cong T \times \mathfrak{v}$. $\square$

Let $\mathfrak{c}^* := \operatorname{Spec} \mathbb{C}[\mathfrak{g}^*]^G \cong \mathfrak{t}^* /\!\!/ W$, where W is the Weyl group. We have a canonical $\mathbb{G}_m$-action on $\mathfrak{c}^*$ coming from the scalar $\mathbb{G}_m$-action on $\mathfrak{g}^*$. It gives rise to a decomposition $\mathfrak{c}^* = \bigoplus_i \mathfrak{c}^*_{d_i}$ into 1-dimensional subspaces, where the integers $d_1 \leq d_2 \leq \cdots \leq d_r$ are the degrees of W. Recall that $d_r = c$ is the Coxeter number of $\mathfrak{g}$.

Let $\mathcal{E} \in \mathrm{Bun}_{\mathcal{G}}$, and write $\mathcal{E}' := \mathcal{E}|_{\mathbb{P}^1\backslash\{0,\infty\}} \in \mathrm{Bun}_{G\times\mathbb{P}^1\backslash\{0,\infty\}}$. The cotangent space $T^*_{\mathcal{E}}\mathrm{Bun}_{\mathcal{G}}$ maps to $\Gamma(\mathbb{P}^1\backslash\{0,\infty\}, \mathfrak{g}^*_{\mathcal{E}'} \otimes \omega_{\mathbb{P}^1\backslash\{0,\infty\}})$, where $\mathfrak{g}^*_{\mathcal{E}'}$ is the bundle on $\mathbb{P}^1\backslash\{0,\infty\}$ associated to $\mathcal{E}'$ and the coadjoint representation $\mathfrak{g}^*$ of G. The G-invariant map $\mathfrak{g}^* \to \mathfrak{c}^*$ gives rise, as $\mathcal{E}$ varies, to a global analogue of the characteristic polynomial called the *(global) Hitchin map* $h^{\mathrm{cl}} : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Hitch}(\mathbb{P}^1\backslash\{0,\infty\})$, where

$$\mathrm{Hitch}\big(\mathbb{P}^1\backslash\{0,\infty\}\big) := \Gamma\big(\mathbb{P}^1\backslash\{0,\infty\}, \mathfrak{c}^* \times^{\mathbb{G}_m} \omega_{\mathbb{P}^1\backslash\{0,\infty\}}\big).$$

Let $\mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}}$ be the image of h^{cl}, so that we write

$$h^{\mathrm{cl}} : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}} \subset \mathrm{Hitch}\big(\mathbb{P}^1\backslash\{0,\infty\}\big).$$

We give an explicit description of $\mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}}$, following [128]. For a point $x \in \mathbb{P}^1$, we let $\mathcal{O}_x$ denote the completed local ring at x and let $F_x = \mathrm{Frac}(\mathcal{O}_x)$ denote its fraction field. Denote by $D_x = \operatorname{Spec} \mathcal{O}_x$ and $D_x^\times = \operatorname{Spec} F_x$ the formal disk and formal punctured disk at x. We write $\omega_{\mathcal{O}_x}$ for the $\mathcal{O}_x$-module $\mathcal{O}_x \cdot dt$ (after choosing a local coordinate t), and ω_F for $F_x \cdot dt$. We have the *local Hitchin map* (see [128, p. 258])

$$h_x^{\mathrm{cl}} : \mathfrak{g}^* \otimes \omega_F \to \mathfrak{c}^* \times^{\mathbb{G}_m} \omega_F^\times =: \mathrm{Hitch}(D_x^\times),$$

where $F = F_x$.

For $i = 0, 1, 2$, let $\mathfrak{p}(i) = \mathfrak{p}_\infty(i) \subset \mathfrak{g}_\infty$ denote the Lie algebra of $I(i) = I_\infty(i)$. Similarly, define $\mathfrak{p}_0(i) \subset \mathfrak{g}_0$ using $I_0^{\mathrm{opp}}(i)$. For an $\mathcal{O}$-lattice $\mathfrak{p} \subset \mathfrak{g} \otimes F$, we define $\mathfrak{p}^\perp := \mathfrak{p}^\vee \otimes_\mathcal{O} \omega_\mathcal{O}$, where $\mathfrak{p}^\vee \subset \mathfrak{g}^* \otimes F$ is the $\mathcal{O}$-dual of $\mathfrak{p}$. The two local Hitchin maps at $x = 0$ and $x = \infty$ give the following two commutative diagrams (see [128, Remark 4.4], [128, Proposition 14]):[2]

$$\begin{array}{ccc} \mathfrak{p}_0(1)^\perp & \longrightarrow & \mathfrak{t}^* \cong \mathfrak{p}_0(1)^\perp/\mathfrak{p}_0(0)^\perp \\ \downarrow & & \downarrow \\ \mathrm{Hitch}(D_0)_{\mathrm{RS}} & \longrightarrow & \mathfrak{c}^* \end{array} \qquad \begin{array}{ccc} \mathfrak{p}(2)^\perp & \longrightarrow & \mathfrak{v}^* \cong \mathfrak{p}(2)^\perp/\mathfrak{p}(1)^\perp \\ \downarrow & & \downarrow \\ \mathrm{Hitch}(D_\infty)_{1/c} & \longrightarrow & \mathfrak{v}^* /\!\!/ T \end{array}$$

where the local Hitchin spaces are defined by (see [128, bottom of p. 263])[3]

$$\mathrm{Hitch}(D_0)_{\mathrm{RS}} = \bigoplus_{1 \le i \le r} \omega_{\mathcal{O}_0}^{d_i}(d_i) \otimes \mathfrak{c}_{d_i}^*, \tag{9.8.1}$$

$$\mathrm{Hitch}(D_\infty)_{1/c} = \bigoplus_{1 \le i < r} \omega_{\mathcal{O}_\infty}^{d_i}(d_i) \otimes \mathfrak{c}_{d_i}^* \bigoplus \omega_{\mathcal{O}_\infty}^{c}(c+1) \otimes \mathfrak{c}_c^*. \tag{9.8.2}$$

The bottom map of the left diagram is obtained by taking the residue at 0. The bottom map of the right diagram is explained in [128, (4.11)]. It involves Kostant's section (see [82])

$$y_p + (\mathfrak{g}^\vee)^{x_p} \xrightarrow{\sim} \mathfrak{c}^*. \tag{9.8.3}$$

This map is $\mathbb{G}_m$-equivariant for a suitable $\mathbb{G}_m$-action on $y_p + (\mathfrak{g}^\vee)^{x_p}$ and the above-mentioned $\mathbb{G}_m$-action on $\mathfrak{c}^*$ (see [96, Proposition 2.2]). Important for us later will be the corollary that $y_p + \mathfrak{g}_\theta^\vee \xrightarrow{\sim} \mathfrak{c}_c^*$ under Kostant's section.

The following result is a $\mathcal{G}(1,2)$-version of a similar formula for $\mathcal{G}(0,1)$ in [128, p. 270].

LEMMA 9.9

We have an isomorphism

$$\mathrm{Hitch}(\mathbb{P}^1)_\mathcal{G} \cong \bigoplus_{1 \le i < r} \Gamma\big(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{d_i}(d_i \cdot 0 + d_i \cdot \infty)\big) \otimes \mathfrak{c}_{d_i}^* \bigoplus \Gamma\big(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{c}\big(c \cdot 0 + (c+1) \cdot \infty\big)\big) \otimes \mathfrak{c}_c^* \cong \mathbb{A}^r \times \mathbb{A}^1. \tag{9.9.1}$$

[2] In [128, Section 4], $\mathfrak{p}$ is an arbitrary parahoric subgroup. We specialize to the case that $\mathfrak{p}$ is the Iwahori. The notation $\mathrm{Hitch}(D_\infty)_{1/c}$ matches [128, p. 272].

[3] In the notation of [128, p. 259], the integer m is equal to the Coxeter number c for us.

Proof
By the same argument as in [128, Lemma 5], we have the description

$$\mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}} \cong \mathrm{Hitch}(D_0)_{\mathrm{RS}} \times_{\mathrm{Hitch}(D_0^\times)} \mathrm{Hitch}\big(\mathbb{P}^1\backslash\{0,\infty\}\big) \times_{\mathrm{Hitch}(D_\infty^\times)} \mathrm{Hitch}(D_\infty)_{1/c}.$$

The explicit descriptions (9.8.1) and (9.8.2) give the first isomorphism in (9.9.1). For the second isomorphism, we note that $\omega_{\mathbb{P}^1}^{d} \cong \mathcal{O}_{\mathbb{P}^1}(-2d)$. Thus $\omega_{\mathbb{P}^1}^{d_i}(d_i\cdot 0 + d_i \cdot \infty) = \mathcal{O}_{\mathbb{P}^1}(-2d_i+2d_i) = \mathcal{O}_{\mathbb{P}^1}$ and $\omega_{\mathbb{P}^1}^{c}(c\cdot 0 + (c+1)\cdot\infty) = \mathcal{O}_{\mathbb{P}^1}(1)$. □

Let $\mu : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathfrak{t}^* \times \mathfrak{v}^*$ be the moment map for the action of $T \times \mathfrak{v}$ on $\mathrm{Bun}_{\mathcal{G}}$.

PROPOSITION 9.10
The following diagram is commutative, all maps are surjective, the bottom map is an isomorphism, and the top map is flat:

$$\begin{array}{ccc} T^*\mathrm{Bun}_{\mathcal{G}} & \xrightarrow{\ \mu\ } & \mathfrak{t}^* \times \mathfrak{v}^* \\ {\scriptstyle h^{\mathrm{cl}}}\big\downarrow & & \big\downarrow \\ \mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}} & \xrightarrow{\ \cong\ } & \mathfrak{c}^* \times \mathfrak{v}^* /\!\!/ T \end{array} \qquad (9.10.1)$$

Proof
The global Hitchin map h^{cl} embeds into the product of the local Hitchin maps h_0^{cl} and h_x^{cl} at 0 and ∞, which establishes the commutativity of (9.10.1).

The explicit description (9.9.1) of $\mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}}$ establishes the isomorphism of the bottom map (see [128, (4.9), Proof of Lemma 19]). The left vertical map of (9.10.1) is surjective by definition. The right vertical map of (9.10.1) is a quotient map and thus surjective. The top map μ of (9.10.1) is surjective because $\mathrm{Bun}_{\mathcal{G}}$ is a principal $(T\times\mathfrak{v})$-bundle over $\mathrm{Bun}_{\mathcal{G}(0,1)}$.

The proof of the last claim is identical to that of [128, Lemma 18], which we repeat. The Hamiltonian reduction $\mu^{-1}(0)/(T\times\mathfrak{v})$ is naturally identified with $T^*\mathrm{Bun}_{\mathcal{G}(0,1)}$. As remarked in the proof of Lemma 9.8, $\mathrm{Bun}_{\mathcal{G}(0,1)}$ is good, and thus $T^*\mathrm{Bun}_{\mathcal{G}(0,1)}$ has dimension 0. This implies that $\dim \mu^{-1}(0) = \dim(T\times\mathfrak{v})$. Let $W \subset T^*\mathrm{Bun}_{\mathcal{G}}$ be the largest open substack such that the fibers of $\mu|_W$ have dimension $\dim(T\times\mathfrak{v})$. Then W is $\mathbb{G}_m$-invariant, and since $\mu^{-1}(0)\subset W$, we have $W = T^*\mathrm{Bun}_{\mathcal{G}}$, so all fibers of $\mu : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathfrak{t}^*\times\mathfrak{v}^*$ have dimension $\dim(T\times\mathfrak{v})$. Since $\mathfrak{t}^*\times\mathfrak{v}^*$ is smooth and $T^*\mathrm{Bun}_{\mathcal{G}}$ is locally a complete intersection, we conclude that μ is flat. □

9.11. Quantization

We recall the descriptions of certain spaces of $\mathfrak{g}^\vee$-opers from [128, Section 5]. (The Lie algebra $\mathfrak{g}^\vee$ will be suppressed from the notation, so that we write Op for what is denoted $\mathrm{Op}_{{}^L\mathfrak{g}}$ in [128].) Recall from Section 3.1 the principal $\mathfrak{sl}_2$-triple $(x_p, 2\rho^\vee, y_p)$ in $\mathfrak{g}^\vee$. The space $\mathrm{Op}(D_x^\times)$ of opers on the formal punctured disk centered at x can be identified with the space of operators

$$d + \big(y_p + (\mathfrak{g}^\vee)^{x_p} \otimes F_x\big)\, dz,$$

where z is a local coordinate at x. The space $\mathrm{Op}(D_0)_{\mathrm{RS}}$ of opers on the formal disk centered at 0 with regular singularities can be identified with the space of operators

$$d + \big(y_p + (\mathfrak{g}^\vee)^{x_p} \otimes \mathcal{O}_0\big)\frac{dq}{q}. \tag{9.11.1}$$

The space $\mathrm{Op}(D_\infty)_{1/c}$ of opers on the formal disk centered at ∞ with slope at most $1/c$ is the space of operators

$$\Big(d + \Big(\frac{y_p}{t} + \frac{1}{t}\mathfrak{b}^\vee \otimes \mathcal{O}_\infty + \frac{1}{t^2}\mathfrak{g}_\theta^\vee \otimes \mathcal{O}_\infty\Big)\, dt\Big)\big/U^\vee(\mathcal{O}_\infty), \tag{9.11.2}$$

where $t = 1/q$. The spaces $\mathrm{Op}(D_0)_{\mathrm{RS}}$ and $\mathrm{Op}(D_\infty)_{1/c}$ are subschemes of $\mathrm{Op}(D_0^\times)$ and $\mathrm{Op}(D_\infty^\times)$, respectively.

In [128, Section 2], a subscheme of opers $\mathrm{Op}(\mathbb{P}^1)_{\mathscr{G}} \subset \mathrm{Op}(\mathbb{P}^1\backslash\{0,\infty\})$ is defined, and according to [128, Lemma 5], we have (cf. [128, p. 272])

$$\mathrm{Op}(\mathbb{P}^1)_{\mathscr{G}} \cong \mathrm{Op}(D_0)_{\mathrm{RS}} \times_{\mathrm{Op}(D_0^\times)} \mathrm{Op}\big(\mathbb{P}^1\backslash\{0,\infty\}\big) \times_{\mathrm{Op}(D_\infty^\times)} \mathrm{Op}(D_\infty)_{1/c}. \tag{9.11.3}$$

The description (9.11.3) is a quantization of (9.9.1).

Let $U(\mathfrak{t})$ and $U(\mathfrak{v})$ denote the universal enveloping algebras of $\mathfrak{t}$ and $\mathfrak{v}$, and let D' be the sheaf of algebras on the smooth site $(\mathrm{Bun}_{\mathscr{G}})_{sm}$ defined by Beilinson and Drinfeld [6, Section 1.2.5]. The following variant of [128, Lemma 21] is the quantization of Proposition 9.10.

PROPOSITION 9.12

We have a commutative diagram of strict morphisms of filtered commutative algebras, where the top map is an isomorphism and the bottom map is flat:

$$\begin{array}{ccc} U(\mathfrak{t})^W \otimes U(\mathfrak{v})^T & \xrightarrow{\ \cong\ } & \mathbb{C}\big[\mathrm{Op}(\mathbb{P}^1)_{\mathscr{G}}\big] \\ \big\downarrow & & \big\downarrow{\scriptstyle h_\nabla} \\ U(\mathfrak{t}) \otimes U(\mathfrak{v}) & \longrightarrow & \Gamma(\mathrm{Bun}_{\mathscr{G}}, D') \end{array} \tag{9.12.1}$$

Proof
The universal enveloping algebras $U(\mathfrak{t})$ and $U(\mathfrak{v})$ have natural filtrations such that the associated graded algebras satisfy $\mathrm{gr}(U(\mathfrak{t})) \cong \mathbb{C}[\mathfrak{t}^*]$ and $\mathrm{gr}(U(\mathfrak{v})) \cong \mathbb{C}[\mathfrak{v}^*]$.

The ring of functions $\mathbb{C}[\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}}]$ has a filtration such that $\mathrm{gr}(\mathbb{C}[\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}}]) \cong \mathbb{C}[\mathrm{Hitch}(\mathbb{P}^1)_{\mathcal{G}}]$, and $\Gamma(\mathrm{Bun}_{\mathcal{G}}, D')$ has a filtration such that $\mathrm{gr}(\Gamma(\mathrm{Bun}_{\mathcal{G}}, D')) \cong \mathrm{Fun}\, T^*\mathrm{Bun}_{\mathcal{G}}$, where we write Fun to denote the commutative pro-algebra of regular functions on an affine ind-stack. The right vertical map h_∇ is defined in [128, (3.3)], and is a quantization of the classical Hitchin map h^{cl}. For these constructions, see [128, p. 254, Section 5.2]. The bottom horizontal map is explained in [128, pp. 255–256]. The top horizontal map is a quantization of the moment map $\mu : T^*\mathrm{Bun}_{\mathcal{G}} \to \mathfrak{t}^* \times \mathfrak{v}^*$ (see [128, Proposition 15]). Thus, taking associate graded algebras of (9.12.1), we recover (9.10.1).

The commutativity of (9.12.1) follows from commutative diagrams (see [128, Proposition 15]) analogous to (9.8.1) and (9.8.2).

By Proposition 9.10, after taking associate graded algebras the top map is an isomorphism and the bottom map is flat; thus the same statements hold in (9.12.1). □

9.13. Proof of Theorem 9.6
Let $\eta : \mathrm{Op}(D_0)_{\mathrm{RS}} \to \mathfrak{c}^*$ be the residue map obtained from the description (9.11.1) and Kostant's isomorphism (9.8.3). Let

$$\varpi : \mathfrak{t}^* \to \mathfrak{t}^* /\!\!/ W = \mathfrak{c}^*$$

be the projection map. We now compute the intersection $\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}} \cap \eta^{-1}(\varpi(h))$ for $h \in \mathfrak{t}^*$, which corresponds to a slice of the isomorphism

$$\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}} \xrightarrow{\cong} \mathfrak{c}^* \times \mathrm{Spec}\, U(\mathfrak{v})^T \tag{9.13.1}$$

given by the top map of Proposition 9.12.

The space $\mathrm{Op}(\mathbb{P}^1 \backslash \{0,\infty\})$ of opers consists of operators of the form

$$\nabla = d + y_p \frac{dq}{q} + v(q)\, dq,$$

where $v(q) \in (\mathfrak{g}^\vee)^{x_p}[q, q^{-1}]$.

Suppose moreover that $\nabla \in \mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}} \cap \eta^{-1}(\varpi(h))$. The condition $\nabla \in \mathrm{Op}(D_0)_{\mathrm{RS}}$ that ∇ has a regular singularity at 0 implies (see (9.11.1)) that $v(q) \in q^{-1}(\mathfrak{g}^\vee)^{x_p}[q]$. Write $v(q) = a/q + v_0(q)$ with $v_0(q) \in (\mathfrak{g}^\vee)^{x_p}[q]$ and $a \in (\mathfrak{g}^\vee)^{x_p}$.

The condition $\eta(\nabla) = \varpi(h)$ says that the residue of ∇ at 0 is $\varpi(h) \in \mathfrak{c}^*$. By Kostant's theorem (see [82, Theorem 7]), the map $\mathfrak{g}^\vee \to \mathfrak{g}^\vee /\!\!/ G^\vee \to \mathfrak{c}^*$ induces the isomorphism (9.8.3). Thus the element a is uniquely determined by $\varpi(h) \in \mathfrak{c}^*$. We denote it by $a = a_h \in (\mathfrak{g}^\vee)^{x_p}$.

Writing $t = 1/q$, the operator becomes

$$\nabla = d - (y_p + a_h)\frac{dt}{t} - v_0\Big(\frac{1}{t}\Big)\frac{dt}{t^2}.$$

The condition $\nabla \in \mathrm{Op}(D_\infty)_{1/c}$ at ∞ implies (see (9.11.2)) that $v_0(\frac{1}{t})$ must be constant and must belong to the root space $\mathfrak{g}^\vee_\theta = \mathbb{C}x_\theta$.

Thus the space of opers $\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}} \cap \eta^{-1}(\varpi(h))$ is the space of operators of the form

$$\nabla = d + (y_p + a_h)\frac{dq}{q} + \alpha x_\theta\, dq, \tag{9.13.2}$$

for $\alpha \in \mathbb{C}$. Thus $\mathrm{Op}(\mathbb{P}^1)_{\mathcal{G}} \cap \eta^{-1}(\varpi(h)) \cong \mathbb{A}^1$. This bijection $\nabla \leftrightarrow \alpha$ corresponds to the isomorphism (9.13.1) under the identification $\mathbb{A}^1 \cong \mathfrak{v}^*/T \cong \mathrm{Spec}(U(\mathfrak{v})^T)$. (The $(r+1)$-dimensional T-module $\mathfrak{v}$ has weights the simple roots $\alpha_1, \ldots, \alpha_r$ and the negative $-\theta$ of the longest root; hence $\mathfrak{v}^*/T \cong \mathbb{A}^1$.)

By construction, the two elements $y_p + a_h$ and $y_p + h$ in $\mathfrak{g}^\vee$ have the same image $\varpi(h)$ in $\mathfrak{g}^\vee /\!\!/ G^\vee \cong \mathfrak{t}^* /\!\!/ W = \mathfrak{c}^*$ and are therefore conjugate by a group element $g \in G^\vee$. Again by Kostant's theorem, $U^\vee$ acts freely on $y_p + \mathfrak{b}^\vee$ via the adjoint action, and the quotient is isomorphic to $y_p + (\mathfrak{g}^\vee)^{x_p}$. Thus $y_p + a_h$ and $y_p + h$ are conjugate by an element in the unipotent subgroup $U^\vee \subset G^\vee$. It follows that $\mathrm{Ad}_g(x_\theta) = x_\theta$.

Recall from Section 9.1 the deformed Frenkel–Gross connection

$$\widetilde{\nabla}^{G^\vee} = d + (y_p + h)\frac{dq}{q} + x_\theta\, dq. \tag{9.13.3}$$

We deduce from $\mathrm{Ad}_g(x_\theta) = x_\theta$ that the two connections (9.13.2) with $\alpha = 1$ and (9.13.3) are gauge equivalent via a *constant* gauge transformation.

To complete the proof of Theorem 9.6, it remains to show that the twisted Kloosterman D-module $\mathrm{TKl}_{G^\vee}$ is isomorphic to the connection (9.13.2) for some $\alpha = \pm 1$. This is achieved in the same manner as in [128, p. 273]. Namely, we compare two automorphic D-modules.

We begin by constructing a Hecke eigen-D-module with Hecke eigenvalue equal to (9.13.2). Let $\phi : \mathfrak{v} \to \mathbb{C}$ be the standard affine character, inducing a character $\varphi : U(\mathfrak{v}) \to \mathbb{C}$. The element $h \in \mathfrak{t}^*$ also defines a character $\varphi_h : U(\mathfrak{t}) \to \mathbb{C}$. The D-module

$$\mathrm{Aut}(h) := \omega_{\mathrm{Bun}_{\mathcal{G}}}^{-1/2} \otimes (D' \otimes_{U(\mathfrak{t})\otimes U(\mathfrak{v}), \varphi_h \otimes \varphi} \mathbb{C})$$

is considered in [128], where D' is the sheaf of critically twisted differential operators on $(\mathrm{Bun}_{\mathcal{G}})_{sm}$, and the tensor product is defined using the bottom map of

Proposition 9.12. Using the flatness statement in Proposition 9.12, Zhu's result (see [128, Corollary 9]) states that $\mathrm{Aut}(h)$ is a Hecke eigen-D-module with the connection (9.13.2) as its Hecke eigenvalue.

Finally, we argue that $\mathrm{Aut}(h)$ is isomorphic to the automorphic D-module of [71] from Definition 9.4. Recall from Section 7.3 the big cell $T \times \mathfrak{v} \cong \mathring{\mathrm{Bun}}_{\mathcal{G}} \subset \mathrm{Bun}_{\mathcal{G}}$, which maps to the basepoint $\star \subset \mathrm{Bun}_{\mathcal{G}(0,1)}$ corresponding to the trivial $\mathcal{G}(0,1)$-bundle. By [128, Remark 6.1], $\omega^{-1/2}_{\mathrm{Bun}_{\mathcal{G}}}$ is canonically trivialized on $\mathring{\mathrm{Bun}}_{\mathcal{G}}$. It follows that the restriction of $\mathrm{Aut}(h)$ to $\mathring{\mathrm{Bun}}_{\mathcal{G}} \cong T \times \mathfrak{v}$ is isomorphic to $\mathsf{M}_T \boxtimes \mathsf{E}^{\phi}$. Furthermore, $\mathrm{Aut}(h)$ is a $(T \times \mathfrak{v}, \mathsf{M}_T \boxtimes \mathsf{E}^{\phi})$-equivariant D-module on $\mathrm{Bun}_{\mathcal{G}}$ (see [126, Section A.4.2] for the definition of equivariant). By [71, Remark 2.5], $\mathrm{Aut}(h)$ is automatically the (intermediate) clean extension of $\mathrm{Aut}(h)|_{\mathring{\mathrm{Bun}}_{\mathcal{G}}}$. Thus $\mathrm{Aut}(h)$ is isomorphic to $A_{\mathcal{G},T}$ specialized at $h \in \mathfrak{t}^*$ for which $\mathrm{TKl}_{G^\vee}$ specialized at $h \in \mathfrak{t}^*$ is an eigenvalue. This shows that $\mathrm{TKl}_{G^\vee}$ specialized at $h \in \mathfrak{t}^*$ is isomorphic to (9.13.2) and thus to the deformed Frenkel–Gross connection (9.13.3), completing the proof. □

10. Equivariant quantum cohomology and weighted geometric crystals

We extend the mirror isomorphism of Theorem 8.3 to the equivariant case. Recall from Section 9 the notation $S := \mathrm{Sym}(\mathfrak{t}) \cong H^*_{T^\vee}(\mathrm{pt})$.

10.1. Equivariant quantum connection

Let $QH^*_{T^\vee}(G^\vee/P^\vee)$ denote the torus-equivariant small quantum cohomology ring of $G^\vee/P^\vee$. It is an algebra over $\mathbb{C}[q_i \mid i \notin I_P] \otimes S$. For $w \in W^P$, we abuse notation by also writing $\sigma_w \in QH^*_{T^\vee}(G^\vee/P^\vee)$ for the equivariant quantum Schubert class. The following equivariant quantum Chevalley formula for a general $G^\vee/P^\vee$ is due to Mihalcea [94].

THEOREM 10.2
For $w \in W^P$, we have in $QH^*_{T^\vee}(G^\vee/P^\vee)$

$$\sigma_i *_q \sigma_w = (\varpi_i^\vee - w \cdot \varpi_i^\vee)\sigma_w + \sum_{\beta^\vee} \langle \varpi_i^\vee, \beta\rangle \sigma_{ws_\beta} + \sum_{\nu^\vee} \langle \varpi_i^\vee, \nu\rangle q_{\eta_P(\nu)} \sigma_{\pi_P(ws_\nu)},$$

where $\varpi_i^\vee \in \mathfrak{t}$ denotes a fundamental weight of $\mathfrak{g}^\vee$, and $\beta^\vee$, $\nu^\vee$ denote roots of $\mathfrak{g}^\vee$. The last two summations are as in Theorem 4.3.

We have a bundle map $QH^*_{T^\vee}(G^\vee/P^\vee) \to \mathfrak{t}^*$. We write $QH^*_h(G^\vee/P^\vee)$ for the fiber of this map over $h \in \mathfrak{t}^*$. The algebra $QH^*_h(G^\vee/P^\vee)$ is again a free $\mathbb{C}[q_i \mid i \notin I_P]$-module with Schubert basis $\{\sigma_w \mid w \in W^P\}$.

Now, assume that $P^\vee \subset G^\vee$ is minuscule. Let $O(1)$ be the line bundle on $G^\vee/P^\vee$ arising from the natural embedding $G^\vee/P^\vee \hookrightarrow \mathbb{P}(V_{\varpi_i^\vee})$. Consider the bundle $QH^*_{T^\vee}(G^\vee/P^\vee)$ over $\mathbb{C}^\times_q \times \mathfrak{t}^*$ extended trivially to $\mathbb{C}^\times_q$. We define the equivariant quantum connection $\mathcal{Q}^{G^\vee/P^\vee}_{T^\vee}$ to be the connection on $H^*_{T^\vee}(G^\vee/P^\vee)$ over $\mathbb{C}^\times_q \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$ by

$$\mathcal{Q}^{G^\vee/P^\vee}_{T^\vee} := d + c_1^T\big(O(1)\big) *_q \frac{dq}{q},$$

where $c_1^T(O(1))$ denotes the equivariant first Chern class of $O(1)$, and $*_q$ denotes equivariant quantum multiplication. We have that $c_1^T(O(1)) = \sigma_{\mathrm{i}} - \varpi_{\mathrm{i}}^\vee \sigma_{\mathrm{id}}$ in $QH^*_{T^\vee}(G^\vee/P^\vee)$, so by Theorem 10.2,

$$c_1^T\big(O(1)\big) *_q \sigma_w = -w \cdot \varpi_i^\vee \sigma_w + \sum_{\beta^\vee} \langle \varpi_i^\vee, \beta\rangle \sigma_{ws_\beta} + \sum_{\nu^\vee} \langle \varpi_i^\vee, \nu\rangle q_{\eta_P(\nu)} \sigma_{\pi_P(ws_\nu)}.$$

Theorem 4.14 has the following equivariant generalization.

THEOREM 10.3
If $P^\vee \subset G^\vee$ is minuscule with corresponding minuscule representation $V_{\varpi_{\mathrm{i}}^\vee}$, then under the isomorphism $L : H^*(G^\vee/P^\vee) \to V_{\varpi_{\mathrm{i}}^\vee}$ of (4.11.1), the equivariant quantum connection $\mathcal{Q}^{G^\vee/P^\vee}_{T^\vee}$ is isomorphic to the pulled back connection $(\mathrm{id}_q \times \mathrm{inv})^* \widetilde{\nabla}^{(G^\vee, \varpi_{\mathrm{i}}^\vee)}$, where $\mathrm{inv} : \mathfrak{t}^* \to \mathfrak{t}^*$ is the map $h \mapsto -h$, and $\mathrm{id}_q : \mathbb{C}^\times_q \to \mathbb{C}^\times_q$ is the identity map.

Proof
The extra term in $c_1^T(O(1)) *_q \sigma_w$, not present in the nonequivariant case is $-w \cdot \varpi_{\mathrm{i}}^\vee \sigma_w$. Evaluating at $h \in \mathfrak{t}^*$, we get the term $-\langle w \cdot \varpi_{\mathrm{i}}^\vee, h\rangle \sigma_w$. This agrees with the calculation $-h \cdot v_w = -\langle w \cdot \varpi_{\mathrm{i}}^\vee, h\rangle v_w$ for $\mathfrak{g}^\vee$ acting on $v_w \in V$. The result then follows from the calculation in Theorem 4.14. $\square$

10.4. Character D-module of a weighted geometric crystal
Define the *weighted character D-module* of the geometric crystal X by

$$\mathrm{WCr}_{(G,P)} := R\pi_*(\mathsf{E}^f \otimes \gamma^* \mathsf{M}_T), \tag{10.4.1}$$

where we recall that $\gamma : X \to T$ is the weight map (6.4.1). It is a D-module over $Z(L_P) \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$. By taking the fiber over $h \in \mathfrak{t}^*$, we obtain the D-module (1.12.1).

Theorem 7.10 has the following generalization.

THEOREM 10.5
Suppose that $P = P_{\mathrm{i}}$ is cominuscule, and identify the bases $Z(L_P) \xrightarrow{\sim} \mathbb{P}^1 \backslash \{0, \infty\}$

via α_{i}. *Then the character D-module* $\mathrm{WCr}_{(G,P)}$ *is isomorphic to a pullback* $(\mathrm{id}_q \times \mathrm{inv})^*\mathrm{TKl}_{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$ *of the twisted Kloosterman D-module.*

Proof
The proof is the same as that of Theorem 7.10, so we sketch the main differences. According to Proposition 7.9, we have $t^{-1} f_T(\tilde{\iota}(x)) = \gamma(x)^{-1}$. Thus adding f_T to the diagram (7.10.2) we can write, with M^{-1} denoting the pullback under inverse of the multiplicative D-module on $\mathbb{P}^1 \backslash \{0,\infty\}$,

$$\begin{aligned}\mathrm{WCr}_{(G,P)} &= R\pi_*(\theta^*\mathsf{E}^{-\phi}) \otimes f_T^*\mathsf{M}_T \otimes \pi^*\mathsf{M}^{-1}) \\ &= R\pi_*\big((f_+, f_0)^*\mathsf{E}^{\phi} \otimes f_T^*\mathsf{M}_T\big) \otimes \mathsf{M}^{-1} \\ &= \mathrm{TKl}_{G^\vee} \otimes \mathsf{M}^{-1},\end{aligned}$$

where we have used the projection formula (see [73, Corollary 1.7.5]) for the second equality. For the third equality, we have used a variant of Lemma 9.3, namely, that $\mathrm{pr}_{2,!}(\mathrm{pr}_1^* A_{\mathscr{g},T} \otimes D_\lambda) = \mathrm{pr}_{2,*}(\mathrm{pr}_1^* A_{\mathscr{g},T} \otimes D_\lambda)$ and $R^i\mathrm{pr}_{2,*}(\mathrm{pr}_1^* A_{\mathscr{g},T} \otimes D_\lambda) = 0$ for $i > 0$ (cf. [71, Section 4.1]). Since M^{-1} is isomorphic to $\mathcal{O}_{\mathbb{P}^1\backslash\{0,\infty\}}$ as D-modules, the conclusion follows. ☐

10.6. The equivariant mirror theorem
Combining Theorems 10.3, 9.6, and 10.5, we obtain the following equivariant strengthening of Theorem 8.3.

THEOREM 10.7
Suppose that P is a cominuscule parabolic subgroup of G, and let $P^\vee$ *be the dual minuscule parabolic subgroup of* $G^\vee$. *We have an isomorphism*

$$\mathrm{WCr}_{(G,P)} \cong \mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$$

of D-modules over $Z(L_P) \times \mathfrak{t}^*$ *relative to* $\mathfrak{t}^*$.

In the case that $G^\vee/P^\vee$ is a Grassmannian, an injection from $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ into $\mathrm{WCr}_{(G,P)}$ is constructed by Marsh and Rietsch [93, Theorems 5.5 and 4.10].

11. The $\hbar$-mirror theorem
Recall that $S = \mathrm{Sym}(\mathfrak{t})$. We introduce an additional parameter $\hbar$ and work with $(D_\hbar \otimes S)$-modules. We shall establish Theorem 11.14, which is a stronger version of Theorem 10.7.

11.1. $D_\hbar \otimes S$*-modules*

The definition of the sheaf $D_{\hbar,X}$ of *ħ-differential operators* on a complex smooth affine algebraic variety X equipped with a $\mathbb{G}_m$-action is recalled in Section 16. An *S-structure* on a $D_{\hbar,X}$-module $\mathcal{M}$ is an action of S on $\mathcal{M}$ that commutes with the $D_{\hbar,X}$-action. Equivalently, $\mathcal{M}$ is a module for the sheaf $D_{\hbar,X} \otimes S$, where elements of S are considered "scalars." For any $D_{\hbar,X}$-module $\mathcal{M}$, the sheaf $\mathcal{M} \otimes S$ is a $(D_{\hbar,X} \otimes S)$-module.

Our basic example is the *multiplicative* $(D_{\hbar,T} \otimes S)$*-module* $\mathsf{M}_T^{1/\hbar}$ on T, defined as follows.

Definition 11.2

For a complex torus T and $\mathfrak{t} = \mathrm{Lie}(T)$, let

$$\mathsf{M}_T^{1/\hbar} := D_{\hbar,T} \otimes S/\langle \xi_k - k \mid k \in \mathfrak{t} \subset S\rangle.$$

We give T the trivial $\mathbb{G}_m$-action and furthermore declare that $k \in \mathfrak{t} \subset S$ has degree 1. This gives $\mathsf{M}_T^{1/\hbar}$ the structure of a *graded* $D_{\hbar,T}$*-module*.

Remark 11.3

The $(D_{\hbar,T} \otimes S)$-module $\mathsf{M}_T^{1/\hbar}$ is a free $(\mathcal{O}_T \otimes S)$-module with basis element "$e^{\ell/\hbar}$," with the action of $\xi_k := (h \mapsto \langle k, h\rangle) \in D_{\hbar,T}$ given by

$$\xi_k \cdot e^{\ell/\hbar} := k e^{\ell/\hbar},$$

for $k \in \mathfrak{t} \subset S$. Here ξ_k should be thought of as "$\hbar\partial_k$" (see Section 16). And "ℓ" should be thought of as the *multivalued function* from T to $S = \mathbb{C}[\mathfrak{t}^*]$ given by

$$\ell(t)(h) := \big\langle \log(t), h\big\rangle, \quad \text{where } h \in \mathfrak{t}^*. \tag{11.3.1}$$

Recall that an *ħ-connection* on a bundle E over X is a $\mathbb{C}$-linear operator $\nabla : \Gamma(X,E) \to \Gamma(X,E) \otimes \Omega_X$ such that $\nabla(gs) = g\nabla(s) + \hbar s \otimes dg$, where $g \in \mathcal{O}_X$ and $s \in \Gamma(X,E)$ are sections. An $\hbar$-connection for $\hbar = 1$ is simply a connection in the usual sense. An alternative description of $\mathsf{M}_T^{1/\hbar}$ is as follows: take the trivial $S[\hbar]$-bundle on T and equip it with the $\hbar$-connection $\hbar d - k$, where $k \in \mathfrak{t} \subset S$.

Suppose that we have $\mathbb{G}_m$-actions on E and X such that the projection $E \to X$ is $\mathbb{G}_m$-equivariant. We then say that the $\hbar$-connection ∇ is *graded* if $\hbar^{-1}\nabla$ is $\mathbb{G}_m$-equivariant, where $\hbar$ is taken to be degree 1 for the $\mathbb{G}_m$-action. Equivalently, if $\nabla = \hbar d + \eta$, then we require that η has degree 1 for the $\mathbb{G}_m$-action.

11.4. Frenkel–Gross connection revisited

Let V be a finite-dimensional $G^\vee$-module, and let $\mu : V \times \mathfrak{t}^* \to V$ denote the action map of $\mathfrak{t}^*$. Let $\mu^* : V \to V \otimes S$ denote the map defined by $\mu^*(v) = v \otimes k$ if $v \in V$

has weight $k \in \mathfrak{t}$. By extending scalars, we obtain a map $\mu^* : V \otimes S \to V \otimes S$. For a $G^\vee$-module V, define the *deformed Frenkel–Gross $\hbar$-connection*

$$\widetilde{\nabla}_{\hbar}^{(G^\vee,V)} := \hbar d + (y_p + \mu^*)\frac{dq}{q} + x_\theta\, dq,$$

acting on the trivial $(V \otimes S[\hbar])$-bundle on $\mathbb{P}^1 \backslash \{0,\infty\}$. Thus for $\hbar = 1$ and $h \in \mathfrak{t}^*$ inducing an evaluation homomorphism $h : S \to \mathbb{C}$, we have that $\widetilde{\nabla}_{\hbar}^{(G^\vee,V)}|_{\hbar=1} \otimes_S \mathbb{C} \cong \widetilde{\nabla}^{(G^\vee,V)} \otimes_S \mathbb{C}$ reduces to (9.1.1).

Declaring that $k \in \mathfrak{t} \subset S$ sits in degree 1, the $\mathbb{G}_m$-action of Section 3.4 extends to this deformation so that the 1-form $(y_p + \mu^*)\frac{dq}{q} + x_\theta\, dq$ has degree 1.

11.5. Equivariant quantum connection revisited

Define the *equivariant $\hbar$-quantum connection*

$$\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee} := \hbar d + c_1^T\big(\mathcal{O}(1)\big) *_q \frac{dq}{q},$$

acting on the trivial bundle over $\operatorname{Spec}\mathbb{C}[q,q^{-1}]$ with fiber the equivariant cohomology $H^*_{T^\vee}(G^\vee/P^\vee) \otimes \mathbb{C}[\hbar]$. Here $c_1^T(\mathcal{O}(1))*_q$ denotes the equivariant quantum cohomology action. Then $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ from Section 10.1 is equal to $\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee}|_{\hbar=1}$.

As in Section 4.5, we define a $\mathbb{G}_m$-action on $QH^*_{T^\vee}(G^\vee/P^\vee)$ by using half the topological degree. As before, $k \in \mathfrak{t} \subset S$ sits in degree 1. The connection 1-form $(\sigma_{\mathrm{i}} *_{q,h} - \varpi_{\mathrm{i}}^\vee)\frac{dq}{q}$ is then homogeneous of degree 1 for the $\mathbb{G}_m$-action.

We then have the following variation of Theorem 10.3.

THEOREM 11.6

If $P^\vee \subset G^\vee$ is minuscule and with corresponding minuscule representation $V_{\varpi_{\mathrm{i}}^\vee}$, then under the isomorphism $L : H^*(G^\vee/P^\vee) \to V_{\varpi_{\mathrm{i}}^\vee}$ of (4.11.1), the equivariant quantum connection $\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee}$ is identified with the deformed Frenkel–Gross $\hbar$-connection $(\mathrm{id}_q \times \mathrm{inv})^* \widetilde{\nabla}_{\hbar}^{(G^\vee,\varpi_{\mathrm{i}}^\vee)}$. This is an isomorphism of graded $\hbar$-connections on $\mathbb{P}^1 \backslash \{0,\infty\} \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$.

PROPOSITION 11.7

For any $\lambda \in \mathbb{C}^\times$, there is an isomorphism

$$\mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee}|_{\hbar=\lambda} \cong [q \mapsto q/\lambda^c]^*[h \mapsto h/\lambda]^* \mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$$

of connections on $\mathbb{P}^1 \backslash \{0,\infty\} \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$.

Proof

Recall that $QH^*_{T^\vee}(G^\vee/P^\vee)$ is a graded ring with the topological degree $\deg(\sigma_w) =$

$2\ell(w)$ and that it follows from Lemma 4.8 that $\deg(q) = 2c$. The gauge transformation $\sigma_w \mapsto \lambda^{\ell(w)}\sigma_w$ then gives the desired isomorphism between the two connections. ☐

11.8. Twisted Kloosterman $D_\hbar$-modules

Define the *exponential $D_{\hbar,\mathbb{A}^1}$-module* by

$$\mathsf{E}^{1/\hbar} := D_{\hbar,\mathbb{A}^1}/D_{\hbar,\mathbb{A}^1}(\hbar\partial_x - 1).$$

Recall the generic affine character $\phi : I(1)/I(2) \to \mathbb{A}^1$, and let $\mathsf{E}^{\phi/\hbar} := \phi^*\mathsf{E}^{1/\hbar}$ denote the pullback.

Let $A^{1/\hbar}_{\mathcal{G},T}$ denote the $(D_\hbar \otimes S)$-module on $\mathrm{Bun}_{\mathcal{G}(1,2)}$ given by taking the $(D_\hbar \otimes S)$-module $\mathsf{M}^{1/\hbar}_T \boxtimes \mathsf{E}^{\phi/\hbar}$ on $T \times I(1)/I(2)$ and pushing it forward to $\mathrm{Bun}_{\mathcal{G}(1,2)}$. We may define the twisted Kloosterman $(D_\hbar \otimes S)$-module $\mathrm{TKl}^{1/\hbar}_{G^\vee}$ on $\mathbb{P}^1\backslash\{0,\infty\}$ as $\mathrm{TKl}_{G^\vee} \otimes \mathbb{C}[\hbar]$. As before, we have associated $(D_\hbar \otimes S)$-modules $\mathrm{TKl}^{1/\hbar}_{(G^\vee,V)}$ and $\mathrm{TKl}^{1/\hbar}_{(G^\vee,\varpi_i^\vee)}$. The $\mathbb{G}_m$-action of Section 7.12 gives the structure of a graded $(D_\hbar \otimes S)$-module.

THEOREM 11.9

Let $\varpi^\vee$ be a minuscule fundamental weight of $G^\vee$. There is a choice of basis element $x_\theta \in \mathfrak{g}^\vee_\theta$ such that we have an isomorphism of graded $(D_{\hbar,\mathbb{P}^1\backslash\{0,\infty\}} \otimes S)$-modules

$$\mathrm{TKl}^{1/\hbar}_{(G^\vee,\varpi^\vee)} \cong \widetilde{\nabla}^{(G^\vee,\varpi^\vee)}_\hbar.$$

As before, we normalize conventions so that ϕ matches with the choice of x_θ from (3.9.1).

Proof

The proof is a variation of the proof of Theorem 9.6. It suffices to show that $\mathrm{TKl}^{1/\hbar}_{G^\vee} \cong \widetilde{\nabla}^{G^\vee}_\hbar$ holds when specializing $\hbar = 1$. Notationwise, the convention is that omitting $\hbar$ from the notation of a $D_\hbar$-module gives the corresponding D-module at $\hbar = 1$. Let $S = \mathrm{Sym}(\mathfrak{t})$, and let $\iota : S^W \to S$ denote the natural inclusion. Consider the automorphic sheaf

$$\mathrm{Aut}_{\mathcal{G},T} := \omega^{-1/2}_{\mathrm{Bun}_{\mathcal{G}}} \otimes \big(D' \otimes_{S^W \otimes U(\mathfrak{v}),\iota\otimes\varphi} (S \otimes \mathbb{C})\big),$$

defined using Proposition 9.12 and the natural isomorphism $U(\mathfrak{t}) \cong S$. The same argument as in the proof of Theorem 9.6 gives that $\mathrm{Aut}_{\mathcal{G},T}$ is a holonomic $(D' \otimes S)$-module. The technology of [128] shows that $\mathrm{Aut}_{\mathcal{G},T}$ is a Hecke eigensheaf on $\mathrm{Bun}_{\mathcal{G}}$. Let $\mathcal{E}$ denote its Hecke eigenvalue, and for a finite-dimensional G-module V, let

$\mathcal{E}^V$ denote its associated bundle. Then $\mathcal{E}^V$ is a $(D_{\mathbb{G}_m} \otimes S)$-module isomorphic to $\nabla^{(G^\vee, V)}$.

On the other hand, as in the proof of Theorem 9.6, $\mathrm{Aut}_{\mathcal{G},T}$ restricted to $\mathring{\mathrm{Bun}}_{\mathcal{G}} \cong T \times \mathfrak{v}$ is isomorphic to $\mathsf{M}_T \boxtimes \mathsf{E}^{\phi}$. Furthermore, $\mathrm{Aut}_{\mathcal{G},T}$ is a $(T \times \mathfrak{v}, \mathsf{M}_T \boxtimes \mathsf{E}^{\phi})$-equivariant D-module on $\mathrm{Bun}_{\mathcal{G}}$. It follows that $\mathrm{Aut}_{\mathcal{G},T} \cong A_{\mathcal{G},T}$. Thus $\mathrm{TKl}_{(G^\vee,V)} \cong \widetilde{\nabla}^{(G^\vee,V)}$ for any V, or equivalently, $\mathrm{TKl}_{G^\vee} \cong \widetilde{\nabla}^{G^\vee}$.

We note that the $\mathbb{G}_m$-actions of Section 3.4 and Section 7.12 are in agreement: they are both induced by the trivial $\mathbb{G}_m$-action on T, the dilation action on $\mathfrak{v} = I(1)/I(2)$, and the action $\zeta \cdot q = \zeta^c q$ of the curve $\mathbb{P}^1 \backslash \{0, \infty\}$ (noting that the Coxeter numbers of G and $G^\vee$ coincide). Thus $\mathrm{TKl}_{G^\vee} \cong \widetilde{\nabla}^{G^\vee}$ as filtered $(D_{\mathbb{P}^1 \backslash \{0,\infty\}} \otimes S)$-modules, where the filtration is induced by the $\mathbb{G}_m$-action on $\mathbb{C}_q^\times$ as explained in Section 16.1. □

11.10. Weighted geometric crystal D-module revisited

We use notation similar to Section 6. Let $\mathrm{WCr}_{(G,P)}^{1/\hbar} := R\pi_*(\mathsf{M}_T^{\gamma/\hbar} \otimes \mathsf{E}^{f/\hbar})$ be the pushforward $(D_{\hbar, Z(L_P)} \otimes S)$-module on $Z(L_P) \otimes S \cong \mathbb{G}_{m,\mathrm{Sym}(t)}$. According to Proposition 6.24, we have that $\pi : X \to Z(L_P)$, $f : X \to \mathbb{A}^1$, and $\gamma : X \to T$ are $\mathbb{G}_m$-equivariant. Thus $\mathrm{WCr}_{(G,P)}^{1/\hbar}$ acquires a natural structure of a graded $(D_{\hbar,Z(L_P)} \otimes S)$-module. In Proposition 16.13, we show that $\mathrm{WCr}_{(G,P)}^{1/\hbar}$ is $\hbar$-torsion-free.

PROPOSITION 11.11

(i) *For any $\lambda \in \mathbb{C}^\times$, there is an isomorphism of $D_{Z(L_P)}$-modules*

$$\mathrm{WCr}_{(G,P)}^{1/\hbar}|_{\hbar=\lambda} \cong [q \mapsto q/\lambda^c]^*[h \mapsto h/\lambda]^* \mathrm{WCr}_{(G,P)},$$

where c is the Coxeter number of G.

(ii) *There is an isomorphism of $(D_{\hbar,Z(L_P)} \otimes S)$-modules*

$$\mathrm{WCr}_{(G,P)}^{1/\hbar} \cong \mathrm{WCr}_{(G,P)} \otimes_{\mathbb{C}} \mathbb{C}[\hbar].$$

Proof

Assertion (i) follows from the homogeneity of the potential f established in Section 6.21 combined with Corollary 6.25 and Lemma 6.29. Note that $\mathsf{M}_T^{\gamma/\hbar}$ is multiplicative and thus invariant under any Kummer pullback.

From (i) we deduce that $\mathrm{WCr}_{(G,P)}^{1/\hbar}$ and $\mathrm{WCr}_{(G,P)} \otimes_{\mathbb{C}} \mathbb{C}[\hbar]$ are isomorphic after localizing $D_{\hbar,\mathbb{P}^1 \backslash \{0,\infty\}}$ at $(\hbar)$. Proposition 16.13 says that $\mathrm{WCr}_{(G,P)}^{1/\hbar}$ is $\hbar$-torsion free, and $\mathrm{WCr}_{(G,P)} \otimes \mathbb{C}[\hbar]$ is also $\hbar$-torsion-free by construction; hence the isomorphism extends to $D_{\hbar,\mathbb{P}^1 \backslash \{0,\infty\}}$. □

The following result has an identical proof to Theorem 10.5.

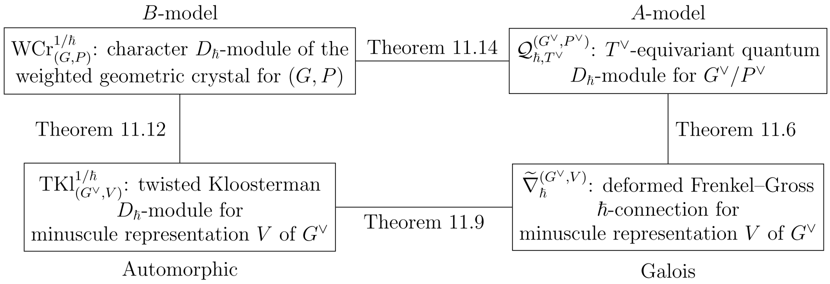

Figure 3. The four $D_\hbar$-modules in the proof of Theorem 11.14.

THEOREM 11.12
Suppose that $P = P_{\mathrm{i}}$ is cominuscule, and identify the bases $Z(L_P) \xrightarrow{\sim} \mathbb{P}^1 \backslash \{0,\infty\}$ via α_{i}. Then the graded character $(D_{\hbar,Z(L_P)} \otimes S)$-module $\mathrm{WCr}^{1/\hbar}_{(G,P)}$ is isomorphic to the graded Kloosterman $(D_{\hbar,\mathbb{P}^1\backslash\{0,\infty\}} \otimes S)$-module $(\mathrm{id}_q \times \mathrm{inv})^* \mathrm{TKl}^{1/\hbar}_{(G^\vee,\varpi^\vee_{\mathrm{i}})}$.

11.13. The $D_\hbar \otimes \mathrm{Sym}(\mathfrak{t})$ mirror theorem
Combining Theorems 11.6, 11.9, and 11.12, we obtain the result given in Figure 3.

THEOREM 11.14
Suppose that P is a cominuscule parabolic subgroup of an almost simple algebraic group G. We have an isomorphism of graded $(D_{\hbar,Z(L_P)} \otimes \mathrm{Sym}(\mathfrak{t}))$-modules

$$\mathrm{WCr}^{1/\hbar}_{(G,P)} \cong \mathcal{Q}^{G^\vee/P^\vee}_{\hbar,T^\vee}.$$

12. Proof of the Peterson isomorphism

We deduce the equivariant Peterson isomorphism (Theorem 12.4) by specializing $\hbar \to 0$ in Theorem 11.14.

12.1. The Gauss–Manin model
Recall that $S = \mathrm{Sym}(\mathfrak{t}) = \mathbb{C}[\mathfrak{t}^*]$. In this subsection, we describe the $(D_{\hbar,\mathbb{P}^1\backslash\{0,\infty\}} \otimes S)$-module $\mathrm{WCr}^{1/\hbar}_{(G,P)}$ more explicitly.

By [73, Proposition 1.5.28(i)] and Proposition 16.7, we may compute $R\pi_*(\mathsf{M}^{\gamma/\hbar}\mathsf{E}^{f/\hbar})$ by computing the sheaf pushforward $\mathrm{GM}^\bullet_\hbar$ along π of the relative de Rham complex $\mathrm{DR}^\bullet_{X/Z(L_P)}(\mathsf{M}^{\gamma/\hbar}\mathsf{E}^{f/\hbar})$. Since $X \cong B^{w_P}_- \times Z(L_P)$, where $B^{w_P}_-$ and $Z(L_P)$ are both affine (and thus also D-affine), it suffices to work with the modules of global sections. The complex $\mathrm{GM}^\bullet_\hbar$ is given by

$$\Omega^0\big(X/Z(L_P)\big) \otimes_{\mathbb{C}} S \to \cdots \to \Omega^{d-1}\big(X/Z(L_P)\big) \otimes_{\mathbb{C}} S \to \Omega^d\big(X/Z(L_P)\big) \otimes_{\mathbb{C}} S,$$

where $d := \dim X - \dim Z(L_P) = \dim B_-^{w_P}$, and $\Omega^k(X/Z(L_P))$ is the module of relative global differentials. Here, the space of global sections of the rank-1 $(D_{\hbar,X} \otimes S)$-module $\mathsf{M}^{\gamma/\hbar}\mathsf{E}^{f/\hbar}$ has been identified with $S[X] = \mathbb{C}[X] \otimes S$ and the $\hbar$-differential is given accordingly by

$$\hbar d + df + \gamma^{-1}\, d\gamma.$$

Here, the differential $\hbar d$ and the forms df and $d\gamma$ are both relative: no differentiation is made in the $Z(L_P)$-direction. The form $d\gamma$ is the differential of the weight map, valued in $\mathfrak{t} = \mathrm{Lie}(T)$.

By Proposition 16.13, we know that $R\pi_*(\mathsf{M}^{\gamma/\hbar}\mathsf{E}^{f/\hbar})$ vanishes except in one degree, so the only nonzero cohomology group of $\mathrm{GM}_\hbar^\bullet$ is

$$\mathrm{GM}_\hbar := \mathrm{coker}\big(\Omega^{d-1}\big(X/Z(L_P)\big) \otimes S \to \Omega^d\big(X/Z(L_P)\big) \otimes S\big).$$

Now, $X \cong B_-^{w_P} \times Z(L_P)$ is an open subset of affine space: specifically, $B_-^{w_P}$ is an open subset of a Schubert cell in G/P. Let $x_1, x_2, \dots, x_d$ be coordinates for this Schubert cell. Let

$$A := \mathrm{Sym}(\mathfrak{t})\big[Z(L_P)\big] = \mathbb{C}\big[X^*\big(Z(L_P)\big)\big] \otimes S = \mathbb{C}[\mathfrak{t}^*, q_i^{\pm 1} \mid i \notin I_P],$$

which is a Laurent polynomial ring over S in $\dim Z(L_P)$ variables, and let $A[B_-^{w_P}] \cong S[X]$. Then $\mathbb{C}[B_-^{w_P}]$ is a localization of $\mathbb{C}[x_1, \dots, x_d]$, and we have isomorphisms of $A[B_-^{w_P}]$-modules

$$\Omega^d\big(X/Z(L_P)\big) \otimes S \cong A[B_-^{w_P}] \cdot \omega,$$
$$\Omega^{d-1}\big(X/Z(L_P)\big) \otimes S \cong \sum_i A[B_-^{w_P}] \cdot \omega_i,$$

where $\omega = \prod_{j=1}^d dx_j$ and $\omega_i = \prod_{j \neq i} dx_j$. Thus the Gauss–Manin module $\mathrm{GM}_\hbar$ can be written explicitly in terms of coordinates by computing the partial derivatives $\frac{\partial f}{\partial x_j} + \gamma^{-1}\frac{\partial \gamma}{\partial x_j}$. Here, $\frac{\partial \gamma}{\partial x_j}$ are the components of the differential $d\gamma$.

The Gauss–Manin module $\mathrm{GM}_\hbar$ is an $A\langle \hbar\partial_{q_i} \mid i \notin I_P\rangle$-module, where $\hbar\partial_{q_i}$ acts via the operator $\hbar\partial_{q_i} + \frac{\partial f}{\partial q_i}$.

Remark 12.2

We may write “f_S” for the weighted *multivalued* potential, that is, $f_S := f + \ell \circ \gamma$, where ℓ is defined as in (11.3.1). The multivaluedness implies that it is not quite an element of $S[X]$. However, its differential $df_S = df + \frac{d\gamma}{\gamma}$ is well defined.

12.3. Peterson isomorphism

Consider the Jacobian ring of the potential f with weight γ:

$$\mathrm{Jac}\big(X/Z(L_P), f, \gamma\big) := \mathrm{Sym}(\mathfrak{t})[X]/\Big(\frac{\partial f}{\partial x_1} + \frac{\partial \gamma}{\gamma \partial x_1}, \dots, \frac{\partial f}{\partial x_d} + \frac{\partial \gamma}{\gamma \partial x_d}\Big).$$

It is independent of the choice of coordinates of $B_-^{w_P}$ because it can be identified with the cokernel of the wedge map with $df + \frac{d\gamma}{\gamma}$ from $\Omega^{d-1}(X/Z(L_P))$ to $\Omega^d(X/Z(L_P))$.

THEOREM 12.4
If $P^\vee$ is minuscule, then we have an isomorphism of $\mathrm{Sym}(\mathfrak{t})[Z(L_P)]$*-algebras*

$$\mathrm{Jac}\big(X/Z(L_P), f, \gamma\big) \cong QH^*_{T^\vee}(G^\vee/P^\vee).$$

Moreover, multiplication by $q\frac{\partial f}{\partial q}$ on the left-hand side corresponds to quantum multiplication by $c_1^T(\mathcal{O}(1)) = \sigma_{\mathrm{i}} - \varpi_{\mathrm{i}}^\vee$ on the right-hand side.

Proof
Recall that $A = \mathrm{Sym}(\mathfrak{t})[Z(L_P)] = \mathbb{C}[\mathfrak{t}^*, q^{\pm 1}]$. By Theorem 11.14, we have an isomorphism of $A\langle \hbar\partial_q\rangle$-modules between $\mathrm{GM}_\hbar$ and the equivariant quantum connection $\mathcal{Q}^{G^\vee/P^\vee}_{\hbar,T^\vee}$, which is the $A\langle \hbar\partial_q\rangle$-module $QH^*_{T^\vee}(G^\vee/P^\vee) \otimes \mathbb{C}[\hbar]$ with the action of $\hbar q\partial_q$ given by $\hbar q\partial_q + (\sigma_{\mathrm{i}} *_q) - \varpi_{\mathrm{i}}$.

At $\hbar = 0$, the map is given by wedging with the relative differential $df + \frac{d\gamma}{\gamma}$, so we have $\mathrm{GM}_0 \cong \mathrm{Jac}(X/Z(L_P), f, \gamma)$ as an $S[q^{\pm 1}]\langle \hbar\partial_q\rangle$-module with the action of $\hbar\partial_q$ given by multiplication by $\frac{\partial f}{\partial q}$ on the right-hand side which we denote by $\mathrm{Jac}(f, \gamma)$ for short.

Under the above isomorphism

$$\alpha : QH^*_{T^\vee}(G^\vee/P^\vee) \cong \mathrm{Jac}(f, \gamma)$$

of $S[q^{\pm 1}]$-modules, quantum multiplication by $\sigma_{\mathrm{i}} - \varpi_{\mathrm{i}}$ corresponds to multiplication by $q\frac{\partial f}{\partial q}$ in $\mathrm{Jac}(f, \gamma)$.

Since $QH^*_{T^\vee}(G^\vee/P^\vee)$ is a free $S[q^{\pm 1}]$-module, we deduce that $\mathrm{Jac}(f, \gamma)$ is also free. Let $\alpha(1_H)$ be the image of the identity 1_H of the ring $H^*_{T^\vee}(G^\vee/P^\vee)$, and let $1_J \in \mathrm{Jac}(f, \gamma)$ denote the identity of the ring $\mathrm{Jac}(f, \gamma)$. It also follows that there exists $\zeta \in \mathrm{Jac}(f, \gamma) \otimes_S \mathbb{C}(\mathfrak{t}^*)$ so that $\alpha(1_H) \cdot \zeta = 1_J$. Let $\zeta\alpha : H^*_{T^\vee}(G^\vee/P^\vee) \cong \mathrm{Jac}(f, \gamma)$ denote the composition of the $S[q^{\pm 1}]$-module isomorphism α with left multiplication by ζ. Then $\zeta\alpha(1_H) = 1_J$ and $\zeta\alpha$ sends quantum multiplication by σ_{i} to multiplication by $q\frac{\partial f}{\partial q}$.

Recall that $S \cong \mathbb{C}[\mathfrak{t}^*]$, so the fraction field is $\mathbb{C}(\mathfrak{t}^*)$. By [94, Corollary 6.5] and [29, Lemma 4.1.3], $QH^*_{T^\vee}(G^\vee/P^\vee) \otimes_S \mathrm{Frac}(S)$ is generated over $\mathrm{Frac}(S)[q^{\pm 1}]$ by σ_{i}, and thus also by $c_1^T(\mathcal{O}(1)) = \sigma_{\mathrm{i}} - \varpi_{\mathrm{i}}^\vee$. We deduce that $\zeta\alpha$ induces a $\mathrm{Frac}(S)[q^{\pm 1}]$-algebra isomorphism after localization. Since the $S[q^{\pm 1}]$-algebras $QH^*_{T^\vee}(G^\vee/P^\vee)$

and $\mathrm{Jac}(f,\gamma)$ are already free as S-modules, it follows that $\zeta\alpha$ is an isomorphism of $S[q^{\pm 1}]$-algebras. $\square$

Recall from (1.17.1) the definition of the Peterson stratum $\mathcal{Y}_P^*$. Rietsch [107] has proved that $\mathrm{Jac}(X/Z(L_P), f, \gamma)$ is isomorphic to $\mathbb{C}[\mathcal{Y}_P^*]$. We thus obtain the following corollary.

COROLLARY 12.5
If $P^\vee$ is minuscule, then we have an isomorphism of $\mathrm{Sym}(\mathfrak{t})[Z(L_P)]$*-algebras*

$$QH^*_{T^\vee}(G^\vee/P^\vee) \cong \mathbb{C}[\mathcal{Y}_P^*].$$

12.6. Example
Consider the case $G^\vee/P^\vee = \mathrm{Gr}(1,n+1) = \mathbb{P}^n$. We have that the equivariant quantum $A\langle\hbar\partial_q\rangle$-module $\mathcal{Q}^{\mathbb{P}^n}_{\hbar,T^\vee}$ is given by the connection

$$\hbar q\frac{d}{dq} + \begin{pmatrix} h_1 & & & q \\ 1 & h_2 & & \\ & \ddots & \ddots & \\ & & 1 & h_{n+1} \end{pmatrix},$$

where $\sum_{i=1}^{n+1} h_i = 0$, and we identify $h = (h_1, h_2, \dots, h_{n+1}) \in \mathfrak{t}^*$ in the usual way. Its dual is isomorphic to $A\langle\hbar\partial_q\rangle/A\langle\hbar\partial_q\rangle L$, where

$$L := \prod_{i=1}^{n+1}\Big(\hbar q\frac{d}{dq} - h_i\Big) - q.$$

This is a hypergeometric differential operator of type ${}_0F_n$. In the notation of [78, Section 3], we see that for $\lambda \in \mathbb{C}^\times$, $\mathcal{Q}^{\mathbb{P}^n}_{\hbar,T^\vee}|_{\hbar=\lambda}$ is the hypergeometric D-module $\mathcal{H}_\hbar(\frac{h_i{}'s}{\lambda}, \emptyset)$. On the other hand, the character $A\langle\hbar\partial_q\rangle$-module $\mathrm{WCr}^{1/\hbar}$ is given by the π_*-pushforward of $\mathsf{M}^{\gamma/\hbar}\mathsf{E}^{f_q/\hbar}$, that is,

$$\int_{x_1\cdots x_{n+1}=q} x_1^{h_1/\hbar}\cdots x_{n+1}^{h_{n+1}/\hbar}\mathsf{E}^{(x_1+\cdots+x_n+x_{n+1})/\hbar}\frac{dx_1\cdots dx_{n+1}}{x_1\cdots x_{n+1}}.$$

The mirror isomorphism $\mathcal{Q}^{\mathbb{P}^n}_{\hbar,T^\vee} \cong \mathrm{WCr}^{1/\hbar}_{(G,P)}$ of Theorem 11.14 follows in this case from a result of Katz [78, Theorem 5.3.1] on convolution of hypergeometric D-modules. In the semiclassical limit $\hbar \to 0$, we recover the equivariant quantum cohomology algebra

$$QH^*_{T^\vee}(\mathbb{P}^n) = \mathbb{C}[x, q^{\pm 1}, \mathfrak{t}^*]\Big/\Big(\prod_{i=1}^{n+1}(x-h_i) = q\Big)$$

from the quantum connection $\mathcal{Q}^{\mathbb{P}^n}_{\hbar,T^\vee}$ on the one hand. And on the other hand, from the potential function $f + \gamma^{-1}\,d\gamma$, and in view of

$$x_i \frac{\partial f}{\partial x_i} + \gamma^{-1} \frac{x_i \partial \gamma}{\partial x_i} = x_i + h_i - \frac{q}{x_1 \cdots x_n},$$

we recover the Jacobi ring $\mathrm{Jac}(f, \gamma)$. By letting $x := x_i + h_i$, which is independent of i, we see that $\frac{\partial f}{\partial x_i} + \gamma^{-1} \frac{\partial \gamma}{\partial x_i} = 0$ is equivalent to $\prod_{i=1}^{n+1}(x - h_i) = q$, in agreement with Theorem 12.4.

13. An enumerative formula

In this section, we calculate the quantum period of minuscule flag varieties (Theorem 13.11). The first coefficient in the q-expansion corresponds to the identity of Corollary 8.4.

13.1. Solution of the geometric crystal D-module

We allow P to be arbitrary until Section 13.4. The Givental [54] integral formula for Whittaker functions (see also [52]) arises in the present context as solutions to $\mathrm{Cr}_{(G,P)}$ via a natural pairing with homology groups. Equivalently, these are special functions that are solutions of the quantum differential equation. The final Section 13.16 treats the case of the classical I_0 and K_0-Bessel functions as an illustration of the main concepts.

The solution complex of $\mathrm{Cr}_{(G,P)}$ is defined (see [73, Section 4.2]) to be

$$\mathrm{Sol}_{(G,P)} := \mathrm{RHom}_D(\mathrm{Cr}^{\mathrm{an}}_{(G,P)}, \mathcal{O}^{\mathrm{an}}_{Z(L_P)}).$$

Recall that $\mathrm{Cr}_{(G,P)} = R\pi_* \mathsf{E}^f$. By [73, Theorem 4.2.5], we can interpret the stalks of $\mathrm{Sol}_{(G,P)}$ as dual to the algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(\mathring{G/P}, \mathsf{E}^{f_t})$ for $t \in Z(L_P)$. Concretely, $\mathrm{Sol}_{(G,P)}$ is the local system of holomorphic horizontal sections of the connection dual to $\mathrm{Cr}_{(G,P)}$.

If P is cominuscule, then by Theorem 8.3, $\mathrm{Cr}_{(G,P)}$ is a coherent D-module; hence $\mathrm{Sol}_{(G,P)}$ is a local system on $Z(L_P)$. For every $t \in Z(L_P)$, we deduce that $H^i_{\mathrm{dR}}(\mathring{G/P}, \mathsf{E}^{f_t})$ is zero unless $i = d = \dim(G/P)$, and that $\dim H^d_{\mathrm{dR}}(\mathring{G/P}, \mathsf{E}^{f_t})$ is constant and equal to $|W^P|$.

13.2. Compact cycles

The following proposition holds for any open Richardson variety so we state and prove it in that generality. For $u \le w$ in W, recall that $\mathcal{R}^w_u$ denotes the open Richardson variety, defined to be the intersection of $B_- \dot{u} B/B$ with $B \dot{w} B/B$.

PROPOSITION 13.3

$H_{\ell(w)-\ell(u)}(\mathcal{R}^w_u)$ is 1-dimensional.

Proof
By Poincaré duality, it is equivalent to treat the cohomology with compact support $H_c^{\ell(w)-\ell(u)}(\mathcal{R}_u^w)$. By [103, Proposition 4.2.1], there is a canonical isomorphism

$$H_c^{\bullet}(\mathcal{R}_u^w) \cong \mathrm{Ext}^{\bullet+\ell(u)-\ell(w)}(M_w, M_u),$$

where M_w and M_u denote the Verma modules in the principal block. Since $\mathcal{R}_u^w$ has real dimension $2(\ell(w)-\ell(u))$, we have $H_c^{\ell(w)-\ell(u)}(\mathcal{R}_u^w) \cong \mathrm{Hom}(M_w, M_u)$. This space is 1-dimensional as follows from the Bernstein–Gelfand–Gelfand correspondence. □

To construct a middle dimension cycle generating $H_{\ell(w)-\ell(u)}(\mathcal{R}_u^w)$, we use that $\mathcal{R}_u^w$ contains many tori. (In fact by Leclerc [89], $\mathbb{C}[\mathcal{R}_u^w]$ contains a cluster algebra and is conjectured to be equal to 1.) We choose any cluster torus $(\mathbb{C}^\times)^{\ell(w)-\ell(u)} \subset \mathcal{R}_u^w$ and consider the middle dimension cycle given by a compact torus $(S^1)^{\ell(w)-\ell(u)}$. We denote integration along this cycle by $\oint$. We can normalize the form ω from [81] which has simple poles along the boundary of $\mathcal{R}_u^w$ such that

$$\oint \frac{\omega}{(2i\pi)^{\ell(w)-\ell(u)}} = 1.$$

In view of Proposition 13.3, the cycle is well defined and independent of the choice of tori.

Recall from Section 6.6 that $G\mathring{/}P \cong \mathcal{R}_{w_P w_0}^{w_0}$, which can be identified with the open projected Richardson variety in G/P. Thus we have shown that the space $H_d(G\mathring{/}P)$ is 1-dimensional and generated by the above compact cycle. For the case of full flag varieties G/B a related construction appears in [108, Section 7.1], and for the Grassmannian in [93, Theorem 4.2].

13.4. Poincaré duality
For $w \in W^P$, we define $\mathrm{PD}(w) := w_0 w w_0^P$, which is still an element of W^P. This is an involution and we have $\ell(\mathrm{PD}(w)) = d - \ell(w)$. Moreover, the Schubert class $\sigma_w \in H^{2\ell(w)}(G^\vee/P^\vee)$ is Poincaré dual to $\sigma_{\mathrm{PD}(w)} \in H^{2\ell(\mathrm{PD}(w))}(G^\vee/P^\vee)$. Since $G^\vee/P^\vee$ is minuscule, a reduced expression for $w \in W^P$ is unique up to commutation relations. It is always (see [21, Section 2.4]) a subexpression in any reduced expression for the longest element $w_P^{-1} = w_0 w_0^P = \mathrm{PD}(1)$ of W^P.

13.5. Givental fundamental solution
Givental has introduced solutions $S_w(\hbar, q)$ of the quantum $\hbar$-connection $\mathcal{Q}_\hbar^{G^\vee/P^\vee}$ in terms of a generating series of gravitational descendants of Gromov–Witten invariants (see [53], [5, Section 4.1], [32, Section 10], [67, Section 5], and [76, Section 2.3]

for details). The functions $S_w(1,q)$, for $w \in W^P$, form a fundamental solution of $\mathcal{Q}^{G^\vee/P^\vee}$ near the regular singular point $q = 0$ (see [48, Section 2]).

The Givental J-function is defined by

$$J^{G^\vee/P^\vee}(\lambda, q) := \sum_{w \in W^P} \langle S_w(\lambda, q), 1 \rangle \sigma_{\mathrm{PD}(w)}.$$

It gives rise to a multivalued holomorphic section

$$J^{G^\vee/P^\vee} : \mathbb{C}^\times_\lambda \times \mathbb{C}^\times_q \to H^*(G/P),$$

which becomes single-valued when factored through the universal cover $H^2(G^\vee/P^\vee) \to \mathbb{C}^\times_q$. Using the notation of [32, Lemma 10.3.3],

$$J^{G^\vee/P^\vee}(\lambda, q) = \exp\Big(\frac{\log q}{\lambda}\sigma_{\mathfrak{i}}\Big)\Big(1 + \sum_{e=1}^{\infty} \sum_{w \in W^P} q^e \Big\langle \frac{\sigma_w}{\lambda - \mathfrak{c}} \Big\rangle_{0,e} \sigma_{\mathrm{PD}(w)}\Big).$$

Intrinsically, the J-function is the solution to the *dual* connection to $\mathcal{Q}_\hbar^{G^\vee/P^\vee}$ that is asymptotic to 1 as q approaches the regular singular point 0 (see [49]).

Example 13.6
For $\mathbb{P}^n$, we have (see [32, Section 10])

$$J^{\mathbb{P}^n}(\lambda, q) = \exp\Big(\frac{\log q}{\lambda}\sigma_{\mathfrak{i}}\Big) \sum_{e=0}^{\infty} q^e \prod_{j=1}^{e} \frac{1}{(\sigma_{\mathfrak{i}} + j\lambda)^{n+1}}. \tag{13.6.1}$$

The case of quadrics is treated in [99, Section 5].

Of particular importance is the component $\langle J^{G^\vee/P^\vee}(\lambda, q), \sigma_{\mathrm{PD}(1)} \rangle = \langle S_{\mathrm{PD}(1)}(\lambda, q), 1 \rangle$ which is a power series in λ^{-1}, q. In Section 4.16, we used the notation $S(q)$ for $S_{\mathrm{PD}(1)}(1, q)$. The single-valuedness follows from considering the kernel of the monodromy operator which is the usual cup product with $\sigma_{\mathfrak{i}}$. Precisely,

$$\big\langle S_{\mathrm{PD}(1)}(\lambda, q), 1 \big\rangle = 1 + \sum_{e=1}^{\infty} q^e \Big\langle \frac{\sigma_{\mathrm{PD}(1)}}{\lambda - \mathfrak{c}}, 1 \Big\rangle_{0,e}. \tag{13.6.2}$$

It is called the *hypergeometric series* of $G^\vee/P^\vee$ in [4] and [5] and called the *quantum period* in [47] and [49]. The sum can be simplified further by expanding $(\lambda - \mathfrak{c})^{-1}$ in power series of λ^{-1} (see [99, Section 5.2] who also consider more generally $\langle S_{\mathrm{PD}(1)}, \sigma_w \rangle$ for any $w \in W^P$).

13.7. Degrees and irregular Hodge filtration

The isomorphism $\mathrm{WCr}_{(G,P)} \cong \mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ from Theorem 11.14 induces for every $t \in Z(L_P)$ an isomorphism

$$\bigoplus_{i=0}^{d} H^{2i}(G^\vee/P^\vee) \cong H^d_{\mathrm{dR}}(G\mathring{/}P, \mathsf{E}^{f_t}). \tag{13.7.1}$$

The cohomology group on the right-hand side is concentrated in a single degree since $\mathrm{WCr}_{(G,P)}$ is a D-module concentrated in a single degree. In this isomorphism, the left-hand side visibly carries a gradation by degree, which can be transported to the right-hand side. We want to spell this out precisely and derive an important proposition.

It is easy to see that the filtration associated to the Jordan decomposition of the linear endomorphism given by the cup product by σ_{i} coincides with the filtration by degree on $H^*(G^\vee/P^\vee)$. The cup product by σ_{i} is the monodromy at $q=0$ of the connection $\mathcal{Q}^{G^\vee/P^\vee}$. Thus we conclude from the mirror isomorphism $\mathcal{Q}^{G^\vee/P^\vee} \cong \mathrm{Cr}_{(G,P)}$ that the filtration by degree is transported on $H^d_{\mathrm{dR}}(G\mathring{/}P, \mathsf{E}^{f_t})$ to the monodromy filtration of $\mathrm{Cr}_{(G,P)}$.

PROPOSITION 13.8

In the isomorphism (13.7.1), the line $\mathbb{C}\cdot\sigma_{PD(1)} = H^{2d}(G^\vee/P^\vee)$ spanned by the top class corresponds to the line spanned by the cohomology class of the volume form ω from Section 6.6. In particular, this cohomology class is nonzero in $H^d_{\mathrm{d}R}(G\mathring{/}P, \mathsf{E}^{f_t})$.

This was previously established for Grassmannians by Marsh and Rietsch [93] and for quadrics by Pech, Rietsch, and Williams [99].

Proof

In view of the above discussion, we only need to analyze the monodromy filtration on $H^d_{\mathrm{dR}}(G\mathring{/}P, \mathsf{E}^{f_t})$ near $\alpha_{\mathrm{i}}(t) = 0$. A convenient way to do so is via the Kontsevich complex $\Omega^\bullet_{f_t}$ of f_t-adapted log-forms, which again involves a resolution $\widetilde{G/P}$ of the singularities of $(G/P, f_t)$. It is established in [38, Corollary 1.4.8] that

$$H^d_{\mathrm{dR}}(G\mathring{/}P, \mathsf{E}^{f_t}) \cong \bigoplus_{p+q=d} H^q(\widetilde{G/P}, \Omega^p_{f_t}).$$

It is possible to write down the monodromy operator and to verify that the decreasing monodromy filtration corresponds to the gradation by $p-q$, following [38], [69], and [79].

Then, by the above, $H^{2d}(G^\vee/P^\vee)$ corresponds under the isomorphism (13.7.1) to the space $H^0(\widetilde{G/P}, \Omega^d_{f_t})$ where by the definition of Ω^d_f, this coincides with the space $H^0(\widetilde{G/P}, \Omega^d(\log))$ of log differential holomorphic top forms. It is known from [81] that H^0 is 1-dimensional and spanned by the form ω. □

In the isomorphism (13.7.1), the left-hand side is of Hodge–Tate type, namely, $H^{2i} = H^{(i,i)}$, because it is spanned by the Schubert classes σ_w which are algebraic. In the mirror isomorphism, $H^*(G^\vee/P^\vee)$ being of Hodge–Tate type translates to $(G\mathring{/}P, f_t)$ being pure in the sense that $\mathrm{Cr}_{(G,P)}$ is a complex supported in one degree. We obtain also parts of [79, Conjecture 3.11] concerning the matching of nc-Hodge structures on both sides of (13.7.1); in particular, on the right-hand side we have identified the irregular Hodge filtration constructed by Deligne [33] and Esnault, Sabbah, and Yu [38]. For the case of certain toric mirror pairs, this matching and much more is established in [102] and [95].

Remark 13.9
We observe that in the case of $\mathbb{P}^n$, the above essentially amounts to a remarkable theorem of Sperber [116] on the slopes of hyper-Kloosterman sums. Thus we are led to conjecture that for almost all primes, the slopes of the minuscule Kloosterman sums $\mathrm{Kl}_{(G,\varpi_{\mathrm{i}})}$ can be read from the cohomology of $G^\vee/P^\vee$. This would follow[4] from a suitable p-adic comparison isomorphism for differential equations of exponential type between Dwork p-adic cohomology and complex Hodge theory, which does not seem to be available in the literature yet. Interestingly, the same Hodge numbers appear in the $(\mathfrak{g}, K)$-cohomology of a certain L-packet of discrete series (see [64]).

13.10. Enumerative formula
We can deduce from the above mirror theorem an integral representation for the quantum period and combinatorial formulas for certain Gromov–Witten invariants. Assertion (ii) below is referred to as the *weak Landau–Ginzburg model* in [101].

THEOREM 13.11
(i) *The quantum period (13.6.2) of $G^\vee/P^\vee$ is equal to the integral of the potential on the middle-dimensional compact cycle of $G\mathring{/}P$,*

$$I_{\mathrm{cpt}}(\lambda, q) := \oint e^{f_q/\lambda} \frac{\omega}{(2i\pi)^{\dim(G/P)}}.$$

[4] Added after submission: Our conjecture can now be established from the main result of Xu and Zhu [125, Theorem 5.3.2(1)] as follows. They construct an overconvergent F-isocrystal which is a p-adic companion of the Kloosterman ℓ-adic sheaf Kl_G of [71], and show that its Newton polygon is the half-sum of positive coroots $\rho^\vee$ for almost all primes. The slopes of the minuscule Kloosterman sums $\mathrm{Kl}_{(G,\varpi_{\mathrm{i}})}$ under the minuscule representation $V_{\varpi_{\mathrm{i}}}$ are therefore $\langle w\varpi_{\mathrm{i}}, \rho^\vee\rangle + \langle \varpi_{\mathrm{i}}, \rho^\vee\rangle$ for $w \in W^P$ (recall from Section 2.4 that the weights of $V_{\varpi_{\mathrm{i}}}$ are the orbit $W \cdot \varpi_{\mathrm{i}}$). For every $d \geq 0$, the number of $w \in W^P$ such that $\langle w\varpi_{\mathrm{i}}, \rho^\vee\rangle + \langle \varpi_{\mathrm{i}}, \rho^\vee\rangle = d$ coincides with the Betti number $\dim H^{2d}(G^\vee/P^\vee)$ (see the proof of Proposition 4.12).

(ii) *For every integer* $e \geq 0$*, the genus-*0 *and degree-*e *Gromov–Witten correlator* $\langle \tau_{ce-2}\sigma_{\mathrm{PD}(1)} \rangle_{0,e}$ *of* $G^\vee/P^\vee$ *is equal to the constant term of* f_1^{ce} *in any cluster chart of* $\overset{\circ}{G/P}$*, divided by* $(ce)!$.

For quadrics, the theorem can be established directly by computing both sides as shown in [99, Section 5.3]. For Grassmannians, the theorem is due to Marsh and Rietsch [93, Theorem 4.2].

Proof
Assertion (ii) follows from (i) by taking residues. Recall that c is the Coxeter number of G. Also note that ω is T-invariant by Lemma 6.26, in particular invariant by $\rho^\vee$. This implies the identity $I_{\mathrm{cpt}}(\lambda, q) = I_{\mathrm{cpt}}(1, q/\lambda^c)$, which is also satisfied by the quantum period (13.6.2).

To establish (i), we observe as consequence of the mirror Theorem 8.3 that $I_{\mathrm{cpt}}(1, q)$ is solution of the quantum connection $\mathcal{Q}^{G^\vee/P^\vee}$. It is a power series in q by Cauchy's residue formula. The same holds for the fundamental solution $S_{\mathrm{PD}(1)}(1, q)$. We can then deduce the desired equality of the two solutions up to scalar from the Frobenius method at the regular singularity $q = 0$. More precisely, we need to consider the equivariant connection $\mathcal{Q}_{T^\vee}^{G^\vee/P^\vee}$ and equivariant Gromov–Witten correlators. For generic $h \in \mathfrak{t}^*$, the monodromy at $q = 0$ is regular semisimple. We then specialize the equivariant parameter to $h = 0$.

To conclude the proof of (i), we need to specialize the solution $\langle S_{\mathrm{PD}(1)}(1, q), 1 \rangle$ as in (13.6.2). It is a power series in q with constant term 1. Moreover, the integral $I_{\mathrm{cpt}}(1, q)$ is against the form ω which implies the identity in view of Proposition 13.8. □

The quantum period typically has infinitely many zeros. As explained by Deligne [34, p. 128], this implies that the irregular Hodge filtration on $H^d_{\mathrm{dR}}(\overset{\circ}{G/P}, \mathsf{E}^{f_t/\hbar})$ does not come from a Hodge structure.

Remark 13.12
The works of Marsh and Rietsch [93] for Grassmannians and Pech, Williams, and Rietsch [99] for quadrics lead us to suggest a more general formula that $\langle S_{\mathrm{PD}(1)}, \sigma_w \rangle$ should be equal to the residue integral $\oint p_w e^{f_q/\lambda} \frac{\omega}{(2i\pi)^{\dim(G/P)}}$, with the Plücker coordinate p_w added. This formula generalizes Theorem 13.11(i), corresponding to the case $w = 1$, and is compatible with the Gamma conjecture and central charges discussed in [49] and [76].

Remark 13.13
In a series of works (see, e.g., [51]), Gerasimov, Lebedev, and Oblezin study the Givental integral from various viewpoints, motivated by Archimedean L-functions, integrable systems of Toda type, and Whittaker functions.

13.14. Projective spaces
For $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$, the Coxeter number is $c = n+1$. We deduce from (13.6.1) that the quantum period $\langle S_{\mathrm{PD}(1)}(\lambda, q), 1\rangle = \langle J^{\mathbb{P}^n}(\lambda, q), \sigma_{\mathrm{PD}(1)}, 1\rangle$ is equal to

$$\sum_{e=0}^{\infty} \frac{1}{(e!)^{n+1}} \Big(\frac{q}{\lambda^{n+1}}\Big)^e = {}_0F_n\Big(\genfrac{}{}{0pt}{}{-}{1 \cdots 1}; \frac{q}{\lambda^{n+1}}\Big).$$

On the other hand,

$$I_{\mathrm{cpt}}(\lambda, q) = \oint e^{\frac{1}{\lambda}(x_1 + \cdots + x_n + \frac{q}{x_1 \cdots x_n})} \frac{dx_1 \cdots dx_n}{(2i\pi)^n x_1 \cdots x_n}.$$

Hence Theorem 13.11 reduces to Erdélyi's integral representation.

Remark 13.15
The quantum period for a general minuscule homogeneous space $G^\vee/P^\vee$ is related to the Bessel functions of matrix argument introduced by Herz (see [91], [112]).

13.16. Classical Bessel functions
For $\mathbb{P}^1 = \mathrm{Gr}(1,2)$, we have $f_q(x) = x + \frac{q}{x}$ for $x \in \mathbb{G}_m = \mathring{\mathbb{P}}^1$, $\omega = \frac{dx}{x}$, and

$$H^i_{\mathrm{dR}}(\mathbb{G}_m, \mathsf{E}^{f_q/\hbar}) = \begin{cases} 0 & \text{if } i = 0, \\ \mathbb{C}\omega \oplus \mathbb{C}x\omega & \text{if } i = 1. \end{cases}$$

Deligne defines an irregular Hodge filtration and shows in [34, p. 127] that $F^1 H^1_{\mathrm{dR}}(\mathbb{G}_m, \mathsf{E}^{f_q/\hbar}) = \mathbb{C}\omega$, which corresponds to Theorem 13.8 above.

The dual space is generated by the two cycles $\oint$ and $\int_0^\infty$, denoted by e_1, $-e_2$ in [34]. Note that the cycle $\int_0^\infty$ depends on q and λ and approaches 0 and ∞ in the direction of rapid decay of the exponential.

The quantum period is ${}_0F_1(\genfrac{}{}{0pt}{}{-}{1}; \frac{q}{\lambda^2}) = I_0(2\sqrt{q}/\lambda) = \oint e^{f_q/\lambda} \frac{\omega}{2i\pi}$. The other integrals are expressed as follows: $\int_0^\infty e^{f_q/\lambda}\omega = 2K_0(2\sqrt{q}/\lambda)$; $\oint e^{f_q/\lambda} x \frac{\omega}{2i\pi} = \sqrt{q} I_1(2\sqrt{q}/\lambda)$; $\int_0^\infty e^{f_q/\lambda} x\omega = -2\sqrt{q} K_1(2\sqrt{q}/\lambda)$. Note that $I_0' = I_1$ and $K_0' = -K_1$.

The determinant of periods

$$\begin{vmatrix} \oint e^{f_q/\lambda}\omega & \oint e^{f_q/\lambda} x\omega \\ \int_0^\infty e^{f_q/\lambda}\omega & \int_0^\infty e^{f_q/\lambda} x\omega \end{vmatrix} = -2i\pi\lambda,$$

established in the last paragraph of [34],[5] corresponds to the Wronskian formula

$$I_\nu(y)K_{\nu+1}(y) + I_{\nu+1}(y)K_\nu(y) = 1/y,$$

for all $y \in \mathbb{R}_{>0}$ and $\nu \in \mathbb{C}$.

More generally, we consider the equivariant version. Let $h \in \mathbb{C}$, and let $h\alpha \in \mathfrak{t}^*$, where α denotes the positive simple root. We consider the integral solutions to $\mathrm{WCr}^{1/\hbar}|_{\hbar=\lambda}$ specialized at $\lambda h\alpha \in \mathfrak{t}^*$,

$$\begin{aligned}\oint \frac{x^{2h}}{q^h} e^{f_q/\lambda} \frac{\omega}{2i\pi} &= q^h \sum_{k=0}^{\infty} \frac{\lambda^{-2k-2h} q^k}{k!\Gamma(k+2h+1)} = \frac{(q/\lambda^2)^h}{\Gamma(1+2h)} {}_0F_1\Big(\genfrac{}{}{0pt}{}{-}{1+2h}; \frac{q}{\lambda^2}\Big) \\ &= I_{2h}(2\sqrt{q}/\lambda),\end{aligned}$$

where compared to Example 6.5, we have $q = t^2$ and the factor $\frac{x^{2h}}{q^h}$ is equal to $(h\alpha)(\gamma(x))$. Similarly, the integral from 0 to ∞ is equal to $K_{2h}(2\sqrt{q}/\lambda)$.

On the quantum connection side, let $\{1, \sigma\}$ be the Schubert basis of $H^*(\mathbb{P}^1)$. Then the equivariant quantum Chevalley formula is

$$\sigma *_q \sigma = q \cdot 1 + (\varpi - s \cdot \varpi) \cdot \sigma = q \cdot 1 + \alpha^\vee \cdot \sigma.$$

Here, s denotes the unique simple reflection. Thus

$$\sigma *_{q,h} \sigma = q \cdot 1 + 2h \cdot \sigma$$

since $\langle \alpha^\vee, \alpha \rangle = 2$. The equivariant quantum connection $\mathcal{Q}^{\mathbb{P}^1}_{\hbar, T^\vee}(h\alpha)$ is

$$\hbar q \frac{d}{dq} + \begin{pmatrix} -\hbar h & q \\ 1 & \hbar h \end{pmatrix}.$$

This is equivalent to the second-order differential operator

$$\Big(\hbar q \frac{d}{dq}\Big)^2 - (q + \hbar^2 h^2),$$

which has solutions specialized to $\hbar = \lambda$ the modified Bessel functions $I_{2h}(2\sqrt{q}/\lambda)$ and $K_{2h}(2\sqrt{q}/\lambda)$. This agrees with Theorem 10.7.

14. Compactified Fano and log Calabi–Yau mirror pairs

Our Theorems 8.3 and 11.14 verify two specific mirror symmetry predictions. In this brief section, the goal is to recast the mirror symmetry of flag varieties in view of recent advances and to provide some evidence for potential generalizations.

[5] The minus sign compared to [34] is because we chose the cycle $\int_0^\infty$ which is $-e_2$.

14.1. Mirror pairs of Fano type

The notion of mirror pairs of Fano type is explained in [79, Section 2.1]. In the context of Rietsch's conjecture that we study in this paper, we have a family of mirror pairs indexed on one side by $H^2(G^\vee/P^\vee)$ and on the other side by $Z(L_P)$. Lemma 8.2 is interpreted as the "mirror map" and can be compared with [76, Lemma 4.2] in the toric case.

The A-model is a triple $(X, g, 1/\omega_X)$ consisting of a projective Fano variety X, a complexified Kähler form g, and an anticanonical section $1/\omega_X$. In the context of Rietsch's conjecture, the variety is $X = G^\vee/P^\vee$, the Kähler class is varying in $H^2(G^\vee/P^\vee)$, and the anticanonical section is the one constructed in [81] (see also [107]). The B-model is another triple $((Y, f), \eta, \omega_Y)$ consisting of a Landau–Ginzburg model, namely, a smooth variety Y with trivial canonical class, a regular function $f : Y \to \mathbb{C}$ (Landau–Ginzburg potential), a Kähler form η, and a nonvanishing canonical section ω_Y (holomorphic volume form).

14.2. D-module version of Rietsch's mirror conjecture

In the context of Rietsch's conjecture, the B-model is as follows. The Landau–Ginzburg model is given by the Berenstein–Kazhdan geometric crystal. The underlying variety is the open Richardson $Y = \overset{\circ}{G/P}$ in G/P, and the Landau–Ginzburg potential is the decoration function f_t of Berenstein and Kazhdan, which depends on the parameter $t \in Z(L_P)$. The volume form ω_Y is again the one constructed in [81].

CONJECTURE 14.3

Let P be a parabolic subgroup of G, and let $P^\vee$ be the dual parabolic subgroup of $G^\vee$. There exists an isomorphism

$$\mathrm{WCr}_{(G,P)}^{1/\hbar} \cong \mathcal{Q}_{\hbar,T^\vee}^{G^\vee/P^\vee}$$

of graded $D_\hbar$-modules over $Z(L_P) \times \mathfrak{t}^*$ relative to $\mathfrak{t}^*$.

Theorem 11.14 establishes Conjecture 14.3 in the case where P is minuscule. The superpotential of $\mathrm{WCr}_{(G,P)}$ (explicitly described in Section 6.8) agrees with the Landau–Ginzburg model defined by Rietsch (see [107, Lemma 5.2]). Thus Conjecture 14.3 is compatible with Rietsch's conjecture in [107].

14.4. Mirror pairs of compactified Landau–Ginzburg models

Following [79, Section 3.2.4], one may also consider quadruples (X, g, ω, f) consisting of a projective Fano variety X, a complexified Kähler form g, a canonical section ω_X, and a potential function f_X. We may then examine, for appropriate choices of Kähler forms, the mirror symmetry between

$$(G/P, g, \omega_{G/P}, f_{G/P}) \qquad \text{and} \qquad (G^\vee/P^\vee, g^\vee, \omega_{G^\vee/P^\vee}, f_{G^\vee/P^\vee}).$$

The A- and B-sides now play a symmetric role. Rietsch's mirror conjecture corresponds to omitting some of the data on both sides. The full mirror conjecture between these compactified mirror pairs involves the matching of a variety of homological data on both sides.

For example, a Fano type mirror pair gives rise to a pair of open Calabi–Yau manifolds by taking the complement of the anticanonical divisor. One obtains triples of a log Calabi–Yau manifold, a Kähler form, and a volume form. In our setting, the log Calabi–Yau varieties are $G^\vee \mathring{/} P^\vee$ and $G\mathring{/}P$, respectively. The volume forms are as before. Thus from the general mirror predictions in [79, Table 2], we expect a matching of the cohomology of the open projected Richardson variety $H^*(G\mathring{/}P)$ and the cohomology of the Langlands dual open projected Richardson variety $H^*(G^\vee \mathring{/} P^\vee)$. We show in the next subsection that this matching holds more generally for arbitrary Richardson varieties.

14.5. Open Richardson varieties

Recall the open Richardson varieties $\mathcal{R}_u^w \subset G/B$, where $u, w \in W$ with $u \le w$, and the special case $\mathcal{R}_{w_0^P}^{w_0} \cong G\mathring{/}P$. They are log Calabi–Yau varieties with canonical volume form (see [81]). We denote by $\check{\mathcal{R}}_u^w \subset G^\vee/B^\vee$ the open Richardson varieties inside the flag variety of the dual group.

PROPOSITION 14.6

For any $i \ge 0$ and $u, w \in W$ with $u \le w$, there is an isomorphism $H^i(\mathcal{R}_u^w) \cong H^i(\check{\mathcal{R}}_u^w)$.

Proof

Since $\mathcal{R}_u^w$ and $\mathcal{R}_u^w$ are smooth complex algebraic varieties of the same dimension, by Poincaré duality, the stated isomorphism is equivalent to the same statement for cohomology with compact support. As in the proof of Proposition 13.3, this is in turn equivalent to the isomorphism $\mathrm{Ext}^\bullet(M_w, M_u) \cong \mathrm{Ext}^\bullet(M_{\mathfrak{g}^\vee,w}, M_{\mathfrak{g}^\vee,u})$, where M_w (resp., $M_{\mathfrak{g}^\vee,w}$) denotes a Verma module in the principal block of category $\mathcal{O}$ for $\mathfrak{g}$ (resp., $\mathfrak{g}^\vee$). By the work of Soergel [115], the principal blocks of category $\mathcal{O}$ for $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are equivalent, and the isomorphism of Ext-groups follows. $\square$

Question 14.7

Can this isomorphism be an indication of mirror symmetry between open Richardson varieties $\mathcal{R}_u^w \subset G/B$ and $\check{\mathcal{R}}_u^w \subset G^\vee/B^\vee$?

15. Proofs from Section 2.5

15.1. Proof of Proposition 2.7

The implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are easy to check directly.

We thank the referee for the following argument showing (2) $\Longrightarrow$ (1), simplifying our original proof. Recall that $Q^\vee$ (resp., $Q_P^\vee$) denotes the coroot lattice spanned by $\alpha_i^\vee, i \in I$ (resp., $\alpha_i^\vee, i \in I_P$). By a result of Peterson and Woodward (see [87, Theorem 10.13(1)]), for each $\lambda_P^\vee \in Q^\vee/Q_P^\vee$ there exists a unique lift $\lambda^\vee \in Q^\vee$ (called the *Peterson–Woodward lift*) of $\lambda_P^\vee$ such that $\langle \alpha, \lambda^\vee \rangle \in \{0, -1\}$ for all $\alpha \in R_P^+$. Since $\mathfrak{i}$ is minuscule and $\beta \in R^+ \setminus R_P^+$, we have $\beta^\vee + Q_P^\vee = \alpha_\mathfrak{i}^\vee + Q_P^\vee \in Q^\vee/Q_P^\vee$. Thus, if $\beta^\vee$ satisfies (2), then it is the Peterson–Woodward lift, which is unique, and we must have $\beta^\vee = \kappa^\vee$.

We show that (3) $\Longrightarrow$ (1). Suppose that G is simply laced. Suppose that $\beta = -w^{-1}(\theta) \in R^+ \setminus R_P^+$, but $\beta \neq \alpha_\mathfrak{i}$. Then $\mathfrak{i}$ is also cominuscule so $\beta = \alpha_\mathfrak{i} + \beta'$, where β' is a nonzero linear combination of α_j for $j \neq \mathfrak{i}$. Since $w \in W^P$, we have $w\alpha_j \in R^+$ for $j \neq \mathfrak{i}$. Thus $w\beta - w\alpha_\mathfrak{i} \in \mathbb{Z}_{\geq 0}R^+ \setminus \{0\}$. Since $w\alpha_\mathfrak{i}$ is a root, it would be impossible for $w\beta = -\theta$.

Suppose that G is of type B_n. Choosing coordinates for R, we have $\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n = \epsilon_1 + \epsilon_2$. We may identify W with the group of signed permutations on $\{1, 2, \ldots, n\}$, and W^P is identified with signed permutations that are increasing, under the order $1 < 2 < \cdots < n < -n < -(n-1) < \cdots < -1$. We have $|W^P| = 2^n$. For example, $w = (2, 4, 5, -3, -1) \in W^P$ and $w^{-1}(\epsilon_1 + \epsilon_2) = -\epsilon_5 + \epsilon_1$. It follows by inspection that $-w^{-1}(\theta) \in R^+ \setminus R_P^+$ implies that $-w^{-1}(\theta) = \epsilon_4 + \epsilon_5 = \alpha_{n-1} + 2\alpha_n$.

Suppose that G is of type C_n. We have $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. The elements of W^P are

$$1, s_1, s_2s_1, s_3s_2s_1, \ldots, s_ns_{n-1}\cdots s_1, s_{n-1}s_ns_{n-1}\cdots s_2s_1, \ldots,$$
$$s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_2s_1,$$

and we have $|W^P| = 2n$. We have $-w^{-1}(\theta) \in R^+ \setminus R_P^+$ if and only if $w = s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_2s_1$, and the statement follows. $\square$

15.2. Proof of Proposition 2.8

We first note the following properties of $w_{P/Q}$.

LEMMA 15.3

(1) $\mathrm{Inv}(w_{P/Q}) = R_P^+ \setminus R_Q^+$,

(2) $\ell(w_{P/Q}) = \langle -2\rho_P, \kappa^\vee \rangle$,

(3) $\ell(w_{P/Q}s_\kappa) = \ell(w_{P/Q}) + \ell(s_\kappa) = \langle 2(\rho - \rho_P), \kappa^\vee \rangle - 1$.

Proof
Let w_P (resp., w_Q) be the maximal element of W_P (resp., W_Q). Then $w_{P/Q}w_Q = w_P$ is length-additive, so $\mathrm{Inv}(w_{P/Q}) = w_Q(\mathrm{Inv}(w_P) \setminus \mathrm{Inv}(w_Q)) = R_P^+ \setminus R_Q^+$, proving (1). Formula (2) follows from Lemma 2.6(2). Since $\mathrm{Inv}(s_\kappa) \cap R_P^+ = \emptyset$, it follows that the product $w_{P/Q}s_\kappa$ is length-additive. Formula (3) follows from (2) and $\kappa \in \tilde{R}$. □

15.3.1. Proof of (1) in Proposition 2.8
It is equivalent to show that $\mathrm{Inv}(w) \supset \mathrm{Inv}(s_\kappa)$. Suppose that $\alpha \in \mathrm{Inv}(s_\kappa)$. Then $\alpha - \langle \alpha, \kappa^\vee \rangle \kappa = s_\kappa \alpha < 0$, where $a = \langle \alpha, \kappa^\vee \rangle > 0$. Thus

$$-a\theta = aw(\kappa) = w(\alpha) - w(s_\kappa \alpha),$$

and it follows that $w\alpha < 0$ because $w(s_\kappa \alpha)$ is a root.

15.3.2. Proof of (2) in Proposition 2.8
After formulas (1) and (3) of Lemma 15.3, it is equivalent to show that $\mathrm{Inv}(ws_\kappa) \supset R_P^+ \setminus R_Q^+$. Let $\alpha \in R_P^+ \setminus R_Q^+$. Then $s_\kappa \alpha = \kappa + \alpha$ by Lemma 2.6. Thus $ws_\kappa \alpha = w\alpha + w\beta = w\alpha - \theta$. Since θ is the highest root, we deduce that $\alpha \in \mathrm{Inv}(ws_\kappa)$.

15.3.3. Proof of (3) in Proposition 2.8
Since $w_{P/Q}^{-1} \in W_P$, it suffices to show that $ws'_\kappa \in W^P$. It suffices to check that $\mathrm{Inv}(ws_\kappa) \cap R_Q^+ = \emptyset$. But s_κ fixes every element in R_Q^+, and $\mathrm{Inv}(w) \cap R_Q^+ = \emptyset$ since $w \in W^P$. The claim follows.

15.3.4. Proof of (4) in Proposition 2.8
The standard parabolic subgroup J is given as follows: for type A_n, we have $J = \{2, 3, \ldots, n-1\}$; for type D_n or E_6, we have $J = [n] \setminus \{2\}$; for type E_7, we have $J = [n] \setminus \{1\}$; for type B_n, we have $J = [n] \setminus \{1, 2\}$; for type C_n, we have $J = \emptyset$. In all cases, it is clear that W_J stabilizes θ.

If $w, v \in W(\kappa)$, then clearly wv^{-1} belongs to the stabilizer of θ. In the simply laced types, this stabilizer is exactly the group W_J. In type B_n, the stabilizer of θ is $W_{[n]\setminus\{2\}}$, but from the description in the proof of Proposition 2.7, it is clear that $wv^{-1} \in W_J$. In type C_n, as noted previously we have $W(\kappa) = \{s_\kappa\}$ which consists of a single element.

The double coset $W_J w W_P$ contains a unique minimal element w', and since $w \in W^P$, we have a length-additive factorization $w = uw'$, where $u \in W_J$. Since $w' \in W^P$ and $(w')^{-1}(\theta) = w^{-1}(\theta) = -\kappa$, we have $w' \in W(\kappa)$.

15.3.5. Proof of final sentence in Proposition 2.8
We assume that $w \in W^P$ satisfies $\mathrm{Inv}(w) \supset \mathrm{Inv}(s_\kappa)$ and $\mathrm{Inv}(ws_\kappa) \supset R_P^+ \setminus R_Q^+$. Suppose first that G is simply laced so that $\kappa = \alpha_{\mathrm{i}}$. Suppose that $-w^{-1}(\theta) = \alpha \neq \alpha_{\mathrm{i}}$. Let $w\alpha_{\mathrm{i}} = -\eta < 0$. Since $w \in W^P$, we have $\alpha \notin R_P^+$. On the other hand, we have $w(\alpha - \alpha_{\mathrm{i}}) = -\theta + \eta < 0$. Again because $w \in W^P$, this shows that $\alpha \notin R^+ \setminus R_P^+$. Thus $\alpha \in R^-$.

Let $\delta = -\alpha \in R^+$. Since $w\delta = \theta$ and $w \in W^P$, we have that $\delta + \lambda$ cannot be a root whenever $0 \neq \lambda \in \sum_{j \in I_P} \mathbb{Z}_{\geq 0}\alpha_j$. If $\delta \in R_P^+$, then it follows that $\delta \in R_P^+ \setminus R_Q^+$. But then $s_\kappa\delta = \delta + \kappa$ implies that $(ws_\kappa)\delta = w\delta + w\kappa > 0$, contradicting the assumption that $\mathrm{Inv}(ws_\kappa) \supset R_P^+ \setminus R_Q^+$.

Thus $\delta \in R^+ \setminus R_P^+$, and again since $w \in W^P$, we may assume that $\delta = \theta$. Thus $w\theta = \theta$, so w lies in the stabilizer $W' \subset W$ of θ. In types E_6, E_7, or D_n, $n \geq 4$, it is easy to see that $\mathrm{Inv}(ws_{\mathrm{i}})$ for $w \in W'$ cannot contain $R_P^+ \setminus R_Q^+$ since W' is a parabolic subgroup that contains the minuscule node i, but does not contain the adjoint node (node 2 in types D_n or E_6 and node 1 in type E_7). In type A_n, the whole claim is easy to check directly, and we conclude that $w \in W(\kappa)$.

Suppose that G is of type B_n. We use notation from the proof of Proposition 2.7. We have $s_\kappa = s_{n-1}s_ns_{n-1}$ and for a signed permutation $w = w_1w_2\cdots w_n \in W^P$, we have $ws_\kappa = w_1w_2\cdots(-w_n)(-w_{n-1})$. Thus the first condition $\mathrm{Inv}(w) \supset \mathrm{Inv}(s_\kappa)$ is equivalent to $w_{n-1}, w_n < 0$. The second condition $\mathrm{Inv}(ws_\kappa) \supset R_P^+ \setminus R_Q^+$ is equivalent to the condition that $\{w_1, w_2, \ldots, w_{n-2}\}$ are all bigger than $-w_n$ and $-w_{n-1}$ under the order $1 < 2 < \cdots n < -n < -(n-1) < \cdots < -1$. It follows that $w_{n-1} = -2$ and $w_n = -1$, so that $w\kappa = -\theta$.

Suppose that G is of type C_n. Then $\ell(s_\kappa) \geq \ell(w)$ for $w \in W^P$ with equality if and only if $w = s_\kappa = s_1s_2\cdots s_{n-1}s_ns_{n-1}\cdots s_2s_1$. The claim follows easily. ☐

16. Background on $D_\hbar$-modules

The main purpose of this section is to establish Proposition 16.13, which is used in Section 12.

16.1. Filtered and graded categories
Let X be a complex smooth affine algebraic variety equipped with a $\mathbb{G}_m$-action. Its structure sheaf $\mathcal{O}_X$ is naturally graded by $\mathbb{G}_m$-homogeneous sections. Denote by $p : T^*X \to X$ the cotangent bundle of X. Denote by D_X the sheaf of differential operators on X. This is a sheaf of noncommutative rings. It is equipped with a *filtration*

$$\cdots \subset D_{X,-1} \subset D_{X,0} \subset D_{X,1} \subset \cdots$$

induced by the gradation of $\mathcal{O}_{T^*X}$ plus the order of the differential operator.

Let $\mathrm{MF}(D_{X,\bullet})$ denote the category of *filtered left $D_{X,\bullet}$-modules* that are quasicoherent as $\mathcal{O}_X$-modules. An object $M_\bullet \in \mathrm{MF}(D_{X,\bullet})$ is equipped with a filtration $\cdots M_{-1} \subset M_0 \subset M_1 \cdots$ satisfying $D_{X,j} M_i \subset M_{i+j}$. The category $\mathrm{MF}(D_{X,\bullet})$ is an additive category, but not an abelian category; it can be made into an exact category by declaring a sequence $0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$ to be exact if $0 \to M'_i \to M_i \to M''_i \to 0$ is exact for all i. (This is stronger than asking for the sequence of underlying unfiltered D_X-modules to be exact.) As shown in [88], one can define the derived category of $\mathrm{MF}(D_{X,\bullet})$; we let $D^b F(D_{X,\bullet})$ denote the bounded derived category of $\mathrm{MF}(D_{X,\bullet})$. There are natural forgetful functors $\mathrm{MF}(D_{X,\bullet}) \to \mathrm{M}(D_X)$ and $D^b F(D_{X,\bullet}) \to D^b(D_X)$ sending a filtered module $M_\bullet$ to the underlying D_X-module M, and a complex $M^\cdot_\bullet$ of filtered modules to the underlying complex $M^\cdot$.

The associated graded of $D_{X,\bullet}$ is the sheaf $\mathrm{gr} D_{X,\bullet} = p_* \mathcal{O}_{T^*X}$ of graded commutative rings on X, where the grading comes from the grading of $\mathcal{O}_X$ together with the declaration that vector fields have degree 1. Since p is affine, we have equivalences of categories

$$\mathrm{M}(\mathcal{O}_{T^*X}) \cong \mathrm{M}(p_* \mathcal{O}_{T^*X}) \qquad \text{and} \qquad D^b(\mathcal{O}_{T^*X}) \cong D^b(p_* \mathcal{O}_{T^*X})$$

between the corresponding categories of quasicoherent $\mathcal{O}_{T^*X}$-modules and quasicoherent $p_* \mathcal{O}_{T^*X}$-modules, and bounded derived categories. We have an associated graded functor, and derived functor

$$\mathrm{gr} : \mathrm{MF}(D_{X,\bullet}) \to \mathrm{M}(\mathcal{O}_{T^*X}) \qquad \text{and} \qquad \mathrm{gr} : D^b F(D_{X,\bullet}) \to D^b(\mathcal{O}_{T^*X}).$$

Definition 16.2

Let $D_{\hbar,X}$ denote the sheaf of graded noncommutative rings with a central section $\hbar$, locally generated by $f \in \mathcal{O}_X$ and sections $\xi \in \Theta_X$ of the tangent sheaf with the relations $[f, \xi] = \hbar(\xi \cdot f)$ and $\xi\eta - \eta\xi = \hbar[\xi, \eta]$. The grading is given by the assignment $\deg(\hbar) = 1$ and the homogeneous degrees of f and ξ induced by the $\mathbb{G}_m$-action.

The sheaf $D_{\hbar,X}/\hbar$ is isomorphic to the sheaf $p_* \mathcal{O}_{T^*X}$, while the localization $D_{\hbar,X}$ at $(\hbar)$ is isomorphic to $D_X[\hbar^{\pm 1}]$. Let $\mathrm{MG}(D_{\hbar,X})$ denote the category of sheaves of graded left $D_{\hbar,X}$-modules that are quasicoherent as graded $\mathcal{O}_X$-modules. To an object $M_\bullet \in \mathrm{MF}(D_{X,\bullet})$ we associate an object

$$M_\bullet \otimes \mathbb{C}[\hbar] =: M^\hbar = \bigoplus_i M_i^\hbar \in \mathrm{MG}(D_{\hbar,X}),$$

by defining $M_i^\hbar = M_i$. The section $\hbar$ acts by the identity, thought of as a map from $M_i^\hbar$ to $M_{i+1}^\hbar$. It is clear that $\otimes\mathbb{C}[\hbar] : \mathrm{MF}(D_{X,\bullet}) \to \mathrm{MG}(D_{\hbar,X})$ is an exact functor.

For the following result, see [88, Section 7] and [109, Section 4].

PROPOSITION 16.3
The functor

$$\otimes\mathbb{C}[\hbar] : \mathrm{MF}(D_{X,\bullet}) \to \mathrm{MG}(D_{\hbar,X}), \qquad M_\bullet \mapsto M^\hbar = M_\bullet \otimes \mathbb{C}[\hbar]$$

is an equivalence between $\mathrm{MF}(D_{X,\bullet})$ *and the full subcategory of* $\hbar$*-torsion-free* $D_{\hbar,X}$*-modules. It induces a derived functor* $\otimes\mathbb{C}[\hbar]$ *giving an equivalence of categories*

$$\otimes\mathbb{C}[\hbar] : D^b F(D_{X,\bullet}) \cong D^b(D_{\hbar,X}).$$

We also have a functor $\otimes_{\mathbb{C}[\hbar]}\mathbb{C} : \mathrm{MG}(D_{\hbar,X}) \to \mathrm{M}(\mathcal{O}_{T^*X})$ and a left derived functor $\overset{L}{\otimes}_{\mathbb{C}[\hbar]} \mathbb{C} : D^b(D_{\hbar,X}) \to D^b(\mathcal{O}_{T^*X})$, setting $\hbar = 0$.

PROPOSITION 16.4
We have commutative diagrams

$$\begin{array}{ccc} \mathrm{MF}(D_{X,\bullet}) & \xrightarrow{\otimes\mathbb{C}[\hbar]} & \mathrm{MG}(D_{\hbar,X}) \\ & \searrow^{\mathrm{gr}} & \downarrow {\scriptstyle \otimes_{\mathbb{C}[\hbar]}\mathbb{C}} \\ & & \mathrm{M}(\mathcal{O}_{T^*X}) \end{array} \qquad \begin{array}{ccc} D^b F(D_{X,\bullet}) & \xrightarrow{\otimes\mathbb{C}[\hbar]} & D^b(D_{\hbar,X}) \\ & \searrow^{\mathrm{gr}} & \downarrow {\scriptstyle \overset{L}{\otimes}_{\mathbb{C}[\hbar]}\mathbb{C}} \\ & & D^b(\mathcal{O}_{T^*X}) \end{array}$$

Example 16.5
Consider $X = \mathbb{G}_m^n \times \mathbb{G}_m$, with coordinates $(x_1, \ldots, x_n, q)$, and equipped with the $\mathbb{G}_m$-action

$$\zeta \cdot (x_1, \ldots, x_n, q) = (\zeta x_1, \ldots, \zeta x_n, \zeta^{n+1} q).$$

The ring $\mathbb{C}[X]$ of Laurent polynomials has a corresponding gradation by homogeneous polynomials. The potential $f = x_1 + \cdots + x_n + \frac{q}{x_1 \cdots x_n}$ has degree 1. The ring of differential operators D_X is filtered by the subspaces $D_{X,i}$, which for each $i \in \mathbb{Z}$ are the linear span of the operators

$$x_1^{a_1} \frac{\partial^{b_1}}{\partial x_1^{b_1}} x_2^{a_2} \frac{\partial^{b_2}}{\partial x_2^{b_2}} \cdots x_n^{a_n} \frac{\partial^{b_n}}{\partial x_n^{b_n}} q^m \frac{\partial^\ell}{\partial q^\ell}, \quad a_1 + a_2 + \cdots + a_n + (n+1)m - n\ell \le i.$$

The D_X-module $D_X / D_X(d - df\wedge)$ that we denote E^f is equipped with a natural filtration and becomes an object of $MF(D_{X,\bullet})$. The ring $D_{\hbar,X}$ is the graded noncommutative ring generated by functions and differential operators ξ_{x_k}, ξ_q with notably the relations

$$[\xi_{x_k}, x_k] = \hbar, \qquad [\xi_q, q] = \hbar.$$

The degrees are given by $\deg(x_k) = 1$, $\deg(\xi_{x_k}) = 0$, $\deg(q) = n+1$, $\deg(\xi_q) = -n$, $\deg(\hbar) = 1$. One can think of ξ_{x_k} as representing "$\hbar\frac{\partial}{\partial x_k}$," and ξ_q as representing "$\hbar\frac{\partial}{\partial q}$." Applying the functor of Proposition 16.3, we have that $\mathsf{E}^f \otimes \mathbb{C}[\hbar]$ becomes the $D_{\hbar,X}$-module $\mathsf{E}^{f/\hbar}$ which we can describe as follows. We have that $\mathsf{E}^{f/\hbar}$ is isomorphic to the quotient of $D_{\hbar,X}$ by the left $D_{\hbar,X}$-ideal generated by the operators $\xi_{x_k} - \frac{\partial f}{\partial x_k}$ and $\xi_q - \frac{\partial f}{\partial q}$. The operators are all homogeneous; hence $\mathsf{E}^{f/\hbar}$ is an element of $MG(D_{\hbar,X})$, and moreover it is $\hbar$-torsion-free, consistently with Proposition 16.3.

16.6. Pushforward functors

In this and the next subsection only, we write $\int_\pi$ to denote the pushforward functor for D-modules, and we reserve π_* for the pushforward functor of quasicoherent sheaves. Let $\pi : X \to Y$ be a $\mathbb{G}_m$-equivariant morphism between complex irreducible smooth varieties X and Y equipped with $\mathbb{G}_m$-actions. We recall results concerning the pushforward functors of D_X-, $D_{\hbar,X}$-, and $\mathcal{O}_{T^*X}$-modules under π. Though we shall not need it, the functors of Proposition 16.4 are also compatible with pullbacks under π.

Let ω_X (resp., ω_Y) denote the canonical line bundles of X (resp., Y). The sheaf ω_X acquires a grading from the $\mathbb{G}_m$-action so that it becomes a filtered right $D_{X,\bullet}$-module. Define

$$D_{Y\leftarrow X} := \pi^{-1}(D_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\pi^{-1}\mathcal{O}_Y} \omega_X,$$

which is a $(\pi^{-1}D_Y, D_X)$-bimodule on X. The module $D_{Y\leftarrow X}$ inherits a filtration from the filtrations of D_Y, ω_Y, and ω_X. We obtain a filtered $(\pi^{-1}D_{Y,\bullet}, D_{X,\bullet})$-bimodule $D_{Y\leftarrow X,\bullet}$ on X, satisfying $\pi^{-1}D_{Y,j} \cdot D_{Y\leftarrow X,i} \cdot D_{X,k} \subset D_{Y\leftarrow X,i+j+k}$. We define the direct image functor by

$$\int_\pi M^\cdot := R\pi_*(D_{Y\leftarrow X,\bullet} \overset{L}{\otimes}_{D_{X,\bullet}} M^\cdot),$$

where $M^\cdot \in D^bF(D_{X,\bullet})$. Similarly, define $\int_\pi : D^b(D_X) \to D^b(D_Y)$ by forgetting filtrations.

PROPOSITION 16.7 ([88, (5.6.1.1)])
The following diagram commutes:

$$\begin{array}{ccc} D^bF(D_{X,\bullet}) & \xrightarrow{\int_\pi} & D^bF(D_{Y,\bullet}) \\ \downarrow & & \downarrow \\ D^b(D_X) & \xrightarrow{\int_\pi} & D^b(D_Y) \end{array}$$

where the vertical arrows are the natural forgetful functors.

Let $T^*Y \times_Y X$ be the pullback of the cotangent bundle T^*Y to X, fitting into the commutative diagram (see [88, (5.0.1)])

$$\begin{array}{ccccc} T^*X & \xleftarrow{\Pi} & T^*Y \times_Y X & \xrightarrow{p_\pi} & X \\ & & \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & T^*Y & \xrightarrow[p_Y]{} & Y \end{array}$$

We have

$$\mathrm{gr} D_{Y\leftarrow X,\bullet} = \pi^*\mathcal{O}_{T^*Y} \otimes_{\mathcal{O}_X} \omega_{X/Y},$$

which has a natural structure of a graded $(\pi^*\mathcal{O}_{T^*Y}, \mathcal{O}_{T^*X})$-bimodule. We now define a functor $\int_\pi : D^b(\mathcal{O}_{T^*X}) \to D^b(\mathcal{O}_{T^*Y})$ by

$$\int_\pi M_0^{\cdot} := \big(R\bar{\pi}_* \circ \Pi^![d]\big)(M_0^{\cdot}), \tag{16.7.1}$$

where $d = \dim X - \dim Y$ and $\Pi^! : D^b(\mathcal{O}_{T^*X}) \to D^b(\mathcal{O}_{T^*Y\times_Y X})$ denotes the upper-shriek functor on derived categories of quasicoherent sheaves.

We will only use (16.7.1) when the map $\pi : X \to Y$ is smooth, in which case we have

$$\Pi^![d](-) = L\Pi^*(-) \otimes_{\mathcal{O}_{T^*Y\times_Y X}} p_\pi^*\omega_{X/Y}, \tag{16.7.2}$$

where $L\Pi^* : D^b(\mathcal{O}_{T^*X}) \to D^b(\mathcal{O}_{T^*Y\times_Y X})$ is the left derived functor of the usual pullback functor Π^* of quasicoherent sheaves.

We have the following compatibility result of pushforwards.

PROPOSITION 16.8 ([88, (5.6.1.2)])
The following diagram commutes:

$$\begin{array}{ccc} D^b F(D_{X,\bullet}) & \xrightarrow{\int_\pi} & D^b F(D_{Y,\bullet}) \\ \Big\downarrow{\scriptstyle \mathrm{gr}} & & \Big\downarrow{\scriptstyle \mathrm{gr}} \\ D^b(\mathcal{O}_{T^*X}) & \xrightarrow{\int_\pi} & D^b(\mathcal{O}_{T^*Y}) \end{array}$$

Finally, we describe the pushforward functor for $D_{X,\hbar}$-modules. We define $D_{Y\leftarrow X,\hbar} := D_{Y\leftarrow X,\bullet} \otimes \mathbb{C}[\hbar]$, which is a graded $(\pi^{-1}D_{Y,\hbar}, D_{X,\hbar})$-bimodule. We define the direct image functor $\int_\pi : D^b(D_{X,\hbar}) \to D^b(D_{Y,\hbar})$ by

$$\int_\pi M^\cdot := R\pi_*(D_{Y\leftarrow X,\hbar} \overset{L}{\otimes}_{D_{X,\hbar}} M^\cdot).$$

PROPOSITION 16.9
The following diagram commutes:

$$\begin{array}{ccc} D^b F(D_{X,\bullet}) & \xrightarrow{\int_\pi} & D^b F(D_{Y,\bullet}) \\ \Big\downarrow{\scriptstyle \otimes\mathbb{C}[\hbar]} & & \Big\downarrow{\scriptstyle \otimes\mathbb{C}[\hbar]} \\ D^b(D_{X,\hbar}) & \xrightarrow{\int_\pi} & D^b(D_{Y,\hbar}) \end{array}$$

Proof
A direct comparison shows that

$$(D_{Y\leftarrow X,\bullet} \overset{L}{\otimes}_{D_{X,\bullet}} M^\cdot) \otimes \mathbb{C}[\hbar] = D_{Y\leftarrow X,\hbar} \overset{L}{\otimes}_{D_{X,\hbar}} \big(M^\cdot \otimes \mathbb{C}[\hbar]\big),$$

as graded $\pi^{-1}(D_{Y,\hbar})$-modules. Similarly, $\otimes\mathbb{C}[\hbar]$ is an exact functor, so it commutes with $R\pi_*$. □

PROPOSITION 16.10
The following diagram commutes:

$$\begin{array}{ccc} D^b(D_{X,\hbar}) & \xrightarrow{\int_\pi} & D^b(D_{Y,\hbar}) \\ \Big\downarrow{\scriptstyle \overset{L}{\otimes}_{\mathbb{C}[\hbar]}\mathbb{C}} & & \Big\downarrow{\scriptstyle \overset{L}{\otimes}_{\mathbb{C}[\hbar]}\mathbb{C}} \\ D^b(\mathcal{O}_{T^*X}) & \xrightarrow{\int_\pi} & D^b(\mathcal{O}_{T^*Y}) \end{array}$$

where the vertical arrows are the natural forgetful functors.

Proof
Combine Proposition 16.4 with Propositions 16.8 and 16.9. □

Example 16.11
Consider $Y = \mathbb{G}_m$, graded by $\deg(q) = n + 1$. The ring $D_{\hbar,Y} = \mathbb{C}[q^{\pm 1}, \hbar]\langle \xi_q \rangle$ satisfies the relation $[\xi_q, q] = \hbar$, and the gradation is given by $\deg(\xi_q) = -n$, $\deg(\hbar) = 1$. The quantum differential operator $(q\xi_q)^{n+1} - q$ is homogeneous of degree $n + 1$. It shall follow from the next subsection that it is isomorphic to the pushforward $\int_p i\mathsf{E}^{f/\hbar}$, where $\mathsf{E}^{f/\hbar}$ is as in Example 16.5 and $\pi : X \to Y$ is the projection onto the second factor.

16.12. Application to the character $D_\hbar$-module
Let $\pi : X \to Z(L_P)$ denote the geometric crystal, and let $f : X \to \mathbb{A}^1$ denote the superpotential. Recall that we defined $\mathbb{G}_m$-actions on X and $Z(L_P)$ in Section 6.21.

PROPOSITION 16.13
The character $(D_{\hbar,Z(L_P)} \otimes \mathrm{Sym}(\mathfrak{t}))$-module $\mathrm{WCr}^{1/\hbar}_{(G,P)} \in D^b(D_{\hbar,Z(L_P)} \otimes \mathrm{Sym}(\mathfrak{t}))$ is $\hbar$-torsion-free and concentrated in a single degree.

Proof
To simplify the notation, we will prove the proposition for $\mathrm{Cr}^{1/\hbar}_{(G,P)} \in D^b(D_{\hbar,Z(L_P)})$ without the weight. Thus let

$$M^\hbar = D_{\hbar,X}/\big(\xi - (\xi \cdot f)\big)$$

denote the cyclic $D_{\hbar,X}$-module generated by a single section $e^{f/\hbar}$. Here $\xi \in \Gamma(X, \Theta_X)$ denotes a vector field on X. We shall show that $N^\hbar := \int_\pi M^\hbar \in D^b(D_{\hbar,Z(L_P)})$ is isomorphic to an $\hbar$-torsion-free $D_{Z(L_P),\hbar}$-module concentrated in a single degree. The condition that $N^\hbar$ is $\hbar$-torsion-free and concentrated in one cohomological degree is equivalent to the condition that the object $N_0 = N^\hbar \overset{L}{\otimes}_{\mathbb{C}[\hbar]} \mathbb{C} \in D^b(\mathcal{O}_{T^*Z(L_P)})$ (see Section 16.1) is concentrated in a single cohomological degree.

Let $M_0 = M^\hbar \otimes_{\mathbb{C}[\hbar]} \mathbb{C} \in \mathrm{M}(\mathcal{O}_{T^*X})$. Then M_0 is isomorphic to $\mathcal{O}_V$, where $V \subset T^*X$ is cut out by the equations $\xi - (\xi \cdot f)$. By Proposition 16.10, we have $N_0 = \int_\pi M_0$. Denote $T^*Z(L_P) \times_{Z(L_P)} X$ by Y. By (16.7.1) and (16.7.2), we have

$$\int_\pi M_0 = R\pi'_*\big(LF^*(M_0) \otimes_{\mathcal{O}_Y} \tilde{\pi}^*\omega_{X/Z(L_P)}\big),$$

where $\tilde{\pi} : Y \to X$ and $\pi' : Y \to T^*Z(L_P)$ are the two projections and $F : Y \to T^*X$ is the natural inclusion. We first show that $LF^*(M_0) \in D^b(\mathcal{O}_W)$ is concentrated in a single cohomological degree. This is equivalent to the condition that

$\mathrm{Tor}^i_{\mathcal{O}_{T^*X}}(\mathcal{O}_Y, \mathcal{O}_V) = 0$ for $i > 0$. It is easy to see that both V and Y are smooth subvarieties of T^*X, and hence Cohen–Macaulay.

The fiber of $Y \cap V$ under $Y \to T^*Z(L_P) \to Z(L_P)$ over a point $q \in Z(L_P)$ can be identified with the critical point set of $f|_{\pi^{-1}(q)}$. Rietsch [107] showed that this critical point set is 0-dimensional, and it follows that $Y \cap V$ is pure of dimension 1. Since $\dim V = \dim X$ and $\dim Y = \dim X + 1$, it follows that the intersection $Y \cap V$ is proper.

If $\mathrm{Tor}^i_{\mathcal{O}_{T^*X}}(\mathcal{O}_Y, \mathcal{O}_V)$ is nonzero, then it is nonzero after localizing to some irreducible component of C of $Y \cap V$. Applying [110, Corollary V.B.6], we obtain $\mathrm{Tor}^i_{\mathcal{O}_{T^*X,C}}(\mathcal{O}_{Y,C}, \mathcal{O}_{V,C}) = 0$ for all $i > 0$, where $\mathcal{O}_{T^*X,C}$ (resp., $\mathcal{O}_{Y,C}$, $\mathcal{O}_{V,C}$) denotes the localization. Thus $\mathrm{Tor}^i_{\mathcal{O}_{T^*X}}(\mathcal{O}_Y, \mathcal{O}_V) = 0$ for $i > 0$, and we deduce that $L^i F^*(M_0) = 0$ for $i > 0$. Since $\pi : X \to Z(L_P)$ is affine, the map π' is also affine, so $R\pi'_*(F^*(M_0))$ is concentrated in a single degree. □

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