Is the finite temperature effective potential effective for dynamics?

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We study the applicability of the finite temperature effective potential in the equation of motion of a homogeneous "misaligned" scalar condensate ϕ and find important caveats that severely restrict its domain of validity: (i) the assumption of local thermodynamic equilibrium is in general not warranted, (ii) we show a direct relation between the effective potential and the thermodynamic entropy density $S = -\partial V_{\mathrm{eff}}(T;\phi)/\partial T$, which entails that for a dynamical $\phi(t)$ the entropy becomes a nonmonotonic function of time, (iii) parametric instabilities in both cases with and without spontaneous symmetry breaking lead to profuse particle production with nonthermal distribution functions, (iv) in the case of spontaneous symmetry breaking spinodal instabilities yield a complex effective potential, internal energy and entropy, an untenable situation in thermodynamics. All these caveats associated with using the effective potential in the equation of motion of the condensate cannot be overcome by finite temperature equilibrium resummation schemes. We argue that the dynamics of the condensate leads to decoupling and freeze-out from local thermodynamic equilibrium, and propose a closed quantum system approach based on unitary time evolution. It yields the correct equations of motion without the caveats of the effective potential, and provides a fully renormalized and thermodynamically consistent framework to study the dynamics of the "misaligned" condensate, with real and conserved energy and entropy amenable to numerical study. The evolution of the condensate leads to profuse stimulated particle production with nonthermal distribution functions. Possible emergent asymptotic nonthermal states and eventual rethermalization are conjectured.

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I. INTRODUCTION

The finite temperature effective potential is a very powerful diagnostic tool to study the phase structure of quantum field theories including thermal and quantum corrections. It is the finite temperature extension of the zero temperature effective potential originally proposed by the pioneering work of Refs. [1–4] to study how radiative corrections modify the symmetry breaking properties of the vacuum. Functional methods provide a systematic formulation of the zero temperature effective potential as the generating functional of single particle irreducible Green's functions at zero four momentum [5–8].

The extension of the effective potential to equilibrium finite temperature was pioneered by Refs. [9,10]. It describes the free energy landscape as a function of the spatially homogeneous and time independent order parameter, the expectation value of a scalar field ϕ, thereby characterizing the different phases of a theory. As such, the finite temperature effective potential plays a fundamental role in cosmology, as it may describe possible cosmological phase transitions [11–15].

A. Motivation and objectives

The finite temperature effective potential was originally introduced and developed with the aim of describing

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equilibrium aspects of spontaneous symmetry breaking including quantum and thermal effects in terms of a free energy as a function of the homogeneous and static order parameter. However, it is often used in the equations of motion of such an order parameter to describe the dynamics of, for example, "misaligned" condensates. A recent study [16] of the zero temperature effective potential, extending the Hamiltonian framework introduced in Refs. [4,17], revealed several important caveats that invalidate its applicability in the equation of motion of the order parameter, namely the condensate or mean field. In the case when the tree-level potential does not feature broken symmetry minima, oscillations of the condensate around its minimum lead to instabilities associated with parametric amplification resulting in the exponential growth of the fluctuations around the mean field with profuse particle production, a physical mechanism similar to that of reheating in cosmology [18–24]. In the case when the tree-level potential admits broken symmetry minima, a different instability emerges when the excursion of the mean field probes a region where the potential features negative second derivatives. This is the spinodal (or tachyonic) instability and again leads to exponential growth of fluctuations around the mean field. In this case the growth of fluctuations is associated with the formation and growth of correlated domains [17,25,26]. In statistical physics this is the hallmark of the process of spinodal decomposition and phase ordering dynamics in phase transitions [27–30].

Both types of instabilities lead to the unambiguous conclusion that the zero temperature effective potential, which by definition and construction is a static function of the mean field, is inadequate to describe the dynamics of the mean field [16].

Motivated by the ubiquity and importance of the effective potential in cosmology and in general to study the phases of a quantum field theory, our objectives in this article are as follows: (i) to critically examine the validity of the finite temperature effective potential in the equation of motion of a homogeneous "misaligned" condensates and (ii), if it is found to be unreliable, to provide an alternative and consistent formulation of the dynamics of the condensate. While ultimately our aim is to study these aspects within the context of an expanding cosmology, in this article we restrict our focus to the case of Minkowski space-time as a first step. Undoubtedly, a critical assessment of the validity of the effective potential in the equation of motion of condensates must start with this simpler case from which much can be learned and whose study will pave the way towards a firmer understanding in cosmology.

B. Brief summary of results

We extend and complement the Hamiltonian formulation of the finite temperature effective potential introduced in Ref. [10], yielding a clear relation to the zero temperature case studied in Ref. [16]. We obtain an exact result: the finite temperature effective potential is the Helmholtz free energy density for the fluctuations around the expectation value of the scalar field ϕ (order parameter). This relation has an important thermodynamic consequence: $S = -\partial V_{\rm eff}[T;\phi]/\partial T$, where S is the thermodynamic entropy density. Therefore, the applicability of the effective potential in a dynamical equation of motion for a "misaligned" condensate ϕ is restricted by fundamental thermodynamic properties of the entropy. In the case of unbroken symmetry we find a nonmonotonic time dependence of the entropy, and in the case of broken symmetry the entropy becomes complex as a consequence of spinodal instabilities. Both cases are untenable in local thermodynamic equilibrium. Implementing a Chapman-Enskog expansion of the Boltzmann equation, we show that the assumption of local thermodynamic equilibrium (LTE) is in general unwarranted as it requires a fine-tuning of couplings to the heat bath. Furthermore, we argue that parametric and spinodal instabilities invalidate the use of an effective potential, which by design and construction is a static equilibrium function of ϕ, in the equation of motion for the condensate. It is argued that the dynamical evolution of the condensate leads to a "freeze out" of the density matrix and decoupling from LTE, and propose a closed quantum system approach to the dynamics. We introduce a method based on unitary time evolution to obtain directly the correct equations of motion for the condensate, which features conservation of energy and entropy, these are always real and without the caveats associated with the effective potential. Parametric and spinodal instabilities lead to an energy transfer between the condensate and the fluctuations resulting in stimulated particle production with nonthermal distribution functions. A fully renormalizable and thermodynamically consistent framework to study the dynamics, amenable to numerical study is provided. Possible asymptotic states and rethermalization are conjectured.

The article is organized as follows: in Sec. II, we briefly review the Hamiltonian approach to the zero temperature effective potential before extending the formulation from Ref. [10] to finite temperature. In this section we relate the static effective potential nonperturbatively to the Helmholtz free energy and thermodynamic entropy of the fluctuations and obtain the well known result for the one-loop effective potential. In Sec. III we analyze the reliability of the static effective potential in the equation of motion for the condensate under the assumption of LTE. In this section we show that LTE is in general not warranted and discuss severe caveats in the use of the effective potential in the equations of motion arising from parametric amplification and spinodal instabilities. In Sec. IV it is argued that the dynamics leads to a "freeze out" from LTE and decoupling from the thermal bath and introduce a closed quantum system approach based on unitary time evolution to obtain the correct equations of motion. These are shown to conserve energy density and entropy, which are manifestly real without the caveats of the effective potential. In this section it is shown that parametric and spinodal instabilities are efficient mechanisms of energy transfer between the condensate and the fluctuations leading to profuse stimulated particle production with nonthermal distributions. We provide a fully renormalized and thermodynamically consistent framework to study the dynamics of the condensate amenable to numerical study. Section V conjectures on the emergence of possible asymptotic states and rethermalization. In Sec. VI we present our conclusions and suggest further avenues of study. An Appendix summarizes the nonequilibrium correlation functions needed to obtain the equations of motion.

II. THE STATIC EFFECTIVE POTENTIAL: ZERO VS FINITE TEMPERATURE

A. Zero temperature

Before we consider the finite temperature effective potential, we briefly summarize the main concepts behind the Hamiltonian approach of Refs. [4,16,17] with the objective of comparing with the finite temperature case discussed below.

Let us consider a scalar theory described by the Hamiltonian

$$\hat H = \int d^3x \left[\frac{\hat\pi^2}{2} + \frac{(\nabla\hat\phi)^2}{2} + V(\hat\phi)\right]; \qquad (2.1)$$

where $\hat\pi$ is the canonical momentum conjugate to the scalar field $\hat\phi$. The Hamiltonian interpretation of the effective potential [4,16,17] defines the effective potential as the expectation value of $\hat H$ in a normalized coherent state $|\Phi\rangle$ in which the field acquires a space-time independent expectation value, namely a mean field ϕ,

$$\phi = \langle\Phi|\hat\phi(\vec x,t)|\Phi\rangle; \qquad \langle\Phi|\hat\pi(\vec x,t)|\Phi\rangle = 0; \qquad (2.2)$$

as

$$V_{\rm eff}(\varphi) \equiv \frac{1}{\mathcal V}\langle\Phi|\hat H(\vec x,t)|\Phi\rangle; \qquad (2.3)$$

where $\mathcal V$ is the spatial volume. Shifting the field operator $\hat\phi$ by its coherent state expectation value ϕ,

$$\hat\phi(\vec x,t) = \phi + \hat\delta(\vec x,t); \qquad \hat\pi(\vec x,t) \equiv \hat\pi_\delta(\vec x,t); \qquad (2.4)$$

the constraints (2.2) imply

$$\langle\Phi|\hat\delta(\vec x,t)|\Phi\rangle = 0; \qquad \langle\Phi|\hat\pi_\delta(\vec x,t)|\Phi\rangle = 0; \qquad (2.5)$$

leading to

$$V_{\rm eff} = V(\phi) + \frac{1}{\mathcal V}\int d^3x \, \langle\Phi|\frac{\hat\pi_\delta^2}{2} + \frac{(\nabla\hat\delta)^2}{2} + \frac{1}{2}V''(\varphi)\hat\delta^2 + \cdots|\Phi\rangle; \qquad (2.6)$$

where the expectation value of the linear terms in $\hat\delta$ and $\hat\pi_\delta$ vanish by the constraints (2.2), and the dots in Eq. (2.6) stand for higher powers of δ leading to higher loop corrections.

In the Hamiltonian formulation quantization proceeds by expanding the fluctuation field $\delta(\vec x,t)$ in the basis of solutions of the Heisenberg field equations for a free field with mass squared $V''[\phi]$, namely

$$\ddot{\delta}(\vec x,t) - \nabla^2\delta(\vec x,t) + V''[\phi]\delta(\vec x,t) = 0; \qquad (2.7)$$

and the field $\delta(\vec x,t)$ is expanded in mode functions,

$$\delta(\vec x,t) = \frac{1}{\sqrt{\mathcal V}}\sum_{\vec k}\left[a_{\vec k}\, g_k(t)\, e^{i\vec k\cdot\vec x} + a^\dagger_{\vec k}\, g^*_k(t)\, e^{-i\vec k\cdot\vec x}\right]; \qquad (2.8)$$

$$\pi_\delta(\vec x,t) = \frac{1}{\sqrt{\mathcal V}}\sum_{\vec k}\left[a_{\vec k}\, \dot g_k(t)\, e^{i\vec k\cdot\vec x} + a^\dagger_{\vec k}\, \dot g^*_k(t)\, e^{-i\vec k\cdot\vec x}\right]; \qquad (2.9)$$

where the mode functions $g_k(t)$ are solutions of the equations [16]

$$\ddot{g}_k(t) + \omega_k^2(t)\, g_k(t) = 0; \qquad \omega_k^2 = k^2 + V''[\phi]; \tag{2.10}$$

with the Wronskian condition

$$\dot{g}_k(t)\, g_k^*(t) - \dot{g}_k^*(t)\, g_k(t) = -i; \tag{2.11}$$

and the annihilation and creation operators are time independent and obey canonical commutation relations. For a space-time constant ϕ, the mode functions are given by

$$g_k(t) = \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}; \qquad \omega_k = \sqrt{k^2 + V''[\phi]}; \tag{2.12}$$

yielding the mode expansion

$$\delta(\vec{x},t) = \sqrt{\hbar}\,\frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[a_{\vec{k}}\, e^{-i\omega_k t} e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\, e^{i\omega_k t} e^{-i\vec{k}\cdot\vec{x}} \right]; \tag{2.13}$$

$$\pi_\delta(\vec{x},t) = -i\sqrt{\hbar}\,\frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\omega_k}{2}} \left[a_{\vec{k}}\, e^{-i\omega_k t} e^{i\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k}}\, e^{i\omega_k t} e^{-i\vec{k}\cdot\vec{x}} \right]; \tag{2.14}$$

and the quadratic Hamiltonian inside the brackets in Eq. (2.6) becomes

$$H = \sum_{\vec{k}} \hbar\omega_k(\phi) \left[a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \right]. \tag{2.15}$$

The constraints (2.5) are implemented by requesting that the coherent state $|\Phi\rangle$ be an eigenstate of the Fock occupation number $n_{\vec{k}} = a^\dagger_{\vec{k}} a_{\vec{k}}$; however, the lowest expectation value of the quadratic Hamiltonian is obtained for the vacuum state for the fluctuations $\hat{\delta}$, namely [16]

$$a_{\vec{k}}|\Phi\rangle = 0; \quad \forall\, \vec{k}; \tag{2.16}$$

leading to the constraint (2.5).

Taking the infinite volume limit with $\sum_{\vec{k}} \to V\int d^3k/(2\pi)^3$ and using (2.16), we find that the effective potential (2.3) is given by

$$V_{\rm eff}(\phi) = V(\phi) + \frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega_k(\phi) + \mathcal{O}(\hbar^2). \tag{2.17}$$

The $\hbar$ in (2.17) originates in the $\sqrt{\hbar}$ in the usual field quantization (2.13), (2.14) and implies that the expression (2.17) is the zero temperature one-loop effective potential. If $|\Phi\rangle$ is an excited eigenstate with $n_k \neq 0$, then the integrand in the second term features an extra contribution $n_k\omega_k(\phi)$ thereby raising the energy [16,17].

In order to compare the above results to the finite temperature case, we introduce the pure state density matrix

$$\rho \equiv |\Phi\rangle\langle\Phi|; \tag{2.18}$$

from which it follows that

$$\phi = \mathrm{Tr}\,\hat{\phi}(\vec{x})\rho; \qquad V_{\rm eff}(\phi) = \frac{1}{V}\mathrm{Tr}\,H\rho; \tag{2.19}$$

and the constraints (2.5) become

$$\mathrm{Tr}\,\hat{\delta}(\vec{x})\rho = 0; \qquad \mathrm{Tr}\,\hat{\pi}_\delta(\vec{x})\rho = 0. \tag{2.20}$$

Before moving on to the finite temperature case, it must be emphasized there are two main assumptions leading up to the zero temperature result (2.17): (i) that the mean field ϕ is time independent, yielding the mode functions given by Eq. (2.12), (ii) that $V''(\phi) > 0$, condition that yields real frequenciesWhen ϕ evolves in time, as in the dynamical case, the $\omega_k(\phi)$.

mode functionsthrough the time dependence oftions (2.10) with$g_k V(t_{00})(\phi$are solutions of the mode equa-$(t))$ that now depends on time$\phi(t$is$)$, and gives rise tonegative for some

parametric instabilities, and ifvalues of $V_{00}(\phi)$ 2 $_{00}(\phi)|$

since for ϕ, there are instabilities for $k < |V$

these wave vectors the frequencies ω_k become purely imaginary. Both instabilities will be addressed in Sec. III within the context of the applicability of the finite temperand have been discussed in greater depth in Ref.ature effective potential in the dynamical evolution of $\phi(t$ot$)$

[16]
which we refer the reader for a more detailed treatment.

B. Finite temperature

The discussion above highlights the interpretation of the zero temperature effective potential as the expectation value of the Hamiltonian in a particular coherent state, defined to be the vacuum for the fluctuations around the mean field ϕ. This formulation does not have a straightforward extrapolation to finite temperature, because the equilibrium density matrix corresponds to a mixed state that describes an ensemble, not a pure state as in (2.18).

The constraints (2.5), which can be implemented straightforwardly in the case of a pure state, must now be imposed in terms of Lagrange multipliers added to the Hamiltonian in the thermal density matrix. This is achieved by following the formulation of the finite temperature effective potential advocated in the seminal articles [9,10]. In particular by implementing the Hamiltonian formulation of Ref. [10],[1] wherein the effective potential is obtained from the Legendre transform of the equilibrium free energy under the constraint that the expectation value of the field is a space-time constant.

To discuss the main arguments in a clear manner, we focus on the simple case of a scalar field with a Lagrangian density

$$\mathcal{L}[\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V[\phi]; \qquad (2.21)$$

yielding the Hamiltonian (2.1). Let us introduce

$$H_J[\phi] = H[\phi] + \int d^3x\, J(\vec{x})\,\phi(\vec{x}); \qquad (2.22)$$

where $J(\vec{x})$ is an external classical source. The canonical partition function is given by

$$Z[T;J] \equiv e^{-\beta F[T;J]} = \mathrm{Tr}\, e^{-\beta H_J[\phi]}; \qquad (2.23)$$

whereThe $F[T;J]$ is the Helmholtz free energy andexpectation value of the field$\beta\phi = (\vec{x}$ $1)=T$in. equilibrium

presence of the source is defined as

$$\phi(\vec{x}) \equiv \frac{\mathrm{Tr}\,\phi(\vec{x})\,e^{-\beta H_J[\phi]}}{\mathrm{Tr}\,e^{-\beta H_J[\phi]}}; \qquad (2.24)$$

which is obtained as a variational derivative with respect to the c-number source, namely

$$\phi(\vec{x}) = -\frac{1}{\beta}\frac{\delta}{\delta J(\vec{x})}\ln Z = \frac{\delta}{\delta J(\vec{x})}F[T;J]; \qquad (2.25)$$

the source $J(\vec{x})$ can be interpreted as a Lagrange multiplier for the constraint $\phi(\vec{x}) = \langle\phi(\vec{x})\rangle$ where the expectation value is obtained with the partition function $Z[T;J]$. The relations (2.24), (2.25) are inverted to yield

$$J(\vec{x}) = J[\phi(\vec{x})]; \qquad (2.26)$$

[1] See the appendix in this reference.

from which the Legendre transform

$$\Omega[T;\phi] = F[T;J[\phi]] - \int d^3x J[\phi](\vec{x})\phi(\vec{x}); \qquad (2.27)$$

yields the generalized Gibbs free energy as a function(al) of temperature and the expectation value $\phi(\vec{x})$. Using the definition (2.24) it is straightforward to find that

$$\frac{\delta}{\delta\phi(\vec{x})}\Omega[T;\phi] = -J[\phi](\vec{x}). \qquad (2.28)$$

From now on we will consider a spatially constant expectation value ϕ, which implies a translationally invariant partition function, and introduce

$$j \equiv \frac{1}{\mathcal{V}}\int d^3x J(\vec{x}); \qquad F[T;J] \equiv \mathcal{V} F[T;j]; \qquad (2.29)$$

with $\mathcal{V}$ the spatial volume, and following Refs. [9,10] define the finite temperature effective potential as

$$\Omega[T;\phi] \equiv \mathcal{V} V_{\rm eff}(T;\phi). \qquad (2.30)$$

From Eqs. (2.27), (2.30) it follows that

$$V_{\rm eff}[T;\phi] = F[T;j[\phi]] - j[\phi]\phi \qquad (2.31)$$

and

$$\frac{dV_{\rm eff}[\phi;T]}{d\phi} = -j. \qquad (2.32)$$

This relation is very illuminating, let us first consider it at tree level, without quantum and thermal corrections, when $V_{\rm eff}[\phi] = V[\phi]$. The relation (2.32) clearly states that j is the external force necessary to maintain the space-time constant ϕ at a value that does not correspond to the minimum of the potential. This force vanishes for ϕ satisfying $dV[\phi]/d\phi = 0$, namely the equilibrium condition. The relation (2.32) must be compared to the classical equation of motion for a spatially constant (homogeneous) field configuration, namely at tree level (dots stand for time derivatives)

$$\frac{d}{d\phi}V[\phi] = -\ddot{\phi}; \qquad (2.33)$$

which when compared with Eq. (2.32) clearly states that in absence of dynamical evolution, the external force j must be applied to maintain ϕ away from the equilibrium value. This observation will be of paramount importance in the discussion of dynamics in the next sections.

As it will become clear in the discussion below, it is more convenient to discuss the effective potential and its dynamical generalization in terms of the fluctuations of the field $\phi(\vec{x})$ around the expectation value $\phi(\vec{x})$, a classical c-number field. Hence, as in the zero temperature case, Eq. (2.4), we introduce the field operator

$$\delta(\vec{x}) = \phi(\vec{x}) - \phi(\vec{x}); \qquad (2.34)$$

and write

$$H_J[\phi] \equiv \int d^3x J(\vec{x})\phi(\vec{x}) + H_J[\delta]; \qquad (2.35)$$

where

$$H_J[\delta] \equiv H[\delta + \phi] + \int d^3x J(\vec{x})\delta(\vec{x}); \qquad (2.36)$$

from which we find

$$e^{-\beta F[T;J]} = e^{-\beta\int d^3x J(\vec{x})\phi(\vec{x})} e^{-\beta F_\delta[T;J]}; \qquad (2.37)$$

with the Helmholtz free energy for the fluctuations around the mean field,

$$F_\delta[T;J] = -\frac{1}{\beta}\ln[\mathrm{Tr}\, e^{-\beta H_J[\delta]}]; \qquad (2.38)$$

and

$$F[T;J] = F_\delta[T;J] + \int d^3x J(\vec{x})\phi(\vec{x}). \qquad (2.39)$$

With this transformation, the relation (2.26) yielding the source $J(\vec{x})$ in terms of the expectation value $\phi(\vec{x})$ is obtained from the constraint

$$\langle \delta(\vec{x}) \rangle = \frac{\mathrm{Tr}\, \delta(\vec{x})\, e^{-\beta H_J[\delta]}}{\mathrm{Tr}\, e^{-\beta H_J[\delta]}} = 0. \qquad (2.40)$$

The Legendre transform (2.27) yields the generalized Gibbs free energy as

$$\Omega[T;\phi] = F_\delta[T;J[\phi]], \qquad (2.41)$$

which upon considering a spatially constant expectation value ϕ yields the effective potential

$$V_{\rm eff}[T;\phi] = F_\delta[T;\phi;j[\phi]]. \qquad (2.42)$$

Namely, the effective potential is the Helmholtz free energy density for the quantum fluctuations around a space-time constant expectation value, with the constraint $\langle \delta(\vec{x}) \rangle = 0$ which defines $j \equiv j[\phi]$. The condition (2.32) yielding

$$\frac{dV_{\rm eff}[T;\phi]}{d\phi} = -j[\phi] \qquad (2.43)$$

is now a consistency condition. This is a main conclusion of this analysis, and while it is a direct result of the analysis in the pioneering work in Refs. [9,10], we emphasize it here because (i) it is an exact result, valid to all orders in couplings and loop expansion, (ii) it has important thermodynamic consequences, in particular

$$V_{\rm eff}[T;\phi] \equiv F_\delta[T;\phi] = U[T;\phi] - T S[T;\phi]; \qquad (2.44)$$

where

$$U[T;\phi] = \frac{1}{V} \frac{\mathrm{Tr}\, H_J\, e^{-\beta H_J[\delta]}}{\mathrm{Tr}\, e^{-\beta H_J[\delta]}} = \frac{\partial}{\partial \beta}\{\beta F_\delta[T;\phi]\}; \qquad (2.45)$$

is the internal energy density, and $S[T;\phi]$ is the thermodynamic entropy density as a function of ϕ, which by the relations (2.44), (2.45) is given by[2]

$$S[T;\phi] = -\frac{\partial}{\partial T} V_{\rm eff}[T;\phi]. \qquad (2.46)$$

These are nonperturbative, exact relations that link directly the effective potential to the thermodynamic internal energy and entropy. In particular, the relation (2.46) is very important because when $V_{\rm eff}[T;\phi]$ is used in a dynamical equation of motion for the mean field ϕ, its time evolution translates into a time evolution of the entropy density, which must be compatible with the fundamental principles of thermodynamics.

Under reversible transformations, namely local thermodynamic equilibrium, the entropy obeys the second law of thermodynamics, it remains constant or increases monotonically. This fundamental aspect will be shown below to be in striking contradiction with the use of the effective potential in dynamical situations.

C. The one-loop effective potential

With the Hamiltonian (2.1) we find

$$H_J[\delta] = \mathcal{V} V[\phi] + \int d^3x \Big\{ \frac{1}{2}\pi^2_\delta(\vec{x}) + \frac{1}{2}(\nabla\delta(\vec{x}))^2 + \frac{1}{2} V''[\phi]\,\delta^2(\vec{x}) + \frac{1}{3!} V'''[\phi]\,\delta^3(\vec{x}) + \cdots + (J(\vec{x}) + V'[\phi])\,\delta(\vec{x}) \Big\}; \qquad (2.47)$$

were primes stand for derivatives with respect to ϕ. Neglecting the terms $\propto \delta^3; \delta^4; \ldots$, which yield higher loop corrections, the constraint $\langle \delta(\vec{x}) \rangle = 0$ is fulfilled by setting $J(\vec{x}) \equiv j$, namely a spatial constant, given by

$$j = -V'[\phi]; \qquad (2.48)$$

thereby cancelling the linear term in δ in (2.47). That the condition (2.48) yields $\langle \delta(\vec{x}) \rangle = 0$ when neglecting the cubic and higher powers of δ is clear, because under these conditions the Hamiltonian is quadratic in δ, describing a simple free field of squared mass $V''[\phi]$ for which the density matrix $e^{-\beta H_J[\delta]}$ is Gaussian with zero mean.

Quantizing the fluctuations by expanding the fluctuation field $\delta(\vec{x};t)$ as in Eq. (2.13) with the frequencies

[2] The partial derivatives are at constant ϕ.

$$\omega_k(\phi) = \sqrt{k^2 + V_0''(\phi)}; \qquad (2.49)$$

implementing the constraint (2.48), and keeping solely the quadratic terms in δ in (2.47) yields

$$H_J[\delta] = \mathcal{V}V[\phi] + \sum_{\vec{k}} \hbar\omega_k(\phi)\left[a^{\dagger}_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right]. \qquad (2.50)$$

The calculation of the Helmholtz free energy now becomes a simple textbook exercise in quantum statistical mechanics, the partition function

$$Z = e^{-\beta F[\phi]} = e^{-\beta \mathcal{V}V[\phi]}\,\mathrm{Tr}\,\Pi_{\vec{k}}\, e^{-\beta H_{\vec{k}}}; \qquad (2.51)$$

with

$$H_{\vec{k}} = \hbar\omega_k(\phi)\left[a^{\dagger}_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right]. \qquad (2.52)$$

The trace is calculated in the occupation number basis yielding

$$\mathrm{Tr}\,\Pi_{\vec{k}}\, e^{-\beta H_{\vec{k}}} = \Pi_{\vec{k}}\left[e^{-\beta\hbar\omega_k(\phi)/2}\sum_{n_{\vec{k}}=0}^{\infty} e^{-\beta\hbar\omega_k(\phi) n_{\vec{k}}}\right]$$

$$= \Pi_{\vec{k}}\,\frac{e^{-\beta\hbar\omega_k(\phi)/2}}{1-e^{-\beta\hbar\omega_k(\phi)}};$$

$$= e^{-\frac{\hbar\beta}{2}\sum_{\vec{k}}\omega_k(\varphi)}\, e^{-\sum_{\vec{k}}\ln[1-e^{-\beta\hbar\omega_k(\phi)}]}; \qquad (2.53)$$

passing to the infinite volume limit with

$$\sum_{\vec{k}} \to \mathcal{V}\int\frac{d^3k}{(2\pi)^3}; \qquad (2.54)$$

we find the one-loop finite temperature effective potential (2.42)

$$V^{(1)}_{\mathrm{eff}}[\phi] = V[\phi] + \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega_k(\phi) + T\int\frac{d^3k}{(2\pi)^3}\ln[1-e^{-\beta\hbar\omega_k(\phi)}]; \qquad (2.55)$$

which is the usual result [9–12]. The $T \to 0$ limit yields the zero temperature one-loop effective potential given by Eq. (2.17), obtained in the previous section via the Hamiltonian approach and the particular coherent state $|$points out that the coherent state$\Phi\rangle$ yielding the constraintis precisely the ground state of the fluctuation(2.5). This analysis clearly$|\Phi\rangle$ with the constraint

(2.16)
Hamiltonian, because in the zero temperature limit, only the ground state contributes to the partition function Z.

An alternative that will prove useful to obtain the equation of motion for the condensate to study dynamics in the next section is to obtain $dV_{\mathrm{eff}}/d\phi$ from Eq. (2.43) where J is determined from solving the constraint

sion$\langle\delta(\vec{x})\rangle = V'_{\mathrm{eff}}[0]\phi$. For example, to zeroth order in the

loop expan- $=$(2.43)$V[\phi]$. To generate a loop expansion for

and the tree-level condition (2.48)J satisfies Eq. we follow

Ref. [10] and write $J = -V_0[\phi] + \hbar J_1 + \hbar^{3/2}J_2 + \quad (2.56)$

field expansionwhere $J_1; J_2$ are of(2.13)$\mathcal{O}$(showing that$(\hbar^0)$, with this expansion and the$\delta \propto \hbar^{1/2}$) it follows

that

$$J_1\hbar\delta \propto \hbar^{3/2}; \qquad J_2\hbar^{3/2}\delta \propto \hbar^2; \qquad (2.57)$$

which are of the same order in $\hbar$ (loop expansion) as $\delta^3 \simeq \hbar^{3/2}; \delta^4 \simeq \hbar^2$.

To generate the loop expansion in a systematic manner, we write the Hamiltonian (2.47) as

$$H_J[\delta] = \mathcal{V}V[\phi] + H_0 + H_I; \qquad (2.58)$$

with

$$H_0 = \int d^3x \left[\frac{1}{2}\pi^2_\delta(\vec{x}) + \frac{1}{2}(\nabla\delta(x))^2 + \frac{1}{2}V''[\varphi]\delta^2(\vec{x})\right] \tag{2.59}$$

and

$$H_I = \int d^3x \left(J\delta(\vec{x}) + V'[\phi]\delta(\vec{x}) + \frac{V'''}{3!}[\phi]\delta^3(\vec{x}) + \frac{V''''}{4!}[\phi]\delta^4(\vec{x}) + \cdots\right). \tag{2.60}$$

Let us define

$$U(\tau) = e^{\frac{\tau}{\hbar}H_0}e^{-\frac{\tau}{\hbar}(H_0+H_I)}, \tag{2.61}$$

from which it follows that

$$e^{-\beta(H_0+H_I)} = e^{-\beta H_0}U(\hbar\beta). \tag{2.62}$$

$U(\tau)$ obeys the differential equation

$$\frac{dU(\tau)}{d\tau} = -\frac{1}{\hbar}H_I(\tau)U(\tau); \qquad U(0) = 1; \tag{2.63}$$

where

$$H_I(\tau) = e^{\frac{\tau}{\hbar}H_0}H_I e^{-\frac{\tau}{\hbar}H_0}. \tag{2.64}$$

The solution of (2.63) is

$$U(\tau) = 1 - \frac{1}{\hbar}\int_0^\tau H_I(\tau_1)d\tau_1 + \frac{1}{\hbar^2}\int_0^\tau d\tau_1\int_0^{\tau_1} H_I(\tau_1)H_I(\tau_2)d\tau_2 + \cdots = T_\tau e^{-\frac{1}{\hbar}\int_0^\tau H_I(\tau_1)d\tau_1}; \tag{2.65}$$

where T_τ is the τ ordering symbol. Therefore, $U(\tau)$ is the time evolution operator in the interaction picture in imaginary time, namely in the Matsubara representation [31]. We can now write the partition function as

$$\mathrm{Tr}\, e^{-\beta H_J[\delta]} = e^{-\beta V F^{(1)}[\phi]}\langle U(\hbar\beta)\rangle_0; \tag{2.66}$$

where

$$e^{-\beta V F^{(1)}[\phi]} = e^{-\beta V V[\phi]}\,\mathrm{Tr}\, e^{-\beta H_0}; \tag{2.67}$$

and the expectation value in the free field theory, defined as

$$\langle U(\hbar\beta)\rangle_0 = \frac{\mathrm{Tr}\, e^{-\beta H_0}U(\hbar\beta)}{\mathrm{Tr}\, e^{-\beta H_0}}; \tag{2.68}$$

can be obtained in a systematic loop expansion. The trace in Eq. (2.67) is precisely given by Eq. (2.53) with the result that $F^{(1)}[\phi]$ is the one-loop finite temperature effective potential $V^{(1)}_{\mathrm{eff}}[\phi]$ given by Eq. (2.55). Therefore, from Eq. (2.42) we find the general form of the effective potential

$$V_{\mathrm{eff}}[\phi] = V^{(1)}_{\mathrm{eff}}[\phi] - \frac{1}{\beta V}\ln[\langle U(\hbar\beta)\rangle_0]. \tag{2.69}$$

This is an exact result where J is determined by the constraint $\langle\delta(\vec{x},t)\rangle = 0$ order by order in the loop expansion.

We note that $\ln[\langle U(\hbar\beta)\rangle_0] = O(\hbar^2)$ because $\langle\delta^m(\vec{x},\tau)\rangle_0 = 0$ for odd values of m. The expectation value (2.68) begins at two loops, and can now be obtained in a loop expansion, with the field δ in the Matsubara interaction picture $\delta(\vec{x},\tau) = e^{\frac{\tau}{\hbar}H_0}\delta(\vec{x},0)e^{-\frac{\tau}{\hbar}H_0}$, namely

$$\delta(\vec{x},\tau) = \sqrt{\frac{\hbar}{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_k}}\left[a_{\vec{k}}e^{-\omega_k\tau}e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}e^{\omega_k\tau}e^{-i\vec{k}\cdot\vec{x}}\right].$$

$$(2.70)$$

However, we need the explicit expression for the source $J(\vec{x})$, which is determined from the constraint

$$\frac{\mathrm{Tr}\, e^{-\beta H_0} U_H(-i\hbar\beta)\,\delta(\vec{0},0)}{\mathrm{Tr}\, e^{-\beta H_0} U(-i\hbar\beta)} = \frac{\langle U(-i\hbar\beta)\,\delta(\vec{0},0)\rangle_0}{\langle U(-i\hbar\beta)\rangle_0} = 0. \qquad (2.71)$$

Up to leading order $\hbar^0$ (neglecting cubic and higher order powers of δ in H_I), the numerator yields

$$-\int d^4x\,\big(J(\vec{x}) + V'[\phi]\big)\langle\delta(\vec{x},\tau)\delta(\vec{0},0)\rangle_0 = 0; \qquad \int d^4x \equiv \int_0^{\hbar\beta} d\tau \int d^3x; \qquad (2.72)$$

since $\langle\delta(\vec{x},\tau)\delta(\vec{0},0)\rangle_0 \neq 0$ it follows that $J(\vec{x}) = j = -V'[\phi]$, which is precisely the relation (2.29), and consistent with the expansion (2.48) with j defined by Eq. (2.56).

Now we obtain the $O(\hbar)$ contribution to j by considering the cubic term in the interaction, yielding

$$-\int d^4x\Big[\big(J(\vec{x}) + V'[\phi]\big)\langle\delta(\vec{x},\tau)\delta(\vec{0},0)\rangle_0 + \frac{V'''[\phi]}{3!}\langle\delta^3(\vec{x},\tau)\delta(\vec{0},0)\rangle_0\Big] = 0; \qquad (2.73)$$

using Wick's theorem

$$\langle\delta^3(\vec{x},\tau)\delta(\vec{0},0)\rangle_0 = 3\langle\delta(\vec{x},\tau)\delta(\vec{0},0)\rangle_0\langle\delta^2(\vec{x},\tau)\rangle_0; \qquad (2.74)$$

the calculation of $\langle\delta^2(\vec{x},\tau)\rangle_0$ is straightforward using the expansion (2.70), leading to the result

$$-\int d^4x\,\langle\delta(\vec{x},\tau)\delta(\vec{0},0)\rangle_0\Big[J(\vec{x}) + V'[\phi] + \frac{\hbar}{2}V'''[\phi]\int\frac{d^3k}{(2\pi)^3}\frac{[1+2n_k(\phi)]}{2\omega_k}\Big] = 0; \qquad n_k(\phi) = \frac{1}{e^{\beta\hbar\omega_k(\phi)}-1}; \qquad (2.75)$$

from which we find

$$j = -V'[\phi] - \frac{\hbar}{2}V'''[\phi]\int\frac{d^3k}{(2\pi)^3}\frac{[1+2n_k(\phi)]}{2\omega_k} = -\frac{d}{d\phi}V_{\mathrm{eff}}^{(1)}[\phi]; \qquad (2.76)$$

given by (2.43) with $V_{\mathrm{eff}}^{(1)}[\phi]$, thus explicitly confirming the relation (2.55) up to order $\hbar$, namely one loop.

This analysis confirms the "recipe" to obtain the one-loop effective potential advocated in Ref. [9]: write $\phi = \phi + \delta$ in the Lagrangian density, expand in the fluctuation up to second order and neglect the linear term in δ, the resulting Lagrangian density describes a free field theory of a scalar field of mass squared $M^2 = V''[\phi]$. The one-loop effective potential is the Helmholtz free energy density of this free field theory.

Although alternative functional methods yield the effective potential in a loop expansion, the main purpose of revisiting and complementing the Hamiltonian framework of Ref. [10], confirming the one-loop results of Refs. [9,10] and explicitly showing that the zero temperature limit coincides with the effective potential obtained from the expectation value of the Hamiltonian in the state (2.16) as shown in Ref. [16], is to highlight the following aspects:

(1) The finite temperature effective potential is obtained for space-time constant expectation values under the assumption of thermal equilibrium. It is identified with the Helmholtz free energy density under a constraint that the expectation value of the fluctuations around a fixed space-time constant mean field vanish. This constraint is implemented by introducing a Lagrange multiplier via an external constant source J coupled linearly to the field. The Lagrange multiplier J represents an external force that keeps the expectation value of the field in equilibrium away from the minimum of the

effective potential. This force vanishes when the expectation value corresponds to the extremum of the effective potential. This is the content of the exact relation (2.32).

(2) The Hamiltonian formulation of both the zero and finite temperature effective potential explicitly shows that the one-loop finite temperature effective potential is the Helmholtz free energy of the free field fluctuations δ around the expectation value ϕ. Up to one loop, this is a free scalar field theory of squared mass $V''[\phi]$, which is a space-time constant, quantized in terms of the usual mode functions

$$e^{i\omega_k t} \text{ of constant frequency } \omega_k(\phi) = \sqrt{k^2 + V''[\phi]}.$$

Furthermore, the distribution function (occupation number) of these quanta is the usual thermal equilibrium Bose-Einstein distribution with frequency $\omega_k(\phi)$. The zero temperature limit is the ground state expectation value of the free field Hamiltonian associated of these fluctuations. It is precisely the one-loop effective potential obtained from the Hamiltonian method in Ref. [16].

(3) The finite temperature effective potential being identified with the Helmoltz free energy density has particular thermodynamic significance because it is directly related to the internal energy density and the entropy density, $S = -\partial V_{\mathrm{eff}}[T;\phi]/\partial T$. This relation is exact and entails that if the effective potential is used in a dynamical equation of motion of the mean field, fundamental thermodynamic properties of the entropy restrict its domain of validity in such equation of motion.

(4) The perturbative method implemented to obtain $j[\phi]$, yielding Eq. (2.76) will be seen below to be very similar to the formulation of the equations of motion from nonequilibrium quantum field theory.

While several of these points seem obvious from the results leading up to the final expression of the effective potential (2.69) with the one-loop result given by Eq. (2.55), when uncritically extrapolated to the dynamical case they will lead to conclusions that are at odds with the fundamental tenets of (local) thermodynamic equilibrium.

III. DYNAMICS: OPEN QUANTUM SYSTEM PERSPECTIVE

As discussed in the previous section, the finite temperature effective potential is a static quantity, designed to explore the free energy landscape in equilibrium at finite temperature as a function of a space-time constant order parameter, namely the expectation value of the scalar field in equilibrium. Yet, it is often used in dynamical situations in an equation of motion for this homogeneous order parameter:

$$\ddot{\phi}(t) + \frac{d}{d\phi} V_{\mathrm{eff}}[\phi(t)] = 0. \qquad (3.1)$$

In this section, we endeavor to understand if and under what circumstances such an equation of motion in terms of $V_{\mathrm{eff}}[\phi]$ is valid. We note that an important consequence of using the static effective potential in the equation of motion (3.1) is that, in this equation the effective potential only depends on time via the time evolution of ϕ, leading to the conserved quantity

$$\frac{1}{2}(\dot{\phi})^2 + V_{\mathrm{eff}}[\phi] = E = \text{constant}. \qquad (3.2)$$

This result is a direct consequence of assuming that the Helmholtz free energy density depends on time solely via the time evolution of ϕ. A consequence of this equation when combined with the exact result (2.46), is that the thermodynamic entropy depends on time via the time dependence of the mean field.

It has important implications: let us consider the unbroken symmetry case in which the minimum of the effective potential is at $\phi = 0$, and that the initial value of ϕ corresponds to a large amplitude with $\dot{\phi} = 0$, hence a large value of E. Then as $\phi(t)$ rolls down the potential hill ϕ and consequently $V_{\mathrm{eff}}[\phi]$, become small, however, the velocity $\dot{\phi}$ has to become large, therefore while V_{eff} is small, its time derivative becomes large, entailing that the Helmholtz free energy and the entropy, which have been obtained in equilibrium are actually changing rapidly in time. This behavior results in a contradiction between the assumptions of thermal equilibrium and the validity of the dynamical equation of motion.

A. Local thermodynamic equilibrium?

Using $V_{\mathrm{eff}}[\phi]$, a static function, in the dynamical equation of motion (3.1) suggests that an underlying (albeit unspelled) assumption is that of LTE. Namely, that the distribution function $n_k(\phi)$ which enters in $dV_{\mathrm{eff}}/d\phi$ [see Eq. (2.76)] is always the Bose-Einstein distribution function at temperature T with the frequencies $\omega_k(t) = \sqrt{k^2 + V''[\phi(t)]}$ which are now time dependent. This implies that the distribution function adjusts to the change in the frequency on time scales much shorter than that of the evolution of the frequency itself. Underpinning this assumption is the concept of treating the dynamics of ϕ as a quantum open system, namely that the scalar field is in contact with other degrees of freedom that constitute a thermal bath, itself in equilibrium at temperature T, with which it exchanges energy momentum via collisional processes. In postulating Eq. (3.1) for the dynamics, the bath itself and its interactions with the scalar field are not specified.

A consistent justification of the assumption of LTE and the applicability of the effective potential as a function of time through the evolution of $\phi(t)$ and the distribution functions $n_k(\phi(t);t)$ would imply solving simultaneously the set of Boltzmann equations for the distribution function with a fully specified collisional term from the coupling to the bath degrees of freedom, along with the equation of motion for $\phi(t)$. Undoubtedly implementing such program is a major undertaking and has not yet been attempted, nor is it our objective in this study. Instead we invoke the usual argument [11,32,33] of comparing the time scales of collisional relaxation with those from the dependence of the distribution function assuming the validity of LTE in its time evolution. Such arguments are ubiquitous in cosmology and underpin the understanding of the validity of LTE during cosmological expansion as well as the freeze-out of species and decoupling from a thermal environment [11,32,33].

In absence of external forces, and in a homogeneous situation the distribution function obeys the Boltzmann equation [11,32,33]

$$\frac{d}{dt} n_k(t) = C[n_k]; \qquad (3.3)$$

where $C[n_k]$ is the collision kernel. The exact distribution function, solution of this Boltzmann equation is written as $n_k(t) = n_{lte}(k;t) + \delta n_k(t)$ where $n_{lte}(k;t)$ is the LTE distribution function, and $\delta n_k(t)$ is the departure from LTE. If $\delta n_k(t)/n_{lte}(k;t) \ll 1$ then LTE is a reliable approximation to the exact distribution function. The departure from LTE is studied within a Chapman-Enskog expansion [34], in terms of the ratios between the relaxation time and the time scale of variation of the distribution function, and the ratio of the mean free path to the spatial scale of variation (Knudsen number). LTE ensues when these ratios are $\ll 1$.

In absence of a specific model for the collision kernel we can resort to the relaxation time approximation for a qualitative (and semiquantitative) estimate [11,32,33],[3]

$$\frac{d}{dt} n_k(t) = -\frac{1}{\tau}\,\delta n_k(t); \qquad \delta n_k(t) = (n_k(t) - n_{lte}(k;t)); \qquad (3.4)$$

where τ is the average time between collisions, i.e., relaxation time or inverse reaction rate $\Gamma = 1/\tau$ [11,32–34],

$$\tau = \frac{1}{n\langle\sigma v\rangle}; \qquad (3.5)$$

with n the density of scatterers, σ the cross section and v the relative velocity. We take the LTE distribution function

$$n_{lte}(k;t) \equiv \frac{1}{e^{\beta\omega_k(t)} - 1}; \qquad (3.6)$$

since this is the distribution function that enters in $V_{eff}[\phi(t)]$. To first order in the Chapman-Enskog expansion [32,34] $n_k(t)$ on the left-hand side of Eq. (3.4) is replaced by $n_{lte}(k;t)$, therefore to this order

$$\delta n_k(t) = -\tau \frac{d}{dt} n_{lte}(k;t); \qquad (3.7)$$

and LTE is a reliable approximation if

$$\frac{\delta n_k(t)}{n_{lte}(k;t)} \ll 1. \qquad (3.8)$$

Let us consider the high temperature limit $\beta\omega_k \ll 1$ where we expect a short relaxation time, yielding

[3] See Sec. IV, Eq. (4.41) and following in Ref. [32].

where up to one loop

$$\delta n_k(t) \simeq T \frac{\dot{\omega}_k(t)}{\omega_k^2(t)} ; \qquad (3.9)$$

therefore LTE in this regime is fulfilled when

$$\frac{\dot{\omega}_k(t)}{\omega_k(t)\,\Gamma_k(t)} \ll 1. \qquad (3.10)$$

Approximating the high temperature limits

$$n \simeq T^3 ; \qquad \langle \sigma v \rangle \simeq \frac{g^2}{T^2} ; \qquad (3.11)$$

with g a dimensionless coupling, and considering longwavelength fluctuations, since we expect these to be the slowest to relax to LTE, the condition (3.8) yields

$$\frac{V'''[\phi]\,\dot{\phi}}{2\,T\,g^2\,V''[\phi]} \ll 1; \qquad (3.12)$$

which obviously depends not only on the coupling g to the bath but also the details of the potential $V[\phi]$ such as mass and couplings. To be specific, let us consider the case with tree-level potential

$$V[\phi] = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 ; \qquad m^2 > 0; \qquad (3.13)$$

with a large amplitude initial condition

$$\dot{\phi}(0) = 0; \qquad \lambda\phi^4(0) \simeq T^4; \qquad \lambda\phi^2(0) \gg m^2; \qquad (3.14)$$

consistently with an initially thermalized state of large energy density $\propto T^4$. As $\phi(t)$ rolls down the potential hill, $\dot{\phi} \simeq \sqrt{\lambda}\,\phi^2(0)$; $V'''[\phi] \simeq \lambda\phi(t) \simeq \lambda\phi(0)$; well before reaching the minimum [for example $V''[\phi(0)]/2 \simeq \lambda\phi^2(0)$] it follows that the ratio $\dot{\phi}(t)$ (3.12) $\simeq$ implies that the condition for the validity of LTE becomes

$$\frac{\lambda^{1/4}}{g^2} \ll 1; \qquad (3.15)$$

which may very well be violated depending on a delicate balance, in other words a fine tuning, of the strengths of couplings. For example, if the collisional kernel refers to collisions among the quanta of the scalar field, with self-interaction given by the potential (3.13) then $g^2 \rightarrow \lambda^2$ with an obvious violation of the LTE condition for weak coupling.

While a more careful treatment of the quantum kinetics combined with the equation of motion for ϕ is required for a thorough assessment of the validity or breakdown of LTE, a program well beyond the scope of this study, this simple analysis highlights that the validity of LTE must be carefully assessed and should not be taken for granted as it may not be fulfilled and may imply fine tuning in generic cases. The reader will recognize that the criterion for the validity of LTE is the usual one invoked in cosmology [11,32,33], where the reaction rate $\Gamma = 1/T$ is compared to the Hubble expansion rate H: in the cosmological case LTE is valid for $H/\Gamma \ll 1$. In the dynamical case under consideration, $\dot{\omega}_k(t)/\omega_k(t)$ replaces H; however, other than this difference, which ultimately is a difference on time scales, the main arguments are indeed similar.

The effective potential being identified with the Helmholtz free energy density implies that

$$V_{eff}[\phi] = U[\phi] - TS[\phi]; \qquad (3.16)$$

$$U = \langle H \rangle / \mathcal{V} = V[\phi] + \frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_k[1 + 2n_k(\phi)] \qquad (3.17)$$

is the internal energy density, and d³k

Z S¼ ð
2πÞ3fð1þnkðϕÞÞlnð1þnkðϕÞÞ−nkðϕÞlnðnkðϕÞÞg

ð3:18Þ

is the entropy density, with the occupation numbers

nkðϕÞ ¼ 1 / eβωkðϕÞ − 1: ð3:19Þ

frequenciesIn the dynamical caseωkðϕðtÞÞ. As a result,

the entropy densityϕðntÞkðdepends on time,

consequentlyϕÞ do depend on time via the the occupation

numbers depends on time, and its time derivative is given

by

S˙ ¼ −V000½2ϕTð2tÞϕ˙ Z ð2dπ3kÞ3 nkðϕÞð1 þ nkðϕÞÞ:
ð3:20Þ

6Consider the tree-level potential (3.13) with V000ðϕϕðtÞÞ ¼¼ 0,

This behavior is actually more general, when the symmetrywithλϕðtÞϕ, whenðtÞ ¼ ϕϕð0is oscillating near the minimum atÞcosðmtÞ, it follows that S˙ ∝ sinð2mtÞ.

is unbroken, ϕ oscillates around the minimum and V000½ϕðtÞϕ˙ðtÞ changes sign, thereby alternating between increasing and decreasing entropy along the trajectory. Namely, the entropy density is a nonmonotonic function of time, a behavior that, a priori is not compatible with a thermodynamic entropy. It may be argued that in the quantum open system approach the entropy of the system may not be a monotonic function of time as the system exchanges energy and momentum with the bath, and that the change in entropy of the system reflects heat transfer to and from the bath, while the total entropy of the system plus the bath increases monotonically or remains constant. However, we emphasize that the nonmonotonicity is in the entropy density, therefore the change in entropy is extensive, therefore such an argument implicitly accepts that the bath itself is not in thermal equilibrium and its dynamics is affected by the system in an extensive manner. Clearly these arguments must be quantified, however, the point remains that the time dependence of the entropy raises relevant questions on the validity of LTE in the dynamical evolution of the mean field.

B. Caveats: Parametric and spinodal instabilities

One of the main objectives of comparing the finite temperature effective potential to the zero temperature effective potential obtained in Ref. [16] is to highlight that the main caveats associated with using the effective potential in the dynamical equation of motion (3.1) discussed in this reference also apply to the one-loop finite temperature effective potential (2.55). After all taking the T → 0 limit in this expression yields the one-loop effective potential obtained in Ref. [16] in the Hamiltonian formulation.

The previous analysis on the validity of LTE, based on a collisional Boltzmann equation, does not include the possibility of instabilities which lead to particle production and nonthermal distribution functions. Two ubiquitous instabilities were studied in detail within the context of the zero temperature effective potential in Ref. [16]: parametric and spinodal, the latter ones associated with spontaneous symmetry breaking. While we refer the reader to this reference for further details, for completeness of presentation we summarize here the main aspects of both instabilities, with the objective of emphasizing that both prevent a formulation of an equilibrium finite temperature effective potential as described in the first section.

Let us first consider the tree-level potential (3.13), yielding V00½ϕðtÞ ¼ m⁴ þ 3λϕ²ðtÞ with m² > 0 and small amplitude oscillations around the minimum at ϕ ¼ 0,

⁴ gkðτÞ þ ½ηk − 2αcosð2τÞgkðτÞ ¼ 0; ð3:24Þ dτ

bands

whose

have

instability

been

namely

$$\phi(t) = \phi(0)\cos(mt). \qquad (3.21)$$

yielding

$$V''(\phi(t)) = m^2 + 3\lambda\phi^2(0)\cos^2(mt). \qquad (3.22)$$

Quantization of the fluctuation field δ [16] with an effective mass squared $V''[\phi(t)]$ given by (3.22) leads to Mathieu's equation [35–37], which features instability bands from parametric amplification describing profuse particle production [16,18–24]. While we refer the reader to Ref. [16] and references therein for a more detailed discussion, for consistency and completeness of presentation we summarize here some of the important aspects of parametric instability in this case. Introducing the dimensionless variables

$$\tau = mt - \frac{\pi}{2}; \quad \alpha = \frac{3\lambda}{4m^2}\phi^2(0); \quad \eta_k = 1 + \kappa^2 + 2\alpha; \quad \kappa = \frac{k}{m}, \qquad (3.23)$$

the mode equations (2.10) become of the form of Mathieu's equation [35–37]

d^2

analyzed in Refs. [16,35–37]. Figure 1 displays two linearly independent solutions in the first instability band, showing the exponential growth from parametric instability.

A general solution $g_k(t)$ is a linear combination of these two linearly independent solutions. The width of each band, labeled by an integer index $n = 2, 3$ is found to be proportional to α_n [16]. Within these instability bands the amplitudes of the mode functions grow as $g_k(t) \propto e^{\nu_k t}$ with real $\nu_k \geq 0$ being the Floquet exponents, the smaller wave vectors feature the largest ν_k and wider bandwidth of the unstable regions [16].

These instabilities and the concomitant particle production clearly indicate that maintaining LTE by collisional coupling to a bath is not a warranted assumption and in general implies fine tuning of couplings to the bath. Furthermore, particle production in the parametrically unstable bands results in nonthermal distribution functions which cannot be approximated by the usual Bose-Einstein distribution functions that emerge in the equilibrium description because particle production is effective within localized bands in momentum. If collisional processes distribute the particles outside the unstable bands into an LTE Bose-Einstein distribution function, such processes must occur on timescales shorter than the inverse of the largest Floquet exponent, again implying a fine-tuning between the coupling to the heat bath and the parameters of the potential. Clearly this is not a generic situation and depends on particular models and couplings to the bath degrees of freedom.

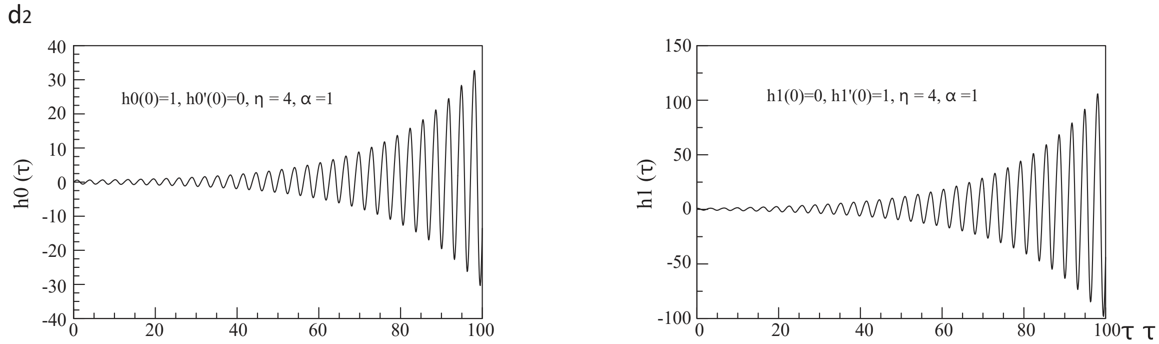

FIG. 1. Two linearly independent solutions of Mathieu's function's Eq. (3.24), $h_0(\tau); h_1(\tau)$ with initial conditions $h_0(0) = 1; h_0'(0) = 0$; $h_1(0) = 0$; $h_1'(0) = 1$, for the first unstable band. A general solution for a mode $g_k(\tau)$ is a complex linear combination of $h_0(\tau)$ and $h_1(\tau)$ satisfying the condition (2.11).

Let us now consider the case in which the tree-level potential leads to spontaneous symmetry breaking, for example the potential (3.13), but with $m^2 = -\mu^2$ with $\mu^2 > 0$, yielding $V''[\phi] = -\mu^2 + 3\lambda\phi^2$. Within the (classical) spinodal region $\phi^2 < \mu^2/3$ it follows that $V''[\phi] < 0$ and there is a band of spinodally unstable wave vectors $k^2 < |V''[\phi]|$ for which the mode functions grow exponentially $|g_k(t)| = s_k e^{\int^t \nu_k(t')dt'} + r_k e^{-\int^t \nu_k(t')dt'}$ with $\nu_k(t) = \sqrt{|V''[\phi(t)]| - k^2} > 0$, and s_k, r_k determined by initial conditions, thereby signalling exponential growth of fluctuations. In this case, for $|\phi|$ within the spinodal region and values of k in the unstable band, the frequencies $\omega_k[\phi] = \sqrt{k^2 + V''[\phi]}$ are purely imaginary and the Helmholtz free energy density, namely the effective potential, the internal energy density, occupation numbers $n_k(\phi)$ and entropy (3.18) are all complex, an untenable situation from the thermodynamic perspective, even when $\phi(t)$ "rolls down" the potential hill very slowly within the spinodal. That equilibrium thermodynamics (or LTE) cannot describe this situation is well known in statistical physics: the spinodal instabilities are associated with the dynamical process of phase separation and the growth of correlated ordered domains [27–29], and has also been studied in quantum field theory [17,25,26]. Particle production in these spinodally unstable bands is followed by particle production by parametric instabilities when the mean field ϕ has rolled down below the spinodal point and oscillates near the (broken symmetry) minimum, again resulting in nonthermal distribution functions as a result of particle production from parametric amplification in the unstable bands.

Both types of instabilities result in profuse particle production and nonthermal distribution functions for the produced particles which are localized in momentum within the unstable bands. A redistribution of particles into thermal distribution functions, the underlying assumption in using the finite temperature effective potential, implies a strong coupling to a thermal bath in such a way that this redistribution occurs on time scales much shorter than the timescales associated with the instabilities. Obviously if and when such a coupling arises is a model dependent, highly fine-tuned and nongeneric case.

C. Partial summary

In the previous sections we have shown that the finite temperature effective potential is associated with the equilibrium Helmholtz free energy density as a function of the mean field ϕ. This is an exact result valid to all orders in couplings and loop expansion with important thermodynamic implications, and assessed whether using the effective potential in the equation of motion for ϕ is warranted.

Based on the following aspects, our conclusions are that the regime of validity of the static effective potential to describe the dynamics of the mean field is very limited.

(1) Using $V_{\rm eff}(\phi(t))$ in the equation of motion with $V_{\rm eff}[\phi]$ obtained in equilibrium quantum field theory, assumes that there is LTE, at a fixed constant temperature, presummably maintained via a coupling to a thermal bath in equilibrium at such temperature. Although the coupling to the thermal bath is in general not specified, we have provided general arguments based on the Boltzmann equation with a collision term in the relaxation time approximation to suggest that LTE is not warranted in many relevant cases, unless there is a fine tuning of couplings to the thermal bath.

(2) We have found severe caveats in the cases both without and with symmetry breaking tree-level potentials. In absence of symmetry breaking, parametric instabilities associated with the oscillatory dynamics of the mean field near the minimum of the potential, leads to profuse particle production, with distribution functions that are not thermal, and more importantly, a nonmonotonic behavior of the entropy. While this latter behavior may be argued to describe an exchange of entropy with an external bath, it runs counter to the main tenets of local equilibrium thermodynamics.

(3) In the case when the tree-level potential admits broken symmetry minima, spinodal instabilities prevent an LTE description of the dynamics. In particular for a band of spinodaly unstable wave vectors when ϕ is within the classical spinodal region within which $V''(\phi) < 0$, the effective frequencies are purely imaginary, fluctuations grow exponentially, yielding a complex effective potential, distribution function $n_k(\phi)$, internal energy, and entropy. Whereas the imaginary part of

the internal energy may be associated with a decay rate of a particular nonequilibrium state [17,26], an imaginary part of the entropy is untenable, and unacceptable in thermodynamics.

(4) The caveats emerging from assuming that the effective potential can be used in the dynamical evolution of the mean field, cannot be overcome by any resummation program in equilibrium quantum field theory, such as, for example resummation of "hard-thermal loops" [38]. Such nonperturbative resummation frameworks cannot possibly address the dynamical instabilities associated with parametric amplification or spinodal decomposition, the latter being a hallmark of the early stages of phase separation, the formation and growth of correlated domains and coarsening during a phase transition [27–29].

IV. DYNAMICS: DECOUPLING AND FREEZE OUT, CLOSED QUANTUM SYSTEM EVOLUTION

The discussion in the previous section outlines several problems inherent in merely using the finite temperature effective potential to describe the behavior of a dynamical expectation value/condensate. Critically, employing this effective potential tacitly assumes a persistent local thermodynamic equilibrium between the condensate and the environment which requires a precise analysis of the couplings to the thermal bath. This obfuscates the problem and prevents one from making simple, model-independent statements about the dynamics of the condensate under such conditions. However, one may consider a closely related scenario wherein a condensate, which was previously in local thermodynamic equilibrium, decouples from the bath and proceeds to evolve in time. In this section we will investigate these dynamics, thereby providing an avenue for studying the behavior of the mean field beyond the time scale when LTE is no longer warranted. This problem is not only both tractable and relevant in its own right, but it will provide a useful comparison to the phenomenologically motivated approach of using the effective potential in the equation of motion [see Eq. (3.1)].

Let us first consider the case when the tree-level potential does not feature spontaneous symmetry breaking, and an initial condition on the mean field such that its velocity is very small and it is up the potential hill, far from the minimum of the tree-level potential so that it does not feature oscillations that lead to parametric amplification and breakdown of LTE in generic cases. As ϕ rolls down the potential hill with a small initial velocity, there is a time interval when the evolution of the mean field is slow and the condition (3.10), or alternatively (3.12), for the validity of LTE is fulfilled. This entails that the instantaneous frequencies $\omega_k(\phi(t))$ are varying slowly on the relaxation time scale, this can be quantified in an adiabatic expansion of the solutions of the mode equations (2.10) [16]. Proposing the Wentzel-Kramers-Brillouin solution

$$g_k(t) = \frac{e^{-i\int_{t_0}^{t}\Omega_k(t')dt'}}{\sqrt{2\Omega_k(t)}}, \tag{4.1}$$

which when inserted into Eq. (2.10) reveals that $\Omega_k(t)$ must satisfy

$$\Omega_k^2(t) = \omega_k^2(t) - \frac{1}{2}\left[\frac{\ddot{\Omega}_k}{\Omega_k} - \frac{3}{2}\left(\frac{\dot{\Omega}_k}{\Omega_k}\right)^2\right]. \tag{4.2}$$

The resulting equation can be solved in an adiabatic expansion

$$\Omega_k^2(t) = \omega_k^2(t)\left[1 - \frac{1}{2}\frac{\ddot{\omega}_k}{\omega_k^3} + \frac{3}{4}\frac{\dot{\omega}_k^2}{\omega_k^4} + \cdots\right]. \tag{4.3}$$

Assuming a slow initial evolution, let us consider the leading (zeroth) adiabatic order, namely

$$g_k^{(0)}(t) = \frac{e^{-i\int_{t_0}^{t}\omega_k(t')dt'}}{\sqrt{2\omega_k(t)}}; \tag{4.4}$$

as the mean field evolves, its velocity increases, and at some timescale t_0 LTE breaks down and the system can no longer remain in thermal contact with the bath. This is the physics of decoupling between the system and the bath. From this timescale onwards, the scalar field evolves independently of the bath, this situation is similar to the decoupling of photons in cosmology, when the mean free path from Thompson scattering is larger than the Hubble radius, the photons evolve freely. Within this context, the time of decoupling is referred to as the "surface of last scattering" and is often approximated to be an instantaneous process.

We model the similar situation as an "instantaneous" decoupling assuming that the density matrix for the fluctuations around the mean field is frozen in the Heisenberg picture, and describes the fluctuations of a free field with the frequencies at the decoupling time t_0. This assumption leads to the following initial conditions to leading adiabatic order from Eq. (4.4):

$$g_k(t_0) = \frac{1}{\sqrt{2\omega_k(t_0)}}; \quad \dot g_k(t_0) = -i\sqrt{\frac{\omega_k(t_0)}{2}};$$

$$\omega_k(t_0) = \sqrt{k^2 + V''(\phi(t_0))}; \tag{4.5}$$

that the mean field is within a region withwhere $\omega_k(t_0)$ are real and positive under the assumption $V''(\phi(t_0)) > 0$,

which is always the case for potentials with unbroken symmetry. The initial conditions (4.5) imply that the Wronskian condition (2.11) on the mode functions $g_k(t)$ is satisfied.

Assuming the validity of LTE up to the decoupling time t_0, the initial density matrix at this time is taken to be given by

$$\rho(t_0) = \frac{e^{-\beta_0 H_0}}{\mathrm{Tr}\, e^{-\beta_0 H_0}}; \tag{4.6}$$

where

$$H_0 = \sum_{\vec{k}} \hbar\omega_k(t_0)\left[a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right]. \tag{4.7}$$

This choice of initial density matrix is consistent with the one-loop effective potential, which is determined by a density matrix describing a free field with a squared mass $V''(\phi) > 0$ as discussed in Sec.(2.50)II B. , in particular the fluctuation Hamiltonian in Eq.

When the tree-level potential features broken symmetry minima and a spinodal region wherein $V''(\phi) < 0$, the situation is much more subtle. The band of wave vectors $0 \le k < |V''(\phi)|$ is spinodally unstable, LTE is not fulfilled regardless of the value of $\dot\phi$; and, as discussed in the previous section, the mode functions within this band feature (nearly) exponential growth in time as a consequence of the instability. The initial conditions (4.5) are valid if $V''(\phi) > 0$, in other words ϕ is outside the spinodal region, however they must be modified if ϕ is within the unstable region where $V''(\phi) < 0$. Nevertheless, we can parametrize the initial conditions on the mode functions within the unstable band at some initial time t_0 as

$$g_k(t_0) = \frac{1}{\sqrt{2W_k}}; \quad \dot g_k(t_0) = -i\sqrt{\frac{W_k}{2}};$$

$$W_k = \sqrt{k^2 + M^2}; \quad M^2 > 0; \tag{4.8}$$

which again imply that the Wronskian condition is fulfilled. The effective mass term $M^2 > 0$ is a parametrization of the initial condition at a time t_0, its actual value depends on the precise "misalignment" mechanism that has resulted in the mean field ϕ to be within the spinodally unstable region and must be specified for particular realization of the dynamics.

A. An explicit example: A "quenched" phase transition

Let us consider the case of a rapid phase transition modeled by a scalar field theory with a time dependent mass term with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{a}{2}(T^2(t) - T_c^2)\phi^2 + \frac{\lambda}{4}\phi^4; \tag{4.9}$$

with $a > 0$ a dimensionless constant, and a time dependent temperature

$$T(t) = T_i\Theta(t_0 - t) + T_f\Theta(t - t_0); \quad T_i > T_c; \quad T_f < T_c. \tag{4.10}$$

This situation describes a sudden phase transition at time $t = t_0$ from an unbroken symmetry case for $t < t_0$ with $T_i > T_c$ to a broken symmetry case for $t > t_0$ with $T_f < T_c$. If for $t < t_0$ the mean field ϕ is oscillating with small amplitude around the equilibrium minimum of the potential at $\phi = 0$ (for $t < t_0$), and at the transition time t_0 is found with a value ϕ_0, the mode functions are of the form $e^{iW_k t}$ with

$$W_k = \sqrt{k^2 + M^2}; \quad M^2 = a(T_i^2 - T_c^2) + 3\lambda\phi_0^2 > 0. \tag{4.11}$$

For $t > t_0$, after the temperature dropped to $T_f < T_c$ the mean field is now within the spinodal region if the initial value ϕ_0 is such that $a(T_f^2 - T_c^2) + 3\lambda\phi_0^2 < 0$, the mode functions at the transition time have precisely the initial conditions (4.8). After this sudden transition, the mean field will begin rolling down the potential hill, and the mode functions $g_k(t)$ that describe the fluctuations will grow nearly exponentially while the mean field is within the spinodal. This simple but relevant example explicitly describes a physical situation in which the mean field is found initially within the spinodal region. The ensuing time evolution of the mode functions exhibit the (nearly) exponential growth associated with the dynamics of the phase transition and the emergence of correlated domains with a growing correlation length [17,26].

This specific example is by no means exhaustive, nor do we dwell here on the "quenching mechanism," but it highlights that in the dynamical case the "misalignment" mechanism by which the initial value of the mean field is found inside the spinodal region must be specified, along with the initial conditions on the mode functions that describe the fluctuations around the mean field.

Under the assumption of instantaneous decoupling and to establish a relation with the thermal density matrix discussed in the previous Sec. II B within the context of the static finite temperature effective potential, let us introduce

$$H_0 \equiv \sum_{\vec k} \hbar W_k \left(a^{\dagger}_{\vec k} a_{\vec k} + \frac{1}{2}\right); \tag{4.13}$$

and the initial density matrix is taken to be given by

$$\rho(t_0) = \frac{e^{-\beta_0 H_0}}{\mathrm{Tr}\, e^{-\beta_0 H_0}}; \tag{4.14}$$

where the frequencies W_k are given by Eq. (4.12) and the time independent annihilation and creation operators are the same that enter in the quantization of the fluctuation field $\delta(\vec x, t)$, given by (2.8).

This particular choice of the initial density matrix is motivated by an "instantaneous decoupling" from LTE

$$g_k(t_0) = \frac{1}{\sqrt{2W_k}}; \quad \dot g_k(t_0) = -i\sqrt{\frac{W_k}{2}}; \quad W_k = \sqrt{k^2 + M^2}; \quad M^2 \equiv \begin{cases} V''(\phi(t_0)); & V''(\phi(t_0)) > 0 \\ -V''(\phi(t_0)); & V''(\phi(t_0)) < 0 \end{cases} \tag{4.12}$$

and has a clear and simple interpretation: it describes a free field theory for the fluctuations with squared mass $M^2 > 0$ given by Eq. (4.12) in thermal equilibrium at a temperature $T_0 = 1/\beta_0$.

The main assumption behind this choice is that the coupling to the thermal bath maintains LTE up to time t_0 at which the time scale of change of the frequencies is much shorter than the relaxation time and the scalar field decouples instantaneously from the bath. From this time onwards the density matrix follows unitary time evolution determined by the dynamics of the scalar field.

Note the similarity with the fluctuation Hamiltonian, the second term on the right hand side of Eq. (2.50) which yields the static one-loop effective potential, however, unlike the frequencies (2.49) that enter in (2.50), which are imaginary within the spinodal region, the W_k that enter in H_0 are always real.

In this (Gaussian) density matrix it follows that

$$\langle a^{\dagger}_{\vec k} \rangle = \mathrm{Tr}\, a^{\dagger}_{\vec k}\, \rho(t_0) = 0; \quad \langle a_{\vec k} \rangle = \mathrm{Tr}\, a_{\vec k}\, \rho(t_0) = 0;$$

$$\langle a^{\dagger}_{\vec k} a_{\vec k'} \rangle = \mathrm{Tr}\, a^{\dagger}_{\vec k} a_{\vec k'} \rho(t_0) \equiv n_k(0)\delta_{\vec k, \vec k'} = \frac{1}{e^{\beta_0 \hbar W_k} - 1}\delta_{\vec k, \vec k'}; \quad \langle a_{\vec k} a_{\vec k'} \rangle = 0; \ \forall\, \vec k, \vec k'; \tag{4.15}$$

and Wick's theorem applies.

B. Equations of motion

After thermal decoupling, the density matrix is frozen in the Heisenberg picture and the time evolution is unitary, For any operator O, Heisenberg's equation of motion become the Schrödinger picture with the unitary time evolution operator.

Both cases, with and without spontaneous symmetry breaking in the tree-level potential can be summarized by the following initial conditions that satisfy the Wronskian condition where we allow the Hamiltonian to depend explicitly on time. The solution of (4.16) is

$$O(\vec{x},t) = U^{-1}(t,t_0)\,O(\vec{x},t_0)\,U(t,t_0), \qquad (4.17)$$

where the unitary time evolution operator (in what follows we set $\hbar = 1$) is given by

$$U^{(+)}(t,t_0) = T e^{-i\int_{t_0}^{t} H(t')dt'}; \quad U^{(-)}(t,t_0) = \tilde{T} e^{i\int_{t_0}^{t} H(t')dt'}. \qquad (4.18)$$

where $T, \tilde{T}$ are the time and anti-time-ordering symbols.

In the Heisenberg picture a density matrix does not depend on time, whereas in the Schrödinger picture its time evolution is given by

$$\rho(t) = U(t,t_0)\,\rho(t_0)\,U^{-1}(t,t_0); \qquad (4.19)$$

namely the density matrix evolves unitarily in time, as a consequence the entropy $S = -\mathrm{Tr}\,\rho(t)\ln(\rho(t))$ is time independent.

With an initial state described by a density matrix $\rho(t_0)$ normalized such that $\mathrm{Tr}\,\rho(t_0) = 1$, expectation values of a Heisenberg field operator are given by

$$\langle O(t)\rangle = \mathrm{Tr}\,O(t)\rho(t_0) = \mathrm{Tr}\,O(t_0)\rho(t). \qquad (4.20)$$

Expectation values and correlation functions are obtained via functional derivatives of the generating functional [39,40]

$$Z[J^+,J^-] \equiv \mathrm{Tr}[U(t,t_0;J^+)\rho(t_0)U^{-1}(t,t_0;J^-)], \qquad (4.21)$$

with respect to the external sources J, where

$$U^{(+)}(t,t_0;J^+) = T e^{-i\int_{t_0}^{t} H(t';J^+)dt'}; \quad U^{(-)}(t,t_0;J^-) = \tilde{T} e^{i\int_{t_0}^{t} H(t';J^-)dt'} \qquad (4.22)$$

with

$$H(t;J) \equiv H(t) + \int d^3x\, J(\vec{x},t)\,O(\vec{x},t). \qquad (4.23)$$

For example correlation functions

$$\langle O^+(\vec{x}_1,t_1)O^+(\vec{x}_2,t_2)\rangle = \mathrm{Tr}(T\,O(\vec{x}_1,t_1)O(\vec{x}_2,t_2))\rho(t_0) = -\frac{\delta^2 Z}{\delta J^+(\vec{x}_1,t_1)\,\delta J^+(\vec{x}_2,t_2)}\bigg|_{J^+=J^-=0}, \qquad (4.24)$$

$$\langle O^-(\vec{x}_2,t_2)O^+(\vec{x}_1,t_1)\rangle = \mathrm{Tr}\,O(\vec{x}_1,t_1)\rho(t_0)O(\vec{x}_2,t_2) = \frac{\delta^2 Z}{\delta J^+(\vec{x}_1,t_1)\,\delta J^-(\vec{x}_2,t_2)}\bigg|_{J^+=J^-=0}, \qquad (4.25)$$

etc. An important result is that

$$\langle O^+(\vec{x},t)\rangle \equiv \mathrm{Tr}\,O(\vec{x},t)\rho(t_0) = i\frac{\delta Z[J^+,J^-]}{\delta J^+(\vec{x},t)}\bigg|_{J^+=J^-=0} = \langle O^-(\vec{x},t)\rangle \equiv \mathrm{Tr}\,\rho(t_0)O(\vec{x},t) = -i\frac{\delta Z[J^+,J^-]}{\delta J^-(\vec{x},t)}\bigg|_{J^+=J^-=0}. \qquad (4.26)$$

This is the Schwinger-Keldysh or in-in formulation of nonequilibrium quantum field theory [39–44].

Let us consider a scalar quantum field theory for a field ϕ as discussed in the previous sections, the generating functional (4.21) in the field representation can be written in a functional integral representation

$$Z[J^+;J^-] = \int D\phi_f D\phi_i D\phi'_i \langle\phi_f|U(t,t_0;J^+)|\phi_i\rangle \times \langle\phi_i|\rho(t_0)|\phi'_i\rangle\langle\phi'_i|U^{-1}(t,t_0;J^-)|\phi_f\rangle; \quad (4.27)$$

in turn the field matrix elements of the evolution operators can be written as path integrals, namely $\langle\phi_f|U(t,t_0;J_+)|\phi_i\rangle \equiv \int D\phi_+ e^{i\int \mathcal{L}[\phi_+;J_+]d^4x}$;

$$\phi_+(t_0) = \phi_i; \qquad \phi_+(t) = \phi_f; \quad (4.28)$$

$$\langle\phi'_i|U^{-1}(t,t_0;J_-)|\phi_f\rangle \equiv \int D\phi_- e^{-i\int \mathcal{L}[\phi_-;J_-]d^4x};$$

$$\phi_-(t_0) = \phi'_i; \qquad \phi_-(t) = \phi_f; \quad (4.29)$$

where

$$\mathcal{L}[\phi;J] = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - J\phi. \quad (4.30)$$

Finally, the functional and path integral representation of the generating functional becomes

$$Z[J_+;J_-] = \int D\phi_f D\phi_i D\phi'_i \int D\phi_+ D\phi_- \times e^{i\int[\mathcal{L}[\phi_+;J_+]-\mathcal{L}[\phi_-;J_-]]d^4x}\rho(\phi_i;\phi'_i;t_0); \quad (4.31)$$

with the boundary conditions on the fields ϕ given by Eqs. (4.28), (4.29) and the notation $\int d^4x \equiv \int_{t_0}^{t} dt' \int d^3x$. The doubling of fields with the $\pm$ branches is a direct consequence of the time evolution of a density matrix, with time evolution forward via $U(t;t_0)$ and backwards with $U^{-1}(t;t_0)$, in contrast to the usual S-matrix or in-out formulation which involves only time evolution forward because it evolves a state rather than a density matrix.

Our objective is to obtain the equation of motion for the expectation value of the scalar field ϕ, namely

$$\mathrm{Tr}\,\phi(\vec{x},t)\rho(t_0) \equiv \phi(t); \quad (4.32)$$

where we consider ϕ to be spatially homogeneous, hence only the zero momentum component of ϕ acquires an expectation value. The equation of motion for ϕ is obtained by following the identity (4.26) which implies that $\langle\phi_+\rangle = \langle\phi_-\rangle = \phi$.

$\frac{1}{2}\left(\frac{\partial\delta^{\pm}}{\partial t}\right)^2$

The equation of motion for $\phi(t)$ is obtained by writing

$$\phi^{\pm}(\vec{x},t) = \phi(t) + \delta^{\pm}(\vec{x},t); \quad (4.33)$$

in the Lagrangian $\mathcal{L}[\phi;J]$ in Eq. (4.30) and requesting that

$$\langle\delta^{\pm}(\vec{x},t)\rangle = 0; \quad (4.34)$$

to all orders in perturbation theory, namely the same constraint as in the static case (2.40).

Upon integration by parts and neglecting surface terms which do not contribute to the equations of motion, and coupling sources only to the fluctuating fields δ, we obtain (dots denote ∂=∂t)

Z ½L½ϕ;δ ;J −L½ϕ;δ ;J d4x ¼ þiZ −ð∇δþÞ2 −V00ðϕðtÞÞδþ2 þJþδþ

$$-i \quad \frac{1}{2} \quad \frac{}{\partial t} \quad -(\nabla\delta^-)^2 - V''$$ Z ∂δ−2 ðϕðtÞÞδ−2 þJ−δ−d4x

Þ þ $\frac{1}{3}$V ðϕðtÞÞδ þ d^4x − ðδ → δ Þ: ð4:35Þ

−iZ ðϕ̈ ðtÞþV0ðϕðtÞÞÞδ þðx;t⃗ 000 þ3 þ −

The currents J in this expression are intended to yield the correlation functions of the fluctuations δ in terms of functional derivatives with respect to them, and should not be confused with the Lagrange multiplier j in the static case of the previous section which enforces the constraint (2.40).

The last line in (4.35) determines the interaction vertices, these are depicted in Fig. 2, just as in the static case, the linear term is considered as part of the interaction. It is instructive to compare to the static case in particular the interaction term in Eq. (2.60), which shows that in the dynamical case ϕ̈ in the linear term in δðx;t⃗ Þ in the interaction term in the last line in (4.35) replaces the Lagrange multiplier J in (2.60). This is in agreement with the discussion right before the classical equation of motion (2.33) comparing it to the constraint equation (2.32).

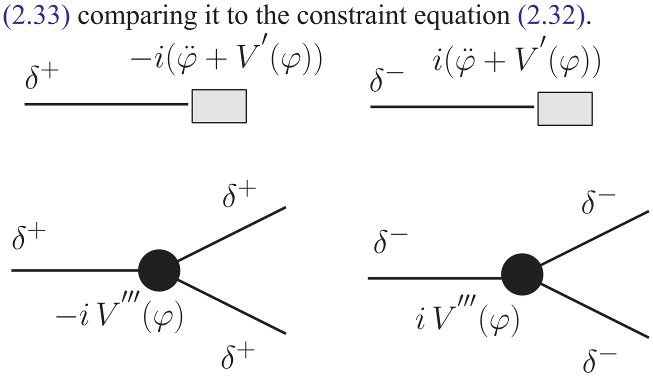

FIG. 2. Interaction vertices from the Lagrangian (4.35) up to tionsand including Oðδ3Þ, the solid lines correspond to the fluctua-∓ ϕ̈ ðtÞþ V0ðϕðtÞÞÞ, the black δ. The gray box stands for ið dot stands for ∓iV000ðϕðtÞÞ.

The equation of motion for the mean field is obtained

action term infrom the condition(4.35)hδto first order, we findð0⃗ ;0Þi ¼ 0. Considering the inter-

− i Z ðϕ̈ ðtÞ þ V0ðϕÞÞhδþð0⃗ ;0Þδþðx;t⃗ Þi þ

ð3ϕðtÞÞhδþð0⃗ ;0Þðδþðx;t⃗ ÞÞ3id4x ¼ 0: ð4:36Þ

V000

The expectation values are obtained in the free field theory defined by the first two lines in (4.35), with the initial

density matrix ρðt0Þ. (4.35) describe a free scalar field The first two lines in theory with a time dependent mass V00ðϕðtÞÞ, yielding the field equations (2.7), and the field expansion (2.8). Using Wick's theorem it follows that

hδþð0⃗ ;0Þðδþðx;t⃗ ÞÞ3i ¼ 3hδþð0⃗ ;0Þδþðx;t⃗ Þihðδþðx;t⃗ ÞÞ2i; ð4:37Þ

and factorizing hδþð0⃗ ;0Þδþðx;t⃗ Þi from the expression

(4.36) we find

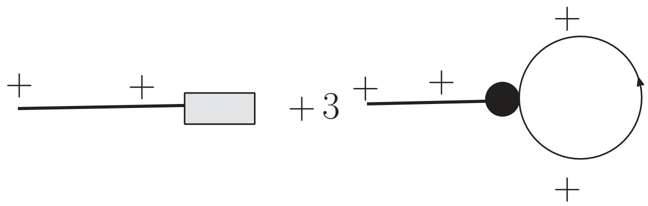

FIG. 3. Equation of motion up to one loop. The solid straight line is the propagatorδᵖðx;t⃗ Þδᵖðx;t⃗ Þi. hδᵖð0⃗ ;0Þδᵖðx;t⃗ Þi, the closed loop is h

1 2

ϕ̈ ðtÞ þ V₀ðϕÞ þ 2_V₀₀₀ðϕðtÞÞhðδₚðx;t⃗ÞÞ i ¼ 0; ð4:38Þ

this equation is depicted symbolically in Fig. 3.

Various correlation functions needed to obtain the equations of motion are summarized in the Appendix.

With the field expansion (2.8) and the correlations (4.15)

it is straightforward to find hðδₚðx;t⃗ ÞÞ₂i ¼ ℏ Z _____ð2π³

Þ₃ jgₖðtÞj₂ð1 þ 2nₖð0ÞÞ; ð4:39Þ

d k

finally yielding the equation of motion

ℏ

ϕ̈ ðtÞ þ V₀ðϕðtÞÞ þ _2 V₀₀₀ðϕðtÞÞ

Z d₃k 2

× ð_____2πÞ₃ jgₖðtÞj ð1 þ

2nₖð0ÞÞ ¼ 0: ð4:40Þ

It is straightforward to check that the same equation of motion is obtained from using the backward branch (δ⁻) contribution from the last line in Eq. (4.35). This is because hhððδδₚₚð0ð⃗ x;t⃗;0ÞÞÞδ⁻₂ði ¼ hðx;t⃗ Þiδis factorized. Also, and of course, the-ðx;t⃗ ÞÞ₂i where now the propagator

same equation is obtained by considering the constraint

hδ-This method to obtain the equations of motion for expec-ð0⃗ ;0Þi ¼ 0. tation values, based on the in-in or Schwinger-Keldysh formulation of nonequilibrium quantum field theory is general and applies to any quantum field theory, furthermore, with few modifications it can be extended to the realm of cosmology [45].

In the case of the scalar field theory defined by the Hamiltonian (2.1) the equation of motion (4.38) can also be obtained directly from the Heisenberg field equation, which follow from the variational principle applied to the full action ϕ ðx;t⃗ Þ −∇²ϕðx;t⃗ Þ þ V⁰ðϕðx;t⃗ ÞÞ ¼ 0; ð4:41Þ

which is obviously fulfilled as an expectation value in the initial density matrix, namely

Trρðt₀Þðϕ ðx;t⃗ Þ −∇₂ϕðx;t⃗ Þ þ V₀ðϕðx;t⃗ ÞÞÞ ¼ 0: ð4:42Þ

Shifting the field operator by the spatially homothe Heisenberg field equationgeneous mean field ϕðx;t⃗ Þ ¼(4.41)ϕðtÞ þ δðx;t⃗ Þ yields, for

ϕ̈ ðtÞ þ V₀ðϕðtÞÞ þ ½δ̈ ðx;t⃗Þ −∇₂δðx;t⃗ Þ þ V₀₀ðϕðtÞÞδðx;t⃗ Þ

1 2

þ _2V₀₀₀ðϕðtÞÞδ ðx;t⃗ Þ þ ¼ 0; ð4:43Þ

using the quantization of the fluctuation via the solution of the free field equations of motion in the background of the mean field, Eqs. (2.7), (2.8), leads to the vanishing of the (third) term inside the bracket in (4.43), yielding the expectation value (4.42)

1 2 ϕ̈ ðtÞ þ V₀ðϕðtÞÞ þ 2 _ V₀₀₀ðϕðtÞÞðTrρðt₀Þδ ðx;t⃗ ÞÞ þ ¼ 0:

ð4:44Þ

With the initial density matrix given by (4.14), the field expansion (2.8), and the expectation values (4.15), it is straightforward to find that

d k

$$\mathrm{Tr}\rho(t_0)\delta^2(\vec{x},t) = \hbar\int\frac{d^3k}{(2\pi)^3}|g_k(t)|^2(1+2n_k(0)). \tag{4.45}$$

Thereby confirming the equation of motion (4.38) obtained via the more general in-in Schwinger-Keldysh formulation. This is not only reassuring, but it also confirms that the equations of motion for the condensate follow from unitary time evolution as it is obtained from the expectationvalue of the Heisenberg equations of motion for the field operators in a time independent density matrix. While this latter derivation is arguably simpler, we have also presented the more rigorous in-in formulation due to its generality and appropriate use in non-equilibrium quantum field theory.

In Refs. [40,46,47] the Schwinger-Keldysh effective action up to one loop was obtained in terms of the fields with labels ± on the two branches, corresponding to forward and backwards time evolution. This action is rewritten in terms of the Keldysh center of mass and relative variables $\Phi = (\phi^+ + \phi^-)/2$; $R = (\phi^+ - \phi^-)$, respectively, and the equations of motion obtained from a variational principle on these variables [40]. Up to one loop it is shown in Refs. [46,47] that the effective equations of motion are of the Langevin type, with a Gaussian stochastic noise, the expectation value of the scalar field is directly determined by the expectation value of the center of mass coordinate with the probability distribution function (Gaussian) of this noise, yielding the equations of motion for the expectation value of the scalar field in the initial density matrix [46,47], which is the method used above to derive the equations of motion from the Heisenberg field equations.

When the mean field ϕ is time independent, namely in the static case, and when ϕ is away from the spinodal region the mode functions are

$$g_k(t) = \frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}; \qquad \omega_k = \sqrt{k^2 + V''(\phi)}; \tag{4.46}$$

[see Eq. (2.12)] and the last two terms in the equation of motion (4.40) become

$$V'(\phi(t)) + \frac{\hbar}{2}V'''(\phi(t))\int\frac{d^3k}{(2\pi)^3}\frac{1+2n_k(\phi)}{2\omega_k} = \frac{d}{d\phi}V_{\rm eff}(\phi); \tag{4.47}$$

in agreement with the static case, Eq. (2.76). However, when $\phi(t)$ is dynamical, the mode functions $g_k(t)$ describe the parametric and spinodal instabilities discussed in the previous sections and the last two terms in the equation of motion (4.40) cannot be identified with a derivative of an effective potential.

As a consequence of the mode equations (2.10), it is straightforward to show that the equation of motion (4.40) yields the conserved quantity

$$\tilde{E} = \overbrace{\frac{1}{2}(\dot\phi(t))^2 + V(\phi(t))}^{\tilde{E}_{cl}} + \overbrace{\frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\left[|\dot g_k(t)|^2 + \omega_k^2(t)|g_k(t)|^2\right](1+2n_k(0))}^{\tilde{E}_{fl}} = \text{constant}; \tag{4.48}$$

as can be easily confirmed by taking $\dot{\tilde{E}}$ and using Eq. (2.10), yielding $\dot\phi$ times Eq. (4.40). The brackets in Eq. (4.48) define the classical ($\tilde{E}_{cl}$), and fluctuation ($\tilde{E}_{fl}$) contributions to the total energy density respectively.

It is important, and enlightening, to compare this conservation law to that obtained from using the static effective potential in the dynamical equation of motion of the homogeneous mean field (3.2). First, we prove that the Eq. (4.48) is the expectation value of the time-independent Hamiltonian (2.1) in the initial density matrix $\rho(t_0)$ up to one-loop order. Let us shift both the field ϕ and its canonical momentum π as

$$\phi(\vec{x},t) = \phi(t) + \delta(\vec{x},t); \qquad \pi(\vec{x},t) = \dot\phi(t) + \pi_\delta(\vec{x},t); \tag{4.49}$$

yielding

$$H[\phi] = V\left[\frac{1}{2}(\dot\phi(t))^2 + V(\phi(t))\right] + H_\delta + \cdots; \tag{4.50}$$

where

$$\hat{H}_\delta = \int d^3x \left[\frac{\hat{\pi}_\delta^2}{2} + \frac{(\nabla\delta)^2}{2} + \frac{V''(\varphi(t))}{2}\hat{\delta}^2\right]; \quad (4.51)$$

and the dots in Eq. (4.50) stand for linear terms with vanishing expectation value in the density matrix $\rho(t_0)$, along with cubic and quartic terms in δ which yield higher loop corrections. Upon quantization of the fluctuation field via the mode expansion (2.8), (2.9) and using the expectation values (4.15), we find

$$\frac{1}{V}\mathrm{Tr}H_\delta\rho(t_0) = \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\left[|\dot{g}_k(t)|^2 + \omega_k^2(t)|g_k(t)|^2\right] \times [1 + 2n_k(0)]; \quad (4.52)$$

therefore, up to one loop $(\mathcal{O}(\hbar))$ we find

$$\tilde{E} = \frac{1}{V}\mathrm{Tr}H[\phi]\rho(t_0); \quad (4.53)$$

namely the constancy of $\tilde{E}$ is the statement that the field Hamiltonian is time independent. In contrast to Eq. (3.2) with the caveats discussed in the previous section in the broken symmetry case, the expectation value of the energy density is constant and always real, and because the time evolution is unitary the entropy density

$$S = \int\frac{d^3k}{(2\pi)^3}\{(1+n_k(0))\ln(1+n_k(0)) - n_k(0)\ln(n_k(0))\}; \quad (4.54)$$

where Eq. (4.54) is obtained using the initial thermal state set by the decoupling and is constant and real.

by Eq.This is in striking contrast to Eq.(3.16) in terms of the internal energy and entropy(3.2) with $V_{\mathrm{eff}}(\phi)$ given densities (3.17), (3.18) each one varying in time, with a nonmonotonic behavior for the entropy and both featuring an imaginary part when the mean field is in the spinodal region.

C. Stimulated particle production

The equation of motion (4.40) and conservation law (4.48) are very similar to the zero temperature case obtained in Ref. [16], with the only difference being the initial occupation number in the one-loop contribution. Following this reference, this similarity suggests us to relate the growth of the mode functions either by parametric amplification or spinodal instabilities to particle production.

1. Unbroken symmetry case

In this case the time-dependent frequencies $\omega_k(t) = \sqrt{k^2 + V''(\phi(t))}$ are always positive, and we introduce the zeroth adiabatic order mode functions

$$\tilde{f}_k(t) = \frac{e^{-i\int_0^t \omega_k(t')dt'}}{\sqrt{2\omega_k(t)}}. \quad (4.55)$$

We expand the exact mode functions $g_k(t)$ in terms of these adiabatic modes by introducing Bogoliubov coefficient functions $\tilde{A}_k(t), \tilde{B}_k(t)$ defined by the following relations

$$g_k(t) = \tilde{A}_k(t)\tilde{f}_k(t) + \tilde{B}_k(t)\tilde{f}_k^*(t); \quad (4.56)$$

$$\dot{g}_k(t) = -i\omega_k(t)[\tilde{A}_k(t)\tilde{f}_k(t) - \tilde{B}_k(t)\tilde{f}_k^*(t)]; \quad (4.57)$$

which can be inverted to obtain the Bogoliubov coefficients

$$\tilde{A}_k(t) = i\tilde{f}_k^*(t)[\dot{g}_k(t) - i\omega_k(t)g_k(t)]; \quad (4.58)$$

$$\tilde{B}_k(t) = -i\tilde{f}_k(t)[\dot{g}_k(t) + i\omega_k(t)g_k(t)]. \quad (4.59)$$

It follows from the Wronskian condition (2.11) that

$$|\tilde{A}_k(t)|^2 - |\tilde{B}_k(t)|^2 = 1. \tag{4.60}$$

The definitions (4.56), (4.57) yield

$$a_{\vec{k}}\, g_k(t) + a^{\dagger}_{-\vec{k}}\, g^*_k(t) = c_{\vec{k}}(t)\tilde{f}_k(t) + c^{\dagger}_{-\vec{k}}(t)\tilde{f}^*_k(t); \tag{4.61}$$

$$a_{\vec{k}}\, \dot{g}_k(t) + a^{\dagger}_{-\vec{k}}\, \dot{g}^*_k(t) = -i\omega_k(t)\left(c_{\vec{k}}(t)\tilde{f}_k(t) - c^{\dagger}_{-\vec{k}}(t)\tilde{f}^*_k(t)\right); \tag{4.62}$$

where

$$c_{\vec{k}}(t) = a_{\vec{k}}\tilde{A}_k(t) + a^{\dagger}_{-\vec{k}}\tilde{B}^*_k(t);$$
$$c^{\dagger}_{\vec{k}}(t) = a^{\dagger}_{\vec{k}}\tilde{A}^*_k(t) + a_{-\vec{k}}\tilde{B}_k(t); \tag{4.63}$$

the condition (4.60) ensures that $c_{\vec{k}}(t), c^{\dagger}_{\vec{k}}(t)$ obey equal time canonical commutation relations. It is straightforward to show that the quadratic Hamiltonian H_δ given by Eq. (4.51) can be written in terms of the time dependent operators $c^{\dagger}_{\vec{k}}(t), c_{\vec{k}}(t)$ as

$$H_\delta = \sum_{\vec{k}} \hbar\omega_k(t)\left[c^{\dagger}_{\vec{k}}(t)c_{\vec{k}}(t) + \frac{1}{2}\right]. \tag{4.64}$$

Following Ref. [16] we define the number of adiabatic particles as

$$\tilde{N}_k(t) = \langle 0|c^{\dagger}_{\vec{k}}(t)c_{\vec{k}}(t)|0\rangle = |\tilde{B}_k(t)|^2; \tag{4.65}$$

where the vacuum state $|0\rangle$ is such that

$$a_{\vec{k}}|0\rangle = 0;\ \forall\ \vec{k}. \tag{4.66}$$

The relation (4.59) and the Wronskian condition (2.11) yield

$$\tilde{N}_k(t) = \frac{1}{2\omega_k(t)}\left[|\dot{g}_k(t)|^2 + \omega^2_k(t)|g_k(t)|^2\right] - \frac{1}{2}; \tag{4.67}$$

from which it follows that

$$\frac{1}{V}\mathrm{Tr}\, H_\delta\, \rho(t_0) = \frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega_k(t)\,[1 + 2\tilde{N}_k(t)]\,(1 + 2n_k(0)). \tag{4.68}$$

With the initial conditions (4.5) $g_k(0) = \frac{1}{\sqrt{2\omega_k(0)}}$; $\dot{g}_k(0) = -i\sqrt{\frac{\omega_k(0)}{2}}$, it follows that

$$\tilde{N}_k(0) = 0; \tag{4.69}$$

therefore the initial state is the vacuum state for the adiabatic particles. The distribution function for the adiabatic particles is given by

$$F_k(t) = \mathrm{Tr}\, c^{\dagger}_{\vec{k}}(t)c_{\vec{k}}(t)\rho(t_0) = \tilde{N}_k(t) + n_k(0)[1 + 2\tilde{N}_k(t)];$$
$$F_k(0) = n_k(0); \tag{4.70}$$

the second term in $F_k(t)$ describes stimulated production of adiabatic particles. In terms of this distribution function, the one-loop contribution to the energy density, Eq. (4.68) can be written in the following illuminating manner,

$$\frac{1}{V}\mathrm{Tr}\, H_\delta\, \rho(t_0) = \frac{\hbar}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega_k(t)\,[1 + 2F_k(t)]. \tag{4.71}$$

We can now gather these results to express the conserved energy density (4.48) in the form

$$\tilde{\mathcal{E}} = \frac{1}{2}\dot{\phi}^2(t) + V(\phi(t)) + \frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega_k(t)[1+2F_k(t)]. \quad (4.72)$$

This expression is remarkably similar to the energy density obtained at zero temperature in Ref. [16], but in terms of the distribution function $F_k(t)$, which describes stimulated particle production instead of the vacuum adiabatic particle number density $\tilde{N}_k(t)$. The conservation of $\tilde{\mathcal{E}}$ along with Eq. (4.70) taken together have an important physical interpretation of the dynamics: a mechanism of energy transfer between the mean field and the quantum fluctuations resulting in the stimulated production of the adiabatic particles with nonthermal distributions. In particular, the exponential growth of the mode functions $g_k(t)$ as a consequence of parametric amplification must result in a drain of the energy stored in the mean field, energy that goes into particle production with nonthermal distributions. The motivation for the choice of the zeroth-order adiabatic mode functions (4.55) now becomes clear: while $\phi(t)$ is oscillating around the minimum, parametric amplification of fluctuations drains energy from the condensate, diminishing its amplitude. This dissipative mechanism entails that asymptotically ϕ will settle at the minimum and the frequencies become slowly varying functions of time approaching an asymptotic limit $\omega_k(\infty) = \sqrt{k^2 + V''(\varphi = 0)}$. In this limit the mode functions $\tilde{f}_k(t) \to e^{-i\omega_k(\infty)t}/\sqrt{2\omega_k(\infty)}$ describing asymptotic "out" particle states.

2. Broken symmetry case

This case is more subtle. Although it is not clear that the fluctuation contribution $\tilde{\mathcal{E}}_{fl}$ in Eq. (4.48) grows as a consequence of the spinodal instabilities, since for spinodally unstable modes $\omega_k^2(t) < 0$, it follows from the mode equations (2.10) that

$$\dot{\tilde{\mathcal{E}}}_{fl} = \frac{\hbar}{2}\frac{d}{dt}V''(\phi(t))\int\frac{d^3k}{(2\pi)^3}|g_k(t)|^2[1+2n_k(0)]. \quad (4.73)$$

As $\phi(t)$ rolls down the potential hill from near the maximum of the potential towards the symmetry breaking minima,the inflection point, namely the end of the spinodal region.$V''(\phi(t))$ increases from a negativevalue to zero at

Therefore, because $|g_k(t)|^2$ grows nearly exponentially in this region, it follows that the fluctuation contribution grows nearly exponentially while $\phi(t)$ traverses the spinodal region. Furthermore, the temperature correction in (4.73) implies an enhancement as compared to the zero temperature case [16], again a manifestation of stimulated production of fluctuations. Because the total energy density remains constant, this energy is drained from the classical contribution $\tilde{\mathcal{E}}_{cl}$ in Eq. (4.48), again, a mechanism of energy transfer from the mean field to the fluctuations implying damping of the amplitude of the mean field. Because $\omega_k(t)$ are imaginary for spinodally unstable wave vectors, we cannot define the adiabatic modes as in the previous case. However, motivated by the argument that the growth of fluctuations implies a damping of the mean field as a consequence of energy transfer to the fluctuations, we follow the treatment of Ref. [16] and introduce K_s as the maximum unstable wave vector while $\phi(t)$ is in the spinodal region. For example for the typical potential $V(\phi) = -m^2\phi^2/2 + \lambda\phi^4/4$ with $m^2 > 0$, it follows that the maximum unstable wave vector is

$$K_s = |V''(0)|. \quad (4.74)$$

For $k \leq K_s$ there is no unambiguous definition of an adiabatic particle number, whereas for $k > K_s$ the mode functions can again be written as in Eqs. (4.56), (4.57) in terms of the zeroth-order adiabatic modes yielding the results obtained above for the case of unbroken symmetry. Therefore, separating the spinodally unstable modes we now write the fluctuation contribution to the energy density (4.48) as

$$\tilde{E}_{fl} = \frac{\hbar}{4\pi^2}\int_0^{K_s}\big[|\dot g_k(t)|^2 + \omega_k^2(t)|g_k(t)|^2\big](1+2n_k(0))k^2dk + \frac{\hbar}{4\pi^2}\int_{K_s}^{\Lambda}\omega_k(t)[1+2F_k(t)]k^2dk, \quad (4.75)$$

where the distribution function $F_k(t)$ is the same as in Eq. (4.70), and we have introduced an upper momentum cutoff $\Lambda \gg |V''(\phi)|$ to discuss renormalization aspects.

Since both spinodal and parametric instabilities lead to an efficient transfer of energy from the condensate to the fluctuations, we expect that at long time, the condensate will oscillate around a minimum below the classical spinodal as the instabilities eventually must shut off by energy conservation. In this asymptotic long time limit the $V''(\phi(t)) > 0$ and the frequencies are real, and the contribution from the modes with $k < K_s$ becomes of the same form as for those with $k > K_s$. Therefore we expect that in the long time limit as ϕ oscillates with small amplitude around a minimum away from and not probing the spinodal region, the mode functions can again be written as in Eqs. (4.56), (4.57), with the interpretation of asymptotic adiabatic particle production, so that the exponential growth from spinodal instabilities is imprinted in the Bogoliubov coefficient functions, thereby describing the production of asymptotic particles.

Therefore, in this limit both contributions in (4.75) have the same form in terms of the stimulated distribution function of produced particles $F_k(t)$.

D. Renormalized dynamical framework

The expression (4.75) for $\tilde{E}_{fl}$ allows us to treat both cases with and without symmetry breaking on the same footing: the case $K_s = |V''(0)| \neq 0$ corresponds to symmetry breaking and $K_s = 0$ to unbroken symmetry. In Ref. [16] the renormalization aspects were studied for the zero temperature case, which can be obtained from the results above by setting $n_k(0) = 0$. Because of the exponential suppression of the high momentum modes in the thermal distribution functions, it follows that the ultraviolet divergences are those of the zero temperature case and, as discussed in detail in Ref. [16], are completely described by the "1" in the bracket in the second term in (4.75), namely the zero point energy.

We proceed to subtract this term from the fluctuation energy and lump it together with $V(\phi(t))$ in the full energy density (4.44), thus defining a new effective potential

$$V_{\rm eff}(\phi) \equiv V(\phi) + \frac{\hbar}{4\pi^2}\int_{K_s}^{\Lambda}\omega_k(t)k^2dk; \quad (4.76)$$

yielding

$$\bar{V}_{\rm eff}(\phi) = V(\phi) + \frac{\hbar}{16\pi^2}\Big[\Lambda^4 + V''(\phi)\Lambda^2 - \frac{1}{4}V''(\phi)^2\ln\frac{4\Lambda^2}{\mu^2} + \frac{1}{4}(V''(\phi))^2\ln\frac{|V''(\phi)|}{\mu^2} - V''(\phi)^2 H\Big[\Big(\frac{K_s^2}{|V''(\phi)|}\Big)^{1/2}\Big]\Big]; \quad (4.77)$$

where μ^2 is a renormalization scale and

$$H[x] = \frac{1}{2}\Big[2x\big(x^2 + \mathrm{sign}V''(\phi)\big)^{3/2} - x\,\mathrm{sign}V''(\phi)\big(x^2 + \mathrm{sign}V''(\phi)\big)^{1/2} - \ln\Big(x + \big(x^2 + \mathrm{sign}V''(\phi)\big)^{1/2}\Big)\Big]; \quad (4.78)$$

with $K_s = 0$ for unbroken symmetry and $K_s = |V''(0)|$ for broken symmetry. In a renormalizable theory, the ultraviolet divergent terms are absorbed into renormalization of the parameters, for example for the bare scalar potential

$$V(\phi) = V_0 + \frac{m_0^2}{2}\phi^2 + \frac{\lambda_0}{4}\phi^4; \quad (4.79)$$

renormalization is achieved by introducing the renormalized parameters

$$m_R^2(\mu) = m_0^2 + \frac{6\lambda_0}{16\pi^2}\Lambda^2 - \frac{6\lambda_0}{32\pi^2} m_0^2 \ln\left[\frac{4\Lambda^2}{\mu^2} - \frac{1}{2}\right], \quad (4.80)$$

$$\lambda_R(\mu) = \lambda_0 - \frac{36\lambda_0^2}{32\pi^2}\ln\left[\frac{4\Lambda^2}{\mu^2} - \frac{1}{2}\right], \quad (4.81)$$

$$V_{0R}(\mu) = V_0 + \frac{\Lambda^4}{16\pi^2} + m_0^2\frac{\Lambda^2}{16\pi^2} - \frac{m_0^4}{64\pi^2}\ln\left[\frac{4\Lambda^2}{\mu^2} - \frac{1}{2}\right], \quad (4.82)$$

and replacing bare by renormalized quantities up to one loop,

$$\bar V_{\text{eff}R}(\phi) = V_R(\phi) + \frac{\hbar}{64\pi^2}\left\{(V''_R(\phi))^2 \ln\frac{|V''_R(\phi)|}{\mu^2} - (V''_R(\phi))^2 H\left(\left|\frac{V''_R(\phi)}{V''_R(\phi_{sR})}\right|^{1/2}\right)\right\}, \quad (4.83)$$

where the subscript R refers to the renormalized quantities in terms of the renormalized mass and coupling. The renormalization group invariance of the effective potential has been discussed in Refs. [3,16]. We note that the argument of the function H, is $(|V''_R(\phi)|/|V''_R(\phi_s)|)^{1/2} > 1$ within the spinodal region for the broken symmetry case when $(\mathrm{sign}(V''_0(\phi)) < 0)$.

This effective potential is manifestly real, unlike the usual effective potential that becomes complex when ϕ is within the spinodal region. After renormalization the total conserved energy density becomes

$$\tilde{\mathcal{E}} = \frac{1}{2}(\dot\phi(t))^2 + \bar V_{\text{eff}R}(\phi) + \frac{\hbar}{4\pi^2}\int_0^{K_s}\left[|\dot g_k(t)|^2 + \omega_k^2(t)|g_k(t)|^2\right](1+2n_k(0))k^2dk + \frac{\hbar}{2\pi^2}\int_{K_s}^{\Lambda}\omega_k(t)F_k(t)k^2dk; \quad (4.84)$$

where everywhere the mass and coupling are the renormalized quantities. The fully renormalized equations of motion are obtained as follows: beginning with the conserved energy density (4.48), and E_{fl} given by Eq. (4.75) subtract from this expression the term with the "1" inside the bracket of the second line, and lump it together with $V(\phi)$ to define $\bar V_{\text{eff}R}(\phi)$ as in Eq. (4.76). Now taking the time derivative of $\mathcal{E}$ yields $\dot\phi$ times the equation of motion, which upon using the equations for the mode functions (2.10) lead to the renormalized equation of motion

$$\ddot\phi(t) + \frac{d}{d\phi}\bar V_{\text{eff}R}(\phi) + \frac{\hbar}{4\pi^2}V'''_{0R}(\phi(t)) \times \int_0^{\Lambda}\left[|g_k(t)|^2(1+2n_k(0)) - \frac{\Theta(k-K_s(\phi))}{2\omega_k}\right]k^2dk = 0; \quad (4.85)$$

where again, everywhere, the mass and coupling are the renormalized ones. Equation (4.85), along with the mode equations (2.10), with initial conditions (4.12) provide a complete description of the dynamics of the mean field (condensate) with the following properties:

(1) The equation of motion (4.85) is consistently renormalized.
(2) The renormalized effective potential $\bar V_R(\phi)$ is manifestly always real for all values of the mean field even within the spinodal region, unlike the usual effective potential which is complex in the case when the tree-level potential features broken symmetry minima.
(3) The energy density is manifestly real and conserved.
(4) The equation of motion for the condensate arises from unitary time evolution of an initial density matrix, as confirmed by obtaining it also from the expectation value of the equations of motion of the Heisenberg field operators in the initial density matrix. Therefore the thermodynamic entropy is constant.[5]

V. DISCUSSION

[5] The thermodynamic entropy should not be confused with the coarse-grained entanglement entropy discussed in Ref. [16].

The dynamics described by the equation of motion (4.85) with the conserved energy (4.84) suggests the emergence of stationary asymptotic states. Let us consider first the case in which the tree-level potential features only one minimum, namely unbroken symmetry, with large amplitude initial conditions on the condensate. As ϕ oscillates around the minimum parametric instabilities lead to profuse particle production, which drains energy from the "classical" part of the energy into the fluctuations, populating parametrically unstable bands in momentum with a nonthermal distribution function. Particle production will continue as long as oscillations continue as demonstrated with the simple Mathieu equation analysis in the previous section. As the energy of the condensate is drained from particle production, the amplitude of oscillations diminishes and the bandwidths of the unstable bands become narrower, suggesting a dissipative mechanism that drives the condensate to the equilibrium minimum but with a highly excited nonthermal population of particles. Eventually this transfer of energy must stop and ϕ settles at the minimum with vanishing velocity, the frequencies $\omega_k(t) \to \omega_k(\infty)$, and the zeroth-order mode functions (4.55) describe asymptotic "out" single particle states. This is an asymptotic fixed point of the dynamics.

Such asymptotic limit will yield the asymptotic value(s) $\phi(\infty)$ as the solution(s) of the renormalized equation of motion (4.85), subject to the constraint of total energy density (4.84) with $\dot{\phi}(\infty) = 0; \ddot{\phi}(\infty) = 0$.

If the tree-level potential features symmetry breaking minima and the initial value of the mean field is large, with a large energy density, then both spinodal and parametric instabilities will be effective in draining energy from the condensate leading to particle production with nonthermal distributions. As the amplitude of the mean field diminishes the mean field can asymptotically settle in a broken symmetry minimum away from the origin, but it is also possible, with a large energy density, that asymptotically the mean field settles in a state with vanishing value. This would imply a restoration of symmetry, which is a possibility for a large energy density, that must be studied numerically and will likely depend on the particular value of parameters. However, in this case the condensate oscillates around a minimum with diminishing amplitude eventually settling at this minimum and again the frequencies $\omega_k(t) \to \omega_k(\infty)$ and the zeroth-adiabatic order mode functions (4.55) describe asymptotic "out" single particle states. In this case the Bogoliubov coefficients and the stimulated distribution function (4.70) include the growth of fluctuations from both, spinodal and parametric instabilities. The asymptotic value(s) $\phi(\infty)$ are again determined by the solutions of the equation of motion (4.85) with the energy constraint (4.84) with $\dot{\phi}(\infty) = 0; \ddot{\phi}(\infty) = 0$.

When the amplitude of oscillations diminishes from the energy transfer to fluctuations via particle production, it is possible that the dynamics "unfreezes" and the coupling to the heat bath or alternative collisional processes become effective again, perhaps leading to a redistribution of the produced quanta and a "rethermalization" on longer timescales. At this stage, this is, of course, a conjecture that can only be assessed with a detailed treatment of the quantum kinetics including the couplings to the bath and or other collisional processes, and merits further and deeper study.

VI. CONCLUSIONS AND FURTHER QUESTIONS

The finite temperature effective potential plays a fundamental role in understanding the phase structure of quantum field theories, including thermal and quantum corrections with ubiquitous applications in cosmological phase transitions. It was originally developed to describe the free energy landscape as a function of an order parameter, which is usually a scalar field condensate, by design and construction it is an equilibrium concept. However, it is often used in the equation of motion for the order parameter, or "misaligned" condensate.

A recent study [16] of the zero temperature effective potential revealed several important caveats that indicate that using the zero temperature effective potential to describe the dynamics of the condensate is in general unwarranted. Motivated by its importance in cosmology, in this article we focus on understanding if and when the finite temperature effective potential is suitable in the equations of motion of a homogeneous condensate. Extending the Hamiltonian formulation we identify the finite temperature effective potential with the Helmholtz free energy of the fluctuations around the condensate. This identification has a profound thermodynamic significance: it allows us to establish a direct relation with the thermodynamic entropy density $S = -\partial V_{eff}[T;\phi]/\partial T$. Therefore, fundamental

When the condensate oscillates around an equilibrium minimum, we find that the entropy is a nonmonotonic function of time, whereas if the tree-level potential feature thermodynamic properties of the entropy severely restrict the applicability of the effective potential in a dynamical equation of motion.

symmetry breaking minima, the effective potential and entropy are complex when the condensate probes the spinodal region with negative second derivative of the tree-level potential. We argue that collisional processes cannot in general maintain local thermodynamic equilibrium unless there is a fine-tuning of couplings, and that the time evolution of the condensate leads to a "freeze-out" of the density matrix and decoupling from the thermal bath. A closed quantum system approach based on unitary time evolution yields the correct and fully renormalized equations of motion for the condensate conserving both energy and entropy, which are manifestly real and without the caveats of the effective potential. These equations imply an efficient energy transfer mechanism between the condensate and fluctuations as a consequence of profuse stimulated particle production via parametric amplification or spinodal instabilities. Particles are produced with nonthermal distribution functions localized in momentum within instability bands either spinodal or parametric, draining energy from the condensate, suggesting the emergence of asymptotic stationary states, the nature of which must be established numerically.

We focused on obtaining the equations of motion consistently up to one loop, which do not include higher order collisional processes, these are of paramount importance if rethermalization is to occur on longer timescales by a redistribution of the created particles. Possible alternative

hδþðx;t⃗Þδþðx⃗ 0;t0Þi ≡ TrTδðx;t3⃗Þδðx⃗ 0;t0Þρðt$_0$Þ

avenues to study these processes would be to implement the effective action approaches introduced in Refs. [48,49].

Although the study in this article is carried out in Minkowski space time, we expect that many of the lessons will remain relevant in an expanding cosmology. In particular the method to obtain the (causal) equations of motion for the condensate including radiative corrections may be adapted from those introduced recently [45] for a different situation within the cosmological context.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: CORRELATION FUNCTIONS

In this appendix we summarize the correlation functions of the fluctuating field $\delta(\vec{x},t)$ that enter in the equation of motion for the mean field $\phi(t)$. With the quantization of the fluctuation field $\delta(\vec{x},t)$ given by Eq. (2.8), and the initial density matrix (4.14) with the expectation values (4.15), we find the following correlation functions:

$$= \hbar \int \frac{d^3k}{(2\pi)^3} \big[g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big]\Theta(t-t') + \big[g_k(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big]\Theta(t'-t) \quad (A1)$$

$$\langle \delta^-(\vec{x},t)\delta^-(\vec{x}',t')\rangle \equiv \mathrm{Tr}\,\tilde{T}\,\delta(\vec{x},t)\delta(\vec{x}',t')\rho(t_0)$$

$$= \hbar \int \frac{d^3k}{(2\pi)^3} \big[g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big]\Theta(t'-t) + \big[g_k(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big]\Theta(t-t') \quad (A2)$$

$$\langle \delta^+(\vec{x},t)\delta^-(\vec{x}',t')\rangle \equiv \mathrm{Tr}\,\delta(\vec{x},t)\rho(t_0)\delta(\vec{x}',t') = \mathrm{Tr}\,\delta(\vec{x}',t')\delta(\vec{x},t)\rho(t_0) = \hbar \int \frac{d^3k}{(2\pi)^3} \big[g_k^*(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big] \quad (A3)$$

$$\langle \delta^-(\vec{x},t)\delta^+(\vec{x}',t')\rangle \equiv \mathrm{Tr}\,\delta(\vec{x}',t')\rho(t_0)\delta(\vec{x},t) = \mathrm{Tr}\,\delta(\vec{x},t)\delta(\vec{x}',t')\rho(t_0) = \frac{\hbar}{(2\pi)^3}\int d^3k \big[g_k(t) g_k(t') e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} n_k(0) + g_k(t') g_k(t) e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n_k(0))\big]. \quad (A4)$$

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