Letter   Editors' Suggestion

Properties of two-level systems in current-carrying superconductors

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We show that in disordered s-wave superconductors, at sufficiently low frequencies ω, the coupling of two-level systems (TLSs) to external ac electric fields increases dramatically in the presence of a dc supercurrent. This giant enhancement manifests in all ac linear and nonlinear phenomena. In particular, it leads to a parametric enhancement of the real part of the ac conductivity and, consequently, of the equilibrium current fluctuations. If the distribution of TLS relaxation times is broad, the conductivity is inversely proportional to ω, and the spectrum of the equilibrium current fluctuations takes the form of $1/f$ noise.

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Introduction. The concept of two-level systems (TLSs) was introduced to explain the unusual thermal and transport properties of insulating and metallic glasses [1,2] (see Refs. [3–8] for a review of the subject). While the microscopic nature of TLSs is still not fully understood, the standard phenomenological model assumes the ability for an atom, or several atoms, to tunnel quantum mechanically at low temperatures between two quasistable atomic configurations, as illustrated in Fig. 1.

The dynamics of TLSs may be described by a 2×2 matrix Hamiltonian of the form

$$\hat{H}_0 = \begin{pmatrix} \epsilon & t \\ t^* & -\epsilon \end{pmatrix}. \tag{1}$$

Here, 2ϵ is the energy difference between the two quasistable configurations and t is the tunneling amplitude between them. Diagonalization of this Hamiltonian yields the eigenvalues of the system

$$\mathcal{E}_\pm = \pm\sqrt{\epsilon^2 + |t|^2}. \tag{2}$$

External perturbations modify the parameters of the Hamiltonian. Usually, it is assumed that the modulation of the interwell splitting ϵ is much bigger than the modulation of t (see, for example, Ref. [7]). Thus, the coupling of TLSs to external low-frequency perturbations is described by the time dependence of the interwell energy splitting $\epsilon(t) = \epsilon_0 + \delta\epsilon(t)$. For example, in the presence of an external electric field $\mathbf{E}(t)$ we have a "dipole" contribution to the energy splitting,

$$\delta\epsilon_d(t) = -\mathbf{E}(t) \cdot \mathbf{d}, \tag{3}$$

where $2\mathbf{d}$ is the dipole moment difference between the two metastable states of the TLS. The magnitude and direction of $\mathbf{d}$ are random. The coupling of this form changes both the eigenvalues $\mathcal{E}_\pm$ as well as the eigenfunctions, and therefore adequately describes both resonant and relaxation absorption mechanisms of the electric field by the TLS.

The TLS has also been suggested as a source of the ubiquitous $1/f$ noise in conductors (see, for example, Refs. [9–12]). Since the two states of the TLS have different electron scattering cross sections, transitions between them change the local conductivity. If the distribution function of TLS relaxation times is broad, the $1/f$ noise of current fluctuations through a voltage-biased sample can be attributed to fluctuations in sample conductance induced by the TLS transitions. This picture has been confirmed by measuring "noise of the noise" in metals [13].

At low temperatures T, the quantum interference of electron waves in metals results in large sensitivity of electron transport to the motion of individual scatterers [14,15]. At relatively weak magnetic fields, the electron system crosses over from the "orthogonal" to the "unitary" ensemble and, as a result, the amplitude of the $1/f$ noise is suppressed by a factor of two [15,16]. Conversely, due to the interaction of TLSs with the Friedel oscillations of the electron density, the properties of TLSs in metals at low temperatures are strongly affected by the quantum interference of the electron wave functions, resulting in a strong dependence of the TLS activation energy on the magnetic field [17]. This phenomenon was revealed by measurements of the magnetic field dependence of the plateaus in the telegraph signal associated with transitions between the two states of a single fluctuator [18,19]. Thus, the strong interaction of TLSs with the local density of electrons in disordered conductors has been verified experimentally.

In disordered s-wave superconductors, the TLSs become the most abundant low-energy excitations at low temperatures, $T \ll \Delta$, with Δ being the superconducting gap, and thus play an important role in the dissipation and decoherence of superconducting systems. Since the typical size of TLSs is believed to be smaller than the superconducting coherence length ξ, their atomic structure and strength of coupling to the electron density should be similar to that in normal metals.[1] In this Letter, we consider the effect of superconducting pairing on the coupling of TLSs to the external ac electric field $\mathbf{E}(t) = \mathrm{Re}(\mathbf{E}_\omega e^{-i\omega t})$. We show that in the presence of a dc supercurrent and at sufficiently small frequencies, the effective interaction between the TLSs and the electromagnetic field is

[1] At $T \ll \Delta$, where the concentration of quasiparticles is low, the relaxation rate of the TLSs in superconductors can be very different from that in normal metals [20,21].

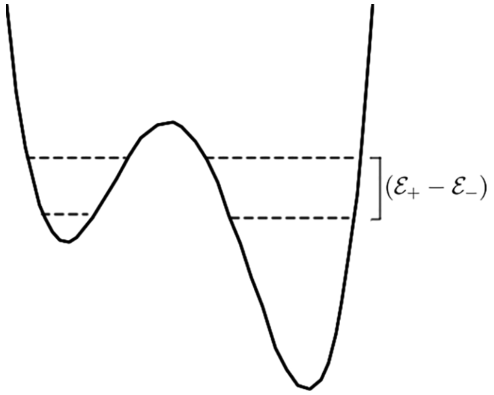

FIG. 1. Qualitative picture of a TLS potential plotted as a function of the configurational coordinate. The dashed lines represent the energy eigenvalues in the two states of the TLSs, which differ by $(\mathcal{E}_+ - \mathcal{E}_-)$.

parametrically enhanced. The physical origin of this effect lies in the large sensitivity of the random Friedel oscillations of the electron density $n(\mathbf{r}, \mathbf{p}_s)$ in disordered superconductors to the changes of the superfluid momentum

$$\mathbf{p}_s(t) = \frac{\hbar}{2}\nabla\chi - \frac{e}{c}\mathbf{A}(t). \tag{4}$$

Here, $\chi(\mathbf{r}, t)$ is the phase of the superconductivity order parameter, and $\mathbf{A}(t)$ is the vector potential. The interwell energy splitting $\epsilon(t)$ of the TLSs is affected by the interaction of the atoms forming TLSs with the electron density. Therefore it acquires a $\mathbf{p}_s$-dependent correction $\delta\epsilon_s$. Since atomic TLSs do not break time-reversal invariance, their energy must be an even function of $\mathbf{p}_s$. Therefore, at small supercurrent densities this correction can be written in the form

$$\delta\epsilon_s(t) = \alpha_s\, p_s^2(t), \quad \alpha_s \equiv \left.\frac{d\epsilon\left(p_s^2\right)}{d\left(p_s^2\right)}\right|_{p_s^2=0}. \tag{5}$$

In the presence of an ac electric field the superfluid momentum becomes time dependent,

$$\dot{\mathbf{p}}_s(t) = e\mathbf{E}(t). \tag{6}$$

This causes the time dependence of the TLS Hamiltonian in Eq. (1). Equation (5) shows that a linear coupling of the external electric field to TLSs mediated by superconductive pairing is possible only in the presence of a dc superfluid momentum $\bar{\mathbf{p}}_s$. In this case, the time-dependent superfluid momentum can be written as $\mathbf{p}_s(t) = \bar{\mathbf{p}}_s + \delta\mathbf{p}_s(t)$, where, according to Eq. (6), $\delta\mathbf{p}_s(t) = \mathrm{Re}(ie\mathbf{E}_\omega e^{-i\omega t}/\omega)$.

The instantaneous relation between the time-dependent superfluid momentum $\mathbf{p}_s(t)$ and the energy splitting described by Eq. (5) remains valid as long as $\hbar\omega \ll \Delta$. Combining Eq. (3) with Eq. (5), linearized with respect to $\delta\mathbf{p}_s$, we find that in this frequency interval, the interaction of TLSs with an ac electric field in current-carrying superconductors has the form $\delta\epsilon(t) = \mathrm{Re}(\delta\epsilon_\omega e^{-i\omega t})$ where

$$\delta\epsilon_\omega = -\left(\boldsymbol{d} - i\frac{e}{\omega}\alpha_s\bar{\mathbf{p}}_s\right)\cdot\mathbf{E}_\omega. \tag{7}$$

Although Eqs. (5) and (7) can be introduced phenomenologically, evaluation of the parameter α_s, which characterizes the sensitivity of TLS level splitting to the superfluid momentum, requires a microscopic consideration.

Below, we develop a microscopic theory of this effect for s-wave superconductors in the diffusive regime, where the coherence length ξ is larger than the electron mean free path ℓ in the normal state, $\xi = \sqrt{\hbar D/\Delta} > \ell > \lambda_F$. Here, $D = v_F\ell/3$ is the diffusion coefficient, v_F is the Fermi velocity, and λ_F is the Fermi wavelength. In this case, the magnitude of the parameter α_s is large because of the high sensitivity of the Friedel oscillations of the electron density to the changes in p_s.

In disordered superconductors, the p_s-dependent correction to the electron density,

$$\delta n(\mathbf{r}, \mathbf{p}_s) = n(\mathbf{r}, 0) - n(\mathbf{r}, \mathbf{p}_s),$$

exhibits random oscillations on the spatial scale of order of λ_F [22,23]. We assume that the typical kinetic and potential energy of electrons in the system are of the same order, $r_s \sim e^2/\hbar v_F \sim 1$, and that the characteristic size of TLSs is of the order of the interatomic spacing, which in metals is of the order of λ_F. In this case, the variance of the p_s-dependent correction to interwell energy splitting may be estimated to within a factor of order unity as

$$\sqrt{\langle[\delta\epsilon_s(t)]^2\rangle} \sim k\epsilon_F\lambda_F^3\sqrt{\langle[\delta n(\mathbf{r}, \mathbf{p}_s(t))]^2\rangle}, \tag{8}$$

where ϵ_F is the Fermi energy, $k \sim 1$, and $\langle\cdots\rangle$ denotes averaging over disorder. Thus, the characteristic values of the TLS parameter α_s in Eqs. (5) and (7) may be estimated using the sensitivity of the variations of the electron density $\delta n(\mathbf{r}, \mathbf{p}_s)$ to the superfluid momentum.

For simplicity, we consider the situation where a superconducting film with a thickness L_z smaller than the skin length is exposed to a monochromatic ac electric field of frequency ω. The spatial modulations of the electron density $n(\mathbf{r})$, which contribute to the TLS level splitting, are caused by the interference of the electronic waves scattered by different impurities. Due to the rapid decay of the Friedel oscillation amplitude with the distance from the impurity, the value of $n(\mathbf{r})$ is determined primarily by impurities closest to the TLS. However, the sensitivity of the electron density to the changes in $\mathbf{p}_s$ is determined by the interference of quantum amplitudes for the electron propagation from the impurities separated from the observation point by distances of order ξ. This can be understood as follows: At $\mathbf{p}_s \neq 0$, the electron amplitudes corresponding to different diffusion paths acquire random phases $\delta\phi_k \sim \int \frac{\mathbf{p}_s}{\hbar}\cdot d\mathbf{l}_k$, where the index k labels different diffusion paths. This changes the interference of contributions of different paths to the electron density. Since $\delta\phi_k$ increases with the length of the path, the sensitivity of the electron density $\delta n(\mathbf{r}, \mathbf{p}_s)$ is controlled by contributions from electron waves traveling from impurities situated at distances of the order of the superconducting correlation length ξ from point $\mathbf{r}$. This is shown qualitatively in Fig. 2. Therefore, the variance of $\delta n(\mathbf{r}, \mathbf{p}_s)$ in the diffusive regime can be obtained using the standard diagram technique of averaging over a random impurity potential [24], valid at $\lambda_F \ll \ell$. To be concrete, we consider thin films of thickness smaller than the coherence length, $\lambda_F < L_z < \xi$. Standard calculations of the Feynman diagrams shown in Fig. 3 that are similar to those in Refs. [22,23,25] yield

$$\sqrt{\langle(\delta n)^2(\mathbf{r}, \mathbf{p}_s)\rangle} = \frac{C}{\lambda_F^3\sqrt{G}}\frac{\Delta}{\epsilon_F}\left(\frac{p_s}{p_d}\right)^2, \quad p_s \ll p_d. \tag{9}$$

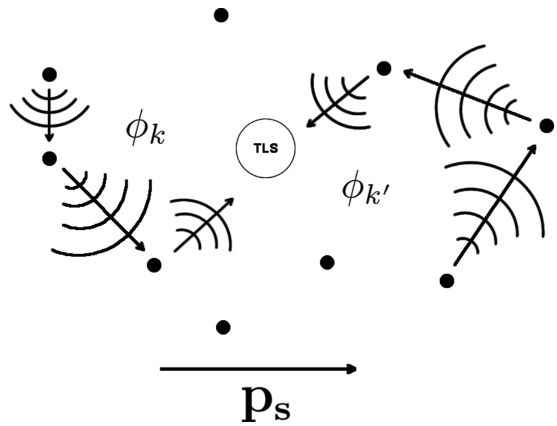

FIG. 2. Illustration of the Friedel oscillations caused by interference of electron waves scattered from multiple impurities. The black dots represent randomly distributed impurities.

Here, $p_d = \sqrt{\hbar\Delta/D}$ is the depairing superfluid momentum, C is a numerical factor of order unity, and

$$G = \frac{8\pi}{3}\frac{\ell L_z}{\lambda_F^2} \tag{10}$$

is the conductance of the film in units of $e^2/2\pi\hbar$.

Using Eq. (9) we obtain the following estimates for the variances of $\delta\epsilon_s(t)$ in Eq. (8), and the parameter α_s in Eqs. (5) and (7),

$$\sqrt{\langle(\delta\epsilon_s)^2\rangle} \sim |\Delta|\frac{1}{\sqrt{G}}\left(\frac{p_s}{p_d}\right)^2 = \frac{1}{\hbar\sqrt{G}}Dp_s^2, \tag{11a}$$

$$\sqrt{\langle\alpha_s^2\rangle} \sim \frac{D}{\hbar\sqrt{G}}, \quad \langle\alpha_s\rangle = 0. \tag{11b}$$

These results show that in situations where the electric field is not orthogonal to $\bar{\mathbf{p}}_s$, the ratio of the second and the first terms in the Parentheses in Eq. (7) is of order

$$\frac{e\alpha_s\bar{p}_s}{\omega d} \sim \frac{\omega*}{\omega}. \tag{12}$$

Here, we introduced the characteristic frequency scale defined by the dc superfluid momentum

$$\omega^* \sim \frac{k}{k_1}\frac{\Delta}{\hbar}\frac{\bar{p}_s}{p_d}\sqrt{\frac{\xi_0}{L_z}}, \tag{13}$$

where $\xi_0 = \hbar v_F/\Delta$ is the coherence length of a pure superconductor, and $k_1 = d/e\lambda_F \sim 1$.

It follows from Eqs. (7) and (12) that

$$\omega < \omega^*. \tag{14}$$

The interaction between the TLSs and the external electric field is dominated by the temporal oscillations of the electron density, rather than by the direct interaction of the external electric field with the dipolar moment of the TLS.

The aforementioned $\bar{\mathbf{p}}_s$-dependent enhancement of the coupling of TLSs to electric fields manifests itself in all ac phenomena. This includes resonant absorption, absorption related to the relaxation mechanism, and all nonlinear effects. The frequency dependences of these effects will be very different from the conventional ones.

Equations (5), (7), and (11) show that in the presence of a supercurrent, the dissipative part of the conductivity tensor becomes anisotropic. The enhancement of TLS coupling to ac electric fields occurs only for the longitudinal geometry, where the ac electric field is parallel to $\bar{\mathbf{p}}_s$. At small ω, the corresponding longitudinal component of the dissipative conductivity, $\sigma_\parallel$, is significantly enhanced as compared to the $\bar{\mathbf{p}}_s = 0$ case. In contrast, the transverse component $\sigma_\perp$ remains roughly the same as that at $\bar{\mathbf{p}}_s = 0$.

As an example, we evaluate the TLS contribution to the dissipative part of the longitudinal microwave conductivity at sufficiently small frequencies, where it is controlled by the Debye relaxation mechanism (see, for example, Refs. [7,26–28]). Energy dissipation in the relaxation mechanism is caused by the time dependence of the TLS energy levels, which creates a nonequilibrium level population, whose relaxation leads to entropy production. In this case the main contribution to the absorption power $P = \sigma_\parallel E_\omega^2/2$ comes from the TLSs with $|\mathcal{E}_+ - \mathcal{E}_-| \lesssim T$, and long relaxation times, $\tau \sim 1/\omega$. The latter condition corresponds to $|t| \ll |\epsilon|$ in Eqs. (1) and (2), which implies $|\mathcal{E}_+ - \mathcal{E}_-| \approx 2|\epsilon|$. Using standard arguments (see, for example, Ref. [27]), we get the following estimate for the dissipative part of the longitudinal conductivity,

$$\sigma_\parallel(\omega) \sim e^2\nu(T)\int d\tau f(\tau)\frac{\lambda_F^2\tau[\omega^2 + \eta\cdot(\omega^*)^2]}{1+(\omega\tau)^2}. \tag{15}$$

Here, $\nu(T)$ is the density of states of TLSs with $\epsilon \sim T$ per unit area, $f(\tau)$ is the distribution function of relaxation times normalized to unity, and $\eta \sim 1$ is a factor of order one. In general, $\nu(T)$ is temperature dependent. However, we note that in glasses this quantity does not depend on T.

At $T \ll \Delta$ the concentration of quasiparticles in s-wave superconductors is exponentially small, and the TLS relaxation times are controlled by their interactions with phonons, similar to the case of dielectrics. The frequency dependence of the conductivity depends on how quickly $f(\tau)$ decays at large values of τ. If $f(\tau \to \infty)$ decays quicker than $1/\tau^2$, then one can introduce a characteristic relaxation time $\tilde{\tau}$. In this case at $p_s = 0$ the conductivity, $\sigma(\omega) \sim e^2\nu\lambda_F^2\omega^2\tilde{\tau}$, is isotropic and quadratic in frequency. However, at $\bar{\mathbf{p}}_s \neq 0$ and $\omega \ll \omega^*$, the longitudinal component of the conductivity tensor, $\sigma_\parallel(\omega)$, exhibits a giant enhancement, proportional to the parameter

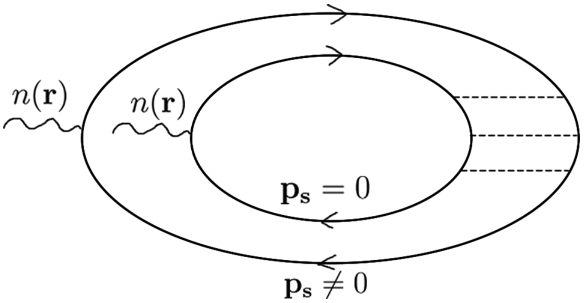

FIG. 3. Feynman diagrams for calculating $\langle(\delta n)^2\rangle$. The dashed lines represent disorder averaging. The solid line correspond to the electron Green's functions, which are matrices in the Gorkov-Nambu space.

$(\omega^*/\omega)^2$. In this case

$$\sigma_\parallel(\omega \to 0) \sim e^2 \nu (\lambda_F \omega^*)^2 \tilde{\tau} \tag{16}$$

becomes frequency independent.

In many disordered systems, especially in glasses, the relaxation time of TLSs has an exponential dependence, $\tau \sim \exp(\zeta)$, on some parameter ζ, which is uniformly distributed. In such cases the distribution function $f(\tau)$ is broad, and has the form (see, for example, Refs. [10,11])

$$f(\tau) \sim 1/\tau. \tag{17}$$

Then, according to Eq. (15) at $\bar{\mathbf{p}}_s = 0$, the conductivity has a linear dependence on the frequency,

$$\sigma(\omega) \sim e^2 \nu \lambda_F^2 \omega. \tag{18}$$

In contrast, at $\bar{p}_s \neq 0$ and $\omega \ll \omega^*$ the longitudinal conductivity is inversely proportional to ω,

$$\sigma_\parallel(\omega) \sim e^2 \nu \frac{(\lambda_F \omega^*)^2}{\omega}. \tag{19}$$

It may seem counterintuitive that, according to Eq. (15), at $\bar{p}_s \neq 0$ the conductivity associated with a localized TLS does not vanish as $\omega \to 0$. The reason for this is as follows. Inelastic processes induce transitions between the two states of TLSs and modulate the supercurrent by changing the superfluid density. At low frequencies, this modulation turns out to be in phase with the oscillations of the external electric field, leading to dissipation.

Since a superconductor carrying a dc supercurrent is in an equilibrium state, one can use the fluctuation-dissipation theorem (FDT) [29] to relate the spectral density of fluctuations of the total current through the system to the dissipative conductivity $\sigma_\parallel(\omega)$,

$$S_s(\omega) = \int dt \overline{\delta I_s(t) \delta I_s(0)} e^{i\omega t} = T \sigma_\parallel(\omega), \quad \hbar\omega < T. \tag{20}$$

Here, the subscript s indicates that the system is in the superconducting state and $\delta I_s(t)$ describes the fluctuation component of the total current in the direction of $\bar{\mathbf{p}}_s$.

It follows from Eqs. (15), and (20) that in the presence of supercurrent, and at $\omega < \omega^*$ the amplitude of the current fluctuations exhibits giant enhancement. In particular, according to Eqs. (19) and (20), in systems where $f(\tau) \sim 1/\tau$, the current correlation function has a $1/f$ form,

$$S_s(\omega) \sim \Gamma \frac{I_s^2}{V\omega}, \quad \Gamma = \frac{\nu T \ell}{\langle N_s \rangle^2}. \tag{21}$$

Here, I_s is the total bias supercurrent passing through the system, and we used the relation between the average supercurrent density and superfluid density per unit area $\langle \mathbf{j}_s \rangle = e \langle N_s \rangle \mathbf{p}_s / m$, which is valid at $p_s \ll p_d$.

The expression for the spectral density of current fluctuations in Eq. (21) describing $1/f$ noise in superconductors is similar to the corresponding expression for the $1/f$ noise of normal current in metals (see, for example, Refs. [10,11]). In both cases the spectral functions are quadratic in the dc current and inversely proportional to the volume of the sample. However, there are significant differences between these two situations.

In normal metals, the $1/f$ noise of the total current is related to the conductance fluctuations and exists only in the presence of a bias current. These current fluctuations are nonequilibrium and therefore do not obey the fluctuation-dissipation theorem. In particular, at $\omega \to 0$ the conductivity of metals becomes frequency independent. In contrast, Eq. (21) describes equilibrium current fluctuations, and the divergence of $S_s(\omega)$ at $\omega \to 0$ is accompanied by the corresponding divergence of the conductivity in Eq. (19).

According to Eqs. (18) and (19), the spectrum of the current fluctuations in superconductors is dominated by the $1/f$ noise for $\omega < \omega^*$. This frequency interval can be significantly larger than the interval in which the $1/f$ noise exceeds the equilibrium Johnson-Nyquist noise in the normal metals.

Equations (5), (7), and (11) can be obtained from an alternative consideration by evaluating the sensitivity of the superfluid density per unit area, N_s, to a change of state of the TLS. It was shown in Refs. [23,30,31] that the rms the amplitude of its mesoscopic fluctuations, of the superfluid density $\delta N_s = N_s - \langle N_s \rangle$, averaged over a region of size ξ, scales with the amplitude of universal conductance fluctuations [14,32] of the sample,

$$\sqrt{\langle (\delta N_s)^2 \rangle} \sim \langle N_s \rangle / G \sim N \lambda_F (\Delta/\epsilon_F). \tag{22}$$

Here, N is the electron density, and we used the expression for average superfluid density per unit area $\langle N_s \rangle \sim N L_z \ell / \xi_0$.

By changing the electron interference pattern, transitions of an individual TLS located at $\boldsymbol{r}_i$ induce time dependence of the superfluid density in the spatial region of the order of the coherence length ξ near $\boldsymbol{r}_i$. We denote the time-dependent part of the superfluid density averaged over this region by $\delta N_s(\mathbf{r}_i, t) \equiv \tilde{N}_{s,i}(t)$. At $r_s \sim 1$ the difference in the electron scattering amplitudes in the two different states of a TLS is of the order of λ_F^2. Therefore, using Eq. (22) the rms value of a random change of the superfluid density caused by a change of state of an individual TLS may be estimated as[2]

$$\sqrt{(\delta \tilde{N}_{s,i})^2} \sim \frac{\sqrt{(\delta N_s)^2}}{\sqrt{G}}. \tag{23}$$

[2]The estimate (23) can be obtained as follows. It was shown in Refs. [14,15] (see also Ref. [23] for a review) that the number of impurities n_0 whose scattering amplitude needs to be changed by $\delta f \sim \lambda_F$ in order to change the conductance by $\delta G \sim e^2/\hbar$ (and consequently the superfluid density by δN_s) is of order $n_0 \sim G$. The qualitative explanation of this fact is as follows. The mesoscopic fluctuations of δN_s originate from the interference of diffusion paths traveling across a region of size ξ. Each diffusive path can be viewed as a tube with a cross-section area $\sim\lambda_F^2$ surrounding a classical diffusive trajectory with a typical length ξ^2/ℓ. Thus, its volume is of the order of $v = \xi^2 \lambda_F^2 / \ell$. The interference pattern changes completely if each diffusion path contains at least one impurity which changes its scattering amplitude. Thus $n_0 \sim \frac{V}{v} = G$, where $V = \xi^2 L_z$ is the volume of a sample of a lateral size ξ. Since the changes of the superfluid density associated with motions of individual scatterers have random signs, the estimate for the rms value of $\delta \tilde{N}_s$, Eq. (23) is obtained by dividing δN_s in Eq. (22) by $\sqrt{n_0}$.

Finally, associating the change of the energy of the TLS with the change of the superfluid energy of a block of size ξ, $\delta\epsilon(\mathbf{p}_s) = \xi^2\delta\tilde{N}_s\mathbf{p}_s^2/m$, we reproduce Eq. (11a).

Using Eqs. (22) and (23) we can also obtain the spectrum of the $1/f$ current noise in Eqs. (19)–(21) without appealing to FDT. This consideration elucidates the microscopic origin of the current noise in current-carrying superconductors, by tracing them to the temporal fluctuations of the superfluid density, $\delta\tilde{N}_s(\mathbf{r}, t)$, which are induced by the TLS transitions.

On the spatial scales larger than ξ, the superfluid density can be written as

$$N_s(\mathbf{r}, t) - \langle N_s\rangle = \sum_i \delta\tilde{N}_{s,i}(t)\xi^2\delta(\mathbf{r} - \mathbf{r}_i). \tag{24}$$

Here, the index i labels individual TLSs. Assuming that transitions of different TLSs are uncorrelated, we get the spectrum of fluctuations of $\delta\tilde{N}_{s,i}(t)$ in the form

$$\int dt\overline{\delta\tilde{N}_{s,i}(t)\tilde{N}_{s,j}(0)}e^{i\omega t} = \delta_{ij}\frac{\tau_i\langle(\delta\tilde{N}_s)^2\rangle}{\pi[1 + (\omega\tau_i)^2]}. \tag{25}$$

The local current density $\mathbf{J}_s(\mathbf{r}, t) = \langle\mathbf{J}_s\rangle + \delta\mathbf{J}_s(\mathbf{r}, t)$ and superfluid momentum $\mathbf{p}_s(\mathbf{r}, t) = \bar{\mathbf{p}}_s + \delta\mathbf{p}_s(\mathbf{r}, t)$ can be determined from the continuity equation $\nabla\cdot\mathbf{J}_s = 0$, which after directing the average superfluid momentum in the x direction, and linearization with respect to small fluctuations, can be written as

$$\langle N_s\rangle\nabla\cdot\delta\mathbf{p}_s(\mathbf{r}, t) + \bar{\mathbf{p}}_s\cdot\nabla N_s(\mathbf{r}, t) = 0. \tag{26}$$

In this form the problem represented by Eqs. (25) and (26) becomes equivalent to the problem of current fluctuations in a current bias normal conductor in which the local conductivity fluctuates in space and time [33]. Solving these equations and averaging the result over the distribution of relaxation time with the distribution function $f(\tau_i) \sim 1/\tau_i$ and over the position of the ith TLS, $\mathbf{r}_i$, we reproduce Eq. (21).

The central assumption we made in this Letter is that the TLSs are situated inside superconductors with $r_s \sim 1$ and that their size is of order interatomic spacing. Therefore the the electron scattering cross-section difference between two the states of TLSs is of order λ_F^2. Recently, other pictures of TLSs have been proposed, including electronic traps [34,35], and localized states associated with spatial fluctuations of the superconductor order parameter in strongly disordered superconductors [36–39]. In these cases, either the characteristic size of the TLS is significantly larger than λ_F, or the TLSs are situated in an insulator close to the superconductor. Enhancements of the microwave absorption rate and the spectrum of equilibrium fluctuations, which are proportional to $\bar{p}_s^2/\omega^2$, exist in these cases as well. However, in these situations, Eq. (15) will acquire additional small preexponential factors.

The enhancement of the linear longitudinal conductivity proportional to $(\omega^*/\omega)^2$ can be several orders of magnitude larger than unity. Because of this, in a wide interval of frequencies, the dissipative properties of current-carrying superconductors are dominated by the coupling of TLSs to the superfluid current. In this frequency interval, the results presented above enable us to evaluate the ac conductivity and the power of $1/f$ current noise up to a factor of order unity. The linear approximation in the external electric field is valid as long as $eE_\omega/\omega \ll \bar{p}_s$. In the nonlinear regime, $p_d > eE_\omega/\omega > \bar{p}_s$, Eq. (5) is still valid, and the enhancement of the nonlinear absorption coefficient is independent of $\bar{p}_s$.

Finally, we would like to mention that the results presented above may be relevant to the physics of superconductor-insulator-superconductor junctions. In the presence of voltage on the junction the current exhibits oscillations in time which are accompanied by oscillations of superfluid velocity in superconducting leads supplying the current to the junction. Therefore there is a contribution to the total dissipation rate of the system associated with TLSs in the bulk of the leads.

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