System Tautochronic Path for Pendulum Vibration Absorber Based on Simple Harmonic Oscillator Transformation

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Centrifugal pendulum vibration absorbers (CPVAs) are essentially collections of pendulums attached to a rotor or rotating component or components within a mechanical system for the purpose of mitigating the typical torsional surging that is inherent to internal combustion engines and electric motors. The dynamic stability and performance of CPVAs are highly dependent on the motion path defined for their pendulous masses. Assemblies of absorbers are tuned by adjusting these paths such that the pendulums respond to problematic orders (multiples of average rotation speed) in a way that smooths the rotational accelerations arising from combustion or other non-uniform rotational acceleration events. For most motion paths, pendulum tuning indeed shifts as a function of the pendulum response amplitude. For a given motion path, the tuning shift that occurs as pendulum amplitude varies produces potentially undesirable dynamic instabilities. Large amplitude pendulum motion that mitigates a high percentage of torsional oscillation while avoiding instabilities brought on by tuning shift introduces complexity and hazards into CPVA design processes. Therefore, identifying pendulum paths whose tuning order does not shift as the pendulum amplitude varies, so-called tautochronic paths, may greatly simplify engineering design processes for generating high-performing CPVAs. To illustrate this new approach and results, a tautochronic cut-out shape producing constant period system motion is obtained for a simplified problem involving a mass sliding in the cut-out of a larger mass that is free to translate horizontally without friction in a constant gravitational field, where the translating base mass replaces the rotating rotor in the centrifugal field.

Keywords: Pendulum vibration absorber, Tautochrone, Isochronous condition

1 Introduction

Automotive Original Equipment Manufacturer (AOEMs) prioritize vehicle comfort and a desirable overall driving experience in new vehicle design. As a result, car companies are highly motivated to identify technologies and techniques to control unwanted vibrations. Centrifugal pendulum vibration absorbers (CPVAs) are now commonly leveraged to address engine generated torsional vibration [1–4]. In typical vehicle CPVA designs, a major challenge is the tuning of pendulums within an absorber assembly by identifying the precise hinge geometries to generate an assembly of pendulums that do not over-respond (and therefore clatter) while at the same time correcting driveline torsional vibration to designated amplitudes [5,6].

Pendulums are order-tuned, meaning their geometry is chosen so that the pendulums respond at a natural frequency that is a specific multiple of average driveline rotation speed. The intuition for order tuning is motivated by considering a simple pendulum in gravity (conceptualized as a mass-less rod connecting a pendulum mass m to a pivot point). The small amplitude undamped resonance of a simple pendulum occurs when a driving force excites the pendulum at a frequency equal to $\sqrt{g/l}$, where g is the acceleration due to gravity, and l is the length of the pendulum rod. By replacing gravitational acceleration g by a centrifugal acceleration term, $R\omega^2$, where ω is the rotor rotation speed (in radians per second) and R is the distance from a rotor center to the pivot point of a mass-less pendulum rod of length l, a correct estimate of the natural frequency of a pendulum is generated as $\omega\sqrt{R/l}$. The pendulum tuning order, $\sqrt{R/l}$, is a positive integer that depends on its installed geometry, and when properly designed to the order of the driveline disturbance, can enable torsional vibration correction at all rotation speeds ω.

An unfortunate reality is that pendulum natural frequencies will typically shift as a function of their swing amplitudes. This shifting resonance complicates pendulum design. In this article, we investigate the design of *tautochronic* pendulums, meaning pendulums that move such that their natural frequencies do not shift as a function of their amplitude of motion. We expect such designs will become increasingly important in improving the stability and performance of a Centrifugal Pendulum Vibration Absorbers (CPVAs) since the nonlinear effects of shifting resonances are mitigated.

CPVAs are passive devices which are used to reduce engine-order torsional vibrations in rotating machines [7]. The dynamic stability and performance of these devices are highly dependent upon the motion path defined for their pendulous masses. We specifically investigate a class of paths that are tautochronic, which implies that their resonance does not vary as the pendulum amplitude grows. In [7], a tautochronic path is derived for a pendulum sliding within a cut-out of a larger mass. The larger mass rolls on frictionless roller bearings. In this paper the tautochronic path for the same problem is obtained through an alternative approach. Sabatini [8] investigated the period of a class of dynamic systems that includes both pendulum motion in a CPVA and pendulum motion in our simplified prototype gravity pendulum. His work led to a general condition that must hold for a tautochronic pendulum path. We show that this condition enforces an equivalence between a nonlinear oscillator and a simple harmonic oscillator, which has the same period of motion for all initial conditions (and hence all

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amplitudes of motion).

This paper expands on the work of Sabatini [8], in which a mathematical condition for tautochronicity is identified for a class of differential equations that includes those that arise in the modeling of the motion of a pendulum in a centrifugal field. The approach is based on a transformation from the physical coordinate to a standard Hamiltonian system. We show that transforming a nonlinear oscillator made tautochronic through path modification actually transforms the nonlinear oscillator into a simple harmonic oscillator. To illustrate the new approach and results, the technique is applied to the simplified problem of determining the cut-out shape that produces tautochronic motion for a mass sliding in the cut-out of a larger mass that is free to translate horizontally without friction. In the simplified problem, centrifugal acceleration is replaced by constant gravitational acceleration and rotation of the rotor inertia is replaced by the translation of the large base mass.

2 Tautochronic Condition

Sabatini in [8,9] presented a method for deriving the tautochronic condition for a class of nonlinear quadratic oscillators with the following form,

$$\ddot{s} + p(s)\dot{s}^2 + q(s) = 0, \tag{1}$$

where s is a physical position coordinate of the oscillator, $p(s)$ and $q(s)$ are smooth odd functions of the coordinate s, and $p(s)$ has positive leading coefficient. According to [10], these types of oscillators have the following form of Lagrangian $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}m(s)\dot{s}^2 - V(s), \tag{2}$$

where the system mass $m(s)$ is position-dependent and $V(s)$ is the system potential energy. In both the centrifugal and gravitation fields, a pendulum vibration absorber system has this same form of Lagrangian and equation of motion (EOM) as that shown in Equation (1) and (2). Specifically, the oscillator coefficients $p(s)$ and $q(s)$ in this physical problem depend on the instantaneous radius of curvature $\rho(s)$ of the absorber mass path, which is assumed to vary as a function of arc-length and thus accommodates a broad range of motion paths including non-circular paths. Similarly, the position-dependent system mass $m(s)$ results from the fact that the center of rotation (rotor) and the center of path curvature for the absorber mass do not share the same point (in general), and therefore, even for a circular path, the radial position of the absorber mass from the center of the rotor varies as a function of the absorbers arc-length displacement. With this general path formulation, we specifically seek the path curvature $\rho(s)$ for the absorber mass path that results in a tautochronic free vibration response of the entire system involving absorber and base mass motion. Following the work of [9], the tautochronic path curvature can be identified with the help of a transformation that transforms Equation (1) into its Hamiltonian form. This coordinate transformation is specifically outlined in the following theorem and then subsequently applied to a pendulum vibration absorber in a uniform gravity field.

2.1 Motion Path Modification to a Simple Harmonic Oscillator.

Theorem 1. *Let s be a function of t satisfying $s(0) = s_0$, $\dot{s}(0) = \dot{s}_0$, and*

$$\ddot{s} + p(s)\dot{s}^2 + q(s) = 0, \tag{3}$$

where ′ · ′ indicates differentiation in t. Suppose $s \in (s_l, s_r)$, $-\infty \le s_l < 0 < s_r \le \infty$ and that p is a bounded, integrable function on (s_l, s_r). Let

$$P(s) = \int_0^s p(x)\ dx \quad and \quad \Phi(s) = \int_0^s \exp P(x)\ dx, \tag{4}$$

and let $u(t) \equiv \Phi(s(t))$; then the initial value problem for s in (3) is equivalent to an initial value problem for $u(t)$ given by $u(0) = \Phi(s_0)$, $\dot{u}(0) = \Phi'(s_0)\dot{s}_0$ and

$$\ddot{u} + h(u) = 0, \tag{5}$$

where $h(u) = \Phi'(s) \cdot q(s)$. When the transformation $\Phi : s \to u$ produces a differential equation of the form indicated in (5) such that the coefficient $h(u(s)) = \Phi'(s) \cdot q(s) = \omega_n^2 \cdot u$, where ω_n^2 is a positive constant, then both $s(t)$ and $u(t)$ must be periodic functions with constant period $T = 2\pi/\omega_n$ for all possible initial conditions s_0 and $\dot{s}_0$. That is, the oscillator's motion is tautochronic.

Proof. For

$$u(t) \equiv \Phi(s(t)), \tag{6}$$

it follows that

$$\dot{u} = \Phi'(s)\dot{s}, \tag{7}$$

$$\ddot{u} = \Phi''(s)\dot{s}^2 + \Phi'(s)\ddot{s}, \tag{8}$$

where $\Phi'(s) = du/ds$ and $\Phi''(s) = d^2u/ds^2$. Then $\ddot{u} + h(u)$ can be divided by $\Phi'(s)$ (because $\Phi'(s) \neq 0$ for all s) and rewritten as

$$\ddot{s} + \left(\frac{\Phi''(s)}{\Phi'(s)}\right)\dot{s}^2 + \frac{h(\Phi(s))}{\Phi'(s)} = \tag{9}$$

$$\ddot{s} + p(s)\dot{s}^2 + q(s) = 0, \tag{10}$$

the last equality following from (3). The initial conditions on u, $u(0) = \Phi(s_0)$, $\dot{u}(0) = \Phi'(s_0)\dot{s}_0$, are an immediate consequence of (6) and (7). This shows that the initial value problem (5) is equivalent to the initial value problem (3).

Next, consider the polar phase plane for the initial value problem $u(0) = \Phi(s(0))$, $\dot{u}(0) = \Phi(s_0)$ and equation (5). Specifically let

$$u = \Gamma(t)\cos\Psi(t) \quad \text{and} \quad \dot{u} = \Gamma(t)\sin\Psi(t). \tag{11}$$

Then the original initial value problem can be written as two first-order differential equations for the polar amplitude $\Gamma(t)$ and the polar angle $\Psi(t)$,

$$\dot{\Gamma}(t) = \sin\Psi(t)\Big(-h\left[\Gamma(t)\cos\Psi(t)\right] + \Gamma(t)\cos\Psi(t)\Big), \tag{12}$$

$$\dot{\Psi}(t) = \frac{-h\left[\Gamma(t)\cos\Psi(t)\right]\cos\Psi(t) - \Gamma(t)\sin^2\Psi(t)}{\Gamma(t)}. \tag{13}$$

From equation (11), it follows that $\Gamma(0) = (u(0)^2 + \dot{u}(0)^2)^{1/2}$. Observe that if

$$\frac{\partial\dot{\Psi}}{\partial\Gamma} = 0, \tag{14}$$

then provided $u(t)$ (and therefore $s(t)$) is periodic, the solution period T is given by

$$T = \int_{\Psi=0}^{2\pi} \left(1/\dot{\Psi}\right)\ d\Psi, \tag{15}$$

which is independent of the choice of $u(0)$ and $\dot{u}(0)$, so that the solution $u(t)$ has a period of oscillation that is tautochronic. The expression $1/\dot{\Psi}(t)$ is an instantaneous frequency of oscillation in the sense that it represents the instantaneous rate of change of the phase angle Ψ.

Using Equation (13), the tautochronic condition (Equation (14)) implies that

$$uh(u) - u^2 h'(u) = 0, \tag{16}$$

where $h'(u) = dh/du$. If $u = 0$, then the tautochronic condition (16) is satisfied trivially. When $u \neq 0$, the equation can be divided by u, and it follows that

$$\frac{dh}{h} = \frac{du}{u}, \tag{17}$$

which implies that

$$\ln h(u) = \ln u + C, \tag{18}$$

so that

$$e^{\ln h(u)} = e^{\ln u} e^{C},$$

and consequently,

$$h(u) = \omega_n^2 u \tag{19}$$

for some positive constant $\omega_n^2 = e^C$. Therefore, the tautochronic condition in (16) implies that $\ddot{u} + \omega_n^2 u = 0$, which indicates u undergoes a simple harmonic oscillation with constant period of motion $T = 2\pi/\omega_n$. This implies that the nonlinear initial value problem in Equation (3) must also be tautochronic with the same period of motion for all amplitudes of periodic motion. □

2.2 Application of the Tautochronic Condition. To identify the tautochronic path for an absorber system, the isochronous condition in Equation (16) needs to be expressed in physical coordinates. This is accomplished by substituting the coordinate transformation $u = \Phi(s)$, which leads to an equation in physical coordinates involving the transformation and the oscillator coefficients $p(s)$ and $q(s)$, as well as the following derivatives, $\Phi'(s)$ and $q'(s)$. Lastly, a derivative of the isochronous condition results in an equivalent condition that only depends on $p(s)$ and $q(s)$ and their derivatives, and thus eliminates the transformation from this condition altogether, enabling a direct application of this in the absorber problem.

To start, we have

$$h(u) = q(\Phi^{-1}(u)) e^{P(\Phi^{-1}(u))}, \tag{20}$$

which after computing a derivative with respect to u, results in

$$h'(u) = \frac{\partial q(\Phi^{-1}(u))}{\partial(\Phi^{-1}(u))} \frac{\partial \Phi^{-1}(u))}{\partial u} e^{P(\Phi^{-1}(u))} + \frac{\partial P(\Phi^{-1}(u))}{\partial(\Phi^{-1}(u))} \frac{\partial \Phi^{-1}(u))}{\partial u} q(\Phi^{-1}(u)) e^{P(\Phi^{-1}(u))}. \tag{21}$$

By using the derivative defined by Equation (7), the following form for $h'(u)$ is obtained

$$h'(u) = q'(\Phi^{-1}(u)) + q(\Phi^{-1}(u)) p(\Phi^{-1}(u)). \tag{22}$$

Now by substituting Equations (20) and (22) in Equation (16), we have

$$\left(q'(\Phi^{-1}(u)) + q((\Phi^{-1}(u)))p((\Phi^{-1}(u)))\right) - \frac{1}{u}\left(q((\Phi^{-1}(u)))e^{P(\Phi^{-1}(u))}\right) = 0. \tag{23}$$

We know from the transformation defined by Equation (6) that $\Phi^{-1}(u) = s$, and therefore the following isochronous condition from [8] can be derived in terms of the physical coordinate s,

$$\sigma = q(s)\Phi'(s) - q'(s)\Phi(s) - \Phi(s)p(s)q(s), \tag{24}$$

where

$$\sigma(s) = 0, \tag{25}$$

is required for the nonlinear oscillator in Equation (1) to exhibit tautochronic motion (i.e., free vibration response that is of constant period).

Notice that Equation (24) involves the transformation $\Phi(s)$ and its derivative $\Phi'(s)$, which depend on the integrals shown in Equation (4). Since $\sigma(s) \equiv 0$, it follows that $\sigma'(s)$ must also be identically zero.

$$\sigma'(s) = q(s)\frac{\Phi''(s)}{\Phi'(s)} - \frac{\Phi(s)}{\Phi'(s)} q''(s) - q(s)p(s) - \frac{\Phi(s)}{\Phi'(s)} q'(s)p(s) - \frac{\Phi(s)}{\Phi'(s)} q(s)p'(s) = 0. \tag{26}$$

(Recall $\Phi'(s) \neq 0$ for all s.) Ultimately, as $\frac{\Phi''(s)}{\Phi'(s)} = p(s)$ and $\frac{\Phi(s)}{\Phi'(s)} \neq 0$, we have

$$q''(s) + p'(s)q(s) + p(s)q'(s) = 0, \tag{27}$$

which is an equivalent tautochronic condition that now conveniently depends explicitly on the position-dependent coefficients $p(s)$ and $q(s)$, and thus eliminates the transformation $\Phi(s)$ and its related integrals involving $p(s)$ (as observed in Equation (24)). Specifically, when applied in the pendulum absorber problem, Equation (27) results in a differential equation in terms of the radius of curvature $\rho(s)$ that can now be directly solved for the tautochronic motion path that results in a tautochronic free vibration response of the pendulum and base mass. In the following sections, we investigate the application of the new isochronous condition (Equation (27)) to identify a system tautochronic motion path for a pendulum vibration absorber in a uniform gravity field and then further compare the properties of this tautochronic response in the physical coordinates versus the transformed coordinates u.

3 Pendulum Vibration Absorber in Gravity Field

Figure 1 shows a pendulum vibration absorber system in a uniform gravity field, which consists of a pendulous mass m that can slide along path cutouts prescribed within a base mass M that is free to translate horizontally (without friction) in a uniform gravity field. The system has two degrees of freedom S and U, which are the arc-length position of the pendulum mass S and the horizontal motion of the base mass U. Mass m is assumed to start at the vertex with an initial speed in the horizontal direction. An arbitrary pendulum path is assumed for the pendulum mass and is parameterized using the local tangent angle ϕ, which can vary as a function of its arc-length position S, and thus accommodates circular and non-circular paths in the formulation. We assume that the vertex occurs at $S = 0$.

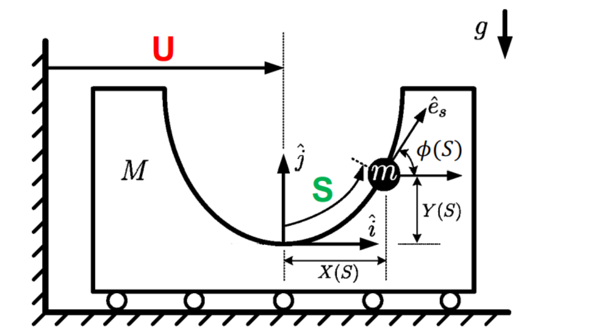

Fig. 1 Pendulum vibration absorber system in a gravity field.

3.1 Dynamic Model. For the gravity problem shown in Figure 1, the system kinetic energy T is

$$T = \frac{1}{2}m(\dot{U}^2 + \dot{S}^2 + 2\dot{U}\dot{S}\cos\phi(S)) + \frac{1}{2}M\dot{U}^2, \quad (28)$$

and the system potential energy is $V = mgY(S)$, where $Y(S)$ is the vertical height of the absorber mass relative to the zero potential line (corresponding with $S = 0$). Then, the system Lagrangian $\mathcal{L} = T - V$ is the following

$$\mathcal{L} = \frac{1}{2}m(\dot{U}^2 + \dot{S}^2 + 2\dot{U}\dot{S}\cos\phi(S)) + \frac{1}{2}M\dot{U}^2 - mgY(S), \quad (29)$$

and the system total energy $C_e = T + V$ is

$$C_e = \frac{1}{2}m(\dot{U}^2 + \dot{S}^2 + 2\dot{U}\dot{S}\cos\phi(S)) + \frac{1}{2}M\dot{U}^2 + mgY(S), \quad (30)$$

where C_e is the total energy constant that depends on the system initial conditions and $\cos\phi(S)$ results from the dot product $(\hat{i}\cdot\hat{e}_s)$ between the local horizontal and path tangent unit vectors (see Figure 1). Furthermore, the system linear momentum C_u is

$$C_u = \frac{\partial T}{\partial \dot{U}} = m\dot{U} + m\dot{S}\cos\phi(S) + M\dot{U}, \quad (31)$$

where C_u is also a constant of motion depending on the system starting conditions. One can eliminate the U dependence in the total energy (Equation (30)) by solving Equation (31) for $\dot{U}$, and then substituting the result into Equation (30). Then, the EOM governing the pendulum motion S can be obtained after computing a time derivative of the resulting energy equation (i.e., $\dot{C}_e = 0$),

$$\ddot{S} + \epsilon\left(\frac{\cos\phi(S)\sin\phi(S)}{\rho\left(1 + \epsilon\sin^2\phi(S)\right)}\right)\dot{S}^2 + g\frac{(1+\epsilon)\sin\phi(S)}{1 + \epsilon\sin^2\phi(S)} = 0, \quad (32)$$

where $\rho = dS/d\phi$ is the local radius of curvature of the pendulum path and $\epsilon = m/M$ is an inertia ratio consisting of the pendulum mass divided by the base mass [2]. In addition, the following substitutions have been made in obtaining Equation (32), specifically $\dot{C}_u = 0$ (conservation of linear momentum) and $dY/dS = \sin\phi(S)$ (see path geometry in Figure 1).

Equation (32) is then non-dimensionalized using the following scheme,

$$s = S/\rho_0, \quad \bar{\rho} = \rho/\rho_0, \quad \text{and} \quad \tau = \omega_0 t,$$

where specifically the dependent coordinate S and the radius of curvature ρ are non-dimensionalized by the initial radius of curvature ρ_0 and the independent coordinate time t is scaled by the small amplitude natural frequency $\omega_0 = \sqrt{g/\rho_0}$. This results in the following non-dimensional EOM for the absorber motion s,

$$s'' + \left(\frac{\epsilon\cos\phi\sin\phi}{\bar{\rho}(\phi)\left(1 + \epsilon\sin^2\phi\right)}\right)s'^2 + \left(\frac{(1+\epsilon)\sin\phi}{1 + \epsilon\sin^2\phi}\right) = 0, \quad (33)$$

where the non-dimensional time τ results in the following time derivative substitutions in Equation (32),

$$\dot{(\,)} = \omega_0(\,)', \quad \ddot{(\,)} = \omega_0^2(\,)'', \quad \text{where} \quad (\,)' = d(\,)/d\tau.$$

Lastly, following a change in dependent variable from s to ϕ, the oscillator in Equation (33) can be put into the standard form (see Equation (1)) for application of the isochronous condition. Specifically, this change in dependent variable results in the following substitutions in Equation (33),

$$s' = \bar{\rho}\phi' \quad \text{and} \quad s'' = \frac{d\bar{\rho}}{d\phi}\phi'^2 + \bar{\rho}\phi'',$$

which results in

$$\phi'' + p(\phi)\phi'^2 + q(\phi) = 0, \quad (34)$$

where the position-dependent coefficients $p(\phi)$ and $q(\phi)$ are

$$p(\phi) = \frac{1}{\rho(\phi)}\frac{d\rho(\phi)}{d\phi} + \frac{\epsilon\cos(\phi)\sin(\phi)}{1 + \epsilon\sin^2(\phi)}, \quad (35)$$

and

$$q(\phi) = \frac{(1+\epsilon)\sin(\phi)}{\rho(\phi)(1 + \epsilon\sin^2(\phi))}, \quad (36)$$

respectively.

3.2 Tautochronic Path for the Pendulum Mass. In this section, we apply the isochronous condition defined by the Equation (27) to the pendulum vibration absorber in a gravity field. This is accomplished after substituting the oscillator coefficients $p(\phi)$ and $q(\phi)$ (Equation (35) and (36)) and their derivatives (with respect to ϕ) into Equation (27). This results in the following first-order differential equation for the non-dimensional path curvature $\bar{\rho}(\phi)$, specifically

$$\cos\phi\left(1 + \epsilon\sin^2\phi\right)\frac{d\bar{\rho}(\phi)}{d\phi} + \sin\phi\left(1 + \epsilon + 3\epsilon\cos^2\phi\right)\bar{\rho}(\phi) = 0, \quad (37)$$

Equation (37) can be solved in closed-form, which results in the following solution for the tautochronic path curvature

$$\bar{\rho}(\phi) = \frac{C\cos\phi}{\left(1 + \epsilon\sin^2\phi\right)^2}, \quad (38)$$

where the constant of integration $C = 1$ is selected so that $\rho = \rho_0$ at the path vertex $\phi = 0$, where ρ_0 is the initial radius of curvature of the path. Equation (38) prescribes the motion path that the pendulum mass should follow to ensure the system will execute tautochronic free vibration when set in motion. In this example, the motion path is specified via the path radius of curvature and specifically indicates how the curvature should vary as a function of the pendulum position $\phi(S)$. Moreover, the tautochronic path curvature in Equation (38) is the same as that derived in [11], which was obtained using the calculus of variations. This verifies the isochronous condition and further demonstrates the utility of this technique, which after obtaining the EOM and coefficients p and q, it directly produces a differential equation in terms of the general path variable to be solved for the tautochronic path.

[2] Note that the commonly used small parameter ϵ in perturbation studies is used here since the pendulum mass is usually much smaller than the base mass in applications (i.e., $\epsilon << 1$). However, it is important to note that we don't assume small ϵ in any of these derivations and therefore a unique system tautochrone exists for any pendulum to base mass ratio.

3.3 Investigation of the Period of Oscillation for the Tautochronic Path.

In this section we investigate the period of oscillation for the gravity problem through a comparison of the instantaneous frequency of the pendulum response in both physical coordinates ϕ and transformed coordinates u. Specifically, we will acquire the explicit form of the transformation $u = \Phi(\phi)$ and $h(u)$ for a pendulum vibration absorber in a gravity field. Of particular interest is the resulting period of motion of the system defined by Equation (34). For this purpose, first we represent this oscillator in polar coordinates to give insight into the instantaneous frequency of oscillation, which can be obtained from the polar angle response. Next, we use the transformation $u = \Phi(\phi)$ to verify the simple harmonic oscillator form of this system when expressed in the u coordinates. Lastly, we simulate both oscillators in polar coordinates to compare their instantaneous frequency of oscillation during free vibration, which is $\dot{\Psi}$ (for the u-coordinates) and $\dot{\psi}$ (for the ϕ-coordinates). Of course the simple harmonic system in the u coordinates will have a constant frequency of oscillation that is independent of amplitude (i.e., initial conditions). However, the physical system in the ϕ coordinates is a nonlinear oscillator, but has properties similar to that of a linear oscillator. Specifically, it has a instantaneous frequency that varies over an oscillation period, but the mean of this variation is equal to the frequency of the u response and is therefore constant and independent of amplitude.

To accomplish this, the tautochronic motion path defined in Equation (38) is used to identify explicit oscillator coefficients $p(\phi)$ and $q(\phi)$. First, we express the physical system response in polar coordinates, using $\phi = R(t)\cos(\psi(t))$ and $\dot{\phi} = R(t)\sin(\psi(t))$, where $R(t)$ is the amplitude and $\psi(t)$ is the polar angle. This enables us to express the EOM in Equation (34) as two first-order differential equations for the polar amplitude $R(t)$ and phase angle $\psi(t)$, which are

$$\dot{R}(t) = \sin\psi(t)\Big(R(t)\cos\psi(t) - q\Big) - pR^2(t)\sin^3\psi(t), \tag{39}$$

$$\dot{\psi}(t) = \frac{-R(t)\sin^2\psi(t)\Big(1 + pR(t)\cos\psi(t)\Big) - q\cos\psi(t)}{R(t)}, \tag{40}$$

where p and q (see Equation (35) and (36)) are evaluated using $\phi = R(t)\cos(\psi(t))$, specifically $p\left[R(t)\cos(\psi(t))\right]$ and $q\left[R(t)\cos(\psi(t))\right]$. In Equation (40), it can be further verified that in physical coordinates, the instantaneous frequency is not independent of amplitude, $d\dot{\psi}/dR \neq 0$. However, as will be further emphasized with simulations, the system response in physical coordinates still executes a constant period free vibration that is independent of amplitude.

For comparison, we derive the transformation $u = \Phi(\phi)$ and the oscillator in u coordinates. This can be accomplished using Equation (4) with the explicit oscillator coefficient $p(\phi)$ (Equation (35)) evaluated with the tautochronic path curvature (Equation (38)), which following two integration steps results in

$$P(\phi) = \int_0^{\phi} p(x)\,dx = \log\cos\phi - \frac{3}{2}\log\left(1 + \epsilon - \epsilon\cos^2\phi\right), \tag{41}$$

and

$$u = \Phi(\phi) = \int_0^{\phi} e^{P(x)}\,dx = \frac{\sin\phi}{\sqrt{1 + \epsilon\sin^2\phi}}. \tag{42}$$

As expected, transforming the oscillator in Equation (34) using $u = \Phi(\phi)$, results in the following simple harmonic oscillator

$$\ddot{u} + (1 + \epsilon)u = 0, \tag{43}$$

where $h(u) = (1 + \epsilon)u$. Specifically, this is a linear oscillator with a constant natural frequency ω_n, where

$$\omega_n = \sqrt{1 + \epsilon}, \tag{44}$$

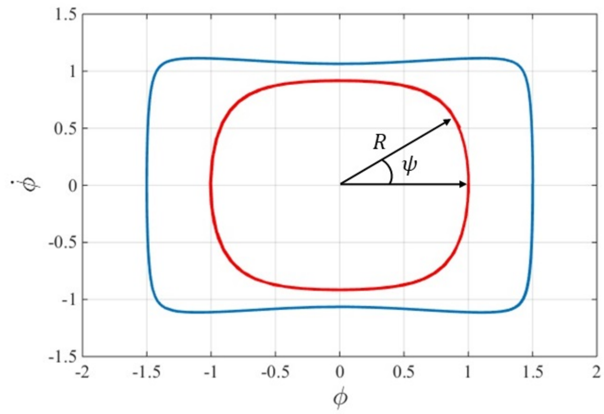

Fig. 2 Phase plane portrait for the tautochronic nonlinear system in physical coordinate ϕ.

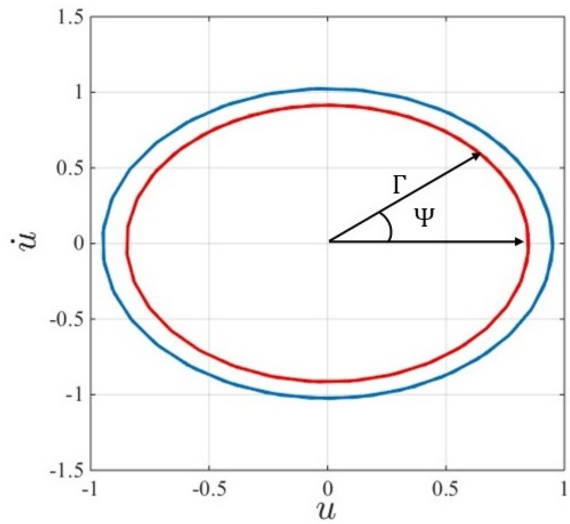

Fig. 3 Phase plane portrait for the tautochronic linear system in u.

which is non-dimensional as a result of the dependent and independent variable scaling outlined in Section 3.1. In accordance with the theorem (see Section 2.1), the isochronous condition in u-coordinates (see Equation (16)) can be immediately verified, specifically after substituting $h(u) = (1 + \epsilon)u$ and its derivative $h'(u) = (1 + \epsilon)$.

Simulation results showing the system response in the phase plane is shown for the physical coordinate ϕ in Figure 2 and for the u coordinate in Figure 3. These results show how the transformation $u = \Phi(\phi)$ nonlinearly stretches the amplitude of the simple harmonic oscillator. Furthermore, Figure 4 shows a comparison of the instantaneous frequency of oscillation over three cycles for both oscillators when $\epsilon = 0.30$. Specifically, the instantaneous frequency is the time rate of change of the polar angles, $\dot{\psi}$ and $\dot{\Psi}$, which corresponds to the ϕ and u phase planes, respectively. The system in physical coordinates is simulated for two different initial conditions including $(\phi(0), \dot{\phi}(0)) = (1.5, 0)$ and $(\phi(0), \dot{\phi}(0)) = (1, 0)$, which in polar coordinates corresponds to $(R(0), \psi(0)) = (1.5, 0)$ and $(R(0), \psi(0)) = (1, 0)$, respectively. Using these initial conditions, Equation (34) and Equations (39)-(40) are used for simulating the physical coordinate responses shown in Figure 2 and 4, respectively. On the other hand, Equation (43) and Equations (12)-(13)

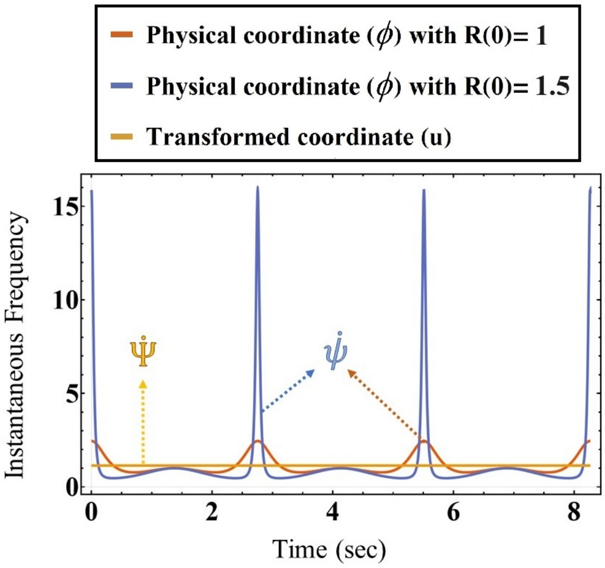

Fig. 4 Instantaneous frequency of both oscillators over three periods of oscillation for $\epsilon = 0.3$.

are used for simulating the linear oscillator response (u coordinates). Furthermore, the coordinate transformation $u = \Phi(\phi)$ in Equation (42), is used to obtain the corresponding initial conditions for the two phase plane trajectories shown in Figure 3, which are $(u(0), \dot{u}(0)) = (0.87, 0)$, $(u(0), \dot{u}(0)) = (0.7, 0)$, and in polar coordinates, is $(\Gamma(0), \Psi(0)) = (0.87, 0)$ and $(\Gamma(0), \Psi(0)) = (0.7, 0)$.

As expected, Figure 4 shows that the instantaneous frequency corresponding to the linear oscillator in u-coordinates is constant $\dot{\Psi} = \sqrt{1+\epsilon}$ for both starting conditions. In addition, as depicted in the figure, the frequency of oscillation does vary for the system in physical coordinates. Specifically, the instantaneous frequency varies both with starting amplitude and during a period of oscillation, which are expected characteristics of a nonlinear oscillator. However, it can be further observed that the average frequency over a period of motion is constant (equal to the frequency of the u system response) and independent of amplitude, which is an intriguing feature of this nonlinear tautochronic oscillator. These free vibration characteristics demonstrate the utility of a pendulum vibration absorber motion path that uses a system tautochrone. A system tautochrone is found to enable constant period free vibration of the nonlinear pendulum response, which can facilitate precise tuning of the pendulum across all amplitudes of operation and thus eliminate nonlinear detuning related performance issues including reduced vibration attenuation and problematic bifurcations that can occur in the system response.

4 Conclusion

Theorem (1) presents a transformation that transforms a class of quadratic nonlinear oscillators which represent the dynamics of pendulum vibration absorber into a simple harmonic oscillator. Consequently, we showed that the initial value problem for the system in physical coordinates, s, is equivalent to an initial value problem in the transformed coordinate, u. Then, stemming from the transformed system, an isochronous condition is derived which comprises the transformation and position dependent coefficients, p and q. Applying the condition to the system leads to a differential equation which solving it culminates in the tautochronic path for the cutout shape. We presented an equivalent isochronous condition that explicitly depends on position dependent coefficients of the nonlinear oscillator and eliminates dependence on the transformation. Then the novel condition is applied to the pendulum vibration absorber problem in a gravity field and ultimately, derived a tautochronic path curvature.

Finally, we conducted an investigation on the period of oscillation to comprehend different aspects of the proposed transformation and the path. For this purpose, we explored the system through analyzing the instantaneous frequency of oscillation, amplitude and phase plane portraits for the system in both physical and transformed coordinates. The results show that for the tautochronic system, the period of the system in physical coordinates, executes the same period of oscillation as the system in the transformed coordinates. Therefore, the free vibration in physical coordinates remarkably shows response characteristics that resembles that of a linear oscillator and thus demonstrates the utility of a system tautochrone motion path to pendulum vibration absorber design, which can facilitate precise tuning of the pendulum across all amplitudes of operation, and thus help in mitigating common performance issues related to the nonlinearity in the system. Future work will investigate the forced vibration response characteristics of this system tautochrone and the bounds on which this tautochronic nonlinear oscillator exhibits linear system response characteristics including the sensitivity of these dynamics to small changes in the physical system (such as errors in the path geometry, system inertia ratio ϵ, and other relevant system design parameters).

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