



List-Avoiding Orientations

Peter Bradshaw¹ · Yaobin Chen² · Hao Ma³ · Bojan Mohar⁴ · Hehui Wu²

Received: 20 September 2022 / Revised: 23 April 2024 / Accepted: 27 April 2024 /

Published online: 11 June 2024

© The Author(s), under exclusive licence to János Bolyai Mathematical Society and Springer-Verlag GmbH Germany, part of Springer Nature 2024

Abstract

Given a graph G with a set $F(v)$ of forbidden values at each $v \in V(G)$, an F -avoiding orientation of G is an orientation in which $\deg^+(v) \notin F(v)$ for each vertex v . Akbari, Dalirrooyfard, Ehsani, Ozeki, and Sherkati conjectured that if $|F(v)| < \frac{1}{2} \deg(v)$ for each $v \in V(G)$, then G has an F -avoiding orientation, and they showed that this statement is true when $\frac{1}{2}$ is replaced by $\frac{1}{4}$. In this paper, we take a step toward this conjecture by proving that if $|F(v)| < \lfloor \frac{1}{3} \deg(v) \rfloor$ for each vertex v , then G has an F -avoiding orientation. Furthermore, we show that if the maximum degree of G is subexponential in terms of the minimum degree, then this coefficient of $\frac{1}{3}$ can be increased to $\sqrt{2} - 1 - o(1) \approx 0.414$. Our main tool is a new sufficient condition for the existence of an F -avoiding orientation based on the Combinatorial Nullstellensatz of Alon and Tarsi.

Keywords Combinatorial Nullstellensatz · List coloring · Orientations · Integer flows

Mathematics Subject Classification 05Cxx

1 Introduction

An *orientation* of a graph G is an assignment of a direction uv or vu to each edge $\{u, v\} \in E(G)$. For an orientation D of a graph G and a vertex $v \in V(G)$, we denote by $E^+(v)$ the arcs outgoing from v in D , and we denote by $E^-(v)$ the arcs incoming to v . We write $\deg_D^+(v) = |E^+(v)|$ and $\deg_D^-(v) = |E^-(v)|$. We write \mathbb{N} for the set

This project was partially funded by NSF RTG grant DMS-1937241.

Supported in part by the NSERC Discovery Grant R611450 (Canada), and by the Research Project N1-0218 of ARRS (Slovenia). On leave from IMFM, Department of Mathematics, University of Ljubljana.

Supported in part by National Natural Science Foundation of China grant 11931006, National Key Research and Development Program of China (Grant No. 2020YFA0713200), and the Shanghai Dawn Scholar Program grant 19SG01.

Extended author information available on the last page of the article

$\{0, 1, 2, \dots\}$ of nonnegative integers. In 1976, Frank and Gyárfás [12] proved that for a graph G and two mappings $a, b : V(G) \rightarrow \mathbb{N}$ satisfying $a(v) \leq b(v)$ for every vertex v , G has an orientation D satisfying $a(v) \leq \deg_D^+(v) \leq b(v)$ for every vertex v if and only if for each subset $U \subseteq V(G)$,

$$\sum_{v \in U} a(v) - d(U) \leq |E(G[U])| \leq \sum_{v \in U} b(v),$$

where $d(U)$ is the number of edges joining U and $\bar{U} = V(G) \setminus U$, and $G[U]$ is the subgraph of G induced by U .

Recently, Akbari, Dalirrooyfard, Ehsani, Ozeki and Sherkati [1] considered the similar problem of finding an orientation of a graph that avoids a certain out-degree at each vertex. Given a graph G and a function $f : V(G) \rightarrow \mathbb{N}$, we say that an orientation D of G is f -avoiding if $\deg_D^+(v) \neq f(v)$ for each $v \in V(G)$. It was proved in [1] that there is an f -avoiding orientation for every 2-connected graph G that is not an odd cycle and for every function $f : V(G) \rightarrow \mathbb{N}$, and that an odd cycle has an f -avoiding orientation if and only if $f(v) \neq 1$ for some vertex v of the cycle. Frank, Tardos, and Sebő [13] considered the same problem modulo 2, or in other words, with parity constraints on the out-degrees of the vertices in G .

In addition to considering orientations that avoid a single given out-degree at each vertex, Akbari et al. considered the graph orientation problem in which each vertex has a list of forbidden out-degrees. Given a graph G and a function $F : V(G) \rightarrow 2^{\mathbb{N}}$, an orientation D of G is said to be F -avoiding if $\deg_D^+(v) \notin F(v)$ for each $v \in V(G)$.

The problem of finding an F -avoiding orientation of a graph G can be represented in the language of *general factors*, introduced by Lovász [23] and defined as follows. Given a graph G and a function $H : V(G) \rightarrow 2^{\mathbb{N}}$, a spanning subgraph $G' \subseteq G$ is an H -factor if $\deg_{G'}(v) \in H(v)$ for each $v \in V(G)$. To represent the problem of finding an F -avoiding orientation of G in the setting of H -factors, we construct a graph $G^{(1)}$ by replacing each edge $e \in E(G)$ with a vertex v_e of degree 2 whose neighbors are the endpoints of e , and we define a function $H : V(G^{(1)}) \rightarrow 2^{\mathbb{N}}$ so that $H(v) = \mathbb{N} \setminus F(v)$ for each $v \in V(G)$ and $H(v_e) = \{1\}$ for each $e \in E(G)$. Then, it is easy to check that the problem of finding an F -avoiding orientation of G is equivalent to the problem of finding an H -factor of $G^{(1)}$. Cornuéjols [5] and Sebő [28] give polynomial-time algorithms for checking whether G has an H -factor whenever $i \notin H(v)$ implies that $i + 1 \in H(v)$ for each $i \in \mathbb{N}$ and $v \in V(G)$, which allow us to check in polynomial time whether G has an F -avoiding orientation whenever the function H defined from F as described above satisfies this condition.

The concept of an F -avoiding orientation is also related to nowhere-zero flows and in particular to Tutte's 3-flow conjecture, which is stated as follows. Given a directed graph D and a positive integer k , a *nowhere-zero k -flow* on D is an assignment $\phi : E(D) \rightarrow \mathbb{Z}_k \setminus \{0\}$ satisfying $\sum_{e \in E^+(v)} \phi(v) \equiv \sum_{e \in E^-(v)} \phi(v) \pmod{k}$ at each vertex $v \in V(G)$. When $k = 3$, a nowhere-zero 3-flow can be interpreted as an orientation of a graph G , and an orientation D of G is a nowhere-zero 3-flow if and only if $\deg_D^+(v) \equiv \deg_D^-(v) \pmod{3}$ holds for each vertex $v \in V(G)$. Tutte's 3-flow conjecture states that every graph G with no edge-cut of size 1 or 3 admits a nowhere-zero 3-flow (see [18, Conjecture (C₂)] or [4]). It has long been known that Tutte's

3-flow conjecture is equivalent to the statement that every 5-regular graph with no edge-cut of size 1 or 3 has an F -avoiding orientation where $F(v) = \{0, 2, 3, 5\}$ at each vertex v (see [4, Unsolved Problem 48] for this equivalent formulation).

It is conjectured that for every graph G , as long as F forbids less than roughly half of the possible out-degrees at each vertex, then G has an F -avoiding orientation.

Conjecture 1.1 [1] *Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If*

$$|F(v)| \leq \frac{1}{2}(\deg_G(v) - 1)$$

for each $v \in V(G)$, then G has an F -avoiding orientation.

If Conjecture 1.1 is true, then the upper bound $\frac{1}{2}(\deg_G(v) - 1)$ is sharp. To show sharpness, take a $2k$ -regular graph G on n vertices with independence number less than $\frac{n}{k+1}$. Then, G has no F -avoiding orientation with $F(v) = \{k, k+1, \dots, 2k-1\}$ for each vertex v . The graph K_{2k+1} is such an example.

One can also compare the size of each forbidden list $F(v)$ to the out-degree of v in some fixed orientation. Since every graph G has an orientation D in which each $v \in V(G)$ satisfies $\deg_D^+(v) \geq \lfloor \deg_G(v)/2 \rfloor$, the following conjecture implies Conjecture 1.1 with an error of at most 1.

Conjecture 1.2 [1] *Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If G has an orientation D such that*

$$\deg_D^+(v) \geq |F(v)| + 1$$

for every $v \in V(G)$, then G has an F -avoiding orientation.

One tool that was used extensively in [1] is the following theorem, called the Combinatorial Nullstellensatz, introduced by Alon and Tarsi in [3] and further developed as a tool by Alon [2].

Theorem 1.3 (Combinatorial Nullstellensatz, [3]) *Let K be a field, and let f be a polynomial in the ring $K[x_1, \dots, x_n]$. Suppose that the degree of f is $t_1 + \dots + t_n$, where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. Then, if S_1, \dots, S_n are subsets of K each satisfying $|S_i| > t_i$, then there exist elements $s_1 \in S_1, \dots, s_n \in S_n$ so that*

$$f(s_1, \dots, s_n) \neq 0.$$

We note that Shirazi and Verstraëte [30] used the Combinatorial Nullstellensatz to prove a result similar to Conjecture 1.1 for F -avoiding subgraphs. They proved that for every graph G , if $F : V(G) \rightarrow 2^{\mathbb{N}}$ satisfies $|F(v)| < \lfloor \deg_G(v)/2 \rfloor$ for each $v \in V(G)$, then G has a spanning subgraph H such that $\deg_H(v) \notin F(v)$ for each $v \in V(G)$. Frank, Lau, and Szabó [11] proved the same result using elementary methods.

In graph theory, the Combinatorial Nullstellensatz is often used for list coloring problems (see e.g. [3, 10, 17] or the monograph [33]). In this case, the polynomial f in the statement of the Combinatorial Nullstellensatz is the *graph polynomial* of a graph G , introduced by Li and Li [21] and defined as follows. For an arbitrary fixed orientation D of G , the *incidence matrix* of D is defined as the matrix $M = (m_{ve} : v \in V(G), e \in E(G))$ for which $m_{ve} = 1$ if $e \in E_G^+(v)$, $m_{ve} = -1$ if $e \in E_G^-(v)$, and $m_{ve} = 0$ otherwise (see e.g. [15, Sect. 2.6]). Then, given a field K , the *graph polynomial* of G is the polynomial in the ring $K[x_v : v \in V(G)]$ defined as $f_D^* = \prod_{e \in E(G)} \left(\sum_{v \in V(G)} m_{ve} x_v \right)$. Equivalently, $f_D^* = \prod (x_u - x_v)$, where the product is taken over all directed arcs $uv \in E(D)$. With this definition, a proper coloring $V(G) \rightarrow K$ can be translated into an assignment of values of K to the variables x_v , and f_D^* takes a nonzero value for a given assignment if and only if the corresponding coloring of G is proper.

One convenient property of the graph polynomial is that its coefficients can be determined solely by counting Eulerian orientations of the graph (see [2] for a complete explanation). These are defined as follows. An orientation D of a graph G is said to be *Eulerian* if $\deg_D^+(v) = \deg_D^-(v)$ holds for all $v \in V(G)$. A subgraph H of G is called *even* if $|E(H)|$ is even and is called *odd* otherwise. Given an orientation D of G , we let $EE(D)$ and $EO(D)$, respectively, denote the number of even and odd subgraphs of G that are Eulerian with respect to D . If G has an orientation D satisfying $EE(D) \neq EO(D)$, then we say that D is an *Alon-Tarsi orientation*. Alon and Tarsi [3] first considered these orientations and proved the following groundbreaking result, which has deep applications in list-coloring: If D is an Alon-Tarsi orientation of G , and if L is a list assignment on G for which $|L(v)| > \deg_D^+(v)$ at each vertex $v \in V(G)$, then G is L -choosable. Motivated by this beautiful result, Jensen and Toft [19] defined the *Alon-Tarsi number* of a graph G , denoted by $AT(G)$, as the minimum value k such that G has an Alon-Tarsi orientation of maximum out-degree less than k . Alon and Tarsi's result can then be rephrased as $\text{ch}(G) \leq AT(G)$, where $\text{ch}(G)$ is the list chromatic number of the graph.

In their work on F -avoiding orientations, Akbari et al. [1] also apply the Combinatorial Nullstellensatz. However, they do not apply this tool to the graph polynomial defined above. Rather, given a graph G , they define a polynomial P in a ring with a set $\{y_e : e \in E(G)\}$ of algebraically independent variables which can take the values 1 or -1 . It turns out that there is a bijection between variable assignments on P and orientations of G . Similarly to the graph polynomial, the coefficients of P can be determined by counting Eulerian orientations in G , and the following theorem relating F -avoiding orientations and Alon-Tarsi orientations holds. In fact, the theorem stated in [1] is slightly weaker, but the following version is an easily proven corollary.

Theorem 1.4 [1] *Let G be a graph, let H be a spanning subgraph of G , and let $F : V(G) \rightarrow 2^{\mathbb{N}}$ be a map. If there exists an Alon-Tarsi orientation D of H such that $|F(v)| \leq \deg_D^+(v)$ for every vertex $v \in V(G)$, then G has an F -avoiding orientation.*

Theorem 1.4 shows a deep connection between the list coloring and F -avoiding orientation problems. We will in fact see in Theorem 3.1 that this connection is a special case of a more general duality between *dual polynomials* obtained from a

common matrix. In the case of Theorem 1.4, the polynomials for list coloring and F -avoiding orientations are related via the incidence matrix of a graph. Furthermore, by using Theorem 1.4 along with a result of Huang, Wong, and Zhu [17], one can immediately prove Conjecture 1.1 for the complete graph K_n . Indeed, Huang et al. [17] proved that if N is a maximum matching in K_n , then $\text{AT}(K_n - N) = \lceil n/2 \rceil$, which implies that $K_n - N$ has an Alon-Tarsi orientation in which each vertex has its out-degree at least $(n - 2) - \lceil n/2 \rceil + 1 = \lfloor \frac{1}{2}n - 1 \rfloor$. Hence, Theorem 1.4 implies that K_n has an F -avoiding orientation whenever $|F(v)| \leq \frac{1}{2}n - 1$ for each vertex v .

Since every graph G has an orientation D in which $\deg_D^+(v) \geq \lfloor \frac{1}{2} \deg_G(v) \rfloor$ for each vertex v , and since $EE(D) > EO(D) = 0$ whenever G is bipartite, Theorem 1.4 also implies that a bipartite graph G has an F -avoiding orientation whenever $|F(v)| \leq \frac{1}{2} \deg_G(v)$ for each v . Furthermore, it is well known that every graph G contains a spanning bipartite subgraph H satisfying $\deg_H(v) \geq \frac{1}{2} \deg_G(v)$ for each vertex v . As a consequence, we have the following result.

Theorem 1.5 [1] *Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$ be a map. If $|F(v)| \leq \frac{1}{4} \deg_G(v)$ for each vertex $v \in V(G)$, then G has an F -avoiding orientation.*

Using a recent result of Lovász, Thomassen, Wu, and Zhang [24] on Tutte's 3-flow conjecture, Akbari et al. [1] also proved that when G is d -regular and d -edge-connected, it is enough to require that $|F(v)| \leq (d - 5)/3$ for each vertex $v \in V(G)$ in order to guarantee an F -avoiding orientation.

Let us now state the main results of this paper. First, we introduce the following theorem, which gives a factor- $\frac{2}{3}$ approximation towards Conjecture 1.2.

Theorem 1.6 *Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If G has an orientation D such that*

$$|F(v)| \leq \frac{2}{3} \deg_D^+(v) - 1$$

for each $v \in V(G)$, then G has an F -avoiding orientation.

Since every graph has an orientation D in which each $v \in V(G)$ satisfies $\deg_D^+(v) \geq \lfloor \frac{1}{2} \deg_G(v) \rfloor$, Theorem 1.6 also implies that Conjecture 1.1 holds with a coefficient of roughly $\frac{1}{3}$, which improves Theorem 1.5 (see our Theorem 4.2). We also show that if G is a graph whose maximum degree is subexponential in terms of its minimum degree, then Conjecture 1.1 holds with an even greater coefficient of roughly $\sqrt{2} - 1 \approx 0.414$.

Theorem 1.7 *Let G be a graph of minimum degree δ and maximum degree $\Delta = e^{o(\delta)}$, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If*

$$|F(v)| \leq (\sqrt{2} - 1 - o(1)) \deg_G(v)$$

for each $v \in V(G)$, then G has an F -avoiding orientation.

Note that in the statement of Theorem 1.7, our function $o(1)$ approaches 0 as $\delta \rightarrow \infty$, and the rate at which $o(1)$ approaches 0 depends on how quickly δ dominates $\log \Delta$.

One of the novel features in this paper is a general setup of dual graph polynomials (see Sect. 3) and the use of multiplied incidence matrix permanents. Special cases of these ideas have been used previously in related problems [6–9, 27, 32], but here we provide a unified general treatment for the first time.

Our paper is organized as follows. In Sect. 2, we prove a sufficient condition (Theorem 2.1) for the existence of an F -avoiding orientation in a graph G , which is the main tool that we use throughout the paper. To prove our sufficient condition, we apply the Combinatorial Nullstellensatz using a polynomial similar to the one used in [1]. In Sect. 3, we show that our polynomial in Sect. 2 has a dual relationship with the traditional graph polynomial via the multiplied incidence matrix of a graph, and we show that this relationship is a special case of a more general duality relation. In Sect. 4, we establish a lemma about fractionally weighted subgraphs, and we use this lemma to prove Theorem 1.6. In Sect. 5, we make use of a randomized approach involving the Lovász Local Lemma to prove Theorem 1.7. Finally, in Sect. 6, we pose some open questions.

2 A Sufficient Condition for an F -Avoiding Orientation

In this section, we give a sufficient condition based on the Combinatorial Nullstellensatz (Theorem 1.3) for the existence of an F -avoiding orientation in a graph.

Henceforth, we make a technical change to the definition of an F -avoiding orientation. Given a graph G with an orientation D and a function $F : V(G) \rightarrow 2^{\mathbb{Z}}$, rather than considering the out-degree of each vertex, we consider the *imbalance* of each vertex $v \in V(G)$, which is defined by Mubayi, Will, and West [26] as the difference $\deg_D^+(v) - \deg_D^-(v)$. We say that D is F -avoiding if $\deg_D^+(v) - \deg_D^-(v) \notin F(v)$ for each $v \in V(G)$. This change is for technical reasons which will become clear later in this section. Since the imbalance of a vertex in a given graph can be uniquely determined from its out-degree and vice versa, and since our main result of this section is only concerned with the size of each forbidden set $F(v)$, our result still holds when the original definition of an F -avoiding orientation is used.

Before stating our condition, we need to establish some notation. We consider a graph G . We order its vertices as v_1, \dots, v_n , and we define an *incidence matrix* $M = (m_{ve} : v \in V(G), e \in E(G))$ for G with respect to the acyclic orientation on G in which each edge $v_i v_j$ is oriented from v_i to v_j if $i < j$. For each vertex $v_i \in V(G)$, we let $E_G^R(v_i)$ denote the edges $v_i v_j \in E(G)$ with $j > i$, and we let $E_G^L(v_i)$ denote the edges $v_i v_j \in E(G)$ with $j < i$. Similarly, we write $\deg_G^L(v_i) = |E_G^L(v_i)|$ and $\deg_G^R(v_i) = |E_G^R(v_i)|$. For each edge $e \in E(G)$, we consider a variable y_e . Given an orientation D of G , and given an edge $e = v_i v_j$ with $i < j$, we set $y_e = 1$ if e is oriented from v_i to v_j in D , and we set $y_e = -1$ otherwise. Given D , we observe that

the imbalance of each vertex v can be expressed as a linear polynomial:

$$\deg_D^+(v) - \deg_D^-(v) = \sum_{e \in E_G^R(v)} y_e - \sum_{e \in E_G^L(v)} y_e = \sum_{e \in E(G)} m_{ve} y_e.$$

Now, suppose that for each vertex $v_i \in V(G)$, we have a list $F(v_i)$ of t_i integers. We would like to find an F -avoiding orientation of G , that is, an orientation D of $E(G)$ such that for each vertex $v \in V(G)$, the imbalance of v satisfies $\deg_D^+(v) - \deg_D^-(v) \notin F(v)$. We fix a field K of characteristic 0, and over K we define the polynomial¹

$$f_0 = \prod_{i=1}^n \prod_{a \in F(v_i)} \left(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e - a \right).$$

By the Combinatorial Nullstellensatz (Theorem 1.3), if f_0 has a nonzero term of degree $\deg f_0$ for which the maximum exponent of each variable y_e is 1, then there exists a vector in $\{-1, 1\}^{E(G)}$ at which f_0 has a nonzero evaluation, and hence G also has an F -avoiding orientation.

For each vertex $v \in V(G)$, we write $t_v = |F(v)|$. We observe that $\deg f_0 = \sum_{v \in V(G)} t_v$ and $t_i = t_{v_i}$ for each $v_i \in V(G)$. Since each term of degree $\deg f_0 = \sum_{v \in V(G)} t_v$ in the monomial expansion of f_0 is obtained by choosing exactly one y_e term from each factor $\left(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e - a \right)$, the terms of maximum degree in the monomial expansion of f_0 are exactly the same as the terms of maximum degree in the monomial expansion of

$$f := \prod_{i=1}^n \left(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e \right)^{t_i} = \prod_{v \in V(G)} \left(\sum_{e \in E(G)} m_{ve} y_e \right)^{t_v}.$$

Therefore, rather than working with f_0 , we work with the simpler polynomial f . By our previous discussion, if f has a nonzero term of degree $\sum_{v \in V(G)} t_v$ for which the maximum exponent of each variable y_e is 1, then G has an F -avoiding orientation.

Given an edge set $A \subseteq E(G)$, we write $y^A = y_{e_1} y_{e_2} \dots y_{e_{|A|}}$, where $A = \{e_1, e_2, \dots, e_{|A|}\}$. We say that a monomial $P \in K[y_e : e \in E(G)]$ is *square-free* if for each edge $e \in E(G)$, y_e^2 does not divide P . Observe that after removing its coefficient, a square-free monomial $P \in K[y_e : e \in E(G)]$ is of the form y^A for some edge set $A \subseteq E(G)$. Therefore, if the monomial expansion of f contains a square-free term with a nonzero coefficient, then by the Combinatorial Nullstellensatz, G has an F -avoiding orientation.

Now, we are ready to state our sufficient condition for when G has an F -avoiding orientation. Although we prove the following theorem within the framework of for-

¹ Here, we consider the integer values $a \in F(v_i)$ as multiples of the multiplicative identity $1 \in K$. Since K has characteristic 0, every element $a \in F(v_i)$ can be expressed as a multiple of 1 in K .

bidden vertex imbalances, it is easy to see that the theorem also holds in the setting of forbidden vertex out-degrees.

Theorem 2.1 *Let $F : V(G) \rightarrow 2^{\mathbb{Z}}$ be an assignment of forbidden imbalances for a graph G . Suppose that there exists an ordering of $V(G)$ and a spanning subgraph H of G such that for each vertex $v \in V(G)$,*

$$|F(v)| \leq \deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v).$$

Then G has an F -avoiding orientation.

Proof In the proof we use the notation introduced above. We also set $E = E(G)$. In order to prove that G has an F -avoiding orientation, we show that the monomial expansion of f over K contains a square-free term and then apply the Combinatorial Nullstellensatz as described above. Given an edge set $A \subseteq E$, we say that y^A is in the support of f if the monomial y^A has nonzero coefficient in the expansion of f . If y^A is in the support of f , we also say that A is in the support of f , and we write $A \in \text{Supp}(f)$. For each j ($1 \leq j \leq n$), we write

$$f_j = \prod_{i=1}^j \left(\sum_{e \in E_G^R(v_i)} y_e - \sum_{e \in E_G^L(v_i)} y_e \right)^{t_i},$$

and we observe that $f = f_n$.

We prove the following stronger claim:

For each j ($1 \leq j \leq n$), there exists an edge set $A_j \subseteq E$ such that

- (a) $A_j \in \text{Supp}(f_j)$,
- (b) if $k > j$, then $A_j \cap E_G^L(v_k) \subseteq E_H^L(v_k)$.

By setting $j = n$ in the claim, we obtain an edge set $A_n \subseteq E$ which is in the support of $f_n = f$. Since A_n is an edge set (and not a multiset), y^{A_n} is square-free and thus satisfies the theorem. When proving the claim, we work in the quotient ring

$$K' = K[y_e : e \in E] / \langle y_e^2 : e \in E \rangle,$$

where $\langle y_e^2 : e \in E \rangle$ is the ideal generated by the squares of the variables y_e . This allows us to ignore all terms in the expansion of each f_j divisible by a square, which is desirable, as we are only interested in finding a square-free term in the expansion of f .

We prove the claim by induction on j . When $j = 1$, we observe that

$$f_1 = \left(\sum_{e \in E_G^R(v_1)} y_e \right)^{t_1} = t_1! \sum_{\substack{A \subseteq E_G^R(v_1) \\ |A|=t_1}} y^A.$$

Since $t_1 = |F(v_1)| \leq \deg_G^L(v_1) - 2 \deg_H^L(v_1) + \deg_H^R(v_1) = \deg_H^R(v_1)$, we can choose A_1 to be any set of t_1 edges of H incident with v_1 . Hence the claim holds for $j = 1$.

Now, let $2 \leq j \leq n$, and suppose the claim holds for $j - 1$. That is, suppose that we have a set A_{j-1} satisfying $y^{A_{j-1}} \in \text{Supp}(f_{j-1})$ and in particular

$$A_{j-1} \cap E_G^L(v_j) \subseteq E_H^L(v_j).$$

We would like to show that the claim holds for j . First, we will show that f_j has a nonzero expansion in our quotient ring, implying the existence of a square-free monomial in the expansion of f_j over K' . Then, we will show that the square-free monomial that we have found in the support of f_j satisfies condition (b). As

$$f_j = f_{j-1} \cdot \left(\sum_{e \in E_G^R(v_j)} y_e - \sum_{e \in E_G^L(v_j)} y_e \right)^{t_j},$$

we can express f_j in the following form:

$$f_j = f_{j-1} \cdot \sum_{a=0}^{t_j} \binom{t_j}{a} (-1)^{t_j-a} \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-a} \left(\sum_{e \in E_G^R(v_j)} y_e \right)^a. \quad (1)$$

When the sum in (1) is restricted to a single value a , each monomial in the expansion of (1) has exactly a variables y_e for which $e \in E_G^R(v_j)$. Since none of these variables y_e appears in f_{j-1} , any nonzero term in the expansion of (1) for a fixed value of a is also a nonzero term in the expansion of (1) when the sum is taken over all values of a .

We restrict our attention to terms in the expansion of (1) that occur when $a = m := \min\{t_j, \deg_H^R(v_j)\}$ in the sum. Hence, we only consider the expansion of

$$f_{j-1} \cdot \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m} \left(\sum_{e \in E_G^R(v_j)} y_e \right)^m. \quad (2)$$

We fix a set $A' \subseteq E_H^R(v_j)$ with m edges and observe that $A' \in \text{Supp} \left(\sum_{e \in E_G^R(v_j)} y_e \right)^m$. Furthermore, we write $B = A_{j-1} \setminus E_G^L(v_j)$, and we write

$$f_{j-1} = y^B g + r,$$

where g and r are polynomials, and r does not have any terms divisible by y^B . Observe that every term of g is a constant times a monomial P of degree $d := \deg(f_{j-1}) - |B| = |A_{j-1} \setminus B| = |A_{j-1} \cap E_G^L(v_j)|$ corresponding to a subset of d edges of $E_G^L(v_j)$. Observe also that $A_{j-1} \cap E_G^L(v_j)$ is in the support of g .

Now, we would like to show that

$$g \cdot \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m} \neq 0, \quad (3)$$

in the quotient ring K' . Observe that if (3) holds in K' , then since $B \cap E_G^L(v_j) = \emptyset$, it also holds that

$$f_{j-1} \cdot \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m} = (y^B g + r) \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m} \quad (4)$$

is nonzero in K' . Also, since no y_e with $e \in E_G^R(v_j)$ divides (4) in K' , it follows that (2) has a nonzero monomial expansion in K' (and hence a square-free term) whenever (3) holds. Therefore, in order to prove condition (a) of our claim, it is enough to prove (3).

If $m = t_j$, then (3) clearly holds, and $g \cdot \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m}$ has a square-free monomial y^S corresponding to an edge subset $S \subseteq E_G^L(v_j)$. Then, the edge set $A_j = A' \cup B \cup S$ corresponds to a square-free monomial in the support of f_j . By construction, the only edges of $A_j \setminus A_{j-1}$ that do not belong to $E(H)$ are those that belong to $E_G^L(v_j)$, which implies condition (b). Hence, the induction step is complete, and the claim is proven.

Otherwise, we have $m = \deg_H^R(v_j)$. We write $E^L = E_G^L(v_j)$ and $b = d + t_j - \deg_H^R(v_j)$. We also expand g as

$$g = \sum_{\substack{D \subseteq E^L \\ |D|=d}} c_D y^D.$$

Then,

$$\begin{aligned} g \left(\sum_{e \in E^L} y_e \right)^{t_j - \deg_H^R(v_j)} &= (t_j - \deg_H^R(v_j))! \left(\sum_{\substack{D \subseteq E^L \\ |D|=d}} c_D y^D \right) \left(\sum_{\substack{U \subseteq E^L \\ |U|=t_j - \deg_H^R(v_j)}} y^U \right) \\ &= (t_j - \deg_H^R(v_j))! \sum_{\substack{Y \subseteq E^L \\ |Y|=b}} \left(\sum_{\substack{D \subseteq Y \\ |D|=d}} c_D \right) y^Y. \end{aligned} \quad (5)$$

Hence, in order to show that (3) holds and prove the first condition of our induction hypothesis, it is enough to show that the expression above is nonzero in our quotient ring.

The expression (5) may be expressed as follows. Let Q be the matrix whose rows are indexed by the b -subsets of E^L and whose columns are indexed by the d -subsets of E^L . Given a set $Y \subseteq E^L$ of size b and a set $D \subseteq E^L$ of size d , we let the (Y, D) -entry of Q be 1 if $D \subseteq Y$ and equal 0 otherwise. In other words, Q is a set-inclusion matrix. Then, the coefficients in (5) are equal to the entries in the vector

$$(t_j - \deg_H^R(v_j))! Q\tau, \quad (6)$$

where τ is the column vector with entries indexed by subsets $D \subseteq E^L$ of size d and whose D -entry equals c_D . Since g is nonzero in our quotient ring, we have that $\tau \neq 0$. Thus, to prove that (3) holds, it suffices to show that (6) is nonzero.

Now, since $d = |A_{j-1} \cap E_G^L(v_j)|$, it follows from condition (b) of the induction hypothesis that $d \leq \deg_H^L(v_j)$. Therefore, $b \leq t_j + \deg_H^L(v_j) - \deg_H^R(v_j)$, which by our initial assumption is at most $|E^L| - \deg_H^L(v_j) \leq |E^L| - d$. Therefore, since $d \leq b \leq |E^L| - d$, it follows that $\binom{|E^L|}{b} \geq \binom{|E^L|}{d}$, and hence it follows from a classical result of Gottlieb [14, Corollary 1(b)] about set-inclusion matrices that Q has full column rank. Hence, the matrix product in (6) is nonzero, and thus (3) holds.

Therefore, $g \cdot \left(\sum_{e \in E_G^L(v_j)} y_e \right)^{t_j-m}$ again has a square-free monomial y^S corresponding to an edge subset $S \subseteq E_G^L(v_j)$. We let $A_j = A' \cup B \cup S$, and then A_j corresponds to a square-free monomial in the support of f_j .

Finally, we check that our edge set A_j satisfies condition (b). By construction, the only edges of $A_j \setminus A_{j-1}$ that do not belong to $E(H)$ are those that belong to $E_G^L(v_j)$, which implies (b). Hence, the induction step is complete, and the claim is proven. As stated at the beginning of the proof, the claim together with the Combinatorial Nullstellensatz implies our theorem. \square

3 Dual Polynomials

In this section, we take a brief aside to show that our polynomial f used in the proof of Theorem 2.1 is related to the traditional graph polynomial and that this relationship is a special case of a more general duality relation. As a corollary, we obtain a simplified proof of Theorem 1.4.

We consider the following setup. Let $A = (a_{ij} : 1 \leq i \leq n, 1 \leq j \leq m)$ be an $n \times m$ matrix, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_m)$ be nonnegative integer vectors. Let $A^{\alpha, \beta}$ be obtained from A by making exactly α_i copies of each row i in A and then subsequently making exactly β_j copies of each column j . We fix a field K and the ring $K[x_i, y_j : 1 \leq i \leq n, 1 \leq j \leq m]$. Given vectors α and β as above, we

define the *dual polynomials*

$$g = \prod_{i=1}^n \left(\sum_{j=1}^m a_{ij} y_j \right)^{\alpha_i},$$

$$g^* = \prod_{j=1}^m \left(\sum_{i=1}^n a_{ij} x_i \right)^{\beta_j}.$$

Note that when only one polynomial is given, its dual is not uniquely determined, since the correspondence between g and g^* depends on the vectors α and β . We say that g and g^* are duals of each other, as the factors of g are obtained from the rows of A , while the factors of g^* are obtained from the columns of A . This relationship is reminiscent to the relationship between dual linear programs, whose respective constraints are given by the rows and the columns of a common matrix.

We write $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ and $y^\beta = \prod_{j=1}^m y_j^{\beta_j}$. We denote the coefficient of a monomial y^β in the polynomial g by $\text{coeff}(y^\beta, g)$. For a vector $u = (u_1, \dots, u_k)$, we write $\|u\|_1 = \sum_{i=1}^k |u_i|$. Finally, if M is a square matrix, then we denote by $\text{per}(M)$ its permanent. We have the following result.

Theorem 3.1 *If $\|\alpha\|_1 = \|\beta\|_1$, then $(\prod_{j=1}^m \beta_j!) \text{coeff}(y^\beta, g) = (\prod_{i=1}^n \alpha_i!) \text{coeff}(x^\alpha, g^*) = \text{per}(A^{\alpha, \beta})$.*

Proof We first show that $(\prod_{j=1}^m \beta_j!) \text{coeff}(y^\beta, g) = \text{per}(A^{\alpha, \beta})$. Consider the monomial expansion of g containing $m^{\|\alpha\|_1}$ terms, each of which is obtained by choosing a term from each of the $\|\alpha\|_1$ factors $\sum_{j=1}^m a_{ij} y_j$ of g and taking the product of these terms. Each of the factors $\sum_{j=1}^m a_{ij} y_j$ of g can be associated with a distinct row in the matrix $A^{\alpha, \mathbf{1}}$ (where $\mathbf{1}$ refers to the all-1 vector of length m). Therefore, each term P in the monomial expansion of g divisible by y^β corresponds to a choice of $\|\alpha\|_1$ entries from $A^{\alpha, \mathbf{1}}$ with one entry from each row and β_j entries in each column j , and the coefficient of P is equal to the product of all $\|\alpha\|_1$ of these entries. Hence, $\text{coeff}(y^\beta, g)$ is equal to $\sum_{\sigma} a_{\sigma}$, where σ runs over all choices of $\|\alpha\|_1$ entries from $A^{\alpha, \mathbf{1}}$ consisting of one entry from each row and β_j entries from each column j , and a_{σ} is the product of all of the values from $A^{\alpha, \mathbf{1}}$ contained in σ . Furthermore, a choice of β_j entries from column j of $A^{\alpha, \mathbf{1}}$ can be represented in $\beta_j!$ ways as a choice of one entry from each of the β_j columns of $A^{\alpha, \beta}$ corresponding to column j of $A^{\alpha, \mathbf{1}}$. Therefore, $(\prod_{j=1}^m \beta_j!) \text{coeff}(y^\beta, g) = \sum_{\tau} a_{\tau}$, where τ runs over all choices of $\|\alpha\|_1$ entries from $A^{\alpha, \beta}$ consisting of one entry from each row and one entry from each column, and a_{τ} is the product of all of the values from $A^{\alpha, \beta}$ contained in τ . Equivalently, $(\prod_{i=1}^n \beta_j!) \text{coeff}(y^\beta, g) = \text{per}(A^{\alpha, \beta})$.

By using the transpose of A and swapping α and β , the same argument shows that $(\prod_{i=1}^n \alpha_i!) \text{coeff}(x^\alpha, g^*) = \text{per}((A^T)^{\beta, \alpha}) = \text{per}(A^{\alpha, \beta})$. \square

We note that considering the permanent of a matrix appears naturally when using the polynomial method. The first such appearances may have been used by DeVos et

al. [6–8]. It was also used in relation to the problem of total weight choosability (see e.g. [9] and [27], [32]). In fact, in [32], Zhu similarly considers the permanent of a matrix obtained by making copies of the rows or columns of a smaller matrix.

In one special case of this matrix setting, A is the incidence matrix M of a graph G , and $\beta \in \{0, 1\}^{|E(G)|}$ indexes an edge set $E' \subseteq E(G)$. If f is defined as in Sect. 2 and $\text{coeff}(y^\beta, f) \neq 0$, then the Combinatorial Nullstellensatz tells us that G has an F -avoiding orientation when $|F(v)| \leq t_v$ for each vertex v of G . Furthermore, the polynomial dual to f is $f^* = \prod_{uv \in E'} (x_u - x_v)$, which is the traditional graph polynomial of $G[E']$. A famous result of Alon and Tarsi [3] tells us that if D is an orientation of a graph H satisfying $\deg_D^+(v) = t_v$ at each vertex $v \in V(H)$, then

$$\left| \text{coeff} \left(\prod_{v \in V(G)} x_v^{t_v}, f^* \right) \right| = |EE(D) - EO(D)|. \quad (\text{AT})$$

Therefore, if there exists an Alon-Tarsi orientation D of $G[E']$ in which $\deg_D^+(v) = t_v$ for each $v \in V(G)$, then (AT) and Theorem 3.1 tell us that

$$\left(\prod_{v \in V(G)} t_v! \right)^{-1} \left| \text{coeff} \left(y^\beta, f \right) \right| = \left| \text{coeff} \left(\prod_{v \in V(G)} x_v^{t_v}, f^* \right) \right| = |EE(D) - EO(D)| \neq 0.$$

Hence, the existence of the Alon-Tarsi orientation D on $G[E']$ implies that G has an F -avoiding orientation whenever $|F(v)| \leq t_v$ for each $v \in V(G)$, proving Theorem 1.4. Theorem 3.1 gives us a sufficient condition for the existence of a nowhere-zero p -flow, for each prime p . In fact, the theorem gives us a stronger conclusion, which we describe below.

Recall that a function $\phi : E(G) \rightarrow \{1, \dots, p-1\}$ on a graph G yields a nowhere-zero p -flow if and only if the congruence $\sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) \equiv 0 \pmod{p}$ holds at each vertex $v \in V(G)$. Generalizing the notion of a nowhere-zero flow, we say that a graph G is \mathbb{Z}_p -connected if for every function $b : V(G) \rightarrow \mathbb{Z}_p$ with $\sum_{v \in V(G)} b(v) = 0$, there exists $\phi : E(G) \rightarrow \mathbb{Z}_p \setminus \{0\}$ such that

$$\sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) = b(v) \quad (7)$$

for each $v \in V(G)$. Since $\sum_{v \in V(G)} \left(\sum_{e \in E^+(v)} \phi(v) - \sum_{e \in E^-(v)} \phi(v) \right) \equiv 0 \pmod{p}$, it follows that if (7) is satisfied at every vertex $v \in V(G) \setminus \{u\}$ for some $u \in V(G)$, then (7) also holds at u .

Theorem 3.2 *Let $p \geq 3$ be a prime, and let G be a graph. Let $G^{(p-2)}$ be obtained from G by replacing each edge with $p-2$ parallel edges, and let H be a spanning subgraph of $G^{(p-2)}$ with exactly $(p-1)(|V(G)|-1)$ edges. If, for some $u \in V(H)$, H has an orientation D in which $\deg_D^+(v) = p-1$ for each $v \in V(G) \setminus \{u\}$ and such that $EE(D) - EO(D) \not\equiv 0 \pmod{p}$, then G is \mathbb{Z}_p -connected.*

Proof Consider an arbitrary orientation of the edges of G . From this orientation, define the incidence matrix $M = (m_{ve} : v \in V(G), e \in E(G))$ of G and consider any $b : V(G) \rightarrow \mathbb{Z}_p$ with $\sum_{v \in V(G)} b(v) = 0$. We fix a vertex $u \in V(G)$, the field \mathbb{Z}_p , and a ring $\mathbb{Z}_p[x_v, y_e : v \in V(G), e \in E(G)]$ in which we define the polynomial

$$g_0 = \prod_{v \in V(G) \setminus \{u\}} \prod_{a \in \mathbb{Z}_p \setminus \{b(v)\}} \left(\sum_{e \in E(G)} m_{ve} y_e - a \right).$$

It is straightforward to show that there is a function $\phi : E(G) \rightarrow \mathbb{Z}_p \setminus \{0\}$ satisfying (7) if and only if there is an assignment from the set $\mathbb{Z}_p \setminus \{0\}$ to each variable y_e that gives g_0 a nonzero evaluation. Furthermore, every term of maximum degree in the expansion of g_0 is also a maximum-degree term in the expansion of

$$g = \prod_{v \in V(G) \setminus \{u\}} \left(\sum_{e \in E(G)} m_{ve} y_e \right)^{p-1} = \prod_{v \in V(G)} \left(\sum_{e \in E(G)} m_{ve} y_e \right)^{\alpha_v},$$

where $\alpha_v = p-1$ for each $v \in V(G) \setminus \{u\}$, and $\alpha_u = 0$. We let $\beta \in \{0, \dots, p-2\}^{|E(G)|}$ be a vector indexing which edges of G , and how many copies of each edge, belong to H . Since the maximum degree terms in g and g_0 are equal, it follows that if $\text{coeff}(y^\beta, g) \neq 0$, then by the Combinatorial Nullstellensatz, g_0 has an assignment for each y_e from $\mathbb{Z}_p \setminus \{0\}$ that makes g_0 nonzero.

Note that $\sum_{v \in V(G)} \alpha_v = (p-1)(|V(G)|-1) = |E(H)| = \sum_{e \in E(G)} \beta_e$. Thus we can apply Theorem 3.1, which shows that $\text{coeff}(y^\beta, g) \neq 0$ if and only if $\text{coeff}(x^\alpha, g^*) \neq 0$, where

$$g^* = \prod_{e \in E(G)} \left(\sum_{v \in V(G)} m_{ve} x_v \right)^{\beta_e} = \prod_{e \in E(H)} \left(\sum_{v \in V(G)} m_{ve} x_v \right).$$

However, since g^* is the graph polynomial of H , and since D is an orientation of H satisfying $\deg_D^+(v) = \alpha_v$ at each $v \in V(H)$ and $EE(D) - EO(D) \neq 0$ in \mathbb{Z}_p , it follows from (AT) that $|\text{coeff}(x^\alpha, g^*)| = |EE(D) - EO(D)| \neq 0$. Therefore, an assignment from $\mathbb{Z}_p \setminus \{0\}$ to each y_e giving g_0 a nonzero evaluation exists. This implies that G is \mathbb{Z}_p -connected. \square

4 A Step Toward Conjectures 1.1 and 1.2

In this section, we prove Theorem 1.6, which establishes relaxations of Conjectures 1.1 and 1.2 with slightly smaller constant factors. Our main tool is Theorem 2.1 along with a fractional rounding method.

Many graph theory problems can be described as integer programming problems. Often, we find a fractional solution, and then use rounding to obtain an integer solution

that approximately solves the original problem. The following lemma shows how this is done.

Lemma 4.1 *Given a graph $G = (V, E)$, let $M = (m_{ve} : v \in V, e \in E)$ be a real-valued matrix in which $m_{ve} \neq 0$ only if $v \in e$. Let $\mathbf{y} \in [0, 1]^E$ be a vector, and let $\mathbf{x} = M\mathbf{y}$. Then, there exists a 01-vector $\mathbf{y}' \in \{0, 1\}^E$ such that $\mathbf{x}' = M\mathbf{y}'$ satisfies $\mathbf{x}'_v \geq \mathbf{x}_v - b_v$ for each $v \in V(G)$, where $b_v = \max\{|m_{ve}| : e \in E\}$. Furthermore, we may choose \mathbf{y}' so that $\mathbf{x}'_v > \mathbf{x}_v - b_v$ whenever $b_v > 0$.*

Proof Given $\mathbf{t} \in [0, 1]^E$, we write $fr(\mathbf{t}) = \{e : 0 < t_e < 1\}$. Let $\mathbf{z} \in [0, 1]^E$ be a solution of $M\mathbf{z} \geq \mathbf{x}$ with minimum size of $fr(\mathbf{z})$. We know that \mathbf{z} exists, because $\mathbf{z} = \mathbf{y}$ satisfies the inequality.

First we show that the subgraph H of G induced by the edges in $fr(\mathbf{z})$ is acyclic. If H contains a cycle C , then we claim that there exists a non-trivial vector $\alpha \in \mathbb{R}^E$ with its support contained in $E(C)$ such that $M\alpha \geq 0$. Indeed, let M' be obtained from M by taking the columns corresponding to $E(C)$ and rows corresponding to $V(C)$. As C is a cycle, M' is a square matrix. Since $m_{ve} = 0$ whenever $v \notin e$, finding our vector α with support contained in $E(C)$ is equivalent to finding a nontrivial vector α' satisfying $M'\alpha' \geq 0$. If M' is singular, then α' may be taken from the nullspace of M' . On the other hand, if M' is invertible, then α' may be taken as the unique solution to $M'\alpha' = e_u$, where e_u is the unit vector corresponding to some vertex $u \in V(C)$. In both cases, we find our vector α' and hence also our vector α .

Now, we can choose a value $p > 0$ such that $\mathbf{z} + p\alpha \in [0, 1]^E$ and $|fr(\mathbf{z} + p\alpha)| < |fr(\mathbf{z})|$. Then, we have $M(\mathbf{z} + p\alpha) = M\mathbf{z} + pM\alpha \geq M\mathbf{z}$, which contradicts our choice of \mathbf{z} . Thus, we conclude that the subgraph H of G induced by $fr(\mathbf{z})$ is acyclic.

Now, if H is non-empty, then H has a leaf v incident to a single edge $e = vu \in E(H)$. If we change the value of \mathbf{z}_e , only the coordinates v and u of $M\mathbf{z}$ change. We can change \mathbf{z}_e to 0 or 1 so that the u -coordinate of $M\mathbf{z}$ increases, while the v -coordinate decreases by less than b_v . We consider $H - e$ and repeat this process until $fr(\mathbf{z})$ is empty, and we write \mathbf{y}' for the new integer vector we obtain from \mathbf{z} . We write $\mathbf{x}' = M\mathbf{y}'$, and by our observation above, $\mathbf{x}'_v \geq \mathbf{x}_v - b_v$ for each $v \in V(G)$, with a strict inequality whenever $b_v > 0$. \square

As a warm-up for our proof of Theorem 1.6, we first prove an easier result, which is an approximation of Conjecture 1.1. By choosing an orientation of G that makes the in-degree and out-degree as equal as possible at each vertex, Theorem 1.6 implies the following result with only a constant error.

Theorem 4.2 *Let G be a graph, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If, for each $v \in V(G)$ we have,*

$$|F(v)| \leq \frac{1}{3} \deg(v) - 1,$$

then G has an F -avoiding orientation.

Proof By Theorem 2.1, we just need to show that there exist an ordering of vertices v_1, \dots, v_n , and a subgraph H of G such that for each vertex $v \in V(G)$, it holds that

$$\deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v) \geq \lfloor \frac{1}{3} \deg(v) \rfloor - 1.$$

We fix an ordering v_1, \dots, v_n of the vertices of G . Also, we define a matrix $M = (m_{ve} : v \in V(G), e \in E(G))$ as follows. For each edge $e = v_i v_j$ with $i < j$, we let the column of M corresponding to e have entries 1 and -2 in the rows corresponding to v_i and v_j , respectively, and we let all other entries in the column be 0. Our task is equivalent to finding a 01-vector $\mathbf{y}' \in \{0, 1\}^E$ such that the vector $\mathbf{x}' = M\mathbf{y}'$ satisfies $\mathbf{x}'_v \geq \lfloor \frac{1}{3} \deg(v) \rfloor - \deg_G^L(v) - 1$. We begin with a vector $\mathbf{y} \in \mathbb{R}^E$ with $1/3$ in each of its entries, and we observe that $\mathbf{x} = M\mathbf{y}$ satisfies

$$\mathbf{x}_v = -\frac{2}{3} \deg_G^L(v) + \frac{1}{3} \deg_G^R(v) = \frac{1}{3} \deg_G(v) - \deg_G^L(v)$$

for each $v \in V(G)$. Hence, by Lemma 4.1, there exists $\mathbf{y}' \in \{0, 1\}^E$ such that the vector $\mathbf{x}' = M\mathbf{y}'$ satisfies

$$\mathbf{x}'_v > \frac{1}{3} \deg(v) - \deg_G^L(v) - 2$$

at each vertex $v \in V(G)$. Equivalently, $\mathbf{x}'_v \geq \lfloor \frac{1}{3} \deg(v) \rfloor - \deg_G^L(v) - 1$ at each v , completing the proof. \square

Using a similar method, we also prove Theorem 1.6, which gives a $\frac{2}{3}$ -approximation of Conjecture 1.2.

Proof of Theorem 1.6 Again, by Theorem 2.1, we just need to show that there exists an ordering v_1, \dots, v_n of $V(G)$, as well as a subgraph H of G such that for each vertex $v \in V(G)$, it holds that

$$\deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v) \geq \lfloor \frac{2}{3} \deg_D^+(v) \rfloor - 1.$$

We fix an ordering v_1, \dots, v_n of $V(G)$, and then define a matrix M as in the proof of Theorem 4.2. To find a subgraph H satisfying the above inequality for each $v \in V(G)$, it is equivalent to find an integer vector $\mathbf{y}' \in \{0, 1\}^E$ such that the vector $\mathbf{x}' = M\mathbf{y}'$ satisfies $\mathbf{x}'_v \geq \lfloor \frac{2}{3} \deg_D^+(v) \rfloor - \deg_G^L(v) - 1$ for each v .

We say that an edge $v_i v_j$ with $i < j$ is a *forward edge* in D if it is oriented from v_i to v_j ; otherwise, we call it a *backward edge* in D . We write $\deg^{L+}(v)$ for the number of backward edges outgoing from v in D , and we write $\deg^{L-}(v)$ for the number of forward edges incoming to v in D . Similarly, we write $\deg^{R+}(v)$ for the number of forward edges outgoing from v in D , and $\deg^{R-}(v)$ for the number of backward edges incoming to v in D .

Let v_1, \dots, v_n be an ordering of $V(G)$ with the minimum possible number of forward edges. We argue that for each $v \in V(G)$, it holds that $\deg^{L+}(v) \geq \deg^{L-}(v)$. Indeed, if this inequality does not hold for v , then we can move v to the first position

in our vertex ordering and decrease the number of forward edges in D by $\deg^{L-}(v) - \deg^{L+}(v) > 0$, a contradiction.

Now, let $y_e = \frac{2}{3}$ for each forward edge e in D , and let $y_e = 0$ for each backward edge e in D . Letting $\mathbf{x} = M\mathbf{y}$, we have

$$\begin{aligned} & \mathbf{x}_v + \deg_G^L(v) - \frac{2}{3} \deg_D^+(v) \\ &= \frac{2}{3} (\deg^{R+}(v) - 2 \deg^{L-}(v)) + (\deg^{L-}(v) + \deg^{L+}(v)) \\ &\quad - \frac{2}{3} (\deg^{L+}(v) + \deg^{R+}(v)) \\ &= \frac{1}{3} (\deg^{L+}(v) - \deg^{L-}(v)) \geq 0. \end{aligned}$$

Therefore, for each $v \in V(G)$, we have $\mathbf{x}_v \geq \frac{2}{3} \deg_D^+(v) - \deg_G^L(v)$. Since $|m_{ve}| \leq 2$ for each entry m_{ev} in M , it follows from Lemma 4.1 that there exists an integer vector $\mathbf{y}' \in \{0, 1\}^E$ such that the vector $\mathbf{x}' = M\mathbf{y}'$ satisfies $\mathbf{x}'_v > \frac{2}{3} \deg_D^+(v) - \deg_G^L(v) - 2$ for each $v \in V(G)$. Equivalently, for each vertex v , we have $\mathbf{x}'_v \geq \lfloor \frac{2}{3} \deg_D^+(v) \rfloor - \deg_G^L(v) - 1$, completing the proof. \square

5 Graphs with a Relaxed Regularity Condition

In this section, we prove Theorem 1.7, which holds for large graphs whose maximum degree is subexponential in terms of their minimum degree. For our proof, we need some preliminary lemmas and definitions. We use the Chernoff bound, a probabilistic tool that can be found in many textbooks (e.g. [25, Chap. 4]).

Lemma 5.1 *Let X be a binomially distributed variable with parameters n and p . Let $\mu = np$, and let $0 < \delta \leq 1$. Then,*

$$\Pr(X > (1 + \delta)\mu) \leq \exp\left(-\frac{1}{3}\delta^2\mu\right)$$

and

$$\Pr(X < (1 - \delta)\mu) \leq \exp\left(-\frac{1}{3}\delta^2\mu\right).$$

Mitzenmacher and Upfal [25] point out that for the first statement of the lemma, it is enough to let $\mu \leq np$, and for the second statement of the lemma, it is enough to let $\mu \geq np$. We use the following corollary of the Chernoff bound.

Lemma 5.2 *Let X be a binomially distributed variable with parameters n and p , and let $0 < \gamma < 1$. Then,*

$$\Pr(|X - pn| \geq \gamma n) \leq 2 \exp\left(-\frac{1}{12}\gamma^2 n\right).$$

Proof First, suppose that $p \geq \frac{1}{2}\gamma$. Since X is a binomially distributed variable with mean of np , we can apply Lemma 5.1 with $\mu = np$ and $\delta = \gamma/2p \leq 1$. Then, by Lemma 5.1, our inequality fails with probability at most $2 \exp\left(-\frac{1}{12} \frac{\gamma^2}{p} n\right) \leq 2 \exp\left(-\frac{1}{12} \gamma^2 n\right)$.

Next, suppose that $p < \frac{1}{2}\gamma$. In this case, we only need to prove the upper bound of the inequality. Since $pn < \gamma n$, we can apply Lemma 5.1 with $\mu = \gamma n$ and $\delta = 1$ to show that the upper bound fails with probability at most $\exp\left(-\frac{1}{3} \gamma n\right) < 2 \exp\left(-\frac{1}{12} \gamma^2 n\right)$. \square

We also use the following well-known symmetric form of the Lovász Local Lemma.

Lemma 5.3 [29] *Let \mathcal{A} be a collection of (bad) events in a probability space. Suppose that each bad event in \mathcal{A} occurs with probability at most p and is independent with all but fewer than D other bad events in \mathcal{A} . If*

$$Dp < 1/e,$$

then with positive probability, no bad event in \mathcal{A} occurs.

Now, we are ready to prove Theorem 1.7. Writing $\alpha = \sqrt{2} - 1$, we can restate Theorem 1.7 as follows.

Theorem 5.4 *Let G be a graph of minimum degree δ and maximum degree $\Delta = e^{o(\delta)}$, and let $F : V(G) \rightarrow 2^{\mathbb{N}}$. If*

$$|F(v)| \leq (\alpha - o(1)) \deg_G(v)$$

for each $v \in V(G)$, then G has an F -avoiding orientation.

In the statement of the theorem, our function $o(1)$ approaches 0 as $\delta \rightarrow \infty$, and the rate at which $o(1)$ approaches 0 depends on how quickly δ dominates $\log \Delta$.

Proof Let $\gamma > 0$ be a fixed value. We randomly choose an ordering of $V(G)$ by using a function ϕ to map each vertex $v \in V(G)$ uniformly at random to the real interval $[0, 1]$ and then letting $V(G)$ be ordered according to the left-right order of the image of $V(G)$ in $[0, 1]$. Since two vertices are mapped to the same value with probability 0, we can assume that all values $\phi(v)$ are distinct.

Consider a vertex $v \in V(G)$, and write $d = \deg_G(v)$. We observe that by Lemma 5.2, it holds with probability at least $1 - 2 \exp\left(-\frac{1}{12} \gamma^2 d\right)$ that

$$(\phi(v) - \gamma)d < \deg_G^L(v) < (\phi(v) + \gamma)d. \quad (8)$$

Now, we probabilistically create a subgraph $H \subseteq G$. If $uv \in E(G)$ and $\phi(u) = x$ and $\phi(v) = y$, then we add the edge uv to H with probability

$$p(x, y) = \begin{cases} \frac{\alpha}{1-\alpha} & \text{if } 0 \leq x \leq \alpha \text{ and } \alpha + \left(\frac{1-\alpha}{\alpha}\right)x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We make some observations about H . First, by construction, a vertex $v \in V(G)$ for which $\phi(v) \leq \alpha$ satisfies $\deg_H^L(v) = 0$, and a vertex $v \in V(G)$ for which $\phi(v) \geq \alpha$ satisfies $\deg_H^R(v) = 0$. Next, consider a vertex $v \in V(G)$, and write $d = \deg_G(v)$. If $\phi(v) \leq \alpha$, then we claim that by our simplified Chernoff bound (Lemma 5.2), it holds with probability at least $1 - 2 \exp(-\frac{1}{12}\gamma^2 d)$ that

$$(\alpha - \phi(v) - \gamma)d \leq \deg_H^R(v) \leq (\alpha - \phi(v) + \gamma)d. \quad (9)$$

Indeed, each neighbor $w \in N(v)$ becomes a right-neighbor of v in H independently with probability

$$\int_0^1 p(\phi(v), y) dy = \frac{\alpha}{1-\alpha} \cdot (1 - \alpha - \frac{1-\alpha}{\alpha} \phi(v)) = \alpha - \phi(v),$$

and hence Lemma 5.2 applies.

On the other hand, if $\phi(v) \geq \alpha$, we claim that Lemma 5.2 implies that, with probability at least $1 - 2 \exp(-\frac{1}{12}\gamma^2 d)$, we have

$$\frac{1}{2}(\phi(v) - \alpha - \gamma)d \leq \deg_H^L(v) \leq \frac{1}{2}(\phi(v) - \alpha + \gamma)d. \quad (10)$$

Indeed, each neighbor w of v becomes a left-neighbor of v in H with probability

$$\int_0^1 p(x, \phi(v)) dx = \left(\frac{\alpha}{1-\alpha} \right)^2 (\phi(v) - \alpha) = \frac{1}{2}(\phi(v) - \alpha),$$

so Lemma 5.2 applies.

Now, depending on the value of $\phi(v)$, we see from (8) and (9) or from (8) and (10) that the distribution on $\deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v)$ is centered around αd . In particular, with probability at least $1 - 4 \exp(-\frac{1}{12}\gamma^2 d)$,

$$(\alpha - 2\gamma)d < \deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v) < (\alpha + 2\gamma)d. \quad (11)$$

Now we apply the Lovász Local Lemma (Lemma 5.3). For each vertex v , we define a bad event B_v to be the event that (11) does not hold for v . We have seen that the probability of each bad event B_v is at most $4 \exp(-\frac{1}{12}\gamma^2 \delta)$. Furthermore, the vertices and edges involved in a given bad event B_v induce a star in G centered at v , and two bad events are independent if and only if their corresponding stars are disjoint. Hence, B_v is dependent with at most Δ^2 other bad events. Hence, by Lemma 5.3, as long as

$$(\Delta^2 + 1) \cdot 4 \exp\left(-\frac{1}{12}\gamma^2 \delta\right) < 1/e,$$

the equation (11) holds for each vertex v with positive probability. Since $\Delta = \exp(o(\delta))$, the inequality holds when δ is sufficiently large, and we hence conclude that with positive probability, our random process chooses a subgraph $H \subseteq G$ that satisfies

(11) at each vertex $v \in V(G)$. Thus, a subgraph H satisfying (11) exists. Therefore, by Theorem 2.1, if each vertex $v \in V(G)$ has a list $F(v)$ of at most $(\alpha - 2\gamma) \deg_G(v)$ forbidden values, then G has an F -avoiding orientation. Letting γ tend to 0 completes the proof. \square

Finally, we show that if we use Theorem 2.1 to prove for some constant β that every regular graph G has an F -avoiding orientation when $|F(v)| \leq \beta \deg(v)$, then $\beta \leq \alpha + o(1)$. In other words, the coefficient $\alpha - o(1)$ in Theorem 5.4 is essentially best possible. To show this, consider the complete graph $G = K_n$, and order its vertices as v_1, \dots, v_n . Suppose that we can use Theorem 2.1 to show that G has an F -avoiding orientation whenever $|F(v)| \leq \beta(n-1)$ at each vertex $v \in V(G)$. We partition $V(K_n)$ into two sets A and B , where A contains all vertices v_i for which $i \leq \lfloor \beta n \rfloor$, and B contains all other vertices. Given an oriented subgraph H of G , we say that the *weight* $w(v)$ of a vertex $v \in V(G)$ is given by

$$w(v) = \deg_G^L(v) - 2 \deg_H^L(v) + \deg_H^R(v).$$

In order to apply Theorem 2.1, the weight of each vertex $v \in V(G)$ must satisfy $w(v) \geq \beta(n-1)$.

Now, let $e(A)$ and $e(B)$ denote the number of edges in $H[A]$ and $H[B]$, respectively, and let $e(A, B)$ denote the number of edges with one endpoint in A and one endpoint in B . We observe that if Theorem 2.1 applies, then the total weight $w(A)$ of all vertices in A satisfies

$$\lfloor \beta n \rfloor \beta(n-1) \leq w(A) = \sum_{i=1}^{\lfloor \beta n \rfloor} (i-1) - e(A) + e(A, B) \leq \frac{1}{2} \lfloor \beta n - 1 \rfloor \lfloor \beta n \rfloor + e(A, B), \quad (12)$$

and the total weight $w(B)$ of all vertices in B satisfies

$$\begin{aligned} (n - \lfloor \beta n \rfloor) \beta(n-1) \leq w(B) &= \sum_{i=\lfloor \beta n \rfloor+1}^n (i-1) \\ &\quad - 2e(A, B) - e(B) \leq \frac{1}{2} \left(n^2 - n - \lfloor \beta n \rfloor^2 - \lfloor \beta n \rfloor \right) - 2e(A, B). \end{aligned} \quad (13)$$

We eliminate the quantity $e(A, B)$ by doubling (12) and adding (13), and then simplifying yields

$$(1 + o(1))n^2 \left(\frac{1}{2} \beta^2 + \beta - \frac{1}{2} \right) \leq 0.$$

This implies that $\beta \leq \alpha + o(1)$. Therefore, the coefficient of $\alpha - o(1)$ appearing in Theorem 5.4 is close to best possible when using Theorem 2.1.

6 Conclusion

It is natural to ask whether Conjecture 1.1 could be proved for every graph G using Theorem 1.4 by finding a suitable orientation D of G , which would give a stronger result. In other words, the following question emerges.

Question 6.1 *Does every graph G have a spanning subgraph with an Alon-Tarsi orientation in which each vertex v has out-degree at least $\frac{1}{2}(\deg_G(v) - 1)$?*

If Question 6.1 has a negative answer, we can still ask the similar question where -1 is replaced with an arbitrary constant $-C$.

If the answer to Question 6.1 is affirmative, then it would imply that every graph G on n vertices and of maximum degree Δ has a subgraph H with at least $|E(G)| - n/2$ edges for which $\text{AT}(H) \leq \lfloor \frac{1}{2}\Delta + \frac{3}{2} \rfloor$. Indeed, given an affirmative answer to Question 6.1 and a graph G , we could first find a subgraph H of G with an Alon-Tarsi orientation D as described in the question. This subgraph necessarily has at least $\sum_{v \in V(G)} \frac{1}{2}(\deg_G(v) - 1) = |E(G)| - n/2$ edges. Then, after reversing the direction given by D for each edge of H , we obtain an Alon-Tarsi orientation of H in which each vertex has out-degree at most $\frac{1}{2}\deg_G(v) + \frac{1}{2}$, implying that $\text{AT}(H) \leq \lfloor \frac{\Delta+3}{2} \rfloor$. This leads us to the following problem.

Question 6.2 *Does there exist a constant C such that every graph G of maximum degree Δ has a matching M such that $\text{AT}(G - M) \leq \frac{1}{2}\Delta + C$?*

If the answer to Question 6.2 is affirmative, it is natural to ask for the optimal value of C . We note that C cannot be smaller than $\frac{1}{2}$, since for any odd n and any maximum matching M in K_n , $\text{AT}(K_n - M) = \frac{n+1}{2}$ (see [17]).

Our requirement that M be a matching rather than an arbitrary set of $n/2$ edges is inspired by the following two results, which both show that if a matching is removed from a graph G , then the upper bound on some coloring parameter of G can be reduced. For planar graphs, Zhu [31] has shown that the Alon-Tarsi number is at most 5 and that this upper bound is tight, but Grytczuk and Zhu [16] have also shown that every planar graph G contains a matching M for which $\text{AT}(G - M) \leq 4$. Furthermore, it is well-known that a graph G of maximum degree Δ satisfies $\chi(G) \leq \Delta + 1$ and that this bound is tight for cliques. However, every graph G of maximum degree Δ contains a matching M for which $\chi(G - M) \leq \lceil \frac{\Delta+1}{2} \rceil$. Indeed, Lovász [22] proved a result on graph coloring which implies, as shown by Jesurum [20], that every graph of maximum degree Δ has a $\lfloor \Delta/k \rfloor$ -defective k -coloring, in which each vertex is adjacent to at most $\lfloor \Delta/k \rfloor$ vertices of the same color. By letting $k = \lceil \frac{\Delta+1}{2} \rceil$, we see that every graph has a 1-defective $\lceil \frac{\Delta+1}{2} \rceil$ -coloring, in which the monochromatically colored edges form a matching. By removing this matching M of monochromatic edges from G , we obtain a graph with chromatic number at most $\lceil \frac{\Delta+1}{2} \rceil$.

References

1. Akbari, S., Dalirrooyfard, M., Ehsani, K., Ozeki, K., Sherkati, R.: Orientations of graphs avoiding given lists on out-degrees. *J. Gr. Theory* **93**(4), 483–502 (2020)

2. Alon, N.: Combinatorial Nullstellensatz. *Comb. Probab. Comput.* **8**(1–2), 7–29 (1999). Recent trends in combinatorics (Mátraháza, 1995)
3. Alon, N., Tarsi, M.: Colorings and orientations of graphs. *Combinatorica* **12**(2), 125–134 (1992)
4. Bondy, J.A., Murty, U.S.R.: *Graph Theory with Applications*. American Elsevier Publishing Co. Inc, New York (1976)
5. Cornuéjols, G.: General factors of graphs. *J. Comb. Theory Ser. B* **45**(2), 185–198 (1988)
6. Crump, I.: Properties of the extended graph permanent. *Commun. Number Theory Phys.* **11**(4), 791–836 (2017)
7. Crump, I., DeVos, M., Yeats, K.: Period preserving properties of an invariant from the permanent of signed incidence matrices. *Ann. Inst. Henri Poincaré D* **3**(4), 429–454 (2016)
8. DeVos, M.: Matrix choosability. *J. Comb. Theory Ser. A* **90**(1), 197–209 (2000)
9. Ding, L., Duh, G.-H., Wang, G., Wong, T.-L., Wu, J., Yu, X., Zhu, X.: Graphs are $(1, \Delta + 1)$ -choosable. *Discret. Math.* **342**(1), 279–284 (2019)
10. Ellingham, M.N., Goddyn, L.: List edge colourings of some 1-factorable multigraphs. *Combinatorica* **16**(3), 343–352 (1996)
11. Frank, A., Chi Lau, L., Szabó, J.: A note on degree-constrained subgraphs. *Discret. Math.* **308**(12), 2647–2648 (2008)
12. Frank, A., Gyárfás, A.: How to orient the edges of a graph? In: *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976)*, Vol. I, volume 18 of *Colloq. Math. Soc. János Bolyai*, pp. 353–364. North-Holland, Amsterdam-New York (1978)
13. Frank, A., Tardos, E., Sebő, A.: Covering directed and odd cuts. *Math. Program. Stud.* **22**, 99–112 (1984). Mathematical programming at Oberwolfach, II (Oberwolfach, 1983)
14. Gottlieb, D.H.: A certain class of incidence matrices. *Proc. Am. Math. Soc.* **17**(6), 1233–1237 (1966)
15. Gross, J.L., Yellen, J.: *Graph theory and its applications. Discrete Mathematics and its Applications (Boca Raton)*, 2nd edn. Chapman & Hall/CRC, Boca Raton, FL (2006)
16. Grytczuk, J., Zhu, X.: The Alon-Tarsi number of a planar graph minus a matching. *J. Comb. Theory Ser. B* **145**, 511–520 (2020)
17. Huang, P.-Y., Wong, T.-L., Zhu, X.: Application of polynomial method to on-line list colouring of graphs. *Eur. J. Comb.* **33**(5), 872–883 (2012)
18. Jaeger, F.: Flows and generalized coloring theorems in graphs. *J. Comb. Theory Ser. B* **26**(2), 205–216 (1979)
19. Jensen, T.R., Toft, B.: *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York (1995). A Wiley-Interscience Publication
20. Jesurum, C.E.: *Coloring with defects*. Master's thesis, MIT (1995)
21. Li, S.-Y.R., Li, W.C.W.: Independence numbers of graphs and generators of ideals. *Combinatorica* **1**(1), 55–61 (1981)
22. Lovász, L.: On decomposition of graphs. *Studia Sci. Math. Hungar.* **1**, 237–238 (1966)
23. Lovász, L.: The factorization of graphs. II. *Acta Math. Acad. Sci. Hungar.* **23**, 223–246 (1972)
24. Lovász, L.M., Thomassen, C., Wu, Y., Zhang, C.-Q.: Nowhere-zero 3-flows and modulo k -orientations. *J. Comb. Theory Ser. B* **103**(5), 587–598 (2013)
25. Mitzenmacher, M., Upfal, E.: *Probability and Computing*, 2nd edn. Cambridge University Press, Cambridge (2017). Randomization and probabilistic techniques in algorithms and data analysis
26. Mubayi, D., Will, T.G., West, D.B.: Realizing degree imbalances in directed graphs. *Discret. Math.* **239**(1–3), 147–153 (2001)
27. Seamone, B.: Bounding the monomial index and $(1, l)$ -weight choosability of a graph. *Discret. Math. Theor. Comput. Sci.* **16**(3), 173–187 (2014)
28. Sebő, A.: General antifactors of graphs. *J. Comb. Theory Ser. B* **58**(2), 174–184 (1993)
29. Shearer, J.B.: On a problem of Spencer. *Combinatorica* **5**(3), 241–245 (1985)
30. Shirazi, H., Verstraëte, J.: A note on polynomials and f -factors of graphs. *Electron. J. Comb.* **15**(1), Note 22, 5 (2008)
31. Zhu, X.: The Alon-Tarsi number of planar graphs. *J. Comb. Theory Ser. B* **134**, 354–358 (2019)
32. Zhu, X.: Every nice graph is $(1, 5)$ -choosable. *J. Comb. Theory Ser. B* **159**, 524–551 (2022)
33. Zhu, X., Balakrishnan, R.: *Combinatorial Nullstellensatz: With Applications to Graph Colouring*. Chapman and Hall/CRC Press, Boca Raton (2021)

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Authors and Affiliations

Peter Bradshaw¹ · Yaobin Chen² · Hao Ma³ · Bojan Mohar⁴ · Hehui Wu²

✉ Hehui Wu
hhwu@fudan.edu.cn

Peter Bradshaw
pb38@illinois.edu

Yaobin Chen
ybchen21@m.fudan.edu.cn

Hao Ma
haoma5@iflytek.com

Bojan Mohar
mohar@sfu.ca

- ¹ Department of Mathematics, University of Illinois Urbana-Champaign, Champaign, USA
- ² Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China
- ³ Faculty of General Education and Foreign Languages, Anhui Institute of Information Technology, Wuhu 241199, China
- ⁴ Department of Mathematics, Simon Fraser University, Burnaby, BC, Canada